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Uniform in Time Lower Bound for Solutions to a Quantum Boltzmann Equation of Bosons

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Abstract

In this paper, we consider a quantum Boltzmann equation, which describes the interaction between excited atoms and a condensate. The collision integrals are taken–over energy manifolds, having the full form of the Bogoliubov dispersion law for particle energy. We prove that nonnegative radially symmetric solutions of the quantum Boltzmann equation are bounded from below by a Gaussian distribution, uniformly in time.

1. Introduction

The discovery of Bose–Einstein condensation (BEC) in trapped ultracold atomic gases in 1995 [4,5] has led to an explosion of research on its properties. A kinetic equation for BECs was first derived by Kirkpatrick and Dorfmann [28,30], using a mean field theory and the Green's function method. Following the path of Kirkpatrick and Dorfmann, several authors have tried to derive kinetic equations to describe the dynamics of BECs [7,9,22,23,25,30,31,40,48]. In the series of papers [19,20,26], C.W. Gardiner, P. Zoller and coauthors formulated the Quantum Kinetic Theory, which is both a genuine kinetic theory and a genuine quantum theory, in terms of the Quantum Kinetic Master Equation (QKME) for bosonic atoms. In the Quantum Kinetic Theory, the significant quantum aspects are restricted to a few modes, the remaining modes being able to be described in the classical way, as in the Boltzmann equation. Indeed, the kinetic aspect of the theory arises from the decorrelation between different momentum bands. The Quantum Kinetic Theory provides a fully quantum mechanical description of the kinetics of a Bose gas, including the regime of a Bose condensation. In particular, the QKME is capable of describing the formation of the Bose condensate. The QKME contains, as limiting cases, the Boltzmann–Norheim (Uehling–Ulenbeck) equation [13,36,46], the Gross–Pitaevskii equation, and the condensate growth term. The condensate growth term is the principal term which gives rise to growth of the condensate, doing this by taking atoms out of the bath of warmer atoms.

Bosons of mass m at temperature T can be regarded as quantum-mechanical wavepackets whose extent is proportional to a thermal de Broglie wavelength

$$\lambda_{dB} = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{\frac{1}{2}},$$

describing the position uncertainty associated with the thermal momentum distribution, in which k_B is the Boltzmann constant and $\hbar$ is the Planck constant. When the gas temperature T is high, the de Broglie wavelength λ_{dB} is very small and the weakly interacting gas is similar to a system of "billiard balls". The dynamics of the density function of the gas $f(t, r, p)$—the probability of finding a particle at time t, position r and momentum p—is described by the Boltzmann–Norheim (Uehling–Ulenbeck) equation

$$\partial_t f(t, r, p) + p \cdot \nabla_r f(t, r, p) = \mathcal{C}_{22}[f](t, r, p), \quad f(0, r, p) = f_0(r, p), \quad (1.1)$$

for $(t, r, p) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, where the collision operator $\mathcal{C}_{22}[f]$ reads

$$\begin{aligned}\mathcal{C}_{22}[f](t, r, p_1) = & \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4)\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \\ & \times [(1 + \vartheta f_1)(1 + \vartheta f_2) f_3 f_4 - f_1 f_2 (1 + \vartheta f_3)(1 + \vartheta f_4)] \, \mathrm{d}p_2 \mathrm{d}p_3 \mathrm{d}p_4, \end{aligned} \quad (1.2)$$

where ϑ is proportional to $\hbar^3$, $\mathcal{E}_p$ is the energy of a particle with momentum p, and we use the short-hand notation $f_j = f(t, r, p_j)$.

The quantum Boltzmann collision operator (1.2) becomes the classical one in the semiclassical limit, as ϑ tends to 0. A consequence of this fact is that at high temperature, the behavior of the Bose gas is, in some sense, quite similar to classical gases. Note that, differenlyt from classical Boltzmann collision operators, where the collision kernels are functions depending on the types of particles considered, the derived collision kernel for the quantum Boltzmann collision operator for bosons is 1.

When the temperature T becomes lower, λ_{dB} becomes smaller. At the BEC transition temperature $T \approx T_{BEC}$, the de Broglie wavelength becomes comparable to the distance between bosons. As a consequence, the atomic wavepackets "overlap" and the atoms become indistinguishable. At this temperature, bosons undergo a quantum-mechanical phase transition and the Bose–Einstein condensate is formed. The gas is said to be at finite temperature if $T_{BEC} > T > 0$K. At this temperature the trapped Bose gas is composed of two distinct components: the high-density *Bose–Einstein Condensate*—localized at the center of the trapping potential- and the low-density cloud of thermally *excited atoms,* spreading over a much wider region. The system of the coupling between the BEC and the excited atoms consists of equations of the wave function $\Psi(t, r)$ of the BEC, which is a function of time and position (t, r) and the density function $f(t, r, p)$, which is a function of time, position, and momentum of the excited atoms (t, r, p). The coupled system that describes the dynamics of BEC and excited atoms then consists of (cf. [22,43]):

- the Gross–Pitaevskii equation that governs the dynamics of the wave function $\Psi(t,r)$ of BEC;
- the Boltzmann equation that models the dynamics of the density distribution $f(t,r,p)$ of excited atoms, which consists of two collision operators:
  - $\mathcal{C}_{12}[f]$ describes the collision of BEC and excited atoms;
  - $\mathcal{C}_{22}[f]$ describes the collision between excited atoms.

For further discussions and a study on such a coupled system, see [22,27,41–43] and the references therein.

1.1. The Model

In this paper, we are interested in the interaction $\mathcal{C}_{12}[f]$ between excited bosons and a condensate; precisely, we study the spatially homogenous quantum Boltzmann equation

$$\frac{\partial f}{\partial t} = n_c(t)\mathcal{C}_{12}[f] \tag{1.3}$$

posed on $\mathbb{R}_+ \times \mathbb{R}^3$, for the density distribution function $f(t,p)$ of excited atoms, coupled with the differential equation

$$\frac{\mathrm{d}n_c}{\mathrm{d}t} = -n_c(t)\int_{\mathbb{R}^3} \mathcal{C}_{12}[f](t,p)\mathrm{d}p, \tag{1.4}$$

posed on $\mathbb{R}_+$, for the density function $n_c(t)$ of the condensate. Here, $\mathcal{C}_{12}[f]$ denotes the collision integral operator that describes the bosons-condensate interaction ([1, 2,7,9,12,25,29,30]), given by

$$\mathcal{C}_{12}[f](t,p) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Big(\mathcal{R}_{p,p_1,p_2}[f] - \mathcal{R}_{p_1,p,p_2}[f] - \mathcal{R}_{p_2,p,p_1}[f]\Big)\, \mathrm{d}p_1\mathrm{d}p_2 \tag{1.5}$$

with

$$\begin{aligned} \mathcal{R}_{p,p_1,p_2}[f] &= \mathcal{K}(p,p_1,p_2)\Big(f_1f_2(1+f) - (1+f_1)(1+f_2)f\Big) \\ \mathcal{K}(p,p_1,p_2) &= K(p,p_1,p_2)\delta(p-p_1-p_2)\delta(\mathcal{E}(p)-\mathcal{E}(p_1)-\mathcal{E}(p_2)), \end{aligned} \tag{1.6}$$

using the short-hand notation $f = f(t,p)$ and $f_j = f(t,p_j)$. Such a simplified model is used, for instance, when the temperature is very low and thus the interaction $\mathcal{C}_{22}[f]$ between bosons themselves is weak and negligible as compared to the interaction $\mathcal{C}_{12}[f]$ (see, for instance, [6–9,12,16]).

In (1.6), $\delta(\cdot)$ denotes the Dirac delta function, and $\mathcal{E}(p)$ denotes the particle energy, which is of the form of the Bogoliubov dispersion relation

$$\mathcal{E}(p) = |p|\sqrt{\kappa_1 + \kappa_2|p|^2}, \qquad \kappa_1 = \frac{gN_o}{m} > 0, \quad \kappa_2 = \frac{1}{4m^2} > 0, \tag{1.7}$$

for m being the mass of the particles, g the interaction coupling constant, and N_o assumed to be a constant. In addition, the kernel $K(p,p_1,p_2)$ in (1.6) is often

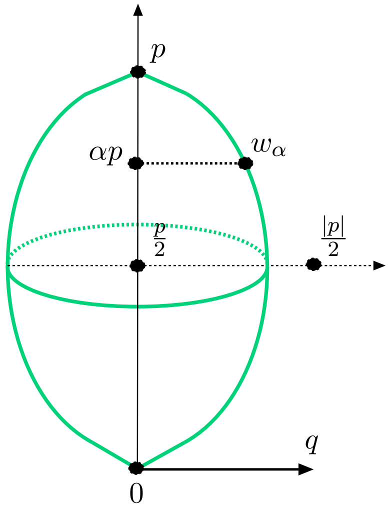

Fig. 1. Illustrated is the oval surface S_p, centered at $\frac{p}{2}$ and having 0 and p as its south and north poles, respectively

referred to as the transition probability kernel ([12,23,24,28,30,40]). In this paper, we shall consider the form

$$K(p, p_1, p_2) = \kappa_0 |p|^\rho |p_1|^\rho |p_2|^\rho, \tag{1.8}$$

for any fixed constant $\rho \in [1, \frac{5}{2}]$ and for some positive constant κ_0. Such a kernel is used, for instance in [12,16,25], when Bose gases are at a sufficiently low temperature. For the sake of presentation, we shall take constants $\kappa_0, \kappa_1, \kappa_2$ to be one. The results in this paper apply to the general case when the constants are positive.

We emphasize that in this paper the full form of energy functions (1.7) is considered, which complicates the analysis in treating the collision integral operator $\mathcal{C}_{12}[f]$. Indeed, the presence of the Dirac delta function in (1.6) reduces the collision integrals over $\mathbb{R}^3 \times \mathbb{R}^3$ to the surface integrals on the so-called energy manifolds, dictated by the conservation laws

$$p = p_1 + p_2, \qquad \mathcal{E}(p) = \mathcal{E}(p_1) + \mathcal{E}(p_2) \tag{1.9}$$

for each $p \in \mathbb{R}^3$; see Figures 1 and 2 for an illustration of these surfaces. In addition to the complication of dealing with the surface integrals, it is certainly not clear whether the second moment of f on these surfaces is bounded, even if the second moment of f in $\mathbb{R}^3$ is bounded. As a matter of fact, for this very reason, the simplified energy functions $\mathcal{E}(p) = c|p|$ or $\mathcal{E}(p) = c|p|^2$ have been used in the literature; see, for instance, [1,2,8,13,15] and the references therein. The former energy law leads to line integrals, whereas the latter reduces to integrals on a sphere, as it is the case for the classical Boltzmann equations (e.g., [16,34,47]). To the best of to our knowledge, the current paper is the first time where such a full energy of the form (1.7) is studied.

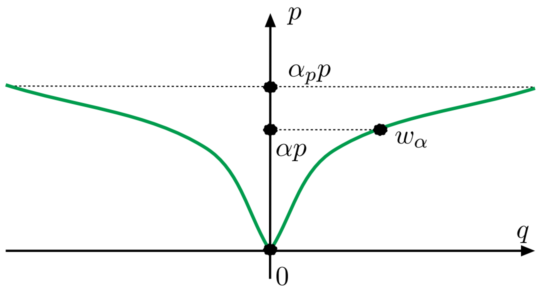

Fig. 2. Sketched is the trace of S_p' on any two dimensional plane containing p

Let us mention that the model (1.3) has also been studied in [6–9], with truncated transition probability kernel near zero or infinity. The propagation of exponential and polynomial energy moments is also studied recently in [3], and the well-posedness theory is developed in [43] for a more general model that in fact contains both $\mathcal{C}_{12}$ and $\mathcal{C}_{22}$. On the other hand, the convergence to equilibrium of a linearized or discrete version of (1.3) is obtained in [11,17]. In this paper, we prove that positive radial solutions to (1.3)–(1.8), if exist, are uniformly bounded below by a Gaussian distribution.

1.2. Related Contexts

Let us also point out that the model (1.3) is also referred to as the phonon Boltzmann equation, proposed by Peierls in 1929 [37,38] to study the interaction of phonon gases. See also [3,11,44] for recent studies. In addition, it also shares a great similarity with three-wave kinetic models used in the weak turbulence theory [14,18,21,32,35,45,49].

1.3. Main Result

Let us now present the main result of this paper. For $m \geq 1$, introduce the function space $\mathbb{L}^1_m(\mathbb{R}^3)$, defined by its finite norm

$$\|f\|_{\mathbb{L}^1_m} := \int_{\mathbb{R}^3} \left(1 + \mathcal{E}(p)^m\right) |f(p)| \mathrm{d}p, \tag{1.10}$$

with $\mathcal{E}(p) = |p|\sqrt{1+|p|^2}$.

Theorem 1.1. *Let $f_0(p) = f_0(|p|)$ be a positive radial initial datum in $\mathbb{L}^1_m(\mathbb{R}^3) \cap C(\mathbb{R}^3)$ for some $m \geq 1$, and let $n_c(0) = n_0$ be a positive initial density constant, so that the Cauchy problem (1.3)–(1.8) has a unique classical positive radial solution $f(t, p) = f(t, |p|)$ in $C([0, \infty), \mathbb{L}^1_m(\mathbb{R}^3) \cap C(\mathbb{R}^3)) \cap C^1([0, \infty), \mathbb{L}^1_m(\mathbb{R}^3) \cap C(\mathbb{R}^3))$ and a unique density function $n_c(t) \in C^1([0, \infty))$ satisfying $n_1 \leq n_c(t) \leq n_2$ for some positive constants n_1, n_2.*

Assume that $f_0(p) \geq \theta_0$ *on* $B_{R_0} = \{|p| \leq R_0\}$ *for some positive constants* θ_0, R_0*. Then, for any time* $T > 0$*, there exist positive constants* θ_1, θ_2 *such that*

$$f(t, p) \geq \theta_1 \exp(-\theta_2 |p|^2), \qquad \forall\, t \geq T, \quad \forall\, p \in \mathbb{R}^3. \tag{1.11}$$

We stress that the existence of positive radial solutions is not studied in this paper. However, such a solution is constructed in [3,43]. The lower bound assumption on $n_c(t)$ means that the condensate is stable and remains present as time evolves, while the upper bound follows from the conservation of mass; see Lemma 2.2. Physically speaking, Theorem 1.1 asserts that the collision operator $\mathcal{C}_{12}[f]$ prevents the excited atoms from falling completely into the condensate. In other words, given a condensate and its thermal cloud, we show that there is some portion of excited atoms which remains outside of the condensate and the density of such atoms remains greater than a Gaussian distribution, uniformly in time $t \geq T$, for any time $T > 0$.

The condition that initial data $f_0(p)$ has positive mass near $\{p = 0\}$ is necessary for such a lower bound by a Gaussian to hold, since otherwise if $f_0(0) = 0$, then $f(t, 0)$ would remain zero for all positive time, as a consequence of $\mathcal{C}_{12}[f](0) = 0$, or

$$\partial_t f(t, 0) = 0, \qquad \forall\, t \geq 0. \tag{1.12}$$

Obtaining lower bounds on solutions to the Boltzmann equation is a classical question, which was first studied by Carleman in his pioneering paper [10]. There, he proved that solutions are bounded from below by

$$\theta_1 \exp(-\theta_2 |p|^{2+\epsilon})$$

for $\epsilon > 0$, using a "spreading property" of the collision operator. This result was later improved by Pulvirenti and Wennberg [39], providing the Gaussian lower bound in the case of hard potentials with cutoff in dimension 3. In [33], Mouhot proved an explicit lower bound on solutions to the full Boltzmann equation on the torus, under the assumption of uniform bounds on certain hydrodynamic quantities, for a broad family of collision kernels including in particular long-range interaction models. The study of lower bounds is an important subject, not only to understand the qualitative behaviour of solutions to the Boltzmann equation, but also to study the convergence to equilibrium using the so-called "entropy-entropy production"method [33,47].

The structure of the paper is as follows. Section 2 is to give the conservation of momentum, energy and the H-theorem of (1.3), while Section 3 provides the technical estimates on the energy surfaces, which are the basic tools of the paper. We derive uniform second-order energy moments in Section 4, and give the proof of the main theorem in Section 5.

2. Conservation Laws and the H-Theorem

In this section, we present a few basic properties of smooth solutions of (1.3).

Lemma 2.1. *For any smooth function $f(p)$, there holds*

$$\int_{\mathbb{R}^3} \mathcal{C}_{12}[f](p)\varphi(p)dp = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \mathcal{R}_{p,p_1,p_2}[f]\Big(\varphi(p) - \varphi(p_1) - \varphi(p_2)\Big)\,\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2$$

for any smooth test function φ.

Proof. By the definition (1.5) of $\mathcal{C}_{12}[f]$, we have

$$\int_{\mathbb{R}^3} \mathcal{C}_{12}[f](p)\varphi(p)dp = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \Big(\mathcal{R}_{p,p_1,p_2}[f] - \mathcal{R}_{p_1,p,p_2}[f] - \mathcal{R}_{p_2,p,p_1}[f]\Big)\varphi(p)\,\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2.$$

By switching the variables $p \leftrightarrow p_1$ and $p \leftrightarrow p_2$ in the second and third integral, respectively, the lemma follows. □

As a consequence, we obtain two important corollaries.

Corollary 2.1 (Conservation of momentum and energy). *Smooth solutions $f(t,p)$ of* (1.3)*, with initial data $f(0,p) = f_0(p)$ satisfy*

$$\int_{\mathbb{R}^3} f(t,p)p\mathrm{d}p = \int_{\mathbb{R}^3} f_0(p)p\mathrm{d}p \tag{2.1}$$

$$\int_{\mathbb{R}^3} f(t,p)\mathcal{E}(p)\mathrm{d}p = \int_{\mathbb{R}^3} f_0(p)\mathcal{E}(p)\mathrm{d}p \tag{2.2}$$

for all $t \geq 0$.

Proof. This follows from Lemma 2.1 by taking $\varphi(p) = p$ or $\mathcal{E}(p)$. □

We note that, unlike the classical Boltzmann equation, Equation (1.3) alone does not conserve the mass. However, the coupled system (1.3)–(1.4) does.

Lemma 2.2 (Conservation of mass). *Smooth solutions $f(t,p)$ and $n_c(t)$ of* (1.3) *and* (1.4)*, with initial data $f(0,p) = f_0(p)$ and $n_c(0) = n_0$, satisfy*

$$n_c(t) + \int_{\mathbb{R}^3} f(t,p)\mathrm{d}p = n_0 + \int_{\mathbb{R}^3} f_0(p)\mathrm{d}p \tag{2.3}$$

for all $t \geq 0$. In particular, the total mass and $n_c(t)$ are uniformly bounded.

Proof. It follows directly from (1.3) and (1.4) that

$$\frac{\mathrm{d}n_c}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,p)\mathrm{d}p = 0,$$

which gives the lemma. □

Lemma 2.3 (H-Theorem). *Smooth solutions $f(t,p)$ of (1.3) satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big[f\log f-(1+f)\log(1+f)\Big]\mathrm{d}p\le 0.$$

In addition, radially symmetric equilibria of (1.3) must have the following form:

$$f(p)=\frac{1}{e^{c\mathcal{E}(p)}-1}, \tag{2.4}$$

for some positive constant c.

Proof. First notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}[f\log f-(1+f)\log(1+f)]\,\mathrm{d}p=\int_{\mathbb{R}^3}\partial_t f\log\left(\frac{f}{f+1}\right)\mathrm{d}p.$$

On the other hand, we write

$$\begin{aligned}\int_{\mathbb{R}^3}\mathcal{C}_{12}[f](p)\varphi(p)\mathrm{d}p=&\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\mathcal{K}(p,p_1,p_2)(1+f)(1+f_1)(1+f_2)\\&\times\left(\frac{f_1}{1+f_1}\frac{f_2}{1+f_2}-\frac{f}{1+f}\right)[\varphi(p)-\varphi(p_1)\\&-\varphi(p_2)]\,\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2.\end{aligned}$$

Using Lemma 2.1 with $\varphi(p)=\log\left(\frac{f(p)}{f(p)+1}\right)$ and the fact that $(a-b)\log(\frac{a}{b})\ge 0$, with equality if and only if $a=b$, we obtain

$$\int_{\mathbb{R}^3}\mathcal{C}_{12}[f](p)\log\left(\frac{f(p)}{f(p)+1}\right)\mathrm{d}p\le 0.$$

This yields the claimed inequality in the H-theorem. In the case of equality, we have

$$\frac{f(p_1)}{f(p_1)+1}\frac{f(p_2)}{f(p_2)+1}-\frac{f(p)}{f(p)+1}=0,$$

or equivalently, setting $h(p)=\log\left(\frac{f(p)}{f(p)+1}\right)$, where h is radially symmetric,

$$h(p_1)+h(p_2)=h(p), \tag{2.5}$$

for all (p,p_1,p_2) so that $\mathcal{K}(p,p_1,p_2)\neq 0$. In particular, by view of the conservation laws (1.9), the function $h(p)$ satisfies $h(p_1+p_2)=h(p_1)+h(p_2)$, for all pairs $(p_1,p_2)\in\mathbb{R}^6$ so that

$$\mathcal{E}(p_1+p_2)=\mathcal{E}(p_1)+\mathcal{E}(p_2).$$

Define $\mathcal{E}^{-1}(a)$ to be the positive number ξ such that $\sqrt{\xi^2+\xi^4}=a$. We then have that

$$h\circ\mathcal{E}^{-1}(a+b)=h\circ\mathcal{E}^{-1}(a)+h\circ\mathcal{E}^{-1}(b)$$

for all p_1 and p_2 such that $|p_1| = \mathcal{E}^{-1}(a)$ and $|p_2| = \mathcal{E}^{-1}(b)$, with the notice that h is radially symmetric. Since a, b may take arbitrary values in $\mathbb{R}$, this yields $h \circ \mathcal{E}^{-1}(a) = -ca$ for some positive constant c and for all $a \geq 0$, or equivalently,

$$h(p) = -c\mathcal{E}(p)$$

for all $p \in \mathbb{R}^3$. This yields (2.4) and hence the H-theorem. □

3. Energy Surfaces

In this section, we study the surface integrals that arise in the collision operator, due to the conservation laws (1.9). Recall the collision kernel

$$\mathcal{K}(p, p_1, p_2) = |p|^\rho |p_1|^\rho |p_2|^\rho \delta(p - p_1 - p_2)\delta(\mathcal{E}(p) - \mathcal{E}(p_1) - \mathcal{E}(p_2)),$$

with $\delta(\cdot)$ being the Dirac delta function. Thus, the volume element $\mathcal{K}(p, p_1, p_2)dp_1dp_2$ or $\mathcal{K}(p_1, p, p_2)dp_1dp_2$ in $\mathbb{R}^6$ is in fact a two-dimensional surface element. Introduce the functions

$$H_p(w) := \mathcal{E}(w - p) + \mathcal{E}(w) - \mathcal{E}(p), \quad G_p(w) := \mathcal{E}(p + w) - \mathcal{E}(w) - \mathcal{E}(p), \tag{3.1}$$

with $\mathcal{E}(w) = |w|\sqrt{1 + |w|^2}$, and the corresponding energy surfaces, dictated by the conservation laws (1.9),

$$S_p := \Big\{w \in \mathbb{R}^d \ : \ H_p(w) = 0\Big\}, \qquad S'_p := \Big\{w \in \mathbb{R}^d \ : \ G_p(w) = 0\Big\}. \tag{3.2}$$

It follows that the collision operators satisfy

$$\begin{aligned} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{R}_{p,p_1,p_2}[f]\, \mathrm{d}p_1\mathrm{d}p_2 &= \int_{S_p} \mathcal{R}_{p,p-p_2,p_2}[f]\frac{\mathrm{d}\sigma(p_2)}{|\nabla H_p(p_2)|} \\ \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{R}_{p_1,p,p_2}[f]\, \mathrm{d}p_1\mathrm{d}p_2 &= \int_{S'_p} \mathcal{R}_{p+p_2,p,p_2}[f]\frac{\mathrm{d}\sigma(p_2)}{|\nabla G_p(p_2)|}. \end{aligned} \tag{3.3}$$

The next two lemmas provide estimates on these surface integrals.

Lemma 3.1. *Let S_p be defined as in (3.2). There are positive constants c_0, C_0 so that*

$$c_0|p|\min\{1, |p|\} \leq \int_{S_p} \frac{\mathrm{d}\sigma(w)}{|\nabla H_p(w)|} \leq C_0|p|\min\{1, |p|\}, \tag{3.4}$$

and for $\gamma \geq 0$,

$$\int_{S_p\cap B(0,\frac{1}{2}|p|)} |w - p|^\gamma |w|^\gamma \frac{\mathrm{d}\sigma(w)}{|\nabla H_p(w)|} \geq c_0|p|^{2\gamma+1} \ \min\{1, |p|\} \tag{3.5}$$

uniformly in $p \in \mathbb{R}^3$. In addition, for any function $F(\cdot)$, we have

$$\int_{S_p} F(|w|)\frac{\mathrm{d}\sigma(w)}{|\nabla H_p(w)|} \leq C_0 \int_0^{|p|} \min\{1, u\}F(u)\, \mathrm{d}u. \tag{3.6}$$

Proof. Recall that S_p is the surface consisting of w so that $H_p(w) = 0$ or

$$\mathcal{E}(w-p) + \mathcal{E}(w) = \mathcal{E}(p)$$

with $\mathcal{E}(w) = |w|\sqrt{1+|w|^2}$. It is clear that S_p is symmetric about $\frac{p}{2}$. We will prove that the surface S_p is of the form as illustrated in Figure 1. First, we note that $\{0, p\} \subset S_p$, and $|w| \le |p|$ and $|w-p| \le |p|$, for all $w \in S_p$, since $\mathcal{E}(w-p) \le \mathcal{E}(p)$, $\mathcal{E}(w) \le \mathcal{E}(p)$, and $\mathcal{E}(p)$ is a nonnegative increasing function.

For $w \in S_p$, we write $w = \alpha p + q$, with $p \cdot q = 0$. Since $|w| \le |p|$ and $|w-p| \le |p|$, $\alpha \in [0,1]$. In addition, recalling (3.1), we compute

$$\nabla_w H_p = (1+2|w-p|^2)\frac{w-p}{\mathcal{E}(w-p)} + (1+2|w|^2)\frac{w}{\mathcal{E}(w)}. \tag{3.7}$$

Thus, $q \cdot \nabla_w H_p > 0$. That is, $H_p(w)$ is strictly increasing in any direction that is orthogonal to p. This, together with the fact that $H_p(\alpha p) < 0$ for $\alpha \in (0,1)$ and $S_p \subset \overline{B(0,|p|)} \cap \overline{B(p,|p|)}$, proves that the surface S_p and the plane

$$\mathcal{P}_\alpha = \Big\{\alpha p + q, \quad p \cdot q = 0\Big\}$$

intersect for each $\alpha \in [0,1]$. In addition, $H_p(\alpha p + q)$ is a radial function in $|q|$, with $q \cdot p = 0$. This asserts that the intersection of S_p and $\mathcal{P}_\alpha$ is precisely the circle centered at αp and of a finite radius $|q_\alpha|$, for each $\alpha \in [0,1]$; see Figure 1.

Surface parametrization. Let $p^\perp$ be in $\mathcal{P}_0 = \{p \cdot q = 0\}$ and let e_θ be the unit vector in $\mathcal{P}_0$ so that the angle between $p^\perp$ and e_θ is θ. We parametrize S_p by

$$S_p = \Big\{w(\alpha,\theta) = \alpha p + |q_\alpha| e_\theta \ : \ \theta \in [0, 2\pi], \ \alpha \in [0,1]\Big\}. \tag{3.8}$$

Since $\partial_\theta e_\theta$ is orthogonal to both p and e_θ, we compute the surface area

$$\begin{aligned} \mathrm{d}\sigma(w) &= |\partial_\alpha w \times \partial_\theta w| \mathrm{d}\alpha \mathrm{d}\theta = |(p + \partial_\alpha |q_\alpha| e_\theta) \times |q_\alpha| \partial_\theta e_\theta| \mathrm{d}\alpha d\theta \\ &= |q_\alpha| |(p + \partial_\alpha |q_\alpha| e_\theta) \times \partial_\theta e_\theta| \mathrm{d}\alpha \mathrm{d}\theta \\ &= |q_\alpha| \sqrt{|p|^2 + |\partial_\alpha |q_\alpha||^2} \mathrm{d}\alpha \mathrm{d}\theta. \end{aligned} \tag{3.9}$$

To compute $\partial_\alpha |q_\alpha|$, we differentiate the equation $H_p(w_\alpha) = 0$, yielding

$$0 = \partial_\alpha w_\alpha \cdot \nabla_w H_p(w_\alpha) = |p| e_p \cdot \nabla_w H_p(w_\alpha) + \partial_\alpha |q_\alpha| e_\theta \cdot \nabla_w H_p(w_\alpha). \tag{3.10}$$

This implies that

$$\partial_\alpha |q_\alpha| = -|p| \frac{e_p \cdot \nabla_w H_p(w_\alpha)}{e_\theta \cdot \nabla_w H_p(w_\alpha)}. \tag{3.11}$$

Therefore, we compute

$$|p|^2 + |\partial_\alpha |q_\alpha||^2 = |p|^2 \frac{|e_p \cdot \nabla_w H_p|^2 + |e_\theta \cdot \nabla_w H_p|^2}{|e_\theta \cdot \nabla_w H_p|^2} = |p|^2 \frac{|\nabla_w H_p|^2}{|e_\theta \cdot \nabla_w H_p|^2},$$

and hence

$$\frac{\mathrm{d}\sigma(w)}{|\nabla_w H_p|} = \frac{|p||q_\alpha|\mathrm{d}\alpha\mathrm{d}\theta}{|e_\theta \cdot \nabla_w H_p|}. \tag{3.12}$$

Surface area. A direct computation yields

$$e_\theta \cdot \nabla_w H_p = |q_\alpha| \left[\frac{1+2|w-p|^2}{\mathcal{E}(w-p)} + \frac{1+2|w|^2}{\mathcal{E}(w)} \right]. \tag{3.13}$$

Recalling that $|w| \le |p|$ and $\mathcal{E}(w) = |w|\sqrt{1+|w|^2}$, and using the fact that $(1 + 2|p|^2)/\mathcal{E}(p)$ is decreasing in $|p|$, we compute

$$\frac{1+2|w-p|^2}{\mathcal{E}(w-p)} + \frac{1+2|w|^2}{\mathcal{E}(w)} \ge \frac{1+2|p|^2}{\mathcal{E}(p)} \ge \min\{1, |p|\}^{-1}.$$

This, (3.12), and (3.13) prove the upper bound on the surface area (3.4). As for the lower bound, it suffices to give an estimate for $\alpha \in [0, 1/2]$, on which $\alpha|p| \le |w| \le |w-p|$. Thus, in this case, we have

$$\frac{1+2|w-p|^2}{\mathcal{E}(w-p)} + \frac{1+2|w|^2}{\mathcal{E}(w)} \le 2\frac{1+2|\alpha p|^2}{\mathcal{E}(\alpha p)} \le C_0 \min\{1, \alpha|p|\}^{-1}.$$

The lower bound on the surface area (3.4) follows.

Surface area in $B(0, \frac{1}{2}|p|)$. In view of (3.11), (3.13), and the identity

$$e_p \cdot \nabla_w H_p = |p|\Big[(\alpha-1)\frac{1+2|w-p|^2}{\mathcal{E}(w-p)} + \alpha\frac{1+2|w|^2}{\mathcal{E}(w)}\Big], \tag{3.14}$$

we have $|\partial_\alpha|q_\alpha|| \le |p|^2|q_\alpha|^{-1}$, which implies

$$|\partial_\alpha|q_\alpha|^2| \le 2|p|^2.$$

Since

$$|w_\alpha|^2 = \alpha^2|p|^2 + |q_\alpha|^2,$$

we then have that

$$\partial_\alpha|w_\alpha|^2 = 2\alpha|p|^2 + \partial_\alpha|q_\alpha|^2.$$

Upon recalling that $\alpha \in [0, 1]$, we have that $|\partial_\alpha|w_\alpha|^2| \le 4|p|^2$ and

$$|w_\alpha|^2 = \int_0^\alpha \partial_\alpha|w_\alpha|^2 \, d\alpha' \le 4\alpha|p|^2,$$

which proves that $w_\alpha \in B(0, \frac{1}{2}|p|)$ for all $\alpha \in [0, \frac{1}{16}]$. The lower bound (3.5) follows.

Surface integral. Let us introduce the radial variable $u = |w_\alpha| = \sqrt{\alpha^2|p|^2 + |q_\alpha|^2}$. We compute $2udu = \partial_\alpha|w_\alpha|^2 d\alpha$. Hence, (3.12) yields

$$\frac{\mathrm{d}\sigma(w)}{|\nabla_w H_p|} = \frac{|p||q_\alpha|}{|e_\theta \cdot \nabla_w H_p|} \frac{2u\mathrm{d}u\mathrm{d}\theta}{\partial_\alpha|w_\alpha|^2}. \tag{3.15}$$

In view of (3.11), we compute

$$\partial_\alpha |w_\alpha|^2 = 2\alpha|p|^2 + 2|q_\alpha|\partial_\alpha|q_\alpha| = 2|p|\frac{\alpha|p|e_\theta\cdot\nabla_w H_p - |q_\alpha|e_p\cdot\nabla_w H_p}{e_\theta\cdot\nabla_w H_p},$$

in which, using (3.13) and (3.14), we compute

$$\alpha|p|e_\theta\cdot\nabla_w H_p - |q_\alpha|e_p\cdot\nabla_w H_p = |p||q_\alpha|\frac{1+2|w-p|^2}{\mathcal{E}(w-p)}.$$

Combining, we obtain

$$\frac{\mathrm{d}\sigma(w)}{|\nabla_w H_p|} = \frac{\mathcal{E}(w-p)u\mathrm{d}u\mathrm{d}\theta}{|p|(1+2|w-p|^2)} \le C_0\min\{1,u\}\mathrm{d}u\mathrm{d}\theta, \tag{3.16}$$

upon recalling that $|w|\le|p|$, $|w-p|\le|p|$ for $w\in S_p$ and $\mathcal{E}(w) = |w|\sqrt{1+|w|^2}$. This proves (3.6). □

Lemma 3.2. *Let S'_p be defined as in (3.2). There are positive constants c_0, C_0 so that for any $F(\cdot)$,*

$$\int_{S'_p} F(|w|)\frac{\mathrm{d}\sigma(w)}{|\nabla G_p(w)|} \le C_0|p|^{-1}\int_0^\infty F(u)\,u\mathrm{d}u, \tag{3.17}$$

and

$$\int_{S'_p} F(|w|)\frac{\mathrm{d}\sigma(w)}{|\nabla G_p(w)|} \ge c_0\min\left\{1,|p|^{-1}\right\}\int_0^\infty F(u)\,u\mathrm{d}u \tag{3.18}$$

for all $p\in\mathbb{R}^3$.

Proof. Recall that S'_p is the surface that consists of w satisfying $\mathcal{E}(p+w) = \mathcal{E}(w)+\mathcal{E}(p)$. First, we compute

$$\begin{aligned} 0 &= \mathcal{E}(p+w)^2 - \Big(\mathcal{E}(p)+\mathcal{E}(w)\Big)^2 \\ &= |p+w|^2 + |p+w|^4 - (|p|^2+|w|^2) - (|p|^4+|w|^4) - 2\mathcal{E}(p)\mathcal{E}(w) \\ &= 2w\cdot p + 2w\cdot p(|p|^2+|w|^2+|p+w|^2) + 2|p|^2|w|^2 - 2\mathcal{E}(p)\mathcal{E}(w). \end{aligned} \tag{3.19}$$

It is clear that $|p|^2|w|^2 < \mathcal{E}(p)\mathcal{E}(w)$. This proves that if $w\in S'_p\backslash\{0\}$, then $w\cdot p > 0$.

Next, recall $G_p(w) := \mathcal{E}(p+w) - \mathcal{E}(w) - \mathcal{E}(p)$, with $\mathcal{E}(p) = |p|\sqrt{1+|p|^2}$. It follows that $G_p(\alpha p) > 0$ for $\alpha > 0$. In addition, we compute

$$\nabla_w G_p = \frac{w+p}{|w+p|}\mathcal{E}'(w+p) - \frac{w}{|w|}\mathcal{E}'(w),$$

and thus the directional derivative of G_p at $w_\alpha = \alpha p + q$, with $p\cdot q = 0$, in the direction of $q\neq 0$ satisfies

$$q\cdot\nabla_w G_p = |q|^2\left[\frac{\mathcal{E}'(p+w_\alpha)}{|p+w_\alpha|} - \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|}\right] < 0,$$

in which we used the fact that $\mathcal{E}'(p)/|p|$ is strictly decreasing in $|p|$. By a view of (3.19), the sign of $G_p(w)$, with $w_\alpha = \alpha p + q$, is the same as that of

$$\begin{aligned}
&\alpha|p|^2\Big(1 + (|p|^2 + |w_\alpha|^2 + |p + w_\alpha|^2)\Big) + |p|^2|w_\alpha|^2 - \mathcal{E}(p)\mathcal{E}(w) \\
&\quad = \alpha|p|^2\Big(1 + 2(|p|^2 + \alpha|p|^2 + |w_\alpha|^2)\Big) \\
&\qquad - \frac{(|p|^2 + |p|^4)(|w_\alpha|^2 + |w_\alpha|^4) - |p|^4|w_\alpha|^4}{\sqrt{|p|^2 + |p|^4}\sqrt{|w_\alpha|^2 + |w_\alpha|^4} + |p|^2|w_\alpha|^2} \\
&\quad = \alpha|p|^2\Big(1 + 2(|p|^2 + \alpha|p|^2 + |w_\alpha|^2)\Big) \\
&\qquad - \frac{|w_\alpha|^2|p|^2 + |w_\alpha|^2|p|^4 + |p|^2|w_\alpha|^4}{\sqrt{|p|^2 + |p|^4}\sqrt{|w_\alpha|^2 + |w_\alpha|^4} + |p|^2|w_\alpha|^2}.
\end{aligned}$$

This yields that $G_p(\alpha p + q) < 0$, as long as

$$\alpha < \frac{(1 + |p|^2) + |w_\alpha|^2}{\sqrt{|p|^2 + |p|^4}\sqrt{\frac{1}{|w_\alpha|^2} + 1} + |p|^2} \frac{1}{\Big(1 + 2(|p|^2 + \alpha|p|^2 + |w_\alpha|^2)\Big)}.$$

Taking $|q| \to \infty$ (and so $|w_\alpha| \to \infty$), we obtain that $\lim_{q\to\infty} G_p(\alpha p + q) < 0$ if and only if

$$\alpha < \alpha_p := \frac{1}{2} \frac{1}{|p|^2 + \sqrt{|p|^2 + |p|^4}}. \tag{3.20}$$

In particular, we note that

$$\alpha_p|p|(1 + |p|) \le C_0, \qquad \forall\, p \in \mathbb{R}^3 \tag{3.21}$$

for some positive constant C_0. Hence, for positive values of α satisfying (3.20), by monotonicity, $G_p(\alpha p) > 0$, and the fact that $G_p(\alpha p + q)$ is radial in $|q|$, there is a unique $|q_\alpha|$ so that $G_p(\alpha p + q) = 0$, for all $|q| = |q_\alpha|$. For $\alpha > \alpha_p$, $G_p(\alpha p + q) > 0$, for all q, with $q \cdot p = 0$.

Surface parametrization. To summarize, the surface S'_p can be described as follows (see Figure 2):

$$S'_p = \Big\{w(\alpha, \theta) = \alpha p + |q_\alpha| e_\theta \;:\; \alpha \in [0, \alpha_p),\ \theta \in [0, 2\pi]\Big\}, \tag{3.22}$$

in which α_p and $|q_\alpha|$ are defined as above and e_θ denotes the unit vector rotating around p and on the orthogonal plane to p.

Surface integral. Recalling (3.15), the surface integral is computed by

$$\frac{d\sigma(w)}{|\nabla_w G_p|} = \frac{|p||q_\alpha|}{|e_\theta \cdot \nabla_w G_p|} \frac{2u\,\mathrm{d}u\,\mathrm{d}\theta}{\partial_\alpha |w_\alpha|^2}, \tag{3.23}$$

with $u = |w_\alpha|$, where, as done in the previous case, we compute

$$\begin{aligned}
\frac{1}{2|p|} e_\theta \cdot \nabla_w G_p \partial_\alpha |w_\alpha|^2 &= \alpha|p| e_\theta \cdot \nabla_w G_p - |q_\alpha| e_p \cdot \nabla_w G_p \\
&= -|p||q_\alpha| \frac{1 + 2|w + p|^2}{\mathcal{E}(w + p)}.
\end{aligned}$$

Combining, we obtain

$$\frac{\mathrm{d}\sigma(w)}{|\nabla_w G_p|} = \frac{\mathcal{E}(w+p)u\mathrm{d}u\mathrm{d}\theta}{|p|(1+2|w+p|^2)}. \tag{3.24}$$

Recalling $\mathcal{E}(w) = |w|\sqrt{1+|w|^2}$, we have

$$\frac{\mathcal{E}(w+p)|w|}{|p|(1+2|w+p|^2)} \le |w||p|^{-1}.$$

On the other hand, by considering $|p| \le 1$ and $|p| \ge 1$ and using the fact that $|w| + |p| \le 2|w+p|$ (on S'_p), we have

$$\frac{\mathcal{E}(w+p)|w|}{|p|(1+2|w+p|^2)} \ge c_0|w|\min\{1, |p|^{-1}\}.$$

This yields the upper and lower bounds on the surface integral. □

4. Moment Estimates

In this section, we shall derive estimates on the energy moment of nonnegative solutions of (1.3). In what follows, we take initial data $f_0(p) = f_0(|p|)$ with finite mass and energy. Thus, thanks to the conservation of mass (2.3) and energy (2.2), mass and energy remain finite for all times. In addition, we recall that $n_c(t)$ remains bounded above and below away from zero.

Proposition 4.1. *Let $f_0(p) = f_0(|p|) \ge 0$ have finite mass and energy. Then, for any $\tau > 0$, nonnegative radial solutions $f(t,p) = f(t,|p|)$ of (1.3) with initial data $f_0(p)$ satisfy*

$$\sup_{t\in[\tau,\infty)} \int_{\mathbb{R}^3} f(t,p)\mathcal{E}^2(p)\mathrm{d}p < +\infty. \tag{4.1}$$

Proof. Take $\varphi = \mathcal{E}^2(p)$ to be the test function in Lemma 2.1. We obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f\mathcal{E}^2\mathrm{d}p \;&=\; n_c(t)\iiint_{\mathbb{R}^9} \mathcal{R}_{p,p_1,p_2}[f]\Big(\mathcal{E}^2(p) \\ &\qquad - \mathcal{E}^2(p_1) - \mathcal{E}^2(p_2)\Big)\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2.\end{aligned}$$

In view of the Dirac delta functions in the collision kernel (1.6), the integral is on the surface dictated by the conditions $p = p_1 + p_2$ and $\mathcal{E}(p) = \mathcal{E}(p_1) + \mathcal{E}(p_2)$. In particular, on the surface, $\mathcal{E}^2(p) - \mathcal{E}^2(p_1) - \mathcal{E}^2(p_2) = 2\mathcal{E}(p_1)\mathcal{E}(p_2)$. Thus, upon recalling that $f \ge 0$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f\mathcal{E}(p)^2\mathrm{d}p &= 2n_c(t)\iiint_{\mathbb{R}^9} \mathcal{R}_{p,p_1,p_2}[f]\mathcal{E}(p_1)\mathcal{E}(p_2)\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2 \\ &= 2n_c(t)\iiint_{\mathbb{R}^9} \mathcal{K}(p,p_1,p_2)\Big(f_1f_2 - (1+f_1+f_2)f\Big)\mathcal{E}(p_1)\mathcal{E}(p_2)\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2 \\ &\le 2n_c(t)\iiint_{\mathbb{R}^9} \mathcal{K}(p,p_1,p_2)\Big(f_1f_2 - f\Big)\mathcal{E}(p_1)\mathcal{E}(p_2)\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2.\end{aligned}$$

Let us set

$$J_1 := 2n_c(t)\iiint_{\mathbb{R}^9}\mathcal{K}(p,p_1,p_2)f_1f_2\mathcal{E}(p_1)\mathcal{E}(p_2)\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2$$
$$J_2 := -2n_c(t)\iiint_{\mathbb{R}^9}\mathcal{K}(p,p_1,p_2)f\mathcal{E}(p_1)\mathcal{E}(p_2)\mathrm{d}p\mathrm{d}p_1\mathrm{d}p_2. \tag{4.2}$$

We first write J_1, J_2 in term of surface integrals. Recalling

$$\mathcal{K}(p,p_1,p_2) = |p|^\rho|p_1|^\rho|p_2|^\rho\delta(p-p_1-p_2)\delta(\mathcal{E}(p)-\mathcal{E}(p_1)-\mathcal{E}(p_2))$$

for $\rho \geq 1$, and following (3.3), we estimate

$$\begin{aligned} J_2 &= -2n_c(t)\iint_{\mathbb{R}^6}\mathcal{K}(p,p_1,p-p_1)f\mathcal{E}(p_1)\mathcal{E}(p-p_1)\mathrm{d}p\mathrm{d}p_1 \\ &= -2n_c(t)\int_{\mathbb{R}^3}\Big(\int_{S_p}|p_1|^\rho|p-p_1|^\rho\mathcal{E}(p_1)\mathcal{E}(p-p_1)\frac{d\sigma(p_1)}{|\nabla H_p(p_1)|}\Big)|p|^\rho f\mathrm{d}p. \end{aligned}$$

Recalling $\mathcal{E}(p) \geq \frac{1}{2}(|p|+|p|^2)$ and using (3.5) in Lemma 3.1, we estimate

$$\begin{aligned} &|p|^\rho\int_{S_p}|p_1|^\rho|p-p_1|^\rho\mathcal{E}(p_1)\mathcal{E}(p-p_1)\frac{d\sigma(p_1)}{|\nabla H_p(p_1)|} \\ &\quad\gtrsim (|p|^{3\rho+3}+|p|^{3\rho+5})\ \min\{1,|p|\} \gtrsim |p|^{3\rho+5}. \end{aligned} \tag{4.3}$$

This proves

$$J_2 \leq -\theta_0\int_{\mathbb{R}^3}|p|^{3\rho+5}f\mathrm{d}p \tag{4.4}$$

for some positive constant θ_0.

Next, we estimate the integral J_1 in (4.2). Again following (3.3), we write

$$\begin{aligned} J_1 &= 2n_c(t)\iint_{\mathbb{R}^6}\mathcal{K}(p_1+p_2,p_1,p_2)f_1f_2\mathcal{E}(p_1)\mathcal{E}(p_2)\mathrm{d}p_1\mathrm{d}p_2 \\ &\lesssim \int_{\mathbb{R}^3}\int_{S'_{p_1}}|p_1+p_2|^\rho|p_1|^\rho|p_2|^\rho f_1f_2\mathcal{E}(p_1)\mathcal{E}(p_2)\frac{\mathrm{d}\sigma(p_2)\mathrm{d}p_1}{|\nabla G_{p_1}(p_2)|} \\ &\lesssim \int_{\mathbb{R}^3}|p_1|^\rho f_1\mathcal{E}(p_1)\Big(\int_{S'_{p_1}}(|p_1|^\rho+|p_2|^\rho)|p_2|^\rho f_2\mathcal{E}(p_2)\frac{\mathrm{d}\sigma(p_2)}{|\nabla G_{p_1}(p_2)|}\Big)\mathrm{d}p_1. \end{aligned}$$

By Lemma 3.2, and the fact that f is radial, the surface integral is estimated by

$$\begin{aligned} &\int_{S'_{p_1}}(|p_1|^\rho+|p_2|^\rho)|p_2|^\rho f_2\mathcal{E}(p_2)\frac{\mathrm{d}\sigma(p_2)}{|\nabla G_{p_1}(p_2)|} \\ &\quad\lesssim C|p_1|^{-1}\int_{\mathbb{R}_+}(|p_1|^\rho+|p_2|^\rho)|p_2|^{\rho+1}f_2\mathcal{E}(p_2)\mathrm{d}(|p_2|) \\ &\quad\lesssim C|p_1|^{-1}\int_{\mathbb{R}^3}(|p_1|^\rho+|p_2|^\rho)|p_2|^{\rho-1}f_2\mathcal{E}(p_2)\mathrm{d}p_2. \end{aligned}$$

Thus, upon recalling $\mathcal{E}(p) = |p|\sqrt{1+|p|^2}$, we obtain

$$
\begin{aligned}
J_1 &\lesssim \iint_{\mathbb{R}^6} (|p_1|^\rho + |p_2|^\rho)|p_1|^{\rho-1}|p_2|^{\rho-1} f_1\mathcal{E}(p_1) f_2\mathcal{E}(p_2) dp_1 dp_2 \\
&\lesssim \Big(\int_{\mathbb{R}^3} |p_1|^{2\rho-1}\mathcal{E}(p_1) f_1 \,\mathrm{d}p_1\Big)\Big(\int_{\mathbb{R}^3} |p_2|^{\rho-1}\mathcal{E}(p_2) f_2 \,\mathrm{d}p_2\Big) \\
&\lesssim \Big(\int_{\mathbb{R}^3} |p|^{2\rho}(1+|p|) f \,\mathrm{d}p\Big)\Big(\int_{\mathbb{R}^3} |p|^{\rho}(1+|p|) f \,\mathrm{d}p\Big).
\end{aligned}
$$

Since $\rho \geq 1$, we note that $|p|^{2\rho} \leq |p|^{2\rho+1} + |p|^2$ and $|p|^\rho \leq |p|^{\rho+1} + |p|$. The above yields

$$
\begin{aligned}
J_1 &\lesssim \Big(\int_{\mathbb{R}^3} (|p|^2 + |p|^{2\rho+1}) f \,\mathrm{d}p\Big)\Big(\int_{\mathbb{R}^3} (|p| + |p|^{\rho+1}) f \,\mathrm{d}p\Big) \\
&\lesssim \int_{\mathbb{R}^3} \mathcal{E}^2(p) f \,\mathrm{d}p + \int_{\mathbb{R}^3} |p|^{2\rho+1} f \,\mathrm{d}p \\
&\quad + \Big(\int_{\mathbb{R}^3} |p|^{2\rho+1} f \,dp\Big)\Big(\int_{\mathbb{R}^3} |p|^{\rho+1} f \,\mathrm{d}p\Big)
\end{aligned}
$$

upon using $|p| \leq \mathcal{E}(p)$ and the conservation of momentum and energy.

To bound J_1 in term of J_2, we note the following interpolation inequality:

$$
\int_{\mathbb{R}^3} |p|^r f \,\mathrm{d}p \leq C_0 \left(\int_{\mathbb{R}^3} |p|^{3\rho+5} f \mathrm{d}p\right)^{\frac{r-2}{3\rho+3}} \left(\int_{\mathbb{R}^3} |p|^2 f \mathrm{d}p\right)^{\frac{3\rho+5-r}{3\rho+3}}, \tag{4.5}
$$

for any r such that $2 \leq r \leq 3\rho + 5$. Applying this inequality into J_1, we obtain

$$
\begin{aligned}
J_1 &\lesssim \int_{\mathbb{R}^3} \mathcal{E}^2(p) f \,\mathrm{d}p + \int_{\mathbb{R}^3} |p|^{2\rho+1} f \,\mathrm{d}p \\
&\quad + \left(\int_{\mathbb{R}^3} |p|^{3\rho+5} f \mathrm{d}p\right)^{\frac{3\rho-2}{3\rho+3}} \left(\int_{\mathbb{R}^3} |p|^2 f \mathrm{d}p\right)^{\frac{3\rho+8}{3\rho+3}}.
\end{aligned}
$$

Now, using the Young's inequality $ab \leq \epsilon a^r + C_\epsilon b^{r'}$, with $1/r + 1/r' = 1$, and the fact that $\int_{\mathbb{R}^3} |p|^2 f \,dp$ is bounded by a constant and by the second energy moment, we obtain

$$
J_1 \leq \epsilon \int_{\mathbb{R}^3} |p|^{3\rho+5} f \mathrm{d}p + C_\epsilon \int_{\mathbb{R}^3} \mathcal{E}^2(p) f \,\mathrm{d}p \tag{4.6}
$$

for any positive ϵ.

Combining (4.4) and (4.6), we have obtained

$$
\frac{d}{dt}\int_{\mathbb{R}^3} f\mathcal{E}^2 \mathrm{d}p \leq -\theta_1 \int_{\mathbb{R}^3} |p|^{3\rho+5} f \mathrm{d}p + C_1 \int_{\mathbb{R}^3} f\mathcal{E}^2 \mathrm{d}p. \tag{4.7}
$$

Let us next expand the first term on the right. Precisely, using (4.5) and the fact that $\int_{\mathbb{R}^3} |p|^2 f \, \mathrm{d}p$ is bounded, we estimate

$$\begin{aligned}\int_{\mathbb{R}^3} \mathcal{E}^2(p) f \, \mathrm{d}p &\le \int_{\mathbb{R}^3} |p|^2 f \, \mathrm{d}p + \int_{\mathbb{R}^3} |p|^4 f \, \mathrm{d}p \\ &\lesssim \Big(\int_{\mathbb{R}^3} |p|^2 f \, \mathrm{d}p \Big)^{\frac{2}{3\rho+3}} + \Big(\int_{\mathbb{R}^3} |p|^{3\rho+5} f \mathrm{d}p \Big)^{\frac{2}{3\rho+3}} \\ &\lesssim \Big(\int_{\mathbb{R}^3} \mathcal{E}^2(p) f \, \mathrm{d}p \Big)^{\frac{2}{3\rho+3}} + \Big(\int_{\mathbb{R}^3} |p|^{3\rho+5} f \mathrm{d}p \Big)^{\frac{2}{3\rho+3}} .\end{aligned}$$

This yields

$$-\int_{\mathbb{R}^3} |p|^{3\rho+5} f \mathrm{d}p \le -\theta_2 \Big(\int_{\mathbb{R}^3} \mathcal{E}^2(p) f \, \mathrm{d}p \Big)^{\frac{3\rho+3}{2}} + C_2 \int_{\mathbb{R}^3} \mathcal{E}^2(p) f \, \mathrm{d}p. \quad (4.8)$$

Hence, (4.7) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \le C_3 \int_{\mathbb{R}^3} f \mathcal{E}^2 \mathrm{d}p \left[1 - \theta_3 \Big(\int_{\mathbb{R}^3} f \mathcal{E}^2 \mathrm{d}p \Big)^{\frac{3\rho+1}{2}} \right] \quad (4.9)$$

for some positive constants C_3, θ_3. Thus, since $f \ge 0$, the standard ODE argument applying to the differential inequality (4.9) yields at once the boundedness of $\int_{\mathbb{R}^3} f \mathcal{E}^2 dp$; for instance, there holds

$$\int_{\mathbb{R}^3} f(t,p) \mathcal{E}^2 \mathrm{d}p \lesssim \max \left\{ \frac{1}{\theta_3^{\frac{2}{3\rho+1}}}, \int_{\mathbb{R}^3} f(\tau,p) \mathcal{E}^2 \mathrm{d}p \right\}$$

for all $t \ge \tau$. The proposition follows. □

Remark 4.1. Following lines similar to the above proof, we can in fact show that energy moments at any order are created and propagated in positive times as obtained in Proposition 4.1 for the second-order energy moment. As a result, we could then drop the restriction $\rho \le \frac{5}{2}$ in the transition probability kernel (1.8), used in (5.10). However, we skip the details as the result will not be used in this paper.

5. Uniform Lower Bound

In this section, we shall prove our main theorem, Theorem 1.1. We recall that there are positive constants n_1, n_2 so that the density function $n_c(t)$ satisfies $n_1 \le n_c(t) \le n_2$ for all $t \ge 0$. Let us write the collision operator as follows:

$$\mathcal{C}_{12}[f] = Q_{\text{gain}}[f] - Q_{\text{loss}}[f], \quad (5.1)$$

where the Gain and Loss operators are defined by

$$\begin{aligned}Q_{\rm gain}[f] &:= \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{K}(p,p_1,p_2) f_1 f_2 \mathrm{d}p_1\mathrm{d}p_2\\ &\quad + 2\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{K}(p_1,p,p_2)(1+f+f_2)f_1 \,\mathrm{d}p_1\mathrm{d}p_2\\ Q_{\rm loss}[f] &:= f\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{K}(p,p_1,p_2)(1+2f_2)\mathrm{d}p_1\mathrm{d}p_2\\ &\quad + 2f\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{K}(p_1,p,p_2) f_2 \,\mathrm{d}p_1\mathrm{d}p_2.\end{aligned}$$

For convenience, we also write

$$Q_{\rm loss}[f] = f\mathcal{L}[f], \tag{5.2}$$

nothing that $\mathcal{L}[f]$ is usually called the collision frequency.

Lemma 5.1. *Suppose that $F(p) \le G(|p|)$, for some radially symmetric function G with*

$$\mathcal{M} = \int_{\mathbb{R}_+} G(u)\ (u^{1+\rho} + u^{1+2\rho})\mathrm{d}u < \infty.$$

Then, there holds

$$\mathcal{L}[F](p) \le C_0\mathcal{M}(1+|p|^{2\rho}) + C_0|p|^{3\rho+1} \tag{5.3}$$

for some positive universal constant C_0.

Proof. We first write the collision integrals in term of surface integrals. Following (3.3), we have

$$\begin{aligned}\mathcal{L}[F] &= \int_{S_p} |p|^\rho |p-p_2|^\rho |p_2|^\rho (1+2F_2)\ \frac{\mathrm{d}\sigma(p_2)}{|\nabla H_p(p_2)|}\\ &\quad + 2\int_{S'_p} |p+p_2|^\rho |p|^\rho |p_2|^\rho F_2\ \frac{\mathrm{d}\sigma(p_2)}{|\nabla G_p(p_2)|}.\end{aligned}$$

Consider the surface integral over S_p. Recall that that $|p_2| \le |p|$ and $|p-p_2| \le |p|$ on S_p. Hence, using Lemma 3.1, we estimate

$$\begin{aligned}&\int_{S_p} |p|^\rho |p-p_2|^\rho |p_2|^\rho (1+2F_2)\ \frac{\mathrm{d}\sigma(p_2)}{|\nabla H_p(p_2)|}\\ &\lesssim |p|^{2\rho}\int_{S_p} (1+2G(|p_2|))|p_2|^\rho\ \frac{\mathrm{d}\sigma(p_2)}{|\nabla H_p(p_2)|}\\ &\lesssim |p|^{2\rho}\int_0^{|p|} (1+G(u))\ \min\{1,u\}u^\rho \mathrm{d}u\\ &\lesssim |p|^{3\rho+1} + |p|^{2\rho}\int_0^{|p|} G(u)u^{\rho+1}\mathrm{d}u,\end{aligned}$$

which is bounded by $C_0|p|^{3\rho+1} + C_0\mathcal{M}|p|^{2\rho}$. Next, we check the integral on S'_p. Lemma 3.2 yields

$$\begin{aligned}\int_{S'_p} &|p+p_2|^\rho |p|^\rho |p_2|^\rho F_2 \, \frac{\mathrm{d}\sigma(p_2)}{|\nabla G_p(p_2)|}\\ &\lesssim \int_{S'_p} (|p|^\rho + |p_2|^\rho)|p|^\rho |p_2|^\rho G(|p_2|) \, \frac{\mathrm{d}\sigma(p_2)}{|\nabla G_p(p_2)|}\\ &\lesssim |p|^{\rho-1} \int_0^\infty (|p|^\rho + u^\rho) G(u) u^{\rho+1} \mathrm{d}u,\end{aligned}$$

which is bounded by $C_0\mathcal{M}(|p|^{\rho-1} + |p|^{2\rho-1}) \le C_0\mathcal{M}(1 + |p|^{2\rho})$. The lemma follows. □

Lemma 5.2. *Let $\delta, \theta > 0$, and F be any nonnegative smooth function so that $F(p) \ge \theta$ on $B_\delta := \{|p| \le \delta\}$. Then, there exists a universal constant $c_0 > 0$ such that*

$$Q_{\text{gain}}[F](p) \ge c_0 |p|^{3\rho+1} \min\{1, |p|\} \theta^2 \tag{5.4}$$

for all $p \in B_{\sqrt{2}\delta}$.

Proof. By definition (5.1) and the assumption on the lower bound on F, we have

$$\begin{aligned} Q_{\text{gain}}[F](p) = &\int_{S_p} K(p, p-p_2, p_2) F(p-p_2) F(p_2) \, \mathrm{d}\sigma(p_2)\\ &+ 2\int_{S'_p} K(p+p_2, p, p_2) F(p+p_2) \Big(F(p) + F(p_2) + 1 \Big) \, \mathrm{d}\sigma(p_2)\\ \gtrsim &\int_{S_p} \mathcal{K}(p, p-p_2, p_2) F(p-p_2) F(p_2) \, \mathrm{d}\sigma(p_2)\\ \gtrsim & |p|^\rho \theta^2 \int_{S_p \cap B(0,\delta) \cap B(p,\delta)} |p-p_2|^\rho |p_2|^\rho \mathrm{d}\sigma(p_2),\end{aligned}$$

in which we note again that $p_2, p - p_2$ are both in B_δ, thanks to the monotonicity of the energy function $\mathcal{E}(p)$.

To proceed, we consider three cases. First, take $p \in B(0, \delta) \setminus B(0, \frac{\delta}{2})$. In this case, $B(\frac{p}{2}, \frac{|p|}{2}) \subset B(0, \delta) \cap B(p, \delta)$, and so we can estimate

$$\begin{aligned} Q_{\text{gain}}[F](p) &\gtrsim |p|^\rho \theta^2 \int_{S_p \cap B(\frac{p}{2}, \frac{|p|}{2})} |p-p_2|^\rho |p_2|^\rho \mathrm{d}\sigma(p_2)\\ &\gtrsim |p|^{3\rho+1} \min\{1, |p|\} \theta^2\end{aligned}$$

for some positive constants c_0, c_1, thanks to the lower bound (3.5), with $\gamma = 1$.

Next, for $p \in B(0, \frac{\delta}{2})$, we note that $B(0, \frac{\delta}{2}) \subset B(0, \delta) \cap B(p, \delta)$. Hence, in this case, we have, by the lower bound (3.5),

$$Q_{\text{gain}}[F](p) \gtrsim |p|^\rho \theta^2 \int_{S_p \cap B(0,\frac{\delta}{2})} |p-p_2|^\rho |p_2|^\rho d\sigma(p_2) \gtrsim |p|^{3\rho+1} \min\{1,|p|\}\theta^2.$$

The lemma is proved for $|p| \le \frac{\delta}{2}$.

Finally, we consider the case when $p \in B(0,\sqrt{2}\delta) \setminus B(0,\delta)$. In this case, we check that $S_p \cap B(0,\delta) \cap B(p,\delta)$ has positive surface area. Indeed, let D_p be the disk that is centered at $\frac{p}{2}$, of radius $\sqrt{\delta^2 - \frac{|p|^2}{4}}$, and is on the plane orthogonal to p. Let x be a point on the boundary of D_p, then $|x - p/2| = \sqrt{\delta^2 - \frac{|p|^2}{4}}$ and $x - p/2$ is orthogonal to p. As a consequence, $|x|^2 = |x-p/2|^2 + |p/2|^2 = \delta^2$ and $|x-p|^2 = |x-p/2|^2 + |p/2|^2$. It is clear that D_p belongs to the intersection $B(0,\delta) \cap B(p,\delta)$ and, since $\sqrt{\delta^2 - \frac{|p|^2}{4}} \ge \frac{|p|}{2}$, the surface S_p crosses the interior of D_p. This proves that $S_p \cap B(0,\delta) \cap B(p,\delta)$ is non–empty. Since $B(0,\delta) \cap B(p,\delta)$ has positive Lebesgue measure, the surface area of $S_p \cap B(0,\delta) \cap B(p,\delta)$ is bounded below from zero by a constant times $|p|$, since any geodesic on the surface starting from 0 to p has a greater length than $|p|$. We can then compute

$$\begin{aligned} Q_{\text{gain}}[F](p) &\gtrsim |p|^\rho \theta^2 \int_{S_p \cap B(0,\delta) \cap B(p,\delta)} |p-p_2|^\rho |p_2|^\rho \mathrm{d}\sigma(p_2) \\ &\gtrsim |p|^{3\rho+1} \min\{1,|p|\}\theta^2, \end{aligned}$$

due to the lower bound (3.5). This completes the proof of the lemma. □

Lemma 5.3. *Let $\delta, \theta > 0$. Suppose that initial data $f_0(p) \ge \theta$ on B_δ, where $B_\delta = \{|p| \le \delta\}$. Let $f(t,p)$ be a solution to (1.3) so that $f(t,p) \le G(t,|p|)$ for all $t \ge 0$ and for some radially symmetric function G so that*

$$\mathcal{M}(t) = \sup_{0 \le s \le t} \int_{\mathbb{R}_+} G(s,u)(|u|^{\rho+1} + |u|^{2\rho+1})\mathrm{d}u < \infty. \tag{5.5}$$

Then, there holds the following uniform lower bound:

$$f(t,p) \ge C_0 t e^{-t\mathcal{M}(t)L_*(\delta)} |p|^{3\rho+1} \min\{1,|p|\}\theta^2, \qquad \forall\, t \ge 0, \tag{5.6}$$

for all $p \in B_{\sqrt{2}\delta}$,

$$L_*(\delta) := c_0(1 + \delta^{3\rho+1}).$$

Here, c_0, C_0 are some universal positive constants independent of $\mathcal{M}$, δ, θ and p.

Proof. Using Lemma 5.1, with $F = f(t,p)$, we obtain

$$\partial_t f(t,p) + L_0(t,|p|) f(t,p) \ge n_1 Q_{\text{gain}}[f](t,p), \tag{5.7}$$

with $L_0(t,|p|) = C_0\mathcal{M}(t)(1+|p|^{2\rho}) + C_0|p|^{3\rho+1}$. Note that $\mathcal{M}(t)$ and hence $L_0(t,|p|)$ are increasing in t. Using the monotonicity and applying the Duhamel's representation to (5.7), we obtain

$$f(t,p) \ge f_0(p) e^{-\int_0^t L_0(s,|p|)ds} + n_1 \int_0^t e^{-\int_\tau^t L_0(s,|p|)ds} Q_{\text{gain}}[f](\tau,p)\mathrm{d}\tau \tag{5.8}$$

for all $t \geq 0$. Since $Q_{\text{gain}}[f](p) \geq 0$ and $L_0(t, |p|)$ is an increasing function in t, it follows that for $p \in B_\delta$, (5.8) yields

$$f(t,p) \quad \geq \quad f_0(p)e^{-tL_0(t,|p|)} \quad \geq \quad \theta e^{-tL_0(t,\delta)}, \qquad t \geq 0. \tag{5.9}$$

Next, for each fixed time $t \geq 0$, we now apply Lemma 5.2 for $F = f(t, p)$, with the new lower bound (5.9) on B_δ, yielding

$$Q_{\text{gain}}[f](t,p) \geq C_0|p|^{3\rho+1}\min\{1,|p|\}\theta^2 e^{-2tL_0(t,\delta)}$$

for all $p \in B_{\sqrt{2}\delta}$. Putting this into (5.8), we obtain

$$\begin{aligned}
f(t,p) &\geq \int_0^t e^{-\int_\tau^t L_0(s,|p|)\mathrm{d}s} Q_{\text{gain}}[f](\tau,p)\mathrm{d}\tau \\
&\geq \int_0^t e^{-\int_\tau^t L_0(t,|p|)\mathrm{d}s} Q_{\text{gain}}[f](\tau,p)\mathrm{d}\tau \\
&\geq \int_0^t e^{-(t-\tau)L_0(t,\delta)} Q_{\text{gain}}[f](\tau,p)\mathrm{d}\tau \\
&\gtrsim |p|^{3\rho+1}\min\{1,|p|\}\theta^2 \int_0^t e^{-(t-\tau)L_0(t,\delta)} e^{-2\tau L_0(t,\delta)}\mathrm{d}\tau \\
&\gtrsim |p|^{3\rho+1}\min\{1,|p|\}\theta^2 e^{-2tL_0(t,\delta)} t \\
&\gtrsim |p|^{3\rho+1}\min\{1,|p|\}\theta^2 e^{-tC_0\mathcal{M}(t)L_*(\delta)} t.
\end{aligned}$$

This completes the proof of (5.6). □

5.1. Proof of Theorem 1.1

We are now ready to give the proof of Theorem 1.1. Let $\theta_0, R_0 > 0$ as in the assumption of Theorem 1.1 so that $f_0(p) \geq 2\theta_0$ on $B_{2R_0} = \{|p| \leq 2R_0\}$. Let τ be sufficiently small so that $f(\tau, p) \geq \theta_0$ on B_{R_0}, thanks to the continuity in time of the (classical) solution $f(t, p)$.

In the proof, we shall apply Lemma 5.3 repeatedly to the solution $f(t, p)$ of (1.3), with $G(t, |p|) = f(t, |p|)$. First, we note that since $f(t, p)$ is radially symmetric, $\mathcal{E}(p) \geq |p|^2$ and $\rho \in [1, \frac{5}{2}]$, we have

$$\int_{\mathbb{R}_+} f(t,|p|)(|p|^{1+\rho} + |p|^{1+2\rho})\mathrm{d}|p| \leq C_0 \int_{\mathbb{R}^3} f(t,p)(1+\mathcal{E}(p)^2)\,\mathrm{d}p \leq C_\tau \tag{5.10}$$

for all $t \geq \tau$, thanks to the conservation of mass and the boundedness of second-order energy moment. This verifies the assumption (5.5) on $G(t, |p|) = f(t, |p|)$, made in Lemma 5.3, with $\mathcal{M}(t) = C_\tau$, which is time-independent.

Fix a positive and sufficiently small $\delta < R_0$, and a positive time t_0 so that

$$t_0 < \frac{1}{4}. \tag{5.11}$$

Since $f_0(\tau, p) \geq \theta_0$ on B_δ, applying Lemma 5.3 to the solution $f(t, p)$ of (1.3) with the initial data $f(\tau, p)$ yields

$$f(\tau + t_0, p) \geq t_0 e^{-t_0 C_\tau L_*(\delta)} C_p \theta_0^2 \tag{5.12}$$

for all $p \in B_{\sqrt{2}\delta}$, in which $L_*(\delta) = c_0(1 + \delta^{3\rho+1})$ and

$$C_p := C_0 |p|^{3\rho+1} \min\{1, |p|\}. \tag{5.13}$$

We stress that C_p does not depend on δ and t_0, and hence the estimate (5.12) can be iterated. Applying again Lemma 5.3 to the solution $f(t, p)$ of (1.3) with the initial data $f(\tau + t_0, p)$ satisfying (5.12), yielding

$$\begin{aligned} f(\tau + t_0 + t_1, p) &\geq t_1 e^{-t_1 L_*(\sqrt{2}\delta)} C_p \Big[t_0 e^{-t_0 L_*(\delta)} C_p \theta_0^2 \Big]^2 \\ &\geq t_1 t_0^2 e^{-t_1 L_*(\sqrt{2}\delta)} e^{-2t_0 L_*(\delta)} \Big(C_p \Big)^{1+2} \theta_0^{2^2} \end{aligned}$$

for arbitrary positive time $t_1 < \frac{1}{4}$ and for all $p \in B_{\sqrt{2}^2\delta}$. For each fixed integer $n \geq 2$, we iteratively apply Lemma 5.3, yielding

$$\begin{aligned} & f(\tau + t_0 + \cdots + t_n, p) \\ & \geq t_n t_{n-1}^2 \cdots t_{n-k}^{2^k} \cdots t_0^{2^n} e^{-t_n L_*(\sqrt{2}^n \delta)} \cdots e^{-2^n t_0 L_*(\delta)} \Big(C_p \Big)^{1+2+\cdots+2^n} \theta_0^{2^{n+1}} \end{aligned}$$

for all $p \in B_{\sqrt{2}^{n+1}\delta}$. By using $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$, the above is reduced to

$$f(\tau + t_0 + \cdots + t_n, p) \geq t_n t_{n-1}^2 \cdots t_{n-k}^{2^k} \cdots t_0^{2^n} \theta_0 \Big(C_p \theta_0 \Big)^{2^{n+1}-1} E_n \tag{5.14}$$

for all $p \in B_{\sqrt{2}^{n+1}\delta}$, in which for convenience we have set

$$E_n := e^{-t_n L_*(\sqrt{2}^n \delta)} \cdots e^{-2^k t_{n-k} L_*(\sqrt{2}^{n-k} \delta)} \cdots e^{-2^n t_0 L_*(\delta)}. \tag{5.15}$$

Case 1: $|p| > \sqrt{2}\delta$**.** Recall that δ, t_0 are fixed. For each p so that $|p| > \sqrt{2}\delta$, we take an integer n satisfying

$$\sqrt{2}^n \delta < |p| \leq \sqrt{2}^{n+1} \delta. \tag{5.16}$$

In particular, $p \in B_{\sqrt{2}^{n+1}\delta}$ and (5.14) holds for arbitrary positive time steps t_k. We now fix an arbitrary time $t \in (\tau, t_*)$, with $t_* = 1/4$. We take $t_k = t_0^k$ and choose t_0 so that $t_0 < \frac{1}{4}$ and

$$\sum_{k=0}^{n} t_k = t.$$

Such a choice of t_0 is possible by the definition of t_*. The lower bound (5.14) then reads

$$f(\tau + t, p) \geq \theta_0 t_n t_{n-1}^2 \cdots t_{n-k}^{2^k} \cdots t_0^{2^n} \Big(C_p \theta_0 \Big)^{2^{n+1}-1} E_n \tag{5.17}$$

for all $t \in (\tau, t_*)$ and all $|p| > \sqrt{2}\delta$, with n being defined by (5.16).

Note in particular that $t_0 \geq T_\tau$ for some positive time T_τ, since $t \geq \tau$. Using this, we can estimate

$$\begin{aligned} t_n t_{n-1}^2 \cdots t_{n-k}^{2^k} \cdots t_0^{2^n} &\geq T_\tau^{n+2(n-1)+\cdots+2^k(n-k)+\cdots+2^n} \\ &\geq T_\tau^{2^n+\sum_{k=0}^n 2^k(n-k)} \geq T_\tau^{2^n(1+\sum_{k=0}^\infty k2^{-k})} = \mathcal{C}_0^{2^n}, \end{aligned}$$

in which $\mathcal{C}_0 = T_\tau^{1+\sum_{k=0}^\infty k2^{-k}}$, which is finite and nonzero.

Next, by the definition (5.13) of C_p, we have $C_p \geq C_\delta$ for some positive constant C_δ, since $|p| > \sqrt{2}\delta$, and hence

$$\theta_0\Big(C_p\,\theta_0\Big)^{2^{n+1}-1} \geq \theta_0\Big(C_\delta\theta_0\Big)^{2^{n+1}-1} \geq \mathcal{C}_1(\mathcal{C}_2)^{2^n}$$

for some positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$, independent of n, p and t.

Finally, we estimate the exponential term E_n defined as in (5.15). Recalling that $4t_0 < 1$, $t_k = t_0^k$ and $L_*(\delta) = c_0(1+\delta^{3\rho+1})$, we have

$$\begin{aligned} e^{-2^k t_{n-k} L_*(\sqrt{2}^{n-k}\delta)} &\geq e^{-2^k t_0^{n-k} c_0(1+\sqrt{2}^{4(n-k)}\delta^{3\rho+1})} \\ &\geq e^{-2^k c_0[t_0^{n-k}+(4t_0)^{n-k}\delta^{3\rho+1}]} = e^{-2^k c_0[1+\delta^{3\rho+1}]}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} E_n &= \exp\left(-\sum_{k=0}^n 2^k t_{n-k} L_*(\sqrt{2}^{n-k}\delta)\right) \geq \exp\left(-c_0[1+\delta^{3\rho+1}]\sum_{k=0}^n 2^k\right) \\ &\geq \exp\Big(-c_0[1+\delta^{3\rho+1}]2^n\Big) = \mathcal{C}_3^{2^n} \end{aligned}$$

for some positive constant $\mathcal{C}_3$, which is independent of n, p, and t.

Putting the above bounds into (5.17), we have obtained

$$f(\tau+t,p) \geq \frac{1}{2}\mathcal{C}_1(\mathcal{C}_0\mathcal{C}_2\mathcal{C}_3)^{2^n} = \theta_1 e^{-\theta_2 2^n} \geq \theta_1 e^{-\theta_3|p|^2} \tag{5.18}$$

for all $t \in [\tau, t_*]$ and all p satisfying (5.16), with $\theta_1 = \mathcal{C}_1$, $\theta_2 = \log\frac{1}{\mathcal{C}_0\mathcal{C}_2\mathcal{C}_3}$ and $\theta_3 = \theta_2/(2\delta^2)$. Here, we stress that the constants θ_j are independent of p and t.

Case 2: $|p| \leq \sqrt{2}\delta$. In this case, using the differential inequalities (5.7) and (5.3), we obtain

$$\partial_t f \geq n_1 Q_{\text{gain}}[f](p) - [C_0\mathcal{M}(1+|p|^{2\rho}) + C_0|p|^{3\rho+1}]f,$$

which yields

$$f(t,p) \geq e^{-[C_0\mathcal{M}(1+|p|^{2\rho})+C_0|p|^{3\rho+1}]t} f_0(p). \tag{5.19}$$

Equation (5.19) implies that for a fixed p_0 and for all $|p| < |p_0|$, there holds

$$f(t,p) \geq e^{-[C_0\mathcal{M}(1+|p_0|^{2\rho})+C_0|p_0|^{3\rho+1}]t} f_0(p). \tag{5.20}$$

Therefore, for each fixed t_0, there exists $c' > 0$: $f(t_0, p) > c'$ for all $|p| < |p_0|$. For each p, by repeating the same argument as in Case 1, in which δ is replace by $|p|/\sqrt{2}$, we can conclude that there exists T_p and $b_{|p|}$ such that for all $t > T_p$, we have

$$f(t, p') > b_{|p|} > 0$$

for all $\sqrt{2}\delta \geq |p'| > |p|$.

Recall that f is continuous in p and $f(t, 0) = f_0(0)$ for all $t \geq 0$, since $\partial_t f(t, 0) = 0$. This proves that there exists a universal constant $r^* > 0$ such that $f(t, p)$ is uniformly bounded from below for all $t_* \geq t \geq 0$ and $|p| \leq r^*$:

$$f(t, p) \geq C_{r^*} > 0, \quad \forall t_* \geq t \geq 0, \quad \forall |p| \leq r^*. \tag{5.21}$$

This yields the lower bound of $f(t, p)$ in the ball $\{|p| \leq \sqrt{2}\delta\}$, for sufficiently small δ.

Iteration. To conclude, we have obtained the Gaussian bound

$$f(t, p) \geq \theta_3 e^{-\theta_4 |p|^2}, \qquad p \in \mathbb{R}^3, \qquad t \in [\tau, \tau + t_*] \tag{5.22}$$

for some universal constants θ_3, θ_4 that are independent of p and t. Here, $t_* = 1/4$. By induction, for each integer $k \geq 1$, we then repeat the above proof, starting with initial data at $t = kt_*$. This yields the same Gaussian bound on the each time interval $[\tau + kt_*, \tau + (k + 1)t_*]$, upon noting that such a bound depends only on the mass and second order energy-moment at $t = kt_*$, which is independent of k^{th} iteration. This proves the Gaussian lower bound for all time $t \geq \tau$, and hence the main theorem.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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