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THE INVISCID LIMIT OF NAVIER–STOKES EQUATIONS FOR VORTEX-WAVE DATA ON $\mathbb{R}^2$*

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This paper is dedicated to Walter Strauss on the occasion of his 80th birthday, as a token of friendship and admiration

Abstract. We establish the inviscid limit of the incompressible Navier–Stokes equations on the whole plane $\mathbb{R}^2$ for initial data having vorticity as a superposition of point vortices and a regular component. In particular, this rigorously justifies the vortex-wave system from the physical Navier–Stokes flows in the vanishing viscosity limit, a model that was introduced by Marchioro and Pulvirenti in the early 90s to describe the dynamics of point vortices in a regular ambient vorticity background. The proof rests on the previous analysis of Gallay in his derivation of the vortex-point system.

Key words. inviscid limit, vortex-wave system, Navier–Stokes

AMS subject classifications. 35Q30, 35Q35

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1. Introduction. In this paper, we are interested in the vanishing viscosity limit of the incompressible Navier–Stokes equations on the plane $\mathbb{R}^2$ for irregular initial data; namely, we consider

$$\begin{aligned} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p^\nu &= \nu \Delta u^\nu, \\ \nabla \cdot u^\nu &= 0 \end{aligned} \tag{1.1}$$

for fluid velocity $u^\nu(x,t) \in \mathbb{R}^2$ and pressure $p^\nu(x,t) \in \mathbb{R}$ at $x \in \mathbb{R}^2$ and $t \geq 0$. The interest is to understand the asymptotic behavior of solutions in the inviscid limit $\nu \to 0$.

It is straightforward to show that in the absence of spatial boundaries, regular solutions of the Navier–Stokes equations converge in strong Sobolev norms to the regular solutions of Euler equations as $\nu \to 0$ (see, e.g., [15, 31, 26]). The convergence (in L^2 for velocity fields) also holds for nonsmooth solutions that include vortex patches [5, 6, 3, 26, 30]. The problem is largely open for less regular data [2, 4], or even for regular data in domains with a boundary (see, e.g., [28, 18, 27, 14] and the references therein).

For initial data whose vorticity consists of a finite sum of point vortices (Dirac masses), Gallay [10] proved that the corresponding Navier–Stokes vorticity indeed converges weakly in the inviscid limit to the sum of point vortices whose centers evolve according to the Helmholtz–Kirchhoff point-vortex system. In this paper, we study the case when initial vorticity consists of one point vortex and a regular part. The case of finitely many point vortices can be treated similarly in combination with [10], where the vortex-point interaction is understood.

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Let us now detail the problem. For velocity field $u^\nu = (u_1^\nu, u_2^\nu)$, let $\omega^\nu = \partial_{x_2} u_1^\nu - \partial_{x_1} u_2^\nu$ be the corresponding vorticity. Taking advantage of the divergence-free condition, we can recover the velocity from vorticity through the so-called Biot–Savart law

$$u^\nu = \nabla^\perp \Delta^{-1} \omega^\nu = K \star \omega^\nu, \qquad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \tag{1.2}$$

where $K(x)$ denotes the Green kernel of $\nabla^\perp \Delta^{-1}$, the $\star$ notation stands for the usual convolution in variable $x \in \mathbb{R}^2$, and $a^\perp = (a_2, -a_1)$ for vectors $a \in \mathbb{R}^2$. It follows from (1.1) that the vorticity solves

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu. \tag{1.3}$$

We solve the vorticity equation (1.3), together with (1.2), for initial data of the form

$$\omega^\nu_{|t=0} = \delta_{z_0}(x) + \omega_0^E(x), \tag{1.4}$$

where δ_{z_0} denotes the Dirac delta function centered at $x = z_0$ and ω_0^E is the regular component of vorticity that has compact support and vanishes in a neighborhood of z_0. The existence and uniqueness for 2D Navier–Stokes equations with such initial data, or in fact, more generally, with initial data of finite measures, are known; see, e.g., [7, 12, 15, 9].

1.1. Vortex-wave system. In the inviscid limit, we do not expect the limiting solutions from (1.3)–(1.4) to satisfy Euler equations, even in a weak sense,[1] but rather the following so-called vortex-wave system coined by Marchioro and Pulvirenti [23, 25] in the early 90s:

$$\begin{aligned} &\partial_t \omega^E + (v^E + H) \cdot \nabla \omega^E = 0, \\ &\dot z(t) = v^E(t, z(t)), \\ &\omega^E_{|t=0} = \omega_0^E, \qquad z(0) = z_0, \end{aligned} \tag{1.5}$$

in which $v^E = K \star \omega^E$ and $H = K(\cdot - z(t))$. That is, in the limit, the regular component of vorticity is transported by the full velocity, while the location of the point vortex is propagated by the velocity v^E generated by the regular vorticity ω^E.

The global weak solutions of (1.5) in $L^1 \cap L^\infty$ were already obtained in [23, 25] (see also [17, 8] for an extension to L^p spaces), while their uniqueness is proved for Lipschitz or even bounded data [29, 16], provided that the ambient velocity is constant in a neighborhood of the point vortex. In particular, let us recall the following theorem.

THEOREM 1.1 (see [16]). *Consider initial data $z_0 \in \mathbb{R}$ and $\omega_0^E \in L^1 \cap L^\infty(\mathbb{R}^2)$. Assume that ω_0^E has compact support and is constant in a neighborhood of z_0. Then, there are a unique global solution $(z(t), \omega^E(t))$ to (1.5) and a positive function $R(t)$ so that $\omega^E(t)$ remains constant in the ball centered at the point vortex $z(t)$ with radius $R(t)$ for all times $t \geq 0$. If we assume in addition that $\omega_0^E \in W^{k,p}$ for $kp > 2$ and $p > 1$, then for any $T \geq 0$, there holds that*

$$\sup_{0 \leq t \leq T} \|\omega^E(t)\|_{W^{k,p}} \leq C_T \tag{1.6}$$

for some constant C_T.

[1] In fact, it is not known whether weak solutions to Euler equations exist with point-vortex data [23, 25].

Theorem 1.1 ensures that $H = K(\cdot - z(t))$ remains regular in the support of $\nabla\omega^E(t)$. The stated regularity (1.6) thus follows from that of Euler equations on $\mathbb{R}^2$ [19].

The vortex-wave system (1.5) can be rigorously derived from Euler equations by replacing the initial Dirac mass δ_{z_0} by $\epsilon^{-2}\chi_\epsilon$, with χ_ϵ being the characteristic function of the ball $\{|x - z_0| \le \epsilon\}$ and taking $\epsilon \to 0$. This was done in [24] (see also [1, 13]). It can also be derived from Navier–Stokes equations in the small viscosity limit, provided that $\nu \le \epsilon^\alpha$ for $\alpha > 0$, as done similarly for the vortex-point system [20, 21, 22]. In this paper, we give a direct derivation of (1.5) as the inviscid limit of the Navier–Stokes flows (1.3) with data (1.4).

1.2. Main result. Consider the viscous problem (1.3) with initial data (1.4). Following [9, 10], we first decompose the vorticity into the so-called regular part $\omega^{E,\nu}$ and irregular part $\omega^{B,\nu}$, both of which are advected by the full velocity vector field $u^\nu = K \star \omega^\nu$. Precisely, we write

$$\omega^\nu = \omega^{E,\nu} + \omega^{B,\nu}, \tag{1.7}$$

where $\omega^{E,\nu}$ and $\omega^{B,\nu}$ solve

$$\begin{aligned} \partial_t\omega^{E,\nu} + u^\nu \cdot \nabla\omega^{E,\nu} &= \nu\Delta\omega^{E,\nu}, \\ \omega^{E,\nu}|_{t=0} &= \omega_0^E \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \partial_t\omega^{B,\nu} + u^\nu \cdot \nabla\omega^{B,\nu} &= \nu\Delta\omega^{B,\nu}, \\ \omega^{B,\nu}(t) \quad &\rightharpoonup \quad \delta_{z_0} \quad \text{as} \quad t \to 0^+. \end{aligned} \tag{1.9}$$

Here and in what follows, the weak convergence for finite measures is understood in the following sense: $\mu_n \rightharpoonup \mu$ if and only if

$$\int_{\mathbb{R}^2} \phi d\mu_n \to \int_{\mathbb{R}^2} \phi d\mu$$

for all the continuous functions ϕ that vanish at infinity. A direct computation shows that the decomposition preserves the mass:

$$\int_{\mathbb{R}^2} \omega^{E,\nu}(x,t)dx = \int_{\mathbb{R}^2} \omega_0^E(x)dx, \qquad \int_{\mathbb{R}^2} \omega^{B,\nu}(x,t)dx = 1 \tag{1.10}$$

for all positive times. We shall prove that in the inviscid limit $\omega^{E,\nu} \to \omega^E$ and $\omega^{B,\nu}$ is concentrated near the point vortex $z(t)$, transported by v^E, yielding weak solutions to the vortex-wave system with the same initial data (ω_0^E, z_0). Precisely, our main theorem reads as follows.

THEOREM 1.2. *Let $z_0 \in \mathbb{R}$, and let $\omega_0^E \in W^{4,4}(\mathbb{R}^2)$, which has compact support and vanishes in a neighborhood of z_0, and let $(z(t), \omega^E(t))$ and $\omega^\nu(t)$ be the unique solution to the vortex-wave system* (1.5) *and to the Navier–Stokes equation* (1.3)*, respectively, with initial data $\omega_0 = \omega_0^E + \delta_{z_0}$. Then, there exists a time $T > 0$, independent of ν, such that the vorticity $\omega^\nu(t)$ can be written as*

$$\omega^\nu(x,t) = \omega^{E,\nu}(x,t) + \omega^{B,\nu}(x,t),$$

where $\omega^{E,\nu}(t), \omega^{B,\nu}(t)$ *satisfy*

$$\sup_{0\le t\le T} \|\omega^{E,\nu}(t) - \omega^E(t)\|_{L^4\cap L^{4/3}(\mathbb{R}^2)} \le C_T\nu,$$
$$\sup_{0\le t\le T} t^{-1}\left\|\omega^{B,\nu}(t,x) - \frac{1}{4\pi\nu t}e^{-\frac{|x-z(t)|^2}{4\nu t}}\right\|_{L^1(\mathbb{R}^2)} \le C_T\nu$$

for some constant C_T *independent of* ν. *In particular,* $\omega^{E,\nu}(t) \to \omega^E(t)$ *strongly in* $L^4\cap L^{4/3}$ *and* $\omega^{B,\nu}(t,\cdot) \rightharpoonup \delta_{z(t)}(\cdot)$ *weakly in the sense of finite measures in the inviscid limit.*

Theorem 1.2 derives the vortex-wave system (1.5) as an inviscid limit of Navier–Stokes flows on the whole plane, complementing the earlier derivation [24, 1, 13] from Euler equations. In addition, we obtain

$$T \ge \min\left\{T_*^-, \frac{1}{5\|\nabla v^E\|_{L^\infty}}\right\},$$

with T_* being the smallest time when the point vortex $z(t)$ meets the support of $\omega^E(s)$ for some $s\in[0,t]$, recalling from Theorem 1.1 that $z(t)$ never meets the support of $\omega^E(t)$ for all times. See Proposition 2.1 and Remark 3.15.

Let us now discuss some difficulties in proving the theorem. First, the initial data containing a Dirac mass are too singular to perform a direct proof from the standard L^2 energy estimates. One then needs to construct a good approximation of solutions to treat the singular part and control the remainder. The difficulty arises due to the presence of a vortex-wave interaction term of the form

$$v^{E,\nu}(t,x)\cdot\nabla_x\left(\frac{1}{4\pi\nu t}e^{-\frac{|x-z(t)|^2}{4\nu t}}\right). \tag{1.11}$$

Formally, this term blows up when x is near the point vortex $z(t)$ and $\nu t\to 0$. To treat this singularity, we follow [10] to work in the vortex scaling variable, construct approximate solutions, and perform weighted energy estimates to control the remainder. However, the weighted energy estimates with the scaling variable $\xi = \frac{x-z(t)}{\sqrt{\nu t}}$ used in [10] are not enough to treat the interaction term (1.11), as it leaves a remainder of order one but not smaller. To overcome this difficulty, we introduce an *approximate viscous* vortex-wave system (section 2), along with the new point vortex $\widetilde{z}(t) = z(t) + O(\nu t)$ and the scaled variable $\xi = \frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}$ in order to close the estimate.

Last, we remark that we assume the initial vorticity to be $\delta_{z_0} + \omega_0^E$, where ω_0^E is smooth and compactly supported away from the point vortex z_0. The regularity is needed in the construction of the high-order approximation of solutions. It would be interesting to further combine our analysis with the viscous approximation near vortex-patch solutions constructed in [30] to treat the case when $\omega_0^E\in L^1\cap L^\infty$.

1.3. Notation. We will denote $A\lesssim B$ to mean that $|A|\le C_0|B|$ for some universal constant $C_0>0$ independent of the viscosity ν. We write $f = O(g)$ to mean that $f\lesssim g$, or simply $O(g)$ to mean that the term can be bounded by $C_0|g|$ for some constant $C_0>0$ independent of ν. We define the norms $\|\cdot\|_{L^4\cap L^{4/3}}$ and $\|\cdot\|_{L^1\cap L^\infty}$ of a function $\omega(x)$ in $\mathbb{R}^2$ to be

$$\|\omega\|_{L^4\cap L^{4/3}} = \|\omega\|_{L^4} + \|\omega\|_{L^{4/3}}, \qquad \|\omega\|_{L^1\cap L^\infty} = \|\omega\|_{L^1} + \|\omega\|_{L^\infty}.$$

We also denote by $\mathfrak{m}(\cdot)$ the Lebesgue measure on $\mathbb{R}^2$.

2. Approximate vortex-wave system. Let $(z(t), \omega^E)$ be the global solution to the vortex-wave system (1.5) with initial data $\omega_0^E \in W^{4,4}$ that has compact support and vanishes in a neighborhood of z_0. We introduce an *approximate viscous* vortex-wave system $(\widetilde{z}(t), \widetilde{\omega}^E)$, given by

$$\begin{aligned}\widetilde{\omega}^E(x,t) &= \omega^E(x,t) + \nu w_{1,a}(x,t),\\ \partial_t \widetilde{z} = \widetilde{v}^E(\widetilde{z}(t),t) &= K \star \widetilde{\omega}^E(\widetilde{z}(t),t), \qquad \widetilde{z}(0) = z_0,\end{aligned} \tag{2.1}$$

where the added vorticity component $w_{1,a}$ solves

$$\partial_t w_{1,a} + \left(v^E + \frac{1}{\sqrt{\nu t}} v^G\left(\frac{x - z(t)}{\sqrt{\nu t}}\right)\right) \cdot \nabla w_{1,a} + v_{1,a} \cdot \nabla \omega^E = \Delta \omega^E \tag{2.2}$$

with zero initial data. Here and in what follows, velocity and vorticity are defined through the Biot–Savart law (1.2). For instance, $v_{1,a} = K \star w_{1,a}$ and $v^G(\xi) = \frac{1}{2\pi}\frac{\xi^\perp}{|\xi|^2}(1 - e^{-|\xi|^2/4})$.

We obtain the following simple proposition.

PROPOSITION 2.1. *Let T_* be defined by*

$$T_* = \inf_{t \geq 0} \Big\{ t \ : \ z(t) \in \cup_{0 \leq s \leq t} \mathrm{supp}(\omega^E(s)) \Big\}, \tag{2.3}$$

with $T_* = \infty$ if $z(t)$ never meets the support of $\omega^E(s)$ for $s \in [0,t]$. Then, for any $T < T_*$, the unique smooth solution $w_{1,a}(t)$ of (2.2) exists on $[0,T]$, has compact support, vanishes in a neighborhood of $z(t)$, and satisfies

$$\mathfrak{m}\left(\mathrm{supp}(w_{1,a}(t))\right) + \|w_{1,a}(t)\|_{W^{2,4}(\mathbb{R}^2)} + \|\partial_t w_{1,a}(t)\|_{L^\infty(\mathbb{R}^2)} + \|v_{1,a}(t)\|_{W^{2,\infty}(\mathbb{R}^2)} \leq C_T \tag{2.4}$$

for $t \in [0,T]$ and for some constant C_T independent of ν. In addition, there holds that

$$|\widetilde{z}(t) - z(t)| \leq C_T \nu t \qquad \text{for any} \quad t \in [0,T]. \tag{2.5}$$

Here, $\mathfrak{m}$ denotes the Lebesgue measure on $\mathbb{R}^2$.

COROLLARY 2.2. *Let T_* be defined as in (2.3). For any $T < T_*$, $\widetilde{\omega}^E(t)$ has compact support, vanishes in a neighborhood of $\widetilde{z}(t)$, and satisfies*

$$\mathfrak{m}\left(\mathrm{supp}(\widetilde{\omega}^E(t))\right) + \|\widetilde{\omega}^E(t)\|_{W^{2,4}(\mathbb{R}^2)} + \|\partial_t \widetilde{\omega}^E(t)\|_{L^\infty(\mathbb{R}^2)} + \|\widetilde{v}^E(t)\|_{W^{2,\infty}(\mathbb{R}^2)} \leq C_T \tag{2.6}$$

for $t \in [0,T]$ and for some constant C_T independent of ν.

Proof. The corollary is a direct consequence of Proposition 2.1 and Theorem 1.1. ∎

Proof of Proposition 2.1. Recall from Theorem 1.1 that $\omega^E(t)$ has compact support and vanishes in a neighborhood of $z(t)$. This remains valid for $w_{1,a}(t)$ for small times, due to the transport structure of (2.2). Precisely, $w_{1,a}(t)$ is supported in $\cup_{0 \leq s \leq t} \mathrm{supp}(\omega^E(s))$. Since $z(t) \notin \mathrm{supp}(\omega^E(t))$ for all positive times, we have $T_* > 0$ by continuity. Thus, for any $T < T_*$, there is a positive distance d_T so that

$$|x - z(t)| \geq d_T > 0 \tag{2.7}$$

for all $x \in \operatorname{supp}(w_{1,a}(t))$ and $0 \le t \le T$, which yields

$$\left| \frac{1}{\sqrt{\nu t}} v^G \left(\frac{x - z(t)}{\sqrt{\nu t}} \right) \right| = \frac{1}{2\pi |x - z(t)|} \left(1 - e^{-\frac{|x-z(t)|^2}{4\nu t}} \right) \le \frac{1}{2\pi |x - z(t)|} \le \frac{1}{2\pi d_T}.$$

Similar estimates hold for derivatives of $v^G(\cdot)$ for x away from $z(t)$. It follows from (2.2) that

$$\begin{aligned} \|w_{1,a}(t)\|_{L^4} &\le \int_0^t \left(\|\Delta \omega^E(s)\|_{L^4} + \|v_{1,a}(s)\|_{L^\infty} \|\nabla \omega^E(s)\|_{L^4} \right) ds \\ &\lesssim \int_0^t (1 + \|v_{1,a}(s)\|_{L^\infty}) ds, \end{aligned}$$

which yields the estimate on $w_{1,a}$, upon using the elliptic estimate $\|v_{1,a}\|_{L^\infty} \lesssim \|w_{1,a}\|_{L^4 \cap L^{4/3}}$ and the fact that $w_{1,a}$ is compactly supported. The derivative estimates follow similarly.

Finally, let us prove the estimate on $\widetilde{z}(t)$. By definition, we write

$$\begin{cases} \widetilde{z}(t) = z_0 + \int_0^t \left(v^E(\widetilde{z}(s), s) + \nu v_{1,a}(\widetilde{z}(s), s) \right) ds, \\ z(t) = z_0 + \int_0^t v^E(z(s), s) ds, \end{cases} \tag{2.8}$$

which gives

$$\begin{aligned} |\widetilde{z}(t) - z(t)| &\le \int_0^t \left| (v^E(\widetilde{z}(s), s) - v^E(z(s), s)) \right| ds + \nu \int_0^t |v_{1,a}(\widetilde{z}(s), s)| ds \\ &\le \int_0^t \|\nabla v^E(s)\|_{L^\infty} |\widetilde{z}(s) - z(s)| \, ds + \nu t \sup_{0 \le s \le t} \|v_{1,a}(s)\|_{L^\infty}. \end{aligned} \tag{2.9}$$

Applying Gronwall's lemma gives (2.5). ∎

3. Inviscid limit for the irregular part. In this section, we give estimates on the irregular part of vorticity $\omega^{B,\nu}$, solving (1.9). Let us recall the equation

$$\begin{aligned} \partial_t \omega^{B,\nu} + u^\nu \cdot \nabla \omega^{B,\nu} &= \nu \Delta \omega^{B,\nu}, \\ \omega^{B,\nu}|_{t=0} &= \delta_{z_0}. \end{aligned} \tag{3.1}$$

Here, $u^\nu = v^{E,\nu} + v^{B,\nu}$ is the velocity field for the full Navier–Stokes equations. Following [10], we introduce the change of variables

$$\xi = \frac{x - \widetilde{z}(t)}{\sqrt{\nu t}}$$

and write

$$v^{B,\nu}(x,t) = \frac{1}{\sqrt{\nu t}} v_2(\xi, t), \qquad \omega^{B,\nu}(x,t) = \frac{1}{\nu t} w_2(\xi, t). \tag{3.2}$$

Here, we recall that $\widetilde{z}(t)$ is the solution to the approximate vortex-wave system, given in (2.1). Note that the change of variables is consistent with the Biot–Savart law: $v_2 = K \star_\xi w_2$. Putting the ansatz into (1.9) for $\omega^{B,\nu}$, we get the following equation:

$$\begin{aligned} \Phi(w_2, v^{E,\nu}) := (t\partial_t - \mathcal{L}) w_2 + \sqrt{\frac{t}{\nu}} (v^{E,\nu}(\widetilde{z}(t) + \xi\sqrt{\nu t}, t) - \partial_t \widetilde{z}(t)) \\ \cdot \nabla_\xi w_2 + \frac{1}{\nu} v_2 \cdot \nabla_\xi w_2 = 0, \end{aligned} \tag{3.3}$$

where $\mathcal{L}$ is defined by

$$\mathcal{L}w_2 := \Delta_\xi w_2 + \frac{1}{2}\xi \cdot \nabla_\xi w_2 + w_2.$$

In the vanishing viscosity limit, we expect that the viscous regular velocity will remain close to the inviscid one: $v^{E,\nu} \to v^E$, and hence the irregular part should tend to the so-called Lamb–Oseen vortex, which is defined by

$$G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}, \qquad v^G(\xi) = \frac{1}{2\pi} \frac{\xi^\perp}{|\xi|^2}\Big(1 - e^{-|\xi|^2/4}\Big).$$

It follows that $\mathcal{L}G = 0$ and $v^G \cdot \nabla_\xi G = 0$. Therefore, the pair $(G(\xi), v^{E,\nu})$ solves (3.3), up to the following error term $tR_1(\xi,t)$, with

$$R(\xi,t) := \frac{1}{\sqrt{\nu t}} \Big(v^{E,\nu}(\tilde{z}(t) + \xi\sqrt{\nu t}, t) - \tilde{v}^E(\tilde{z}(t), t) \Big) \cdot \nabla G, \tag{3.4}$$

which does not vanish in the inviscid limit, upon recalling that $\partial_t \tilde{z}(t) = \tilde{v}^E(\tilde{z}(t), t)$. Roughly speaking, $R = \mathcal{O}(1)$ in the small viscosity limit.

We shall construct better approximate solutions to (3.3). Here, we stress that (3.3) involves two unknown functions $w_2, v^{E,\nu}$ which are coupled through the full velocity u^ν. To leading order, let us take $v^{E,\nu}_{app} = \tilde{v}^E$, with $\tilde{v}^E$ solving the approximate vortex-wave system (2.1) and

$$w_{2,\mathrm{app}}(\xi,t) = G(\xi) + (\nu t) w_{2,a}(\xi,t), \tag{3.5}$$

where $w_{2,a}$ is to be defined later. The pair $(w_{2,\mathrm{app}}, v^{E,\nu}_{app})$ thus solves (3.3), leaving an error of the form

$$\begin{aligned}\Phi(w_{2,\mathrm{app}}, v^{E,\nu}_{app}) &= t(\Lambda + \nu(1-\mathcal{L}))w_{2,a} + \nu t^2 \partial_t w_{2,a} + \nu t^2 v_{2,a} \cdot \nabla w_{2,a} \\ &\quad + \sqrt{\nu} t^{3/2} (\tilde{v}^E(\tilde{z}(t) + \xi\sqrt{\nu t}, t) - \tilde{v}^E(\tilde{z}(t), t)) \cdot \nabla w_{2,a} + tR_1(\xi,t),\end{aligned} \tag{3.6}$$

where $R_1(\xi,t)$ is defined as in (3.4) with $v^{E,\nu}_{app} = \tilde{v}^E$, and

$$\Lambda w := v^G \cdot \nabla_\xi w + v \cdot \nabla_\xi G, \qquad v = K \star w.$$

To treat the order one remainder $R_1(\xi,t)$, we first solve $(\Lambda + \nu(1-\mathcal{L}))w_{2,a} = -R_1$ to leading order in ν. We recall the following proposition from [10, Lemma 5 and Remark 1].

PROPOSITION 3.1. *Let $z = z(\xi)$ be a function of the form*

$$z(\xi) = a_1(r)\cos(2\theta) + a_2(r)\sin(2\theta) + a_3(r)\cos(3\theta) + a_4(r)\sin(3\theta)$$

for $\xi = re^{i\theta}$. Assume that the coefficients satisfy

$$\sum_{i=1}^4 (|a_i(r)| + |a_i'(r)|) \le C_0 P(r) e^{-r^2/4} \qquad \forall r > 0$$

for some polynomial $P(r)$. Then, for any $\nu > 0$, there exists a unique solution w^ν to the elliptic equation

$$\Lambda w^\nu + \nu(1-\mathcal{L})w^\nu = z$$

such that

$$|w^\nu(\xi)| + |\nabla w^\nu(\xi)| \le C_\gamma e^{-\gamma|\xi|^2/4}$$

for any $\gamma \in (0,1)$ and for some constant C_γ that is independent of ν.

3.1. Vortex-wave reaction term. In this section, we show that the leading term in the reaction term in (3.4) satisfies the assumption of Proposition 3.1. Precisely, we introduce

$$R_1(\xi,t) = \frac{1}{\sqrt{\nu t}}(\widetilde{v}^E(\widetilde{z}(t)+\xi\sqrt{\nu t},t) - \widetilde{v}^E(\widetilde{z}(t),t))\cdot \nabla G. \tag{3.7}$$

We have the following lemma.

Lemma 3.2. *For any $T>0$, there is a constant C_T so that*

$$|R_1(\xi,t) - A_0(\xi,t)| \le C_T(\nu t)|\xi|^4 e^{-|\xi|^2/4},$$

where

$$\begin{aligned} A_0(\xi,t) = {}& \frac{1}{16\pi^2}|\xi|^2 e^{-|\xi|^2/4}\int_{\mathbb{R}^2}\frac{\sin(2\psi)}{|\widetilde{z}(t)-y|^2}\widetilde{\omega}^E(y,t)dy \\ & - \frac{1}{16\pi^2}\sqrt{\nu t}|\xi|^3 e^{-|\xi|^2/4}\int_{\mathbb{R}^2}\frac{\sin(3\psi)}{|\widetilde{z}(t)-y|^3}\widetilde{\omega}^E(y,t)dy. \end{aligned} \tag{3.8}$$

Here, ψ denotes the angle between ξ and $\widetilde{z}(t)-y$.

Proof. Recalling (3.7) and $G=\frac{1}{4\pi}e^{-|\xi|^2/4}$, and using the Biot–Savart law (1.2), we have

$$\begin{aligned} R_1(\xi,t) &= \frac{-1}{8\pi\sqrt{\nu t}}(\widetilde{v}^E(\widetilde{z}(t)+\xi\sqrt{\nu t},t) - \widetilde{v}^E(\widetilde{z}(t),t))\cdot \xi e^{-|\xi|^2/4} \\ &= \frac{-e^{-|\xi|^2/4}}{16\pi^2\sqrt{\nu t}}\int_{\mathbb{R}^2}\xi\cdot\left(\frac{(\widetilde{z}(t)+\xi\sqrt{\nu t}-y)^\perp}{|\widetilde{z}(t)+\xi\sqrt{\nu t}-y|^2} - \frac{(\widetilde{z}(t)-y)^\perp}{|\widetilde{z}(t)-y|^2}\right)\widetilde{\omega}^E(y,t)dy \\ &= \frac{-e^{-|\xi|^2/4}}{16\pi^2\sqrt{\nu t}}\int_{\mathbb{R}^2}\xi\cdot(\widetilde{z}(t)-y)^\perp\left(\frac{1}{|\widetilde{z}(t)+\xi\sqrt{\nu t}-y|^2} - \frac{1}{|\widetilde{z}(t)-y|^2}\right)\widetilde{\omega}^E(y,t)dy \\ &=: A_1(\xi,t)+A_2(\xi,t), \end{aligned}$$

where $A_1(\xi,t), A_2(\xi,t)$ denote the integrals over $\{|\xi|\sqrt{\nu t}\le \frac12|\widetilde{z}(t)-y|\}$ and $\{|\xi|\sqrt{\nu t}\ge \frac12|\widetilde{z}(t)-y|\}$, respectively. Let us first treat $A_1(\xi,t)$. Applying Lemma A.2 for $|\xi|\sqrt{\nu t}\le \frac12|\widetilde{z}(t)-y|$, we have

$$\frac{1}{|\widetilde{z}(t)+\xi\sqrt{\nu t}-y|^2} - \frac{1}{|\widetilde{z}(t)-y|^2} = \frac{1}{|\widetilde{z}(t)-y|^2}\sum_{n=1}^\infty(-1)^n\frac{|\xi|^n\sqrt{\nu t}^n}{|\widetilde{z}(t)-y|^n}\frac{\sin((n+1)\psi)}{\sin(\psi)}.$$

Here, ψ is the angle between ξ and $\widetilde{z}(t)-y$. Thus, we get

$$\begin{aligned} &\xi\cdot(\widetilde{z}(t)-y)^\perp\left(\frac{1}{|\widetilde{z}(t)+\xi\sqrt{\nu t}-y|^2} - \frac{1}{|\widetilde{z}(t)-y|^2}\right) \\ &= \sum_{n=2}^\infty(-1)^{n+1}(\nu t)^{\frac{n-1}{2}}\frac{|\xi|^n}{|\widetilde{z}(t)-y|^n}\sin(n\psi) \\ &= -(\nu t)^{1/2}\frac{|\xi|^2}{|\widetilde{z}(t)-y|^2}\sin(2\psi) + (\nu t)\frac{|\xi|^3}{|\widetilde{z}(t)-y|^3}\sin(3\psi) \\ &\qquad + \frac{1}{\sqrt{\nu t}}\sum_{n\ge 4}(-1)^{n+1}\frac{(|\xi|\sqrt{\nu t})^n}{|\widetilde{z}(t)-y|^n}\sin(n\psi), \end{aligned}$$

in which we can estimate

$$\left|\frac{1}{\sqrt{\nu t}}\sum_{n\geq 4}(-1)^{n+1}\frac{(|\xi|\sqrt{\nu t})^n}{|\widetilde{z}(t)-y|^n}\sin(n\psi)\right|\leq 2\frac{(\nu t)^{3/2}|\xi|^4}{|\widetilde{z}(t)-y|^4},$$

since $|\xi|\sqrt{\nu t}\leq \frac{1}{2}|\widetilde{z}(t)-y|$. Hence, we have

$$\begin{aligned}A_1(\xi,t) &= \frac{|\xi|^2 e^{-|\xi|^2/4}}{16\pi^2}\int_{|\xi|\sqrt{\nu t}\leq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^2}\sin(2\psi)\widetilde{\omega}^E(y,t)dy\\ &\quad -\frac{\sqrt{\nu t}|\xi|^3 e^{-|\xi|^2/4}}{16\pi^2}\int_{|\xi|\sqrt{\nu t}\leq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^3}\sin(3\psi)\widetilde{\omega}^E(y,t)dy\\ &\quad +\mathcal{O}(\nu t|\xi|^4 e^{-|\xi|^2/4})\int_{|\xi|\sqrt{\nu t}\leq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^4}\sin(4\psi)\widetilde{\omega}^E(y,t)dy.\end{aligned}$$

We note that all the integrals above are bounded by $\|\widetilde{\omega}^E(t)\|_{L^1}$, since $\widetilde{z}(t)$ is bounded away from the support of $\widetilde{\omega}^E(t)$ by Corollary 2.2. Therefore, defining $A_0(\xi,t)$ as in (3.8), we can write

$$\begin{aligned}A_1(\xi,t) &= A_0(\xi,t)-\frac{|\xi|^2 e^{-|\xi|^2/4}}{16\pi^2}\int_{|\xi|\sqrt{\nu t}\geq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^2}\sin(2\psi)\widetilde{\omega}^E(y,t)dy\\ &\quad +\frac{\sqrt{\nu t}|\xi|^3 e^{-|\xi|^2/4}}{16\pi^2}\int_{|\xi|\sqrt{\nu t}\geq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^3}\sin(3\psi)\widetilde{\omega}^E(y,t)dy\\ &\quad +\mathcal{O}(\nu t|\xi|^4 e^{-|\xi|^2/4}).\end{aligned}$$

It remains to treat the integral over the domain $\{|\xi|\sqrt{\nu t}>\frac{1}{2}|\widetilde{z}(t)-y|\}$. Since $\widetilde{z}(t)$ is bounded away from the support of $\widetilde{\omega}^E(t)$, the above (explicitly written) integrals vanish for $|\xi|\sqrt{\nu t}\leq c_T$ for all $t\in[0,T]$ for some constant c_T. On the other hand, for $|\xi|\sqrt{\nu t}\geq c_T$, we have

$$\left|\frac{|\xi|^2 e^{-|\xi|^2/4}}{16\pi^2}\int_{|\xi|\sqrt{\nu t}\geq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^2}\sin(2\psi)\widetilde{\omega}^E(y,t)dy\right|\leq C_T\nu t|\xi|^4 e^{-|\xi|^2/4}\|\widetilde{\omega}^E(t)\|_{L^1}$$

for some constant C_T. Similarly, we also have $A_2(\xi,t)=0$ for $|\xi|\sqrt{\nu t}\leq c_T$ for all $t\in[0,T]$ for some constant c_T, while for $|\xi|\sqrt{\nu t}\geq c_T$, we have

$$\begin{aligned}|A_2(\xi,t)| &\leq |A_1(\xi,t)|+|A(\xi,t)|\\ &\leq C_T|\xi|^2(1+\nu t|\xi|^2)e^{-|\xi|^2/4}\|\widetilde{\omega}^E(t)\|_{L^1}+C_T(\nu t)^{-1/2}|\xi|e^{-|\xi|^2/4}\|\widetilde{v}^E\|_{L^\infty}\\ &\leq C_T(\nu t)|\xi|^4 e^{-|\xi|^2/4},\end{aligned}$$

upon using Corollary 2.2 to bound $\widetilde{v}^E$ and $\widetilde{\omega}^E$. The lemma follows. ◻

3.2. Construction of an approximation solution. We now construct $w_{2,a}$ that solves the following elliptic equation:

$$\Lambda w_{2,a}+\nu(1-\mathcal{L})w_{2,a}=-A_0(\xi,t), \tag{3.9}$$

with $A_0(\xi,t)$ defined as in (3.8). We have the following.

LEMMA 3.3. *There exists a solution $w_{2,a}$ to (3.9) so that, for any $\gamma \in (0,1)$, there holds that*

$$|w_{2,a}(t,\xi)| + |\nabla w_{2,a}(\xi,t)| \le C_\gamma e^{-\gamma|\xi|^2/4}$$

uniformly in $\nu > 0$. In particular, we have

$$\|v_{2,a}(t)\|_{L^\infty} + \int_{\mathbb{R}^2} |w_{2,a}(\xi,t)|^2 e^{|\xi|^2/4} d\xi + \int_{\mathbb{R}^2} |\nabla w_{2,a}(\xi,t)|^2 e^{|\xi|^2/4} d\xi \lesssim 1. \tag{3.10}$$

Proof. For each $y \in \mathbb{R}^2$, we introduce

$$\begin{aligned} B_0(\xi,y,t) = &\frac{-1}{16\pi^2}|\xi|^2 e^{-|\xi|^2/4}\frac{\sin(2\psi)}{|\widetilde{z}(t)-y|^2}\widetilde{\omega}^E(y,t) \\ &+ \frac{1}{16\pi^2}\sqrt{\nu t}|\xi|^3 e^{-|\xi|^2/4}\frac{\sin(3\psi)}{|\widetilde{z}(t)-y|^3}\widetilde{\omega}^E(y,t), \end{aligned} \tag{3.11}$$

recalling ψ the angle between ξ and $\widetilde{z}(t)-y$. If follows from (3.8) that $A_0(\xi,t) = \int_{\mathbb{R}^2} B_0(\xi,y,t)\ dy$. It is clear that for each y, $B_0(\xi,y,t)$ satisfies the assumption of Proposition 3.1, and hence we can define

$$W_{2,a}(\xi,y,t) := \Big(\Lambda + \nu(1-\mathcal{L})\Big)^{-1} B_0(\xi,y,t),$$

stressing that $y \in \mathbb{R}^2$ and $t \ge 0$ play a role as independent parameters. The solution $w_{2,a}$ is thus defined by the average of $W_{2,a}(\xi,y,t)$ with respect to y. The pointwise estimates follow directly from Proposition 3.1 and the estimates on $\widetilde{\omega}^E$. Taking $\gamma > 1/2$ and using the elliptic estimate $\|v_{2,a}\|_{L^\infty} \lesssim \|w_{2,a}\|_{L^1\cap L^\infty}$, we obtain the estimates (3.10). ∎

3.3. Estimating the error term. Construct $w_{2,a}$ as in Lemma 3.3. Then, $w_{2,\mathrm{app}} = G(\xi) + \nu t w_{2,a}$ and $v^{E,\nu}_{\mathrm{app}} = \widetilde{v}^E$ approximately solves (3.3) in the following sense.

PROPOSITION 3.4. *For any $\gamma \in (0,1)$, there holds that*

$$\left|\Phi(w_{2,app}, v^{E,\nu}_{app})(\xi,t)\right| \le C_\gamma \nu t^{3/2} e^{-\gamma|\xi|^2/4} \tag{3.12}$$

for some constant C_γ.

Proof. Fix a $\gamma \in (0,1)$. Using (3.9) into (3.6), we write

$$\begin{aligned} \Phi(w_{2,\mathrm{app}}, v^{E,\nu}_{app})(\xi,t) &= \nu t^2 v_{2,a}\cdot\nabla w_{2,a} + \sqrt{\nu}t^{3/2}(\widetilde{v}^E(\widetilde{z}(t)+\xi\sqrt{\nu t},t) - \widetilde{v}^E(\widetilde{z}(t),t))\cdot\nabla w_{2,a} \\ &\quad + \nu t^2\partial_t w_{2,a} + t(R_1(\xi,t) - A_0(\xi,t)) \\ &=: \sum_{i=1}^4 \Phi_i(\xi,t). \end{aligned}$$

Let us estimate each term on the right-hand side. Using Proposition 2.1 and Lemma 3.3, we get

$$|\Phi_1(\xi,t)| \le \nu t^2\|v_{2,a}(t)\|_{L^\infty}|\nabla w_{2,a}(\xi,t)| \lesssim \nu t^2 e^{-\gamma|\xi|^2/4}.$$

Similarly, using Corollary 2.2, we bound

$$|\widetilde{v}^E(\xi\sqrt{\nu t}+\widetilde{z}(t),t) - \widetilde{v}^E(\widetilde{z}(t),t)| \lesssim |\xi|\sqrt{\nu t}\|\nabla\widetilde{v}^E\|_{L^\infty}$$

and hence

$$\begin{aligned}|\Phi_2(\xi,t)| &\le \sqrt{\nu}t^{3/2}|\widetilde{v}^E(\xi\sqrt{\nu t}+\widetilde{z}(t),t)-\widetilde{v}^E(\widetilde{z}(t),t)||\nabla w_{2,a}(\xi,t)|\\ &\lesssim \nu t^2|\xi|e^{-\gamma'|\xi|^2/4}\\ &\lesssim \nu t^2 e^{-\gamma|\xi|^2/4},\end{aligned}$$

upon taking γ' from Lemma 3.3 so that $\gamma'>\gamma$.

Next, we treat $\Phi_3(\xi,t)=\nu t^2\partial_t w_{2,a}$. Since $\sqrt{t}\partial_t$ commutes with Λ and $\mathcal{L}$, (3.9) gives

$$(\nu(1-\mathcal{L})+\Lambda)\,(\sqrt{t}\partial_t w_{2,a}) = -\sqrt{t}\partial_t A_0(\xi,t).$$

To apply Proposition 3.1, it suffices to prove that

$$\sqrt{t}|\partial_t A_0(\xi,t)|\lesssim |\xi|^2(1+|\xi|)e^{-|\xi|^2/4}. \tag{3.13}$$

Indeed, we recall from (3.11) that

$$\begin{cases}A_0(\xi,t)=\int_{\mathbb{R}^2}B_0(\xi,y,t)dy,\\ B_0(\xi,y,t)=\frac{-1}{16\pi^2}|\xi|^2e^{-|\xi|^2/4}\frac{\sin(2\psi)}{|\widetilde{z}(t)-y|^2}\widetilde{\omega}^E(y,t)+\frac{1}{16\pi^2}\sqrt{\nu t}|\xi|^3e^{-|\xi|^2/4}\dfrac{\sin(3\psi)}{|\widetilde{z}(t)-y|^3}\widetilde{\omega}^E(y,t),\end{cases} \tag{3.14}$$

where ψ is the angle between ξ and $\widetilde{z}(t)-y$. By Corollary 2.2, $\widetilde{\omega}^E(t)$ and $\partial_t\widetilde{\omega}^E(t)$ are both bounded, compactly supported, and vanishing in a neighborhood of $\widetilde{z}(t)$. In particular, $|\widetilde{z}(t)-y|$ is bounded below away from zero for y in the support of $\widetilde{\omega}^E(t)$. The estimate (3.13) thus follows, upon recalling that $\partial_t\widetilde{z}(t)=\widetilde{v}^E(\widetilde{z}(t),t)$ and $\widetilde{v}^E$ is bounded (Corollary 2.2). Arguing similarly as in Lemma 3.3, we obtain

$$|\sqrt{t}\partial_t w_{2,a}(\xi,t)|\le C_\gamma e^{-\gamma|\xi|^2/4}.$$

Finally, the last term $\Phi_4(\xi,t)=t(R_1(\xi,t)-A_0(\xi,t))$ is already treated in Lemma 3.2. This concludes the proof. ∎

3.4. Equations for the remainder. Having introduced the approximate solutions $w_{2,\text{app}}$ and $v^{E,\nu}_{\text{app}}$, let us now study the remainder. Precisely, we search for solutions of (3.3) in the following form:

$$\begin{cases}w_2=G(\xi)+(\nu t)w_{2,a}+(\nu t)\bar{w}_2,\\ v^{E,\nu}=\widetilde{v}^E+\nu^{3/2}\bar{v}_1,\end{cases} \tag{3.15}$$

in which $\widetilde{v}^E$ and $w_{2,a}$ are constructed in the previous sections. Putting this ansatz into (3.3), we have

$$\begin{aligned}&(t\partial_t-\mathcal{L}+1)\bar{w}_2+\frac{1}{\nu}\Lambda\bar{w}_2+\sqrt{\frac{t}{\nu}}(\widetilde{v}^E-\dot{\widetilde{z}})\cdot\nabla\bar{w}_2+t(\bar{v}_2\cdot\nabla w_{2,a}+v_{2,a}\cdot\nabla\bar{w}_2)\\ &+\frac{1}{\sqrt{t}}(\bar{v}_1\cdot\nabla G)+\nu\sqrt{t}(\bar{v}_1\cdot\nabla w_{2,a})+t(\bar{v}_2\cdot\nabla\bar{w}_2)+\nu\sqrt{t}(\bar{v}_1\cdot\nabla\bar{w}_2)\\ &+\frac{1}{\nu t}\Phi(w_{2,app},v^{E,\nu}_{app})=0,\end{aligned} \tag{3.16}$$

in which we stress that $\widetilde{v}^E$ and $\bar{v}_1$ are functions of (x,t), while $G,w_{2,a}$, and $\bar{w}_2$ are functions of ξ,t. Again, velocity and vorticity are defined through the Biot–Savart law in their respective variables.

Our goal is to derive estimates for the remainder solution $(\bar w_2, \bar v_1)$ in suitable function spaces. Precisely, we shall work with the following weighted L^2 norm:

$$\|\omega\|^2_{L^2_p} := \int_{\mathbb{R}^2} |\omega(\xi)|^2 p(\xi) d\xi, \qquad p(\xi) = e^{|\xi|^2/4}.$$

The weight function is natural in view of the following lemma.

LEMMA 3.5. *The operator $\mathcal{L}$ is self-adjoint in L^2_p, while Λ is skew-symmetric in L^2_p. In particular, we have $\mathcal{L} \le 0$ and*

$$\langle \Lambda\omega, \omega\rangle_{L^2_p} = 0$$

for any $\omega(\xi)$ in the domain of Λ.

Proof. The lemma follows from a direct calculation; see [11, Lemma 4.8]. ∎

LEMMA 3.6 (elliptic estimates). *Let $\bar v_2 = K \star_\xi \bar w_2$ be the velocity obtained from $\bar w_2$ by the Biot–Savart law. There holds that*

$$\|\bar v_2\|_{L^\infty} \lesssim \|\bar w_2\|_{L^2_p} + \|\bar w_2\|^{1/2}_{L^2_p} \|\nabla \bar w_2\|^{1/2}_{L^2_p}.$$

Proof. By Hölder's inequality and Sobolev embeddings, we have

$$\begin{aligned}\|\bar v_2\|_{L^\infty} &\lesssim \|\bar w_2\|^{1/2}_{L^{4/3}} \|\bar w_2\|^{1/2}_{L^4} \lesssim \|\bar w_2\|^{1/2}_{L^2_p} \left(\|\bar w_2\|_{L^2_p} + \|\nabla \bar w_2\|_{L^2_p}\right)^{1/2} \\ &\lesssim \|\bar w_2\|^{1/2}_{L^2_p} \left(\|\bar w_2\|^{1/2}_{L^2_p} + \|\nabla \bar w_2\|^{1/2}_{L^2_p}\right) \\ &= \|\bar w_2\|_{L^2_p} + \|\bar w_2\|^{1/2}_{L^2_p} \|\nabla \bar w_2\|^{1/2}_{L^2_p}.\end{aligned}$$

The proof is complete. ∎

3.5. Estimates for the remainder. This section is devoted to proving the following proposition.

PROPOSITION 3.7. *There are a positive constant κ and a positive time T so that*

$$\begin{aligned} t\frac{d}{dt}\|\bar w_2(t)\|^2_{L^2_p} &+ \kappa(\|(1+|\xi|)\bar w_2(t)\|^2_{L^2_p} + \|\nabla \bar w_2(t)\|^2_{L^2_p}) \\ &\lesssim t\|\bar w_2(t)\|^5_{L^2_p} + \nu t \|\bar v_1(t)\|^4_{L^\infty} + t^{-1}\|\bar v_1(t)\|^2_{L^\infty} \end{aligned} \tag{3.17}$$

uniformly in ν and in $t \in [0, T]$.

The proposition follows from weighted energy estimates. To proceed, using (3.16) for $t\partial_t \bar w_2$, we compute

$$t\frac{d}{dt}\|\bar w_2(t)\|^2_{L^2_p} = \int_{\mathbb{R}^2} (t\partial_t \bar w_2(\xi,t))\bar w_2(\xi,t) p(\xi) d\xi = \sum_{i=1}^9 \mathcal{E}_i(t), \tag{3.18}$$

where

$$\begin{cases} \mathcal{E}_1(t) = \int_{\mathbb{R}^2} p(\xi)(\mathcal{L}\bar{w}_2 - \bar{w}_2)(\xi,t)d\xi, \\ \mathcal{E}_2(t) = -\frac{1}{\nu}\int_{\mathbb{R}^2} \Lambda\bar{w}_2(\xi,t)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_3(t) = -\sqrt{\frac{t}{\nu}}\int_{\mathbb{R}^2}((\widetilde{v}^E - \dot{\widetilde{z}})\cdot\nabla\bar{w}_2)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_4(t) = -t\int_{\mathbb{R}^2}(\bar{v}_2\cdot\nabla w_{2,a} + v_{2,a}\cdot\nabla\bar{w}_2)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_5(t) = -t\int_{\mathbb{R}^2}(\bar{v}_2\cdot\nabla\bar{w}_2)\,\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_6(t) = -\nu\sqrt{t}\int_{\mathbb{R}^2}(\bar{v}_1\cdot\nabla\bar{w}_2)\,\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_7(t) = -\frac{1}{\nu t}\int_{\mathbb{R}^2}\Phi_{\text{app}}(\xi,t)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_8(t) = -\frac{1}{\sqrt{t}}\int_{\mathbb{R}^2}(\bar{v}_1\cdot\nabla G)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_9(t) = -\nu\sqrt{t}\int_{\mathbb{R}^2}(\bar{v}_1\cdot\nabla w_{2,a})\,\bar{w}_2(\xi,t)p(\xi)d\xi. \end{cases}$$

Let us estimate each term $\mathcal{E}_i$. Thanks to Lemma 3.5, we have $\mathcal{E}_2(t) = 0$, while $\mathcal{E}_1(t) \le -\|\bar{w}_2(t)\|^2_{L^2_p}$. In fact, the following lemma gives a better coercive estimate for $\mathcal{E}_1(t)$.

LEMMA 3.8 (diffusive term). *There holds that*

$$\mathcal{E}_1(t) \le -\frac{1}{24}\Big(\|\nabla\bar{w}_2(t)\|^2_{L^2_p} + \|(1+|\xi|)\bar{w}_2(t)\|^2_{L^2_p}\Big).$$

Proof. Recalling $\mathcal{L} = 1 + \frac{1}{2}\xi\cdot\nabla + \Delta$ and integrating by parts, we compute

$$\begin{aligned} &\int_{\mathbb{R}^2}(\mathcal{L}\bar{w}_2 - \bar{w}_2)(\xi,t)p(\xi)\bar{w}_2(\xi,t)d\xi \\ &= \int_{\mathbb{R}^2}\left(\Delta\bar{w}_2 + \frac{1}{2}\xi\cdot\nabla\bar{w}_2\right)\bar{w}_2(\xi,t)p(\xi)d\xi \\ &= -\int_{\mathbb{R}^2}|\nabla\bar{w}_2|^2p(\xi)d\xi - \int_{\mathbb{R}^2}\bar{w}_2(\nabla p\cdot\nabla\bar{w}_2)d\xi + \frac{1}{4}\int_{\mathbb{R}^2}\left(\xi\cdot\nabla(|\bar{w}_2|^2)\right)p(\xi,t)d\xi \\ &= -\int_{\mathbb{R}^2}|\nabla\bar{w}_2|^2p(\xi,t)d\xi - \int_{\mathbb{R}^2}\bar{w}_2(\nabla p\cdot\nabla\bar{w}_2)d\xi - \frac{1}{2}\int_{\mathbb{R}^2}|\bar{w}_2|^2p(\xi,t)d\xi - \frac{1}{4}\int_{\mathbb{R}^2}|\bar{w}_2|^2(\xi\cdot\nabla p)d\xi. \end{aligned}$$

The second integral is treated by

$$-\int_{\mathbb{R}^2}\bar{w}_2(\nabla p\cdot\nabla\bar{w}_2)d\xi \le \frac{3}{4}\int_{\mathbb{R}^2}|\nabla\bar{w}_2|^2p(\xi,t) + \frac{1}{3}\int_{\mathbb{R}^2}\frac{|\nabla p|^2}{p^2}|\bar{w}_2|^2p(\xi)d\xi.$$

Recalling now the weight function $p(\xi) = e^{|\xi|^2/4}$, we obtain the lemma at once. ∎

LEMMA 3.9. *There holds that*

$$\mathcal{E}_3(t) \lesssim t\|\xi\bar{w}_2(t)\|^2_{L^2_p}.$$

Proof. Integrating by parts and using the fact that $\widetilde{v}^E - \dot{\widetilde{z}}$ is divergence-free, we have

$$\begin{aligned} \mathcal{E}_3(t) &= -\sqrt{\frac{t}{\nu}}\int_{\mathbb{R}^2}\Big((\widetilde{v}^E - \dot{\widetilde{z}})\cdot\nabla\bar{w}_2\Big)\bar{w}_2(\xi,t)p(\xi)d\xi \\ &= \frac{1}{2}\sqrt{\frac{t}{\nu}}\int_{\mathbb{R}^2}(\widetilde{v}^E - \dot{\widetilde{z}})\cdot\nabla p(\xi)|\bar{w}_2(\xi,t)|^2d\xi. \end{aligned}$$

Recalling $\dot{\widetilde{z}} = \widetilde{v}^E(\widetilde{z}(t),t)$ and using Corollary 2.2, we estimate

$$|\widetilde{v}^E(\xi\sqrt{\nu t} + \widetilde{z}(t),t) - \dot{\widetilde{z}}(t)| = |\widetilde{v}^E(\xi\sqrt{\nu t} + \widetilde{z}(t),t) - \widetilde{v}^E(\widetilde{z}(t),t)| \lesssim \sqrt{\nu t}|\xi|.$$

The lemma follows, upon using $\nabla p = \frac{1}{2}\xi p(\xi)$. ∎

LEMMA 3.10. *There holds that*

$$\mathcal{E}_4(t) \lesssim t\left(\|\bar{w}_2(t)\|_{L^2_p}^2 + \|\nabla \bar{w}_2(t)\|_{L^2_p}^2\right).$$

Proof. We write $\mathcal{E}_4(t) = -t\left(\mathcal{E}_{41}(t) + \mathcal{E}_{42}(t)\right)$, where

$$\begin{cases} \mathcal{E}_{41}(t) = \int_{\mathbb{R}^2} \left(\bar{v}_2 \cdot \nabla w_{2,a}\right) \bar{w}_2(\xi,t) p(\xi) d\xi, \\ \mathcal{E}_{42}(t) = \int_{\mathbb{R}^2} \left(v_{2,a} \cdot \nabla \bar{w}_2\right) \bar{w}_2(\xi,t) p(\xi) d\xi. \end{cases}$$

Using Hölder's inequality, we estimate

$$|\mathcal{E}_{41}(t)| \le \|\bar{v}_2(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L^2_p} \left(\int_{\mathbb{R}^2} |\nabla w_{2,a}(\xi,t)|^2 p(\xi) d\xi\right)^{1/2},$$

in which the integral is bounded by Lemma 3.3. As for $\|\bar{v}_2(t)\|_{L^\infty}$, we use the elliptic estimate and Sobolev embedding, giving

$$\|\bar{v}_2\|_{L^\infty}^2 \lesssim \|\bar{w}_2\|_{L^{4/3}} \|\bar{w}_2\|_{L^4} \lesssim \|\bar{w}_2\|_{L^{4/3}} \|\bar{w}_2\|_{L^2}^{1/2} (\|\bar{w}_2\|_{L^2} + \|\nabla \bar{w}_2\|_{L^2})^{1/2}.$$

Recalling the weight function $p = e^{|\xi|^2/4}$, we have $\|\bar{w}_2\|_{L^{4/3}} \lesssim \|\bar{w}_2\|_{L^2_p}$. Thus, we get

$$\|\bar{v}_2\|_{L^\infty}^2 \lesssim \|\bar{w}_2\|_{L^2_p}^{3/2} (\|\bar{w}_2\|_{L^2_p} + \|\nabla \bar{w}_2\|_{L^2_p})^{1/2} \lesssim \|\bar{w}_2\|_{L^2_p}^2 + \|\nabla \bar{w}_2\|_{L^2_p}^2, \tag{3.19}$$

and so

$$|\mathcal{E}_{41}(t)| \lesssim \|\bar{w}_2(t)\|_{L^2_p} (\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla \bar{w}_2(t)\|_{L^2_p}) \lesssim \|\bar{w}_2(t)\|_{L^2_p}^2 + \|\nabla \bar{w}_2(t)\|_{L^2_p}^2.$$

On the other hand, the estimate on $\mathcal{E}_{42}(t)$ is direct, since $v_{2,a}$ is bounded. The lemma follows. ∎

LEMMA 3.11. *There holds that*

$$\mathcal{E}_5(t) \lesssim t\left(\|\bar{w}_2(t)\|_{L^2_p}^2 + \|\bar{w}_2(t)\|_{L^2_p}^5 + \|\nabla \bar{w}_2(t)\|_{L^2_p}^2\right).$$

Proof. By Hölder's inequality and (3.19), we get

$$\begin{aligned} |\mathcal{E}_5(t)| &= t\left|\int_{\mathbb{R}^2} \left(\bar{v}_2 \cdot \nabla \bar{w}_2\right) \bar{w}_2(\xi,t) p(\xi) d\xi\right| \\ &\le t \|\bar{v}_2(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L^2_p} \|\nabla \bar{w}_2(t)\|_{L^2_p} \\ &\lesssim t\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla \bar{w}_2(t)\|_{L^2_p}\right)^{1/4} \|\bar{w}_2(t)\|_{L^2_p}^{7/4} \|\nabla \bar{w}_2(t)\|_{L^2_p}. \end{aligned}$$

The lemma follows upon using Young's inequality. ∎

LEMMA 3.12. *There holds that*

$$\mathcal{E}_6(t) \lesssim \nu t \|\bar{v}_1(t)\|_{L^\infty}^4 + \nu t \|\bar{w}_2(t)\|_{L^2_p}^4 + \nu \|\nabla \bar{w}_2(t)\|_{L^2_p}^2.$$

Proof. Again by Hölder's inequality, we get

$$\begin{aligned} |\mathcal{E}_6(t)| &= \nu\sqrt{t}\left|\int_{\mathbb{R}^2} \left(\bar{v}_1 \cdot \nabla \bar{w}_2\right) \bar{w}_2(\xi,t) p(\xi) d\xi\right| \\ &\lesssim \nu t^{1/2} \|\bar{v}_1(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L^2_p} \|\nabla \bar{w}_2(t)\|_{L^2_p}, \end{aligned}$$

which yields the lemma upon using Young's inequality. ∎

LEMMA 3.13. *There holds that*

$$\mathcal{E}_7(t) \lesssim t^{1/2}\|\bar{w}_2(t)\|_{L^2_p}.$$

Proof. Using the estimates from (3.12) for a fixed $\gamma \in \left(\frac{1}{2}, 1\right)$ and Hölder's inequality, we get

$$\begin{aligned}|\mathcal{E}_7(t)| &\le (\nu t)^{-1}\int_{\mathbb{R}^2} |\Phi_{\mathrm{app}}(\xi,t)||\bar{w}_2(\xi,t)|p(\xi)d\xi \\ &\le (\nu t)^{-1}\int_{\mathbb{R}^2} (\nu t^{3/2})C_\gamma e^{-\gamma|\xi|^2/4}|\bar{w}_2(\xi,t)|p(\xi)d\xi \\ &\le C_\gamma t^{1/2}\left(\int_{\mathbb{R}^2} e^{-2\gamma|\xi|^2/4}p(\xi)d\xi\right)^{1/2}\left(\int_{\mathbb{R}^2} |\bar{w}_2(\xi,t)|^2 p(\xi)d\xi\right)^{1/2} \\ &\lesssim t^{1/2}\|\bar{w}_2(t)\|_{L^2_p},\end{aligned}$$

where we used $\gamma > 1/2$. This concludes the proof. ∎

LEMMA 3.14. *There hold that*

$$\mathcal{E}_8(t) \lesssim t^{-1/2}\|\bar{v}_1(t)\|_{L^\infty}\|\bar{w}_2(t)\|_{L^2_p}, \qquad \mathcal{E}_9(t) \lesssim \nu t^{1/2}\|\bar{v}_1(t)\|_{L^\infty}\|\bar{w}_2(t)\|_{L^2_p}.$$

Proof. We recall that

$$\mathcal{E}_8(t) = -\frac{1}{\sqrt{t}}\int_{\mathbb{R}^2}(\bar{v}_1(\xi,t)\cdot\nabla G(\xi))\bar{w}_2(\xi,t)p(\xi)d\xi,$$

where $G(\xi) = \frac{1}{4\pi}e^{-|\xi|^2/4}$ and $p(\xi) = e^{|\xi|^2/4}$. We have

$$|\mathcal{E}_8(t)| \lesssim t^{-1/2}\|\bar{v}_1(t)\|_{L^\infty}\int_{\mathbb{R}^2}|\xi||\bar{w}_2(\xi,t)|d\xi \lesssim t^{-1/2}\|\bar{v}_1(t)\|_{L^\infty}\|\bar{w}_2(t)\|_{L^2_p}.$$

The proof for $\mathcal{E}_9(t)$ is identical, upon recalling the pointwise bound on $\nabla w_{2,a}$ from Lemma 3.3. ∎

Proof of Proposition 3.7. We are now ready to prove Proposition 3.7. Collecting and combining all the estimates from the previous lemmas, we get

$$\begin{aligned}&t\frac{d}{dt}\|\bar{w}_2(t)\|^2_{L^2_p} + \kappa(\|(1+|\xi|)\bar{w}_2(t)\|^2_{L^2_p} + \|\nabla\bar{w}_2(t)\|^2_{L^2_p}) \\ &\lesssim t\left(\|(1+|\xi|)\bar{w}_2(t)\|^2_{L^2_p} + \|\bar{w}_2(t)\|^5_{L^2_p} + \|\nabla\bar{w}_2(t)\|^2_{L^2_p}\right) + t^{1/2}\|\bar{w}_2(t)\|_{L^2_p} \\ &\quad + \nu t\|\bar{v}_1(t)\|^4_{L^\infty} + \nu t\|\bar{w}_2(t)\|^4_{L^2_p} + \nu\|\nabla\bar{w}_2(t)\|^2_{L^2_p} + t^{-1/2}\|\bar{v}_1(t)\|_{L^\infty}\|\bar{w}_2(t)\|_{L^2_p}\end{aligned} \tag{3.20}$$

for $\kappa = 1/24$. Taking t and ν sufficiently small and using Young's inequality, we obtain

$$\begin{aligned}&t\frac{d}{dt}\|\bar{w}_2(t)\|^2_{L^2_p} + \frac{\kappa}{2}(\|(1+|\xi|)\bar{w}_2(t)\|^2_{L^2_p} + \|\nabla\bar{w}_2(t)\|^2_{L^2_p}) \\ &\qquad\lesssim t\|\bar{w}_2(t)\|^5_{L^2_p} + \nu t\|\bar{v}_1(t)\|^4_{L^\infty} + t^{-1}\|\bar{v}_1(t)\|^2_{L^\infty}.\end{aligned} \tag{3.21}$$

This completes the proof of the proposition. ∎

Remark 3.15. The constraint on the smallness of times T is precisely due to the term $\mathcal{E}_3(t)$ treated in Lemma 3.9. The remaining terms are treated using the standard Young's inequality. Hence, we in fact obtain

(3.22)
$$t\frac{d}{dt}\|\bar w_2(t)\|^2_{L^2_p} + \kappa\Big(\|\bar w_2(t)\|^2_{L^2_p} + \|\nabla \bar w_2(t)\|^2_{L^2_p} + (1-5t\|\nabla v^E(t)\|_{L^\infty})\|\xi \bar w_2(t)\|^2_{L^2_p}\Big)$$
$$\lesssim t(\|\bar w_2(t)\|^2_{L^2_p} + \|\bar w_2(t)\|^5_{L^2_p}) + \nu t\|\bar v_1(t)\|^4_{L^\infty} + t^{-1}\|\bar v_1(t)\|^2_{L^\infty}$$

for all positive times, as long as the estimates from Proposition 2.1 and Corollary 2.2 on the approximate vortex-wave solutions are valid. This yields a lower bound on the smallness of T so that $\sup_{0\le t\le T} 5t\|\nabla v^E(t)\|_{L^\infty} \le 1$.

Remark 3.16. One may try to improve the time interval by introducing a new weight function, as done similarly in [10], $p_{new}(\xi) = p(\xi)(1+\nu t q(\xi,t))$, where $q(\xi,t)$ solves

$$v^G(\xi)\cdot\nabla_\xi q = \frac{1}{\sqrt{\nu t}}\Big(v^E(z(t)+\xi\sqrt{\nu t},t) - v^E(z(t),t)\Big)\cdot \xi,$$

whose solution is, however, unclear for large $\xi\sqrt{\nu t}$.

4. Inviscid limit for the regular part. In the previous section, we have proved the a priori estimate for $\omega^{B,\nu}$ and $v^{E,\nu}$ in the weighted energy space with the rescaled variable $\xi = \frac{x-\widetilde z(t)}{\sqrt{\nu t}}$. In this section, we derive estimates on the regular vorticity component $\omega^{E,\nu}$, which solves

$$\partial_t \omega^{E,\nu} + u^\nu\cdot\nabla\omega^{E,\nu} = \nu\Delta\omega^{E,\nu} \tag{4.1}$$

with the initial data ω^E_0. We write

$$\begin{cases} \omega^{E,\nu}(t,x) = \widetilde\omega^E(t,x) + \nu^{3/2}\bar w_1(t,x),\\ v^{E,\nu}(t,x) = \widetilde v^E(t,x) + \nu^{3/2}\bar v_1(t,x),\\ v^{B,\nu}(t,x) = \frac{1}{\sqrt{\nu t}} v^G\left(\frac{x-\widetilde z(t)}{\sqrt{\nu t}}\right) + \sqrt{\nu t}(v_{2,a}+\bar v_2)\left(\frac{x-\widetilde z(t)}{\sqrt{\nu t}},t\right),\\ u^\nu(t,x) = v^{E,\nu}(t,x) + v^{B,\nu}(t,x), \end{cases} \tag{4.2}$$

where $(\widetilde z(t),\widetilde\omega^E)$ is the solution to the viscous vortex-wave system introduced in section 2, while v^G and $v_{2,a}$ are constructed in section 3. Here, we note that the form of the common velocity $u^\nu(t,x)$ is compatible with the form in (3.15) and (3.2) in the scaled variable ξ. The velocity $\bar v_2$ is kept the same as in the previous section, with ξ replaced by $\frac{x-\widetilde z(t)}{\sqrt{\nu t}}$ and $\bar v_2 = K\star_\xi \bar w_2$. It is natural to work in the original variables (x,t) instead of (ξ,t), since $\omega^{E,\nu}(t)$ solves (4.1) with regular initial data ω^E_0. Hence, one does not expect $\omega^{E,\nu}$ to have the localized behavior near the point vortex. Roughly speaking, we want to get an a priori bound on $\|\bar v_1(t)\|_{L^\infty}$ (in terms of $\bar w_2(t)$) on a time interval independent of ν. Precisely, we shall prove the following proposition.

PROPOSITION 4.1. *Let $\bar w_1$ solve* (4.1) *and* (4.2)*. There exists a positive time T, independent of $\nu > 0$, such that*

$$\|\bar w_1(t)\|_{L^4\cap L^{4/3}} \lesssim \int_0^t s^{3/2}(\|\bar w_2(t)\|_{L^2_p} + \|\nabla\bar w_2(t)\|_{L^2_p})ds + \nu^{1/2}t$$

for $t\in[0,T]$.

4.1. Equations for the remainder. In this subsection, we derive the equations for the remainder $\bar{w}_1$ as well as $\bar{v}_2$ appearing in (4.1) and (4.2). Putting the ansatz (4.2) into (4.1) and using (2.2), we obtain the following transport-diffusion equation for $\bar{w}_1$:

$$\partial_t \bar{w}_1 + u^\nu \cdot \nabla \bar{w}_1 - \nu \Delta \bar{w}_1 = f(x,t),$$

where $f(x,t)$ are given by

$$\begin{aligned} f(x,t) = &-\frac{1}{\nu\sqrt{t}}\left(v^G\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}\right) - v^G\left(\frac{x-z(t)}{\sqrt{\nu t}}\right)\right)\cdot \nabla w_{1,a} - \bar{v}_1 \cdot \nabla \widetilde{\omega}^E \\ &- \frac{\sqrt{t}}{\nu}\bar{v}_2 \cdot \nabla \widetilde{\omega}^E \\ &- \sqrt{\nu}(v_{1,a}\cdot \nabla w_{1,a}) + \frac{1}{2\pi\nu^{3/2}}\frac{(x-z(t))^\perp}{|x-z(t)|^2}e^{-\frac{|x-z(t)|^2}{4\nu t}}\cdot \nabla\omega^E + \sqrt{\nu}\Delta w_{1,a}. \end{aligned} \tag{4.3}$$

4.2. Estimating the forcing term $\boldsymbol{f(x,t)}$. In this subsection, we prove the following proposition.

PROPOSITION 4.2. *Let $f(x,t)$ be defined as in (4.3). There holds that*

$$\|f(t)\|_{L^4\cap L^{4/3}} \lesssim \|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} + t^{3/2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla \bar{w}_2(t)\|_{L^2_p}\right) + \sqrt{\nu}.$$

We will give a proof at the end of this subsection, after proving some useful lemmas. First, let us write f as

$$f(x,t) = f_1(x,t) + f_2(x,t) + f_3(x,t),$$

where

$$\begin{cases} f_1(x,t) = -\frac{1}{\nu\sqrt{t}}\left(v^G\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}\right) - v^G\left(\frac{x-z(t)}{\sqrt{\nu t}}\right)\right)\cdot\nabla w_{1,a} - \sqrt{\nu}(v_{1,a}\cdot\nabla w_{1,a}) \\ \quad + \frac{1}{2\pi\nu^{3/2}}\frac{(x-z(t))^\perp}{|x-z(t)|^2}e^{-\frac{|x-z(t)|^2}{4\nu t}}\cdot\nabla\omega^E + \sqrt{\nu}\Delta w_{1,a}, \\ f_2(x,t) = -\bar{v}_1\cdot\nabla\widetilde{\omega}^E, \\ f_3(x,t) = -\frac{\sqrt{t}}{\nu}\bar{v}_2\cdot\nabla\widetilde{\omega}^E. \end{cases}$$

In what follows, we bound $\|f_i(t)\|_{L^4\cap L^{4/3}}$ for each $i \in \{1,2,3\}$.

LEMMA 4.3. *There holds that*

$$\|f_1(t)\|_{L^4\cap L^{4/3}} \lesssim \sqrt{\nu}$$

uniformly in $\nu > 0$.

Proof. First, we see that

$$\left\|-\sqrt{\nu}(v_{1,a}\cdot\nabla w_{1,a}) - \frac{1}{2\pi\nu^{3/2}}\frac{(x-z(t))^\perp}{|x-z(t)|^2}e^{-\frac{|x-z(t)|^2}{4\nu t}}\cdot\nabla\omega^E + \sqrt{\nu}\Delta\omega_{1,a}\right\|_{L^4\cap L^{4/3}} \lesssim \sqrt{\nu}$$

thanks to the fact that ω^E is supported away from $z(t)$ and $\widetilde{z}(t)$, and $w_{1,a}$ is bounded in $W^{2,4}$, by Proposition 2.1. Now, for the first term in f_1, it suffices to prove that

$$\frac{1}{\sqrt{\nu t}}\left|v^G\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}\right) - v^G\left(\frac{x-z(t)}{\sqrt{\nu t}}\right)\right| \lesssim \nu t \qquad \forall x\in \operatorname{supp}(w_{1,a}). \tag{4.4}$$

As long as the above claim is proved, we would get

$$\begin{aligned}&\left\|\frac{1}{\nu\sqrt{t}}\left(v^G\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}\right)-v^G\left(\frac{x-z(t)}{\sqrt{\nu t}}\right)\right)\cdot\nabla w_{1,a}\right\|_{L^4\cap L^{4/3}}\\&\quad\lesssim\sqrt{\nu}\|\nabla w_{1,a}(t)\|_{L^4\cap L^{4/3}(\mathrm{supp}(w_{1,a}))}\lesssim\sqrt{\nu}\end{aligned}$$

by Proposition 2.1.

Now we shall prove the inequality (4.4). To this end, let us denote

$$\eta_1 = x-\widetilde{z}(t) \qquad \text{and} \qquad \eta_2 = x-z(t). \tag{4.5}$$

The left-hand side of (4.4) can be rewritten as

$$\frac{1}{\sqrt{\nu t}}\left(v^G\left(\frac{\eta_1}{\sqrt{\nu t}}\right)-v^G\left(\frac{\eta_2}{\sqrt{\nu t}}\right)\right)=\frac{1}{2\pi}\left(V_1(\eta_1,\eta_2)+V_2(\eta_1,\eta_2)\right), \tag{4.6}$$

where

$$\begin{cases}V_1(\eta_1,\eta_2)=\left(\frac{\eta_1^\perp}{|\eta_1|^2}-\frac{\eta_2^\perp}{|\eta_2|^2}\right),\\ V_2(\eta_1,\eta_2)=\left(\frac{\eta_2^\perp}{|\eta_2|^2}e^{-|\eta_2|^2/4\nu t}-\frac{\eta_1^\perp}{|\eta_1|^2}e^{-|\eta_1|^2/4\nu t}\right).\end{cases}$$

When $x\in\mathrm{supp}(\widetilde{\omega}^E(t))$, by the properties established in section 2, we have a positive constant c_T, independent of ν, such that

$$|x-z(t)|\geq c_T \qquad \text{and} \qquad |x-\widetilde{z}(t)|\geq c_T \qquad \forall t\in[0,T]. \tag{4.7}$$

This implies that $|\eta_1|\geq c_T$ and $|\eta_2|\geq c_T$, upon recalling the notations (4.5). Thus, we get

$$\begin{aligned}|V_1(\eta_1,\eta_2)|&=\left|\frac{\eta_1^\perp}{|\eta_1|^2}-\frac{\eta_2^\perp}{|\eta_2|^2}\right|\leq\left|\frac{\eta_1^\perp}{|\eta_1|^2}-\frac{\eta_2^\perp}{|\eta_1|^2}\right|+\left|\frac{\eta_2^\perp}{|\eta_1|^2}-\frac{\eta_2^\perp}{|\eta_2|^2}\right|\\&\leq\frac{|\eta_1-\eta_2|}{|\eta_1|^2}+|\eta_2|\frac{\left||\eta_2|^2-|\eta_1|^2\right|}{|\eta_1|^2|\eta_2|^2}\\&\leq c_T^{-2}|\eta_1-\eta_2|+\frac{1}{|\eta_1|^2|\eta_2|}\left||\eta_2|-|\eta_1|\right|(|\eta_1|+|\eta_2|)\lesssim|\eta_1-\eta_2|\\&=|(x-\widetilde{z}(t))-(x-z(t))|=|\widetilde{z}(t)-z(t)|\lesssim\nu t\qquad\text{(by the estimate (2.5)).}\end{aligned}$$

Hence,

$$|V_1(\eta_1,\eta_2)|\lesssim\nu t. \tag{4.8}$$

Now, for $V_2(\eta_1,\eta_2)$, note that we shall only consider $x\in\mathrm{supp}(\widetilde{\omega}^E(t))$, in which we get (4.7). In this case, we get

$$|V_2(\eta_1,\eta_2)|\leq|\eta_2|^{-1}e^{-|\eta_2|^2/4\nu t}+|\eta_1|^{-1}e^{-|\eta_1|^2/4\nu t}\leq 2c_T^{-1}e^{-c_T^2/4\nu t}\lesssim\nu t. \tag{4.9}$$

Combining (4.6), (4.8), and (4.9), we get the desired inequality (4.4). The bound for the first term is complete. This concludes the proof. ∎

LEMMA 4.4. *There holds that*

$$\|f_2(t)\|_{L^4\cap L^{4/3}}\lesssim\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}}.$$

Proof. We have

$$\|f_2(t)\|_{L^4\cap L^{4/3}} = \|\bar{v}_1(t)\cdot\nabla\widetilde{\omega}^E(t)\|_{L^4\cap L^{4/3}} \le \|\bar{v}_1(t)\|_{L^\infty}\|\nabla\widetilde{\omega}^E(t)\|_{L^4\cap L^{4/3}} \lesssim \|\bar{w}_1(t)\|_{L^4\cap L^{4/3}}$$

by Corollary 2.2 and Lemma A.1. The proof is complete. ∎

LEMMA 4.5. *There holds that*

$$\|f_3(t)\|_{L^4\cap L^{4/3}} \lesssim t^{3/2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right).$$

Proof. We recall that

$$f_3(x,t) = \frac{\sqrt{t}}{\nu}\bar{v}_2(\xi,t)\cdot\nabla\widetilde{\omega}^E(t,x), \qquad \xi = \frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}.$$

We shall only consider $x\in\operatorname{supp}(\widetilde{\omega}^E(t))$. Since $\widetilde{\omega}^E(t)$ is supported away from $\widetilde{z}(t)$, there exists $d_T>0$ such that

$$|x-\widetilde{z}(t)|\ge d_T \qquad \text{for} \quad x\in\operatorname{supp}(\widetilde{\omega}^E(t)). \tag{4.10}$$

Since $\int_{\mathbb{R}^2}\bar{w}_2(\xi,t)d\xi = 0$, by Lemma A.1, we get

$$\begin{aligned}\|(1+|\xi|^2)\bar{v}_2(t)\|_{L^\infty} &\lesssim \|(1+|\xi|^2)\bar{w}_2(t)\|_{L^4} + \|(1+|\xi|^2)\bar{w}_2(t)\|_{L^{4/3}}\\ &\lesssim \|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}.\end{aligned}$$

This implies that, for x in the support of $\widetilde{\omega}^E(t)$, we get

$$|\bar{v}_2(t,\xi)| \lesssim \frac{1}{1+|\xi|^2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right) \lesssim (\nu t)\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right).$$

Thus, we get

$$\|f_3(t)\|_{L^4\cap L^{4/3}} \lesssim \frac{\sqrt{t}}{\nu}\|\bar{v}_2(\xi,t)\cdot\nabla\widetilde{\omega}^E(t,x)\|_{L^4\cap L^{4/3}} \lesssim t^{3/2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right).$$

The proof is complete. ∎

We conclude this subsection by proving Proposition 4.2.

Proof of Proposition 4.2. The proof follows as a direct consequence of the previous lemmas for f_i, $i\in\{1,2,3\}$, in this subsection. ∎

4.3. A priori estimates for the remainder. In this subsection, we give a proof for our main proposition (Proposition 4.1), stated at the beginning of section 4. We recall from section 4.1 that $\bar{w}_1$ solves the heat transport equation

$$\partial_t\bar{w}_1 + u^\nu\cdot\nabla\bar{w}_1 - \nu\Delta\bar{w}_1 = f(x,t).$$

A standard $L^4\cap L^{4/3}$ estimate for the heat transport equation yields

$$\begin{aligned}\frac{d}{dt}\left(\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}}\right) &\lesssim \|f(t)\|_{L^4\cap L^{4/3}}\\ &\lesssim \|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} + t^{3/2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right) + \sqrt{\nu},\end{aligned}$$

using Proposition 4.2. Now, applying Gronwall's lemma for the above inequality, we have

$$\begin{aligned}\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} &\lesssim \int_0^t\left(s^{3/2}(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}) + \sqrt{\nu}\right)ds\\ &\lesssim \int_0^t s^{3/2}(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p})ds + \nu^{1/2}t.\end{aligned} \tag{4.11}$$

The proof is complete.

5. Proof of inviscid limit.

In this section, we conclude the proof for the inviscid limit, using the a priori estimates obtained from the previous sections. Let us first prove the following proposition, before proving our main theorem, stated in the first part of this paper.

PROPOSITION 5.1. *There exists a time $T > 0$, independent of the viscosity ν, such that*

$$\sup_{0\le t\le T}\Big(\|\bar w_2(t)\|_{L^2_p} + \|\bar w_1(t)\|_{L^4\cap L^{4/3}}\Big) \lesssim 1$$

uniformly in ν.

Proof. First, we recall the following estimates for $\|\bar w_2(t)\|_{L^2_p}$ and $\|\bar w_1(t)\|_{L^4\cap L^{4/3}}$ proven in Propositions 3.7 and 4.1:

$$\begin{aligned}&\frac{d}{dt}\|\bar w_2(t)\|^2_{L^2_p} + \frac{\kappa}{t}(\|(1+|\xi|)\bar w_2(t)\|^2_{L^2_p} + \|\nabla \bar w_2(t)\|^2_{L^2_p})\\ &\qquad\lesssim \|\bar w_2(t)\|^5_{L^2_p} + \nu\|\bar v_1(t)\|^4_{L^\infty} + t^{-2}\|\bar v_1(t)\|^2_{L^\infty},\\ &\|\bar w_1(t)\|_{L^4\cap L^{4/3}} \lesssim \int_0^t s^{3/2}\Big(\|\bar w_2(t)\|_{L^2_p} + \|\nabla \bar w_2(t)\|_{L^2_p}\Big)ds + \nu^{1/2}t.\end{aligned} \tag{5.1}$$

Let

$$\mathcal{G}(t) = \|\bar w_2(t)\|^2_{L^2_p} + \int_0^t s^{-1}(\|\bar w_2(s)\|^2_{L^2_p} + \|\nabla\bar w_2(s)\|^2_{L^2_p})ds.$$

From the inequality (5.1), it is straightforward that

$$\|\bar w_1(t)\|_{L^4\cap L^{4/3}} \lesssim t^{5/2}\mathcal{G}(t)^{1/2} + \nu^{1/2}t. \tag{5.2}$$

Thus, we have

$$\begin{aligned}\mathcal{G}'(t) &= \frac{d}{dt}\|\bar w_2(t)\|^2_{L^2_p} + t^{-1}\Big(\|\bar w_2(t)\|^2_{L^2_p} + \|\nabla\bar w_2(t)\|^2_{L^2_p}\Big)\\ &\lesssim \|\bar w_2(t)\|^5_{L^2_p} + \nu\|\bar v_1(t)\|^4_{L^\infty} + t^{-2}\|\bar v_1(t)\|^2_{L^\infty} \qquad \text{(by (5.1))}\\ &\lesssim \mathcal{G}(t)^{5/2} + \nu\|\bar w_1(t)\|^4_{L^4\cap L^{4/3}} + t^{-2}\|\bar w_1(t)\|^2_{L^4\cap L^{4/3}}\\ &\lesssim \mathcal{G}(t)^{5/2} + \nu\Big(t^{5/2}\mathcal{G}(t)^{1/2} + \nu^{1/2}t\Big)^4 + t^{-2}\Big(t^{5/2}\mathcal{G}(t)^{1/2}+\nu^{1/2}t\Big)^2 \qquad \text{(by (5.2))}\\ &\lesssim \mathcal{G}(t)^{5/2} + \nu t^{10}\mathcal{G}(t)^2 + \nu^3t^4 + t^3\mathcal{G}(t) + \nu.\end{aligned}$$

By standard ODE theory, we have a time $T > 0$, which is independent of $\nu > 0$, such that $\mathcal{G}(t)$ is uniformly bounded for $t \in [0, T]$. Since $\mathcal{G}(t) \ge \|\bar w_2(t)\|_{L^2_p}$, the proof for $\|\bar w_2(t)\|_{L^2_p}$ is complete. The bound $\|\bar w_1(t)\|_{L^4\cap L^{4/3}} \lesssim 1$ follows from the inequality (5.2). ∎

We conclude this section by proving our main theorem, stated in the first part of this paper.

Proof of Theorem 1.2. We have proved that $\|\bar w_2(t)\|_{L^2_p}$ is uniformly bounded in ν. We recall from section 3 that

$$\omega^{B,\nu}(t,x) = \frac{1}{\nu t}w_2(\xi,t) = \frac{1}{\nu t}\left(G(\xi) + (\nu t)w_{2,a} + (\nu t)\bar w_2\right) = \frac{1}{\nu t}G(\xi) + w_{2,a} + \bar w_2,$$

where $G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}$ and $\xi = (x - \widetilde{z}(t))/\sqrt{\nu t}$. We compute

$$\begin{aligned}\left\| \omega^{B,\nu}(t,x) - \frac{1}{4\pi\nu t} e^{-\frac{|x-\widetilde{z}(t)|^2}{4\nu t}} \right\|_{L^1_x} &= \|w_{2,a}(\xi,t) + \bar{w}_2(\xi,t)\|_{L^1_x} \\ &= \nu t \|w_{2,a}(t) + \bar{w}_2(t)\|_{L^1_\xi} \\ &\lesssim (\nu t)\left(\|w_{2,a}(t)\|_{L^2_p} + \|\bar{w}_2(t)\|_{L^2_p} \right) \\ &\lesssim (\nu t). \end{aligned} \tag{5.3}$$

For ease of notation, we denote by $G_{\widetilde{z}(t)}(x)$ and $G_{z(t)}(x)$ the Gaussians $\frac{1}{4\pi\nu t} e^{-\frac{|x-\widetilde{z}(t)|^2}{4\nu t}}$ and $\frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}}$, respectively. Our goal now is to compare the two Gaussians in the L^1 norm. To this end, let us denote $A = \frac{|x-\widetilde{z}(t)|^2}{4\nu t}$ and $B = \frac{|x-z(t)|^2}{4\nu t}$. We have

$$G_{\widetilde{z}(t)}(x) - G_{z(t)}(x) = e^{-A} - e^{-B} = e^{-B}\left(e^{B-A} - 1\right).$$

We have

$$\begin{aligned} B - A &= (4\nu t)^{-1}\left(|x - z(t)|^2 - |x - \widetilde{z}(t)|^2\right) = (4\nu t)^{-1}\left(2x\cdot(\widetilde{z}(t) - z(t)) + |z(t)|^2 - |\widetilde{z}(t)|^2\right) \\ &\lesssim (4\nu t)^{-1}\left(|x||\widetilde{z}(t) - z(t)| + |\widetilde{z}(t) - z(t)|\right) \\ &\lesssim |x| + 1 \qquad (\text{since} \quad |\widetilde{z}(t) - z(t)| \lesssim \nu t) \\ &\le |x - z(t)| + |z(t)| + 1 \lesssim \frac{|x - z(t)|}{\sqrt{\nu t}} + 1. \end{aligned}$$

Here, we used the standard fact of the vortex-wave system that $|z(t)| \lesssim 1$ for any fixed interval of time. Indeed, one can see that $|z(t)| \le |z_0| + \int_0^t |v^E(z(s),s)|ds \le |z_0| + t\|v^E\|_{L^\infty}$. Hence, we get

$$|G_{\widetilde{z}(t)}(x) - G_{z(t)}(x)| \lesssim e^{-\frac{|x-z(t)|^2}{4\nu t} + M_T \frac{|x-z(t)|}{\sqrt{\nu t}}} \qquad \text{for some} \quad M_T > 0. \tag{5.4}$$

Integrating both sides of the inequality (5.4) in $x \in \mathbb{R}^2$, we have

$$\|G_{z(t)} - G_{\widetilde{z}(t)}\|_{L^1_x} \lesssim \int_{\mathbb{R}^2} e^{-\frac{|x-z(t)|^2}{4\nu t} + M_T \frac{|x-z(t)|}{\sqrt{\nu t}}} dx.$$

Making the change of variables $y = \frac{x-z(t)}{\sqrt{\nu t}}$ in the above integral, we thus obtain

$$\|G_{z(t)} - G_{\widetilde{z}(t)}\|_{L^1_x} \lesssim \nu t. \tag{5.5}$$

Combining the inequalities (5.3) and (5.5), we get

$$\left\| \omega^{B,\nu}(t,x) - \frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}} \right\|_{L^1_x} \lesssim \nu t.$$

The inequality $\|\omega^{E,\nu}(t) - \omega^E(t)\|_{L^4 \cap L^{4/3}} \lesssim \nu$ follows directly from the expansion (4.2), the inequality (5.2), and the uniform bound of $\mathcal{G}(t)$. The proof is complete. ∎

Appendix A. In this section, we collect several useful lemmas used in this paper.

LEMMA A.1 (elliptic estimates). *Let $v = K \star \omega$ be the velocity vector field obtained from the vorticity ω on $\mathbb{R}^2$. Define the norm $\|\cdot\|_{L^4 \cap L^{4/3}} = \|\cdot\|_{L^4} + \|\cdot\|_{L^{4/3}}$. The following inequalities hold:*

$$\|v\|_{L^\infty} \lesssim \|\omega\|_{L^4 \cap L^{4/3}}, \qquad \|v\|_{L^\infty} \lesssim \|\omega\|_{L^1 \cap L^\infty}.$$

Moreover, if $\int_{\mathbb{R}^2}\omega(x)dx = 0$*, then*

$$\|(1+|x|^2)v\|_{L^\infty} \lesssim \|(1+|x|^2)\omega\|_{L^4\cap L^{4/3}}.$$

Proof. From the Biot–Savart law (1.2), we estimate

(A.1)
$$\begin{aligned}|v(x)| &\lesssim \int_{\mathbb{R}^2}\frac{|\omega(y)|}{|x-y|}dy = \left(\int_{|x-y|\le R} + \int_{|x-y|\ge R}\right)\frac{|\omega(y)|}{|x-y|}dy\\ &\lesssim \left(\int_{|x-y|\le R}|x-y|^{-4/3}dy\right)^{3/4}\|\omega\|_{L^4} + \left(\int_{|x-y|\ge R}|x-y|^{-4}dy\right)^{1/4}\|\omega\|_{L^{4/3}}\\ &\lesssim R^{1/2}\|\omega\|_{L^4} + R^{-1/2}\|\omega\|_{L^{4/3}}.\end{aligned}$$

Thus, choosing $R = \frac{\|\omega\|_{L^{4/3}}}{\|\omega\|_{L^4}}$, we have $\|v\|_{L^\infty}\lesssim \|\omega\|^{1/2}_{L^{4/3}}\|\omega\|^{1/2}_{L^4}$, which gives the first inequality. As for the second, we use $\|\omega\|_{L^p}\le \|\omega\|^{1/p}_{L^1}\|\omega\|^{1-1/p}_{L^\infty}$.

It remains to check the last inequality. We shall check it for v_2, the second component of v. The estimate on v_1 is similar. First, we check

$$|x||v_2(x)| \lesssim \int_{\mathbb{R}^2}\frac{1}{|x-y|}|y||\omega(y)|dy. \tag{A.2}$$

By the Biot–Savart law and $\int_{\mathbb{R}^2}\omega(y)dy = 0$, we have

$$|v_2(x)| = \frac{1}{2\pi}\left|\int_{\mathbb{R}^2}\frac{x_1-y_1}{|x-y|^2}\omega(y)dy\right| \le \frac{1}{2\pi}\int_{\mathbb{R}^2}\left|\frac{x_1-y_1}{|x-y|^2} - \frac{x_1}{|x|^2}\right||\omega(y)|dy.$$

Now we have

$$\frac{x_1-y_1}{|x-y|^2} - \frac{x_1}{|x|^2} = \frac{1}{|x|^2|x-y|^2}\left(|x|^2(x_1-y_1) - x_1|x-y|^2\right).$$

It follows that $|x|^2(x_1-y_1) - x_1|x-y|^2 \le 4|x||y||x-y|$. Hence,

$$|x|\Big[\frac{x_1-y_1}{|x-y|^2} - \frac{x_1}{|x|^2}\Big] \le \frac{4|y|}{|x-y|},$$

which gives (A.2). Now, multiplying both sides of (A.2) by $|x|$, we have

$$\begin{aligned}|x|^2|v_2(x)| &\lesssim \int_{\mathbb{R}^2}\frac{|x||y|}{|x-y|}|\omega(y)|dy \le \int_{\mathbb{R}^2}\frac{|y|+|x-y|}{|x-y|}|y||\omega(y)|dy\\ &= \int_{\mathbb{R}^2}\frac{1}{|x-y|}|y|^2|\omega(y)|dy + \int_{\mathbb{R}^2}|y||\omega(y)|dy.\end{aligned}$$

Let us first treat the first term in the above. Repeating the argument of (A.1) for $\omega = |y|^2|\omega(y)|$, we have

$$\int_{\mathbb{R}^2}\frac{1}{|x-y|}|y|^2|\omega(y)|dy \lesssim \|(1+|y|^2)\omega(y)\|_{L^4\cap L^{4/3}}.$$

For the second term, using Hölder's inequality, we get

$$\int_{\mathbb{R}^2}|y||\omega(y)|dy = \int_{\mathbb{R}^2}\frac{|y|}{1+|y|^2}(1+|y|^2)|\omega(y)|dy \lesssim \|(1+|y|^2)\omega(y)\|_{L^{4/3}}.$$

Thus,

$$|x|^2|v_2(x)| \lesssim \|(1+|x|^2)\omega\|_{L^4\cap L^{4/3}}.$$

The lemma follows. ∎

LEMMA A.2. *Let $z_1, z_2 \in \mathbb{C}$, and let ψ be the angle between z_1 and z_2. Assuming that $|z_1| < |z_2|$ and $\sin(\psi) \neq 0$, there holds that*

$$\frac{1}{|z_1+z_2|^2} - \frac{1}{|z_2|^2} = \frac{1}{|z_2|^2}\sum_{n=1}^{\infty}(-1)^n \frac{|z_1|^n}{|z_2|^n}\frac{\sin((n+1)\psi)}{\sin(\psi)}.$$

Proof. Let $\frac{z_1}{z_2} = z = re^{i\psi}$. We have

$$\frac{1}{|z_1+z_2|^2} - \frac{1}{|z_2|^2} = \frac{1}{|z_2|^2}\left(\frac{1}{|1+z|^2} - 1\right).$$

Now, for $|z| < 1$, we have

$$\begin{aligned}\frac{1}{|1+z|^2} &= \frac{1}{(1+z)(1+\bar z)} = (1 - z + z^2 - \cdots)(1 - \bar z + \bar z^2 + \cdots)\\ &= 1 - (z+\bar z) + (z^2 + z\bar z + \bar z^2) - (z^3 + z^2\bar z + z\bar z^2 + \bar z^3) + \cdots .\end{aligned}$$

Now, for each n, we have

$$z^n + z^{n-1}\bar z + \cdots + z\bar z^{n-1} + \bar z^n = \frac{z^{n+1} - \bar z^{n+1}}{z - \bar z} = r^n\frac{\sin((n+1)\psi)}{\sin\psi}.$$

This concludes the proof. ∎

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