



The Inviscid Limit of Navier–Stokes Equations for Analytic Data on the Half-Space

TOAN T. NGUYEN & TRINH T. NGUYEN

Communicated by P. CONSTANTIN

Abstract

In their classical work, SAMMARTINO AND CAFLISCH (Commun Math Phys 192(2):433–461, 1998a; Commun Math Phys 192(2):463–491, 1998b) proved the inviscid limit of the incompressible Navier–Stokes equations for well-prepared data with analytic regularity in the half-space. Their proof is based on the detailed construction of Prandtl’s boundary layer asymptotic expansions. In this paper, we give a direct proof of the inviscid limit for general analytic data without having to construct Prandtl’s boundary layer correctors. Our analysis makes use of the boundary vorticity formulation and the abstract Cauchy–Kovalevskaya theorem on analytic boundary layer function spaces that capture unbounded vorticity.

1. Introduction

In this paper, we are interested in the inviscid limit of the Navier–Stokes equations for incompressible fluids

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u \\ \nabla \cdot u &= 0 \end{aligned} \tag{1.1}$$

posed on the half space $\mathbb{T} \times \mathbb{R}_+$, with the classical no-slip boundary condition

$$u|_{z=0} = 0. \tag{1.2}$$

In the inviscid limit: $\nu \rightarrow 0$, one would expect that solutions u^ν converge to solutions of the corresponding Euler equations for incompressible fluids, however, the inviscid limit problem for the no-slip boundary condition (1.2) is open due to the appearance of boundary layers and the creation of unbounded vorticity near the boundary. On the one hand, the friction causes the fluid to stick to the boundary, the no-slip condition (1.2). On the other hand, the inviscid flow allows the fluid

to slip along the boundary. The rapid change of the tangential velocity on the boundary gives rise to transition or boundary layers in the small viscosity limit. As a consequence, the vorticity is of order δ_v^{-1} on the boundary, in which δ_v denotes the thickness of boundary layers, within which the velocity changes rapidly. This leads to possible large energy production, due to the large convection $u \cdot \nabla u$ within the boundary layers.

In his seminal paper, PRANDTL [33] postulated a boundary layer ansatz that balances the inertial and viscous forces in the dynamics of (1.1), leading to the well-known Prandtl's boundary layer equation, a simplification of complicated Navier–Stokes equations in a thin layer of thickness $\delta_v = \sqrt{\nu}$. In the half space, the Prandtl's ansatz reads

$$u(t, x, z) = u_E(t, x, z) + u_P\left(t, x, \frac{z}{\sqrt{\nu}}\right) + o(1)_{L^\infty}, \quad (1.3)$$

where u_E solves the corresponding Euler equations, and u_P is introduced to correct the no-slip boundary condition of the Navier–Stokes equations which does not satisfy by Euler solutions u_E . Here, the velocity field $u_E(t, x, 0) + u_P(t, x, Z)$ solves the well-known Prandtl's boundary layer equation, and the remainder $o(1)_{L^\infty}$ is expected to converge to zero in L^∞ as $\nu \rightarrow 0$. Formally, it is even possible to write a higher order asymptotic expansion for u^ν in terms of powers of $\sqrt{\nu}$. Since then, the Prandtl boundary layers have been intensively studied in the mathematical literature. Notably, solutions to the Prandtl equations have been constructed for monotonic data [1, 29, 31, 32] or data with Gevrey or analytic regularity ([11, 22, 35], among others). In the case of non-monotonic data with Sobolev regularity, the Prandtl boundary layer equations are ill-posed [8, 12, 20].

The validity of Prandtl's Ansatz (1.3), and hence the inviscid limit, were established in [35, 36] for well-prepared data with analytic regularity (precisely, analytic data that initially satisfy the boundary layer expansion (1.3) with remainder of order $\sqrt{\nu}$). A similar result is also obtained in [26, 37]. The Ansatz (1.3), with a specific boundary layer shear profile, has been recently justified for data with Gevrey regularity [10]. When only data with Sobolev regularity are assumed, except for data with special symmetry data with special symmetry ([13, 30] and the references therein), the asymptotic expansions (1.3) are false due to the strong instability of boundary layers [14–17, 19], including those boundary layers that are spectrally stable to the Euler equations. Finally, we mention that the inviscid limit holds for other types of boundary conditions other than the no-slip one (1.2); see, for instance, [9, 21, 28], and also the review papers [3, 27]. In particular, [28] proves the inviscid limit for Navier-type boundary conditions via energy estimates without dealing directly with the boundary layer asymptotic expansions.

On the other hand, turning back to the no-slip case (1.2), Kato [23] constructed a much thinner boundary layer of thickness $\delta_v = \nu$, as compared to the Prandtl's thickness $\delta_v = \sqrt{\nu}$, aiming precisely to control the large convection near the boundary. This leads to his well-known criteriom, which asserts that the inviscid limit holds in the energy norm if and only if the energy dissipation near the boundary vanishes in the inviscid limit: precisely,

$$\nu \int_0^T \iint_{\{z \lesssim \nu\}} |\nabla u|^2 \, dx dz dt \longrightarrow 0, \quad \text{as } \nu \rightarrow 0. \quad (1.4)$$

(See also its many variants, for instance, [3, 5–7, 24], among others.)

In this paper, we give a direct proof of the inviscid limit for data with analytic regularity without having to construct Prandtl’s boundary layer correctors. The proof relies on the boundary vorticity formulation, the pointwise bounds on the Green function, and the abstract Cauchy–Kovalevskaya theorem on boundary layer function spaces.

1.1. Boundary Vorticity Formulation

We shall work with the boundary vorticity formulation and the solution representation as in the recent work by MAEKAWA [25, 26]; see also [2]. Precisely, let $\omega = \partial_z u_1 - \partial_x u_2$ be the corresponding vorticity in $\mathbb{T} \times \mathbb{R}_+$. Then, the vorticity equation reads

$$\partial_t \omega - \nu \Delta \omega = -u \cdot \nabla \omega \quad (1.5)$$

with $u = \nabla^\perp \Delta^{-1} \omega$. Here and throughout the paper, Δ^{-1} denotes the inverse of the Laplacian operator with the Dirichlet boundary condition: precisely, $\phi = \Delta^{-1} \omega$ solves $\Delta \phi = \omega$ on the half-space $\mathbb{T} \times \mathbb{R}_+$, with $\phi|_{z=0} = 0$.

To ensure the no-slip boundary condition, we impose $\partial_t u_1 = 0$ on the boundary. This leads to

$$0 = \partial_t u_1 = \partial_z \Delta^{-1} \partial_t \omega = \partial_z \Delta^{-1} (\nu \Delta \omega - u \cdot \nabla \omega)$$

on the boundary. Introduce ω_* so that $\Delta \omega_* = 0$ with $\omega_* = \omega$ on the boundary. This yields $\partial_z \Delta^{-1} \Delta \omega = \partial_z (\omega - \omega_*) = (\partial_z + |\partial_x|) \omega$, in which $|\partial_x|$ denotes the Dirichlet-to-Neumann operator on the half space. Thus, the boundary condition on vorticity reads

$$\nu (\partial_z + |\partial_x|) \omega|_{z=0} = [\partial_z \Delta^{-1} (u \cdot \nabla \omega)]|_{z=0}. \quad (1.6)$$

Throughout this paper, we shall deal with the Navier–Stokes solutions that solve (1.5)–(1.6), together with the Biot–Savart law $u = \nabla^\perp \Delta^{-1} \omega$. Such a solution will be constructed via the Duhamel’s integral representation, treating the nonlinearity as a source term.

1.2. Analytic Boundary Layer Function Spaces

In this paper, we shall deal with analytic boundary layer spaces introduced in [17, 18]. Precisely, we consider holomorphic functions on the pencil-like complex domain

$$\Omega_\sigma = \left\{ z \in \mathbb{C} : |\Im z| < \min\{\sigma \Re z, \sigma\} \right\} \quad (1.7)$$

for $\sigma > 0$. Let $\delta = \sqrt{\nu}$ be the classical boundary layer thickness. We introduce the analytic boundary layer function spaces $\mathcal{B}^{\sigma, \delta}$ that consists of holomorphic functions on Ω_σ with a finite norm

$$\|f\|_{\sigma, \delta} = \sup_{z \in \Omega_\sigma} |f(z)| e^{\beta \Re z} \left(1 + \delta^{-1} \phi_P(\delta^{-1} z)\right)^{-1} \quad (1.8)$$

for some small $\beta > 0$, and for boundary layer weight function

$$\phi_P(z) = \frac{1}{1 + |\Re z|^P}$$

for some fixed constant $P > 1$. Here, we suppress the dependence on β, P as they are fixed throughout the paper. We expect that the vorticity function $\omega(t, x, z)$, for each fixed t, x , will be in $\mathcal{B}^{\sigma, \delta}$, precisely describing the behavior near the boundary and near infinity. In fact, there is an additional initial layer of thickness $\delta_t = \sqrt{vt}$ that appears near the boundary. To capture this, we introduce the time-dependent boundary layer norm

$$\|f\|_{\sigma, \delta(t)} = \sup_{z \in \Omega_\sigma} |\omega(z)| e^{\beta \Re z} \left(1 + \delta_t^{-1} \phi_P(\delta_t^{-1} z) + \delta^{-1} \phi_P(\delta^{-1} z)\right)^{-1}, \quad (1.9)$$

with $\delta_t = \sqrt{vt}$, $\delta = \sqrt{v}$, and with the same boundary layer weight function $\phi_P(\cdot)$. By convention, the norm $\|\cdot\|_{\sigma, \delta(0)}$ at time $t = 0$ is replaced by $\|\cdot\|_{\sigma, \delta}$, the boundary layer norm with precisely one boundary layer behavior of thickness δ , and $\|\cdot\|_{\sigma, 0}$ denotes the norm without the boundary layer behavior.

For functions depending on two variables $f(x, z)$, we introduce the partial Fourier transform in variable x :

$$f(x, z) = \sum_{\alpha \in \mathbb{Z}} f_\alpha(z) e^{i \alpha x}$$

and introduce the following analytic norm:

$$\|f\|_{\rho, \sigma, \delta(t)} = \sum_{\alpha \in \mathbb{Z}} e^{\rho |\alpha|} \|f_\alpha\|_{\sigma, \delta(t)},$$

for $\rho, \sigma > 0$. We denote by $\mathcal{B}^{\rho, \sigma, \delta(t)}$ the corresponding spaces. In Section 2, we shall recall some basic properties of such analytic function spaces.

1.3. Main Results

Our main results are

Theorem 1.1. *Let $M_0 > 0$ and let ω_0 be in $\mathcal{B}^{\rho_0, \sigma_0, \delta}$ for $\rho, \sigma > 0$ and for $\delta = \sqrt{v}$, with $\|\omega_0\|_{\rho_0, \sigma_0, \delta} \leq M_0$. Then, there is a positive time T so that the solution $\omega(t)$ to the Navier–Stokes equations (1.5)–(1.6), with the initial data $\omega(0) = \omega_0$, exists in $C^1([0, T]; \mathcal{B}^{\rho, \sigma, \delta(t)})$ for $0 < \rho < \rho_0$ and $0 < \sigma < \sigma_0$. In particular, there is a C_0 so that the vorticity $\omega(t)$ satisfies*

$$|\omega(t, x, z)| \leq C_0 e^{-\beta z} \left(1 + \delta_t^{-1} \phi_P(\delta_t^{-1} z) + \delta^{-1} \phi_P(\delta^{-1} z)\right) \quad (1.10)$$

for $(t, x, z) \in [0, T] \times \mathbb{T} \times \mathbb{R}_+$, with $\delta_t = \sqrt{vt}$ and $\delta = \sqrt{v}$.

Theorem 1.2. *Let $M_0 > 0$ and let u_0^v be divergence-free analytic initial data so that $u_0^v = 0$ on the boundary and $\omega_0^v = \nabla \times u_0^v$ is in $\mathcal{B}^{\rho_0, \sigma_0, \delta}$ for $\rho, \sigma > 0$ and for $\delta = \sqrt{v}$, with $\|\omega_0^v\|_{\rho_0, \sigma_0, \delta} \leq M_0$. Then, the inviscid limit holds for Navier–Stokes solutions with the initial data u_0 . Precisely, there are unique local solutions $u^v(t)$ to the Navier–Stokes equations (1.1)–(1.2), for small $v > 0$, and a unique solution $u^0(t)$ to the corresponding Euler equations, with initial data $u^0(0) = \lim_{v \rightarrow 0} u_0^v$, so that*

$$\sup_{t \in [0, T]} \|u^v(t) - u^0(t)\|_{L^p} \rightarrow 0$$

for $2 \leq p < \infty$, as $v \rightarrow 0$.

As mentioned, the proof of the main theorems is direct, using the vorticity formulation (1.5)–(1.6). In fact, the existence of analytic solutions is proved, without having to derive the pointwise bounds on the Green function for the Stokes problem; see Sections 3.4 and 4.1. However, in order to prove the propagation of boundary layer behaviors as described in (1.10), the detailed estimates on the Green function are crucial. The main results apply in particular for well-prepared analytic data that satisfy the Prandtl’s ansatz (1.3). For general analytic data, beside the Prandtl’s layers, the initial layers whose thickness is of order $\sqrt{v t}$ appear as captured in (1.10).

Finally, we mention that the analysis avoids dealing directly with the Prandtl’s layers and Prandtl’s asymptotic expansions, and hence appears robust to resolve the inviscid limit problem (for analytic data) in domains with curved boundaries.

The paper proceeds with some basic properties of the analytic boundary layer norms and elliptic estimates in Section 2. The main analysis of the paper is presented in Section 3 where we study in details the Stokes problem with boundary data and sources in the boundary layer function spaces. The nonlinear iteration and the proof of the main theorems are given in Section 4.

2. Analytic Function Spaces

In this section, we shall prove some basic properties of the analytic norms as well as the elliptic estimates that yield bounds on velocity in term of vorticity. These norms and estimates can be found in [18]. See also [35, 36].

2.1. Analytic Spaces

Let $f(x, z)$ be holomorphic functions on $\mathbb{T} \times \Omega_\sigma$, with Ω_σ being the pencil-like complex domain defined as in (1.7). For $\rho, \sigma > 0$ and $p \geq 1$, we introduce the analytic function spaces denoted by $\mathcal{L}_{\rho, \sigma}^p$ with the finite norm

$$\|f\|_{\mathcal{L}_{\rho, \sigma}^p} := \sum_{\alpha \in \mathbb{Z}} e^{\rho|\alpha|} \|f_\alpha\|_{L_\sigma^p}, \quad \|f_\alpha\|_{L_\sigma^p} := \sup_{0 \leq \theta < \sigma} \left(\int_{\partial \Omega_\theta} |f_\alpha(z)|^p |dz| \right)^{1/p}, \quad (2.1)$$

in which $f_\alpha = f_\alpha(z)$ denotes the Fourier transform of $f(x, z)$. In the case when $p = \infty$, we replace the L^p norm by the sup norm over Ω_σ . Recalling the analytic boundary layer space $B^{\rho, \sigma, \delta(t)}$ introduced in Section 1.2, we have

Lemma 2.1. *There holds the embedding $B^{\rho, \sigma, \delta(t)} \subset \mathcal{L}_{\rho, \sigma}^1$.*

Proof. For the holomorphic functions $f_\alpha(z)$ satisfying

$$|f_\alpha(z)| \leq \|f_\alpha\|_{\sigma, \delta(t)} e^{-\beta \Re z} \left(1 + \delta_t^{-1} \phi_P(\delta_t^{-1} z) + \delta^{-1} \phi_P(\delta^{-1} z)\right),$$

it is clear that $\|f_\alpha\|_{L_\sigma^1} \leq \|f_\alpha\|_{\sigma, \delta(t)}$. By taking the summation over $\alpha \in \mathbb{Z}$, the lemma follows. \square

Lemma 2.2. *For any $0 < \sigma' < \sigma$, $0 < \rho' < \rho$, and $\psi(z) = \frac{z}{1+z}$, there hold*

$$\|fg\|_{\mathcal{L}_{\rho, \sigma}^1} \leq \|f\|_{\mathcal{L}_{\rho, \sigma}^\infty} \|g\|_{\mathcal{L}_{\rho, \sigma}^1} \text{ and} \quad (2.2)$$

$$\|\partial_x f\|_{\mathcal{L}_{\rho', \sigma}^1} \leq \frac{C}{\rho - \rho'} \|f\|_{\mathcal{L}_{\rho, \sigma}^1}, \quad \|\psi(z) \partial_z f\|_{\mathcal{L}_{\rho', \sigma'}^1} \leq \frac{C}{\sigma - \sigma'} \|f\|_{\mathcal{L}_{\rho, \sigma}^1}. \quad (2.3)$$

The same estimates hold for boundary layer norms $\|\cdot\|_{\rho, \sigma, \delta}$ replacing $\|\cdot\|_{\mathcal{L}_{\rho, \sigma}^1}$ in the above three inequalities.

Proof. By definition, we write

$$fg(x, z) = \sum_{\alpha \in \mathbb{Z}} e^{i\alpha x} \sum_{\beta \in \mathbb{Z}} f_{\alpha-\beta}(z) g_\beta(z),$$

and hence, we estimate

$$\begin{aligned} \|fg\|_{\mathcal{L}_{\rho, \sigma}^1} &= \sum_{\alpha \in \mathbb{Z}} e^{\rho|\alpha|} \left\| \sum_{\beta \in \mathbb{Z}} f_{\alpha-\beta}(\cdot) g_\beta(\cdot) \right\|_{L_\sigma^1} \\ &\leq \sum_{\alpha \in \mathbb{Z}} \sum_{\beta \in \mathbb{Z}} e^{\rho|\alpha-\beta|} e^{\rho|\beta|} \|f_{\alpha-\beta}\|_{L_\sigma^\infty} \|g_\beta\|_{L_\sigma^1}, \end{aligned}$$

which proves the first inequality. As for the second, we compute

$$\|\partial_x f\|_{\mathcal{L}_{\rho', \sigma}^1} \leq \sum_{\alpha} \|f_\alpha\|_{L_\sigma^1} |\alpha| e^{\rho'|\alpha|}.$$

Using the fact that $(\rho - \rho')|\alpha| e^{(\rho' - \rho)|\alpha|}$ is bounded, the second inequality follows.

Finally, we check the third inequality. By the Cauchy integral formula, we have

$$\partial_z f_\alpha(z) = \frac{1}{2\pi i} \int_{C(z, R_z)} \frac{f_\alpha(y)}{(y - z)^2} dy,$$

where $C(z, R_z)$ is the circle, centered at z and of radius R_z so that $C(z, R_z) \in \Omega_\sigma$. Let us take

$$R_z = c_0(\sigma - \sigma') \begin{cases} \Re(z) & \text{if } \Re(z) < 1 \\ 1 & \text{if } \Re(z) \geq 1 \end{cases}$$

for some small and positive c_0 . Thus, using the parametrization $y = z + e^{iw}R_z$ with $0 \leq w \leq 2\pi$, we get

$$\begin{aligned}\partial_z f_\alpha(z) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f_\alpha(z + R_z e^{iw})}{R_z^2 e^{2iw}} (R_z i e^{iw}) dw \\ &= \frac{1}{2\pi R_z} \int_0^{2\pi} f_\alpha(z + R_z e^{iw}) e^{-iw} dw.\end{aligned}$$

Now, for any $0 \leq \theta' < \sigma'$, we compute

$$\begin{aligned}\int_{\partial\Omega_{\theta'}} |\psi(\Re z) \partial_z f_\alpha(z)| |dz| &\leq \int_{\partial\Omega_{\theta'}} \int_0^{2\pi} \frac{\psi(\Re z)}{2\pi R_z} |f_\alpha(z + R_z e^{iw})| |dw| |dz| \\ &\leq \frac{C_0}{\sigma - \sigma'} \int_{\partial\Omega_{\theta'}} \int_0^{2\pi} |f_\alpha(z + R_z e^{iw})| |dw| |dz| \\ &\leq \frac{2\pi C_0}{\sigma - \sigma'} \sup_{0 \leq w \leq 2\pi} \int_{\partial\Omega_{\theta'}} |f_\alpha(z + R_z e^{iw})| |dz|.\end{aligned}$$

It remains to show that the above integral is bounded by $2\|f\|_{\mathcal{L}_{\rho,\sigma}^1}$. To this end, it suffices to show that for each fixed $w \in [0, 2\pi]$, there is a positive constant $\theta < \sigma$ so that

$$z + R_z e^{iw} \in \partial\Omega_\theta, \quad \forall z \in \partial\Omega_{\theta'}. \quad (2.4)$$

Case 1: $\Re(z) \leq 1$. Recalling $R_z = c_0(\sigma - \sigma')\Re(z)$ and $\Im(z) = \theta'\Re(z)$ on $\partial\Omega_{\theta'}$, we compute

$$\begin{aligned}\Re(z + R_z e^{iw}) &= \Re(z) + R_z \cos(w) = \Re(z)(1 + c_0(\sigma - \sigma') \cos w) \\ \Im(z + R_z e^{iw}) &= \Im(z) + R_z \sin(w) = \Re(z)(\theta' + c_0(\sigma - \sigma') \sin w).\end{aligned}$$

Hence, $z + R_z e^{iw} \in \partial\Omega_\theta$ for $\theta = \frac{\theta' + c_0(\sigma - \sigma') \sin w}{1 + c_0(\sigma - \sigma') \cos w}$. We now check $\theta < \sigma$. Indeed, we need

$$\theta' + c_0(\sigma - \sigma') \sin w < \sigma(1 + c_0(\sigma - \sigma') \cos w),$$

which is equivalent to

$$\sigma - \theta' > c_0(\sigma - \sigma')(\sin w - \sigma \cos w).$$

Since $\theta' < \sigma'$ and c_0 can be taken arbitrarily small (independent of σ, σ'), the above inequality follows.

Case 2: $\Re(z) \geq 1$. Similarly, using $R_z = c_0(\sigma - \sigma')$ and $\Im(z) = \theta'$ on $\partial\Omega_{\theta'}$, we compute

$$\Im(z + R_z e^{iw}) = \Im(z) + R_z \sin w = \theta' + c_0(\sigma - \sigma') \sin w = \tilde{\theta},$$

where $\tilde{\theta} < \sigma$ for sufficiently small c_0 and for $\theta' < \sigma' < \sigma$. This proves (2.4). \square

2.2. Elliptic Estimates

Next, we recall the elliptic estimates, which are adapted from [17, 18].

Proposition 2.3. *Let ϕ be the solution of $-\Delta\phi = \omega$ with the zero Dirichlet boundary condition, and set $u = \nabla^\perp\phi$. Then, there holds*

$$\|u_1\|_{\mathcal{L}_{\rho,\sigma}^\infty} + \|u_2\|_{\mathcal{L}_{\rho,\sigma}^\infty} \leq C\|\omega\|_{\mathcal{L}_{\rho,\sigma}^1}, \quad (2.5)$$

$$\|\partial_x u_1\|_{\mathcal{L}_{\rho,\sigma}^\infty} + \|\nabla u_2\|_{\mathcal{L}_{\rho,\sigma}^\infty} + \|\psi^{-1}u_2\|_{L_{\rho,\sigma}^\infty} \leq C\|\omega\|_{\mathcal{L}_{\rho,\sigma}^1} + C\|\partial_x\omega\|_{\mathcal{L}_{\rho,\sigma}^1}, \quad (2.6)$$

$$\|\nabla u_1\|_{\mathcal{L}_{\rho,\sigma}^1} + \|\nabla u_2\|_{\mathcal{L}_{\rho,\sigma}^1} \leq C\|\omega\|_{\mathcal{L}_{\rho,\sigma}^1}, \quad (2.7)$$

with $\psi(z) = z/(1+z)$, for some constant C .

Proof. Taking the Fourier transform, it suffices to study the classical one-dimensional Laplace equation

$$\partial_z^2\phi_\alpha - \alpha^2\phi_\alpha = \omega_\alpha \quad (2.8)$$

on Ω_σ , with the Dirichlet boundary condition $\phi_\alpha(0) = 0$, and $\alpha > 0$. For real values z , the solution ϕ_α of (2.8) is explicitly given by

$$\phi_\alpha(z) = \int_0^z G_-(y, z)\omega_\alpha(y)dy + \int_z^\infty G_+(y, z)\omega_\alpha(y)dy$$

with

$$G_\pm(y, z) = -\frac{1}{2\alpha} \left(e^{\pm\alpha(z-y)} - e^{-\alpha(y+z)} \right).$$

This expression may be extended to complex values of z . Indeed, for $z \in \Omega_\sigma$, there is a positive θ so that $z \in \partial\Omega_\theta$. We then write $\partial\Omega_\theta = \gamma_-(z) \cup \gamma_+(z)$, consisting of complex numbers $y \in \partial\Omega_\theta$ so that $\Re y < \Re z$ and $\Re y > \Re z$, respectively. Then, we write

$$\phi_\alpha(z) = \int_{\gamma_-(z)} G_-(y, z)\omega_\alpha(y)dy + \int_{\gamma_+(z)} G_+(y, z)\omega_\alpha(y)dy. \quad (2.9)$$

We note in particular that for $y \in \gamma_\pm(z)$, there holds

$$|G_\pm(y, z)| \leq \alpha^{-1} e^{-\alpha|y-z|}.$$

This proves that

$$|\phi_\alpha(z)| \leq \int_{\partial\Omega_\theta} \alpha^{-1} e^{-\alpha|y-z|} |\omega_\alpha(y)| |dy| \leq \alpha^{-1} \int_{\partial\Omega_\theta} |\omega_\alpha(y)| |dy|, \quad (2.10)$$

which by definition yields $\sup_{\Omega_\sigma} |\alpha\phi_\alpha(z)| \leq \|\omega_\alpha\|_{L_\sigma^1}$. The same proof holds for $\partial_z\phi_\alpha(z)$. This completes the proof of (2.5). The estimate (2.6) follows by treating ∂_x as multiplication by α in the Fourier space.

Finally, taking L^1 norm of the estimate (2.10) and upon noting that the kernel $\alpha e^{-\alpha|y-z|}$ is bounded in L^1 norm, we obtain the estimate (2.7) for $\alpha^2\phi_\alpha$. The second derivative in z , we use $\partial_z^2\phi_\alpha = \alpha^2\phi_\alpha + \omega_\alpha$. This completes the proof of the lemma. \square

2.3. Bilinear Estimates

Lemma 2.4. *For any ω and $\tilde{\omega}$, denoting by v the velocity related to ω , we have*

$$\begin{aligned}\|v \cdot \nabla \tilde{\omega}\|_{\mathcal{L}_{\rho,\sigma}^1} &\leq C\|\omega\|_{\mathcal{L}_{\rho,\sigma}^1}\|\tilde{\omega}_x\|_{\mathcal{L}_{\rho,\sigma}^1} + C(\|\omega\|_{\mathcal{L}_{\rho,\sigma}^1} + \|\omega_x\|_{\mathcal{L}_{\rho,\sigma}^1})\|\psi(z)\partial_z \tilde{\omega}\|_{\mathcal{L}_{\rho,\sigma}^1} \\ \|v \cdot \nabla \tilde{\omega}\|_{\rho,\sigma,\delta} &\leq C\|\omega\|_{\rho,\sigma,\delta}\|\tilde{\omega}_x\|_{\rho,\sigma,\delta} + C(\|\omega\|_{\rho,\sigma,\delta} + \|\omega_x\|_{\rho,\sigma,\delta})\|\psi(z)\partial_z \tilde{\omega}\|_{\rho,\sigma,\delta}\end{aligned}$$

Proof. We write

$$(v \cdot \nabla)\tilde{\omega} = v_1 \partial_x \tilde{\omega} + v_2 \partial_z \tilde{\omega}.$$

Using (2.2) and (2.5), the first term is clear. On the other hand, the second term is due to (2.2) and (2.6). Finally, for the boundary layer norms, we note that

$$\|fg\|_{\rho,\sigma,\delta} \leq \|f\|_{\mathcal{L}_{\rho,\sigma}^\infty} \|g\|_{\rho,\sigma,\delta}.$$

The lemma thus follows, upon using Proposition 2.3 and noting that $\|f\|_{\mathcal{L}_{\rho,\sigma}^1} \leq \|f\|_{\rho,\sigma,\delta}$. \square

3. The Stokes Problem

In this section, we study the inhomogenous Stokes problem

$$\begin{aligned}\omega_t - v\Delta\omega &= f(t, x, z), \quad \text{in } \mathbb{T} \times \Omega_\sigma, \\ v(\partial_z + |\partial_x|)\omega &= g(t, x), \quad \text{on } z = 0,\end{aligned}\tag{3.1}$$

together with the initial data $\omega|_{t=0} = \omega_0$. Let e^{vtB} denote the semigroup of the corresponding Stokes problem: namely, the heat equation $\partial_t \omega - v\Delta\omega = 0$ on $\mathbb{T} \times \Omega_\sigma$ with the homogenous boundary condition $v(\partial_z + |\partial_x|)\omega|_{z=0} = 0$. Solutions to the linear Stokes problem are then constructed via Duhamel's integral representation, which will be proved in the next subsection, and we have

$$\omega(t) = e^{vtB}\omega_0 + \int_0^t e^{v(t-s)B}f(s) \, ds + \int_0^t \Gamma(v(t-s))g(s) \, ds\tag{3.2}$$

in which $\Gamma(vt) = e^{vtB}(g\mathcal{H}_{\mathbb{T} \times \{y=0\}}^1)$, where $\mathcal{H}_{\mathbb{T} \times \{y=0\}}^1$ is the one-dimensional Hausdorff measure restricted on the boundary; precisely, see (3.22)–(3.23) for the explicit construction of e^{vtB} and $\Gamma(vt)$ in term of the Green function for the Stokes problem.

In this section, we shall derive uniform bounds for the Stokes semigroup in analytic spaces, with the analytic norm

$$\|\omega\|_{\rho,\sigma,\delta(t)} = \sum_{\alpha \in \mathbb{Z}} e^{\rho|\alpha|} \|\omega_\alpha\|_{\sigma,\delta(t)},$$

with the boundary layer norm defined by

$$\|\omega_\alpha\|_{\sigma,\delta(t)} = \sup_{z \in \Omega_\sigma} |\omega_\alpha(z)| e^{\beta \Re z} \left(1 + \delta_t^{-1} \phi_P(\delta_t^{-1} z) + \delta^{-1} \phi_P(\delta^{-1} z)\right)^{-1},\tag{3.3}$$

in which the boundary thicknesses are $\delta_t = \sqrt{vt}$ and $\delta = \sqrt{v}$. As for the initial data, the norm is measured by $\|\omega_\alpha\|_{\sigma, \delta(0)}$, which consists of precisely one boundary layer behavior whose thickness is $\delta = \sqrt{v}$. We introduce

$$|||\omega(t)|||_{\rho, \sigma, \delta(t), k} = \sum_{j+\ell \leq k} \|\partial_x^j (\psi(z) \partial_z)^\ell \omega(t)\|_{\rho, \sigma, \delta(t)}$$

and

$$|||\omega|||_{\mathcal{W}_{\rho, \sigma}^{k, 1}} = \sum_{j+\ell \leq k} \|\partial_x^j (\psi(z) \partial_z)^\ell \omega(t)\|_{\mathcal{L}_{\rho, \sigma}^1}.$$

We also denote $|||g|||_{\rho, k}$ the corresponding analytic norm for $g = g(x)$. We obtain the following key proposition:

Proposition 3.1. *Let e^{vtB} be the semigroup for the linear Stokes problem, and $\Gamma(vt)$ be the operator $e^{vtB}(g\mathcal{H}_{\mathbb{T} \times \{y=0\}}^1)$, where $\mathcal{H}_{\mathbb{T} \times \{y=0\}}^1$ is the one-dimensional Hausdorff measure restricted on the boundary. Then, ∂_x commutes with both e^{vtB} and $\Gamma(vt)$. In addition, for any $k \geq 0$, and for any $0 \leq s < t \leq T$, there hold*

$$\begin{aligned} |||e^{vtB} f|||_{\rho, \sigma, \delta(t), k} &\lesssim |||f|||_{\rho, \sigma, \delta(0), k}, \\ |||e^{v(t-s)B} f|||_{\rho, \sigma, \delta(t), k} &\lesssim \sqrt{\frac{t}{s}} |||f|||_{\rho, \sigma, \delta(s), k}, \text{ and} \\ |||\Gamma(v(t-s))g|||_{\rho, \sigma, \delta(t), k} &\lesssim \sqrt{\frac{t}{t-s}} |||g|||_{\rho, k} \end{aligned}$$

uniformly in the inviscid limit. Similarly, we also obtain

$$|||e^{vtB} f|||_{\mathcal{W}_{\rho, \sigma}^{k, 1}} \lesssim |||f|||_{\mathcal{W}_{\rho, \sigma}^{k, 1}}, \quad |||\Gamma(vt)g|||_{\mathcal{W}_{\rho, \sigma}^{k, 1}} \lesssim |||g|||_{\rho, k}$$

uniformly in the inviscid limit.

3.1. Duhamel Principle

We first treat the Stokes problem on $\mathbb{T} \times \mathbb{R}_+$. By taking the Fourier transform in x , the problem is reduced to

$$\begin{aligned} \partial_t \omega_\alpha - v \Delta_\alpha \omega_\alpha &= f_\alpha(t, x, z), & \text{in } \mathbb{R}_+ \\ v(\partial_z + |\alpha|) \omega_\alpha &= g_\alpha(t), & \text{on } z = 0, \end{aligned} \tag{3.4}$$

in which ω_α denotes the Fourier transform of ω with respect to x , and $\Delta_\alpha = \partial_z^2 - \alpha^2$. Let $G_\alpha(t, z; y)$ be the corresponding Green function of the linear Stokes problem (3.4). That is, for each fixed $y \geq 0$, the function $G_\alpha(t, z; y)$ solves

$$\begin{aligned} (\partial_t - v \Delta_\alpha) G_\alpha(t, z; y) &= 0, & \text{in } \mathbb{R}_+ \\ v(\partial_z + |\alpha|) G_\alpha(t, z; y) &= 0, & \text{on } z = 0, \end{aligned} \tag{3.5}$$

together with the initial data $G_\alpha(0, z; y) = \delta_y(z)$. The Green function will be constructed so that $G_\alpha(t, \cdot; y) \in L^1$ for each t, y . we then have

Lemma 3.2. (Duhamel’s principle). *For any $T > 0$, and for any $f_\alpha \in L^\infty(0, T; L^1(\mathbb{R}_+))$ and $g_\alpha \in L^\infty(0, T)$, the unique solution to the linear Stokes problem (3.4), with the initial data $\omega_\alpha(0, z) = \omega_{0,\alpha}(z)$ in $L^1(\mathbb{R}_+)$, satisfies*

$$\begin{aligned}\omega_\alpha(t, z) &= \int_0^\infty G_\alpha(t, y; z) \omega_{0,\alpha}(y) \, dy + \int_0^t G_\alpha(t-s, 0; z) g_\alpha(s) \, ds \\ &\quad + \int_0^t \int_0^\infty G_\alpha(t-s, y; z) f_\alpha(s, y) \, dy \, ds.\end{aligned}\tag{3.6}$$

Proof. Using (3.4), we compute

$$\begin{aligned}&\int_0^t \int_0^\infty G_\alpha(t-s, y; z) f_\alpha(s, y) \, dy \, ds \\ &= \int_0^t \int_0^\infty G_\alpha(t-s, y; z) (\partial_s + \nu\alpha^2 - \nu\partial_y^2) \omega_\alpha(s, y) \, dy \, ds \\ &= \int_0^t \int_0^\infty (\partial_s + \nu\alpha^2 - \nu\partial_y^2) G_\alpha(t-s, y; z) \omega_\alpha(s, y) \, dy \, ds \\ &\quad + \int_0^\infty G_\alpha(0, y; z) \omega_\alpha(t, y) \, dy \\ &\quad - \int_0^\infty G_\alpha(t, y; z) \omega_{0,\alpha}(y) \, dy \\ &\quad + \nu \int_0^t \left(G_\alpha(t-s, y; z) \partial_y \omega_\alpha - \partial_y G_\alpha(t-s, y; z) \omega_\alpha \right)_{|y=0} \, ds.\end{aligned}$$

The lemma follows, upon using the initial data and boundary conditions on $G_\alpha(t, y; z)$. \square

3.2. The Green Function for the Stokes Problem

In this section, we derive sufficient pointwise bounds on the temporal Green function for the linear Stokes problem (3.4). Precisely, we prove the following:

Proposition 3.3. *Let $G_\alpha(t, y; z)$ be the Green function of the Stokes problem (3.4). There holds*

$$G_\alpha(t, y; z) = H_\alpha(t, y; z) + R_\alpha(t, y; z),\tag{3.7}$$

in which $H_\alpha(t, y; z)$ is exactly the one-dimensional heat kernel with the homogenous Neumann boundary condition and $R_\alpha(t, y; z)$ is the residual kernel due to the boundary condition. Precisely, There hold

$$\begin{aligned}H_\alpha(t, y; z) &= \frac{1}{\sqrt{\nu t}} \left(e^{-\frac{|y-z|^2}{4\nu t}} + e^{-\frac{|y+z|^2}{4\nu t}} \right) e^{-\alpha^2 \nu t} \text{ and} \\ |\partial_z^k R_\alpha(t, y; z)| &\lesssim \mu_f^{k+1} e^{-\theta_0 \mu_f |y+z|} + (\nu t)^{-\frac{k+1}{2}} e^{-\theta_0 \frac{|y+z|^2}{\nu t}} e^{-\frac{1}{8} \alpha^2 \nu t}\end{aligned}$$

for $y, z \geq 0$, $k \geq 0$, and for some $\theta_0 > 0$ and for $\mu_f = |\alpha| + \frac{1}{\sqrt{\nu}}$.

Remark 3.4. We note that the residual term $R_\alpha(t, y; z)$ contains a term without viscous dissipation $e^{-\alpha^2 v t}$, and this is precisely due to the $|\alpha|$ term in the boundary condition in the linear Stokes problem (3.5). Observe that $\omega_\alpha = \alpha e^{-\alpha z}$ is an exact stationary solution to the linear homogenous Stokes problem (3.4).

Remark 3.5. By the reflection method (e.g., [26]), the residual Green kernel can be explicitly defined by

$$R_\alpha(t, y; z) = 2e^{-\alpha^2 v t} (\alpha^2 + \alpha \partial_z) (-\Delta_\alpha^{-1}) G(vt, y + z), \quad (3.8)$$

with $G(t, z) = \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t}$. The pointwise bounds as derived in Proposition 3.3 are in particular useful in propagating unbounded vorticity with boundary layer behaviors.

We proceed the construction of the Green function via the resolvent equation. Namely, for each fixed $y \geq 0$, let $G_{\lambda, \alpha}(y, z)$ be the L^1 solution to the resolvent problem

$$\begin{aligned} (\lambda - v \Delta_\alpha) G_{\lambda, \alpha}(y, z) &= \delta_y(z) \\ v(\partial_z + |\alpha|) G_{\lambda, \alpha}(y, 0) &= 0. \end{aligned} \quad (3.9)$$

We then obtain

Lemma 3.6. *Let $\mu = v^{-1/2} \sqrt{\lambda + \alpha^2 v}$, having positive real part. There holds*

$$G_{\lambda, \alpha}(y, z) = H_{\lambda, \alpha}(y, z) + R_{\lambda, \alpha}(y, z), \quad (3.10)$$

in which $H_{\lambda, \alpha}(y, z)$ denotes the resolvent kernel of the heat problem with homogeneous Neumann boundary condition; namely,

$$H_{\lambda, \alpha}(y, z) = \frac{1}{2v\mu} \left(e^{-\mu|y-z|} + e^{-\mu|y+z|} \right), \quad R_{\lambda, \alpha}(y, z) = \frac{\alpha(\alpha + \mu)}{\lambda\mu} e^{-\mu|y+z|}.$$

In particular, $G_{\lambda, \alpha}(y, z)$ is meromorphic with respect to λ in $\mathbb{C} \setminus \{-\alpha^2 v - \mathbb{R}_+\}$ with a pole at $\lambda = 0$.

Proof. The construction is standard, upon noting that $G_{\lambda, \alpha}(y, z)$ is a linear combination of $e^{\pm \mu z}$ and satisfies the following jump conditions across $z = y$:

$$[G_{\lambda, \alpha}(y, z)]|_{z=y} = 0, \quad [v\partial_z G_{\lambda, \alpha}(y, z)]|_{z=y} = 1.$$

The lemma follows. \square

Proof of Proposition 3.3. The temporal Green function $G_\alpha(t, z; y)$ can then be constructed via the inverse Laplace transform

$$G_\alpha(t, y; z) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} G_{\lambda, \alpha}(y, z) d\lambda, \quad (3.11)$$

in which the contour of integration Γ is taken such that it remains on the right of the (say, L^2) spectrum of the linear operator $\lambda - v \Delta_\alpha$, which is $-\alpha^2 v - \mathbb{R}_+$.

In view of (3.10), we set $H_\alpha(t, y; z)$ and $R_\alpha(t, y; z)$ to be the corresponding temporal Green function of $H_{\lambda, \alpha}(y, z)$ and $R_{\lambda, \alpha}(y, z)$, respectively. It follows that $H_\alpha(t, y; z)$ is the temporal Green function of the one-dimensional heat problem with the homogenous Neumann boundary condition, yielding

$$H_\alpha(t, y; z) = \frac{1}{\sqrt{4\pi vt}} \left(e^{-\frac{|y-z|^2}{4vt}} + e^{-\frac{|y+z|^2}{4vt}} \right) e^{-v\alpha^2 t}.$$

It remains to compute the residual Green function $R_\alpha(t, y; z)$:

$$R_\alpha(t, y; z) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha(\alpha + \mu)}{\lambda \mu} d\lambda. \quad (3.12)$$

Note that the z -derivative of $R_\alpha(t, y; z)$ gains an extra μ in the above integral.

Case 1: $\alpha^2 v \leq 1$. By the Cauchy's theory, we may decompose the contour of integration as $\Gamma = \Gamma_{\pm} \cup \Gamma_c$, having

$$\begin{aligned} \Gamma_{\pm} &:= \left\{ \lambda = -\frac{1}{2}v\alpha^2 + v(a^2 - b^2) + 2viab \pm iM, \quad \pm b \in \mathbb{R}_+ \right\}, \\ \Gamma_c &:= \left\{ \lambda = -\frac{1}{2}v\alpha^2 + va^2 + Me^{i\theta}, \quad \theta \in [-\pi/2, \pi/2] \right\} \end{aligned} \quad (3.13)$$

for some positive number M and for $a = \frac{|y+z|}{2vt}$. Since $\alpha^2 v \leq 1$, we take M large enough so that the pole $\lambda = 0$ remains on the left of the contour Γ . It is clear that $|\lambda| \gtrsim 1$ on Γ .

On Γ_c , we note that

$$\begin{aligned} \Re \mu &= v^{-1/2} \Re \sqrt{\frac{1}{2}v\alpha^2 + va^2 + Me^{i\theta}} \geq v^{-1/2} \sqrt{\frac{1}{2}v\alpha^2 + va^2} \geq a, \\ \Re \mu &= v^{-1/2} \Re \sqrt{\frac{1}{2}v\alpha^2 + va^2 + Me^{i\theta}} \geq v^{-1/2} \sqrt{M}. \end{aligned}$$

This implies that $\Re \mu \geq \frac{a}{2} + \theta_0 \mu_f$ for some positive constant θ_0 , recalling $\mu_f = \alpha + 1/\sqrt{v}$ and $\alpha \leq v^{-1/2}$. In particular, $|\mu| \gtrsim \mu_f \geq \alpha$. This proves that

$$\begin{aligned} \left| \int_{\Gamma_c} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha(\alpha + \mu)}{\lambda \mu} d\lambda \right| &\lesssim \int_{-\pi/2}^{\pi/2} e^{Mt} e^{a^2 vt} e^{-\frac{a}{2}|y+z|} e^{-\theta_0 \mu_f |y+z|} \mu_f d\theta \\ &\lesssim \mu_f e^{-\theta_0 \mu_f |y+z|} e^{a^2 vt} e^{-\frac{a}{2}|y+z|} \\ &\lesssim \mu_f e^{-\theta_0 \mu_f |y+z|}, \end{aligned}$$

in which we used $e^{a^2 vt} e^{-\frac{a}{2}|y+z|} = 1$ by definition of a . As for derivatives, we estimate

$$\begin{aligned} \left| \int_{\Gamma_c} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha(\alpha + \mu)}{\lambda} d\lambda \right| &= \left| \int_{\Gamma_c} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha}{v(\mu - \alpha)} d\lambda \right| \\ &\lesssim v^{-1} \int_{-\pi/2}^{\pi/2} e^{Mt} e^{a^2 vt} e^{-\frac{a}{2}|y+z|} e^{-\theta_0 \mu_f |y+z|} d\theta \\ &\lesssim \mu_f^2 e^{-\theta_0 \mu_f |y+z|}, \end{aligned}$$

upon recalling $\mu_f = \alpha + 1/\sqrt{v}$.

Meanwhile, on Γ_{\pm} , we note that

$$\Re \mu = \Re \sqrt{\frac{1}{2}\alpha^2 + (a+ib)^2 \pm i\sqrt{v}^{-1}M} \geq \Re \sqrt{(a+ib)^2} = a,$$

upon noting that the sign of b and $\pm M$ is the same on Γ_{\pm} . Similarly, we note that $\Re \mu \gtrsim M/\sqrt{v}$. By the definition of a , we have

$$|e^{\lambda t} e^{-\mu|y+z|}| \leq e^{-\frac{1}{2}v\alpha^2 t} e^{-\frac{|y+z|^2}{4vt}} e^{-vb^2 t},$$

and together with the fact that $\lambda = v(\mu^2 - \alpha^2)$, we compute

$$\frac{\alpha(\alpha + \mu)}{\lambda \mu} d\lambda = \frac{2i\alpha(a+ib)}{\mu(\mu - \alpha)} db.$$

Since $\alpha^2 v \lesssim 1$, we have $\alpha \lesssim |\mu|$. In addition, we have $(a+ib)^2 \pm i\sqrt{v}^{-1}M = \mu^2 - \alpha^2$ on Γ_{\pm} with b having the same sign as does $\pm M$. This implies that $|a+ib|^2 \leq |\mu^2 - \alpha^2| \lesssim |\mu|^2$. Putting the above computations together, we obtain

$$\begin{aligned} \left| \int_{\Gamma_{\pm}} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha(\alpha + \mu)}{\lambda \mu} d\lambda \right| &\leq C_0 e^{-\frac{1}{2}v\alpha^2 t} e^{-\frac{|y+z|^2}{4vt}} \int_{\mathbb{R}} e^{-vb^2 t} \frac{\alpha|a+ib|}{|\mu(\mu - \alpha)|} db \\ &\leq C_0 e^{-\frac{1}{2}v\alpha^2 t} e^{-\frac{|y+z|^2}{4vt}} \int_{\mathbb{R}} e^{-vb^2 t} db \\ &\leq C_0(vt)^{-1/2} e^{-\frac{1}{2}v\alpha^2 t} e^{-\frac{|y+z|^2}{4vt}}. \end{aligned}$$

As for the derivatives, we estimate

$$\begin{aligned} \left| \int_{\Gamma_{\pm}} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha(\alpha + \mu)}{\lambda} d\lambda \right| &\leq C_0 e^{-\frac{1}{2}v\alpha^2 t} e^{-\frac{|y+z|^2}{4vt}} \int_{\mathbb{R}} e^{-vb^2 t} (a + |b|) db \\ &\leq C_0(vt)^{-1} \left(1 + \frac{|x+y|}{\sqrt{vt}} \right) e^{-\frac{1}{2}v\alpha^2 t} e^{-\frac{|y+z|^2}{4vt}} \\ &\leq C_0(vt)^{-1} e^{-\frac{1}{2}v\alpha^2 t} e^{-\frac{|y+z|^2}{8vt}}. \end{aligned}$$

Case 2: $\alpha^2 v \geq 1$. Take $a = \frac{|y+z|}{2\sqrt{vt}}$ as in the previous case. Consider first the case when $|a - \alpha| \geq \frac{1}{2}\alpha$. In this case, we move the contour of integration to

$$\Gamma_1 := \left\{ \lambda = -v\alpha^2 + v(a^2 - b^2) + 2vib, \quad \pm b \in \mathbb{R}_+ \right\},$$

which may pass the pole at $\lambda = 0$ (precisely, it does when $a = \alpha$). By Cauchy's theory, we have

$$R_{\alpha}(t, y; z) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha(\alpha + \mu)}{\lambda \mu} d\lambda + \text{Res}_0,$$

in which the residue at the pole $\lambda = 0$ is given by

$$\text{Res}_0 = 2\alpha e^{-\alpha|y+z|} \tag{3.14}$$

when $a < \alpha$. We take $\text{Res}_0 = 0$ when $a > \alpha$. Note that the residue does not decay in time. This accounts for the contribution of the inhomogenous Neumann boundary condition. Since $\alpha^2 v \geq 1$, we have $\mu_f = \alpha + 1/\sqrt{v} \leq 2\alpha$, and hence

$$\text{Res}_0 \leq 2\mu_f e^{-\frac{1}{2}\mu_f|y+z|}.$$

As for the integral term, we note that $\mu = a + ib$, and hence

$$\frac{\alpha(\alpha + \mu)}{\lambda\mu} d\lambda = \frac{2iv\alpha(\alpha + \mu)}{\lambda} db.$$

Note that Γ_1 cuts the real axis at $v(a^2 - \alpha^2)$ and the imaginary axis at $2vab_0$ (when $a > \alpha$), with $b_0 = \pm\sqrt{a^2 - \alpha^2}$. This, in particular, yields $|\lambda| \gtrsim v\alpha(a + \alpha)$ and $|\mu| = v^{-1/2}|\sqrt{\lambda + \alpha^2}v| \lesssim v^{-1/2}|\lambda|^{1/2}$. Moreover,

$$\left| \frac{\alpha(\alpha + \mu)}{\lambda\mu} d\lambda \right| \leq db.$$

We thus obtain

$$\begin{aligned} \left| \int_{\Gamma_1} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha(\alpha + \mu)}{\lambda\mu} d\lambda \right| &\leq C_0 e^{-v\alpha^2 t} e^{-\frac{|y+z|^2}{4vt}} \int_{\mathbb{R}} e^{-vb^2 t} db \\ &\leq C_0 (vt)^{-1/2} e^{-v\alpha^2 t} e^{-\frac{|y+z|^2}{4vt}}. \end{aligned} \quad (3.15)$$

It remains to consider the case when $|a - \alpha| \leq \frac{1}{2}\alpha$ and $\alpha^2 v \geq 1$. We note in particular that $\frac{1}{2}\alpha \leq a \leq \frac{3}{2}\alpha$. In this case, we simply modify the contour of integration by taking

$$\Gamma_1 := \left\{ \lambda = -\frac{1}{8}v\alpha^2 + v(a^2 - b^2) + 2vab, \quad \pm b \in \mathbb{R}_+ \right\}.$$

Observe that the contour Γ_1 leaves the origin on the left, with $|\lambda| \gtrsim v\alpha^2$. The integral is thus estimated exactly as done in (3.15). The derivative estimates follow as in the previous case.

The proof of Proposition 3.3 is complete. \square

3.3. The Green Function on Ω_σ

The Green function constructed in Proposition 3.3 can be easily extended to the complex domain Ω_σ defined by

$$\Omega_\sigma = \left\{ z \in \mathbb{C} : \quad |\Im z| < \min\{\sigma|\Re z|, \sigma\} \right\},$$

for some small $\sigma > 0$. Indeed, in view of (3.8), the Green function involves precisely the heat kernel $G(t, z) = \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t}$, which is extended to the complex domain. In addition, we note that for $z \in \Gamma_\sigma$, there holds $\Im z \leq \sigma\Re z$, which implies that

$$|e^{-z^2/4t}| \leq e^{-|\Re z|^2/4t + |\Im z|^2/4t} \leq e^{-(1-\sigma^2)|\Re z|^2/4t}.$$

Similar estimates hold for the other terms in the Green function $G_\alpha(t, y; z) = H_\alpha(t, y; z) + R_\alpha(t, y; z)$, yielding

$$\begin{aligned} H_\alpha(t, y; z) &\lesssim \frac{1}{\sqrt{vt}} \left(e^{-(1-\sigma^2) \frac{|\Re y - \Re z|^2}{4vt}} + e^{-(1-\sigma^2) \frac{|\Re y + \Re z|^2}{4vt}} \right) e^{-\frac{1}{8}\alpha^2 vt}, \\ R_\alpha(t, y; z) &\lesssim \mu_f e^{-\theta_0(1-\sigma)\mu_f |\Re y + \Re z|}, \end{aligned} \quad (3.16)$$

for $y, z \in \Gamma_\sigma$, and for some $\theta_0 > 0$ and for $\mu_f = |\alpha| + \frac{1}{\sqrt{v}}$.

The solution $\omega_\alpha(t, z)$ to the Stokes problem can now be constructed on Ω_σ in a manner similar to that which was done for (2.9). Precisely, for any $z \in \Omega_\sigma$, let θ be the positive constant so that $z \in \partial\Omega_\theta$. The Duhamel principle (3.6) then becomes

$$\begin{aligned} \omega_\alpha(t, z) &= \int_{\partial\Omega_\theta} G_\alpha(t, y; z) \omega_{0,\alpha}(y) \, dy + \int_0^t G_\alpha(t-s, 0; z) g_\alpha(s) \, ds \\ &\quad + \int_0^t \int_{\partial\Omega_\theta} G_\alpha(t-s, y; z) f_\alpha(s, y) \, dy \, ds, \end{aligned} \quad (3.17)$$

which is well-defined for $z \in \Omega_\sigma$; having the Green function $G_\alpha(t, y; z)$ satisfies the pointwise estimates (3.16), similar to those on the real line. For this reason, it suffices to derive convolution estimates for real values y, z .

3.4. Convolution Estimates

We now derive convolution estimates. We start with the analytic L^1 norms. For $k \geq 0$, we introduce

$$\|\omega_\alpha\|_{\mathcal{W}_\sigma^{k,1}} = \sum_{j=0}^k \|(\psi(z)\partial_z)^j \omega_\alpha\|_{L_\sigma^1}.$$

We prove the following:

Proposition 3.7. *Let $T > 0$ and let $G_\alpha(t, y; z)$ be the Green function of the Stokes problem (3.4), constructed in Proposition 3.3. Then, for any $0 \leq s < t \leq T$ and $k \geq 0$, there is a universal constant C_T so that*

$$\begin{aligned} \left\| \int_0^\infty G_\alpha(t, y; \cdot) \omega_\alpha(y) \, dy \right\|_{\mathcal{W}_\sigma^{k,1}} &\leq C_T \|\omega_\alpha\|_{\mathcal{W}_\sigma^{k,1}}, \\ \left\| \int_0^\infty G_\alpha(t-s, y; \cdot) \omega_\alpha(y) \, dy \right\|_{\mathcal{W}_\sigma^{k,1}} &\leq C_T \|\omega_\alpha\|_{\mathcal{W}_\sigma^{k,1}}, \end{aligned}$$

uniformly in the inviscid limit.

Proof. We shall prove the convolution for real values y, z . For the complex extension, see Section 3.3. Recall from Proposition 3.3 that $G_\alpha(t, y; z) = H_\alpha(t, y; z) + R_\alpha(t, y; z)$, with

$$\begin{aligned} H_\alpha(t, y; z) &= \frac{1}{\sqrt{vt}} \left(e^{-\frac{|y-z|^2}{4vt}} + e^{-\frac{|y+z|^2}{4vt}} \right) e^{-\alpha^2 vt}, \\ |\partial_z^k R_\alpha(t, y; z)| &\lesssim \mu_f^k e^{-\theta_0 \mu_f |y+z|} + (vt)^{-\frac{k+1}{2}} e^{-\theta_0 \frac{|y+z|^2}{vt}} e^{-\frac{1}{8}\alpha^2 vt} \end{aligned}$$

for $k \geq 0$. In particular, $\|G_\alpha(t, y; \cdot)\|_{L_\sigma^1} \lesssim 1$, for each fixed y, t . The L^1 convolution estimate is thus straightforward. We now check the estimates for derivatives. We estimate

$$\begin{aligned} \psi(z) \partial_z R_\alpha(t, y; z) &\lesssim |z| \mu_f^2 e^{-\theta_0 \mu_f |y+z|} + \frac{|z|}{\sqrt{vt}} e^{-\theta_0 \frac{|y+z|^2}{vt}} e^{-\frac{1}{8} \alpha^2 vt} \\ &\lesssim \mu_f e^{-\frac{1}{2} \theta_0 \mu_f |y+z|} + \frac{1}{\sqrt{vt}} e^{-\theta_0 \frac{|y+z|^2}{2vt}} e^{-\frac{1}{8} \alpha^2 vt}. \end{aligned}$$

That is, $\psi(z) \partial_z R_\alpha(t, y; z)$ obeys essentially the same bound as does $R_\alpha(t, y; z)$. The convolution estimates for derivatives of $R_\alpha(t, y; z)$ follow.

Next, we treat the integral involving $H_\alpha(t, y; z)$. Precisely, we set $H(t, y; z) = \frac{1}{\sqrt{vt}} e^{-\frac{|y-z|^2}{4vt}}$. Note that $\partial_z H(t, y; z) = -\partial_y H(t, y; z)$. Hence, we compute

$$\begin{aligned} &\int_0^\infty \psi(z) \partial_z H(t-s, y; \cdot) \omega_\alpha(y) \, dy \\ &= \int_0^{z/2} \psi(z) \partial_z H(t-s, y; \cdot) \omega_\alpha(y) \, dy - \psi(z) H(t-s, z/2; z) \omega_\alpha(z/2) \\ &\quad + \int_{z/2}^\infty \psi(z) H(t-s, y; \cdot) \partial_y \omega_\alpha(y) \, dy. \end{aligned}$$

We now estimate each term on the right. Since $\psi(z) \leq 2\psi(y)$ for $y \geq z/2$, the last integral on the right is already estimated in the previous case with $\omega_\alpha(y)$ replaced by $\psi(y) \partial_y \omega_\alpha(y)$. As for the first integral, since $y \leq z/2$, we compute

$$\begin{aligned} \psi(z) \partial_z H(t-s, y; z) &\lesssim \frac{z}{1+z} (v(t-s))^{-1} e^{-\frac{|y-z|^2}{8v(t-s)}} \\ &\lesssim |y-z| (v(t-s))^{-1} e^{-\frac{|y-z|^2}{8v(t-s)}} \\ &\lesssim (v(t-s))^{-1/2} e^{-\frac{|y-z|^2}{16v(t-s)}}. \end{aligned}$$

Thus, the integral over $[0, z/2]$ is again already estimated in the previous case. Finally, we compute

$$\begin{aligned} |\psi(z) H(t-s, z/2; z) \omega_\alpha(z/2)| &\lesssim z (v(t-s))^{-1/2} e^{-\frac{|z|^2}{16v(t-s)}} |\omega_\alpha(z/2)| \\ &\lesssim |\omega_\alpha(z/2)|, \end{aligned}$$

whose L_σ^1 norm is clearly bounded by $\|\omega_\alpha\|_{L_\sigma^1}$. \square

3.5. Convolution Estimates with Boundary Layer Behaviors

In this section, we provide the convolution estimates of the Green function against functions in the boundary layer spaces, whose norm is defined by

$$\|\omega_\alpha\|_{\sigma, \delta(t)} = \sup_{z \in \Omega_\sigma} |\omega_\alpha(z)| e^{\beta \Re z} \left(1 + \delta_t^{-1} \phi_P(\delta_t^{-1} z) + \delta^{-1} \phi_P(\delta^{-1} z)\right)^{-1}, \quad (3.18)$$

for $t > 0$ and $\beta > 0$, in which the boundary thicknesses are $\delta_t = \sqrt{\nu t}$ and $\delta = \sqrt{\nu}$ and for boundary layer weight $\phi_P(z) = \frac{1}{1+|\Re z|^P}$, $P > 1$. We also introduce the boundary norm for derivatives:

$$\|\omega_\alpha\|_{\sigma, \delta(t), k} = \sum_{j=0}^k \|(\psi(z)\partial_z)^j \omega_\alpha\|_{\sigma, \delta(t)}$$

for $k \geq 0$. In the case $t = 0$, the norm $\|\cdot\|_{\sigma, \delta(0)}$ is defined to consist of precisely one boundary layer with thickness $\delta = \sqrt{\nu}$.

We prove the following:

Proposition 3.8. *Let $T > 0$ and let $G_\alpha(t, y; z)$ be the Green function of the Stokes problem (3.4), constructed in Proposition 3.3. Then, for any $0 \leq s < t \leq T$ and $k \geq 0$, there is a universal constant C_T so that*

$$\begin{aligned} \left\| \int_0^\infty G_\alpha(t, y; \cdot) \omega_\alpha(y) dy \right\|_{\sigma, \delta(t), k} &\leq C_T \|\omega_\alpha\|_{\sigma, \delta(0), k}, \\ \left\| \int_0^\infty G_\alpha(t-s, y; \cdot) \omega_\alpha(y) dy \right\|_{\sigma, \delta(t), k} &\leq C_T \sqrt{\frac{t}{s}} \|\omega_\alpha\|_{\sigma, \delta(s), k} \end{aligned}$$

uniformly in the inviscid limit.

We shall prove the convolution estimates for real values y, z . The complex extension follows from the similar estimates on the Green function obtained in (3.16). As a consequence, Proposition 3.8 is a direct combination of the following two lemmas:

Lemma 3.9. *Let $R(t, y; z) := \mu_f e^{-\mu_f |y+z|}$, with $\mu_f = \alpha + \frac{1}{\sqrt{\nu}}$. Then, for any s, t , and $k \geq 0$, there is a universal constant C_0 so that*

$$\left\| \int_0^\infty R(t-s, y; z) \omega_\alpha(y) dy \right\|_{\sigma, \delta(t), k} \leq C_0 \|\omega_\alpha\|_{\sigma, \delta(s), k}.$$

Proof. The estimate for $k = 0$ follows directly from the fact that ω_α belongs to L^1 : $\|\omega_\alpha\|_{L^1} \leq C_0 \|\omega_\alpha\|_{\sigma, \delta(s)}$. The convolution in fact belongs to the boundary layer space with finite norm $\|\cdot\|_{\sigma, \delta(0)}$. As for derivatives, we note that $|\psi(z)\partial_z R(t, y; z)| \leq z\mu_f^2 e^{-\mu_f |y+z|} \leq C_0 \mu_f e^{-\frac{1}{2}\mu_f z}$. The derivative estimates thus follow identically. \square

Lemma 3.10. *Let $H(t, y; z) := (vt)^{-1/2} e^{-\frac{|y+z|^2}{M\nu t}}$, for some positive M , and let $T > 0$. Then, for any $0 \leq s < t \leq T$, $\varepsilon_0 > 0$, and $k \geq 0$, there is a universal constant C_T so that*

$$\begin{aligned} \left\| \int_0^\infty H(t, y; \cdot) \omega_\alpha(y) dy \right\|_{\sigma, \delta(t), k} &\leq C_T \|\omega_\alpha\|_{\sigma, \delta(0), k}, \\ \left\| \int_0^\infty H(t-s, y; \cdot) \omega_\alpha(y) dy \right\|_{\sigma, \delta(t), k} &\leq C_T \sqrt{\frac{t}{s}} \|\omega_\alpha\|_{\sigma, \delta(s), k} \end{aligned}$$

uniformly in the inviscid limit.

Proof. It suffices to prove the convolution for $H(t, y; z) = (\nu t)^{-1/2} e^{-\frac{|y-z|^2}{M\nu t}}$. We start with the case $k = 0$. Let $0 \leq s < t$. For $|y - z| \geq M\beta\nu(t - s)$, it is clear that

$$e^{-\frac{|y-z|^2}{M\nu(t-s)}} e^{-\beta|y|} \leqq e^{-\beta|z|} e^{-|y-z|\left(\frac{|y-z|}{M\nu(t-s)} - \beta\right)} \leqq e^{-\beta|z|},$$

whereas, for $|y - z| \leq M\beta\nu(t - s)$, we note that

$$e^{-M\beta^2\nu(t-s)} e^{-\beta|y|} \leqq e^{-\beta|y-z|} e^{-\beta|y|} \leqq e^{-\beta|z|}.$$

That is, the exponential decay $e^{-\beta z}$ is recovered at an expense of a slowly growing term in time $e^{M\beta^2\nu(t-s)}$ which is bounded in finite time. Precisely, this proves

$$e^{-\frac{|y-z|^2}{M\nu(t-s)}} e^{-\beta y} \leqq e^{M\beta^2\nu(t-s)} e^{-\beta|z|}, \quad \forall y, z \in \mathbb{R}. \quad (3.19)$$

It remains to study the integral

$$\int_0^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \left(1 + \delta_s^{-1} \phi_P(\delta_s^{-1} y) + \delta^{-1} \phi_P(\delta^{-1} y)\right) dy. \quad (3.20)$$

First, without the boundary layer behavior, the integral is clearly bounded. We now treat the boundary layer terms. Using the fact that $\phi_P(\cdot)$ is decreasing, we have

$$\begin{aligned} & \int_{z/2}^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \delta^{-1} \phi_P(\delta^{-1} y) dy \\ & \leqq C_0 \delta^{-1} \phi_P(\delta^{-1} z) \int_{z/2}^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} dy \\ & \leqq C_0 \delta^{-1} \phi_P(\delta^{-1} z), \end{aligned}$$

and, upon noting that $y/\delta_s \geq z/2\delta_t$, we obtain

$$\begin{aligned} & \int_{z/2}^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \delta_s^{-1} \phi_P(\delta_s^{-1} y) dy \\ & \leqq C_0 \delta_s^{-1} \phi_P(\delta_t^{-1} z) \int_{z/2}^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} dy \\ & \leqq C_0 \sqrt{t/s} \delta_t^{-1} \phi_P(\delta_t^{-1} z). \end{aligned}$$

By contrast on $y \in (0, \frac{z}{2})$, we have $|y - z| \geq \frac{z}{2}$ and $\phi_P \leqq 1$. Hence, we have

$$\begin{aligned} & \int_0^{z/2} (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \delta_s^{-1} \phi_P(\delta_s^{-1} y) dy \\ & \leqq e^{-\frac{|z|^2}{8M\nu(t-s)}} \int_0^{z/2} (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{2M\nu(t-s)}} \delta_s^{-1} \phi_P(\delta_s^{-1} y) dy \\ & \leqq C_0 e^{-\frac{|z|^2}{8M\nu t}} \min \left\{ \delta_s^{-1}, \delta_{t-s}^{-1} \right\}. \end{aligned}$$

Note that $\min\{\delta_s^{-1}, \delta_{t-s}^{-1}\} \leq 2\delta_t^{-1}$. Hence, the above integral is bounded in $\|\cdot\|_{\sigma, \delta(t)}$ norm. Similarly, we estimate

$$\int_0^{z/2} (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \delta^{-1} \phi_P(\delta^{-1}y) dy \leq C_0 e^{-\frac{|z|^2}{8M\nu(t-s)}} \delta^{-1}.$$

To estimate this, we will prove that

$$e^{-\frac{|z|^2}{8M\nu t}} \leq C_0 e^{\varepsilon_0 t} \phi_P(\delta^{-1}z) \quad (3.21)$$

for arbitrarily small ε_0 (and hence, C_0 depends on ε_0). Indeed, when $|z| \geq \varepsilon_0 \sqrt{\nu t}$, it is clear that $e^{-\frac{|z|^2}{8M\nu t}} \lesssim \phi_P(\delta^{-1}z)$. On the other hand, when $|z| \leq \varepsilon_0 \sqrt{\nu t}$, we note that $e^{z/\delta} \leq e^{\varepsilon_0 t}$, which implies that $1 \leq e^{\varepsilon_0 t} e^{-z/\delta} \leq C_0 e^{\varepsilon_0 t} \phi_P(\delta^{-1}z)$. The estimate (3.21) follows, and hence the claimed estimate for $k = 0$.

Next, we consider the derivative estimate. Note that $\partial_z H(t, y; z) = -\partial_y H(t, y; z)$. Hence, we compute

$$\begin{aligned} & \int_0^\infty \psi(z) \partial_z H(t-s, y; \cdot) \omega_\alpha(y) dy \\ &= \int_0^{z/2} \psi(z) \partial_z H(t-s, y; \cdot) \omega_\alpha(y) dy - \psi(z) H(t-s, z/2; z) \omega_\alpha(z/2) \\ &+ \int_{z/2}^\infty \psi(z) H(t-s, y; \cdot) \partial_y \omega_\alpha(y) dy. \end{aligned}$$

We now estimate each term on the right. Since $\psi(z) \leq 2\psi(y)$ for $y \geq z/2$, the last integral on the right is already estimated in the previous case with $\omega_\alpha(y)$ replaced by $\psi(y) \partial_y \omega_\alpha(y)$. As for the first integral, since $y \leq z/2$, we compute

$$\begin{aligned} \psi(z) \partial_z H(t-s, y; z) &\lesssim \frac{z}{1+z} (\nu(t-s))^{-1} e^{-\frac{|y-z|^2}{2M\nu(t-s)}} \\ &\lesssim |y-z| (\nu(t-s))^{-1} e^{-\frac{|y-z|^2}{2M\nu(t-s)}} \\ &\lesssim (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{4M\nu(t-s)}}. \end{aligned}$$

Thus, the integral over $[0, z/2]$ is again already estimated in the previous case. Finally, we compute

$$\begin{aligned} & |\psi(z) H(t-s, z/2; z) \omega_\alpha(z/2)| \\ &\lesssim z (\nu(t-s))^{-1/2} e^{-\frac{|z|^2}{4M\nu(t-s)}} \left(1 + \delta_s^{-1} \phi_P(\delta_s^{-1}z) + \delta^{-1} \phi_P(\delta^{-1}z) \right) \\ &\lesssim 1 + \delta_s^{-1} \phi_P(\delta_s^{-1}z) + \delta^{-1} \phi_P(\delta^{-1}z) \\ &\lesssim 1 + \sqrt{t/s} \delta_t^{-1} \phi_P(\delta_t^{-1}z) + \delta^{-1} \phi_P(\delta^{-1}z), \end{aligned}$$

in which the last inequality is again due to the decreasing property of ϕ_P and the fact that $\delta_s \leq \delta_t$. This completes the proof of the lemma. \square

3.6. Semigroup Bounds in Analytic Spaces

In this section, we shall prove Proposition 3.1 on deriving uniform bounds for the Stokes semigroup in analytic spaces, with the analytic norm

$$\|\omega\|_{\rho, \sigma, \delta(t)} = \sum_{\alpha \in \mathbb{Z}} e^{\rho|\alpha|} \|\omega_\alpha\|_{\sigma, \delta(t)}.$$

We first write the Stokes semigroup e^{tB} and the operator $\Gamma(t)$ in the Fourier series:

$$e^{vtB}\omega = \sum_{\alpha \in \mathbb{Z}} e^{i\alpha x} (e^{vtB}\omega)_\alpha, \quad \Gamma(vt)g = \sum_{\alpha \in \mathbb{Z}} e^{i\alpha x} (\Gamma(vt)g)_\alpha, \quad (3.22)$$

in which

$$(e^{vtB}\omega)_\alpha = \int_0^\infty G_\alpha(t, y; z) \omega_\alpha(y) dy, \quad (\Gamma(vt)g)_\alpha = G_\alpha(t, 0; z) g_\alpha, \quad (3.23)$$

with the Green kernel $G_\alpha(t, y; z)$ constructed in Proposition 3.3. The convolution estimates obtained in Proposition 3.8 yield

$$\|(e^{v(t-s)B}\omega)_\alpha\|_{\sigma, \delta(t), k} \leq C_0 \sqrt{\frac{t}{s}} \|\omega_\alpha\|_{\sigma, \delta(s), k}.$$

These prove the claimed estimates on e^{vtB} . As for the trace operator, using Proposition 3.3, we have

$$\begin{aligned} G_\alpha(t-s, 0; z) &\lesssim (v(t-s))^{-1/2} e^{-\frac{z^2}{4v(t-s)}} + \mu_f e^{-\mu_f z} \\ &\lesssim (v(t-s))^{-1/2} e^{-\frac{z^2}{4vt}} + \mu_f e^{-\mu_f z}. \end{aligned} \quad (3.24)$$

By definition of the boundary layer norm, $\|G_\alpha(t-s, 0; z)\|_{\sigma, \delta(t)} \lesssim \sqrt{\frac{t}{t-s}}$. The proof of Proposition 3.1 is complete.

4. Proof of the Main Theorems

As mentioned in the introduction, we construct the solutions to the Navier–Stokes equation via the vorticity formulation

$$\begin{aligned} \partial_t \omega - v \Delta \omega &= -u \cdot \nabla \omega \\ v(\partial_z + |x|) \omega|_{z=0} &= [\partial_z \Delta^{-1}(u \cdot \nabla \omega)]|_{z=0}, \end{aligned} \quad (4.1)$$

in which $u = \nabla^\perp \Delta^{-1} \omega$, with Δ^{-1} being the inverse of Laplacian with the Dirichlet boundary condition. For convenience, we set $N = u \cdot \nabla \omega$. The solution to (4.1) is then constructed via Duhamel's principle as follows:

$$\omega(t) = e^{vtB}\omega_0 - \int_0^t e^{v(t-s)B} N(s) ds + \int_0^t \Gamma(v(t-s))(\partial_z \Delta^{-1} N(s))|_{z=0} ds, \quad (4.2)$$

with $\omega_0 \in \mathcal{B}^{\rho_0, \sigma_0, \delta}$, for some $\rho_0, \sigma_0 > 0$.

4.1. Nonlinear Iteration

Let us fix positive numbers γ , ζ , and ρ_0 , and introduce the following nonlinear iterative norm for vorticity:

$$A(\gamma) = \sup_{0 < \gamma t < \rho_0} \sup_{\rho < \rho_0 - \gamma t} \left\{ |||\omega(t)|||_{\mathcal{W}_{\rho,\rho}^{1,1}} + |||\omega(t)|||_{\mathcal{W}_{\rho,\rho}^{2,1}} (\rho_0 - \rho - \gamma t)^\zeta \right\}, \quad (4.3)$$

recalling

$$|||\omega(t)|||_{\mathcal{W}_{\rho,\rho}^{k,1}} = \sum_{j+\ell \leq k} \|\partial_x^j (\psi(z) \partial_z)^\ell \omega(t)\|_{L_{\rho,\rho}^1}.$$

Here, for the sake of presentation, we take the same analyticity radius in x and z ; namely, $\sigma = \rho < \rho_0$. Thanks to Lemma 2.1, $\omega_0 \in \mathcal{W}_{\rho,\rho}^{k,1}$, for any $k \geq 0$.

We shall show that the vorticity norm remains finite for sufficiently large γ . The weight $(\rho_0 - \rho - \gamma t)^\zeta$, with a small $\zeta > 0$, is standard to avoid time singularity when recovering the loss of derivatives [4, 34]. Let $\rho < \rho_0 - \gamma t$. Thanks to Lemma 2.4, we have

$$\begin{aligned} |||N(t)|||_{\mathcal{W}_{\rho,\rho}^{0,1}} &\lesssim |||\omega(t)|||_{\mathcal{W}_{\rho,\rho}^{1,1}}^2 \leqq A(\gamma)^2 \\ |||N(t)|||_{\mathcal{W}_{\rho,\rho}^{1,1}} &\lesssim |||\omega(t)|||_{\mathcal{W}_{\rho,\rho}^{1,1}} |||\omega(t)|||_{\mathcal{W}_{\rho,\rho}^{2,1}} \leqq A(\gamma)^2 (\rho_0 - \rho - \gamma t)^{-\zeta}. \end{aligned} \quad (4.4)$$

In addition, using the elliptic estimates, we have

$$|||(\partial_z \Delta^{-1} N(t))|_{z=0}|||_{\rho,k} \lesssim |||N(t)|||_{\mathcal{W}_{\rho,\rho}^{k,1}}. \quad (4.5)$$

Now, using the Duhamel integral formula (4.2), we estimate

$$\begin{aligned} |||\omega(t)|||_{\mathcal{W}_{\rho,\rho}^{k,1}} &\leqq |||e^{vtB} \omega_0|||_{\mathcal{W}_{\rho,\rho}^{k,1}} + \int_0^t |||e^{v(t-s)B} N(s)|||_{\mathcal{W}_{\rho,\rho}^{k,1}} ds \\ &\quad + \int_0^t |||\Gamma(v(t-s))(\partial_z \Delta^{-1} N(s))|_{z=0}|||_{\mathcal{W}_{\rho,\rho}^{k,1}} ds. \end{aligned}$$

In view of Proposition 3.1, the term from the initial data is already estimated, giving $|||e^{vtB} \omega_0|||_{\mathcal{W}_{\rho,\rho}^{k,1}} \leqq \|\omega_0\|_{\mathcal{W}_{\rho,\rho}^{k,1}}$. As for the integral terms, we estimate

$$\begin{aligned} \int_0^t |||e^{v(t-s)B} N(s)|||_{\mathcal{W}_{\rho,\rho}^{1,1}} ds &\leqq C_0 \int_0^t |||N(s)|||_{\mathcal{W}_{\rho,\rho}^{1,1}} ds \\ &\leqq C_0 A(\gamma)^2 \int_0^t (\rho_0 - \rho - \gamma s)^{-\zeta} ds \\ &\leqq C_0 \gamma^{-1} A(\gamma)^2. \end{aligned}$$

Similarly,

$$\int_0^t |||\Gamma(v(t-s))(\partial_z \Delta^{-1} N(s))|_{z=0}|||_{\mathcal{W}_{\rho,\rho}^{1,1}} ds \leqq C_0 \int_0^t |||N(s)|||_{\mathcal{W}_{\rho,\rho}^{1,1}} ds,$$

which is again bounded by $C_0\gamma^{-1}A(\gamma)^2$. Next, we give estimates for $k = 2$. Noting that $\rho < \rho_0 - \gamma t \leq \rho_0 - \gamma s$, we take $\rho' = \frac{\rho + \rho_0 - \gamma s}{2}$ and compute

$$\begin{aligned} \int_0^t |||e^{\nu(t-s)B}N(s)|||_{\mathcal{W}_{\rho,\rho}^{2,1}} ds &\leq C_0 \int_0^t |||N(s)|||_{\mathcal{W}_{\rho,\rho}^{2,1}} ds \\ &\leq C_0 \int_0^t \frac{1}{\rho' - \rho} |||N(s)|||_{\mathcal{W}_{\rho',\rho'}^{1,1}} ds \\ &\leq C_0 A(\gamma)^2 \int_0^t (\rho_0 - \rho - \gamma s)^{-1-\zeta} ds \\ &\leq C_0 \gamma^{-1} A(\gamma)^2 (\rho_0 - \rho - \gamma t)^{-\zeta}. \end{aligned}$$

The same computation holds for the trace operator $\Gamma(\nu t)$, yielding

$$A(\gamma) \leq C_0 \|\omega_0\|_{\mathcal{W}_{\rho,\rho}^{2,1}} + C_0 \gamma^{-1} A(\gamma)^2.$$

By taking γ sufficiently large, the above yields the uniform bound on the iterative norm in term of initial data. This yields the local solution in $L_{\rho,\rho}^1$ for $t \in [0, T]$, with $T = \gamma^{-1}\rho_0$.

4.2. Propagation of Boundary Layers

It remains to prove that the constructed solution has the boundary layer behavior as expected, having already constructed solutions in $L_{\rho,\rho}^1$ spaces. Indeed, we now introduce the following nonlinear iterative norm for vorticity:

$$B(\gamma) = \sup_{0 < \gamma t < \rho_0} \sup_{\rho < \rho_0 - \gamma t} \left\{ |||\omega(t)|||_{\rho,\delta(t),1} + |||\omega(t)|||_{\rho,\delta(t),2} (\rho_0 - \rho - \gamma t)^\zeta \right\}, \quad (4.6)$$

with the boundary layer norm

$$|||\omega(t)|||_{\rho,\delta(t),k} = \sum_{j+\ell \leq k} \|\partial_x^j (\psi(z) \partial_z)^\ell \omega(t)\|_{\rho,\rho,\delta(t)}.$$

Thanks to Lemma 2.4, we estimate

$$\begin{aligned} |||N(t)|||_{\rho,\delta(t),0} &\lesssim |||\omega(t)|||_{\rho,\delta(t),1}^2 \leq B(\gamma)^2 \\ |||N(t)|||_{\rho,\delta(t),1} &\lesssim |||\omega(t)|||_{\rho,\delta(t),1} |||\omega(t)|||_{\rho,\delta(t),2} \leq B(\gamma)^2 (\rho_0 - \rho - \gamma t)^{-\zeta}. \end{aligned} \quad (4.7)$$

Now, using the Duhamel integral formula (4.2), we estimate

$$\begin{aligned} |||\omega(t)|||_{\rho,\delta(t),k} &\leq |||e^{\nu t B} \omega_0|||_{\rho,\delta(t),k} + \int_0^t |||e^{\nu(t-s)B} N(s)|||_{\rho,\delta(t),k} ds \\ &\quad + \int_0^t |||\Gamma(\nu(t-s))(\partial_z \Delta^{-1} N(s))|_{z=0}|||_{\rho,\delta(t),k} ds. \end{aligned}$$

In view of Proposition 3.1, the term from the initial data is already estimated, giving $|||e^{vtB}\omega_0|||_{\rho,\delta(t),k} \leq \|\omega_0\|_{\rho,\delta(0),k}$. In addition,

$$\begin{aligned} & \int_0^t |||\Gamma(v(t-s))(\partial_z \Delta^{-1} N(s))|_{z=0}|||_{\rho,\delta(t),k} ds \\ & \lesssim \int_0^t \sqrt{t}(t-s)^{-1/2} \|(\partial_z \Delta^{-1} N(s))|_{z=0}\|_{\rho,k} ds \\ & \lesssim \int_0^t \sqrt{t}(t-s)^{-1/2} |||N(s)|||_{\mathcal{W}_{\rho,\rho}^{k,1}} ds \\ & \lesssim \sup_{0 \leq s \leq t} |||N(s)|||_{\mathcal{W}_{\rho,\rho}^{k,1}} \int_0^t \sqrt{t}(t-s)^{-1/2} ds, \end{aligned}$$

in which $\|N(s)\|_{\mathcal{W}_{\rho,\rho}^{k,1}}$ is bounded, thanks to the iteration obtained in the previous subsection. It remains to estimate

$$\begin{aligned} & \int_0^t |||e^{v(t-s)B}N(s)|||_{\rho,\delta(t),1} ds \\ & \leq C_0 \int_0^t \sqrt{\frac{t}{s}} |||N(s)|||_{\rho,\delta(s),1} ds \leq C_0 B(\gamma)^2 \int_0^t \sqrt{\frac{t}{s}} (\rho_0 - \rho - \gamma s)^{-\zeta} ds \\ & \leq C_0 B(\gamma)^2 \left(\int_0^{t/2} + \int_{t/2}^t \right) \sqrt{\frac{t}{s}} (\rho_0 - \rho - \gamma s)^{-\zeta} ds \\ & \leq C_0 B(\gamma)^2 \left(t(\rho_0 - \rho - \frac{1}{2}\gamma t)^{-\zeta} + \frac{1}{\gamma} (\rho_0 - \rho - \frac{1}{2}\gamma t)^{1-\zeta} \right) \\ & \leq C_0 \gamma^{-1} B(\gamma)^2 (\rho_0 - \rho)^{-\zeta}, \end{aligned}$$

in which we used $\gamma t \leq \rho_0$ and $\gamma t < \rho_0 - \rho$. Next, noting that $\rho < \rho_0 - \gamma t \leq \rho_0 - \gamma s$, we take $\rho' = \frac{\rho + \rho_0 - \gamma s}{2}$ and compute

$$\begin{aligned} & \int_0^t |||e^{v(t-s)B}N(s)|||_{\rho,\delta(t),2} ds \\ & \leq C_0 \int_0^t \sqrt{\frac{t}{s}} |||N(s)|||_{\rho,\delta(s),2} ds \leq C_0 \int_0^t \sqrt{\frac{t}{s}} \frac{1}{\rho' - \rho} |||N(s)|||_{\rho',\delta(s),1} ds \\ & \leq C_0 B(\gamma)^2 \int_0^t \sqrt{\frac{t}{s}} (\rho_0 - \rho - \gamma s)^{-1-\zeta} ds \\ & \leq C_0 B(\gamma)^2 \left(\int_0^{t/2} + \int_{t/2}^t \right) \sqrt{\frac{t}{s}} (\rho_0 - \rho - \gamma s)^{-1-\zeta} ds \\ & \leq C_0 B(\gamma)^2 \left(t(\rho_0 - \rho - \frac{1}{2}\gamma t)^{-1-\zeta} + \frac{1}{\gamma} (\rho_0 - \rho - \gamma t)^{-\zeta} \right) \\ & \leq C_0 \gamma^{-1} B(\gamma)^2 (\rho_0 - \rho - \gamma t)^{-\zeta}. \end{aligned}$$

This proves the boundedness of the iterative norm $B(\gamma)$, and hence the propagation of the boundary layer behaviors. Theorem 1.1 follows.

4.3. The Inviscid Limit

Let us now prove Theorem 1.2. Since $\omega_0^\nu \in \mathcal{B}^{\rho_0, \sigma_0, \delta}$, the velocity u_0^ν and its conormal derivatives $\partial_x^k u_0^\nu, (\psi(z)\partial_z)^j u_0^\nu$, for $k, j \geq 0$ and $\psi = \frac{z}{1+z}$, are all uniformly bounded on $\mathbb{T} \times \mathbb{R}_+$, thanks to the elliptic estimates; see Proposition 2.3. In particular, the limit of u_0^ν exists in the classical sense: $u^0(0) = \lim_{\nu \rightarrow 0} u_0^\nu$. From Theorem 1.1, we check the validity of Kato's condition:

$$\begin{aligned} & \nu \int_0^T \iint_{\mathbb{T} \times \mathbb{R}_+} |\nabla u(t, x, z)|^2 \, dx dz dt \\ &= \nu \int_0^T \iint_{\mathbb{T} \times \mathbb{R}_+} |\omega(t, x, z)|^2 \, dx dz dt \\ &\leq Cv \int_0^T \iint_{\mathbb{T} \times \mathbb{R}_+} e^{-2\beta z} \left(1 + \delta_t^{-1} \phi_P(\delta_t^{-1} z) + \delta^{-1} \phi_P(\delta^{-1} z)\right)^2 \, dx dz dt \\ &\leq Cv \int_0^T \left(1 + \delta_t^{-1} + \delta^{-1}\right) dt \leq C_T \sqrt{\nu}, \end{aligned}$$

which tends to zero as $\nu \rightarrow 0$. This proves that $\sup_{t \in [0, T]} \|u^\nu(t) - u^0(t)\|_{L^2} \rightarrow 0$. The L^p convergence follows from the interpolation between L^2 and L^∞ norms and the fact that u^ν is bounded in L^∞ thanks to the elliptic estimate (2.5).

Acknowledgements. The authors are indebted to Claude Bardos and Emmanuel Grenier for their many fruitful discussions. The research was supported in part by the NSF under grants DMS-1405728 and DMS 1764119.

Conflict of interest The authors declare that they have no conflict of interest.

Human and Animals Rights This research does not involve Human Participants and/or Animals. The manuscript complies to the Ethical Rules applicable for the Archive for Rational Mechanics and Analysis journal.

References

1. ALEXANDRE, R., WANG, Y.-G., XU, C.-J., YANG, T.: Well-posedness of the Prandtl equation in Sobolev spaces. *J. Am. Math. Soc.* **28**(3), 745–784 (2015)
2. ANDERSON, C.R.: Vorticity boundary conditions and boundary vorticity generation for two-dimensional viscous incompressible flows. *J. Comput. Phys.* **80**(1), 72–97 (1989)
3. BARDOS, C.W., TITI, E.S.: Mathematics and turbulence: where do we stand? *J. Turbul.* **14**(3), 42–76 (2013)
4. CAFLISCH, R.E.: A simplified version of the abstract Cauchy–Kowalewski theorem with weak singularities. *Bull. Am. Math. Soc. (N.S.)* **23**(2), 495–500 (1990)
5. CONSTANTIN, P., ELGINDI, T., IGNATOVA, M., VICOL, V.: Remarks on the inviscid limit for the Navier–Stokes equations for uniformly bounded velocity fields. *SIAM J. Math. Anal.* **49**(3), 1932–1946 (2017)
6. CONSTANTIN, P., KUKAVICA, I., VICOL, V.: On the inviscid limit of the Navier–Stokes equations. *Proc. Am. Math. Soc.* **143**(7), 3075–3090 (2015)
7. CONSTANTIN, P., VICOL, V.: Remarks on high reynolds numbers hydrodynamics and the inviscid limit. *J. Nonlinear Sci.* **28**(2), 711–724 (2018)

8. GÉRARD-VARET, D., DORMY, E.: On the ill-posedness of the Prandtl equation. *J. Am. Math. Soc.* **23**(2), 591–609 (2010)
9. GERARD-VARET, D., LACAVE, C., NGUYEN, T.T., ROUSSET, F.: The vanishing viscosity limit for 2d navier–stokes in a rough domain. *J. Math. Pures Appl.* (2017) (**to appear**)
10. GERARD-VARET, D., MAEKAWA, Y., MASMOUDI, N.: Gevrey stability of Prandtl expansions for 2d Navier–Stokes. (2016). [arXiv:1607.06434](https://arxiv.org/abs/1607.06434)
11. GERARD-VARET, D., MASMOUDI, N.: Well-posedness for the Prandtl system without analyticity or monotonicity. *Ann. Sci. Éc. Norm. Supér.* (4), **48**(6), 1273–1325 (2015)
12. GÉRARD-VARET, D., NGUYEN, T.T.: Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.* **77**(1–2), 71–88 (2012)
13. GIE, G.-M., KELLIHER, J., LOPES FILHO, M.C., MAZZUCATO, A., NUSSENZVEIG LOPES, H.J.: The vanishing viscosity limit for some symmetric flows. arXiv preprint, 2017
14. GRENIER, E.: On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* **53**(9), 1067–1091 (2000)
15. GRENIER, E., NGUYEN, T.T.: Green function for linearized Navier–Stokes around a boundary layer profile: near critical layers (2017). [arXiv:1705.05323](https://arxiv.org/abs/1705.05323)
16. GRENIER, E., NGUYEN, T.T.: On nonlinear instability of Prandtl’s boundary layers: the case of Rayleigh’s stable shear flows (2017). [arXiv:1706.01282](https://arxiv.org/abs/1706.01282)
17. GRENIER, E., NGUYEN, T.T.: Sharp bounds on linear semigroup of Navier–Stokes with boundary layer norms (2017). [arXiv:1703.00881](https://arxiv.org/abs/1703.00881)
18. GRENIER, E., NGUYEN, T.T.: *The stability of Prandtl’s boundary layers* (2017) (**in preparation**)
19. GRENIER, E., NGUYEN, T.T.: Sublayer of prandtl boundary layers. *Arch. Ration. Mech. Anal.* (2018) (**to appear**)
20. GUO, Y., NGUYEN, T.T.: A note on Prandtl boundary layers. *Comm. Pure Appl. Math.* **64**(10), 1416–1438 (2011)
21. IFTIMIE, D., SUEUR, F.: Viscous boundary layers for the Navier–Stokes equations with the Navier slip conditions. *Arch. Ration. Mech. Anal.* **199**(1), 145–175 (2011)
22. IGNATOVA, M., VICOL, V.: Almost global existence for the Prandtl boundary layer equations. *Arch. Ration. Mech. Anal.* **220**(2), 809–848 (2016)
23. KATO, T.: Remarks on zero viscosity limit for nonstationary Navier–Stokes flows with boundary. In: *Seminar on Nonlinear Partial Differential Equations (Berkeley, Calif., 1983)*, volume 2 of *Math. Sci. Res. Inst. Publ.*, pp. 85–98. Springer, New York, 1984
24. KELLIHER, J.P.: On Kato’s conditions for vanishing viscosity. *Indiana Univ. Math. J.* **56**(4), 1711–1721 (2007)
25. MAEKAWA, Y.: Solution formula for the vorticity equations in the half plane with application to high vorticity creation at zero viscosity limit. *Adv. Differ. Equ.* **18**(1–2), 101–146 (2013)
26. MAEKAWA, Y.: On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure Appl. Math.* **67**(7), 1045–1128 (2014)
27. MAEKAWA, Y., MAZZUCATO, A.: The inviscid limit and boundary layers for navier–stokes flows. *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*, pp. 1–48 (Eds. A. Novotny and Y. Giga). Springer, 2017
28. MASMOUDI, N., ROUSSET, F.: Uniform regularity for the Navier–Stokes equation with Navier boundary condition. *Arch. Ration. Mech. Anal.* **203**(2), 529–575 (2012)
29. MASMOUDI, N., WONG, T.K.: Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Commun. Pure Appl. Math.* **68**(10), 1683–1741 (2015)
30. MAZZUCATO, A., TAYLOR, M.: Vanishing viscosity plane parallel channel flow and related singular perturbation problems. *Anal. PDE*, 1(1):35–93, 2008
31. OLENIK, O.A.: On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid. *J. Appl. Math. Mech.* **30**, 951–974 (1967), 1966

32. OLENIK, O.A., SAMOKHIN, V.N.: *Mathematical models in boundary layer theory, volume 15 of Applied Mathematics and Mathematical Computation*. Chapman & Hall/CRC, Boca Raton, FL, 1999
33. PRANDTL, L.: Über flüssigkeits-bewegung bei sehr kleiner reibung pp. 484–491 (1904)
34. SAFONOV, M.V.: The abstract Cauchy-Kovalevskaya theorem in a weighted Banach space. *Commun. Pure Appl. Math.* **48**(6), 629–637 (1995)
35. SAMMARTINO, M., CAFLISCH, R.E.: Zero viscosity limit for analytic solutions, of the Navier–Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Commun. Math. Phys.* **192**(2), 433–461 (1998a)
36. SAMMARTINO, M., CAFLISCH, R.E.: Zero viscosity limit for analytic solutions of the Navier–Stokes equation on a half-space. II. Construction of the Navier–Stokes solution. *Commun. Math. Phys.* **192**(2), 463–491 (1998b)
37. WANG, C., WANG, Y., ZHANG, Z.: Zero-viscosity limit of the Navier–Stokes equations in the analytic setting. *Arch. Ration. Mech. Anal.* **224**(2), 555–595 (2017)

TOAN T. NGUYEN and TRINH T. NGUYEN

Department of Mathematics, Penn State University, State College, PA, 16803, USA.

e-mail: nguyen@math.psu.edu

e-mail: tnx5114@psu.edu

(Received December 4, 2017 / Accepted May 26, 2018)

Published online June 12, 2018

© Springer-Verlag GmbH Germany, part of Springer Nature (2018)