



Contents lists available at [ScienceDirect](#)

Journal of Functional Analysis

www.elsevier.com/locate/jfa



Linear inviscid damping and enhanced viscous dissipation of shear flows by using the conjugate operator method



Emmanuel Grenier ^a, Toan T. Nguyen ^b, Frédéric Rousset ^{c,*},
Avy Soffer ^{d,e}

^a *Equipe Projet Inria NUMED, INRIA Rhône Alpes, Unité de Mathématiques Pures et Appliquées, UMR 5669, CNRS et École Normale Supérieure de Lyon, 46, allée d'Italie, 69364 Lyon Cedex 07, France*

^b *Department of Mathematics, Penn State University, State College, PA 16803, United States of America*

^c *Laboratoire de Mathématiques d'Orsay (UMR 8628), Université Paris-Sud et Institut Universitaire de France, 91405 Orsay Cedex, France*

^d *School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, PR China*

^e *Rutgers University, United States of America*

ARTICLE INFO

Article history:

Received 24 April 2018

Accepted 20 September 2019

Available online 10 October 2019

Communicated by Fang Hua-Lin

Keywords:

Inviscid damping

Enhanced dissipation

Linear stability of shear flows

Conjugate operator method

ABSTRACT

We study the large time behavior of solutions to two-dimensional Euler and Navier-Stokes equations linearized about shear flows of the mixing layer type in the unbounded channel $\mathbb{T} \times \mathbb{R}$. Under a simple spectral stability assumption on a self-adjoint operator, we prove a local form of the linear inviscid damping that is uniform with respect to small viscosity. We also prove a local form of the enhanced viscous dissipation that takes place at times of order $\nu^{-1/3}$, ν being the small viscosity. To prove these results, we use a Hamiltonian approach, following the conjugate operator method developed in the study of Schrödinger operators,

* Corresponding author.

E-mail addresses: Emmanuel.Grenier@ens-lyon.fr (E. Grenier), nguyen@math.psu.edu (T.T. Nguyen), frédéric.rousset@math.u-psud.fr (F. Rousset), soffer@math.rutgers.edu (A. Soffer).

combined with a hypocoercivity argument to handle the viscous case.

© 2019 Elsevier Inc. All rights reserved.

1. Introduction

In this paper, we are interested in the long time behavior of solutions to the two-dimensional incompressible Euler and Navier Stokes equations in the vanishing viscosity limit linearized near a stationary shear flow. More precisely, we shall study the following linearized incompressible Euler and Navier-Stokes systems respectively,

$$\partial_t v + (U_s \cdot \nabla)v + (v \cdot \nabla)U_s + \nabla p = 0, \quad \nabla \cdot v = 0, \quad (1.1)$$

and

$$\partial_t v + (U_s \cdot \nabla)v + (v \cdot \nabla)U_s + \nabla p - \nu \Delta v = 0, \quad \nabla \cdot v = 0, \quad (1.2)$$

for $(x, y) \in \mathbb{T} \times \mathbb{R}$ and $\nu > 0$ small, where U_s is a smooth stationary shear flow under the form

$$U_s = \begin{pmatrix} U(y) \\ 0 \end{pmatrix}. \quad (1.3)$$

We shall study at this linearized level the inviscid damping and the enhanced viscous dissipation. In the particular case of the Couette flow, $U(y) = y$, these are well known phenomena that go back to observations by Kelvin and Orr in fluid mechanics and can be justified from an explicit computation in the Fourier space. By denoting by $(v_\alpha^1, v_\alpha^2)_{\alpha \in \mathbb{Z}}$ the Fourier coefficients (taking Fourier series in the x variable) of the velocity $v = (v^1, v^2)$, the inviscid damping is the property that for $\alpha \neq 0$, and for smooth enough initial data, we have in the large time for the solution of (1.1)

$$\|v_\alpha^1(t)\| \lesssim \frac{1}{\alpha t}, \quad \|v_\alpha^2(t)\| \lesssim \frac{|\alpha|}{(\alpha t)^2} \quad (1.4)$$

where throughout the paper $\|\cdot\|$ will stand for the L^2 norm in the $y \in \mathbb{R}$ variable. The main reason for this decay is the mixing phenomenon produced by the free transport operator $y\partial_x$. This property is also true for the solution of (1.2) uniformly with respect to ν . The enhanced dissipation is the property that for the solution of (1.2), we have

$$\|\omega_\alpha(t)\| \lesssim e^{-\nu t^3} \quad (1.5)$$

for ω_α denoting the Fourier coefficients of the vorticity $\omega = \partial_x v^2 - \partial_y v^1$. This shows that the solution of (1.2) is damped by the combination of mixing and viscosity at the

time scale $\nu^{-\frac{1}{3}}$ which is much shorter than the viscous time scale ν^{-1} . For more details, we refer to the introduction of [4]. Note that outstanding results that prove that these properties are still true for solutions of the nonlinear equations close to the Couette flow in strong enough norms have been obtained recently [3–5].

The generalization of these properties to nontrivial shear flows has also received a lot of attention. Small in some sense perturbations of the Couette flow have been studied in [27]. Then (possibly degenerate) monotonic shear flows in bounded channels (that is to say in $\mathbb{T} \times [0, 1]$) were studied in [24,26]. The Kolmogorov flow, that is to say the shear flow $U(y) = \sin y$ in a doubly periodic channel has been studied in [19,25,15]. The related case of radial vortices has been also much studied recently [2,8,17].

In this paper, we shall focus on mixing layers type shear flows in $\mathbb{T} \times \mathbb{R}$. Precisely, we assume that U is smooth and satisfies

$$\forall y \in \mathbb{R}, U'(y) > 0, \quad \lim_{y \rightarrow \pm\infty} U(y) = U_{\pm}, \quad \frac{U''}{U} \in L^{\infty}, \quad \forall y \in \mathbb{R}, \frac{U''}{U}(y) < 0 \quad (\mathbf{H1})$$

for some constants U_{\pm} . Let us comment on the above assumptions on U''/U . A smooth shear flow that satisfies the two first properties necessarily has an inflection point. In view of Rayleigh's inflection point theorem, we therefore have to be careful in order to ensure its linear stability. The classical shear flows for which this can be ensured are the shear flows in the class \mathcal{K}_+ (we use the notation of [18], such flows were already studied by Tollmien [23]) for which we assume that there exists a unique inflection point y_s and that $-U''(y)/(U(y) - U(y_s))$ is bounded and positive. By changing x into $x - ct$, with $c = U(y_s)$, we can always change U into $U - U(y_s)$ in (1.1) and (1.2) so that the last two assumptions in (H1) are verified. We will also make the following mild assumption. Let us set $m = (-U''/U)^{\frac{1}{2}}$ which is well defined (as a real positive function) and smooth thanks to (H1). Assume that

$$\forall k, \exists C_k > 0, \quad |m^{(k)}| \leq C_k m, \quad \lim_{|y| \rightarrow +\infty} m = 0, \quad m \in L^2, \quad \frac{U''}{U - U_{\pm}} \in L^{\infty} \cap L^2, \quad (\mathbf{H2})$$

with $m^{(k)}$ being the k th-order derivatives of m .

Finally, we will make an assumption that ensures the spectral stability of U for (1.1). This means that it excludes the existence of nontrivial solutions of (1.1) such that

$$\omega(t, x, y) = e^{\lambda t} e^{i\alpha x} \Omega(y), \quad \alpha \in \mathbb{Z}^*, \quad \operatorname{Re} \lambda > 0, \quad \Omega \in L^2(\mathbb{R}). \quad (1.6)$$

Note that of course in the presence of such instabilities estimates like (1.4) cannot be true. Let us consider the Schrödinger operator

$$\mathcal{L} = -\partial_y^2 - m^2, \quad D(\mathcal{L}) = H^2(\mathbb{R}) \quad (1.7)$$

and define λ_0 as the infimum of the spectrum of this self-adjoint operator (note that because of (H2), its essential spectrum is $[0, +\infty[$). We assume

$$\lambda_0 > -1. \quad (\mathbf{H3})$$

Note that this assumption is almost sharp. Indeed, if $\lambda_0 < -1$, we get from Theorem 1.5 of [18] that there exist growing modes of the form (1.6) for every $\alpha \in (0, \sqrt{-\lambda_0})$ and in particular for $\alpha = 1$ so that U is unstable on $\mathbb{T} \times \mathbb{R}$.

The main examples of shear profiles $U(y)$ for which assumptions **(H1)-(H3)** are verified are shear flows under the form

$$U(y) = V\left(\frac{y}{L}\right)$$

where we can take V under the form

$$V(z) = \tanh z, \quad \text{or} \quad V(z) = \int_0^z \frac{1}{(1+s^2)^k} ds$$

for k sufficiently large. Assumptions **(H1)** and **(H2)** are easily verified, while Assumption **(H3)** is verified if L is sufficiently large. In the case of the hyperbolic tangent the lowest eigenvalue of \mathcal{L} is explicitly known; precisely, we have $\lambda_0(L) = -\frac{1}{L^2}$ (the associated eigenfunction being $1/\cosh(y/L)$). Hence, **(H3)** is verified as soon as $L > 1$. Again, this is sharp, since if $L < 1$, we get from [18] that the mixing layer is unstable.

The aim of this paper is to show that for shear flows satisfying **(H1)-(H3)** appropriate local versions of (1.4) and (1.5) hold. One of the main purposes of this paper is also to introduce an Hamiltonian approach to prove the inviscid damping, with sharp decay in time following the conjugate operator method, which has been well developed in the study of Hamiltonian operators; for instance, see [1,6,9,14]. This approach is different from the one in [24,26,15] where shear flows in bounded channels are considered. The approach was based on a direct proof of the limiting absorption principle from resolvent constructions. This is also different from the approach of [19] that relies more on an abstract argument like the RAGE theorem and gives qualitative results. Here, we are able to get sharp quantitative estimates. Our approach will rely on a suitable symmetrized version of the linearized Euler equation in vorticity form that we introduce in the next section. We shall then use on this symmetrized formulation the tools developed in the study of abstract Schrödinger equations $i\partial_t \psi = H\psi$. The two main steps in Mourre's method are establishing the Mourre Estimate, and then using abstract theorems to derive various properties of H . The first step is to find a self-adjoint operator A such that the Mourre estimate

$$iE_I[H, A]E_I \geq \theta E_I^2 + K$$

with θ positive, and K a compact operator holds. Here I denotes a finite interval in the support of the spectrum of H , and E_I is the spectral projection of H on that interval. There are also some technical assumptions on the common domain of H and A , and

on the multiple commutators of H with A , which we do not list here. They will not be an issue in this paper. Such A is called conjugate operator by Mourre. From this estimate, it follows that in the interval I , H can have at most finitely many eigenvalues with L^2 eigenvectors. Finally, and most importantly, it follows that in any subinterval, where we can have the above estimate with $K = 0$, the spectrum of H is purely absolutely continuous, that local decay estimate holds and that the corresponding resolvent estimate holds:

$$\| \langle A \rangle^{-a} (E - H)^{-1} \langle A \rangle^{-a} \| < C$$

as an operator from L^2 to L^2 , for all E in the subinterval, provided $a > 1/2$. In this paper, we will be able to get the Mourre estimate with $K = 0$ on any sufficiently small interval in (U_-, U_+) due to the absence of eigenvalues (see Lemma 2.2).

This result had many applications over the last 40 years, and many refinements. The original proof of the local decay estimate by Mourre [20] was done by deriving Differential inequalities for the weighted resolvents. A new, time dependent approach, which also gave time dependent propagation estimates, including minimal and maximal velocity bounds was later developed in [14], [22]. It is this later method that we use in this work. The starting point is the identity

$$\frac{d}{dt} \langle \psi(t), A\psi(t) \rangle = \langle \psi(t), i[H, A]\psi(t) \rangle$$

if $i\partial_t\psi = H\psi$. This identity is then truncated in a suitable way in order to use the Mourre estimate.

A classical example of conjugate operator when H is a differential operator with real symbol $\omega(k)$ is

$$A = \frac{1}{2}(x \cdot \nabla \omega(p) + \nabla \omega(p) \cdot x), \quad p =: -i\nabla_x.$$

The Mourre estimate then holds at every point where the group velocity $\nabla \omega$ does not vanish. In some cases, for example if ω is radial monotone, it suffices to use $A = \frac{1}{2}(x \cdot p + p \cdot x)$. Here H will be a compact perturbation of H_0 which is just the multiplication by the real monotone function $U(y)$, therefore, the conjugate operator will be just $A = i\partial_y$.

The paper is organized as follows. In the two next sections, we describe our mains results. Sections 4 and 5 are devoted to the proof of the main results. Finally, section 6 is devoted to the proof of some technical lemmas.

Throughout the paper we use the notation $\|\cdot\|$ for the $L^2(\mathbb{R})$ norm and $\langle \cdot, \cdot \rangle$ for the real L^2 scalar product:

$$\langle f, g \rangle = \operatorname{Re} \int_{\mathbb{R}} f(y) \bar{g}(y) dy.$$

We also use the notation $\langle A \rangle = (1 + A^2)^{\frac{1}{2}}$ for the symmetric operator $A = i\partial_y$ on L^2 . In addition, for $\alpha \in \mathbb{Z}$, we write $\nabla_\alpha = (\partial_y, i\alpha)^T$ and $\Delta_\alpha = \partial_y^2 - \alpha^2$.

2. Inviscid damping

2.1. Symmetric form of the equation

We shall work with the vorticity form of the equation (1.1). Set $\omega = \partial_x v^2 - \partial_y v^1$, then ω solves

$$\partial_t \omega + U(y) \partial_x \omega - v^2 U''(y) = 0$$

and v^2 can be recovered from ω by $\Delta v^2 = \partial_x \omega$.

Let $\alpha \in \mathbb{Z}$ be the corresponding Fourier variable of x . Taking the Fourier transform in x , we rewrite the above equation in the Fourier space as:

$$i\partial_t \omega_\alpha = \alpha L_0(y) \omega_\alpha, \quad L_0 := U(y) - U''(y) \Delta_\alpha^{-1}, \quad (2.1)$$

in which $\Delta_\alpha = \partial_y^2 - \alpha^2$. When $\alpha = 0$, the problem is reduced to $\partial_t \omega_\alpha = 0$ and therefore no mixing occurs. We shall thus only consider the case when $\alpha \neq 0$. We shall moreover focus on the case $\alpha > 0$. The case $\alpha < 0$ can be handled from the same arguments as below by reversing the direction of propagation. It is then convenient to use a change of time scale in (2.1), we set

$$\omega_\alpha(t, y) = \tilde{\omega}_\alpha(\alpha t, y) \quad (2.2)$$

so that dropping the tilde and the subscript α , we obtain

$$i\partial_t \omega = L_0(y) \omega, \quad L_0 = U(y) - U''(y) \Delta_\alpha^{-1}. \quad (2.3)$$

For convenience, we write

$$L_0 = U(y) \left(1 + m(y)^2 \Delta_\alpha^{-1} \right), \quad m(y) := (-U''(y)/U(y))^{\frac{1}{2}}.$$

Let us introduce the operator

$$\Sigma = 1 + m \Delta_\alpha^{-1} m.$$

Note that Σ depends on α , through Δ_α^{-1} , but we omit to write this dependence explicitly. We observe that Σ is a bounded symmetric operator on L^2 and that $m \Delta_\alpha^{-1} m$ is a compact operator, upon noting that m tends to zero at infinity thanks to (H2). Moreover, mainly thanks to (H3), we also have the following lemma whose proof is given in Section 4.

Lemma 2.1. *Assuming **(H1)-(H3)**, there exists a constant $c_0 > 0$ such that for every $\alpha \in \mathbb{Z}^*$, in the sense of symmetric operators, we have*

$$\Sigma \geq c_0 > 0.$$

We can thus write $\Sigma = S^2$ for some bounded symmetric coercive operator S on L^2 . Moreover, we also have that $S - 1$ is compact since

$$S - 1 = m\Delta_\alpha^{-1}m(1 + S)^{-1}.$$

By setting $\omega = mS^{-1}\psi$, we finally find

$$i\partial_t\psi = H\psi, \quad H = SU(y)S, \quad S = (1 + m\Delta_\alpha^{-1}m)^{\frac{1}{2}} \quad (2.4)$$

with the initial condition $\psi_{/t=0} = \psi_0 = Sm^{-1}\omega_0$. Note that we will always assume that $\psi_0 \in L^2$, which in terms of ω_0 means that ω_0 is decaying sufficiently fast so that $\frac{1}{m}\omega_0 \in L^2$

We also point out that H actually depends on α in a smooth way (since we focus on $|\alpha| \geq 1$). We omit this dependence for notational convenience. All the estimates that we shall give in the following are uniform with respect to α .

2.2. Spectral properties of H and conjugate operator

Let $\sigma(H)$ be the spectrum of H on L^2 . A first useful property is that:

Lemma 2.2. *Assuming **(H1)-(H3)**, we have $\sigma(H) = [U_-, U_+]$ and there is no embedded eigenvalue in $[U_-, U_+]$.*

Again the proof of Lemma 2.2 will be given in Section 4. To exclude eigenvalues and embedded eigenvalues we will adapt the arguments of [18], [19], [21] for the Rayleigh equation in bounded domains.

As an immediate Corollary, we get from the abstract RAGE Theorem that

Corollary 2.3. *For any compact operator C on $L^2(\mathbb{R})$, there holds*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \|Ce^{-itH}\psi_0\|^2 dt = 0$$

for any $\psi_0 \in L^2$.

In terms of the original vorticity function ω , since $\omega = mS^{-1}\psi(t)$ with S being a bounded operator, we observe that for every $\varepsilon > 0$, the operator $C = \langle i\partial_y \rangle^{-\varepsilon}mS^{-1}$ is compact on L^2 , and hence the above result gives

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \|\langle i\partial_y \rangle^{-\varepsilon} \omega(t)\|^2 dt = 0$$

for every ω_0 such that $\frac{1}{m}\omega_0 \in L^2$, where ω solves (2.3). In particular, this yields some sort of time decay for the velocity.

We shall now use the conjugate operator method to get quantitative and more precise versions of this result. We will use $A = i\partial_y$ as a conjugate operator in order to exploit that $U' > 0$. Note that A is a symmetric operator on L^2 .

Observe that $U : \mathbb{R} \rightarrow (U_-, U_+)$ is a diffeomorphism. We can thus define a smooth function F on (U_-, U_+) by $F(u) = U'(U^{-1}(u))$. This yields $F(U(y)) = U'(y)$, for all $y \in \mathbb{R}$. Note that for every compact interval $I \subset (U_-, U_+)$, there exists $\theta_I > 0$ such that $F \geq \theta_I$ on I .

The crucial property that we will prove in Section 4 is the following:

Lemma 2.4. *Assume (H1)-(H3). For every compact interval $I \subset (U_-, U_+)$, there exists a compact operator K such that for every $g \in \mathcal{C}_c^\infty((U_-, U_+), \mathbb{R}_+)$ with the support contained in I , there holds*

$$g(H)i[H, A]g(H) \geq \theta_I g(H)^2 + g(H)Kg(H) \quad (2.5)$$

where $A = i\partial_y$, $[H, A] = HA - AH$, $\theta_I = \min_I F(u) > 0$, and $g(H)$ is defined through the usual functional calculus.

The above lemma is also true with $g(H) = 1_I(H)$ the spectral projection onto I . We have stated the estimate in this way since it will be the one that is the most useful for us. Note that in our simple setting, the commutator $[H, A]$ and the higher iterates are bounded operators.

This localized commutator estimate was introduced in [20] and is well known to have many interesting consequences on the structure of the spectrum of H . Since we know from Lemma 2.2 that there are no eigenvalues, we can get for example from [20], [10], [1] that the limiting absorption principle holds for every interval $I \subset]U_-, U_+[$ and that there is no singular continuous spectrum. Note that when $m \in L^2$, the operator $m\Delta_\alpha^{-1}m$ is in the trace class. This follows directly from the expression of the kernel which is given by

$$\mathcal{K}(y_1, y_2) = \frac{1}{|\alpha|} e^{-|\alpha| |y_1 - y_2|} m(y_1)m(y_2).$$

Thus, H is a trace class perturbation of the multiplication operator by U . In addition, it follows from Kato's Theorem [16] that the continuous spectrum of H is $\sigma_{ac}(H) = [U_-, U_+]$. By combining these facts, we get, again from the Kato's theorem, that the wave operators exist and are complete, which in particular implies the following scattering result:

Corollary 2.5. *Assuming **(H1)-(H3)**, for every $\psi_0 \in L^2$, there exists $\psi_+ \in L^2$ such that*

$$\lim_{t \rightarrow +\infty} \|e^{-itH}\psi_0 - e^{-itU(y)}\psi_+\| = 0.$$

Again, this can be translated into a scattering result in the original unknowns in a weighted L^2 space. In the following we shall focus on the consequences of Lemma 2.4 on time dependent quantitative propagation estimates that are more flexible and in particular that can be also performed for (1.2) for small positive ν .

2.3. Main inviscid result

Our main result for (2.4) is the following:

Theorem 2.1. *Assume **(H1)-(H3)**. For every $k \in \mathbb{N}^*$ and for every compact interval I_0 in $]U_-, U_+[$, there exists a constant $C > 0$ such that for any initial data $\psi_0 \in H^k$, the solution $\psi(t)$ to (2.4) satisfies the estimate*

$$\|\langle A \rangle^{-k} g_{I_0}(H)\psi(t)\| \leq \frac{C}{1+t^k} \|\langle A \rangle^k \psi_0\|,$$

uniformly in $t \geq 0$ and $\alpha \in \mathbb{Z}^*$, where $A = i\partial_y$ and g_{I_0} is any smooth and compactly supported function in I_0 .

The above result can be easily translated in the original velocity coordinates. Indeed, for $\alpha \in \mathbb{Z}^*$, we have

$$\|v_\alpha^1(t)\| \leq \|\langle A \rangle^{-1} \omega_\alpha(\alpha t)\| = \|\langle A \rangle^{-1} mS^{-1} \psi(\alpha t)\| \lesssim \|\langle A \rangle^{-1} \psi(\alpha t)\|$$

since $\langle A \rangle^{-1} mS^{-1} \langle A \rangle$ is a bounded operator. In a similar way, we have

$$\|v_\alpha^2(t)\| \lesssim |\alpha| \|\langle A \rangle^{-2} \psi(\alpha t)\|.$$

Therefore, we obtain the following

Corollary 2.6. *Assume that the initial vorticity is of the form $\omega_0 = mS^{-1} g_{I_0}(H)\psi_0$, for any $\psi_0 \in H^2$ and for any compact interval I_0 in (U_-, U_+) . Then, the solution v to (1.1) satisfies the following estimates*

$$\|v_\alpha^1(t)\| \leq \frac{C}{1+(|\alpha|t)} \|\psi_0\|_{H^1}, \quad \|v_\alpha^2(t)\| \leq \frac{C|\alpha|}{1+(|\alpha|t)^2} \|\psi_0\|_{H^2},$$

uniformly in $t \geq 0$ and $\alpha \in \mathbb{Z}^*$, with v_α being the Fourier transform of v with respect to variable x .

The fact that the finite edges of the spectrum of H at U_{\pm} are not covered is a well-known limitation of the Mourre's theory [20]. In our case, this is a real difficulty that comes from the fact that $U'(y)$ tends to zero at infinity and hence there is no positive lower bound in the Mourre's estimate (2.5).

3. Uniform mixing and enhanced dissipation

We shall now describe our results for the viscous equations (1.2). Again we write the equation in the vorticity form and take the Fourier transform in x , leading to

$$\partial_t \omega + i\alpha L_0(y)\omega - \nu \Delta_\alpha \omega = 0 \quad (3.1)$$

where L_0 is defined in (2.1). As in the inviscid case, we shall focus on the case $\alpha \neq 0$. Let us again set $\omega = mS^{-1}\psi$ to obtain for ψ

$$\partial_t \psi + i\alpha H\psi - \nu S \frac{1}{m} \Delta_\alpha mS^{-1}\psi = 0.$$

Note that we shall not perform the time scaling (2.2), as it is not well adapted to the viscous term. In addition, the dissipation term is no longer symmetric. Nevertheless, it is symmetric up to a very small error. Precisely, we can write the above equation under the form

$$\partial_t \psi + i\alpha H\psi - \nu \Delta_\alpha \psi = \nu R\psi \quad (3.2)$$

in which

$$\begin{aligned} R &= S \frac{1}{m} \Delta_\alpha mS^{-1} - \Delta_\alpha = S \frac{1}{m} \partial_y^2 mS^{-1} - \partial_y^2 \\ &= S \frac{m''}{m} S^{-1} + 2S \frac{m'}{m} \partial_y S^{-1} + (S-1) \partial_y^2 S^{-1} + \partial_y^2 S^{-1} (1-S). \end{aligned} \quad (3.3)$$

As we will see, the right hand-side does not have much influence on the dynamics for times $\nu t \ll 1$.

We shall use the form (3.2) to state our main results. At first we shall establish that the estimates of Theorem 2.1 can be generalized to (3.2) up to the viscous dissipation time scale ν^{-1} . Precisely, we have

Theorem 3.1. *Assume (H1)-(H3). For every $k \in \mathbb{N}^*$ and for every compact interval $I_0 \subset (U_-, U_+)$, there exist positive constants C, M_0 such that for every initial data $\psi_0 \in H^k$ and every $\nu \in (0, 1]$, the solution $\psi(t)$ to (3.2) satisfies the estimate*

$$\|\langle A \rangle^{-k} g_{I_0}(H)\psi(t)\| \leq C \left(\frac{1}{1 + (|\alpha|t)^k} \|\langle A \rangle^k \psi_0\| + (\nu t)^{\frac{1}{2}} e^{M_0 \nu t} \|\psi_0\| \right),$$

uniformly in $t \geq 0$ and $\alpha \in \mathbb{Z}^*$, where $A = i\partial_y$ and g_{I_0} is any smooth and compactly supported function in I_0 .

Note that the above result shows that the estimates of Theorem 2.1 remain valid up to a correction term that is very small as long as $\nu t \ll 1$. One can think that the study of the stability of stationary shear flows $U_s = [U(y), 0]$ in the Navier-Stokes equation is not really pertinent for times larger than ν^{-1} . Indeed, $U_s(y)$ is not an exact stationary solution of the nonlinear Navier-Stokes equation, though it is classical in fluid mechanics to add a small stationary forcing term in the equation so that $U_s(y)$ becomes an exact solution (see [7] for example). The exact shear solution of Navier-Stokes equations (without a forcing) is time dependent, $U_s = [U(t, y), 0]$, with $U(t, y)$ solving the heat equation

$$\partial_t U - \nu \partial_y^2 U = 0, \quad U|_{t=0} = U(y).$$

As long as $\nu t \ll 1$, it does not make much a difference to replace $U(t, y)$ by $U(y)$. Nevertheless, for $\nu t \gtrsim 1$, the stationary profile $U(y)$ is no longer a good approximation, and in particular the derivatives $\partial_y^l U(t, y)$ are damped by the diffusion. This was taken into account for example in the papers [25,15,19]. Let us also point out that our assumptions **(H1)-(H3)** ensure the spectral stability of the shear flows to the Euler equations, but no assumptions were made to ensure the stability to the Navier-Stokes equations for all times (noting that since the channel $\mathbb{T} \times \mathbb{R}$ has no boundary, the result of [12,13] does not apply).

Our last main result is the following local enhanced dissipation for (3.2).

Theorem 3.2. *Assume **(H1)-(H3)**. For every compact interval $I_0 \subset (U_-, U_+)$, there are positive constants C_0, M_0, c_0 such that for every initial data $\psi_0 \in H^1$ and every $\nu \in (0, 1]$, the solution $\psi(t)$ to (3.2) satisfies the estimate*

$$N(t) \leq C_0 \left(e^{-c_0 \nu^{\frac{1}{3}} t} N(0) + (\nu^{\frac{1}{3}} + (\nu t)^{\frac{1}{2}} e^{M_0 \nu t}) (\|\psi_0\| + \|\alpha \psi_0\|) \right)$$

uniformly in $t \geq 0$ and $\alpha \in \mathbb{Z}^*$, where

$$N(t) = \|g_{I_0}(H)\psi(t)\| + \|\alpha g_{I_0}(H)\psi(t)\| + \nu^{\frac{1}{3}} \|\partial_y g_{I_0}(H)\psi(t)\|$$

and g_{I_0} is any smooth and compactly supported function in I_0 .

From the above estimate we see that after localization in a strict spectral subspace of H the solution of (3.2) is damped at the time scale $\nu^{-\frac{1}{3}}$ which is much smaller than the usual viscous dissipation scale ν^{-1} .

4. Proof of the inviscid results

In this section, we shall prove the results stated in Section 2.

4.1. Proof of Lemma 2.1

First, we observe that the essential spectrum of $\Sigma = 1 + m\Delta_\alpha^{-1}m$ on L^2 is reduced to 1 because of the decay assumptions on m in **(H2)**. Thus, it suffices to show that Σ has only positive eigenvalues. Let us assume by contradiction that $\lambda \leq 0$ is an eigenvalue of Σ . That is, there exists a nonzero $\psi \in L^2$ such that

$$\Sigma\psi = \lambda\psi.$$

Set $u = \Delta_\alpha^{-1}m\psi$. Then, $u \in H^2$ and

$$-\lambda(-\Delta_\alpha u) + (\mathcal{L} + \alpha^2)u = 0$$

where $\mathcal{L} = -\partial_y^2 - m^2$ as defined in (1.7). Taking the scalar product with u and integrating by parts, we get from **(H3)** that

$$-\lambda\|\nabla_\alpha u\|^2 + (\lambda_0 + \alpha^2)\|u\|^2 \leq 0$$

with $\nabla_\alpha = (\partial_y, i\alpha)^T$. Since $-\lambda \geq 0$, $\alpha \in \mathbb{Z}^*$, and $\lambda_0 + \alpha^2 > 0$, we get that $u = 0$, which is a contradiction. Lemma 2.1 follows.

4.2. Proof of Lemma 2.2

Since H is a compact perturbation of the multiplication operator by $U(y)$, we first get that $\sigma_{ess}(H) = [U_-, U_+]$. To exclude eigenvalues and embedded eigenvalues we will adapt the arguments of [19,18,21] for the Rayleigh equation in bounded domains. To proceed, let $c \in \mathbb{R}$ be an eigenvalue of H . That is, there exists a nonzero $\psi \in L^2(\mathbb{R})$ such that

$$H\psi = c\psi.$$

Case 1: $c \in \mathbb{R} \setminus [U_-, U_+]$. In view of (2.4), we get that the vorticity $\omega = mS^{-1}\psi \in L^2$ and solves

$$(U - c)\omega = U''\Delta_\alpha^{-1}\omega.$$

Setting $\phi = \Delta_\alpha^{-1}\omega$, we note that $\phi \in H^2$ and solves the Rayleigh equation

$$-\partial_y^2\phi + \frac{U''}{U - c}\phi = -\alpha^2\phi. \quad (4.1)$$

Note that since $c \notin [U_-, U_+]$, $U''/(U - c)$ is not singular. This means that $-\alpha^2 < 0$ is an eigenvalue of the one-dimensional Schrödinger operator $-\partial_y^2 + \frac{U''}{U - c}$. Since the essential

spectrum of this operator is $[0, +\infty[$, we obtain that the bottom of the spectrum is an eigenvalue $\lambda \leq -\alpha^2 < 0$ and that the corresponding eigenvector can be taken positive. Therefore, there exists $v \in H^2$, $v > 0$ such that

$$-\partial_y^2 v - \lambda v = -\frac{U''}{U - c} v. \quad (4.2)$$

Observe that $U''/(U - c)$ belongs to L^1 , since $1/(U - c)$ is bounded and $U'' = -Um^2 \in L^1$, upon recalling from Assumption **(H2)** that U is bounded and $m \in L^2$. By using the Green's function of $-\partial_y^2 - \lambda$, we get from (4.2) that

$$v = -G_{\sqrt{-\lambda}} * \frac{U''}{U - c} v, \quad G_{\sqrt{-\lambda}}(y) = -\frac{1}{\sqrt{-\lambda}} e^{-\sqrt{-\lambda} |y|}.$$

In particular, $v \in L^1$, since

$$\|v\|_{L^1} \lesssim \left\| \frac{U''}{U - c} v \right\|_{L^1} \lesssim \|v\|_{L^\infty} \lesssim \|v\|_{H^1} < +\infty.$$

Finally, we rewrite (4.2) as

$$-\partial_y((U - c)\partial_y v) + \partial_y(U' v) = \lambda(U - c)v.$$

Therefore, we obtain after integration that $\lambda \int_{\mathbb{R}} (U - c)v = 0$, which is a contradiction since $v > 0$ and $U - c$ has a constant sign.

Case 2: $c \in \{U_-, U_+\}$. In this case, we have $U''/(U - c) \in L^\infty \cap L^2$ from Assumption **(H2)**. Hence, again we have $v \in L^1$, since

$$\|v\|_{L^1} \lesssim \left\| \frac{U''}{U - c} v \right\|_{L^1} \lesssim \|v\|_{L^2} < +\infty.$$

We thus arrive at the same contradiction as in the previous case.

Case 3: $c \in (U_-, U_+)$. Let y_c be the point (which is unique since $U' > 0$) such that $U(y_c) = c$ and set $I_- = (-\infty, y_0)$ and $I_+ = (y_0, +\infty)$. As in the previous cases, we get that there exists a nontrivial $\phi \in H^2(\mathbb{R})$ that solves the Rayleigh equation (4.1) on I_{\pm} .

We first prove that we must have $\phi(y_c) \neq 0$. Indeed, assuming otherwise that $\phi(y_c) = 0$ and proceeding as above, we get that at least one of the self-adjoint operators $L_{\pm} = -\partial_y^2 + \frac{U''}{U - c}$ with domain $H^2(I_{\pm}) \cap H_0^1(I_{\pm})$ (which are well defined thanks to the Hardy inequality and the fact that $U' > 0$) has a negative eigenvalue $-\alpha^2$. Therefore, we again find that for one of the intervals I_{\pm} , there exist a negative eigenvalue λ_{\pm} and a positive eigenfunction v_{\pm} such that

$$-\partial_y((U - c)\partial_y v_{\pm}) + \partial_y(U' v_{\pm}) = \lambda_{\pm}(U - c)v_{\pm}, \quad y \in I_{\pm}.$$

We can then also integrate on I_{\pm} to obtain

$$\lambda_{\pm} \int_{I_{\pm}} (U - c)v_{\pm} = 0$$

upon recalling that $U(y_c) = c$ and $v_{\pm}(y_c) = 0$. This yields a contradiction, since $U - c$ and v_{\pm} have a constant sign on I_{\pm} . This proves that $\phi(y_c) \neq 0$.

Next, since $\phi \in H^2(\mathbb{R})$ and solves (4.1), we have $\frac{U''}{U-c}\phi \in L^2$. Together with $\phi(y_c) \neq 0$, we must have $U''(y_c) = 0$. Consequently, we have proven that if $c \in (U_-, U_+)$ is an embedded eigenvalue, we must have $c = U(y_c)$ with $U''(y_c) = 0$. Since we assume that U''/U is strictly negative, we must also have $U(y_c) = 0$ and therefore the only remaining possibility for an embedded eigenvalue is $c = 0$. Going back to the expression of H in (2.4), we immediately see that 0 is not an eigenvalue of H since S is invertible thanks to Lemma 2.1.

4.3. Proof of Lemma 2.4

We shall now turn to the proof of Lemma 2.4. Recall that $H = SUS$ with $S = (1 + m\Delta_{\alpha}^{-1}m)^{\frac{1}{2}}$. Let us write

$$H = U + (S - 1)U(S - 1) + (S - 1)U + U(S - 1)$$

in which we note that $S - 1$ is a compact operator on L^2 , upon noting that $(1 + S)^{-1}$ is bounded, $m\Delta_{\alpha}^{-1}m$ is compact on L^2 , and $S - 1 = m\Delta_{\alpha}^{-1}m(1 + S)^{-1}$. Take $A = i\partial_y$ as the conjugate operator. We obtain

$$i[H, A] = U' + K_1 = F(U) + K_1$$

with $F(U(y)) = U'(y)$ and K_1 a compact operator on L^2 .

Let I be a compact interval in (U_-, U_+) and let g be in $\mathcal{C}_c^{\infty}((U_-, U_+), \mathbb{R}_+)$ with the support contained in I . We then take $\tilde{I} \subset (U_-, U_+)$ to be a slightly bigger interval such that there exists a smooth \tilde{g} with the support contained in \tilde{I} and $\tilde{g} = 1$ on I . Since F is bounded below away from zero on the support of \tilde{g} , we get that there exists a positive constant θ_I such that

$$\tilde{g}(U)i[H, A]\tilde{g}(U) \geq \theta_I \tilde{g}(U)^2 + \tilde{g}(U)K_1\tilde{g}(U). \quad (4.3)$$

We can then write

$$\begin{aligned} g(H)i[H, A]g(H) &= g(H)\tilde{g}(H)i[H, A]\tilde{g}(H)g(H) \\ &= g(H)\tilde{g}(U)i[H, A]\tilde{g}(U)g(H) \\ &\quad + g(H)\left((\tilde{g}(H) - \tilde{g}(U))i[H, A]\tilde{g}(H) + \tilde{g}(U)i[H, A](\tilde{g}(H) - g(U))\right)g(H). \end{aligned}$$

Thus, using (4.3), we get

$$g(H)\tilde{g}(U)i[H, A]\tilde{g}(U)g(H) \geq \theta_I g(H)\tilde{g}(U)^2g(H) \geq \theta_I g(H)^2 + g(H)(\tilde{g}(U)^2 - \tilde{g}(H)^2)g(H).$$

To conclude, it suffices to use that if $f \in \mathcal{C}_c^\infty(\mathbb{R})$, then $f(H) - f(U)$ is a compact operator. We refer to Lemma 6.2, ii).

4.4. Local decay estimates

We shall now prove a propagation estimate that will be crucial for the proof of Theorem 2.1.

Lemma 4.1. *Let I be a compact interval in (U_-, U_+) such that Lemma 2.4 holds. Let $J \subset I$ and g_J be in $\mathcal{C}_c^\infty((U_-, U_+), \mathbb{R}_+)$, having its support contained in J and satisfying*

$$g_J(H)i[H, A]g_J(H) \geq \frac{\theta_I}{2}g_J(H)^2, \quad (4.4)$$

with θ_I as in Lemma 2.4. Then, for every $k \in \mathbb{N}$, there exists a constant C_k so that

$$\|\langle A \rangle^{-k}g_J(H)\psi(t)\| \leq \frac{C_k}{\theta_I^{k+\frac{1}{2}}} \frac{1}{1+t^k} \|\langle A \rangle^k g_J(H)\psi_0\|, \quad (4.5)$$

for every $t \geq 0$ and for every $\psi_0 \in H^k$, where ψ solves (2.4).

Proof. Take $\chi(\xi) = \frac{1}{2}(1 - \tanh \xi)$ and observe that χ has the property that

$$\chi' = -\phi^2, \quad |\phi^{(m)}(\xi)| \leq C_m \phi(\xi), \quad \forall \xi \in \mathbb{R}, \quad \forall m \in \mathbb{N}^* \quad (4.6)$$

where $\phi = 1/(\sqrt{2} \cosh \xi)$. Following the method of [14], we shall use a localized energy estimate. Set $A_{t,s} = \frac{1}{s}(A - a - \theta t)$ for $A = i\partial_y$, $a \in \mathbb{R}$, $s \geq 1$ and $\theta = \frac{\theta_I}{4}$. In what follows, χ and ϕ stand for $\chi(A_{t,s})$, $\phi(A_{t,s})$, respectively, and g_J for $g_J(H)$. These are self-adjoint operators on L^2 , and g_J commutes with H . In addition, all the estimates are uniform in a and $s \geq 1$, and they do not depend on the subinterval J .

Using the equation (2.4) and symmetry properties, we observe that

$$\frac{d}{dt} \|\chi^{\frac{1}{2}}g_J\psi\|^2 = \frac{d}{dt} \langle \chi g_J\psi, g_J\psi \rangle = \frac{\theta}{s} \|\phi g_J\psi\|^2 + \langle i[\chi, H]g_J\psi, g_J\psi \rangle. \quad (4.7)$$

To evaluate the right hand side, we use the commutation formula from [14,11,10], which we recall in Lemma 6.1. For every $p \geq 1$, we get

$$\langle i[\chi, H]f, f \rangle = -\frac{1}{s} \langle \phi^2 i[A, H]f, f \rangle + \sum_{j=2}^{p-1} \frac{1}{j!} \frac{1}{s^j} \langle \chi^{(j)} i ad_A^j H f, f \rangle + \frac{1}{s^p} \langle R_p f, f \rangle \quad (4.8)$$

with $ad_A H = [H, A]$, $ad_A^j H = [ad_A^{j-1} H, A]$, and

$$\|R_p\| \leq C_p \|ad_A^p H\| \leq C_p$$

where $\|\cdot\|$ stands here for the operator norm from L^2 to L^2 . In the next computation, we continue to denote by R_p any bounded operator which is bounded by a harmless constant. For the first term on the right of (4.8), we use again the commutation formula to get

$$\begin{aligned} \frac{1}{s} \langle \phi^2 i[A, H]f, f \rangle &= \frac{1}{s} \langle \phi i[A, H]f, \phi f \rangle \\ &= \frac{1}{s} \langle i[A, H]\phi f, \phi f \rangle + \sum_{j=1}^{p-2} \frac{1}{j!} \frac{1}{s^{j+1}} \langle \phi^{(j)} i ad_A^{j+1} H f, \phi f \rangle + \frac{1}{s^p} \langle R_p f, f \rangle. \end{aligned}$$

For the terms in the above sum, we can use repeatedly the commutation formula to get in the end that

$$\frac{1}{s} \langle \phi^2 i[A, H]f, f \rangle = \frac{1}{s} \langle i[A, H]\phi f, \phi f \rangle + \sum_{j=1}^{p-2} \frac{1}{s^{j+1}} \sum_{k,l} \langle R_k \phi_l f, \phi f \rangle + \frac{1}{s^p} \langle R_p f, f \rangle$$

where in the above sum k, l runs in finite sets and ϕ_l stands for some derivatives of ϕ , which in particular satisfies the estimate $|\phi_l| \lesssim |\phi|$ by using (4.6).

In a similar way, to estimate the other terms in (4.8), we observe that $\chi^{(j)} = -(\phi^2)^{(j-1)}$ can be expanded as a sum of terms under the form $\phi_k \tilde{\phi}_m$ where $\phi_k, \tilde{\phi}_m$ and their derivatives are controlled by ϕ . By using again the commutation formula as many times as necessary, this allows to write an expansion under the form

$$\sum_{j=2}^{p-1} \frac{1}{j!} \frac{1}{s^j} \langle (\phi^2)^{(j-1)} ad_A^j H f, f \rangle = \sum_{j=2}^{p-1} \frac{1}{s^j} \sum_{k,l,m} \langle R_k \phi_l f, \tilde{\phi}_m f \rangle + \frac{1}{s^p} \langle R_p f, f \rangle.$$

In particular, we get from (4.8) and the above expansion formula that for every f (assuming $s \geq 1$)

$$\langle i[\chi, H]f, f \rangle \leq -\frac{1}{s} \langle i[A, H]\phi f, \phi f \rangle + \frac{C_p}{s^2} \|\phi f\|^2 + \frac{C_p}{s^p} \|f\|^2. \quad (4.9)$$

From (4.7), we thus find that

$$\begin{aligned} \frac{d}{dt} \|\chi^{\frac{1}{2}} g_J \psi\|^2 &\leq \frac{1}{s} \left(\theta \|\phi g_J \psi\|^2 - \langle i[A, H]\phi g_J \psi, \phi g_J \psi \rangle \right) + \frac{C_p}{s^2} \|\phi g_J \psi\|^2 + \frac{C_p}{s^p} \|g_J \psi\|^2 \\ &\leq \frac{1}{s} \left(\theta - \frac{\theta_I}{2} \right) \|\phi g_J \psi\|^2 + \frac{C_p}{s^2} \|\phi g_J \psi\|^2 + \frac{C_p}{s^p} \|g_J \psi\|^2 \end{aligned}$$

where we have used (4.4) in the last inequality. Consequently, we can choose $\theta = \theta_I/4$ and s sufficiently large ($s \geq \frac{16C_p}{\theta_I}$) to obtain

$$\frac{d}{dt} \|\chi^{\frac{1}{2}} g_J \psi(t)\|^2 \leq \frac{C_p}{s^p} \|g_J \psi(t)\|^2 \leq \frac{C_p}{s^p} \|g_J \psi_0\|^2 \quad (4.10)$$

upon using

$$\frac{d}{dt} \|g_J(H)\psi\|^2 = 0.$$

Integrating (4.10) between 0 and t and recalling $\chi = \chi(A_{t,s})$, we find that for every t ,

$$\left\| \chi^{\frac{1}{2}} \left(\frac{A - a - \theta t}{s} \right) g_J \psi(t) \right\|^2 \leq \left\| \chi^{\frac{1}{2}} \left(\frac{A - a}{s} \right) g_J \psi_0 \right\|^2 + \frac{C_p t}{s^p} \|g_J \psi_0\|^2$$

uniformly for all $a \in \mathbb{R}$ and $s \geq 1$, with $\theta = \frac{\theta_I}{4}$. In particular for $\theta_I t \geq \frac{1}{\theta_I^2}$, we can take $s = C_p(\theta_I t)^{\frac{1}{2}}$ and $a = -\frac{\theta_I}{8}t$ to obtain

$$\left\| \chi^{\frac{1}{2}} \left(\frac{A - \frac{\theta_I}{8}t}{C_p(\theta_I t)^{\frac{1}{2}}} \right) g_J \psi(t) \right\|^2 \leq \left\| \chi^{\frac{1}{2}} \left(\frac{A + \frac{\theta_I}{8}t}{C_p(\theta_I t)^{\frac{1}{2}}} \right) g_J \psi_0 \right\|^2 + \frac{C_p}{\theta_I(\theta_I t)^{\frac{p}{2}-1}} \|g_J \psi_0\|^2. \quad (4.11)$$

To conclude, for $k \geq 0$, we write that

$$\begin{aligned} \|\langle A \rangle^{-k} g_J \psi(t)\| &\leq \left\| \langle A \rangle^{-k} \chi^{\frac{1}{2}} \left(\frac{A - \frac{\theta_I}{8}t}{C_p(\theta_I t)^{\frac{1}{2}}} \right) g_J \psi(t) \right\| \\ &\quad + \left\| \langle A \rangle^{-k} \left(1 - \chi^{\frac{1}{2}} \left(\frac{A - \frac{\theta_I}{8}t}{C_p(\theta_I t)^{\frac{1}{2}}} \right) \right) g_J \psi(t) \right\|. \end{aligned} \quad (4.12)$$

Let us first estimate the second term on the right. By using $\|g_J(H)\psi(t)\| = \|g_J(H)\psi_0\|$, it suffices to bound in the operator norm

$$\left\| \langle A \rangle^{-k} \left(1 - \chi^{\frac{1}{2}} \left(\frac{A - \frac{\theta_I}{8}t}{C_p(\theta_I t)^{\frac{1}{2}}} \right) \right) \right\| \lesssim \frac{1}{(\theta_I t)^k}.$$

Indeed, the estimate is clear, when $A \geq \theta_I t/16$, due to the factor $\langle A \rangle^{-k}$. In the case when $A \leq \theta_I t/16$, we observe that $1 - \chi^{\frac{1}{2}}$ term can be bounded by $e^{-C(\theta_I t)^{\frac{1}{2}}}$, which is again bounded by the algebraic decay.

Let us now bound the first term on the right of (4.12). Using (4.11) and choosing p sufficiently large, we thus get

$$\|\langle A \rangle^{-k} g_J \psi(t)\| \lesssim \left\| \chi^{\frac{1}{2}} \left(\frac{A + \frac{\theta_I}{8}t}{C_p(\theta_I t)^{\frac{1}{2}}} \right) \langle A \rangle^{-k} \langle A \rangle^k g_J \psi_0 \right\| + \frac{1}{\theta_I^{\frac{1}{2}}} \frac{1}{(\theta_I t)^k} \|g_J \psi_0\|.$$

In the above, the first term on the right is bounded by $C_p(\theta_I t)^{-k} \|\langle A \rangle^k g_J \psi_0\|$ by considering $A \leq -\theta_I t/16$ and $A \geq -\theta_I t/16$ and using the fact that $\chi(\xi)$ decays exponentially to zero as $\xi \rightarrow +\infty$.

Thus, we have obtained

$$\theta_I^{k+\frac{1}{2}} \|\langle A \rangle^{-k} g_J \psi(t)\| \lesssim \frac{1}{1+t^k} \|\langle A \rangle^k g_J \psi_0\|$$

for $\theta_I^3 t \geq 1$. The estimate for $\theta_I^3 t \leq 1$ is clear. The lemma follows. \square

4.5. Proof of Theorem 2.1

Our aim is now to deduce Theorem 2.1 from Proposition 4.1. In order to conclude, we only need to prove that thanks to the spectral properties that we have already established, we can use Lemma 2.4 to localize the problem in a suitable way so that the assumption (4.4) will be matched. Let us take I_0 any closed interval included in (U_-, U_+) and take I such that $I_0 \subset \dot{I}$ and that Lemma 2.4 holds. In particular, for every point $E \in I_0$ and every positive number δ , we can take $g_{E,\delta}$ a smooth function supported in $(E - 2\delta, E + 2\delta)$ and equal to one on $[E - \delta, E + \delta]$. For $g_{E,\delta}(H)$ and for δ small enough so that $(E - 2\delta, E + 2\delta) \subset I$, Lemma 2.4 yields

$$g_{E,\delta}(H) i[H, A] g_{E,\delta}(H) \geq \theta_I g_{E,\delta}(H)^2 + g_{E,\delta}(H) K g_{E,\delta}(H).$$

Let us show that we can take δ sufficiently small such that (4.4) holds for $g_{E,\delta}(H)$. Indeed, since K is compact, we can approximate it by a finite rank operator in the operator norm. Thus, it suffices to prove that for every $\varepsilon > 0$, we get $g_{E,\delta} K g_{E,\delta} \geq -\varepsilon g_{E,\delta}^2$, for sufficiently small δ and for $K = a \otimes b$ a rank one operator. In this case, we then have

$$\langle g_{E,\delta} K g_{E,\delta} f, f \rangle = \langle g_{E,\delta} f, a \rangle \langle g_{E,\delta} f, b \rangle,$$

and therefore by Cauchy-Schwarz

$$\langle g_{E,\delta} K g_{E,\delta} f, f \rangle \geq -\|g_{E,2\delta} a\| \|g_{E,2\delta} b\| \|g_{E,\delta} f\|^2$$

where $g_{E,2\delta}$ is a smooth function supported in $(E - 4\delta, E + 4\delta)$ that is one on the support of $g_{E,\delta}$. The result follows by using that for $c = a, b \in L^2$, thanks to the spectral measure, we can write

$$\|g_{E,2\delta} c\|^2 = \int_{[U_-, U_+]} |g_{E,2\delta}(\lambda)|^2 \langle dE_\lambda c, c \rangle$$

and by using the Lebesgue theorem, upon noting that the measure $\langle dE_\lambda c, c \rangle$ is continuous, thanks to Lemma 2.2. This proves that (4.4), and hence, (4.5) hold for $J = (E - 2\delta, E + 2\delta)$.

Finally, we can cover I_0 by a finite number of such intervals J with $\overline{J} \subset I$ sufficiently small such that (4.4) holds. Take a partition of unity associated to this covering of I_0 . For each J , the estimate (4.5) holds (noting that the constants in the estimate are independent of J). Taking an initial data under the form $g_{I_0}(H)\psi_0$ supported in I_0 , we can then sum the estimate to obtain the final result, Theorem 2.1. Note however that the constants in the final estimate do depend on I_0 and might blow up at the edges of the spectrum of H .

5. Viscous case

We shall now prove Theorem 3.1 and Theorem 3.2. We use the form (3.2) of the equation. To estimate the remainder R defined as in (3.3), we can use again that both S, S^{-1} are bounded operators and

$$S - 1 = m\Delta_\alpha^{-1}m(1 + S)^{-1} = m\Delta_\alpha^{-1}m(1 + (1 + m\Delta_\alpha^{-1}m)^{\frac{1}{2}})^{-1}.$$

Thus, in view of (3.3), we can write

$$R = R^0 + \partial_y R^1, \quad \|R^0\| + \|R^1\| \leq C_R \quad (5.1)$$

for some constant C_R that is independent of ν .

5.1. Basic energy estimate

As a preliminary, we first establish that

Proposition 5.1. *There are positive constants M_0, C such that for every $\nu \in (0, 1]$, the solution of (3.2) satisfies the estimates*

$$\|\psi(t)\| \leq e^{M_0\nu t} \|\psi_0\|, \quad \nu \int_0^t \|\nabla_\alpha \psi\|^2 \leq \|\psi_0\|^2 (1 + C\nu t e^{2M_0\nu t}) \quad (5.2)$$

uniformly for all $t \geq 0$ and $\alpha \in \mathbb{Z}$. Here, $\nabla_\alpha = (\partial_y, i\alpha)^T$.

Note that the above estimates are uniform in α . In addition, when α is large enough, the estimates can be improved in the sense that we could take $M_0 = 0$. However, we shall not use the improvement.

Proof. The proposition is an easy consequence of the fact that H is symmetric. Indeed, taking integration by parts and using (3.2) and (5.1), we obtain that

$$\frac{1}{2} \frac{d}{dt} \|\psi\|^2 + \nu \|\nabla_\alpha \psi\|^2 \leq C\nu (\|\psi\|^2 + \|\psi\| \|\partial_y \psi\|).$$

Using the Young inequality, we thus get

$$\frac{d}{dt} \|\psi\|^2 + \nu \|\nabla_\alpha \psi\|^2 \leq C\nu \|\psi\|^2. \quad (5.3)$$

The first estimate in (5.2) follows from the Gronwall inequality, while the second is obtained by integrating in time the above inequality. \square

5.2. Proof of Theorem 3.1

We proceed as in the proof of Theorem 2.1. We first choose I_0 and I as in Section 4.5 and cover I_0 with a finite number of small intervals such that on each small interval the estimate (4.5) holds. Let us take J to be any of these small intervals. We now proceed as in the proof of Lemma 4.1 by computing

$$\frac{d}{dt} \|\chi^{\frac{1}{2}} g_J \psi\|^2$$

with $g_J = g_J(H)$ and

$$\chi = \chi(A_{\alpha t, \alpha s}), \quad A_{\alpha t, \alpha s} = \frac{A - a - \theta \alpha t}{\alpha s}.$$

Note that the only difference here is that we have replaced t and s by αt and αs , since we did not perform the change of the time scale as in (2.2). We again focus on $\alpha > 0$. As similarly done in (4.7), we obtain for the solution $\psi(t)$ to (3.2)

$$\begin{aligned} \frac{d}{dt} \|\chi^{\frac{1}{2}} g_J \psi\|^2 &= \frac{d}{dt} \langle \chi g_J \psi, g_J \psi \rangle \\ &= \frac{\theta}{s} \|\phi g_J \psi\|^2 + \langle i\alpha[\chi, H]g_J \psi, g_J \psi \rangle + 2\nu \langle \chi g_J \Delta_\alpha \psi, g_J \psi \rangle + 2\nu \langle \chi g_J R \psi, g_J \psi \rangle. \end{aligned} \quad (5.4)$$

We now estimate each term on the right. The first two terms are estimated exactly as done in the proof of Lemma 4.1 or precisely in (4.9), yielding

$$\frac{\theta}{s} \|\phi g_J \psi\|^2 + \langle i\alpha[\chi, H]g_J \psi, g_J \psi \rangle \leq \frac{1}{s} \left(\theta - \frac{\theta_I}{2} + \frac{C_p}{\alpha s} \right) \|\phi g_J \psi\|^2 + \frac{\alpha C_p}{(\alpha s)^p} \|g_J \psi\|^2 \quad (5.5)$$

where C_p is independent of α and s (and ν , of course). Next for the third term on the right of (5.4), we can integrate by parts (observe that χ commutes with ∂_y) to obtain

$$\begin{aligned} &\langle \chi g_J \Delta_\alpha \psi, g_J \psi \rangle \\ &= -\|\nabla_\alpha \chi^{\frac{1}{2}} g_J \psi\|^2 + \langle \chi^{\frac{1}{2}} [g_J, \partial_y] \partial_y \psi, \chi^{\frac{1}{2}} g_J \psi \rangle - \langle \chi^{\frac{1}{2}} [g_J, \partial_y] \psi, \chi^{\frac{1}{2}} \partial_y g_J \psi \rangle \\ &= -\|\nabla_\alpha \chi^{\frac{1}{2}} g_J \psi\|^2 + \langle \chi^{\frac{1}{2}} [[g_J, \partial_y], \partial_y] \psi, \chi^{\frac{1}{2}} g_J \psi \rangle - 2 \langle \chi^{\frac{1}{2}} [g_J, \partial_y] \psi, \chi^{\frac{1}{2}} \partial_y g_J \psi \rangle. \end{aligned}$$

Using the Lemma 6.2 i) to estimate the commutators, we find

$$\langle \chi g_J \Delta_\alpha \psi, g_J \psi \rangle \leq -\|\nabla_\alpha \chi^{\frac{1}{2}} g_J \psi\|^2 + C\|\psi\|(\|\chi^{\frac{1}{2}} g_J \psi\| + \|\partial_y \chi^{\frac{1}{2}} g_J \psi\|).$$

In a similar way, using the decomposition (5.1) and integrating by parts, we get

$$\langle \chi g_J R \psi, g_J \psi \rangle \leq C\|\psi\|(\|\partial_y \chi^{\frac{1}{2}} g_J \psi\| + \|\chi^{\frac{1}{2}} g_J \psi\|).$$

Using $\|\chi^{\frac{1}{2}} g_J\| \lesssim 1$ and the Young inequality, we thus obtain

$$\langle \chi g_J \Delta_\alpha \psi, g_J \psi \rangle + \langle \chi g_J R \psi, g_J \psi \rangle \leq -\frac{1}{2}\|\nabla_\alpha \chi^{\frac{1}{2}} g_J \psi\|^2 + C\|\psi\|^2 \quad (5.6)$$

for some constant C that is independent of ν .

Consequently, putting (5.5) and (5.6) into (5.4), and choosing again $\theta = \theta_I/4$ and s large so that $\alpha s \geq 4C_p/\theta_I$, we obtain

$$\frac{d}{dt}\|\chi^{\frac{1}{2}} g_J \psi\|^2 \leq \frac{\alpha C_p}{(\alpha s)^p} \|g_J \psi\|^2 + C\nu\|\psi\|^2 \leq \frac{\alpha C_p}{(\alpha s)^p} \|g_J \psi\|^2 + C\nu e^{2M_0\nu t} \|\psi_0\|^2 \quad (5.7)$$

where the last estimate comes from Proposition 5.1. On the other hand, using the same commutator estimates as above (now with $\chi = 1$), we also get that

$$\frac{d}{dt}\|g_J \psi\|^2 + \nu\|\nabla_\alpha g_J \psi\|^2 \leq C\nu\|\psi\|^2 \leq C\nu e^{2M_0\nu t} \|\psi_0\|^2 \quad (5.8)$$

which, after an integration in time, yields $\|g_J \psi(t)\| \leq \|g_J \psi_0\| + C(\nu t)^{\frac{1}{2}} e^{M_0\nu t} \|\psi_0\|$. Hence, the inequality (5.7) now becomes

$$\frac{d}{dt}\|\chi^{\frac{1}{2}} g_J \psi\|^2 \leq \frac{\alpha C_p}{(\alpha s)^p} \|g_J \psi_0\|^2 + C\nu e^{2M_0\nu t} \|\psi_0\|^2. \quad (5.9)$$

Finally, for times t such that $\theta_I^3 \alpha t \geq 1$, we integrate (5.9) over $(0, t)$ and take $\alpha s = C_p(\alpha \theta_I t)^{\frac{1}{2}}$ and $a = -\frac{\theta_I \alpha t}{8}$. Recalling $\chi = \chi(A_{\alpha t, \alpha s})$, we obtain

$$\begin{aligned} \left\| \chi^{\frac{1}{2}} \left(\frac{A - \frac{\theta_I}{8} \alpha t}{C_p(\theta_I \alpha t)^{\frac{1}{2}}} \right) g_J \psi(t) \right\|^2 &\leq \left\| \chi^{\frac{1}{2}} \left(\frac{A + \frac{\theta_I}{8} \alpha t}{C_p(\theta_I \alpha t)^{\frac{1}{2}}} \right) g_J \psi_0 \right\|^2 \\ &\quad + \frac{C_p \alpha t}{(\theta_I \alpha t)^{\frac{1}{2}}} \|g_J \psi_0\|^2 + C\nu t e^{2M_0\nu t} \|\psi_0\|^2. \end{aligned}$$

Note that in this estimate C_p is independent of J and θ_I , while C might depend on the compact intervals I_0 and I . From this estimate, we easily deduce in the same way as done in the proof of Lemma 4.1 that

$$\theta_I^{k+\frac{1}{2}} \|\langle A \rangle^{-k} g_J \psi(t)\| \lesssim \frac{1}{(\alpha t)^k} \|\langle A \rangle^k g_J \psi_0\| + C_N(\nu t)^{\frac{1}{2}} e^{M_0\nu t} \|\psi_0\|^2,$$

for times t so that $\theta_I^3 \alpha t \geq 1$. When $\theta_I^3 \alpha t \leq 1$, the estimate is clear. Thus, summing up over a finite number of such small intervals J , we complete the proof.

5.3. Proof of the enhanced dissipation, Theorem 3.2

We again consider a finite number of small intervals J covering I_0 as above so that

$$g_J(H)i[H, A]g_J(H) \geq \frac{\theta_I}{2}g_J(H)^2, \quad (5.10)$$

for $g_J \in \mathcal{C}_c^\infty((U_-, U_+), \mathbb{R}_+)$ with support contained in J . We shall estimate $g_J(H)\psi$, with ψ solving (3.2). We compute

$$\partial_t g_J \psi + i\alpha H g_J \psi - \nu \Delta_\alpha g_J \psi = \nu \mathcal{C}_\nu \psi \quad (5.11)$$

where $\mathcal{C}_\nu = \mathcal{C}_\nu^0 + \partial_y \mathcal{C}_\nu^1$, with

$$\mathcal{C}_\nu^0 = [[g_J, \partial_y], \partial_y] + g_J R^0 + [g_J, \partial_y] R^1, \quad \mathcal{C}_\nu^1 = 2[g_J, \partial_y] + g_J R^1,$$

Here, R^0, R^1 are as in (5.1). In particular, from the commutator estimates, we obtain

$$\|\mathcal{C}_\nu^0\| + \|\mathcal{C}_\nu^1\| \leq C. \quad (5.12)$$

Take again $A = i\partial_y$. The starting point is to compute

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \langle Ag_J \psi, \alpha g_J \psi \rangle &= -\alpha \langle \partial_t g_J \psi, Ag_J \psi \rangle \\ &= \alpha^2 \langle iH g_J \psi, Ag_J \psi \rangle - \nu \alpha \langle \Delta_\alpha g_J \psi, Ag_J \psi \rangle - \nu \alpha \langle \mathcal{C}_\nu \psi, Ag_J \psi \rangle. \end{aligned}$$

The crucial term in the above identity is the first one on the right hand-side. Indeed, thanks to (5.10), we have

$$\alpha^2 \langle iH g_J \psi, Ag_J \psi \rangle = -\frac{1}{2} \alpha^2 \langle i[H, A]g_J \psi, g_J \psi \rangle \leq -\frac{\theta_I}{4} \alpha^2 \|g_J \psi\|^2.$$

For the viscous terms on the right hand-side, we estimate

$$\nu |\alpha| |\langle \Delta_\alpha g_J \psi, Ag_J \psi \rangle| \lesssim \nu \|\Delta_\alpha g_J \psi\| \|\alpha \partial_y g_J \psi\|$$

and after an integration by parts

$$\nu |\alpha| |\langle \mathcal{C}_\nu \psi, Ag_J \psi \rangle| \lesssim \nu \|\alpha \psi\| (\|\partial_y g_J \psi\| + \|\partial_y^2 g_J \psi\|).$$

This yields

$$\begin{aligned}
& -\frac{d}{dt} \langle Ag_J\psi, \alpha g_J\psi \rangle + \frac{\theta_I}{2} \|\alpha g_J\psi\|^2 \\
& \lesssim \nu \|\Delta_\alpha g_J\psi\| \|\alpha \partial_y g_J\psi\| + \nu \|\alpha\psi\| (\|\partial_y g_J\psi\| + \|\partial_y^2 g_J\psi\|).
\end{aligned} \tag{5.13}$$

Next, using (5.8), we have

$$\frac{d}{dt} \|\alpha g_J\psi(t)\|^2 + \nu \|\alpha \nabla_\alpha g_J\psi(t)\|^2 \leq C\nu \|\alpha\psi\|^2. \tag{5.14}$$

It remains to estimate $\|Ag_J\psi\|^2$. Similarly as done above, we get

$$\frac{1}{2} \frac{d}{dt} \|Ag_J\psi\|^2 + \nu \|\nabla_\alpha \partial_y g_J\psi\|^2 \lesssim \nu (\|\psi\| + \|\partial_y \mathcal{C}_\nu^1 \psi\|) \|\partial_y g_J\psi\| + \alpha \| [A, H] g_J\psi \| \|\partial_y g_J\psi\|$$

in which

$$\| [A, H] g_J\psi \| \lesssim \|g_J\psi\|, \quad \|\partial_y \mathcal{C}_\nu^1 \psi\| \lesssim \|\psi\| + \|\partial_y \psi\|.$$

Thus, using the Young inequality, we obtain

$$\frac{d}{dt} \|Ag_J\psi\|^2 + \nu \|\nabla_\alpha \partial_y g_J\psi\|^2 \lesssim \nu (\|\psi\|^2 + \|\partial_y \psi\|^2) + \|\alpha g_J\psi\| \|\partial_y g_J\psi\|. \tag{5.15}$$

To conclude, we shall combine the estimates (5.8), (5.13), (5.14), and (5.15) in a suitable way. We introduce

$$Q(t) = \Gamma^4 (\|g_J\psi(t)\|^2 + \|\alpha g_J\psi(t)\|^2) - \Gamma \nu^{\frac{1}{3}} \langle Ag_J\psi, \alpha g_J\psi \rangle + \nu^{\frac{2}{3}} \|Ag_J\psi(t)\|^2$$

where $\Gamma \geq 1$ is a large parameter (independent of ν and α) that we will choose later. We first observe that if Γ is sufficiently large, $Q(t)$ is equivalent to a weighted H^1 norm. Namely,

$$Q(t) \approx \|g_J\psi(t)\|^2 + \|\alpha g_J\psi(t)\|^2 + \nu^{\frac{2}{3}} \|\partial_y g_J\psi(t)\|^2.$$

We now add up the estimates (5.8), (5.13), (5.14), and (5.15) with the corresponding weight as in $Q(t)$ and use the Young inequality to obtain

$$\frac{d}{dt} Q(t) + c_0 \nu^{\frac{1}{3}} Q(t) \leq C_0 \nu e^{2M_0 \nu t} (\|\psi_0\|^2 + \|\alpha\psi_0\|^2) + C_0 \nu^{\frac{5}{3}} \|\partial_y \psi\|^2 \tag{5.16}$$

for some positive constants C_0, c_0 . Indeed, the left hand side is clear, upon recalling that $|\alpha| \geq 1$. Let us check the right hand side. In view of (5.13), we estimate

$$\begin{aligned}
\Gamma \nu^{\frac{4}{3}} \|\Delta_\alpha g_J\psi\| \|\alpha \partial_y g_J\psi\| & \leq \Gamma^{-1} \nu^{\frac{5}{3}} \|\partial_y^2 g_J\psi\|^2 + \nu \|\alpha^2 g_J\psi\|^2 \\
& + C_0 \Gamma^2 \nu (\Gamma + \nu^{\frac{2}{3}}) \|\alpha \partial_y g_J\psi\|^2
\end{aligned}$$

$$\begin{aligned} \Gamma \nu^{\frac{4}{3}} \|\alpha \psi\| (\|\partial_y g_J \psi\| + \|\partial_y^2 g_J \psi\|) &\leq \Gamma^{-1} \nu^{\frac{5}{3}} \|\partial_y^2 g_J \psi\|^2 + \nu \|\partial_y g_J \psi\|^2 \\ &\quad + C_0 \Gamma^2 \nu (\Gamma + \nu^{\frac{2}{3}}) \|\alpha \psi\|^2 \end{aligned}$$

in which each term on the right, except the last term involving $\|\alpha \psi\|^2$, is absorbed into the left hand side, precisely the corresponding viscous term, of (5.15), (5.14), and (5.8), upon taking Γ large enough. Similarly, in view of (5.15), we estimate

$$\nu^{\frac{2}{3}} \|\alpha g_J \psi\| \|\partial_y g_J \psi\| \lesssim \nu^{\frac{1}{3}} \|\alpha g_J \psi\|^2 + \nu \|\partial_y g_J \psi\|^2$$

which is again controlled by the left hand side of (5.13) and the viscous term in (5.8), respectively. Thus, we have obtained

$$\frac{d}{dt} Q(t) + c_0 \nu^{\frac{1}{3}} Q(t) \lesssim \nu (\|\psi\|^2 + \|\alpha \psi\|^2) + \nu^{\frac{5}{3}} \|\partial_y \psi\|^2.$$

This yields (5.16), upon using (5.2).

Finally, we integrate the differential inequality (5.16) and use (5.2) again to obtain

$$Q(t) \leq e^{-c_0 \nu^{\frac{1}{3}} t} Q(0) + C_0 (\nu^{\frac{2}{3}} + C \nu t e^{2M_0 \nu t}) (\|\psi_0\|^2 + \|\alpha \psi_0\|^2).$$

Theorem 3.2 follows.

6. Technical lemmas

In this section, we shall recall some commutator estimates used throughout the paper. These results can be found, for instance, in [10,11,14]. The main idea is to use the Helffer-Sjöstrand formula to express the functional calculus of a self-adjoint operator.

Let us start with almost analytic extensions. Let us introduce S^ρ for $\rho \in \mathbb{R}$ the set of \mathcal{C}^∞ functions on \mathbb{R} such that

$$|f^{(m)}(x)| \leq C_m \langle x \rangle^{\rho-m}, \quad \forall x \in \mathbb{R}, \quad \forall m \in \mathbb{N}.$$

We also set

$$\|f\|_\rho = \sup_{x \in \mathbb{R}, m \in \mathbb{N}} \langle x \rangle^{m-\rho} |f^{(m)}(x)|.$$

An almost analytic extension of f is a function \tilde{f} on \mathbb{C} such that

$$\begin{aligned} \tilde{f}|_{\mathbb{R}} &= f, \\ \text{supp } \tilde{f} &\subset \left\{ x + iy, |y| \leq 2\langle x \rangle, x \in \text{supp } f \right\}, \\ |\partial_{\bar{z}} \tilde{f}(z)| &\leq C \langle x \rangle^{\rho-N-1} |y|^N \end{aligned} \tag{6.1}$$

for some N fixed and large enough. As an example, one can take

$$\tilde{f}(x + iy) = \left(\sum_{r=0}^N f^{(r)}(x) \frac{(iy)^r}{r!} \right) \chi \left(\frac{y}{\langle x \rangle} \right)$$

where $\chi(s)$ is a smooth function which is equal to 1 for $|s| \leq 1$ and 0 for $|s| \geq 2$.

Now, let T be a self-adjoint operator (not necessarily bounded, though we only need the results for bounded operators). For any $f \in S^\rho$, we can define the operator $f(T)$ by

$$f(T) = \lim_{R \rightarrow +\infty} \frac{i}{2\pi} \int_{\mathbb{C} \cap |\operatorname{Re} z| \leq R} \partial_{\bar{z}} \tilde{f}(z) (z - T)^{-1} dL(z) \quad (6.2)$$

where $dL(z) = dx dy$ is the Lebesgue measure on \mathbb{C} identified to \mathbb{R}^2 . Observe that when $\rho < 0$, the above integral converges in the operator norm.

Lemma 6.1. [10, 11, 14] *For $k \geq 1$, let $f \in S^\rho$ with $\rho < k$, and let B be a bounded self-adjoint operator on L^2 such that the iterated commutators $\operatorname{ad}_T^j B$, $j \leq k$, are also bounded. Then, there holds the expansion*

$$[f(T), B] = \sum_{j=1}^{k-1} \frac{1}{j!} f^{(j)}(T) \operatorname{ad}_T^j B + R_k(f, T, B)$$

with

$$\|R_k(f, T, B)\| \leq C_k(f) \|\operatorname{ad}_T^k B\|$$

where $C_k(f)$ depends only on k and $\|f\|_\rho$.

In addition, we also use the following:

Lemma 6.2. *Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$, $A = i\partial_y$, and H be the bounded and self-adjoint operator defined as in (2.4). Then, we have*

- (i) $[A, f(H)]$ is a bounded operator.
- (ii) $f(H) - f(U)$ is a compact operator.

Proof. We start with proving (i). Thanks to (6.2) for $f(H)$, we have

$$[A, f(H)] = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) (z - H)^{-1} [A, H] (z - H)^{-1} dL(z),$$

with \tilde{f} an almost analytic extension of f . Since $[A, H]$ is bounded, the result follows directly from the facts that f is compactly supported and that the integral converges in the operator norm thanks to (6.1).

Let us next prove ii). In a similar way, we write

$$f(H) - f(U) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) ((z - H)^{-1} - (z - U)^{-1}) dL(z).$$

Since $H = U + K$ with K compact, the above yields

$$f(H) - f(U) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) (z - H)^{-1} K(z - U)^{-1} dL(z).$$

Again, the integral converges in the operator norm, since $(z - H)^{-1} K(z - U)^{-1}$ is a compact operator for every $z \notin \mathbb{R}$. The result follows. \square

References

- [1] W.O. Amrein, A. Boutet de Monvel, V. Georgescu, *C_0 -Groups, Commutator Methods and Spectral Theory of N -Body Hamiltonians*, Modern Birkhäuser Classics, Birkhäuser/Springer, Basel, 1996; 2013 reprint of the 1996 edition.
- [2] J. Bedrossian, M. Coti Zelati, Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows, *Arch. Ration. Mech. Anal.* 224 (3) (2017) 1161–1204.
- [3] J. Bedrossian, N. Masmoudi, Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations, *Publ. Math. Inst. Hautes Études Sci.* 122 (2015) 195–300.
- [4] J. Bedrossian, N. Masmoudi, V. Vicol, Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the two dimensional Couette flow, *Arch. Ration. Mech. Anal.* 219 (3) (2016) 1087–1159.
- [5] J. Bedrossian, V. Vicol, F. Wang, The Sobolev stability threshold for 2d shear flows near Couette, *J. Nonlinear Sci.* (2016).
- [6] A. Boutet de Monvel-Berthier, V. Georgescu, A. Soffer, N -body Hamiltonians with hard-core interactions, *Rev. Math. Phys.* 6 (4) (1994) 515–596.
- [7] P.G. Drazin, W.H. Reid, *Hydrodynamic Stability*, second edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004.
- [8] T. Gallay, Enhanced dissipation and axisymmetrization of two-dimensional viscous vortices, arXiv, 2017.
- [9] V. Georgescu, M. Larenas, A. Soffer, Abstract theory of pointwise decay with applications to wave and Schrödinger equations, *Ann. Henri Poincaré* 17 (8) (2016) 2075–2101.
- [10] C. Gérard, A proof of the abstract limiting absorption principle by energy estimates, *J. Funct. Anal.* 254 (11) (2008) 2707–2724.
- [11] S. Golénia, T. Jecko, A new look at Mourre's commutator theory, *Complex Anal. Oper. Theory* 1 (3) (2007) 399–422.
- [12] E. Grenier, Y. Guo, Toan T. Nguyen, Spectral instability of general symmetric shear flows in a two-dimensional channel, *Adv. Math.* 292 (2016) 52–110.
- [13] E. Grenier, Y. Guo, Toan T. Nguyen, Spectral instability of characteristic boundary layer flows, *Duke Math. J.* 165 (16) (2016) 3085–3146.
- [14] W. Hunziker, I.M. Sigal, A. Soffer, Minimal escape velocities, *Comm. Partial Differential Equations* 24 (11–12) (1999) 2279–2295.
- [15] S. Ibrahim, Y. Maekawa, N. Masmoudi, On pseudospectral bound for non-selfadjoint operators and its application to stability of Kolmogorov flows, arXiv, 2017.
- [16] T. Kato, Perturbation of continuous spectra by trace class operators, *Proc. Japan Acad.* 33 (1957) 260–264.

- [17] T. Li, D. Wei, Z. Zhang, Pseudospectral and spectral bounds for the Oseen vortices operator, arXiv:1701.06269.
- [18] Z. Lin, Instability of some ideal plane flows, SIAM J. Math. Anal. 35 (2) (2003) 318–356.
- [19] Z. Lin, M. Xu, Metastability of Kolmogorov flows and inviscid damping of shear flows, arXiv, 2017.
- [20] E. Mourre, Absence of singular continuous spectrum for certain selfadjoint operators, Comm. Math. Phys. 78 (3) (1980/1981) 391–408.
- [21] D.H. Sattinger, On the Rayleigh problem in hydrodynamic stability, SIAM J. Appl. Math. 15 (1967) 419–425.
- [22] I.M. Sigal, A. Soffer, Local decay estimates and velocity bounds Princeton, preprint, 1988.
- [23] W. Tollmien, Ein Allgemeines Kriterium der Instabilität laminarer Geschwindigkeitsverteilungen, Nachr. Ges. Wiss. Göttingen Math.-Phys. 50 (1935) 79–114.
- [24] D. Wei, Z. Zhang, W. Zhao, Linear inviscid damping for a class of monotone shear flow in Sobolev spaces, Comm. Pure Appl. Math. 71 (4) (2018) 617–687.
- [25] D. Wei, Z. Zhang, W. Zhao, Linear inviscid damping and enhanced dissipation for the Kolmogorov flow, arXiv, 2017.
- [26] D. Wei, Z. Zhang, W. Zhao, Linear inviscid damping and vorticity depletion for shear flows, Ann. PDE 5 (1) (2019), Art. 3.
- [27] C. Zillinger, Linear inviscid damping for monotone shear flows in a finite periodic channel, boundary effects, blow-up and critical Sobolev regularity, Arch. Ration. Mech. Anal. 221 (3) (2016) 1449–1509.