



On Global Stability of Optimal Rearrangement Maps

HUY Q. NGUYEN & TOAN T. NGUYEN

Communicated by P. CONSTANTIN

Abstract

We study the nonlocal vectorial transport equation $\partial_t y + (\mathbb{P}y \cdot \nabla)y = 0$ on bounded domains of \mathbb{R}^d where \mathbb{P} denotes the Leray projector. This equation was introduced to obtain the unique optimal rearrangement of a given map y_0 as the infinite time limit of the solution with initial data y_0 (Angenent et al.: SIAM J Math Anal 35:61–97, 2003; McCann: A convexity theory for interacting gases and equilibrium crystals. Thesis (Ph.D.)-Princeton University, ProQuest LLC, Ann Arbor, MI, p 163, 1994; Brenier: J Nonlinear Sci 19(5):547–570, 2009). We rigorously justify this expectation by proving that for initial maps y_0 sufficiently close to maps with strictly convex potential, the solutions y are global in time and converge exponentially quickly to the optimal rearrangement of y_0 as time tends to infinity.

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^d equipped with the Lebesgue measure. Two L^2 maps $y_1, y_2 : \Omega \rightarrow \mathbb{R}^d$ are rearrangements of each other if they define the same image measure of the Lebesgue measure, i.e.,

$$\int_{\Omega} f(y_1(x))dx = \int_{\Omega} f(y_2(x))dx$$

for all compactly supported continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$. A celebrated theorem due to Brenier [5] asserts that for each L^2 map $y_0 : \Omega \rightarrow \mathbb{R}^d$ there exists a unique rearrangement y^* with *convex potential*, i.e. $y^* = \nabla p^*$ for some convex function p^* . Moreover, among all possible rearrangements of y_0 , y^* minimizes the quadratic cost function

$$\int_{\Omega} |y(x) - x|^2 dx.$$

We shall refer to y^* as the *optimal rearrangement* of y_0 . Finding the unique optimal rearrangement y^* for a given map y_0 is thus among the main concerns in optimal transport theory. As an attempt to get the optimal rearrangement y^* of y_0 as an *equilibrium state in the infinite time* of a dynamical system that could be efficiently solved by computer, Angenent, Haker, and Tannenbaum [1] (see also McCann [12, 13] and Brenier [6]) proposed the following nonlocal vectorial transport model (AHT):

$$\begin{aligned}\partial_t y + u \cdot \nabla y &= 0, \\ u &= \mathbb{P}y,\end{aligned}\tag{1.1}$$

where $y = y(x, t) \in \mathbb{R}^d$, $x \in \Omega \subset \mathbb{R}^d$, $t \geq 0$, and \mathbb{P} denotes the classical Leray projector onto the space of divergence-free vector fields. Throughout the paper, we take either $\Omega = \mathbb{T}^d$ (periodic domain) or a bounded domain in \mathbb{R}^d , $d \geq 2$ with smooth boundary. The Leray projector $u = \mathbb{P}y$ is defined as follows: for a given map $y : \Omega \rightarrow \mathbb{R}^d$, we construct the potential p that solves

$$\begin{cases} \Delta p = \nabla \cdot y & \text{in } \Omega \\ \frac{\partial p}{\partial n} = y \cdot n & \text{on } \partial\Omega, \end{cases}\tag{1.2}$$

where n is the unit outward normal to $\partial\Omega$. Then we define

$$\mathbb{P}y = y - \nabla p.$$

As a consequence of the definition, the velocity $u = \mathbb{P}y$ is tangent to the boundary,

$$u \cdot n = 0 \quad \text{on } \partial\Omega.\tag{1.3}$$

When $\Omega = \mathbb{T}^d$, the Leray projector $\mathbb{P} = (\mathbb{P}_{ij})$ is simply a Fourier multiplier matrix of order 0 with

$$\mathbb{P}_{ij}(\xi) = 1 - \frac{\xi_i \xi_j}{|\xi|^2}, \quad \xi \in \mathbb{Z}^d.$$

Interestingly, the AHT model (1.1) can also be obtained as the zero inertial limit of generalized (damped) Euler-Boussinesq equations in convection theory [6–8]. In addition, by specifying $y(x) = (0, \rho(x))$, (1.1) reduces to the incompressible porous media (IPM) equations

$$\begin{cases} \partial_t \rho + (u \cdot \nabla) \rho = 0, & x \in \Omega \subset \mathbb{R}^2, \\ u + \nabla p = (0, \rho)^T. \end{cases}\tag{1.4}$$

Here ρ plays the role of fluid density. Stability of the special solution $\rho_*(x_1, x_2) = x_2$ of (1.4) has been proved in [10] for $\Omega = \mathbb{R}^2$, \mathbb{T}^2 , and in [9] for $\Omega = \mathbb{T} \times (-l, l)$ which posses two horizontal boundaries. The presence of boundaries, though only flat boundaries, makes the proof in [9] more involved.

Following [6], let us explain why (1.1) is expected to capture the optimal rearrangement of initial maps as steady states in infinite time. First, since the velocity u is divergence-free and tangent to the boundary, we have

$$\frac{d}{dt} \int_{\Omega} f(y(x, t)) dx = 0 \quad \forall t > 0$$

for any compactly supported continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Integrating this in time we obtain that each $y(t)$, $t > 0$ is a rearrangement of $y(0)$. Second, it is readily checked that the balance law

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} |y - x|^2 dx = - \int_{\Omega} |u|^2 dx$$

holds. In particular, steady states must be gradients since their Leray projections vanish. Conversely, all gradients are clearly steady states of (1.1). Now if y is global and the infinite-time limit y_{∞} of y exists (in a sufficiently strong topology) then the integral $\int_0^{\infty} \int_{\Omega} |u|^2 dx dt$ is finite. Consequently, u vanishes as $t \rightarrow \infty$ and thus y_{∞} must be a gradient, $y_{\infty} = \nabla p_{\infty}$. If we have in addition that p_{∞} is a convex function, then coupling with the fact that y_{∞} is a rearrangement of $y(0)$ we conclude by virtue of the aforementioned theorem of Bernier that y_{∞} is the unique optimal rearrangement of $y(0)$. The remaining issues in the above argument are global existence and long time behavior for (1.1). On the other hand, the objects that we expect (1.1) to capture in infinite time are maps with convex potential. A natural problem then, is:

Are maps with convex potential globally stable?

Our goal in the present paper is to prove that maps with *strictly convex* potential are globally stable. Precisely, our main theorem reads as follow:

Theorem 1.1. *Let $s > 1 + \frac{d}{2}$ be an integer with $d \geq 2$. Let Ω be a C^{∞} bounded domain in \mathbb{R}^d . Consider $y_* = \nabla p_*$ for some strictly convex function $p_* : \Omega \rightarrow \mathbb{R}$ whose Hessian satisfies*

$$\nabla^2 p_*(x) \geq \theta_0 Id \quad \forall x \in \Omega, \quad \theta_0 > 0. \quad (1.5)$$

Then, there exists a small positive number ε depending only on θ_0 and $\|y_\|_{H^{s+1}(\Omega)}$ such that for all $y_0 \in H^s(\Omega)$ with $\|y_0 - y_*\|_{H^s(\Omega)} \leq \varepsilon$, problem (1.1) has a unique global solution y . In addition, there is a positive constant C depending only on θ_0 and $\|y_*\|_{H^{s+1}(\Omega)}$ so that*

$$\|y(t) - y_*\|_{H^s(\Omega)} \leq C \|y_0 - y_*\|_{H^s(\Omega)} \quad (1.6)$$

and

$$\|\mathbb{P}y(t)\|_{H^s(\Omega)} \leq C \|y_0 - y_*\|_{H^s(\Omega)} e^{-\frac{\theta_0}{C}t} \quad (1.7)$$

for all $t \geq 0$. Moreover, there exists a strictly convex function $p_{\infty} : \Omega \rightarrow \mathbb{R}$ such that

$$\|y(t) - \nabla p_{\infty}\|_{H^{s-1}(\Omega)} \leq C e^{-\frac{\theta_0}{C}t} \quad \forall t \geq 0. \quad (1.8)$$

In particular, ∇p_{∞} is the optimal rearrangement of y_0 .

Remark 1.2. The domain Ω need not be C^∞ but only $C^{[s]+n_0}$ for a sufficiently large integer n_0 .

For periodic perturbations, our result reads as follow:

Theorem 1.3. Let $s > 1 + \frac{d}{2}$ be a real number with $d \geq 2$. Let Ω be the box $[0, L]^d$, $L > 0$. Consider $y_* = \nabla p_*$ for some function $p_* : \Omega \rightarrow \mathbb{R}$ whose Hessian is L -periodic and satisfies

$$\nabla^2 p_*(x) \geq \theta_0 Id \quad \forall x \in \Omega, \quad \theta_0 > 0. \quad (1.9)$$

Then, there exists a small positive number ε depending only on θ_0 and $\|y_*\|_{H^{s+1}(\Omega)}$ such that the following holds. For all $y_0 \in H^s(\Omega)$ such that $y_0 - y_*$ is L -periodic and $\|y_0 - y_*\|_{H^s(\Omega)} \leq \varepsilon$, problem (1.1) has a unique global solution y . In addition, there is a positive constant C depending only on θ_0 and $\|y_*\|_{H^{s+1}(\Omega)}$ so that $y(t) - y_*$ is L -periodic,

$$\|y(t) - y_*\|_{H^s(\Omega)} \leq C \|y_0 - y_*\|_{H^s(\Omega)} \quad (1.10)$$

and

$$\|\mathbb{P}y(t)\|_{H^s(\Omega)} \leq C \|y_0 - y_*\|_{H^s(\Omega)} e^{-\frac{\theta_0}{4}t} \quad (1.11)$$

for all $t \geq 0$. Moreover, there exists a strictly convex function $p_\infty : \Omega \rightarrow \mathbb{R}$ such that

$$\|y(t) - \nabla p_\infty\|_{H^{s-1}(\Omega)} \leq C e^{-\frac{\theta_0 t}{4}} \quad \forall t \geq 0. \quad (1.12)$$

In particular, ∇p_∞ is the optimal rearrangement of y_0 .

Remark 1.4. In Theorem 1.3, the steady solution $y_* = \nabla p_*$ need not be periodic, only its Hessian $\nabla^2 p_*$ is required to be periodic and positive. A typical example is $p_* = |x|^2 + q_*(x)$ where q_* is L -periodic with sufficiently small $\|\nabla^2 q_*\|_{L^\infty(\Omega)}$. The initial map y_0 need not be periodic, but if the initial perturbation $y_0 - y_*$ is periodic then it remains so for all positive times.

The estimates (1.8) and (1.12) exhibit the exponential convergence towards the optimal rearrangement of y_0 provided that y_0 is sufficiently close to a map with strictly convex potential. This justifies the efficiency of the AHT model (1.1). Theorem 1.1 also provides the first class of time-dependent global solutions to this nonlocal vectorial transport equation for which the issues of global regularity and finite-time blowup remain open.

Let $y_* = \nabla p_*$ be a steady state of (1.1) where p_* satisfies the strict convexity condition (1.5). Introduce the perturbation $z = y - y_*$. Noticing that $\mathbb{P}y_* = 0$, equation (1.1) yields

$$\begin{aligned} \partial_t z + u \cdot \nabla y_* + u \cdot \nabla z &= 0, \\ u &= \mathbb{P}z, \end{aligned} \quad (1.13)$$

where $u \cdot n = 0$ on $\partial\Omega$. In order to obtain the global stability, some form of decay is needed. Since z is transported, it is not expected to decay. Our idea is to obtain

decay for the divergence-free part u of z . Indeed, taking Leray's projection of (1.13) one finds that u obeys

$$\partial_t u + \mathbb{P}(u \cdot \nabla y_*) + \mathbb{P}(u \cdot \nabla z) = 0. \quad (1.14)$$

An L^2 energy estimate combined with the strict convexity of p_* and the fact that \mathbb{P} is self-adjoint in L^2 shows that u decays exponentially when measured in L^2 . We need however decay of high Sobolev norms of u in order to close the nonlinear iteration. In performing a direct H^s energy estimate for u at the level of (1.14), there are at least two difficulties:

- (i) the term $u \cdot \nabla z$ would induce a loss of derivatives due to the presence of ∇z ;
- (ii) to reveal the damping mechanism due to $\nabla y_* = \nabla^2 p_* \geq \theta_0 \text{Id}$, one needs to make appear the term $D^s u \cdot \nabla y_*$ where D^s denotes any partial derivatives of order s . However, in the presence of boundaries, D^s do not commute with \mathbb{P} . Moreover, in general the commutator $[D^s, \mathbb{P}]$ does not exhibit a gain of derivative, and hence is of the same order as the damping term.

To handle (i) we commute \mathbb{P} with $u \cdot \nabla$ as follows:

$$\partial_t u + \mathbb{P}(u \cdot \nabla y_*) + u \cdot \nabla u + [\mathbb{P}, u \cdot \nabla]z = 0. \quad (1.15)$$

The new nonlinear term $u \cdot \nabla u$ is now an advection term, and thus does not induce any loss of derivatives. However, a gain of one derivative in $[\mathbb{P}, u \cdot \nabla]z$ is then needed. As mentioned in (ii), such a gain is not true in general for $[\mathbb{P}, \partial_j]$. Interestingly, if one replaces partial derivatives ∂_j with $u \cdot \nabla$, this holds even in domains with boundary, provided only that u is tangent to the boundary. This is the content of the next theorem, which is of independent interest. Throughout this paper we denote

$$\mathbb{T}^d := (\mathbb{R}/L\mathbb{Z})^d.$$

Theorem 1.5. *Let $s > 1 + \frac{d}{2}$ be an integer with $d \geq 2$. Consider $\Omega = \mathbb{T}^d$ or Ω a bounded domain in \mathbb{R}^d with smooth boundary. Let \mathbb{P} denote the Leray projector associated to Ω . Then, for any vector fields $u, z \in H^s(\Omega; \mathbb{R}^d)$ with $u \cdot n|_{\partial\Omega} = 0$ when $\partial\Omega \neq \emptyset$, the commutator estimate*

$$\|[\mathbb{P}, u \cdot \nabla]z\|_{H^s(\Omega)} \leq C \|u\|_{H^s(\Omega)} \|z\|_{H^s(\Omega)} \quad (1.16)$$

holds for some universal constant C .

Regarding the difficulty (ii), we observe that “tangential derivatives” commute nicely with the Leray projector while “normal derivatives” do not. We then introduce a boundary adapted system of derivatives \mathcal{D}^s (see Section 2.1) which are defined everywhere and become the usual tangential and normal derivatives when restricted to the boundary. Next, to avoid the commutator $[D^s, \mathbb{P}]$ when dealing with the nonlocal term $\mathbb{P}(u \cdot \nabla y_*)$, we write

$$\int_{\Omega} \mathcal{D}^s u \cdot \mathcal{D}^s \mathbb{P}(u \cdot \nabla y_*) dx = \int_{\Omega} \mathcal{D}^s u \cdot \mathcal{D}^s (u \cdot \nabla y_*) dx + \int_{\Omega} \mathcal{D}^s u \cdot \mathcal{D}^s (\mathbb{P} - \text{Id})(u \cdot \nabla y_*) dx$$

and notice a special structure in the second integral. This allows us to prove a *hierarchy of estimates* for the velocity u , ordered by the *number of normal derivatives* in \mathcal{D}^s , and hence to close our nonlinear iteration.

For the proof of Theorem 1.1 we will need the local well-posedness of the AHT model (1.1) in Sobolev spaces.

Theorem 1.6. *Let Ω be a bounded domain in \mathbb{R}^d , $d \geq 2$ with smooth boundary or periodic boundary conditions. Let $s > 1 + \frac{d}{2}$ be an integer. Then for any initial data $z_0 \in H^s(\Omega)$, there exist a positive time T depending only on $\|z_0\|_{H^s(\Omega)}$ and a unique solution $z \in C([0, T]; H^s(\Omega))$ of (1.1).*

Local well-posedness of (1.1) in Hölder spaces $C^{1,\alpha}(\Omega)$ has been obtained in [1]. Since the velocity u has the same Sobolev regularity as the unknown z , the proof of Theorem 1.6 is standard via energy methods, and thus will be skipped.

The paper is organized as follows. Section 2 is devoted to various commutator estimates involving the Leray projector. Theorem 1.3 is proved in Section 3, and Theorem 1.1 is proved in Section 4.

Throughout this paper, we denote by ∂_j , $j \in \{1, \dots, d\}$ the j th partial derivative and by D^m any partial derivatives of order $m \in \mathbb{N}$.

2. Commutator Estimates

2.1. A Boundary Adapted System of Derivatives

For simplicity, we assume from now on that Ω is a C^∞ domain. Let $\delta(x) = \text{dist}(x, \partial\Omega)$ be the distance function. There exists a small number $\kappa > 0$ such that δ is C^∞ in the neighborhood of the region

$$\Omega_{3\kappa} = \{x \in \Omega : \delta(x) \leq 3\kappa\},$$

and $\nabla\delta(x) \neq 0$ for any $x \in \Omega_{3\kappa}$. See Section 3.2, Chapter III in [4]. Note that the unit outward normal $n(x) = -\nabla\delta(x)$ for $x \in \partial\Omega$. We thus can extend n to $\Omega_{3\kappa}$ by setting

$$n(x) = -\frac{\nabla\delta(x)}{|\nabla\delta(x)|}, \quad x \in \Omega_{3\kappa}.$$

For each $x \in \Omega_{3\kappa}$, we can choose $\tau(x) = \{\tau_j(x) : j = 1, \dots, d-1\}$ an orthonormal basis of $(n(x))^\perp$ in \mathbb{R}^d such that $\tau_j \in C^\infty(\Omega_{3\kappa})$.

Next we fix a cutoff function $\chi_1 : \overline{\Omega} \rightarrow [0, 1]$ satisfying

$$\chi_1 \equiv 1 \quad \text{in a neighborhood of } \Omega_{2\kappa}, \quad \chi_1 \equiv 0 \quad \text{in } \overline{\Omega} \setminus \Omega_{3\kappa}. \quad (2.1)$$

For a vector field $v : \Omega \rightarrow \mathbb{R}^d$ we define its weighted normal and tangential components respectively by

$$v_n(x) = \chi_1(x)v(x) \cdot n(x), \quad v_{\tau_j}(x) = \chi_1(x)v(x) \cdot \tau_j(x), \quad j = 1, \dots, d-1$$

for $x \in \Omega$. In particular, $v = v_n n + \sum_{j=1}^{d-1} v_{\tau_j} \tau_j$ in $\Omega_{2\kappa}$. In the special case of gradient vectors ∇f where $f : \Omega \rightarrow \mathbb{R}$, we write

$$\partial_n f = (\nabla f)_n, \quad \partial_{\tau_j} f = (\nabla f)_{\tau_j}, \quad j = 1, \dots, d-1.$$

Both $\partial_n f$ and $\partial_{\tau_j} f$ are defined over Ω and become the usual normal and tangential derivatives when restricted to the boundary. Note in addition that

$$\nabla f = n \partial_n f + \sum_{j=1}^{d-1} \tau_j \partial_{\tau_j} f \quad \text{in } \Omega_{2\kappa}. \quad (2.2)$$

For a vector field $v : \Omega \rightarrow \mathbb{R}^d$ we write $\partial_n v = (\nabla v) \cdot n$ and similarly for $\partial_{\tau_j} v$. Then we have

$$|\nabla v|^2 = \sum_{i=1}^d |\nabla v_i|^2 = \sum_{i=1}^d (|\partial_n v_i|^2 + \sum_{j=1}^{d-1} |\partial_{\tau_j} v_i|^2) = |\partial_n v|^2 + \sum_{j=1}^{d-1} |\partial_{\tau_j} v|^2 \quad (2.3)$$

for $x \in \Omega_{2\kappa}$.

Lemma 2.1. For $v : \Omega \rightarrow \mathbb{R}^d$ and $f : \Omega \rightarrow \mathbb{R}$ we have

$$\partial_n v \cdot n + \sum_{j=1}^{d-1} \partial_{\tau_j} v \cdot \tau_j = \operatorname{div} v \quad (2.4)$$

and

$$\partial_n^2 f + \sum_{j=1}^{d-1} \partial_{\tau_j}^2 f = \Delta f + \nabla f \cdot (n \cdot \nabla) n + \sum_{j=1}^{d-1} \nabla f \cdot (\tau_j \cdot \nabla) \tau_j \quad (2.5)$$

at any $x \in \Omega_{2\kappa}$.

Proof. We first notice that since $\chi_1 \equiv 1$ in $\Omega_{2\kappa}$. If R denotes the matrix whose first $d-1$ columns are $\tau_1, \dots, \tau_{d-1}$ and whose d th column is n , then R is orthonormal; that is, $RR^T = \operatorname{Id}$. Using this and the above definitions of ∂_n and ∂_{τ_j} we have

$$\begin{aligned} & \partial_n v \cdot n + \sum_{j=1}^{d-1} \partial_{\tau_j} v \cdot \tau_j \\ &= \sum_{k,\ell=1}^d \partial_k v_\ell n_k n_\ell + \sum_{j=1}^{d-1} \sum_{k,\ell=1}^d \partial_k v_\ell \tau_{j,k} \tau_{j,\ell} \\ &= \sum_{k,\ell=1}^d \partial_k u_\ell \sum_{j=1}^d R_{k,j} R_{j,\ell}^T \\ &= \sum_{k,\ell=1}^d \partial_k u_\ell \delta_{k,\ell} \\ &= \operatorname{div} u. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 & \partial_n^2 f + \sum_{j=1}^{d-1} \partial_{\tau_j}^2 f \\
 &= \sum_{k,\ell=1}^d \partial_k \partial_\ell f n_k n_\ell + \sum_{j=1}^{d-1} \sum_{k,\ell=1}^d \partial_k \partial_\ell f \tau_{j,k} \tau_{j,\ell} + \nabla f \cdot (n \cdot \nabla) n + \sum_{j=1}^{d-1} \nabla f \cdot (\tau_j \cdot \nabla) \tau_j \\
 &= \sum_{k,\ell=1}^d \partial_k \partial_\ell f \delta_{k,\ell} + \nabla f \cdot (n \cdot \nabla) n + \sum_{j=1}^{d-1} \nabla f \cdot (\tau_j \cdot \nabla) \tau_j \\
 &= \Delta f + \nabla f \cdot (n \cdot \nabla) n + \sum_{j=1}^{d-1} \nabla f \cdot (\tau_j \cdot \nabla) \tau_j.
 \end{aligned}$$

□

2.2. Proof of Theorem 1.5

In this section, we study the commutator term $[\mathbb{P}, u \cdot \nabla]z$ appearing in (1.15) and prove Theorem 1.5. We start with the periodic case $\Omega = \mathbb{T}^d$, $d \geq 2$.

Lemma 2.2. *Let $s > 1 + \frac{d}{2}$ be a real number and let \mathbb{P} be the Leray projector. Then, for any $u, z \in H^s(\mathbb{T}^d)$, there holds*

$$\|[\mathbb{P}, u \cdot \nabla]z\|_{H^s(\mathbb{T}^d)} \leq C \|u\|_{H^s} \|z\|_{H^s(\mathbb{T}^d)}$$

for some universal constant C .

Proof. For any two functions $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$, we have the following Bony's decomposition (see [2,3]):

$$fg = T_f g + T_g f + R(f, g),$$

where $T_f g$ and $T_g f$ are paraproducts so that the following hold:

- Paraproduct estimates:

$$\|T_f g\|_{H^r} \leq C \|f\|_{L^\infty} \|g\|_{H^r} \quad \forall r \in \mathbb{R}. \quad (2.6)$$

- Reminder estimates:

$$\|R(f, g)\|_{H^{r_1+r_2-\frac{d}{2}}} \leq C \|f\|_{H^{r_1}} \|g\|_{H^{r_2}} \quad (2.7)$$

for all r_1 and r_2 in \mathbb{R} satisfying $r_1 + r_2 > 0$.

- Commutator estimates:

$$\|[T_f, m(D)]g\|_{H^{r-k+\alpha}} \leq C \|f\|_{W^{\alpha,\infty}} \|g\|_{H^r} \quad (2.8)$$

for any homogeneous operator $m(D)$ of order k , $\alpha \in (0, 1]$, and $r \in \mathbb{R}$.

Recall that on \mathbb{T}^d , the Leray projector $\mathbb{P} = (\mathbb{P}_{ij})$ is a Fourier multiplier matrix of order 0. In addition, we note that $[\mathbb{P}, \nabla] = 0$. Denoting by v_k the k th component of the vector v , we have

$$\begin{aligned}
 ([\mathbb{P}, u \cdot \nabla]z)_k &= (\mathbb{P}(u \cdot \nabla z))_k - (u \cdot \nabla \mathbb{P}z)_k \\
 &= \mathbb{P}_{kj}(u_i \partial_i z_j) - u_i \partial_i \mathbb{P}_{kj} z_j \\
 &= \mathbb{P}_{kj}(u_i \partial_i z_j) - u_i \mathbb{P}_{kj} \partial_i z_j \\
 &= \mathbb{P}_{kj}(T_{u_i} \partial_i z_j) + \mathbb{P}_{kj}(T_{\partial_i z_j} u_i) + \mathbb{P}_{kj} R(u_i, \partial_i z_j) \\
 &\quad - T_{u_i} \mathbb{P}_{kj} \partial_i z_j - T_{\mathbb{P}_{kj} \partial_i z_j} u_i - R(u_i, \mathbb{P}_{kj} \partial_i z_j) \\
 &= [\mathbb{P}_{kj}, T_{u_i}] \partial_i z_j + \mathbb{P}_{kj}(T_{\partial_i z_j} u_i) + \mathbb{P}_{kj} R(u_i, \partial_i z_j) \\
 &\quad - T_{\mathbb{P}_{kj} \partial_i z_j} u_i - R(u_i, \mathbb{P}_{kj} \partial_i z_j).
 \end{aligned}$$

Here and in what follows, the summation in i, j is understood. Since \mathbb{P} is a Fourier multiplier of order 0 and T_{u_i} has order 0, the commutator estimate (2.8) yields

$$\|[\mathbb{P}_{kj}, T_{u_i}] \partial_i z_j\|_{H^s} \leq C \|u\|_{W^{1,\infty}} \|\nabla z\|_{H^{s-1}} \leq C \|u\|_{H^s} \|z\|_{H^s}$$

where we used the embedding $H^s \subset W^{1,\infty}$ for $s > 1 + \frac{d}{2}$. The paraproduct terms $\mathbb{P}_{kj}(T_{\partial_i z_j} u_i)$ and $T_{\mathbb{P}_{kj} \partial_i z_j} u_i$ are estimated in H^s by means of the paraproduct rule (2.6), the embedding $H^s \subset W^{1,\infty}$, and the fact that \mathbb{P} is continuous from H^s to H^s . Finally, the reminder terms can be treated using (2.7) as follows:

$$\|R(u_i, \partial_i z_j)\|_{H^s} \leq \|R(u_i, \partial_i z_j)\|_{H^{s+s-1-\frac{d}{2}}} \leq C \|u\|_{H^s} \|\nabla z\|_{H^{s-1}} \leq C \|u\|_{H^s} \|z\|_{H^s},$$

which gives the lemma. \square

We next turn to the case when Ω has a boundary. Fix an integer $s > 1 + \frac{d}{2}$. By definition, we write $z = \mathbb{P}z + \nabla f$, where f solves

$$\begin{cases} \Delta f = \operatorname{div} z & \text{in } \Omega, \\ \frac{\partial f}{\partial n} = z \cdot n & \text{on } \partial\Omega. \end{cases} \quad (2.9)$$

In particular, the standard elliptic regularity theory yields $\|f\|_{H^{s+1}(\Omega)} \leq C \|z\|_{H^s(\Omega)}$. Similarly, we write $(u \cdot \nabla)z = \mathbb{P}((u \cdot \nabla)z) + \nabla g$, where g solves

$$\begin{cases} \Delta g = \operatorname{div}((u \cdot \nabla)z) & \text{in } \Omega, \\ \frac{\partial g}{\partial n} = (u \cdot \nabla)z \cdot n & \text{on } \partial\Omega. \end{cases} \quad (2.10)$$

Combining things, we have

$$\begin{aligned}
 [\mathbb{P}, u \cdot \nabla]z &= \mathbb{P}((u \cdot \nabla)z) - (u \cdot \nabla)\mathbb{P}z \\
 &= (u \cdot \nabla)z - \nabla g - (u \cdot \nabla)(z - \nabla f) \\
 &= (u \cdot \nabla)(\nabla f) - \nabla g.
 \end{aligned} \quad (2.11)$$

We shall bound the H^s norm of $[\mathbb{P}, u \cdot \nabla]z$, using the elliptic estimate

$$\|h\|_{H^s(\Omega)} \leq C \|\operatorname{div} h\|_{H^{s-1}(\Omega)} + C \|\operatorname{curl} h\|_{H^{s-1}(\Omega)} + C \|h \cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \quad (2.12)$$

for $h = [\mathbb{P}, u \cdot \nabla]z$, where the terms on the right hand side are estimated in the following lemmas:

Lemma 2.3. *There exists a positive constant C such that*

$$\|\operatorname{div}([\mathbb{P}, u \cdot \nabla]z)\|_{H^{s-1}(\Omega)} \leq C \|u\|_{H^s(\Omega)} \|z\|_{H^s(\Omega)} \quad (2.13)$$

and

$$\|\operatorname{curl}([\mathbb{P}, u \cdot \nabla]z)\|_{H^{s-1}(\Omega)} \leq C \|u\|_{H^s(\Omega)} \|z\|_{H^s(\Omega)}. \quad (2.14)$$

Proof. In view of (2.11), we compute, using (2.9),

$$\operatorname{div}((u \cdot \nabla)(\nabla f)) = \nabla u : (\nabla \otimes \nabla) f + u \cdot \nabla \operatorname{div} z.$$

On the other hand, using equation (2.10), we have

$$\operatorname{div}(\nabla g) = \Delta g = \operatorname{div}((u \cdot \nabla)z) = \nabla u : (\nabla z)^T.$$

Combining, the above we have

$$\operatorname{div}([\mathbb{P}, u \cdot \nabla]z) = \nabla u : [(\nabla \otimes \nabla) f - (\nabla z)^T]. \quad (2.15)$$

The estimate (2.13) thus follows directly from (2.15), upon using the fact that $H^{s-1}(\Omega)$ is an algebra and the elliptic estimates $\|f\|_{H^{s+1}(\Omega)} \leq C \|z\|_{H^s(\Omega)}$.

Next, in view of (2.11), we write

$$[\mathbb{P}, u \cdot \nabla]z = \nabla(u \cdot \nabla f - g) - \nabla u_k \partial_k f,$$

which gives $\operatorname{curl}([\mathbb{P}, u \cdot \nabla]z) = \nabla u_k \times \partial_k \nabla f$. The estimate (2.14) then follows from elliptic estimates as before. \square

Lemma 2.4. *There exists a positive constant C such that*

$$\|[\mathbb{P}, u \cdot \nabla]z \cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C \|u\|_{H^s(\Omega)} \|z\|_{H^s(\Omega)}. \quad (2.16)$$

Proof. We use the decomposition $u = u_n n + \sum_{j=1}^{d-1} u_{\tau_j} \tau_j$ in $\Omega_{2\kappa}$. Then we compute

$$\begin{aligned} (u \cdot \nabla)z \cdot n &= (u \cdot \nabla)(z \cdot n) - (u \otimes z) : \nabla n \\ &= u_n (n \cdot \nabla)(z \cdot n) + \sum_{j=1}^{d-1} u_{\tau_j} (\tau_j \cdot \nabla)(z \cdot n) - (u \otimes z) : \nabla n \end{aligned}$$

in $\Omega_{2\kappa}$. Since $u \cdot n = 0$ on $\partial\Omega$, we have $u_n = 0$ on $\partial\Omega$. Taking the trace of the above equation on $\partial\Omega$ and recalling (2.10), we get

$$\nabla g \cdot n = (u \cdot \nabla)z \cdot n = \sum_{j=1}^{d-1} u_{\tau_j} \partial_{\tau_j} (z \cdot n) - (u \otimes z) : \nabla n \quad \text{on } \partial\Omega.$$

Similarly, on $\partial\Omega$, we have

$$(u \cdot \nabla)(\nabla f) \cdot n = \sum_{j=1}^{d-1} u_{\tau_j} \partial_{\tau_j} (\nabla f \cdot n) - (u \otimes \nabla f) : \nabla n.$$

Recalling (2.11) and using the boundary condition (2.9), which gives $\partial_{\tau_j}(\nabla f \cdot n) = \partial_{\tau_j}(z \cdot n)$ on $\partial\Omega$, we obtain

$$[\mathbb{P}, u \cdot \nabla]z \cdot n = (u \cdot \nabla)(\nabla f) \cdot n - \nabla g \cdot n = [u \otimes (z - \nabla f)] : \nabla n$$

on $\partial\Omega$. Using the trace inequality, we bound

$$\begin{aligned} \|[\mathbb{P}, u \cdot \nabla]z \cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} &\leq C \|u \otimes (z - \nabla f)\|_{H^s(\Omega)} \\ &\leq C \|u\|_{H^s(\Omega)} (\|z\|_{H^s(\Omega)} + \|f\|_{H^{s+1}(\Omega)}), \end{aligned}$$

which gives (2.16), upon recalling the elliptic estimates $\|f\|_{H^{s+1}} \leq C \|z\|_{H^s}$. \square

2.3. Commutators Between the Leray Projector and Tangential Derivatives

Proposition 2.5. *Let $m \geq 2$ be an integer. There exists a constant $C > 0$ such that*

$$\|[P, \mathbb{P}]u\|_{L^2(\Omega)} \leq C \|u\|_{H^{m-1}(\Omega)}$$

for any $P \in \{\Pi_{j=1}^m \partial_{\sigma_j} : \sigma_j \in \{\tau_1, \dots, \tau_{d-1}\}\}$ and any vector field $u \in H^{m-1}(\Omega)$.

Proof. Without loss of generality we consider $P = \partial_{\tau_1}^m$. In view of the identity

$$[\partial_{\tau_1}^{q+1}, \mathbb{P}]u = [\partial_{\tau_1}^q, \mathbb{P}]\partial_{\tau_1}u + \partial_{\tau_1}^q[\partial_{\tau_1}, \mathbb{P}]u, \quad q \geq 1,$$

and by induction in m , it suffices to prove that

$$\|[\partial_{\tau_1}, \mathbb{P}]\partial_{\tau_1}u\|_{L^2(\Omega)} \leq C \|u\|_{H^1(\Omega)} \quad (2.17)$$

and

$$\|[\partial_{\tau_1}, \mathbb{P}]u\|_{H^j(\Omega)} \leq C \|u\|_{H^j(\Omega)} \quad \forall j \geq 1. \quad (2.18)$$

To this end, for any vector field v , we write $\mathbb{P}v = v - \nabla f$ and $\mathbb{P}(\partial_{\tau_1}v) = \partial_{\tau_1}v - \nabla g$ where f and g solve

$$\begin{cases} \Delta f = \operatorname{div} v & \text{in } \Omega, \\ \frac{\partial f}{\partial n} = v \cdot n & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta g = \operatorname{div}(\partial_{\tau_1}v) & \text{in } \Omega, \\ \frac{\partial g}{\partial n} = (\partial_{\tau_1}v) \cdot n & \text{on } \partial\Omega \end{cases}$$

respectively. Then

$$[\partial_{\tau_1}, \mathbb{P}]v = \nabla g - \partial_{\tau_1}\nabla f = \nabla(g - \partial_{\tau_1}f) - [\partial_{\tau_1}, \nabla]f. \quad (2.19)$$

We compute

$$\begin{aligned} \Delta \partial_{\tau_1}f &= \partial_{\tau_1}\Delta f + \Delta(\chi_1 \tau_1) \cdot \nabla f + 2\nabla(\chi_1 \tau_1) : \nabla \nabla f, \\ \operatorname{div}(\partial_{\tau_1}v) &= \partial_{\tau_1}\operatorname{div} v + \nabla v : (\nabla(\chi_1 \tau_1))^T, \end{aligned}$$

where χ_1 is defined as in (2.1). As a consequence, $h := g - \partial_{\tau_1} f$ satisfies

$$\Delta h = \nabla v : (\nabla(\chi_1 \tau_1))^T - \Delta(\chi_1 \tau_1) \cdot \nabla f - 2\nabla(\chi_1 \tau_1) : \nabla \nabla f \quad \text{in } \Omega. \quad (2.20)$$

Regarding the boundary condition, we have

$$\begin{aligned} \partial_{\tau_1}(v \cdot n) &= \partial_{\tau_1} v \cdot n + \nabla n : (v \otimes \tau_1), \\ \partial_n(\partial_{\tau_1} f) &= \nabla \nabla f : (n \otimes \tau_1) + \nabla \tau_1 : (\nabla f \otimes n), \\ \partial_{\tau_1}(\partial_n f) &= \nabla \nabla f : (n \otimes \tau_1) + \nabla n : (\nabla f \otimes \tau_1) \end{aligned}$$

in Ω . This yields

$$\begin{aligned} \partial_n h &= \partial_n g - \partial_{\tau_1} \partial_n f - \nabla \tau_1 : (\nabla f \otimes n) + \nabla n : (\nabla f \otimes \tau_1) \\ &= (\partial_{\tau_1} v) \cdot n - \partial_{\tau_1}(v \cdot n) - \nabla \tau_1 : (\nabla f \otimes n) + \nabla n : (\nabla f \otimes \tau_1) \quad (2.21) \\ &= -\nabla n : (v \otimes \tau_1) - \nabla \tau_1 : (\nabla f \otimes n) + \nabla n : (\nabla f \otimes \tau_1) \quad \text{on } \partial\Omega. \end{aligned}$$

In addition, elliptic estimates combined with trace inequalities

$$\|v\|_{H^{\ell-\frac{3}{2}}(\partial\Omega)} \leq C\|v\|_{H^{\ell-1}(\Omega)} \quad \forall \ell \geq 2$$

yield

$$\|f\|_{H^\ell(\Omega)} \leq C\|v\|_{H^{\ell-1}(\Omega)} \quad \forall \ell \geq 2. \quad (2.22)$$

Proof of (2.18). In view of (2.20), (2.21) we deduce using elliptic estimates, trace inequalities and (2.22) that for any $\ell \geq 2$,

$$\begin{aligned} \|h\|_{H^\ell(\Omega)} &\leq C\|\nabla v : (\nabla(\chi_1 \tau_1))^T - \Delta(\chi_1 \tau_1) \cdot \nabla f - 2\nabla(\chi_1 \tau_1) : \nabla \nabla f\|_{H^{\ell-2}(\Omega)} \\ &\quad + C\|-\nabla n : (v \otimes \tau_1) - \nabla \tau_1 : (\nabla f \otimes n) + \nabla n : (\nabla f \otimes \tau_1)\|_{H^{\ell-\frac{3}{2}}(\partial\Omega)} \\ &\leq C\|\nabla v : (\nabla(\chi_1 \tau_1))^T - \Delta(\chi_1 \tau_1) \cdot \nabla f - 2\nabla(\chi_1 \tau_1) : \nabla \nabla f\|_{H^{\ell-2}(\Omega)} \quad (2.23) \\ &\quad + C\|-\nabla n : (v \otimes \tau_1) - \nabla \tau_1 : (\nabla f \otimes n) + \nabla n : (\nabla f \otimes \tau_1)\|_{H^{\ell-1}(\Omega)} \\ &\leq C\|v\|_{H^{\ell-1}(\Omega)} + C\|f\|_{H^\ell(\Omega)} \\ &\leq C'\|v\|_{H^{\ell-1}(\Omega)}. \end{aligned}$$

Note that the trace inequality used in the second inequality in (2.23) does not hold when $\ell = 1$. Now for any $j \geq 1$ using (2.19), (2.22) and (2.23) with $\ell = j+1 \geq 2$ together with the estimate

$$\|[\partial_{\tau_1}, \nabla]f\|_{H^j(\Omega)} \leq C\|f\|_{H^{j+1}(\Omega)}$$

we obtain

$$\begin{aligned} \|[\partial_{\tau_1}, \mathbb{P}]v\|_{H^j(\Omega)} &\leq \|h\|_{H^{j+1}(\Omega)} + \|[\partial_{\tau_1}, \nabla]f\|_{H^j(\Omega)} \\ &\leq C\|v\|_{H^j(\Omega)} + C\|f\|_{H^{j+1}(\Omega)} \\ &\leq C'\|v\|_{H^j(\Omega)} \end{aligned}$$

which is the desired estimate (2.18) if we set $v = u$.

Proof of (2.17). Again, we use the equations (2.20), (2.21) with $v = \partial_{\tau_1} u$ and H^1 elliptic estimate for the Neumann problem to have

$$\begin{aligned} \|h\|_{H^1(\Omega)} &\leq C \|\nabla v : (\nabla(\chi_1 \tau_1))^T - \Delta(\chi_1 \tau_1) \cdot \nabla f - 2\nabla(\chi_1 \tau_1) : \nabla \nabla f\|_{H^{-1}(\Omega)} \\ &\quad + C \| -\nabla n : (v \otimes \tau_1) - \nabla \tau_1 : (\nabla f \otimes n) + \nabla n : (\nabla f \otimes \tau_1) \|_{H^{-\frac{1}{2}}(\partial\Omega)} \\ &\leq C' \|v\|_{L^2(\Omega)} + C' \|f\|_{H^1(\Omega)} + C' \|v\|_{H^{-\frac{1}{2}}(\partial\Omega)} + C' \|\nabla f\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \end{aligned}$$

Since $v = \partial_{\tau_1} u$ we have $\|v\|_{L^2(\Omega)} \leq C \|u\|_{H^1(\Omega)}$ and

$$\|v\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C \|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C' \|u\|_{H^1(\Omega)}.$$

Moreover, using (2.2) and the Neumann boundary condition for f we can write

$$\nabla f = \sum_{j=1}^{d-1} \tau_j \partial_{\tau_j} f + n \partial_n f = \sum_{j=1}^{d-1} \tau_j \partial_{\tau_j} f + n (\partial_{\tau_1} u \cdot n) \quad \text{on } \partial\Omega.$$

This implies

$$\begin{aligned} \|\nabla f\|_{H^{-\frac{1}{2}}(\partial\Omega)} &\leq C \sum_{j=1}^{d-1} \|\partial_{\tau_j} f\|_{H^{-\frac{1}{2}}(\partial\Omega)} + C \|\partial_{\tau_1} u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \\ &\leq C' \|f\|_{H^{\frac{1}{2}}(\partial\Omega)} + C \|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C'' \|f\|_{H^1(\Omega)} + C \|u\|_{H^1(\Omega)}. \end{aligned}$$

Thus, we obtain

$$\|h\|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)} + C \|f\|_{H^1(\Omega)}.$$

The H^1 elliptic estimate for f gives

$$\begin{aligned} \|f\|_{H^1(\Omega)} &\leq C \|\partial_{\tau_1} u\|_{L^2(\Omega)} + C \|\partial_{\tau_1} u \cdot n\|_{H^{-\frac{1}{2}}(\partial\Omega)} \\ &\leq C \|\partial_{\tau_1} u\|_{L^2(\Omega)} + C' \|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C'' \|u\|_{H^1(\Omega)}. \end{aligned}$$

Consequently

$$\|h\|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)}$$

which combined with the commutator estimate

$$\|[\partial_{\tau_1}, \nabla] f\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \leq C \|f\|_{H^1(\Omega)}$$

completes the proof of (2.17). \square

Next we fix a cutoff function $\chi_2 : \overline{\Omega} \rightarrow [0, 1]$ satisfying

$$\chi_2 \equiv 0 \quad \text{in } \Omega_\kappa, \quad \chi_2 \equiv 1 \quad \text{in } \overline{\Omega} \setminus \Omega_{2\kappa}. \quad (2.24)$$

Proposition 2.6. *Let $m \geq 1$ be an integer. There exists a constant $C > 0$ such that*

$$\|[\chi_2 D^m, \mathbb{P}]u\|_{L^2(\Omega)} \leq C \|u\|_{H^{m-1}(\Omega)}$$

for any vector field $u \in H^{m-1}(\Omega)$.

Proof. Without loss of generality we consider $D^m = \partial_1^m$. We have

$$\begin{aligned} [\chi_2 \partial_1^m, \mathbb{P}]u &= \chi_2 \partial_1^m(u - \nabla f) - [\chi_2 \partial_1^m u - \nabla g] \\ &= \chi_2 \partial_1^m \nabla f - \nabla g \\ &= \nabla(\chi_2 \partial_1^m f - g) - \nabla \chi_2 \partial_1^m f. \end{aligned}$$

where f and g solve

$$\begin{cases} \Delta f = \operatorname{div} u & \text{in } \Omega, \\ \frac{\partial f}{\partial n} = u \cdot n & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta g = \operatorname{div}(\chi_2 \partial_1^m u) & \text{in } \Omega, \\ \frac{\partial g}{\partial n} = (\chi_2 \partial_1^m u) \cdot n & \text{on } \partial\Omega \end{cases}$$

respectively. By elliptic estimates for f we have

$$\|\nabla \chi_2 \partial_1^m f\|_{L^2(\Omega)} \leq C \|u\|_{H^{m-1}(\Omega)}. \quad (2.25)$$

Setting $h = \chi_2 \partial_1^m \nabla f - g$, we compute

$$\begin{aligned} \Delta h &= \Delta \chi_2 \partial_1^m f + 2\nabla \chi_2 \cdot \nabla \partial_1^m f - \nabla \chi_2 \cdot \partial_1^m u \\ &= \Delta \chi_2 \partial_1^m f + 2\operatorname{div}(\nabla \chi_2 \partial_1^m f) - 2\Delta \chi_2 \partial_1^m f - \partial_j(\nabla \chi_2 \cdot \partial_1^{m-1} u) \\ &\quad + \nabla \partial_1 \chi_2 \cdot \partial_1^{m-1} u. \end{aligned}$$

On the other hand, since $\chi_2 \equiv 0$ near $\partial\Omega$, $h \equiv 0$ near $\partial\Omega$. Thus, standard elliptic estimates give

$$\|h\|_{H^1(\Omega)} \leq C \|u\|_{H^{m-1}(\Omega)}. \quad (2.26)$$

A combination of (2.25) and (2.26) concludes the proof. \square

3. Proof of Theorem 1.3

Let us start with a priori estimates for the perturbation $z = y - y_*$, which solves

$$\partial_t z + u \cdot \nabla y_* + u \cdot \nabla z = 0, \quad u = \mathbb{P}z \quad (3.1)$$

in $\Omega = [0, L]^d$. We consider p_* such that $\nabla y_* = \nabla^2 p_*$ is L -periodic. In what follows, u and z are L -periodic smooth solutions to (3.1), and s is real number satisfying $s > 1 + \frac{d}{2}$. We shall denote by \mathbb{T}^d the box Ω with periodic boundary conditions. We have the following:

Lemma 3.1. *We have*

$$\frac{1}{2} \frac{d}{dt} \|z(t)\|_{H^s(\mathbb{T}^d)}^2 \leq C_1(1 + \|z(t)\|_{H^s(\mathbb{T}^d)}) \|u(t)\|_{H^s(\mathbb{T}^d)} \|z(t)\|_{H^s(\mathbb{T}^d)}, \quad (3.2)$$

for some constant C_1 depending only on s, d , and $\|y_*\|_{H^{s+1}}$.

Proof. Denote $J^s = (1 - \Delta)^{\frac{s}{2}}$. Note that $\|f\|_{H^s(\mathbb{T}^d)} = \|J^s f\|_{L^2(\mathbb{T}^d)}$. Applying J^s to (3.1), multiplying the resulting equation by $J^s z$ and integrating in space we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} |J^s z|^2 dx \\ &= - \int_{\mathbb{T}^d} \left[J^s z \cdot J^s(u \cdot \nabla z) + J^s(u \cdot \nabla y_*) \cdot J^s z \right] dx \\ &= - \int_{\mathbb{T}^d} \left[J^s z \cdot ([J^s, u] \cdot \nabla z) dx + J^s z \cdot (u \cdot \nabla J^s z) + J^s(u \cdot \nabla y_*) \cdot J^s z \right] dx \\ &= - \int_{\mathbb{T}^d} \left[J^s z \cdot ([J^s, u] \cdot \nabla z) + \frac{1}{2} u \cdot \nabla |J^s z|^2 dx + J^s(u \cdot \nabla y_*) \cdot J^s z \right] dx \\ &= - \int_{\mathbb{T}^d} \left[J^s z \cdot ([J^s, u] \cdot \nabla z) + J^s(u \cdot \nabla y_*) \cdot J^s z \right] dx \end{aligned}$$

where we used the fact that $u = \mathbb{P}z$ is divergence free. Since $s > 1 + \frac{d}{2}$, the Kato-Ponce's commutator estimate yields

$$\|[J^s, u] \cdot \nabla z\|_{L^2} \leq C \|u\|_{H^s} \|z\|_{H^s}.$$

On the other hand, we have

$$\left| \int_{\mathbb{T}^d} J^s(u \cdot \nabla y_*) \cdot J^s z \right| \leq C \|y_*\|_{H^{s+1}} \|u\|_{H^s} \|z\|_{H^s}.$$

Putting this all together leads to (3.2). \square

Lemma 3.2. *Let θ_0 be the constant as in (1.5). Then it holds that*

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\mathbb{T}^d)}^2 + \theta_0 \|u(t)\|_{L^2(\mathbb{T}^d)}^2 \leq C_2 \|u(t)\|_{L^2(\mathbb{T}^d)}^2 \|z(t)\|_{H^s(\mathbb{T}^d)} \quad (3.3)$$

$$\frac{d}{dt} \|u(t)\|_{H^s(\mathbb{T}^d)}^2 + \theta_0 \|u(t)\|_{H^s(\mathbb{T}^d)}^2 \leq C_2 \|u(t)\|_{H^s(\mathbb{T}^d)}^2 \|z(t)\|_{H^s(\mathbb{T}^d)} + C_2 \|u(t)\|_{L^2(\mathbb{T}^d)}^2 \quad (3.4)$$

for some constant C_2 depending only on s, d , and $\|y_*\|_{H^{s+1}}$.

Proof. Applying the Leray projection to the first equation in (3.1) gives

$$\partial_t u + \mathbb{P}(u \cdot \nabla y_*) + \mathbb{P}(u \cdot \nabla z) = 0. \quad (3.5)$$

A direct H^s energy estimate for u from this equation requires a control for z in H^{s+1} , and thus the estimates cannot be closed. To resolve the issue, we first commute with u to have

$$\partial_t u + \mathbb{P}(u \cdot \nabla y_*) + u \cdot \nabla u + [\mathbb{P}, u \cdot \nabla] z = 0 \quad (3.6)$$

We now perform an H^s energy estimate for this equation. As in the proof of Lemma 3.1 we have

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} |J^s u|^2 dx + \int_{\mathbb{T}^d} J^s u \cdot J^s \mathbb{P}(u \cdot \nabla y_*) dx \\
 &= - \int_{\mathbb{T}^d} J^s u \cdot ([J^s, u] \cdot \nabla u) dx - \int_{\mathbb{T}^d} J^s u \cdot (u \cdot \nabla J^s u) dx \\
 &\quad - \int_{\mathbb{T}^d} J^s u \cdot J^s ([\mathbb{P}, u \cdot \nabla] z) dx \\
 &= - \int_{\mathbb{T}^d} J^s u \cdot ([J^s, u] \cdot \nabla u) dx - \frac{1}{2} \int_{\mathbb{T}^d} u \cdot \nabla |J^s u|^2 dx \\
 &\quad - \int_{\mathbb{T}^d} J^s u \cdot J^s ([\mathbb{P}, u \cdot \nabla] z) dx \\
 &= - \int_{\mathbb{T}^d} J^s u \cdot ([J^s, u] \cdot \nabla u) dx - \int_{\mathbb{T}^d} J^s u \cdot J^s ([\mathbb{P}, u \cdot \nabla] z) dx,
 \end{aligned}$$

where the fact that u is divergence-free was used to cancel out the term $\int_{\mathbb{T}^d} u \cdot \nabla |J^s u|^2 dx$. In addition, using the facts that $[J^s, \mathbb{P}] = 0$ and \mathbb{P} is self-adjoint in L^2 , we obtain

$$\begin{aligned}
 \int_{\mathbb{T}^d} J^s u \cdot J^s \mathbb{P}(u \cdot \nabla y_*) &= \int_{\mathbb{T}^d} J^s u \cdot \mathbb{P} J^s (u \cdot \nabla y_*) = \int_{\mathbb{T}^d} J^s u \cdot J^s (u \cdot \nabla y_*) \\
 &= \int_{\mathbb{T}^d} J^s u \cdot (J^s u \cdot \nabla y_*) + \int_{\mathbb{T}^d} J^s u \cdot ([J^s, \nabla y_*] \cdot u).
 \end{aligned} \tag{3.7}$$

It follows from the convexity assumption (1.9) that

$$\int_{\mathbb{T}^d} J^s u \cdot (J^s u \cdot \nabla y_*) = \int_{\mathbb{T}^d} J^s u \cdot (J^s u \cdot \nabla^2 p_*) \geq \theta_0 \|u\|_{H^s}^2. \tag{3.8}$$

Using Kato-Ponce's commutator estimate gives

$$\| [J^s, u] \cdot \nabla u \|_{L^2} \leq C \|u\|_{H^s} \|u\|_{H^s}$$

and

$$\| [J^s, \nabla y_*] \cdot u \|_{L^2} \leq C \|\nabla y_*\|_{W^{1,\infty}} \|u\|_{H^{s-1}} + C \|\nabla y_*\|_{H^s} \|u\|_{L^\infty} \leq C \|y_*\|_{H^{s+1}} \|u\|_{H^{s-1}}.$$

By virtue of Theorem 1.5 for $\Omega = \mathbb{T}^d$, we have

$$\| [\mathbb{P}, u \cdot \nabla] z \|_{H^s} \leq C \|u\|_{H^s} \|z\|_{H^s}.$$

Combining, and using $\|u\|_{H^s} \leq \|z\|_{H^s}$, we thus obtain

$$\frac{d}{dt} \|u(t)\|_{H^s}^2 + 2\theta_0 \|u(t)\|_{H^s}^2 \leq C_2 \|u(t)\|_{H^s}^2 \|z(t)\|_{H^s} + C_3 \|y_*\|_{H^{s+1}} \|u(t)\|_{H^s} \|u(t)\|_{H^{s-1}}, \tag{3.9}$$

in which, using interpolation and Young's inequality, the last term can be estimated by

$$\|u(t)\|_{H^s} \|u(t)\|_{H^{s-1}} \leq \|u(t)\|_{H^s}^{2-\frac{1}{s}} \|u(t)\|_{L^2}^{\frac{1}{s}} \leq \gamma \|u(t)\|_{H^s}^2 + C_\gamma \|u(t)\|_{L^2}^2$$

for $\gamma > 0$. Taking γ sufficiently small, we get the claimed H^s estimates in (3.4) directly from (3.9).

Finally, by means of the Sobolev embedding $\|\nabla z\|_{L^\infty} \leq C\|z\|_{H^s}$, the fact that \mathbb{P} is self-adjoint in L^2 , and the convexity assumption (1.5), an L^2 energy estimate for (3.5) gives the estimate (3.3). This ends the proof of the lemma. \square

For L -periodic smooth solutions (u, z) defined on the maximal interval $[0, T^*)$ of (3.1), which exists thanks to the local existence theory in Theorem 1.6, let us introduce the bootstrap norm

$$\mathcal{N}(t) := \sup_{0 \leq \tau \leq t} \left(\|z(\tau)\|_{H^s(\mathbb{T}^d)}^2 + M^2 e^{\frac{\theta_0}{2}\tau} \|u(\tau)\|_{L^2(\mathbb{T}^d)}^2 + M e^{\frac{\theta_0}{2}\tau} \|u(\tau)\|_{H^s(\mathbb{T}^d)}^2 \right) \quad (3.10)$$

for some fixed and large $M > 0$ and for $t < T^*$.

Proposition 3.3. *There exist positive constants ε, C_* , depending only on θ_0 and $\|y_*\|_{H^{s+1}}$, such that whenever $\mathcal{N}(0) < \varepsilon$ we have $\mathcal{N}(t) \leq C_* \mathcal{N}(0)$ for all $t < T^*$.*

Proof. We shall prove that

$$\mathcal{N}(t) \leq C_0 \mathcal{N}(0) + C_0 \mathcal{N}(t)^{3/2} \quad (3.11)$$

for all $t < T^*$. The proposition follows directly from the standard continuous induction.

As for the claim (3.11), we integrate (3.2) in time and use the definition of $\mathcal{N}(t)$, yielding

$$\begin{aligned} \|z(t)\|_{H^s}^2 &\leq \|z(0)\|_{H^s}^2 + C_1 \int_0^t (1 + \|z(\tau)\|_{H^s}) \|u(\tau)\|_{H^s} \|z(\tau)\|_{H^s} d\tau \\ &\leq \|z(0)\|_{H^s}^2 + C_1 M^{-1/2} (1 + \mathcal{N}(t)^{1/2}) \mathcal{N}(t) \int_0^t e^{-\theta_0 \tau/4} d\tau \quad (3.12) \\ &\leq \|z(0)\|_{H^s}^2 + C_3 M^{-1/2} (1 + \mathcal{N}(t)^{1/2}) \mathcal{N}(t). \end{aligned}$$

Next taking M sufficiently large so that $\theta_0 M > C_2$, we obtain from (3.3) and (3.4) that

$$\frac{d}{dt} (M \|u(t)\|_{L^2}^2 + \|u(t)\|_{H^s}^2) + \theta_0 (M \|u(t)\|_{L^2}^2 + \|u(t)\|_{H^s}^2) \leq C_4 M \|u(t)\|_{H^s}^2 \|z(t)\|_{H^s}$$

where $C_4 = C_2(2 + M)$. This yields

$$\begin{aligned} &M^2 \|u(t)\|_{L^2}^2 + M \|u(t)\|_{H^s}^2 \\ &\leq e^{-\theta_0 t} \left[M^2 \|u(0)\|_{L^2}^2 + M \|u(0)\|_{H^s}^2 \right] + C_4 M \int_0^t e^{-\theta_0(t-\tau)} \|u(\tau)\|_{H^s}^2 \|z(\tau)\|_{H^s} d\tau \\ &\leq e^{-\theta_0 t} \left[M^2 \|u(0)\|_{L^2}^2 + M \|u(0)\|_{H^s}^2 \right] + C_4 \mathcal{N}(t)^{3/2} \int_0^t e^{-\theta_0(t-\tau)} e^{-\theta_0 \tau/2} d\tau \quad (3.13) \\ &\leq e^{-\theta_0 t/2} \left[M^2 \|u(0)\|_{L^2}^2 + M \|u(0)\|_{H^s}^2 \right] + C_5 \mathcal{N}(t)^{3/2} e^{-\theta_0 t/2}. \end{aligned}$$

Combining with (3.12) and choosing again M large, if needed, we obtain the claim (3.11) and hence the proposition. \square

With the ε and C_* given in Proposition 3.3, we have proven that $\|z(t)\|_{H^s(\mathbb{T}^d)} \leq C_* \mathcal{N}(0) \leq C_* \varepsilon$ and $\|u(t)\|_{H^s(\mathbb{T}^d)} \leq C \|y_0 - y_*\|_{H^s(\mathbb{T}^d)} e^{-\frac{\theta_0}{4}t}$ for all time $t < T^*$. Consequently, the solution z of (3.1) is global in time and enjoys the same bounds for all $t > 0$. Using equation (3.1) and the estimates (1.6) and (1.7), we deduce that $\partial_t z \in L^1(0, \infty; H^{s-1}(\mathbb{T}^d))$. This yields

$$\lim_{t \rightarrow \infty} z(x, t) = z_0(x) + \int_0^\infty \partial_t z(x, \tau) d\tau := z_\infty(x) \quad \text{in } H^{s-1}(\mathbb{T}^d),$$

and thus

$$\lim_{t \rightarrow \infty} y(x, t) = y_\infty(x) := z_\infty(x) + y_*(x) \quad \text{in } H^{s-1}(\Omega), \quad \Omega = [0, L]^d.$$

Furthermore,

$$\begin{aligned} \|y(t) - y_\infty\|_{H^{s-1}(\mathbb{T}^d)} &= \|z(t) - z_\infty\|_{H^{s-1}(\mathbb{T}^d)} \\ &= \left\| \int_t^\infty u \cdot \nabla z(\tau) d\tau \right\|_{H^{s-1}(\mathbb{T}^d)} \leq C e^{-\frac{\theta_0}{4}t} \end{aligned} \quad (3.14)$$

for all $t \geq 0$. Using the Leray projection we write

$$y(x, t) = u(x, t) + \nabla p(x, t), \quad y_\infty(x) = u_\infty(x) + \nabla p_\infty(x),$$

where $u_\infty = \mathbb{P}y_\infty : \Omega \rightarrow \mathbb{R}^d$ and $p : \Omega \rightarrow \mathbb{R}$. In view of the Pythagorean identity

$$\|y(t) - y_\infty\|_{L^2(\Omega)}^2 = \|u(t) - u_\infty\|_{L^2(\Omega)}^2 + \|\nabla p(t) - \nabla p_\infty\|_{L^2(\Omega)}^2,$$

we find that each term on the right hand side converges to 0 as $t \rightarrow \infty$. This, together with the fact that $u(t) \rightarrow 0$ in $H^s(\mathbb{T}^d)$, implies that $u_\infty \equiv 0$. Thus, $y_\infty = \nabla p_\infty$ is a gradient and in view of (3.14) we have

$$\|y(t) - \nabla p_\infty\|_{H^{s-1}(\mathbb{T}^d)} \leq C e^{-\frac{\theta_0}{4}t}$$

for all $t \geq 0$. As a consequence of this, (1.9) and the bound $\|y - y_*\|_{L^\infty(0, \infty; H^s(\mathbb{T}^d))} \leq C_* \varepsilon$, if ε is sufficiently small then $\nabla^2 p_\infty > 0$. Thus, p_∞ is (strictly) convex and ∇p_∞ is the optimal rearrangement of y_0 by virtue of Brenier's theorem ([5]). This ends the proof of Theorem 1.3.

4. Proof of Theorem 1.1

We now turn to the case when Ω is a bounded domain with smooth boundary. Recall that the perturbation $z = y - y_*$ obeys

$$\partial_t z + u \cdot \nabla y_* + u \cdot \nabla z = 0, \quad u = \mathbb{P}z. \quad (4.1)$$

In what follows, we fix an integer $s > 1 + \frac{d}{2}$.

Lemma 4.1. *There exists $C > 0$ depending only on $\|y_*\|_{H^{s+1}(\Omega)}$ such that*

$$\frac{1}{2} \frac{d}{dt} \|z(t)\|_{H^s(\Omega)}^2 \leq C_1(1 + \|z(t)\|_{H^s(\Omega)}) \|u(t)\|_{H^s(\Omega)} \|z(t)\|_{H^s(\Omega)}. \quad (4.2)$$

Proof. First of all, an L^2 estimate for (4.1) gives

$$\frac{1}{2} \frac{d}{dt} \|z(t)\|_{L^2(\Omega)}^2 = - \int_{\Omega} z \cdot (u \cdot \nabla y_*) dx \leq \|\nabla y_*\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|z\|_{L^2(\Omega)}, \quad (4.3)$$

where we used the fact that $u \cdot n|_{\partial\Omega} = 0$ to have $\int_{\Omega} z \cdot (u \cdot \nabla z) = 0$ upon integration by parts. Now let s be an integer greater than $1 + d/2$. Recall that D^s denotes any partial derivatives of order s . Applying D^s to equation (3.1) and arguing as in the proof of Lemma 3.1 yields

$$\frac{1}{2} \frac{d}{dt} \|D^s z(t)\|_{L^2(\Omega)}^2 \leq C_1(1 + \|z(t)\|_{H^s(\Omega)}) \|u(t)\|_{H^s(\Omega)} \|z(t)\|_{H^s(\Omega)} \quad (4.4)$$

upon using the commutator estimate (see [11] page 129)

$$\|D^s(fg) - fD^s g\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^\infty(\Omega)} \|g\|_{H^{s-1}(\Omega)} + C\|f\|_{H^s(\Omega)} \|g\|_{L^\infty(\Omega)} \quad (4.5)$$

and the fact that $u \cdot n|_{\partial\Omega} = 0$ when taking integration by parts in integral involving the highest derivatives $u \cdot \nabla |D^s z|^2$. Combining (4.3) and (4.4) leads to the estimate (4.2). \square

Next let us recall equation (3.5) for u

$$\partial_t u + \mathbb{P}(u \cdot \nabla y_*) + \mathbb{P}(u \cdot \nabla z) = 0. \quad (4.6)$$

Repeating verbatim the proof of (3.3) we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 + \theta_0 \|u(t)\|_{L^2(\Omega)}^2 \leq C_2 \|u(t)\|_{L^2(\Omega)}^2 \|z(t)\|_{H^s(\Omega)}. \quad (4.7)$$

We will need decay of the H^s norm of u . Let us note that the proof of (3.4) in Lemma 3.2 does carry over to domains with boundary since the Leray projector does not commute with D^s , as used in (3.7) for $\Omega = \mathbb{T}^d$. To treat the boundary and the nonlocality of \mathbb{P} , we use the derivatives ∂_{τ_j} and ∂_n introduced in Section 2.3. These derivatives are defined everywhere in Ω and become the usual tangential and normal derivative when restricted to the boundary. The trade-off is that ∂_{τ_j} and ∂_n do not commute with usual partial derivatives, leading to commutators that are of lower order.

For $k \in \{0, 1, \dots, s\}$ we set

$$\mathcal{D}_k^s = \left\{ \prod_{j=1}^s \partial_{\sigma_j} : \sigma_j \in \{\tau_1, \dots, \tau_{d-1}, n\} \text{ and } \#\{j : \sigma_j = n\} = k \right\}.$$

In other words, each derivative in \mathcal{D}_k^s has exactly k normal derivatives and $s - k$ tangential derivatives. We also define the norms

$$\|v\|_{s,k} = \left(\sum_{j=0}^k \sum_{P \in \mathcal{D}_j^s} \|Pv\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

for $v : \Omega \rightarrow \mathbb{R}^d$.

Due to the presence of χ_1 in ∂_n and ∂_{τ_j} , the norms $\|u\|_{s,k}$ control u near the boundary.

4.1. Interior Estimates for u

The next lemma provides a control of u in the interior.

Lemma 4.2. *There exists $C > 0$ depending only on $\|y_*\|_{H^{s+1}(\Omega)}$ such that*

$$\frac{1}{2} \frac{d}{dt} \|\chi_2 u\|_{H^s}^2 + \theta_0 \|\chi_2 u\|_{H^s}^2 \leq C \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + C \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)} \quad (4.8)$$

where χ_2 is defined in (2.24).

Proof. As in (3.5), we commute \mathbb{P} with $u \cdot \nabla$ in the last term of equation (4.6) to have

$$\partial_t u + \mathbb{P}(u \cdot \nabla y_*) + u \cdot \nabla u + [\mathbb{P}, u \cdot \nabla]z = 0. \quad (4.9)$$

Set $P = \chi_2 \partial_1^s$. Applying P to (4.9), then multiplying the resulting equation by Pu and integrating over Ω , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |Pu|^2 dx + \int_{\Omega} Pu \cdot P\mathbb{P}(u \cdot \nabla y_*) dx \\ &= - \int_{\Omega} Pu \cdot ([P, u] \cdot \nabla u) dx - \int_{\Omega} Pu \cdot (u \cdot [P, \nabla]u) dx \\ & \quad - \int_{\Omega} Pu \cdot P([\mathbb{P}, u \cdot \nabla]z) dx. \end{aligned} \quad (4.10)$$

where we used the fact that

$$\int_{\Omega} Pu \cdot (u \cdot Pu) dx = \frac{1}{2} \int_{\Omega} u \cdot \nabla |Pu|^2 dx = 0$$

since $\nabla \cdot u = 0$ in Ω and $u \cdot n|_{\partial\Omega} = 0$. We now treat each term on the right-hand side of (4.10). It is readily seen that

$$\begin{aligned} \|[P, u] \cdot \nabla u\|_{L^2(\Omega)} &\leq C \|u\|_{H^s(\Omega)}^2, \\ \|u \cdot [P, \nabla]u\|_{L^2(\Omega)} &\leq C \|u\|_{H^{s-1}(\Omega)}^2. \end{aligned} \quad (4.11)$$

In addition, Theorem 1.5 applied to Ω gives

$$\|P([\mathbb{P}, u \cdot \nabla]z)\|_{L^2(\Omega)} \leq C\|\mathbb{P}, u \cdot \nabla]z\|_{H^s(\Omega)} \leq C\|u\|_{H^s(\Omega)}\|z\|_{H^s(\Omega)}. \quad (4.12)$$

Putting together (4.10), (4.11), (4.12) and using the estimate $\|u\|_{H^s} \leq C\|z\|_{H^s}$ we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |Pu|^2 dx + \int_{\Omega} Pu \cdot P\mathbb{P}(u \cdot \nabla y_*) dx \leq C\|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)}. \quad (4.13)$$

As for the second term on the left-hand side of (4.13), we commute P with \mathbb{P} and then with ∇y_* to have

$$\begin{aligned} & \int_{\Omega} Pu \cdot P\mathbb{P}(u \cdot \nabla y_*) dx \\ &= \int_{\Omega} Pu \cdot [P, \mathbb{P}](u \cdot \nabla y_*) dx + \int_{\Omega} Pu \cdot \mathbb{P}([P, \nabla y_*]u) dx \\ & \quad + \int_{\Omega} Pu \cdot \mathbb{P}(\nabla y_* \cdot Pu) dx \\ &= \int_{\Omega} Pu \cdot [P, \mathbb{P}](u \cdot \nabla y_*) dx + \int_{\Omega} Pu \cdot \mathbb{P}([P, \nabla y_*]u) dx \\ & \quad + \int_{\Omega} [\mathbb{P}, P]u \cdot (\nabla y_* \cdot Pu) dx + \int_{\Omega} Pu \cdot (\nabla y_* \cdot Pu) dx. \end{aligned} \quad (4.14)$$

By virtue of Proposition 2.6,

$$\|[\mathbb{P}, P]u\|_{H^s(\Omega)} \leq C\|u\|_{H^{s-1}(\Omega)}$$

and

$$\|[P, \mathbb{P}](u \cdot \nabla y_*)\|_{L^2(\Omega)} \leq C\|u \cdot \nabla y_*\|_{H^{s-1}(\Omega)} \leq C\|u\|_{H^{s-1}(\Omega)}\|y_*\|_{H^s(\Omega)}.$$

The local commutator $[P, \nabla y_*]u$ can be bounded as

$$\|[P, \nabla y_*]u\|_{L^2(\Omega)} \leq C\|y_*\|_{H^{s+1}(\Omega)}\|u\|_{H^{s-1}(\Omega)}. \quad (4.15)$$

On the other hand, the convexity condition (1.5) yields

$$\int_{\Omega} Pu \cdot (\nabla y_* \cdot Pu) dx \geq \theta_0 \|Pu\|_{L^2(\Omega)}^2.$$

We then deduce from (4.14) that

$$\int_{\Omega} Pu \cdot P\mathbb{P}(u \cdot \nabla y_*) dx \geq \theta_0 \|Pu\|_{L^2(\Omega)}^2 - C\|y_*\|_{H^{s+1}(\Omega)}\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} \quad (4.16)$$

which combined with (4.13) leads to (4.17). The same estimates hold for mixed derivatives $\chi_2 D^s$ where D^s is any partial derivative of order s . \square

4.2. Estimates for Tangential Derivatives of u

Lemma 4.3. *There exists $C > 0$ depending only on $\|y_*\|_{H^{s+1}(\Omega)}$ such that*

$$\frac{1}{2} \frac{d}{dt} \|u\|_{s,0}^2 + \theta_0 \|u\|_{s,0}^2 \leq C \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + C \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)}. \quad (4.17)$$

Proof. The proof follows along the same lines as in Lemma 4.2 upon taking $P \in \mathcal{D}_0^s$ and using Proposition 2.5 in place of Proposition 2.6. \square

4.3. Estimates for Mixed Derivatives of u

The next lemma concerns $\|u\|_{s,1}$.

Lemma 4.4. *There exists $M_1 > 0$ such that*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u\|_{s,1}^2 + \frac{\theta_0}{2} \|u\|_{s,1}^2 \\ & \leq M_1 \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + M_1 \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)} + M_1 \|u\|_{s,1} \|u\|_{s,0}. \end{aligned} \quad (4.18)$$

Proof. Let $P \in \mathcal{D}_1^s$. Assume without loss of generality that $P = \partial_{t_1}^{s-1} \partial_n$. Commuting equation (4.9) with P gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |Pu|^2 dx + \int_{\Omega} Pu \cdot P\mathbb{P}(u \cdot \nabla y_*) dx \\ & = - \int_{\Omega} Pu \cdot ([P, u] \cdot \nabla u) dx - \int_{\Omega} Pu \cdot (u \cdot [P, \nabla]u) dx \\ & \quad - \int_{\Omega} Pu \cdot P([P, u \cdot \nabla]z) dx. \end{aligned} \quad (4.19)$$

Arguing as in the proof of Lemma 4.3, we find that the right-hand side is bounded by $C \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)}$. Now we write using the definition of \mathbb{P} that

$$\int_{\Omega} Pu \cdot P\mathbb{P}(u \cdot \nabla y_*) dx = \int_{\Omega} Pu \cdot P(u \cdot \nabla y_*) dx - \int_{\Omega} Pu \cdot P \nabla f dx$$

where f solves

$$\begin{cases} \Delta f = \operatorname{div}(u \cdot \nabla y_*) & \text{in } \Omega, \\ \partial_n f = (u \cdot \nabla y_*) \cdot n & \text{on } \partial\Omega. \end{cases} \quad (4.20)$$

Commuting P with ∇y_* gives

$$\int_{\Omega} Pu \cdot P(u \cdot \nabla y_*) dx = \int_{\Omega} Pu \cdot (Pu \cdot \nabla y_*) dx + \int_{\Omega} Pu \cdot [P, \nabla y_* \cdot]u dx$$

where the local commutator $[P, \nabla y_* \cdot]u$ satisfies

$$\|[P, \nabla y_*] \cdot u\|_{L^2(\Omega)} \leq C \|y_*\|_{H^{s+1}(\Omega)} \|u\|_{H^{s-1}(\Omega)}$$

and by the convexity assumption (1.5),

$$\int_{\Omega} Pu \cdot (Pu \cdot \nabla y_*) \, dx \geq \theta_0 \|Pu\|_{L^2(\Omega)}^2.$$

The rest of this proof is devoted to the control of $\int_{\Omega} Pu \cdot P \nabla f \, dx$. First, since $\chi_1 \equiv 1$ in $\Omega_{2\kappa} \supset \text{supp}(1 - \chi_2)$, in view of (2.2), the decomposition

$$\nabla g = (1 - \chi_2)n\partial_n g + (1 - \chi_2)\tau_j \partial_{\tau_j} g + \chi_2 \nabla g \quad (4.21)$$

holds in Ω for any scalar $g : \Omega \rightarrow \mathbb{R}$. Using this with $g = Pf$, we write

$$\begin{aligned} & \int_{\Omega} Pu \cdot P \nabla f \, dx \\ &= \int_{\Omega} Pu \cdot \nabla Pf \, dx + \int_{\Omega} Pu \cdot [P, \nabla]f \, dx \\ &= \int_{\Omega} (1 - \chi_2)(Pu \cdot n)\partial_n Pf \, dx + \int_{\Omega} (1 - \chi_2)(Pu \cdot \tau_j)\partial_{\tau_j} Pf \, dx \\ & \quad + \int_{\Omega} \chi_2 Pu \cdot \nabla Pf \, dx + \int_{\Omega} Pu \cdot [P, \nabla]f \, dx \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (4.22)$$

Due to the presence of the local commutator $[P, \nabla]f$, it is readily seen that

$$|I_4| \leq C \|u\|_{H^s} \|f\|_{H^s(\Omega)} \leq C' \|u\|_{H^s} \|u\|_{H^{s-1}(\Omega)}. \quad (4.23)$$

As for I_3 , we integrate by parts noticing that $\text{div } u = 0$ in Ω and $\chi_2 \equiv 0$ near $\partial\Omega$ to obtain

$$I_3 = \int_{\Omega} \chi_2 Pu \cdot \nabla Pf \, dx = - \int_{\Omega} [\text{div}, \chi_2 P]u Pf, \quad (4.24)$$

which implies

$$|I_3| \leq C \|u\|_{H^s(\Omega)} \|u\|_{H^{s-1}(\Omega)}. \quad (4.25)$$

Estimate for I_1 . We first note that

$$\begin{aligned} Pu \cdot n &= \partial_{\tau_1}^{s-1} (\partial_n u \cdot n) - [\partial_{\tau_1}^{s-1}, n \cdot] \partial_n u \\ &= -\partial_{\tau_1}^{s-1} (\partial_{\tau_j} u \cdot \tau_j) - [\partial_{\tau_1}^{s-1}, n \cdot] \partial_n u \\ &= -(\partial_{\tau_1}^{s-1} \partial_{\tau_j} u) \cdot \tau_j - [\partial_{\tau_1}^{s-1}, \tau_j] \partial_{\tau_j} u - [\partial_{\tau_1}^{s-1}, n \cdot] \partial_n u \end{aligned}$$

This implies

$$\|Pu \cdot n\|_{L^2(\Omega)} \leq C \|u\|_{s,0} + C \|u\|_{H^{s-1}(\Omega)}. \quad (4.26)$$

On the other hand, it follows from (4.20) that

$$\begin{cases} \Delta \partial_{\tau_1}^{s-1} f = [\Delta, \partial_{\tau_1}^{s-1}]f + [\partial_{\tau_1}^{s-1}, \text{div}](u \cdot \nabla y_*) + \text{div } \partial_{\tau_1}^{s-1}(u \cdot \nabla y_*) := g_1 & \text{in } \Omega, \\ \partial_n \partial_{\tau_1}^{s-1} f = [\partial_n, \partial_{\tau_1}^{s-1}]f + \partial_{\tau_1}^{s-1}\{(u \cdot \nabla y_*) \cdot n\} := g_2 & \text{on } \partial\Omega. \end{cases} \quad (4.27)$$

It is easy to see that

$$\|[\Delta, \partial_{\tau_1}^{s-1}]f\|_{L^2(\Omega)} + \|\partial_{\tau_1}^{s-1}, \operatorname{div}\| (u \cdot \nabla y_*)\|_{L^2(\Omega)} \leq C\|u\|_{H^{s-1}(\Omega)}.$$

In addition, (2.3) gives

$$\begin{aligned} & \|\operatorname{div} \partial_{\tau_1}^{s-1} (u \cdot \nabla y_*)\|_{L^2(\Omega)} \\ & \leq C\|\nabla \partial_{\tau_1}^{s-1} (u \cdot \nabla y_*)\|_{L^2(\Omega)} \\ & \leq C\|\partial_n \partial_{\tau_1}^{s-1} (u \cdot \nabla y_*)\|_{L^2(\Omega)} + C\|\partial_{\tau_j} \partial_{\tau_1}^{s-1} (u \cdot \nabla y_*)\|_{L^2(\Omega)} \\ & \leq C\|u\|_{s,1} + C\|u\|_{s,0}. \end{aligned}$$

Consequently

$$\|g_1\|_{L^2(\Omega)} \leq C\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,0} + C\|u\|_{s,1}.$$

Using the trace inequality and arguing as above we obtain that

$$\|g_2\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,0} + C\|u\|_{s,1}.$$

Then the H^2 elliptic estimate for (4.27) leads to

$$\begin{aligned} \|\partial_{\tau_1}^{s-1} f\|_{H^2(\Omega)} & \leq C\|g_1\|_{L^2(\Omega)} + C\|g_2\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ & \leq C\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,0} + C\|u\|_{s,1}. \end{aligned} \quad (4.28)$$

Next we write

$$\begin{aligned} I_1 & = \int_{\Omega} (1 - \chi_2)(Pu \cdot n) \partial_n P f \, dx \\ & = \int_{\Omega} (1 - \chi_2)(Pu \cdot n) P \partial_n f \, dx + \int_{\Omega} (1 - \chi_2)(Pu \cdot n) [\partial_n, P] f \, dx \\ & = \int_{\Omega} (1 - \chi_2)(Pu \cdot n) \partial_{\tau_1}^{s-1} \partial_n^2 f \, dx + \int_{\Omega} (1 - \chi_2)(Pu \cdot n) [\partial_n, P] f \, dx \\ & = \int_{\Omega} (1 - \chi_2)(Pu \cdot n) \partial_n^2 \partial_{\tau_1}^{s-1} f \, dx + \int_{\Omega} (1 - \chi_2)(Pu \cdot n) [\partial_{\tau_1}^{s-1}, \partial_n^2] f \, dx \\ & \quad + \int_{\Omega} (1 - \chi_2)(Pu \cdot n) [\partial_n, P] f \, dx. \end{aligned}$$

In view of (4.26) and (4.28) we deduce that

$$|I_1| \leq C\|u\|_{H^{s-1}(\Omega)}^2 + C\|u\|_{s,0}^2 + C\|u\|_{s,1}\|u\|_{s,0}. \quad (4.29)$$

Estimate for I_2 . We first write

$$\begin{aligned} I_2 & = \int_{\Omega} (1 - \chi_2)(Pu \cdot \tau_j) \partial_{\tau_j} \partial_{\tau_1}^{s-1} \partial_n f \, dx \\ & = \int_{\Omega} (1 - \chi_2)(Pu \cdot \tau_j) \partial_n \partial_{\tau_j} \partial_{\tau_1}^{s-1} f \, dx + \int_{\Omega} (1 - \chi_2)(Pu \cdot \tau_j) [\partial_{\tau_j} \partial_{\tau_1}^{s-1}, \partial_n] f \, dx \end{aligned}$$

where

$$\left| \int_{\Omega} (1 - \chi_2)(Pu \cdot \tau_j) [\partial_{\tau_j} \partial_{\tau_1}^{s-1}, \partial_n] f dx \right| \leq C \|u\|_{H^s(\Omega)} \|u\|_{H^{s-1}(\Omega)}. \quad (4.30)$$

On the other hand, by Hölder's and Young's inequalities

$$\begin{aligned} & \left| \int_{\Omega} (1 - \chi_2)(Pu \cdot \tau_j) \partial_n \partial_{\tau_j} \partial_{\tau_1}^{s-1} f dx \right| \\ & \leq C \|Pu \cdot \tau_j\|_{L^2(\Omega)} \|\partial_n \partial_{\tau_j} \partial_{\tau_1}^{s-1} f\|_{L^2(\Omega)} \\ & \leq C \|Pu\|_{L^2(\Omega)} \|\partial_{\tau_j} \partial_{\tau_1}^{s-1} f\|_{H^1(\Omega)} \\ & \leq \frac{\theta_0}{2} \|Pu\|_{L^2(\Omega)}^2 + C' \|\partial_{\tau_j} \partial_{\tau_1}^{s-1} f\|_{H^1(\Omega)}^2. \end{aligned} \quad (4.31)$$

Using again equation (4.20) we find

$$\begin{cases} \Delta \partial_{\tau_j} \partial_{\tau_1}^{s-1} f = [\Delta, \partial_{\tau_j} \partial_{\tau_1}^{s-1}] f + [\partial_{\tau_j} \partial_{\tau_1}^{s-1}, \operatorname{div}](u \cdot \nabla y_*) + \operatorname{div} \partial_{\tau_j} \partial_{\tau_1}^{s-1} (u \cdot \nabla y_*) & \text{in } \Omega, \\ \partial_n \partial_{\tau_j} \partial_{\tau_1}^{s-1} f = [\partial_n, \partial_{\tau_j} \partial_{\tau_1}^{s-1}] f + \partial_{\tau_j} \partial_{\tau_1}^{s-1} \{(u \cdot \nabla y_*) \cdot n\} & \text{on } \partial\Omega. \end{cases} \quad (4.32)$$

Multiplying the first equation by $\partial_{\tau_j} \partial_{\tau_1}^{s-1} f$ then integrating over Ω and using the second equation to cancel out the leading boundary term, we deduce that $h = \partial_{\tau_j} \partial_{\tau_1}^{s-1} f$ satisfies

$$\begin{aligned} \int_{\Omega} |\nabla h|^2 dx &= - \int_{\Omega} h \left\{ [\Delta, \partial_{\tau_j} \partial_{\tau_1}^{s-1}] f + [\partial_{\tau_j} \partial_{\tau_1}^{s-1}, \operatorname{div}](u \cdot \nabla y_*) \right\} dx \\ &\quad + \int_{\partial\Omega} h [\partial_n, \partial_{\tau_j} \partial_{\tau_1}^{s-1}] f dS \\ &= I_{2a} + I_{2b}. \end{aligned}$$

We observe that $\|h\|_{L^2(\Omega)} \leq C \|u\|_{H^{s-1}(\Omega)}$ and

$$\|[\Delta, \partial_{\tau_j} \partial_{\tau_1}^{s-1}] f + [\partial_{\tau_j} \partial_{\tau_1}^{s-1}, \operatorname{div}](u \cdot \nabla y_*)\|_{L^2(\Omega)} \leq C \|u\|_{H^s(\Omega)},$$

hence

$$|I_{2a}| \leq C \|u\|_{H^s(\Omega)} \|u\|_{H^{s-1}(\Omega)}.$$

On the other hand, by virtue of the trace inequality and interpolation, the surface integral is controlled as

$$\begin{aligned} |I_{2b}| &\leq C \|f\|_{H^s(\partial\Omega)}^2 \leq C' \|f\|_{H^{s+\frac{1}{2}}(\Omega)}^2 \\ &\leq C'' \|f\|_{H^s(\Omega)} \|f\|_{H^{s+1}(\Omega)} \leq C''' \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)}. \end{aligned}$$

It follows that

$$\|h\|_{H^1(\Omega)}^2 \leq \|h\|_{L^2(\Omega)}^2 + \|\nabla h\|_{L^2(\Omega)}^2 \leq C \|u\|_{H^s(\Omega)} \|u\|_{H^{s-1}(\Omega)}.$$

Plugging this into (4.31) and recalling (4.30) we deduce that

$$|I_2| \leq \frac{\theta_0}{2} \|Pu\|_{L^2(\Omega)}^2 + C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}. \quad (4.33)$$

Putting together the above considerations we arrive at

$$\frac{1}{2} \frac{d}{dt} \|Pu\|_{L^2}^2 + \frac{\theta_0}{2} \|Pu\|_{L^2}^2 \leq C\|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}.$$

Then summing over all $P \in \mathcal{D}_1^s$ yields

$$\frac{1}{2} \frac{d}{dt} \|u\|_{s,1}^2 + \frac{\theta_0}{2} \|u\|_{s,1}^2 \leq M_1 \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + M_1 \|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}$$

which combined with (4.17) for tangential derivatives leads to the desired estimate (4.18). \square

Lemma 4.5. *For each $k \in \{1, 2, \dots, s\}$ there exists $M_k > 0$ such that*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{s,k}^2 + \frac{\theta_0}{2} \|u\|_{s,k}^2 \\ \leq M_k \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + M_k \|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} + M_k \|u\|_{s,k}\|u\|_{s,k-1}. \end{aligned} \quad (4.34)$$

Proof. The base case $k = 1$ has been proved in Lemma 4.4. Assume (4.34) for some $k \in \{1, 2, \dots, s-1\}$ we prove it for $k+1$ in place of k . Let $P \in \mathcal{D}_{k+1}^s$. We assume without loss of generality that $P = \partial_{\tau_1}^{s-k-1} \partial_n^{k+1}$. Commuting equation (4.9) with P gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |Pu|^2 dx + \int_{\Omega} Pu \cdot P\mathbb{P}(u \cdot \nabla y_*) dx \\ = - \int_{\Omega} Pu \cdot ([P, u] \cdot \nabla u) dx - \int_{\Omega} Pu \cdot (u \cdot [P, \nabla]u) dx \\ - \int_{\Omega} Pu \cdot P([\mathbb{P}, u \cdot \nabla]z) dx. \end{aligned} \quad (4.35)$$

As in the proof of Lemma 4.4 it suffices to treat the damping term

$$\int_{\Omega} Pu \cdot P\mathbb{P}(u \cdot \nabla y_*) dx = \int_{\Omega} Pu \cdot P(u \cdot \nabla y_*) dx - \int_{\Omega} Pu \cdot P\nabla f dx$$

where f solves (4.20):

$$\begin{cases} \Delta f = \operatorname{div}(u \cdot \nabla y_*) & \text{in } \Omega, \\ \partial_n f = (u \cdot \nabla y_*) \cdot n & \text{on } \partial\Omega. \end{cases} \quad (4.36)$$

Commuting P with ∇y_* gives

$$\int_{\Omega} Pu \cdot P(u \cdot \nabla y_*) dx = \int_{\Omega} Pu \cdot (Pu \cdot \nabla y_*) dx + \int_{\Omega} Pu \cdot [P, \nabla y_*] \cdot u dx$$

where the local commutator $[P, \nabla y_*] \cdot u$ satisfies

$$\|[P, \nabla y_*] \cdot u\|_{L^2(\Omega)} \leq C \|y_*\|_{H^{s+1}(\Omega)} \|u\|_{H^{s-1}(\Omega)},$$

and by the convexity assumption (1.5),

$$\int_{\Omega} Pu \cdot (Pu \cdot \nabla y_*) \, dx \geq \theta_0 \|Pu\|_{L^2(\Omega)}^2.$$

Then it remains to prove that

$$\int_{\Omega} Pu \cdot P \nabla f \, dx \leq C \|u\|_{H^s(\Omega)} \|u\|_{H^{s-1}(\Omega)} + C \|u\|_{s,k+1} \|u\|_{s,k}. \quad (4.37)$$

To this end, let us write using the decomposition (4.21) that for $k \geq 1$,

$$\begin{aligned} P \nabla f &= \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \partial_n^2 f + [\partial_{\tau_1}^{s-k-1} \partial_n^{k+1}, \nabla] f \\ &= (1 - \chi_2) \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \partial_n^2 f + \chi_2 \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \partial_n^2 f + [\partial_{\tau_1}^{s-k-1} \partial_n^{k+1}, \nabla] f. \end{aligned}$$

The commutator is a lower order term in the sense that

$$\|[\partial_{\tau_1}^{s-k-1} \partial_n^{k+1}, \nabla] f\|_{L^2(\Omega)} \leq C \|f\|_{H^s(\Omega)} \leq C \|u\|_{H^{s-1}(\Omega)},$$

leading to the bound

$$\int_{\Omega} Pu \cdot [\partial_{\tau_1}^{s-k-1} \partial_n^{k+1}, \nabla] f \, dx \leq C \|Pu\|_{L^2(\Omega)} \|u\|_{H^{s-1}(\Omega)} \leq C \|u\|_{H^s(\Omega)} \|u\|_{H^{s-1}(\Omega)}.$$

Integration by parts as in (4.25) yields

$$\left| \int_{\Omega} Pu \cdot \chi_2 \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \partial_n^2 f \, dx \right| \leq C \|u\|_{H^s(\Omega)} \|u\|_{H^{s-1}(\Omega)}.$$

In the main term $(1 - \chi_2) \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \partial_n^2 f$, since the support of $(1 - \chi_2)$ is contained in $\Omega_{2\kappa}$, we can use (2.5) and (4.36) to write

$$\begin{aligned} \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \partial_n^2 f &= -\nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \partial_{\tau_j}^2 f + \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \operatorname{div}(u \cdot \nabla y_*) \\ &\quad + \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \left[\nabla f \cdot (n \cdot \nabla) n + \nabla f \cdot (\tau_j \cdot \nabla) \tau_j \right] \\ &= -\nabla \partial_n^{k-1} \partial_{\tau_1}^{s-k-1} \partial_{\tau_j}^2 f - \nabla [\partial_{\tau_1}^{s-k-1}, \partial_n^{k-1}] \partial_{\tau_j}^2 f + \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \operatorname{div}(u \cdot \nabla y_*) \\ &\quad + \nabla \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \left[\nabla f \cdot (n \cdot \nabla) n + \nabla f \cdot (\tau_j \cdot \nabla) \tau_j \right] \end{aligned}$$

in $\Omega_{2\kappa}$, where the sums over j were taken. Since the commutator $[\partial_{\tau_1}^{s-k-1}, \partial_n^{k-1}] \partial_{\tau_j}^2 f$ is bounded in $H^1(\Omega)$ by $C \|f\|_{H^s(\Omega)} \leq C \|u\|_{H^{s-1}(\Omega)}$ we obtain

$$\left| \int_{\Omega} (1 - \chi_2) Pu \cdot \nabla [\partial_{\tau_1}^{s-k-1}, \partial_n^{k-1}] \partial_{\tau_j}^2 f \, dx \right| \leq C \|u\|_{H^s(\Omega)} \|u\|_{H^{s-1}(\Omega)}. \quad (4.38)$$

Consequently,

$$\|g_1\|_{H^{k-1}(\Omega)} \leq C\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,k}. \quad (4.43)$$

As for g_2 we first use the trace theorem to have

$$\begin{aligned} & \|[\partial_n, \partial_{\tau_1}^{s-k-1} \partial_{\tau_j}]f\|_{H^{k-\frac{1}{2}}(\partial\Omega)} \\ & \leq C\|[\partial_n, \partial_{\tau_1}^{s-k-1} \partial_{\tau_j}]f\|_{H^k(\Omega)} \leq C\|f\|_{H^s(\Omega)} \leq C\|u\|_{H^{s-1}(\Omega)}. \end{aligned}$$

The fact that $k \geq 1$ was used in the first inequality. Then we write

$$\partial_{\tau_1}^{s-k-1} \partial_{\tau_j} (u \cdot \nabla y_*) = (\partial_{\tau_1}^{s-k-1} \partial_{\tau_j} u) \cdot \nabla y_* + [\partial_{\tau_1}^{s-k-1} \partial_{\tau_j}, \nabla y_*]u$$

where the commutator can be bounded using the trace theorem as follows

$$\|[\partial_{\tau_1}^{s-k-1} \partial_{\tau_j}, \nabla y_*]u\|_{H^{k-\frac{1}{2}}(\partial\Omega)} \leq C\|[\partial_{\tau_1}^{s-k-1} \partial_{\tau_j}, \nabla y_*]u\|_{H^k(\Omega)} \leq C\|u\|_{H^{s-1}(\Omega)}.$$

In addition,

$$\begin{aligned} \|(\partial_{\tau_1}^{s-k-1} \partial_{\tau_j} u) \cdot \nabla y_*\|_{H^{k-\frac{1}{2}}(\partial\Omega)} & \leq C\|\partial_{\tau_1}^{s-k-1} \partial_{\tau_j} u\|_{H^{k-\frac{1}{2}}(\partial\Omega)} \\ & \leq C\|\partial_{\tau_1}^{s-k-1} \partial_{\tau_j} u\|_{H^k(\Omega)} \\ & \leq C\|u\|_{s,k}. \end{aligned}$$

Thus,

$$\|g_2\|_{H^{k-\frac{1}{2}}(\partial\Omega)} \leq \|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,k}. \quad (4.44)$$

Combining (4.42), (4.43) and (4.44) leads to the bound (4.40), which implies that

$$\begin{aligned} I_1 & \leq C\|Pu\|_{L^2(\Omega)}\|u\|_{H^{s-1}(\Omega)} + C\|Pu\|_{L^2(\Omega)}\|u\|_{s,k} \\ & \leq C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,k+1}\|u\|_{s,k}. \end{aligned} \quad (4.45)$$

Estimate for I_2 . Decomposing $\nabla = \tau_j \partial_{\tau_j} + n \partial_n$ in $\Omega_{2\kappa} \supset \text{supp}(1 - \chi_2)$ gives $I_2 = I_{2a} + I_{2b}$ where

$$\begin{aligned} I_{2a} & = \int_{\Omega} (1 - \chi_2) \{(\partial_{\tau_1}^{s-k-1} \partial_n^{k+1} u) \cdot \tau_j\} \{\partial_{\tau_j} \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \text{div}(u \cdot \nabla y_*)\} dx, \\ I_{2b} & = \int_{\Omega} (1 - \chi_2) \{(\partial_{\tau_1}^{s-k-1} \partial_n^{k+1} u) \cdot n\} \{\partial_n \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \text{div}(u \cdot \nabla y_*)\} dx. \end{aligned}$$

We notice that there are at most k normal derivatives in $\partial_{\tau_j} \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \text{div}(u \cdot \nabla y_*)$, hence

$$|I_{2a}| \leq C\|u\|_{s,k+1}\|u\|_{s,k}.$$

As for I_{2b} we write using (2.4) that

$$\begin{aligned} (\partial_{\tau_1}^{s-k-1} \partial_n^{k+1} u) \cdot n & = \partial_{\tau_1}^{s-k-1} \partial_n^k (\partial_n u \cdot n) + [\partial_{\tau_1}^{s-k-1} \partial_n^k, n] \partial_n u \\ & = -\partial_{\tau_1}^{s-k-1} \partial_n^k (\partial_{\tau_j} u \cdot \tau_j) + [\partial_{\tau_1}^{s-k-1} \partial_n^k, n] \partial_n u. \end{aligned}$$

It is readily seen that

$$\begin{aligned} & \left| \int_{\Omega} (1 - \chi_2) \{ [\partial_{\tau_1}^{s-k-1} \partial_n^k, n \cdot] \partial_n u \} \{ \partial_n \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \operatorname{div}(u \cdot \nabla y_*) \} dx \right| \\ & \leq C \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)}. \end{aligned}$$

On the other hand, there are at most k normal derivatives in $\partial_{\tau_1}^{s-k-1} \partial_n^k (\partial_{\tau_j} u \cdot \tau_j)$, and thus

$$\begin{aligned} & \left| \int_{\Omega} (1 - \chi_2) \{ \partial_{\tau_1}^{s-k-1} \partial_n^k (\partial_{\tau_j} u \cdot \tau_j) \} \{ \partial_n \partial_{\tau_1}^{s-k-1} \partial_n^{k-1} \operatorname{div}(u \cdot \nabla y_*) \} dx \right| \\ & \leq C \|u\|_{s,k} \|u\|_{s,k+1}. \end{aligned}$$

All together we have prove that

$$|I_2| \leq C \|u\|_{H^s(\Omega)} \|u\|_{H^{s-1}(\Omega)} + C \|u\|_{s,k+1} \|u\|_{s,k} \quad (4.46)$$

In view of (4.45) and (4.46) we finish the proof of (4.37), and hence the proof of (4.34) with $k+1$ in place of k . \square

4.4. H^s Estimate for u

We have proved in Lemmas 4.3, 4.4 and 4.5 that

$$\frac{1}{2} \frac{d}{dt} \|u\|_{s,0}^2 + \theta_0 \|u\|_{s,0}^2 \leq M_0 \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + M_0 \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)} \quad (4.47)$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{s,k}^2 + \frac{\theta_0}{2} \|u\|_{s,k}^2 & \leq M_k \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + M_k \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)} \\ & \quad + M_k \|u\|_{s,k} \|u\|_{s,k-1} \end{aligned} \quad (4.48)$$

for all $k \in \{1, 2, \dots, s\}$. Applying Young's inequality yields

$$M_k \|u\|_{s,j} \|u\|_{s,j-1} \leq \frac{\theta_0}{4} \|u\|_{s,j-1}^2 + M'_j \|u\|_{s,j-1}^2, \quad 1 \leq j \leq s.$$

It follows from this and (4.48) with $k = s$ and $k = s-1$ that

$$\begin{aligned} & \frac{d}{dt} \|u\|_{s,s}^2 + \frac{\theta_0}{2} \|u\|_{s,s}^2 \\ & \leq 2M_s \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + 2M_s \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)} + 2M'_s \|u\|_{s,s-1}^2 \end{aligned} \quad (4.49)$$

and

$$\begin{aligned} & \frac{d}{dt} \|u\|_{s,s-1}^2 + \frac{\theta_0}{2} \|u\|_{s,s-1}^2 \\ & \leq 2M_{s-1} \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + 2M_{s-1} \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)} \\ & \quad + 2M'_{s-1} \|u\|_{s,s-2}^2. \end{aligned} \quad (4.50)$$

Let $N_{s-1} > 0$ be such that $\frac{\theta_0}{2} N_{s-1} - 2M'_{s-1} = \frac{\theta_0}{2}$. Multiplying (4.50) by N_{s-1} then adding the resulting inequality to (4.49) we obtain

$$\begin{aligned} \frac{d}{dt} (\|u\|_{s,s}^2 + N_{s-1} \|u\|_{s,s-1}^2) + \frac{\theta_0}{2} (\|u\|_{s,s}^2 + \|u\|_{s,s-1}^2) &\leq N'_{s-1} \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} \\ &+ N'_{s-1} \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)} + N'_{s-1} \|u\|_{s,s-2}^2 \end{aligned}$$

for some $N'_{s-1} > 0$. Continuing this process, one can find $s+1$ positive constants B and N_j , $0 \leq j \leq s-1$ such that

$$\begin{aligned} \frac{d}{dt} (\|u\|_{s,s}^2 + \sum_{j=0}^{s-1} N_j \|u\|_{s,j}^2) \\ + \frac{\theta_0}{2} \sum_{j=0}^s \|u\|_{s,j}^2 \leq B \|u\|_{H^s(\Omega)}^2 \|z\|_{H^s(\Omega)} + B \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)}. \end{aligned}$$

Setting

$$Z^2(u) = \|u\|_{s,s}^2 + \sum_{j=0}^{s-1} N_j \|u\|_{s,j}^2$$

and

$$2\theta_1 = \frac{\theta_0}{2 \max_{0 \leq j \leq s-1} \{1, N_j\}},$$

we arrive at

$$\frac{d}{dt} Z^2(u) + 2\theta_1 Z^2(u) \leq B \|u(t)\|_{H^s(\Omega)}^2 \|z(t)\|_{H^s(\Omega)} + B \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)}. \quad (4.51)$$

Set

$$W^2(u) = Z^2(u) + \|\chi_2 u\|_{H^s(\Omega)}^2 \quad (4.52)$$

where χ_2 is given by (2.24). Combining (4.51) with (4.8) one can find a constant $C > 0$ such that

$$\frac{d}{dt} W^2(u) + 2\theta_1 W^2(u) \leq C \|u(t)\|_{H^s(\Omega)}^2 \|z(t)\|_{H^s(\Omega)} + C \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)}. \quad (4.53)$$

To recover the H^s estimate for u from the preceding estimate on $W(u)$, we prove the next lemma.

Lemma 4.6. *There exists $A > 0$ depending only on s such that*

$$\frac{1}{A} W^2(u) \leq \|u\|_{H^s(\Omega)}^2 \leq A W^2(u) + A \|u\|_{L^2(\Omega)}^2 \quad (4.54)$$

for any H^s vector field $u : \Omega \rightarrow \mathbb{R}^d$.

Proof. First, the inequality

$$Z^2(u) \leq A \|u\|_{H^s(\Omega)}^2 + A \|\chi_2 u\|_{H^s(\Omega)}^2$$

is obvious if A is sufficiently large.

Next recall from (4.21) and (2.3) that for any $w : \Omega \rightarrow \mathbb{R}^2$ it holds that

$$|\nabla w|^2 \leq |\partial_n w|^2 + \sum_{j=1}^d |\partial_{\tau_j} w|^2 + \|\chi_2 \nabla w\|_{L^2(\Omega)}^2. \quad (4.55)$$

In the rest of this proof, the sum over $j \in \{1, \dots, d-1\}$ will be omitted. Let D^{s-1} be an arbitrary partial derivative of order $s-1$. Without loss of generality, assume $D^{s-1} = \partial_1 D^{s-2}$ for some partial derivative D^{s-2} of order $s-2$. Applying (4.55) with $w = D^{s-1}u$ gives

$$\|\nabla D^{s-1}u\|_{L^2(\Omega)}^2 \leq \|\partial_n \partial_1 D^{s-2}u\|_{L^2(\Omega)}^2 + \|\partial_{\tau_j} \partial_1 D^{s-2}u\|_{L^2(\Omega)}^2 + \|\chi_2 \nabla w\|_{L^2(\Omega)}^2.$$

We thus have replaced one partial derivative with one normal and one tangential derivative. To continue, we commute ∂_n with ∂_1 to have

$$\begin{aligned} \|\partial_n \partial_1 D^{s-2}u\|_{L^2(\Omega)}^2 &\leq 2\|\partial_1 \partial_n D^{s-2}u\|_{L^2(\Omega)}^2 + 2\|[\partial_n, \partial_1] D^{s-2}u\|_{L^2(\Omega)}^2 \\ &\leq 2\|\partial_1 \partial_n D^{s-2}u\|_{L^2(\Omega)}^2 + C\|u\|_{H^{s-1}(\Omega)}^2. \end{aligned}$$

Similarly for $\|\partial_{\tau_j} \partial_1 D^{s-2}u\|_{L^2(\Omega)}^2$ we obtain

$$\begin{aligned} \|\nabla D^{s-1}u\|_{L^2(\Omega)}^2 &\leq 2\|\nabla \partial_n D^{s-2}u\|_{L^2(\Omega)}^2 \\ &\quad + 2\|\nabla \partial_{\tau_j} D^{s-2}u\|_{L^2(\Omega)}^2 + C\|u\|_{H^{s-1}(\Omega)}^2 + \|\chi_2 u\|_{H^s(\Omega)}^2. \end{aligned}$$

Now applying (4.55) with $w = \partial_n D^{s-2}u$ and $w = \partial_{\tau_j} D^{s-2}u$ leads to

$$\begin{aligned} \|\nabla D^{s-1}u\|_{L^2(\Omega)}^2 &\leq 2\|\partial_n \partial_n D^{s-2}u\|_{L^2(\Omega)}^2 + 2\|\partial_{\tau_j} \partial_n D^{s-2}u\|_{L^2(\Omega)}^2 + 2\|\partial_n \partial_{\tau_j} D^{s-2}u\|_{L^2(\Omega)}^2 \\ &\quad + 2\|\partial_{\tau_j} \partial_{\tau_j} D^{s-2}u\|_{L^2(\Omega)}^2 + C\|u\|_{H^{s-1}(\Omega)}^2 + \|\chi_2 u\|_{H^s(\Omega)}^2. \end{aligned}$$

Next we write $D^{s-2} = \partial_j D^{s-3}$ with $j \in \{1, \dots, d\}$ and continue the process until no partial derivatives are left on the right-hand side, yielding

$$\|\nabla D^{s-1}u\|_{L^2(\Omega)}^2 \leq CZ^2(u) + C\|u\|_{H^{s-1}(\Omega)}^2 + \|\chi_2 u\|_{H^s(\Omega)}^2.$$

This combined with the interpolation inequality $\|u\|_{H^{s-1}} \leq C\|u\|_{H^s}^\alpha \|u\|_{L^2}^{1-\alpha}$, $\alpha \in (0, 1)$ and a Young inequality implies the desired estimate (4.54). \square

By interpolation and Young's inequality, the last term on the right-hand side of (4.53) is bounded as

$$C\|u(t)\|_{H^s(\Omega)}\|u(t)\|_{H^{s-1}(\Omega)} \leq \gamma\|u(t)\|_{H^s(\Omega)}^2 + C_\gamma\|u(t)\|_{L^2(\Omega)}^2$$

for any $\gamma > 0$. Using (4.54) and choosing γ sufficiently small so that $A\gamma < \theta_1$, we deduce from (4.53) that

$$\frac{d}{dt} W^2(u) + \theta_1 W^2(u) \leq C_2\|u(t)\|_{H^s(\Omega)}^2 \|z(t)\|_{H^s(\Omega)} + C_2\|u(t)\|_{L^2(\Omega)}^2 \quad (4.56)$$

for some $C_2 > 0$.

4.5. Proof of Theorem 1.1

We define the bootstrap norm by (3.10)

$$\mathcal{N}(t) := \sup_{0 \leq \tau \leq t} \left(\|z(\tau)\|_{H^s(\Omega)}^2 + M^2 e^{\frac{\theta_1}{2}\tau} \|u(\tau)\|_{L^2(\Omega)}^2 + M e^{\frac{\theta_1}{2}\tau} \|u(\tau)\|_{H^s(\Omega)}^2 \right) \quad (4.57)$$

for some large $M > 0$ to be fixed and for $t < T^*$, the maximal time of existence which is positive thanks to the local existence theory, Theorem 1.6.

Proposition 4.7. *There exist positive constants ε, C_* , depending only on θ_0 and $\|y_*\|_{H^{s+1}}$, such that whenever $\mathcal{N}(0) < \varepsilon$, we have $\mathcal{N}(t) \leq C_* \mathcal{N}(0)$ for all $t < T^*$.*

Proof. As in the proof of Proposition 3.3, it suffices to prove that

$$\mathcal{N}(t) \leq C_0 \mathcal{N}(0) + C_0 \mathcal{N}(t)^{3/2} \quad (4.58)$$

for all $t < T^*$. Repeating verbatim the proof of (3.12) we obtain from (4.2) that

$$\|z(t)\|_{H^s}^2 \leq \|z(0)\|_{H^s}^2 + C_3 M^{-1/2} (1 + \mathcal{N}(t)^{1/2}) \mathcal{N}(t). \quad (4.59)$$

Next we combine the estimates (4.7) and (4.56) and argue as in (3.13) to get

$$M^2 e^{\theta_1 t/2} \|u(t)\|_{L^2}^2 + M e^{\theta_1 t/2} W^2(u) \leq \left[M^2 \|u(0)\|_{L^2}^2 + M \|u(0)\|_{H^s}^2 \right] + C_5 \mathcal{N}(t)^{3/2}. \quad (4.60)$$

Then (4.58) follows from (4.59), (4.60) and (4.54). \square

Finally, the proof of Theorem 1.3 follows along the same lines as that of Theorem 1.1.

Acknowledgements. The authors thank Yann Brenier for introducing them the AHT model studied in this paper, Robert McCann for informing them the references [12, 13], and Dong Li for constructive comments. HN's research was supported by NSF Grant DMS-1907776. TN's research was supported in part by the NSF under Grant DMS-1764119 and by the 2018-2019 AMS Centennial Fellowship. Part of this work was done while TN was visiting the Department of Mathematics and the Program in Applied and Computational Mathematics at Princeton University.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. ANGENENT, S., HAKER, S., TANNENBAUM, A.: Minimizing flows for the Monge–Kantorovich problem. *SIAM J. Math. Anal.* **35**, 61–97, 2003
2. BAHOURI, H., CHEMIN, J.-Y., DANCHIN, R.: *Fourier Analysis and Nonlinear Partial Differential Equations*, vol. 343. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2011
3. BONY, J.-M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)* **14**(2), 209–246, 1981
4. BOYER, F., FABRIE, P.: *Mathematical Tools for the Study of the Incompressible Navier–Stokes Equations and Related Models*, vol. 183. Applied Mathematical Sciences. Springer, Berlin, 2013
5. BRENIER, Y.: Polar factorization and monotone rearrangement of vector-valued functions. *Commun. Pure Appl. Math.* **44**(4), 375–417, 1991
6. BRENIER, Y.: Optimal transportation, convection, magnetic relaxation and generalized Boussinesq equations. *J. Nonlinear Sci.* **19**(5), 547–570, 2009
7. BRENIER, Y.: On the hydrostatic and Darcy limits of the convective Navier–Stokes equations. *Chin. Ann. Math. Ser. B* **30**(6), 683–696, 2009
8. BRENIER, Y.: Rearrangement, convection, convexity and entropy. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **371**(2005), 20120343, 2013
9. CASTRO, A., CÓRDOBA, D., LEAR, D.: Global existence of quasi-stratified solutions for the confined IPM equation. *Arch. Ration. Mech. Anal.* **232**, 437–471, 2018
10. ELGINDI, T.M.: On the asymptotic stability of stationary solutions of the inviscid incompressible porous medium equation. *Arch. Ration. Mech. Anal.* **225**(2), 573–599, 2017
11. MAJDA, A., BERTOZZI, A.: *Vorticity and Incompressible Flow*, vol. 27. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002
12. McCann, R.: A convexity theory for interacting gases and equilibrium crystals. *Thesis (Ph.D.)*–Princeton University, ProQuest LLC, Ann Arbor, MI, p. 163, 1994
13. MCCANN, R.: A convexity principle for interacting gases. *Adv. Math.* **128**(1), 153–179, 1997

HUY Q. NGUYEN
 Department of Mathematics,
 Brown University,
 Providence
 RI

02912 USA.

e-mail: hnguyen@math.brown.edu

and

TOAN T. NGUYEN
 Department of Mathematics,
 Penn State University,
 State College
 PA

16803 USA.

e-mail: nguyen@math.psu.edu

(Received January 9, 2019 / Accepted June 23, 2020)

Published online July 6, 2020

© Springer-Verlag GmbH Germany, part of Springer Nature (2020)