Derivative estimates for screened Vlasov-Poisson system around Penrose-stable equilibria

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Abstract

In this paper, we establish derivative estimates for the Vlasov-Poisson system with screening interactions around Penrose-stable equilibria on the phase space $\mathbb{R}^d_x \times \mathbb{R}^d_v$, with dimension $d \geq 3$. In particular, we establish the optimal decay estimates for higher derivatives of the density of the perturbed system, precisely like the free transport, up to a log correction in time. This extends the recent work [13] by Han-Kwan, Nguyen and Rousset to higher derivatives of the density. The proof makes use of several key observations from [13] on the structure of the forcing term in the linear problem, with induction arguments to classify all the terms appearing in the derivative estimates.

1 Introduction

1.1 The system

In this paper, we consider the screened Vlasov-Poisson system on the phase space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, with the dimension $d \geq 3$:

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 \tag{1.1}$$

Here $f = f(t, x, v) \geq 0$ is the probability distribution of charged particles in plasma,

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv$$

is the electric charge density, and

$$E(t, x) = -\nabla_x (1 - \Delta_x)^{-1} (\rho - 1)$$

is electric field. This model has been investigated in physical literatures [6, 7, 2] and also recent mathematical works [5, 13]. We refer the readers to [19, 20, 15, 9, 17] for global existence and regularity results. The system (1.1) has

$$(f, \rho, E) = (\mu(v), 1, 0)$$

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as a steady solution for any smooth and decaying function $\mu(v) \geq 0$ with the normalized condition

$$\int_{\mathbb{R}^d} \mu(v) dv = 1.$$

We assume that $\mu(v)$ satisfies the Penrose stability condition ($\widehat{\nabla_v \mu}$ denotes the Fourier transform):

$$\inf_{\Im\tau \leq 0} \inf_{\xi \in \mathbb{R}^d} \left| 1 - \int_0^\infty e^{-i\tau s} \frac{1}{1+|\xi|^2} i\xi \cdot \widehat{\nabla_v \mu}(s\xi) ds \right| \geq \kappa \qquad \text{for} \quad \kappa > 0, \tag{1.2}$$

which is classical in the study of Landau damping [18, 4], where the authors justify the asymptotic stability for homogeneous equilibria that satisfies (1.2) on the phase space $\mathbb{T}^d \times \mathbb{R}^d$, under analytic and Gevrey perturbations for the Vlasov-Poisson system. The stability condition (1.2) is also natural in studying the quasineutral limit of the Vlasov-Poisson equations [14, 11, 10] and the long time estimates for the Vlasov-Maxwell equations [12].

1.2 Previous works

In [5], Bedrossian, Masmoudi and Mouhot justify the asymptotic stability of the equilibria when $\mu(v)$ satisfies the condition (1.2). The proof is inspired by [4] for Landau damping on the confined phase space $\mathbb{T}^d \times \mathbb{R}^d$. Using the dispersive mechanism in Fourier space to control the plasma echo resonance, the authors in [5] prove that the Fourier mode of the density $\hat{\rho}(t,\xi)$ decays like $\frac{1}{(t|\xi|)^{N-\delta}}$ if the initial perturbation is in Sobolev space of high regularity order N and some $\delta \in (0, N)$. The decay is far from being optimal, as the dispersion in the physical space was not taken into account.

In the recent work [13], Han-Kwan, Nguyen, and Rousset revisit the asymptotic stability of equilibria that satisfy the condition (1.2). They obtain the decay estimates for the perturbed electric charge density $\rho(t)$ as follows:

$$\|\rho(t)\|_{L^1} + \langle t\rangle \|\nabla_x \rho(t)\|_{L^1} + \langle t\rangle^d \|\rho(t)\|_{L^\infty} + \langle t\rangle^{d+1} \|\nabla_x \rho(t)\|_{L^\infty} \lesssim \varepsilon_0 \log(1+t), \qquad \langle t\rangle = \sqrt{t^2+1}.$$

Unlike [5], which relies on the nonlinear energy estimates, the authors in [13] use direct dispersive mechanism in the physical space, which is like the free transport up to $\log(t)$. This is achieved by a pointwise dispersive estimate, directly on the resolvent kernel for the linearized system around non-zero stable equilibria $\mu(v)$. This is followed by solving the equations by characteristic methods, inspired from the classical work of Bardos and Degond [3]. At the same time, the authors in [13] are able to propagate C^1 norm for the initial data and thus allow more general perturbations. We note that the dispersive mechanism of the free transport operator $\partial_t + v \cdot \nabla_x$ on $\mathbb{R}^d \times \mathbb{R}^d$ is also one of the key ingredients in the classical results of Bardos and Degond [3] in 1985, where they study the asymptotic stability of Vlasov-Poisson around vacuum ($\mu(v) = 0$).

Regarding the stability of vacuum (when $\mu(v) = 0$) for the unscreened Vlasov-Poisson systems, we refer the readers to the work [16] for the extension of Bardos-Degond results for optimal decays of higher derivatives. In [21], Smulevici obtains the spatial decay by the vector field methods. In [22], Wang justifies the stability of vacuum for Vlasov-Poisson by Fourier methods. In [8], Choi, Ha and Lee justify the same result for 2D screened Vlasov-Poisson.

2 Main results

2.1 Main theorem

In this paper, we will give decay estimates for higher derivatives of $\rho(t)$, under the assumption that the initial perturbation f_0 is small in suitable Sobolev space and for $\mu(v)$ satisfying decaying assumption: Given any $m \in \mathbb{N}$ and $M > 0$, there exists $C_{m,M} > 0$ such that

$$|\nabla_v^m \mu(v)| \leq C_{m,M} \langle v \rangle^{-M} \quad \text{for all} \quad v \in \mathbb{R}^d. \tag{2.1}$$

The equation for the perturbed quantities around the equilibria of (1.1) reads

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu = -E \cdot \nabla_v f, \\ E = -\nabla_x (1 - \Delta_x)^{-1} \rho, \\ \rho = \int_{\mathbb{R}^d} f dv. \end{cases} \tag{2.2}$$

Our main theorem is as follows:

Theorem 2.1. *Let $N > 1$ be an integer. Let $\mu(v) \geq 0$ be sufficiently smooth and satisfy the Penrose stability condition* (1.2)*, the decaying bound* (2.1) *and the normalized condition $\int_{\mathbb{R}^d} \mu(v) dv = 1$. There exists $\varepsilon_0 > 0$ such that for all $f_0(x, v)$ satisfying the smallness assumption*

$$\max_{0 \leq k \leq N} \left(\|\nabla_{x,v}^k f_0\|_{L^1_{x,v}} + \|\nabla_{x,v}^k f_0\|_{L^1_x L^\infty_v} \right) \leq \varepsilon_0,$$

the solution $f(t, x, v)$ to the equations (2.2) *with initial data $f|_{t=0} = f_0(x, v)$ satisfies*

$$\max_{0 \leq k \leq N} \left(\langle t \rangle^k \|\nabla_x^k \rho(t)\|_{L^1} + \langle t \rangle^{k+d} \|\nabla_x^k \rho(t)\|_{L^\infty} \right) \lesssim \varepsilon_0 \log(1 + t).$$

where $\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv$. Here the norm $\| \cdot \|_{L^p_x L^q_v}$ is defined by

$$\|g\|_{L^p_x L^q_v} = \left(\int_{\mathbb{R}^d} \|g(x, v)\|^p_{L^q_v} dx \right)^{1/p}.$$

2.2 Motivation

The improved decays for higher derivatives of $\rho(t)$ can be seen from the free transport equation

$$\partial_t f_{\text{free}} + v \cdot \nabla_x f_{\text{free}} = 0, \qquad f|_{t=0} = f_0.$$

The solution is given by

$$f_{\text{free}} = f_0(x - tv, v), \qquad \rho_{\text{free}}(t, x) = \int_{\mathbb{R}^d} f_0(x - tv, v) dv.$$

Making the change of variables $w = x - tv$, we obtain

$$\rho_{\text{free}}(t, x) = \int_{\mathbb{R}^d} f_0 \left(w, \frac{x - w}{t} \right) t^{-d} dw. \tag{2.3}$$

Hence

$$\|\rho_{\text{free}}(t)\|_{L^\infty} \le t^{-d}\|f_0\|_{L^1_x L^\infty_v} \quad \text{and} \quad \|\rho_{\text{free}}(t)\|_{L^1} \le \|f_0\|_{L^1_x L^1_v}.$$

Thus for the free transport equations, $\rho_{\text{free}}(t)$ decays like t^{-d} in L^∞. This dispersive mechanism for the free transport (which can be seen as linearized Vlasov-Poisson around zero) is one of the key ingredients in the classical work [3] by Bardos and Degond.

Now we discuss the decay for one derivative of $\rho_{\text{free}}(t)$. Taking ∇_x on both sides of (2.3), we get

$$\nabla_x \rho_{\text{free}}(t) = t^{-d}t^{-1}\int_{\mathbb{R}^d} \nabla_v f_0\left(w, \frac{x-w}{t}\right) dw,$$

and hence

$$\|\nabla_x\rho_{\text{free}}(t)\|_{L^\infty} \le t^{-(d+1)}\|\nabla_v f_0\|_{L^1_x L^\infty_v} \quad \text{and} \quad \|\nabla_x\rho_{\text{free}}(t)\|_{L^1} \le t^{-1}\|\nabla_v f_0\|_{L^1_x L^1_v}.$$

This implies that $\nabla_x\rho_{\text{free}}(t)$ decays like t^{-d-1} in L^∞ and t^{-1} in L^1. This decaying mechanism is in fact extended to the Vlasov-Poisson system around vacuum by Hwang, Rendall and Velazquez [16]. In particular, the authors in [16] establish the improved decay estimates

$$\|\nabla_x^k \rho(t)\|_{L^\infty} \lesssim (1+t)^{-d-k} \qquad \text{for any} \quad k \ge 0 \tag{2.4}$$

for small initial data near vacuum.

The natural question is whether the estimate (2.4) still holds for solution to the screened Vlasov-Poisson under small perturbation around nonzero homogeneous equilibria $\mu(v)$ that satisfies Penrose condition (1.2). In this paper, we prove that this is essentially the case, namely $\nabla_x^k\rho(t)$ decays like $\nabla_x^k\rho_{\text{free}}(t)$, up to a log in time correction.

2.3 Outline of the proof

The set up: Using the characteristics

$$\begin{cases} \frac{d}{ds}X_{s,t}(x,v) &= V_{s,t}(x,v), \quad X_{t,t}(x,v) = x, \\ \frac{d}{ds}V_{s,t}(x,v) &= E(s, X_{s,t}(x,v)), \quad V_{t,t}(x,v) = v, \end{cases} \tag{2.5}$$

the solution $f(t,x,v)$ to the perturbation equation (2.2) can be written as

$$f(t,x,v) = f_0(X_{0,t}(x,v), V_{0,t}(x,v)) - \int_0^t E(s, X_{s,t}(x,v)) \cdot \nabla_v\mu(V_{s,t}(x,v))ds.$$

Integrating both sides in $v \in \mathbb{R}^d$ and using the fact that $E = -\nabla_x(1-\Delta_x)^{-1}\rho$, we get

$$\rho(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla_x(1-\Delta_x)^{-1}\rho(s, x-(t-s)v)\cdot\nabla_v\mu(v)dvds + S(t,x) \tag{2.6}$$

where $S(t,x)$ is given by

$$S(t,x) = \int_{\mathbb{R}^d} f_0(X_{0,t}(x,v), V_{0,t}(x,v))dv$$
$$+ \int_0^t \int_{\mathbb{R}^d} E(s, x-(t-s)v) \cdot \nabla_v \mu(v) dv ds - \int_0^t \int_{\mathbb{R}^d} E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v)) dv ds. \tag{2.7}$$

Taking spacetime Fourier transform, one can express $\widetilde{\rho}(\tau,\xi)$ as

$$\widetilde{\rho}(\tau,\xi) = \frac{1}{1-\widetilde{K}(\tau,\xi)} \widetilde{S}(\tau,\xi), \qquad \text{where} \quad \widetilde{K}(\tau,\xi) = \int_0^\infty e^{-i\tau t} \frac{i\xi}{1+|\xi|^2} \cdot \widehat{\nabla_v \mu}(t\xi) dt. \tag{2.8}$$

The Penrose stability condition (1.2) is equivalent to

$$\inf_{\Im(\tau)\le 0} \inf_{\xi\in\mathbb{R}^d} |1-\widetilde{K}(\tau,\xi)| \ge \kappa \quad \text{for some} \quad \kappa > 0,$$

which is to avoid the singularity in (2.8). Following [13], we can write in the original (t,x) variables as follows:

$$\rho(t) = S(t) + \int_0^t G(t-s) \star_x S(s) ds \tag{2.9}$$

where

$$G(t,x) = \mathcal{F}^{-1}_{(\tau,\xi)\to(t,x)} \left(\frac{\widetilde{K}(\tau,\xi)}{1-\widetilde{K}(\tau,\xi)} \right). \tag{2.10}$$

One of the main results in [13] was to derive pointwise estimates for the resolvent kernel G.

Sketch of the proof of the main theorem:

Our first step is to derive higher derivative estimates for the Green kernel $G(t)$, by using Paley-Littlewood decomposition and localized frequency bounds of G. Making use of the decay bounds for $\nabla_x^k G(t)$, we are able to propagate the decay of $\nabla_x^k \rho(t)$ by the forcing term $S(t)$:

$$\|\nabla_x^k \rho(t)\|_{L^1} \lesssim \langle t\rangle^{-k} \log(1+t) \|S\|_{Y_t^N} \qquad \text{and} \quad \|\nabla_x^k \rho(t)\|_{L^\infty} \lesssim \langle t\rangle^{-d-k} \log(1+t) \|S\|_{Y_t^N}. \tag{2.11}$$

where

$$\|S\|_{Y_t^N} = \max_{0\le k\le N} \sup_{0\le s\le t} \left(\langle s\rangle^k \|\nabla_x^k S(s)\|_{L^1} + \langle s\rangle^{d+k} \|\nabla_x^k S(s)\|_{L^\infty} \right).$$

Thus, it suffices to bound the derivatives of the forcing term $S(t,x)$, defined in (2.7). Inspired by [13, 3], we show that the trajectories $(X_{s,t}(x,v), V_{s,t}(x,v))$ are closed to the characteristics of the free transport $(x-(t-s)v, v)$. To bound $\nabla_x^k S(t,x)$, we also need to use induction argument to decompose $\nabla_x^k S(t,x)$ into quantities involving $\nabla_x^k E$, $\nabla_x^k \rho$ and lower order derivative terms involving characteristic trajectories.

2.4 Organization of the paper

The paper is organized as follows: In Section 3, we justify the estimate (2.11) for the density $\rho(t)$. In Section 4, we prove the decay estimates for the characteristics. In Section 5, we bound the forcing term $S(t,x)$, by determining the forms of its derivatives (Lemma 5.2 and 5.7), and then justify the decay estimates for each of the terms (Proposition 5.3 and Proposition 5.9).

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2.6 Notations

For any complex numbers A,B, we write $A \lesssim B$ or $A = O(B)$, to mean that there exists a universal constant $C_0 > 0$ such that $|A| \le C_0|B|$.
For a vector field $F(x) = (F_1(x), \cdots, F_q(x)) \in \mathbb{R}^q$ with $x \in \mathbb{R}^m$, we denote $\nabla_x^k F$ to be the set of derivatives

$$\{\partial^\alpha F_j : \quad 1 \le j \le q, \quad |\alpha| = k\}.$$

Moreover, for two vector functions F, G defined on $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, we denote $(\nabla_x^u F)(\nabla_y^v G)$ to be the set of all products XY, where $X \in \nabla_x^u F$ and $Y \in \nabla_y^v G$.
For any two vectors $a = (a_i)_{1\le i\le p}, b = (b_i)_{1\le i\le p}$, we denote $a \le b$ if $a_i \le b_i$ for all $1 \le i \le d$.
We also denote $\langle t\rangle = (1+t^2)^{1/2}$. It is obvious that $\langle t\rangle \lesssim 1+t \lesssim \langle t\rangle$ for all $t \ge 0$.
For a function $f(x)$ with $x \in \mathbb{R}^d$, we denote $\hat{f}(\xi)$ to be the standard Fourier transform of f, given by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx.$$

For a function $h(t,x)$ with $x \in \mathbb{R}^d$ and $t \ge 0$, the space-time Fourier transform of h is denoted by $\widetilde{h}(\tau,\xi)$, and is given by

$$\widetilde{h}(\tau,\xi) = \int_0^\infty \int_{\mathbb{R}^d} h(t,x)e^{-i\tau t}e^{-ix\cdot\xi}dxdt.$$

We also use the Paley-Littlewood decomposition on $\mathbb{R}^d$. Let $\chi \in [0,1]$ be a smooth compactly supported function on the annulus $\frac{1}{4} \le |\xi| \le 4$ and equal to one on the annulus $\frac{1}{2} \le |\xi| \le 2$. Define

$$\chi_q(\xi) = \chi\left(\frac{\xi}{2^q}\right) \qquad \text{for} \quad q \in \mathbb{Z}$$

For $u \in \mathcal{S}'(\mathbb{R}^d)$, we have the Paley-Littlewood decomposition

$$u = \sum_{q\in\mathbb{Z}} u_q, \qquad \text{where} \quad \hat{u}_q(\xi) = \hat{u}(\xi)\chi_q(\xi).$$

3 Linear estimates

3.1 Dispersive estimates for the Green kernel

Now let G be the kernel defined as in (2.10). Using Paley-Littlewood decomposition, we can decompose G as

$$G = \sum_{q\in\mathbb{Z}} G_q.$$

Now, we recall the following decaying bounds on the localized-in-frequency Green kernel G_q in [13].

Lemma 3.1. *Let $\mu(v)$ be smooth, satisfy the bound* (2.1) *and the Penrose stability condition* (1.2)*. For any $K > 0$, there exists $A \geq 1$ such that for every $\delta \in (0, 1]$, and for every $q \in \mathbb{Z}$ with $2^q \geq A$:*

$$\|G_q(t)\|_{L^1} \lesssim \frac{2^{q(1+\delta)}}{1+2^{2q}} \cdot \frac{1}{(1+2^q t)^K} \quad \textit{and} \quad \|G_q(t)\|_{L^\infty} \lesssim \frac{2^{q(d+1+\delta)}}{1+2^{2q}} \cdot \frac{1}{(1+2^q t)^K}.$$

Moreover, for $2^q \leq A$, one has

$$\|G_q(t)\|_{L^1} \lesssim \frac{2^q}{(1+2^q t)^K} \quad \textit{and} \quad \|G_q(t)\|_{L^\infty} \lesssim \frac{2^{q(d+1)}}{(1+2^q t)^K}.$$

Making use of the decay bounds on the Green kernels $\{G_q\}_{q\in\mathbb{Z}}$, we establish the following bound on the derivatives of $G(t)$:

Theorem 3.2. *For any integer $k \geq 0$, there holds*

$$\|\nabla_x^k G(t)\|_{L^1} \lesssim t^{-k-1} \qquad \textit{and} \quad \|\nabla_x^k G(t)\|_{L^\infty} \lesssim t^{-d-1-k}$$

for $t > 0$, where G is the kernel defined in (2.10)*.*

Proof. We take K so that $K > k + d + 1$. We bound $\|\nabla_x^k G(t)\|_{L^1}$ as follows:

$$\begin{aligned}
\|\nabla_x^k G(t)\|_{L^1} &\lesssim \sum_{2^q \leq A} 2^{kq}\|G_q(t)\|_{L^1} + \sum_{2^q \geq A} 2^{kq}\|G_q(t)\|_{L^1} \\
&\lesssim \sum_{2^q \leq A} \frac{2^{q(k+1)}}{(1+2^q t)^K} + \sum_{2^q \geq A} \frac{2^{q(k+d+1+\delta)}}{(1+2^{2q})(1+2^q t)^K}.
\end{aligned}$$

For the first term, we see that

$$\begin{aligned}
\sum_{2^q \leq A} \frac{2^{q(k+1)}}{(1+2^q t)^K} &\lesssim \left(\sum_{2^q \leq t^{-1}} + \sum_{t^{-1} \leq 2^q \leq 1} + \sum_{1 \leq 2^q \leq A} \right) \frac{2^{q(k+1)}}{(1+2^q t)^K} \\
&\lesssim \sum_{2^q \leq t^{-1}} 2^{q(k+1)} + t^{-K} \sum_{2^q \leq 1} 2^{q(k+1-K)} + t^{-K} \sum_{1 \leq 2^q \leq A} 2^{q(k+1-K)} \\
&\lesssim t^{-(k+1)} + t^{-K} \lesssim t^{-(k+1)}.
\end{aligned}$$

The second term is treated as follows:

$$\sum_{2^q \ge A} \frac{2^{q(k+d+1+\delta)}}{(1+2^{2q})(1+2^q t)^K} \lesssim t^{-K} \sum_{2^q \ge A} 2^{q(k+d+1+\delta-2-K)} \lesssim t^{-K} \lesssim t^{-k-1} \qquad \text{since} \quad K > k+1+d.$$

The decay bound for $\|\nabla_x^k G(t)\|_{L^1}$ is complete. Now we bound $\|\nabla_x^k G(t)\|_{L^\infty}$ as follows:

$$\begin{aligned}
\|\nabla_x^k G(t)\|_{L^\infty} &\lesssim \sum_{2^q \le A} 2^{kq} \|G_q(t)\|_{L^\infty} + \sum_{2^q \ge A} 2^{kq} \|G_q(t)\|_{L^\infty} \\
&\lesssim \sum_{2^q \le A} \frac{2^{q(d+1+k)}}{(1+2^q t)^K} + \sum_{2^q \ge A} \frac{2^{q(d+1+\delta+k)}}{1+2^{2q}} \cdot \frac{1}{(1+2^q t)^K}.
\end{aligned}$$

For the first term, we see that

$$\begin{aligned}
\sum_{2^q \le A} \frac{2^{q(d+1+k)}}{(1+2^q t)^K} &\lesssim \left(\sum_{2^q \le t^{-1}} + \sum_{t^{-1} \le 2^q \le 1} + \sum_{1 \le 2^q \le A} \right) \frac{2^{q(d+1+k)}}{(1+2^q t)^K} \\
&\lesssim \sum_{2^q \le t^{-1}} 2^{q(d+1+k)} + t^{-K} \sum_{t^{-1} \le 2^q \le 1} 2^{q(d+1+k-K)} + t^{-K} \sum_{1 \le 2^q \le A} 2^{q(d+1+k-K)} \\
&\lesssim t^{-(d+1+k)} + t^{-K} \lesssim t^{-(d+1+k)}.
\end{aligned}$$

For the second term, we see that

$$\sum_{2^q \ge A} \frac{2^{q(d+1+\delta+k)}}{1+2^{2q}} \cdot \frac{1}{(1+2^q t)^K} \lesssim t^{-K} \sum_{2^q \ge A} 2^{q(d+1+\delta+k-K-2)} \lesssim t^{-K} \lesssim t^{-(d+1+k)}.$$

The proof is complete. $\square$

3.2 Decay estimates for the density and electric field of the linearized problem

In this section, we drive decay estimates for higher derivatives of the density $\rho(t)$, defined in (2.9) and and the electric field $E = -\nabla_x (1 - \nabla_x)^{-1} \rho$. For $N > 0$, we define

$$\|S\|_{Y_t^N} = \max_{0 \le k \le N} \sup_{0 \le s \le t} \Big(\langle s \rangle^k \|\nabla_x^k S(s)\|_{L^1} + \langle s \rangle^{d+k} \|\nabla_x^k S(s)\|_{L^\infty} \Big). \tag{3.1}$$

By definition, we see that

$$\|\nabla_x^k S(s)\|_{L^1} \le \langle s \rangle^{-k} \|S\|_{Y_t^N} \qquad \text{and} \quad \|\nabla_x^k S(s)\|_{L^\infty} \le \langle s \rangle^{-(k+d)} \|S\|_{Y_t^N}$$

for $0 \le k \le N$ and $0 \le s \le t$.

Theorem 3.3. *Let $N \ge 1$ be an integer. The density $\rho(t)$ defined in (2.9) satisfies the bound*

$$\max_{0 \le k \le N} \Big\{ \langle t \rangle^k \|\nabla_x^k \rho(t)\|_{L^1} + \langle t \rangle^{d+k} \|\nabla_x^k \rho(t)\|_{L^\infty} \Big\} \lesssim \log(1+t) \|S\|_{Y_t^N} \qquad \text{for} \quad t > 0. \tag{3.2}$$

Proof. Fix k so that $0 \le k \le N$. First we estimate $\nabla_x^k \rho$ in L^1. Applying ∇_x^k to both sides of (2.9), we get

$$\nabla_x^k \rho(t) = \nabla_x^k S(t) + \int_0^{t/2} \nabla_x^k G(t-s) \star_x S(s) ds + \int_{t/2}^t G(t-s) \star_x \nabla_x^k S(s) ds.$$

This implies

$$\begin{aligned}
\|\nabla_x^k \rho(t)\|_{L^1} &\le \|\nabla_x^k S(t)\|_{L^1} + \int_0^{t/2} \|\nabla_x^k G(t-s)\|_{L^1} \|S(s)\|_{L^1} ds + \int_{t/2}^t \|G(t-s)\|_{L^1} \|\nabla_x^k S(s)\|_{L^1} ds \\
&\lesssim \langle t \rangle^{-k} \|S\|_{Y_t^N} + \|S\|_{Y_t^N} \left(\int_0^{t/2} \frac{1}{(t-s)^{k+1}} ds + \int_{t/2}^t \frac{1}{(1+t-s)} \langle s \rangle^{-k} ds \right) \\
&\lesssim \|S\|_{Y_t^N} \left(\langle t \rangle^{-k} + t^{-k} + \frac{\log(1+t)}{(1+t)^k} \right)
\end{aligned}$$

Hence

$$\langle t \rangle^k \|\nabla_x^k \rho(t)\|_{L^1} \lesssim \log(1+t) \|S\|_{Y_t^N}.$$

Now we estimate the L^∞ norm of $\nabla_x^k \rho$. We have

$$\begin{aligned}
\|\nabla_x^k \rho(t)\|_{L^\infty} &\le \|\nabla_x^k S(t)\|_{L^\infty} + \int_0^{t/2} \|\nabla_x^k G(t-s)\|_{L^\infty} \|S(s)\|_{L^1} ds + \int_{t/2}^t \|G(t-s)\|_{L^1} \|\nabla_x^k S(s)\|_{L^\infty} ds \\
&\lesssim \|S\|_{Y_t^N} \langle t \rangle^{-k-d} + \|S\|_{Y_t^N} \left(\int_0^{t/2} \frac{1}{(t-s)^{k+d+1}} ds + \int_{t/2}^t \frac{1}{(1+t-s)} \langle s \rangle^{-k-d} ds \right) \\
&\lesssim \frac{\log(1+t)}{(1+t)^{d+k}} \|S\|_{Y_t^N},
\end{aligned}$$

and hence

$$\langle t \rangle^{d+k} \|\nabla_x^k \rho(t)\|_{L^\infty} \lesssim \log(1+t) \|S\|_{Y_t^N}.$$

The proof is complete. ∎

Theorem 3.4. *Assume that*

$$\max_{0 \le k \le N} \left\{ \langle t \rangle^{d+k} \|\nabla_x^k \rho(t)\|_{L^\infty} + \langle t \rangle^k \|\nabla_x^k \rho(t)\|_{L^1} \right\} \lesssim \varepsilon \log(1+t),$$

there holds

$$\max_{0 \le k \le N} \left\{ \langle t \rangle^{d+k} \|\nabla_x^k E(t)\|_{L^\infty} + \langle t \rangle^k \|\nabla_x^k E(t)\|_{L^1} \right\} \lesssim \varepsilon \log(1+t).$$

Proof. Since $E = -\nabla_x (1 - \Delta_x)^{-1} \rho$, we have

$$\nabla_x^k E = -\nabla_x (1 - \Delta_x)^{-1} (\nabla_x^k \rho)$$

By the standard elliptic estimate, we get

$$\begin{cases} \|\nabla_x^k E(t)\|_{L^\infty} & \lesssim \|\nabla_x^k \rho(t)\|_{L^\infty} \lesssim \varepsilon \langle t \rangle^{-d-k} \log(1+t), \\ \|\nabla_x^k E(t)\|_{L^1} & \lesssim \|\nabla_x^k \rho(t)\|_{L^1} \lesssim \varepsilon \langle t \rangle^{-k} \log(1+t). \end{cases}$$

The proof is complete. ∎

4 Decay estimates for the characteristics

4.1 Main theorem

From the equations (2.5), the characteristics $X_{s,t}(x,v)$ and $V_{s,t}(x,v)$ can be written as follows:

$$\begin{cases} X_{s,t}(x,v) & = x-(t-s)v+\int_s^t(\tau-s)E(\tau,X_{\tau,t}(x,v))d\tau, \\ V_{s,t}(x,v) & = v-\int_s^t E(\tau,X_{\tau,t}(x,v))d\tau. \end{cases}$$

Following [13], we define $(Y_{s,t}(x,v),W_{s,t}(x,v))$ so that

$$\begin{cases} X_{s,t}(x,v) = x-(t-s)v+Y_{s,t}(x-tv,v), \\ V_{s,t}(x,v) = v+W_{s,t}(x-tv,v). \end{cases} \tag{4.1}$$

Hence, we get

$$\begin{cases} Y_{s,t}(x,v) & = \int_s^t(\tau-s)E(\tau,x+\tau v+Y_{\tau,t}(x,v))d\tau, \\ W_{s,t}(x,v) & = -\int_s^t E(\tau,x+\tau v+Y_{\tau,t}(x,v))d\tau. \end{cases} \tag{4.2}$$

Our main theorem in this section is as follows:

Theorem 4.1. *Assume that*

$$\max_{0\le k\le N}\Big(\langle t\rangle^{d+k}\|\nabla_x^k E(t)\|_{L^\infty}\Big)\lesssim \varepsilon\log(1+t) \qquad \textit{for all} \quad t>0, \tag{4.3}$$

there holds, for all $s\in[0,t]$, the following inequalities:

$$\max_{0\le k\le N}\|\nabla_v^k Y_{s,t}\|_{L^\infty}\lesssim \varepsilon\frac{\log(1+s)}{(1+s)^{d-2}} \tag{4.4}$$

and

$$\max_{0\le k\le N}\|\nabla_v^k W_{s,t}\|_{L^\infty}\lesssim \varepsilon\frac{\log(1+s)}{(1+s)^{d-1}}. \tag{4.5}$$

Before giving the proof, we recall the Faa di Bruno's formula in [1], which allows us to compute higher order derivatives of $Y_{s,t}$ and $W_{s,t}$ by the generalized chain rule:

Lemma 4.2. *Let $u:\mathbb{R}^d\to\mathbb{R}^m$ and $F:\mathbb{R}^m\to\mathbb{R}$ be smooth functions. For each multi-index $\alpha\in\mathbb{N}^d$, we have*

$$\partial^\alpha(F\circ u)=\sum_{\mu,\nu}C_{\mu,\nu}\partial^\mu F\prod_{1\le|\beta|\le|\alpha|,1\le j\le m}(\partial^\beta u^j)^{\nu_{\beta_j}}$$

where $C_{\mu,\nu}$ are non negative integers, and the sum is taken over μ,ν such that $1\le|\mu|\le|\alpha|$, $\nu_{\beta_j}\in\mathbb{N}^\star$,

$$\sum_{1\le|\beta|\le|\alpha|}\nu_{\beta_j}=\mu_j \qquad \textit{for} \quad 1\le j\le m, \qquad \textit{and} \qquad \sum_{1\le|\beta|\le|\alpha|,1\le j\le m}\beta\nu_{\beta_j}=\alpha.$$

Proof of Theorem 4.1: Fix $k \in \{0, 1, \cdots, N\}$. We shall prove that

$$\langle s\rangle^{d-2}\|\nabla_v^k Y_{s,t}\|_{L^\infty} + \langle s\rangle^{d-1}\|\nabla_v^k W_{s,t}\|_{L^\infty} \lesssim \varepsilon \log(1+s).$$

In [13], the authors prove the above statement when $k \in \{0, 1\}$, thus we shall only consider $k \geq 2$. Let us first justify the bound for the case $k = 2$. Applying $\partial^2_{v_iv_j}$ to both sides of (4.2), we have

$$\begin{aligned}\partial^2_{v_iv_j} Y_{s,t}(x,v) = &\int_s^t (\tau - s)\partial^2_{x_ix_j} E(\tau, x + \tau v + Y_{\tau,t}(x,v)) \left(\tau + \partial_{v_j} Y_{\tau,t}(x,v)\right) (\tau + \partial_{v_i} Y_{\tau,t}(x,v))\, d\tau \\ &+ \int_s^t (\tau - s)\partial_{x_i} E(\tau, x + \tau v + Y_{\tau,t}(x,v))\partial^2_{v_iv_j} Y_{\tau,t}(x,v) d\tau.\end{aligned}$$

This implies

$$\begin{aligned}|\partial^2_{v_iv_j} Y_{s,t}(x,v)| &\leq \int_s^t (\tau - s)\|\nabla_x^2 E(\tau)\|_{L^\infty}(\tau + \|\nabla_v Y_{\tau,t}\|_{L^\infty})^2 d\tau + \int_s^t (\tau - s)\|\nabla_x E(\tau)\|_{L^\infty} \sup_{0\leq s\leq t}\|\nabla_v^2 Y_{s,t}\|_{L^\infty} d\tau \\ &\lesssim \int_s^t (\tau - s)\cdot \varepsilon\log(1+\tau)\langle\tau\rangle^{-d-2}(\tau+\varepsilon)^2 d\tau + \sup_{0\leq s\leq t}\|\nabla_v^2 Y_{s,t}\|_{L^\infty}\int_s^t (\tau - s)\cdot\varepsilon\log(1+\tau)\langle\tau\rangle^{-d-1} d\tau.\end{aligned}$$

Hence we get

$$\|\nabla_v^2 Y_{s,t}\|_{L^\infty} \lesssim \varepsilon\frac{\log(1+s)}{(1+s)^{d-2}} + \left(\varepsilon\frac{\log(1+s)}{(1+s)^{d-1}}\right)\sup_{0\leq s\leq t}\|\nabla_v^2 Y_{s,t}\|_{L^\infty}.$$

Thus, as long as $\varepsilon\frac{\log(1+s)}{(1+s)^{d-1}}$ is small, we have

$$\|\nabla_v^2 Y_{s,t}\|_{L^\infty} \lesssim \varepsilon\frac{\log(1+s)}{(1+s)^{d-2}}.$$

Now we estimate $\nabla_v^2 W_{s,t}$. Applying $\partial^2_{v_iv_j}$ to both sides of(4.2), we get

$$\begin{aligned}\partial^2_{v_iv_j} W_{s,t}(x,v) = &-\int_s^t \left(\partial^2_{x_ix_j} E(\tau, x+\tau v + Y_{\tau,t}(x,v))(\tau + \partial_{v_j} Y_{\tau,t}(x,v))\right)(\tau + \partial_{v_i} Y_{\tau,t}(x,v))\, d\tau \\ &- \int_s^t \partial_{x_i} E(\tau, x + \tau v + Y_{\tau,t}(x,v))\left(\partial^2_{v_iv_j} Y_{\tau,t}(x,v)\right) d\tau.\end{aligned}$$

This implies

$$|\partial^2_{v_iv_j} W_{s,t}(x,v)| \leq \int_s^t \|\nabla_x^2 E(\tau)\|_{L^\infty}(\tau + \|\nabla_v Y_{\tau,t}\|_{L^\infty})^2 + \int_s^t \|\nabla_x E(\tau)\|_{L^\infty}\|\nabla_v^2 Y_{\tau,t}\|_{L^\infty} d\tau$$

Using the fact that $\langle t\rangle^d\|E(t)\|_{L^\infty} + \langle t\rangle^{d+1}\|\nabla_x E(t)\|_{L^\infty} \lesssim \varepsilon\log(1+t)$ and $\|\nabla_v^2 Y_{\tau,t}\|_{L^\infty} \lesssim \varepsilon$, we get

$$\begin{aligned}|\partial^2_{v_iv_j} W_{s,t}(x,v)| &\lesssim \varepsilon\int_s^t \frac{\log(1+\tau)}{(1+\tau)^{d+2}}(\tau+\varepsilon)^2 d\tau + \varepsilon^2\int_s^t \frac{\log(1+\tau)}{(1+\tau)^{d+1}} d\tau \\ &\lesssim \varepsilon\int_s^t \frac{\log(1+\tau)}{(1+\tau)^d} d\tau + \varepsilon^2\frac{\log(1+s)}{(1+s)^d} \lesssim \varepsilon\frac{\log(1+s)}{(1+s)^{d-1}}.\end{aligned}$$

Now for a general multi-index α with $k = |\alpha| \geq 3$, we proceed by induction on k, by assuming that the decay estimates (4.4) and (4.5) are true for all index with length less than k. Applying the chain rule (4.2) for $Y_{s,t}$, we get

$$\begin{aligned}
\partial_v^\alpha Y_{s,t} &= \int_s^t (\tau - s) \sum_{(\mu,\nu)\in I} C_{\mu,\nu} \partial_x^\mu E \prod_{1\leq|\beta|\leq|\alpha|, 1\leq j\leq d} \left(\partial_v^\beta (x_j + \tau v_j + Y_{\tau,t}^j(x,v))\right)^{\nu_{\beta_j}} d\tau \\
&= \int_s^t (\tau - s) \sum_{(\mu,\nu)\in I} C_{\mu,\nu} \partial_x^\mu E \prod_{1\leq|\beta|\leq|\alpha|, 1\leq j\leq d} \left(\partial_v^\beta (\tau v_j + Y_{\tau,t}^j(x,v))\right)^{\nu_{\beta_j}} d\tau \\
&= \int_s^t (\tau - s) \sum_{(\mu,\nu)\in I} C_{\mu,\nu} \partial_x^\mu E \prod_{1\leq|\beta|\leq|\alpha|, 1\leq j\leq d} \tau^{\nu_{\beta_j}} \left(\partial_v^\beta v_j\right)^{\nu_{\beta_j}} d\tau \\
&\quad + \int_s^t (\tau - s) \sum_{(\mu,\nu)\in I} C_{\mu,\nu} \partial_x^\mu E \prod_{1\leq|\beta|\leq|\alpha|, 1\leq j\leq d} \left(\partial_v^\beta Y_{\tau,t}^j(x,v)\right)^{\nu_{\beta_j}} d\tau.
\end{aligned} \tag{4.6}$$

where

$$I = \left\{ (\mu,\nu) \in \mathbb{N}^{2d} : \quad \sum_{1\leq|\beta|\leq|\alpha|} \nu_{\beta_j} = \mu_j \quad \text{for} \quad 1 \leq j \leq d, \sum_{1\leq|\beta|\leq|\alpha|, 1\leq j\leq d} \beta \nu_{\beta_j} = \alpha \right\}.$$

Now we fix μ, ν and estimate each term appearing in (4.6). The first term can be estimated as follows:

$$\begin{aligned}
\int_s^t (\tau - s) \partial_x^\mu E \prod_{1\leq|\beta|\leq|\alpha|, 1\leq j\leq d} \tau^{\nu_{\beta_j}} (\partial_v^\beta v_j)^{\nu_{\beta_j}} d\tau &\lesssim \int_s^t (\tau - s) \frac{\varepsilon \log(1+\tau)}{(1+\tau)^{d+|\mu|}} \tau^{\sum_{j=1}^d \nu_{\beta_j}} d\tau \\
&\lesssim \int_s^t \tau^{\mu_j + 1} \frac{\varepsilon \log(1+\tau)}{(1+\tau)^{d+|\mu|}} d\tau \\
&\leq \int_s^t \varepsilon \frac{\log(1+\tau)}{(1+\tau)^{d-1}} d\tau \lesssim \varepsilon \frac{\log(1+s)}{(1+s)^{d-2}}.
\end{aligned}$$

Now we estimate the second term in (4.6), which is

$$\int_s^t (\tau - s) \partial_x^\mu E \prod_{1\leq|\beta|\leq|\alpha|, 1\leq j\leq d} \left(\partial_v^\beta Y_{\tau,t}^j(x,v)\right)^{\nu_{\beta_j}} d\tau. \tag{4.7}$$

We consider two cases:
Case 1: $\beta_j < |\alpha|$ for all $1 \leq j \leq d$.
In this case, we use the induction hypothesis on $\partial_v^\beta Y_{\tau,t}$, which is $|\partial_v^\beta Y_{\tau,t}^j(x,v)| \lesssim \varepsilon$. Hence (4.7) can be bounded by

$$\int_s^t (\tau - s) \|\partial_x^\mu E(\tau)\|_{L^\infty} \varepsilon^{\sum_{j=1}^d \nu_{\beta_j}} d\tau \lesssim \int_s^t \tau \frac{\varepsilon \log(1+\tau)}{(1+\tau)^{d+|\mu|}} \varepsilon^{|\mu|} d\tau \lesssim \varepsilon \frac{\log(1+s)}{(1+s)^{d-2}}.$$

Case 2: There exists $j_0 \in \{1, 2, \cdots, d\}$ such that $\beta_{j_0} = |\alpha|$.
In this case, since $|\beta| \le |\alpha|$, we have $\beta_j = 0$ for all $j \neq j_0$. Hence the term is reduced to

$$\int_s^t (\tau - s)\partial_x^\mu E \cdot (\partial_{v_{j_0}}^{|\alpha|} Y_{\tau,t}^j)^{\nu_{|\alpha|}} O(\varepsilon) d\tau,$$

where we use the fact that $\|Y_{s,t}\|_{L^\infty} \lesssim \varepsilon$. Moreover, since $\sum_{j=1}^d \beta\nu_{\beta_j} = \alpha$, we have $\nu_{|\alpha|} \le 1$. Hence

$$\begin{aligned}
&\int_s^t (\tau - s)\partial_x^\mu E \cdot (\partial_{v_{j_0}}^{|\alpha|} Y_{\tau,t}^j)^{\nu_{|\alpha|}} O(\varepsilon) \lesssim \varepsilon \int_s^t \tau \frac{\varepsilon \log(1+\tau)}{(1+\tau)^{d+|\mu|}} \sup_{0\le s\le t} \|\partial_v^{|\alpha|} Y_{s,t}\|_{L^\infty} d\tau \\
&= \sup_{0\le s\le t} \|\partial_v^{|\alpha|} Y_{s,t}\|_{L^\infty} \int_s^t \varepsilon^2 \frac{\log(1+\tau)}{(1+\tau)^{d+|\mu|-1}} d\tau \lesssim \varepsilon^2 \sup_{0\le s\le t} \|\nabla_v^{|\alpha|} Y_{s,t}\|_{L^\infty}.
\end{aligned}$$

Thus, we get the inequality of the form

$$\|\nabla_v^\alpha Y_{s,t}\|_{L^\infty} \lesssim \varepsilon \frac{\log(1+s)}{(1+s)^{d-2}} + \varepsilon^2 \sup_{0\le s\le t} \|\nabla_v^{|\alpha|} Y_{s,t}\|_{L^\infty}.$$

Hence for ε small, we get

$$\|\nabla_v^\alpha Y_{s,t}\|_{L^\infty} \lesssim \varepsilon \frac{\log(1+s)}{(1+s)^{d-2}} \quad \text{for} \quad \text{all} \quad 0 \le s \le t.$$

The proof is complete.

4.2 Straightening the characteristics

Next, we recall the following lemma about straightening the characteristics from [13].

Lemma 4.3. *Let $Y_{s,t}$ be defined as in* (4.1) *and* (4.2)*. Assume that E satisfies the decay estimates* (4.3)*, there holds*

$$\|\nabla_x Y_{s,t}\|_{L^\infty} \lesssim \frac{\varepsilon \log(1+s)}{(1+s)^{d-1}}.$$

Proof. By the definition of $Y_{s,t}$, we have

$$Y_{s,t}(x,v) = \int_s^t (\tau - s) E(\tau, x + \tau v + Y_{\tau,t}(x,v)) d\tau$$

Hence we get

$$\begin{aligned}
|\nabla_x Y_{s,t}(x,v)| &\le \int_s^t (\tau - s)\|\nabla_x E(\tau)\|_{L^\infty}(1 + \|\nabla_x Y_{\tau,t}\|_{L^\infty}) d\tau \\
&\lesssim \int_s^t (\tau - s) \frac{\varepsilon \log(1+\tau)}{(1+\tau)^{d+1}} d\tau + \sup_{0\le\tau\le t} \|\nabla_x Y_{\tau,t}\|_{L^\infty} \int_s^t (\tau - s) \frac{\varepsilon \log(1+\tau)}{(1+\tau)^{d+1}} d\tau \\
&\lesssim \varepsilon \frac{\log(1+s)}{(1+s)^{d-1}} + \left\{ \varepsilon \frac{\log(1+s)}{(1+s)^{d-1}} \right\} \sup_{0\le s\le t} \|Y_{s,t}\|_{L^\infty}.
\end{aligned}$$

Hence as long as ε is small enough, we have

$$|\nabla_x Y_{s,t}(x,v)| \lesssim \frac{\varepsilon \log(1+s)}{(1+s)^{d-1}}$$

The proof is complete. ∎

Lemma 4.4. *For $0 \le s \le t$, there exists a C^1 map $(x,v) \to \Psi_{s,t}(x,v)$ such that*

$$X_{s,t}(x, \Psi_{s,t}(x,v)) = x - (t-s)v$$

for all $x, v \in \mathbb{R}^d$. Moreover, if E satisfies the estimates (4.3), there holds

$$\langle s\rangle^d |\Psi_{s,t}(x,v) - v| + \langle s\rangle^{d-1} |\nabla_v(\Psi_{s,t}(x,v) - v)| \lesssim \varepsilon \log(1+s)$$

for all $x, v \in \mathbb{R}^d$ and $0 \le s \le t$.

Proof. We write

$$X_{s,t}(x,v) = x - (t-s)(v + \Phi_{s,t}(x,v))$$

and will show that $(x,v) \to (x, v + \Phi_{s,t}(x,v))$ is a C^1 differomorphism. To this end, we prove that

$$\langle s\rangle^d \|\Phi_{s,t}\|_{L^\infty} + \langle s\rangle^d \|\nabla_x \Phi_{s,t}\|_{L^\infty} + \langle s\rangle^{d-1} \|\nabla_v \Phi_{s,t}\|_{L^\infty} \lesssim \varepsilon \log(1+s)$$

We have

$$\begin{aligned}
\Phi_{s,t}(x,v) &= -\frac{1}{t-s}(X_{s,t}(x,v) - (t-s)v) \\
&= -\frac{1}{t-s} \int_s^t (\tau - s) E(\tau, x - (t-\tau)v + Y_{\tau,t}(x - vt, v)) d\tau \qquad (4.8) \\
&\lesssim \frac{1}{t-s} \int_s^t (\tau - s) \|E(\tau)\|_{L^\infty} d\tau \lesssim \int_s^t \|E(\tau)\|_{L^\infty} d\tau.
\end{aligned}$$

Since $E = -\nabla_x (1 - \Delta_x)^{-1} \rho$, we have

$$\|E(\tau)\|_{L^\infty} \lesssim \|\nabla_x \rho(\tau)\|_{L^\infty} \lesssim \frac{\varepsilon \log(1+\tau)}{(1+\tau)^{d+1}}.$$

Thus we have

$$|\Phi_{s,t}(x,v)| \lesssim \int_s^t \frac{\varepsilon \log(1+\tau)}{(1+\tau)^{d+1}} d\tau \lesssim \varepsilon \frac{\log(1+s)}{(1+s)^d}.$$

Thus $\langle s\rangle^d \|\Phi_{s,t}\|_{L^\infty} \lesssim \varepsilon \log(1+s)$. Next, applying ∇_x to both sides of (4.8), we have

$$\|\nabla_x \Phi_{s,t}\|_{L^\infty} \lesssim \frac{1}{t-s} \int_s^t (\tau - s) \|\nabla_x E(\tau)\|_{L^\infty} (1 + \|\nabla_x Y_{\tau,t}\|_{L^\infty}) \, d\tau.$$

Now using lemma 4.3, we get

$$\|\nabla_x \Phi_{s,t}\|_{L^\infty} \lesssim \int_s^t \frac{\varepsilon \log(1+\tau)}{(1+\tau)^{d+1}} d\tau \lesssim \varepsilon \langle s\rangle^{-d} \log(1+s).$$

Now we bound $\|\nabla_v \Phi_{s,t}\|_{L^\infty}$. Applying ∇_v to both sides of (4.8), we get

$$\|\nabla_v \Phi_{s,t}\|_{L^\infty} \lesssim \frac{1}{t-s}\int_s^t (\tau - s)\|\nabla_x E(\tau)\|_{L^\infty}\left\{(t-\tau) + t\|\nabla_x Y_{\tau,t}\|_{L^\infty}\right\} d\tau.$$

Again, using Lemma 4.3, we have

$$\begin{aligned}
\|\nabla_v \Phi_{s,t}\|_{L^\infty} &\lesssim \frac{1}{t-s}\int_s^t (\tau-s)\frac{\varepsilon\log(1+\tau)}{(1+\tau)^{d+1}}\left((t-\tau) + \frac{\varepsilon t\log(1+\tau)}{(1+\tau)^{d-1}}\right) d\tau \\
&\lesssim \varepsilon\frac{\log(1+s)}{(1+s)^{d-1}} + \varepsilon\int_s^t \frac{t(\tau-s)}{t-s}\frac{\log(1+\tau)^2}{(1+\tau)^{d-1}} d\tau \\
&\lesssim \varepsilon\frac{\log(1+s)}{(1+s)^{d-1}} + \varepsilon\int_s^t (\tau-s)\frac{\log(1+\tau)^2}{(1+\tau)^{d-1}} d\tau + \varepsilon\int_s^t (\tau-s)^2\frac{\log(1+\tau)^2}{(1+\tau)^{d-1}} d\tau \\
&\lesssim \varepsilon\frac{\log(1+s)}{(1+s)^{d-1}}.
\end{aligned}$$

Hence, the map $(x, v) \to (x, v + \Phi_{s,t}(x, v))$ is a C^1 differomorphism. Thus there exists a C^1 differomorphism $v \to \Psi_{s,t}(x, v)$ such that

$$X_{s,t}(x, \Psi_{s,t}(x, v)) = x - (t-s)v.$$

Combining this with $X_{s,t}(x, v) = x - (t - s)(v + \Phi_{s,t}(x, v))$, we have

$$\begin{cases} |\Psi_{s,t}(x,v) - v| & \lesssim \|\Phi_{s,t}\|_{L^\infty} \lesssim \varepsilon\langle s\rangle^{-d}\log(1+s), \\ |\nabla_v(\Psi_{s,t}(x,v) - v)| & \lesssim \|\nabla_v\Phi_{s,t}\|_{L^\infty} \lesssim \varepsilon\langle s\rangle^{-(d-1)}\log(1+s). \end{cases}$$

The proof is complete. □

5 Decay estimates for the forcing term

In this section, we derive decay estimates for the derivatives of the forcing term, appearing in the equation (2.6). The forcing term $S(t, x)$ is given by

$$\begin{aligned}
S(t,x) = &\int_{\mathbb{R}^d} f_0(X_{0,t}(x,v), V_{0,t}(x,v))dv + \int_0^t\int_{\mathbb{R}^d} E(s, x-(t-s)v)\cdot\nabla_v\mu(v)dvds \\
&- \int_0^t\int_{\mathbb{R}^d} E(s, X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v))dvds \\
&= \mathcal{I}(t,x) + \mathcal{R}_L(t,x) - \mathcal{R}_{NL}(t,x),
\end{aligned} \tag{5.1}$$

where

$$\begin{cases} \mathcal{I}(t,x) & = \int_{\mathbb{R}^d} f_0(X_{0,t}(x,v), V_{0,t}(x,v))dv, \\ \mathcal{R}_L(t,x) & = \int_0^t\int_{\mathbb{R}^d} E(s, x-(t-s)v)\cdot\nabla_v\mu(v)dvds, \\ \mathcal{R}_{NL}(t,x) & = \int_0^t\int_{\mathbb{R}^d} E(s, X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v))dvds. \end{cases}$$

Fix $N \geq 1$, we will give decay estimates for $\|S\|_{Y_t^N}$, under the decaying assumptions on derivatives of E, $Y_{s,t}$ and $W_{s,t}$. We also recall the smallness assumption for the initial perturbation $f_0(x,v)$ as follows:

$$\max_{0\leq k\leq N} \|\nabla_{x,v}^k f_0\|_{L_x^1 L_v^\infty} \leq \varepsilon_0. \tag{5.2}$$

Our main theorem is as follows:

Theorem 5.1. *Let $N > 1$ be an integer. Assume that $E, W_{s,t}, Y_{s,t}$ satisfy the decay estimates* (4.3), (4.5) *and* (4.4) *and f_0 satisfies the smallness assumption* (5.2), *there holds*

$$\sup_{0\leq k\leq N} \Big(\langle t\rangle^k \|\nabla_x^k \mathcal{I}(t)\|_{L^1} + \langle t\rangle^{d+k} \|\nabla_x^k \mathcal{I}(t)\|_{L^\infty} \Big) \lesssim \varepsilon_0, \tag{5.3}$$

and

$$\sup_{0\leq k\leq N} \Big(\langle t\rangle^k \|\nabla_x^k \mathcal{R}(t)\|_{L^1} + \langle t\rangle^{d+k} \|\nabla_x^k \mathcal{R}(t)\|_{L^\infty} \Big) \lesssim \varepsilon^2.$$

5.1 Decay estimates for the initial data term

First, we estimate $\mathcal{I}(t,x)$, under suitable smallness assumption (5.2) on the initial data f_0. In [13], the authors prove that

$$\|\mathcal{I}(t)\|_{L^1} + \langle t\rangle^d \|\mathcal{I}(t)\|_{L^\infty} + \langle t\rangle \|\nabla_x \mathcal{I}(t)\|_{L^1} + \langle t\rangle^{d+1} \|\nabla_x \mathcal{I}(t)\|_{L^\infty} \lesssim \varepsilon_0.$$

Hence, we shall establish the bound (5.3) for $k \geq 2$. First, we establish the following lemma

Lemma 5.2. *Let $k \geq 2$ and*

$$\mathcal{I}(t,x) = \int_{\mathbb{R}^d} f_0(X_{0,t}(x,v), V_{0,t}(x,v)) dv.$$

The term $\nabla_x^k \mathcal{I}(t,x)$ can be written as a sum of many terms, which are all in the form

$$\int_{\mathbb{R}^d} \nabla_{x,v}^\alpha f_0 \cdot (\nabla_v^{\beta_1} Y_{0,t})^{k_1} \cdots (\nabla_v^{\beta_r} Y_{0,t})^{k_r} \cdot (\nabla_v^{\gamma_1} W_{0,t})^{s_1} \cdots (\nabla_v^{\gamma_t} W_{0,t})^{s_t} \frac{dw}{t^{d+k}} \tag{5.4}$$

where α, $(\beta_1, k_1), \cdots, (\beta_r, k_2)$, and $(\gamma_1, s_1), \cdots, (\gamma_t, s_t)$ satisfy

$$1 \leq |\alpha| \leq k \qquad \text{and} \qquad (\beta_1 k_1 + \cdots + \beta_r k_r) + (\gamma_1 s_1 + \cdots + \gamma_t s_t) \leq k. \tag{5.5}$$

Proof. From (4.1), we get

$$\mathcal{I}(t,x) = \int_{\mathbb{R}^d} f_0(x - tv + Y_{0,t}(x-tv, v), v + W_{0,t}(x-tv, v)) dv.$$

Let $w = x - tv$, we get

$$\mathcal{I}(t,x) = \int_{\mathbb{R}^d} f_0 \left(w + Y_{0,t}(w, \frac{x-w}{t}), \frac{x-w}{t} + W_{0,t}(w, \frac{x-w}{t}) \right) \frac{dw}{t^d}.$$

By a direct calculation, we have

$$\begin{aligned}\partial^2_{x_ix_j}\mathcal{I}(t,x) &= \int_{\mathbb{R}^3}\left(\partial_{x_ix_j}f_0\partial_{v_j}Y_{0,t}+\partial_{x_iv_j}f_0(1+\partial_{v_j}W_{0,t})\right)\left(\partial_{v_i}Y_{0,t}\right)\frac{dw}{t^{d+2}}+\int_{\mathbb{R}^3}\left(\partial_{x_i}f_0\right)\left(\partial_{v_iv_j}Y_{0,t}\right)\frac{dw}{t^{d+2}}\\ &+\int_{\mathbb{R}^3}\left(\partial_{x_jv_i}f_0\partial_{v_j}Y_{0,t}+\partial_{v_iv_j}f_0(1+\partial_{v_j}W_{0,t})\right)\left(\partial_{v_i}W_{0,t}\right)\frac{dw}{t^{d+2}}+\int_{\mathbb{R}^3}\left(\partial_{v_i}f_0\right)\left(\partial_{v_iv_j}W_{0,t}\right)\frac{dw}{t^{d+2}}.\end{aligned} \tag{5.6}$$

It is clear from the above that (5.6) that

$$\begin{aligned}\nabla^2_x\mathcal{I}(t,x) &= \int_{\mathbb{R}^d}\{(\nabla^2_xf_0)(\nabla_vY_{0,t})+(\nabla_{x,v}f_0)(\nabla_vY_{0,t})+(\nabla_xf_0)(\nabla^2_vY_{0,t})+(\nabla^2_{x,v}f_0)(\nabla_vY_{0,t})\\ &+(\nabla^2_vf_0)(\nabla_vW_{0,t})+(\nabla^2_vf_0)(\nabla_vW_{0,t})^2+(\nabla_vf_0)(\nabla^2_vW_{0,t})\}\frac{dw}{t^{d+2}}\end{aligned}$$

which satisfies the hypothesis for $k=2$. Now by induction, we assume that this statement is true for k, and we shall prove it for $k+1$. Applying ∂_{x_i} to the term (5.4) and using the product rules, we have three types of terms that appear, namely:

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^d}\frac{d}{dx_i}(\nabla^\alpha_{x,v}f_0)(\nabla^{\beta_1}_vY_{0,t})^{k_1}\cdots(\nabla^{\beta_r}_vY_{0,t})^{k_r}\cdot(\nabla^{\gamma_1}_vW_{0,t})^{s_1}\cdots(\nabla^{\gamma_t}_vW_{0,t})^{s_t}\frac{dw}{t^{d+k}}\\ I_2 &= \int_{\mathbb{R}^d}(\nabla^\alpha_{x,v}f_0)(\nabla_vY_{0,t})(\nabla^{\beta_1}_vY_{0,t})^{k_1-1}\cdots(\nabla^{\beta_r}_vY_{0,t})^{k_r}\cdot(\nabla^{\gamma_1}_vW_{0,t})^{s_1}\cdots(\nabla^{\gamma_t}_vW_{0,t})^{s_t}\frac{dw}{t^{d+k+1}}\\ I_3 &= \int_{\mathbb{R}^d}(\nabla^\alpha_{x,v}f_0)(\nabla^{\beta_1}_vY_{0,t})^{k_1}\cdots(\nabla^{\beta_r}_vY_{0,t})^{k_r}\cdot(\nabla_vW_{0,t})(\nabla^{\gamma_1}_vW_{0,t})^{s_1-1}\cdots(\nabla^{\gamma_t}_vW_{0,t})^{s_t}\frac{dw}{t^{d+k+1}}\end{aligned}$$

Here, I_1, I_2, I_3 appear when ∂_{x_i} hits $\nabla^\alpha_{x,v}f_0$, $(\nabla^{\beta_1}_vY_{0,t})^{k_1}$ and $(\nabla^{\gamma_1}_vW_{0,t})^{s_1}$ respectively. Note that we assume that the derivative hits the above terms on just the pairs (β_1,k_1) or (γ_1,s_1), as this is up to a permutation of indices.

Treating I_1:

By a direct calculation, we see that I_1 can be written as

$$\int_{\mathbb{R}^d}\left(\nabla^{\alpha+1}_{x,v}f_0\cdot\nabla_vY_{0,t}+\nabla^{\alpha+1}_{x,v}f_0(1+\nabla_vW_{0,t})\right)(\nabla^{\beta_1}_vY_{0,t})^{k_1}\cdots(\nabla^{\beta_r}_vY_{0,t})^{k_r}\cdot(\nabla^{\gamma_1}_vW_{0,t})^{s_1}\cdots(\nabla^{\gamma_t}_vW_{0,t})^{s_t}\frac{dw}{t^{d+k+1}}$$

which satisfies the induction hypothesis for $|\alpha|+1=k+1$.

Treating I_2:

For I_2, we check the condition (5.5) for the new indices and multi-indices, which is

$$|\alpha|\le k+1 \qquad \text{and} \quad 1+\beta_1(k_1-1)+(\beta_2k_2+\cdots+\beta_rk_r)+(\gamma_1s_1+\cdots+\gamma_ts_t)\le k+1$$

The statement $|\alpha|\le k+1$ is true, as $|\alpha|\le k$. For the second condition, we note that, by the induction hypothesis:

$$(\beta_1k_1+\beta_2k_2+\cdots+\beta_rk_r)+(\gamma_1s_1+\cdots+\gamma_ts_t)\le k.$$

Adding both sides of the above by $1-\beta_1$, the new left hand side can be bounded by $k+1-\beta_1\le k+1$, since $\beta_1\ge 0$. The proof is complete. Finally, the term I_3 is treated exactly as I_2, and we skip the details. $\square$

Theorem 5.3. *Assume that*

$$\max_{0\le k\le N} \|\nabla^k_{x,v} f_0\|_{L^1_x L^\infty_v} \le \epsilon_0$$

and $E, Y_{s,t}, W_{s,t}$ satisfy the decay estimates (4.3), (4.4), *and* (4.5) *for $0 \le k \le N$. Then*

$$\mathcal{I}(t,x) = \int_{\mathbb{R}^d} f_0(X_{0,t}(x,v), V_{0,t}(x,v))dv$$

satisfies the following decay estimate:

$$\sup_{0\le k\le N} \Big(\langle t\rangle^k \|\nabla^k_x \mathcal{I}(t)\|_{L^1} + \langle t\rangle^{d+k}\|\nabla^k_x \mathcal{I}(t)\|_{L^\infty}\Big) \lesssim \varepsilon_0.$$

Proof. By the above lemma, it suffices to prove that

$$\langle t\rangle^k \|\nabla^k_x \mathcal{J}(t)\|_{L^1} + \langle t\rangle^{d+k}\|\nabla^k_x \mathcal{J}(t)\|_{L^\infty} \lesssim \varepsilon_0.$$

where

$$\begin{cases} \mathcal{J} = \int_{\mathbb{R}^d} \nabla^\alpha_{x,v} f_0 \cdot (\nabla^{\beta_1}_v Y_{0,t})^{k_1} \cdots (\nabla^{\beta_r}_v Y_{0,t})^{k_r} \cdot (\nabla^{\gamma_1}_v W_{0,t})^{s_1} \cdots (\nabla^{\gamma_t}_v W_{0,t})^{s_t} \dfrac{dw}{t^{d+k}} \\ 1 \le |\alpha| \le k \qquad \text{and} \qquad (\beta_1 k_1 + \cdots + \beta_r k_r) + (\gamma_1 s_1 + \cdots + \gamma_t s_t) \le k. \end{cases}$$

We have

$$\mathcal{J}(t,x) \lesssim t^{-k} \int_{\mathbb{R}^d} |\nabla^\alpha_{x,v} f_0|(X_{0,t}(x,v), V_{0,t}(x,v))dv$$

Hence, we get

$$t^k \|\mathcal{J}(t)\|_{L^1} + t^{d+k}\|\mathcal{J}(t)\|_{L^\infty} \lesssim \varepsilon_0.$$

The proof is complete. □

5.2 Decay estimates for the reaction term

In this section, we estimate the derivatives of the reaction term

$$\mathcal{R} = \mathcal{R}_L - \mathcal{R}_{NL} = \int_0^t \int_{\mathbb{R}^d} E(s, x-(t-s)v)\cdot\nabla_v \mu(v)dvds - \int_0^t \int_{\mathbb{R}^d} E(s, X_{s,t}(x,v))\cdot\nabla_v \mu(V_{s,t}(x,v))dvds$$

appearing as a forcing term in (5.1). For a general time-dependent vector field $E(s)$ and a smooth decaying function μ, we also denote $\mathcal{T}$ to be

$$\mathcal{T}(E,\mu) = \int_0^t \int_{\mathbb{R}^d} E(s, x-(t-s)v)\cdot\nabla_v \mu(v)dvds - \int_0^t \int_{\mathbb{R}^d} E(s, X_{s,t}(x,v))\cdot\nabla_v \mu(V_{s,t}(x,v))dvds \quad (5.7)$$

Our main theorem is as follows:

Theorem 5.4. *Let $N > 1$ be an integer. Assume that $E, W_{s,t}, Y_{s,t}$ satisfy the decay estimates* (4.3), (4.5) *and* (4.4) *for all $0 \le k \le N$, there holds*

$$\max_{0\le k\le N} \Big(\langle t\rangle^k \|\nabla^k_x \mathcal{R}(t)\|_{L^1} + \langle t\rangle^{d+k}\|\nabla^k_x \mathcal{R}(t)\|_{L^\infty}\Big) \lesssim \varepsilon^2.$$

First, we recall the following proposition from [13]. We also give a detailed proof for the readers convenience.

Proposition 5.5. *Assuming that $E, Y_{s,t}, W_{s,t}$ satisfies the decaying estimates* (4.3),(4.4) *and* (4.5) *respectively, we have*

$$\|\mathcal{T}(E,\mu)\|_{L^1} + \langle t\rangle^d \|\mathcal{T}(E,\mu)\|_{L^\infty} \lesssim \varepsilon^2.$$

Proof. Making the change of variables $v \to \Psi_{s,t}(x,v)$ so that $X_{s,t}(x,\Psi_{s,t}(x,v)) = x-(t-s)v$ (see Lemma 4.4), we have

$$\begin{aligned}
\mathcal{T}(E,\mu) &= \int_0^t\int_{\mathbb{R}^d} E(s,x-(t-s)v)\cdot\nabla_v\mu(v)dvds - \int_0^t\int_{\mathbb{R}^d} E(s,X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v))dvds \\
&= \int_0^t\int_{\mathbb{R}^d} E(s,x-(t-s)v)\cdot\nabla_v\mu(v)dvds \\
&\quad - \int_0^t\int_{\mathbb{R}^d} E(s,x-(t-s)v)\cdot\nabla_v\mu(V_{s,t}(x,\Psi_{s,t}(x,v))\det(\nabla_v\Psi_{s,t}(x,v))dvds
\end{aligned}$$

Hence, one can rewrite $\mathcal{T}$ as $\mathcal{T}_1+\mathcal{T}_2$, where

$$\begin{aligned}
\mathcal{T}_1 &= \int_0^t\int_{\mathbb{R}^d} E(s,x-(t-s)v)\cdot\nabla_v\left\{\mu(v)-\mu(V_{s,t}(x,\Psi_{s,t}(x,v)))\right\}dvds \\
\mathcal{T}_2 &= \int_0^t\int_{\mathbb{R}^d} E(s,x-(t-s)v)\cdot\nabla_v\mu(V_{s,t}(x,\Psi_{s,t}(x,v))\left\{1-\det(\nabla_v\Psi_{s,t}(x,v))\right\}dvds
\end{aligned}$$

Bounding $\mathcal{T}_1(t)$:
We have

$$\begin{aligned}
\mathcal{T}_1(t,x) &= \int_0^t\int_{\mathbb{R}^d} E(s,x-(t-s)v)\cdot\nabla_v\left\{\mu(v)-\mu(V_{s,t}(x,\Psi_{s,t}(x,v)))\right\}dvds \\
&\lesssim \int_0^t\int_{\mathbb{R}^d} |E(s,x-(t-s)v)\cdot|\langle v\rangle^{-M}\cdot|v-V_{s,t}(x,\Psi_{s,t}(x,v))|dvds \\
&\lesssim \int_0^t\int_{\mathbb{R}^d} |E(s,x-(t-s)v)|\langle v\rangle^{-M}\left\{|v-V_{s,t}(x,v)|+|V_{s,t}(x,v)-V_{s,t}(x,\Psi_{s,t}(x,v))|\right\}dvds \\
&\lesssim \int_0^t\int_{\mathbb{R}^d} |E(s,x-(t-s)v)|\langle v\rangle^{-M}\left\{|v-V_{s,t}(x,v)|_{L^\infty}+\|\nabla_v V_{s,t}\|_{L^\infty}|v-\Psi_{s,t}(x,v)|\right\}dvds
\end{aligned}$$

Using the fact that

$$|v-V_{s,t}(x,v)| \lesssim \frac{\varepsilon\log(1+s)}{(1+s)^{d-1}}, \quad |v-\Psi_{s,t}(x,v)| \lesssim \frac{\varepsilon\log(1+s)}{(1+s)^d} \quad \text{and} \quad \|\nabla_v V_{s,t}\|_{L^\infty} \lesssim 1, \tag{5.8}$$

we get

$$\mathcal{T}_1(t,x) \lesssim \int_0^t\int_{\mathbb{R}^d} |E(s,x-(t-s)v)|\langle v\rangle^{-M}\left(\frac{\varepsilon\log(1+s)}{(1+s)^{d-1}}+\frac{\varepsilon\log(1+s)}{(1+s)^d}\right)dvds.$$

Hence

$$\begin{aligned}\|\mathcal{T}_1(t)\|_{L^1} &\lesssim \int_0^t \frac{\varepsilon \log(1+s)}{(1+s)^{d-1}} \int_{\mathbb{R}^d} \|E(s)\|_{L^1} \langle v\rangle^{-M} dv \\ &\lesssim \int_0^t \frac{\varepsilon^2 \log(1+s)^2}{(1+s)^{d-1}} ds \lesssim \varepsilon^2,\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{T}_1(t)\|_{L^\infty} &\lesssim \int_0^t \int_{\mathbb{R}^d} \|E(s)\|_{L^\infty} \langle v\rangle^{-M} \left(\frac{\varepsilon \log(1+s)}{(1+s)^{d-1}} + \frac{\varepsilon \log(1+s)}{(1+s)^d}\right) dvds \\ &\lesssim \int_0^t \frac{\varepsilon^2 \log(1+s)^2}{(1+s)^{2d-1}} ds \lesssim \varepsilon^2 \log(1+t) \int_0^t \frac{\log(1+s)}{(1+s)^{2d-1}} ds \\ &\lesssim \varepsilon^2 \frac{\log(1+t)^2}{(1+t)^{2d-2}} \lesssim \varepsilon^2 \langle t\rangle^{-d}.\end{aligned}$$

Thus

$$\|\mathcal{T}_1(t)\|_{L^1} + \langle t\rangle^d \|\mathcal{T}_1(t)\|_{L^\infty} \lesssim \varepsilon^2.$$

Bounding $\mathcal{T}_2(t)$:
Since $v \to \Psi_{s,t}(x,v)$ is a differomorphism, making the change of variables $\Psi_{s,t}^{-1} : \Psi_{s,t}(x,v) \to v$ gives

$$\begin{aligned}\mathcal{T}_2(t,x) &= \int_0^t \int_{\mathbb{R}^d} E(s, x-(t-s)v) \cdot \nabla_v \mu(V_{s,t}(x,v))(1 - \det(\nabla_v \Psi_{s,t}(x,v)) \det(\nabla_v(\Psi_{s,t}^{-1}(x,v)))dvds \\ &\lesssim \int_0^t \int_{\mathbb{R}^d} |E(s, x-(t-s)v)| \cdot |\nabla_v \mu| (V_{s,t}(x,v)) \cdot \left|\det(\nabla_v \Psi_{s,t}(x,v))^{-1} - 1\right| dvds\end{aligned}$$

Using the inequality $|\nabla_v(\Psi_{s,t}(x,v) - v)| \lesssim \frac{\varepsilon \log(1+s)}{(1+s)^{d-1}}$, we have

$$\mathcal{T}_2(t,x) \lesssim \int_0^t \int_{\mathbb{R}^d} |E(s, x-(t-s)v)| \cdot |\nabla_v \mu|(V_{s,t}(x,v)) \cdot \frac{\varepsilon \log(1+s)}{(1+s)^{d-1}} dsdv$$

Hence we have

$$\begin{aligned}\|\mathcal{T}_2(t)\|_{L^1} &\lesssim \int_0^t \left\{\varepsilon \frac{\log(1+s)}{(1+s)^{d-1}} \|E(s)\|_{L^1} \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} |\nabla_v \mu|(V_{s,t}(x,v)) dv\right\} ds \\ &\lesssim \int_0^t \varepsilon^2 \frac{\log(1+s)^2}{(1+s)^d} ds \lesssim \varepsilon^2.\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{T}_2(t)\|_{L^\infty} &\lesssim \int_0^t \|E(s)\|_{L^\infty} \cdot \frac{\varepsilon \log(1+s)}{(1+s)^{d-1}} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla_v \mu|(V_{s,t}(x,v)) dvds \\ &\lesssim \int_0^t \varepsilon^2 \frac{\log(1+s)^2}{(1+s)^{2d-1}} ds \lesssim \varepsilon^2 \log(1+t) \int_0^t \frac{\log(1+s)}{(1+s)^{2d-1}} ds \lesssim \varepsilon^2 \frac{\log(1+t)}{(1+t)^{2d-2}} \\ &\lesssim \varepsilon^2 \langle t\rangle^{-d}\end{aligned}$$

This implies that

$$\|\mathcal{T}_2(t)\|_{L^1} + \langle t\rangle^d \|\mathcal{T}_2(t)\|_{L^\infty} \lesssim \varepsilon^2.$$

The proof is complete. $\square$

Lemma 5.6. *There holds*

$$\begin{aligned}\partial_j \mathcal{T}(E,\mu) = &\frac{1}{t}\left(\mathcal{T}(s\partial_j E,\mu) + \mathcal{T}(E,\partial_j\mu)\right)\\ &+ \frac{1}{t}\sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} \partial_{v_j} Y_{s,t}(x-tv,v)\cdot\nabla_x E_k(s,X_{s,t}(x,v))(\partial_k\mu)(V_{s,t}(x,v))dvds\\ &+ \frac{1}{t}\sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} E_k(s,X_{s,t}(x,v))(\partial_{v_j} W_{s,t})(x-tv,v)\cdot\nabla_v(\partial_k\mu)(V_{s,t}(x,v))dvds.\end{aligned}$$

Proof. We recall that

$$\mathcal{R}_{NL} = \int_0^t \int_{\mathbb{R}^d} E(s,X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v))dvds.$$

Using the identities (4.1), we have

$$\mathcal{R}_{NL} = \int_0^t \int_{\mathbb{R}^d} E(s,x-(t-s)v + Y_{s,t}(x-tv,v))\cdot\nabla_v\mu(v + W_{s,t}(x-tv,v))dvds.$$

Making the change of variable $w = x - tv$, we obtain

$$\begin{aligned}\mathcal{R}_{NL} &= \int_0^t \int_{\mathbb{R}^d} E\left(s, w + \frac{s}{t}(x-w) + Y_{s,t}(w,\frac{x-w}{t})\right)\cdot\nabla_v\mu\left(\frac{x-w}{t} + W_{s,t}(w,\frac{x-w}{t})\right) t^{-d} dwds\\ &= t^{-d}\sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} E_k\left(s, w + \frac{s}{t}(x-w) + Y_{s,t}(w,\frac{x-w}{t})\right)\partial_k\mu\left(\frac{x-w}{t} + W_{s,t}(w,\frac{x-w}{t})\right) dwds\end{aligned}$$

Similarly, for $\mathcal{R}_L$ we have

$$\mathcal{R}_L = \int_0^t \int_{\mathbb{R}^d} E(s,x-(t-s)v)\cdot\nabla_v\mu(v)dvds = t^{-d}\sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} E_k\left(s, w + \frac{s}{t}(x-w)\right)\partial_k\mu\left(\frac{x-w}{t}\right) dwds$$

The lemma follows by a direct calculation. The proof is complete. $\square$

Now we establish the following lemma, by induction on the degree of derivatives.

Lemma 5.7. *Let $n \geq 2$, then $\nabla_x^n \mathcal{R}(t,x)$ can be written as a sum of terms, are all either of the form*

$$\frac{1}{t^n}\mathcal{T}(s^k\nabla_x^k E, f(\mu))$$

or the form

$$\frac{1}{t^n}\int_0^t\int_{\mathbb{R}^d}\left\{(\nabla_v^{m_1}Y_{s,t})^{k_1}\cdots(\nabla_v^{m_a}Y_{s,t})^{k_a}\right\}\cdot\left\{(\nabla_v^{n_1}W_{s,t})^{l_1}\cdots(\nabla_v^{n_b}W_{s,t})^{l_b}\right\}$$
$$\times\left\{(s^{t_1}\nabla_x^{u_1}E)\cdots(s^{t_c}\nabla_x^{u_c}E)\right\}f(\mu)dvds$$

where $f(\mu)$ is some expression only depending on $\mu(v)$ or its derivatives. Moreover, the indices satisfy the following set of conditions:

- *No loss of derivative condition:* $k\le n$, $\max\{\{m_1,k_1,\cdots,m_a,k_a\}\cup\{n_1,l_1,\cdots,n_b,l_b\}\cup\{t_1,u_1,\cdots,t_c,u_c\}\}\le n$.
- *E-decay condition:* $(t_1,t_2,\cdots,t_c)\le(u_1,u_2,\cdots,u_c)$.
- *Y-W show-up condition:* $\min\{a,b\}\ge 1$.
- *E-show up condition:* $c\ge 1$.
- *W-E decay condition: If $b=0$ then $t_h+1\le u_h$ for some $1\le h\le c$.*

Let us call the first form to be type-I and the second one is type-II.

Remark 5.8. *The conditions on the indices are important for the decay estimates. The first condition means that we do not lose derivatives in the estimates. The second condition requires a good decay for the quantities $s^{t_i}\nabla_x^{u_i}E$. The third condition means that at least $Y_{s,t}$ or $W_{s,t}$ (or their derivatives) shows up in the expression. The forth condition means that E or its derivatives must show up in the expression. Finally, the last condition means that if $W_{s,t}$ (or its derivatives) does not show up, then we have more gradient of E to control the power of s. The reason for the last condition will be clear when we estimate* (5.11) *later in the paper.*

Proof. The lemma is proved by induction on n. Assuming the lemma is true for n, we justify the above claim for $n+1$.
Gradient of type-I terms:
Applying ∂_j to the term $\mathcal{T}(s^k\nabla_x^kE,f(\mu))$ and using Lemma 5.6, we have

$$\frac{1}{t^n}\partial_jT(s^k\nabla_x^kE,f(\mu))=\frac{1}{t^{n+1}}\Big(T(s^{k+1}\nabla_x^k\partial_jE,f(\mu))+T(s^k\nabla_x^kE,\partial_jf(\mu))\Big)$$
$$+\frac{1}{t^{n+1}}\int_0^t\int_{\mathbb{R}^d}\Big(\big(\partial_{v_j}Y_{s,t}(x-tv,v)\big)\cdot\nabla_x(s^k\nabla_x^kE)\,(s,X_{s,t}(x,v))\,(\nabla_vf(\mu))\,(V_{s,t}(x,v))\Big)dvds$$
$$+\frac{1}{t^{n+1}}\int_0^t\int_{\mathbb{R}^d}(s^k\nabla_x^kE)\,(s,X_{s,t}(x,v)))\left\{\partial_{v_j}W_{s,t}(x-tv,v)\cdot\nabla_v^2f(\mu)(V_{s,t}(x,v))\right\}dvds.$$

We can see that all of the above terms are either type-I or type-II, and satisfy the induction hypothesis with order $n+1$.

Gradient of type-II form:
Now we consider the type-II term:

$$\mathcal{D} = \frac{1}{t^n}\int_0^t \int_{\mathbb{R}^d} \left\{(\nabla_v^{m_1} Y_{s,t})^{k_1}\cdots(\nabla_v^{m_a} Y_{s,t})^{k_a}\right\}\cdot\left\{(\nabla_v^{n_1} W_{s,t})^{l_1}\cdots(\nabla_v^{n_b} W_{s,t})^{l_b}\right\}$$
$$\times\left\{(s^{t_1}\nabla_x^{u_1}E)\cdots(s^{t_c}\nabla_x^{u_c}E)\right\} f(\mu)dvds.$$

Here $(\nabla_v^{m_i}Y_{s,t})^{k_i}, (\nabla_v^{n_j}W_{s,t})^{l_j}$ is evaluated at $(x-tv, v)$ and $\nabla_x^{u_r}E$ is evaluated at $(s, X_{s,t}(x,v))$. Making the change of variables $w = x - tv$, we can rewrite $\mathcal{D}$ as

$$t^{-d}\frac{1}{t^n}\int_0^t \int_{\mathbb{R}^d} \left\{(\nabla_v^{m_1} Y_{s,t})^{k_1}\cdots(\nabla_v^{m_a} Y_{s,t})^{k_a}\right\}\left\{(\nabla_v^{n_1} W_{s,t})^{l_1}\cdots(\nabla_v^{n_b} W_{s,t})^{l_b}\right\}$$
$$\times\left\{(s^{t_1}\nabla_x^{u_1}E)\cdots(s^{t_c}\nabla_x^{u_c}E)\right\} f(\mu)dwds.$$

where $(\nabla_v^{m_i}Y_{s,t})^{k_i}, (\nabla_v^{n_j}W_{s,t})^{l_j}$ is evaluated at $(w, \frac{x-w}{t})$ and $\nabla_x^{u_r}E$ is evaluated at

$$\left(s, Y_{s,t}(w, \frac{x-w}{t}) + w + \frac{s}{t}(x-w)\right)$$

Applying ∇_x to $\mathcal{D}$ and using the product rules, we have

$$\begin{cases}
\mathcal{D}_1 & = t^{-d}\frac{1}{t^n}\int_0^t\int_{\mathbb{R}^d} k_1\left((\nabla_v^{m_1}Y_{s,t})^{k_1-1}\left\{\nabla_v^{m_1+1}Y_{s,t}\right\}\frac{1}{t}\right)\cdots(\nabla_v^{m_a}Y_{s,t})^{k_a} \\
& \quad \left\{(\nabla_v^{n_1}W_{s,t})^{l_1}\cdots(\nabla_v^{n_b}W_{s,t})^{l_b}\right\}\left\{(s^{t_1}\nabla_x^{u_1}E)\cdots(s^{t_c}\nabla_x^{u_c}E)\right\}f(\mu)dwds, \\
\mathcal{D}_2 & = t^{-d}\frac{1}{t^n}\int_0^t\int_{\mathbb{R}^d}\left\{(\nabla_v^{m_1}Y_{s,t})^{k_1}\cdots(\nabla_v^{m_a}Y_{s,t})^{k_a}\right\} \\
& \quad \left(l_1\left\{(\nabla_v^{n_1}W_{s,t})^{l_1-1}(\nabla_v^{n_1+1}W_{s,t})\right\}\frac{1}{t}\right)\cdots(\nabla_v^{n_b}W_{s,t})^{l_b} \\
& \quad \left\{(s^{t_1}\nabla_x^{u_1}E)\cdots(s^{t_c}\nabla_x^{u_c}E)\right\}f(\mu)dwds, \\
\mathcal{D}_3 = & t^{-d}\frac{1}{t^n}\int_0^t\int_{\mathbb{R}^d}\left\{(\nabla_v^{m_1}Y_{s,t})^{k_1}\cdots(\nabla_v^{m_a}Y_{s,t})^{k_a}\right\}\left\{(\nabla_v^{n_1}W_{s,t})^{l_1}\cdots(\nabla_v^{n_b}W_{s,t})^{l_b}\right\} \\
& \left(\frac{1}{t}\left\{s^{t_1}(\nabla_x^{u_1+1}E)(\nabla_vY_{s,t})\right\}+\frac{1}{t}\left\{s^{t_1+1}\nabla_x^{u_1+1}E\right\}\right)\cdots(s^{t_c}\nabla_x^{u_c}E)f(\mu)dwds.
\end{cases}$$

Now making the change of variables $v = \frac{x-w}{t}$, we get

$$\begin{cases}
\mathcal{D}_1 & = \frac{1}{t^{n+1}}\int_0^t\int_{\mathbb{R}^d} k_1\left((\nabla_v^{m_1}Y_{s,t})^{k_1-1}\left\{\nabla_v^{m_1+1}Y_{s,t}\right\}\right)\cdots(\nabla_v^{m_a}Y_{s,t})^{k_a} \\
& \quad \left\{(\nabla_v^{n_1}W_{s,t})^{l_1}\cdots(\nabla_v^{n_b}W_{s,t})^{l_b}\right\}\left\{(s^{t_1}\nabla_x^{u_1}E)\cdots(s^{t_c}\nabla_x^{u_c}E)\right\}f(\mu)dvds, \\
\mathcal{D}_2 & = \frac{1}{t^{n+1}}\int_0^t\int_{\mathbb{R}^d}\left\{(\nabla_v^{m_1}Y_{s,t})^{k_1}\cdots(\nabla_v^{m_a}Y_{s,t})^{k_a}\right\} \\
& \quad \left(l_1\left\{(\nabla_v^{n_1}W_{s,t})^{l_1-1}(\nabla_v^{n_1+1}W_{s,t})\right\}\right)\cdots(\nabla_v^{n_b}W_{s,t})^{l_b} \\
& \quad \left\{(s^{t_1}\nabla_x^{u_1}E)\cdots(s^{t_c}\nabla_x^{u_c}E)\right\}f(\mu)dvds, \\
\mathcal{D}_3 = & \frac{1}{t^{n+1}}\int_0^t\int_{\mathbb{R}^d}\left\{(\nabla_v^{m_1}Y_{s,t})^{k_1}\cdots(\nabla_v^{m_a}Y_{s,t})^{k_a}\right\}\left\{(\nabla_v^{n_1}W_{s,t})^{l_1}\cdots(\nabla_v^{n_b}W_{s,t})^{l_b}\right\} \\
& \left\{s^{t_1}(\nabla_x^{u_1+1}E)(\nabla_vY_{s,t})+s^{t_1+1}\nabla_x^{u_1+1}E\right\}\cdots(s^{t_c}\nabla_x^{u_c}E)f(\mu)dvds.
\end{cases}$$

From the above expressions, it is straightforward that the new terms are of type-II, which satisfy the induction hypothesis with $n+1$. The proof is complete. $\square$

Proposition 5.9. *Let $\mathcal{R}$ be defined as in* (5.7)*. There holds, for $n \geq 2$, the decaying estimates*

$$\langle t\rangle^n \|\nabla_x^n \mathcal{R}(t)\|_{L^1} + \langle t\rangle^{d+n} \|\nabla_x^k \mathcal{R}(t)\|_{L^\infty} \lesssim \varepsilon^2.$$

Proof. By Lemma 5.7, $\nabla_x^n \mathcal{R}(t,x)$ can be decomposed as a sum of many terms, all are either of the form $\mathcal{R}_1$ or $\mathcal{R}_2$, where

$$\begin{cases} \mathcal{R}_1 & = \frac{1}{t^n} \mathcal{T}(s^k \nabla_x^k E, f(\mu)) \\ \mathcal{R}_2 & = \frac{1}{t^n} \int_0^t \int_{\mathbb{R}^d} \left\{ (\nabla_v^{m_1} Y_{s,t})^{k_1} \cdots (\nabla_v^{m_a} Y_{s,t})^{k_a} \right\} \cdot \left\{ (\nabla_v^{n_1} W_{s,t})^{l_1} \cdots (\nabla_v^{n_b} W_{s,t})^{l_b} \right\} \\ & \quad \left\{ (s^{t_1} \nabla_x^{u_1} E) \cdots (s^{t_c} \nabla_x^{u_c} E) \right\} f(\mu) dv ds \end{cases}$$

where

$$k \leq n \qquad \text{and} \qquad \max\{\{m_1, k_1, \cdots, m_a, k_a\} \cup \{n_1, l_1, \cdots, n_b, l_b\} \cup \{t_1, u_1, \cdots, t_c, u_c\}\} \leq n.$$

Bounding $\mathcal{R}_1(t)$:
We have

$$\mathcal{R}_1(t) = \frac{1}{t^n} \mathcal{T}(s^k \nabla_x^k E, f(\mu))$$

with $k \leq n$ and $f(\mu)$ is a decay function in v which only depends on μ or its derivatives. Applying Proposition 5.5, we only need to check the assumption

$$\|s^k \nabla_x^k E(s)\|_{L^\infty} \lesssim \frac{\varepsilon \log(1+s)}{(1+s)^d},$$

which is true, since $\|\nabla_x^k E(s)\|_{L^\infty} \lesssim \frac{\varepsilon \log(1+s)}{(1+s)^{d+k}}$. The proof for $\mathcal{R}_1$ is complete.

Bounding $\mathcal{R}_2(t)$:
Now we bound $\mathcal{R}_2(t)$, where

$$\mathcal{R}_2(t,x) = \frac{1}{t^n} \int_0^t \int_{\mathbb{R}^d} \left\{ (\nabla_v^{m_1} Y_{s,t})^{k_1} \cdots (\nabla_v^{m_a} Y_{s,t})^{k_a} \right\} \cdot \left\{ (\nabla_v^{n_1} W_{s,t})^{l_1} \cdots (\nabla_v^{n_b} W_{s,t})^{l_b} \right\}$$
$$\left\{ (s^{t_1} \nabla_x^{u_1} E) \cdots (s^{t_c} \nabla_x^{u_c} E) \right\} f(\mu) dv ds$$

Here $(\nabla_v^{m_i} Y_{s,t})^{k_i}, (\nabla_v^{n_j} W_{s,t})^{l_j}$ is evaluated at $(x - tv, v)$ and $\nabla_x^{u_r} E$ is evaluated at $(s, X_{s,t}(x,v))$. We also recall the following set of conditions, proved in Lemma 5.7:

$$\begin{cases} k \leq n \quad \text{and} \quad \max\{\{m_1, k_1, \cdots, m_a, k_a\} \cup \{n_1, l_1, \cdots, n_b, l_b\} \cup \{t_1, u_1, \cdots, t_c, u_c\}\} \leq n, \\ (t_1, t_2, \cdots, t_c) \leq (u_1, u_2, \cdots, u_c), \\ \min\{a, b\} \geq 1, \\ c \geq 1 \\ \text{If} \quad b = 0 \quad \text{then} \quad t_h + 1 \leq u_h \quad \text{for some} \quad 1 \leq h \leq c. \end{cases} \tag{5.9}$$

First, we will show that $\|\mathcal{R}_2(t)\|_{L^1} \lesssim \varepsilon^2 \langle t \rangle^{-n}$.
Using the third condition in (5.9) and the fact that

$$\max_{0\le k\le n}\left(\|\nabla_v^k Y_{s,t}\|_{L^\infty}+\|\nabla_v^k W_{s,t}\|_{L^\infty}\right)\lesssim \varepsilon\frac{\log(1+s)}{(1+s)^{d-2}},$$

we get

$$\mathcal{R}_2(t,x)\lesssim t^{-n}\int_0^t\int_{\mathbb{R}^d}\varepsilon\frac{\log(1+s)}{(1+s)^{d-2}}\prod_{i=1}^{c}\left(s^{t_i}\|\nabla_x^{u_i}E(s)\|_{L^\infty}\right)|f(\mu)|(V_{s,t}(x,v))dvds. \tag{5.10}$$

Now using the second and the forth condition in (5.9), and the decay of $\nabla_x^{u_i}E$, we get

$$s^{t_i}\|\nabla_x^{u_i}E(s)\|_{L^\infty}\lesssim\frac{\varepsilon\log(1+s)}{(1+s)^{d+u_i-t_i}}\lesssim\frac{\varepsilon\log(1+s)}{(1+s)^d}.$$

Applying the above inequality to (5.10), we obtain

$$\begin{aligned}\mathcal{R}_2(t,x)&\lesssim t^{-n}\int_0^t\int_{\mathbb{R}^d}\varepsilon^2\cdot\frac{\log(1+s)}{(1+s)^{d-2}}\cdot\frac{\log(1+s)}{(1+s)^d}|f(\mu)|(V_{s,t}(x,v))dvds\\&\lesssim\varepsilon^2t^{-n}\int_0^t\int_{\mathbb{R}^d}\frac{\log(1+s)^2}{(1+s)^{2d-2}}|f(\mu)|(V_{s,t}(x,v))dvds.\end{aligned}$$

Hence, using the decaying assumption of μ, we obtain

$$\|\mathcal{R}_2(t)\|_{L^1}\lesssim\varepsilon^2t^{-n}\int_0^t\frac{\log(1+s)^2}{(1+s)^{2d-2}}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|f(\mu)|(V_{s,t}(x,v))dxdv\right)ds\lesssim\varepsilon^2t^{-n}.$$

The decaying bound for $\|\mathcal{R}_2(t)\|_{L^1}$ is complete.
We split the integral in $\mathcal{R}_2$ into $\int_0^{t/2}+\int_{t/2}^t$, so that $\mathcal{R}_2=\mathcal{R}_3+\mathcal{R}_4$, where

$$\begin{cases}\mathcal{R}_3(t,x) & =\frac{1}{t^n}\int_{t/2}^t\int_{\mathbb{R}^d}\left\{(\nabla_v^{m_1}Y_{s,t})^{k_1}\cdots(\nabla_v^{m_a}Y_{s,t})^{k_a}\right\}\cdot\left\{(\nabla_v^{n_1}W_{s,t})^{l_1}\cdots(\nabla_v^{n_b}W_{s,t})^{l_b}\right\}\\ & \left\{(s^{t_1}\nabla_x^{u_1}E)\cdots(s^{t_c}\nabla_x^{u_c}E)\right\}f(\mu)dvds\\ \mathcal{R}_4(t,x) & =\frac{1}{t^n}\int_0^{t/2}\int_{\mathbb{R}^d}\left\{(\nabla_v^{m_1}Y_{s,t})^{k_1}\cdots(\nabla_v^{m_a}Y_{s,t})^{k_a}\right\}\cdot\left\{(\nabla_v^{n_1}W_{s,t})^{l_1}\cdots(\nabla_v^{n_b}W_{s,t})^{l_b}\right\}\\ & \left\{(s^{t_1}\nabla_x^{u_1}E)\cdots(s^{t_c}\nabla_x^{u_c}E)\right\}f(\mu)dvds\end{cases}$$

For $\mathcal{R}_3(t,x)$, by the same argument for $\mathcal{R}_1(t,x)$, we have the pointwise bound

$$\begin{aligned}\mathcal{R}_3(t,x)&\lesssim\varepsilon^2t^{-n}\int_{t/2}^t\frac{\log(1+s)^2}{(1+s)^{2d-2}}\int_{\mathbb{R}^d}|f(\mu)|(V_{s,t}(x,v))dvds\\&\lesssim\varepsilon^2t^{-n-d}\int_{t/2}^t\frac{\log(1+s)^2}{(1+s)^{d-2}}ds\lesssim\varepsilon^2t^{-(n+d)}\end{aligned}$$

Thus $\|\mathcal{R}_3(t)\|_{L^\infty}\lesssim\varepsilon^2t^{-(n+d)}$.
Now to bound $\|\mathcal{R}_4(t)\|_{L^\infty}$, we use the inequalities

$$\max_{0\le k\le n}\left(\langle s\rangle^{d-2}\|\nabla_v^kY_{s,t}\|_{L^\infty}+\langle s\rangle^{d-1}\|\nabla_v^kW_{s,t}\|_{L^\infty}\right)\lesssim\varepsilon\log(1+s)$$

to get

$$\begin{aligned}
\mathcal{R}_4(t,x) &\lesssim t^{-n}\int_0^{t/2}\int_{\mathbb{R}^d}\left(\frac{\varepsilon\log(1+s)}{(1+s)^{d-2}}\right)^a\left(\frac{\varepsilon\log(1+s)}{(1+s)^{d-1}}\right)^b \\
&\qquad\times\prod_{i=1}^{c}\left(s^{t_i}|\nabla_x^{u_i}E|(s,X_{s,t}(x,v))\right)\cdot|f(\mu)|(V_{s,t}(x,v))dvds \\
&\lesssim t^{-n}\varepsilon^{a+b}\int_0^{t/2}\int_{\mathbb{R}^d}\frac{(\log(1+s))^{a+b}}{(1+s)^{a(d-2)+b(d-1)}} \\
&\qquad\times\prod_{i=1}^{c}\left\{s^{t_i}|\nabla_x^{u_i}E|(s,X_{s,t}(x,v))\right\}\cdot|f(\mu)|(V_{s,t}(x,v))dvds
\end{aligned}$$

Using the change of variable $v\to\Psi_{s,t}(x,v)$ so that $X_{s,t}(x,\Psi_{s,t}(x,v))=x-(t-s)v$, we have

$$\begin{aligned}
\mathcal{R}_4(t,x) \lesssim t^{-n}\varepsilon^{a+b}&\int_0^{t/2}\int_{\mathbb{R}^d}\frac{(\log(1+s))^{a+b}}{(1+s)^{a(d-2)+b(d-1)}}\prod_{i=1}^{c}\left\{s^{t_i}|\nabla_x^{u_i}E|(s,x-(t-s)v)\right\} \\
&\cdot|f(\mu)|(V_{s,t}(x,\Psi_{s,t}(x,v)))|\det(\nabla_v\Psi_{s,t}(x,v))|dvds
\end{aligned}$$

Now making the change of variables $w=x-(t-s)v$, we have

$$\begin{aligned}
\mathcal{R}_4(t,x) &\lesssim t^{-n}\varepsilon^{a+b}\int_0^{t/2}(t-s)^{-d}\int_{\mathbb{R}^d}\frac{(\log(1+s))^{a+b}}{(1+s)^{a(d-2)+b(d-1)}}\prod_{i=1}^{c}\left\{s^{t_i}|\nabla_x^{u_i}E|(s,w)\right\} \\
&\quad\cdot|f(\mu)|\left(V_{s,t}(x,\Psi_{s,t}(x,\frac{x-w}{t})\right)dwds \\
&\lesssim t^{-(n+d)}\varepsilon^{a+b}\int_0^{t/2}\frac{\log(1+s)^{a+b}}{(1+s)^{a(d-2)+b(d-1)}}\cdot\min_{1\le i\le c}\left\{s^{t_i}\|\nabla_x^{u_i}E(s)\|_{L^1}\right\}ds \\
&\lesssim t^{-(n+d)}\varepsilon^{a+b}\int_0^{t/2}\frac{\log(1+s)^{a+b}}{(1+s)^{a(d-2)+b(d-1)}}\cdot\min_{1\le i\le c}\left\{s^{t_i}\frac{\varepsilon\log(1+s)}{(1+s)^{u_i}}\right\}ds \\
&\lesssim t^{-(n+d)}\varepsilon^{a+b+1}\int_0^{t/2}\frac{(\log(1+s))^{a+b+1}}{(1+s)^{\{a(d-2)+b(d-1)+\max_{1\le i\le c}(u_i-t_i)\}}}ds.
\end{aligned}$$

Now using the third condition in (5.9), we get $\varepsilon^{a+b+1}\lesssim\varepsilon^2$ as long as ε is small. Hence we get

$$\|\mathcal{R}_4(t)\|_{L^\infty}\lesssim\varepsilon^2 t^{-(n+d)}\int_0^{t/2}\frac{(\log(1+s))^{a+b+1}}{(1+s)^{\{a(d-2)+b(d-1)+\max_{1\le i\le c}(u_i-t_i)\}}}ds. \tag{5.11}$$

Thus, it suffices to prove that

$$\mathcal{C}=\int_0^{t/2}\frac{(\log(1+s))^{a+b+1}}{(1+s)^{\{a(d-2)+b(d-1)+\max_{1\le i\le c}(u_i-t_i)\}}}ds\lesssim 1.$$

We consider two cases:
Case 1: $b=0$.

In this case, we use use the last condition listed in (5.9) to get $\max_{1\le i\le c}(u_i - t_i) \ge 1$, and hence

$$\mathcal{C} \lesssim \int_0^{t/2} \frac{(\log(1+s))^{a+1}}{(1+s)^{a(d-2)+1}} \lesssim \int_0^{t/2} \frac{(\log(1+s))^{a+1}}{(1+s)^{a(d-2)+1}} ds \lesssim 1.$$

Case 2: $b \ge 1$.
In this case, we estimate $\mathcal{C}$ as follows:

$$\mathcal{C} \lesssim \int_0^{t/2} \frac{(\log(1+s))^{a}}{(1+s)^{a(d-2)}} \cdot \frac{(\log(1+s))^{b+1}}{(1+s)^{b(d-1)}} ds \lesssim \int_0^{t/2} \frac{(\log(1+s))^{b+1}}{(1+s)^{b(d-1)}} ds \lesssim 1.$$

The proof is complete. □

Finally, we give the proof for the main Theorem 2.1.
Proof of Theorem 2.1. Let

$$\mathcal{N}(t) = \sup_{0\le s\le t} \max_{0\le k\le N} \frac{\left(\langle s\rangle^k \|\nabla_x^k \rho(s)\|_{L^1} + \langle s\rangle^{k+d} \|\nabla_x^k \rho(s)\|_{L^\infty}\right)}{\log(1+s)}.$$

Now we fix a constant $M_0 > 0$, which will be chosen later. We prove that $\mathcal{N}(t) \le M_0\varepsilon_0$ for all $t \ge 0$, as long as ε_0 is small enough. Let

$$T_\star = \sup\{T > 0 : \quad \mathcal{N}(t) \le M_0\varepsilon_0 \quad \text{for all} \quad 0 \le t \le T\} \tag{5.12}$$

We shall prove that $T_\star = \infty$ by contradiction argument. Assuming that $T_\star < \infty$, we have

$$\mathcal{N}(T_\star) = \varepsilon_0 \quad \text{and} \quad \sup_{0\le t\le T} \mathcal{N}(t) < \varepsilon_0 \quad \text{for all} \quad T < T_\star.$$

For $t \in (0, T_\star)$, by Theorem 3.3 and 5.3, there exists $C_0, C_1 > 0$ such that

$$\mathcal{N}(t) \le C_0 \|S\|_{Y_t^N} \le C_0C_1(\varepsilon_0 + \varepsilon_0^2)$$

for ε_0 small enough. Let $t \to T_\star$, we have

$$M_0\varepsilon_0 \le C_0C_1(\varepsilon_0 + \varepsilon_0^2)$$

which is false, as long as $M_0 > 2C_0C_1$ and ε_0 small. The proof is complete.

References

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[2] R. Balescu. *Statistical mechanics of charged particles.* Monographs in Statistical Physics and Thermodynamics, Vol. 4. Interscience Publishers John Wiley & Sons, Ltd. London-New York-Sydney, 1963.

[3] C. Bardos and P. Degond. Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2(2):101–118, 1985.

[4] J. Bedrossian, N. Masmoudi, and C. Mouhot. Landau damping: paraproducts and Gevrey regularity. *Ann. PDE*, 2(1):Art. 4, 71, 2016.

[5] J. Bedrossian, N. Masmoudi, and C. Mouhot. Landau damping in finite regularity for unconfined systems with screened interactions. *Comm. Pure Appl. Math.*, 71(3):537–576, 2018.

[6] T. J. M. Boyd and J. J. Sanderson. *The physics of plasmas.* Cambridge University Press, Cambridge, 2003.

[7] P.-H. Chavanis. Statistical mechanics of violent relaxation in stellar systems. In *Multiscale problems in science and technology (Dubrovnik, 2000)*, pages 85–116. Springer, Berlin, 2002.

[8] S.-H. Choi, S.-Y. Ha, and H. Lee. Dispersion estimates for the two-dimensional Vlasov-Yukawa system with small data. *J. Differential Equations*, 250(1):515–550, 2011.

[9] R. T. Glassey. *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

[10] D. Han-Kwan. Quasineutral limit of the Vlasov-Poisson system with massless electrons. *Comm. Partial Differential Equations*, 36(8):1385–1425, 2011.

[11] D. Han-Kwan and M. Iacobelli. The quasineutral limit of the Vlasov-Poisson equation in Wasserstein metric. *Commun. Math. Sci.*, 15(2):481–509, 2017.

[12] D. Han-Kwan, T. T. Nguyen, and F. Rousset. Long time estimates for the Vlasov-Maxwell system in the non-relativistic limit. *Comm. Math. Phys.*, 363(2):389–434, 2018.

[13] D. Han-Kwan, T. T. Nguyen, and F. Rousset. Asymptotic stability of equilibria for screened vlasov-poisson systems via pointwise dispersive estimates, 2019.

[14] D. Han-Kwan and F. Rousset. Quasineutral limit for Vlasov-Poisson with Penrose stable data. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(6):1445–1495, 2016.

[15] E. Horst. On the asymptotic growth of the solutions of the Vlasov-Poisson system. *Math. Methods Appl. Sci.*, 16(2):75–86, 1993.

[16] H. J. Hwang, A. Rendall, and J. J. L. Velázquez. Optimal gradient estimates and asymptotic behaviour for the Vlasov-Poisson system with small initial data. *Arch. Ration. Mech. Anal.*, 200(1):313–360, 2011.

[17] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.*, 105(2):415–430, 1991.

[18] C. Mouhot and C. Villani. On Landau damping. *Acta Math.*, 207(1):29–201, 2011.

[19] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Differential Equations*, 95(2):281–303, 1992.

[20] J. Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Comm. Partial Differential Equations*, 16(8-9):1313–1335, 1991.

[21] J. Smulevici. Small data solutions of the Vlasov-Poisson system and the vector field method. *Ann. PDE*, 2(2):Art. 11, 55, 2016.

[22] X. Wang. Decay estimates for the $3d$ relativistic and non-relativistic vlasov-poisson systems, 2018.