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A brief introduction to amenable equivalence relations

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1. Introduction

The notion of an *amenable equivalence relation* was introduced by Zimmer [**23**] [**24**] in the course of his analysis of orbit equivalence relations in ergodic theory. More recently it played an important role in Monod's striking family of examples of nonamenable groups which do not contain nonabelian free subgroups. Recall that a group is *amenable* if it supports a mean which is translation invariant. Such groups were first studied by von Neumann who observed that amenable groups can not contain nonabelian free subgroups. He famously asked if this was the only obstruction to a group's amenability.

Monod's example[1] can be described as follows. If A is a subring of $\mathbf{R}$, define $H(A)$ to be the group of all piecewise $\mathrm{PSL}_2(A)$ homeomorphisms of the real projective line which fix the point at infinity.

THEOREM 1.1. [**18**] *If A is any dense subring of $\mathbf{R}$, then $H(A)$ is nonamenable. Moreover, if $f, g \in H(\mathbf{R})$, then either $\langle f, g\rangle$ is metabelian or else contains an infinite rank free abelian subgroup. In particular, $H(\mathbf{R})$ does not contain a nonabelian free subgroup.*

Subsequently, Lodha and the author constructed a finitely presented nonamenable subgroup of $H(\mathbf{Z}[1/\sqrt{2}])$ [**16**]. Lodha has since shown that the group of [**16**] is moreover of type F_∞ [**15**].

At least from a group-theoretic perspective, the most novel aspect of [**18**] was the use of Zimmer's notion of an amenable equivalence relation in the proof of the nonamenability of the groups $H(A)$. The purpose of this article is to give a brief survey of the theory of amenable and hyperfinite equivalence relations and illustrate how it can be used to show that certain discrete groups are nonamenable.

The subject matter falls within the broader scope of what is sometimes called *measurable group theory* — the study of groups through the analysis of their action on measure spaces and using analytical tools. This is in contrast with *geometric group theory*, where groups are studied through their actions which preserve an

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[1]This is a slight and, at least for our purposes, inessential variation of Monod's example from [**18**].

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underlying geometry and by employing geometric techniques and concepts. Measurable group theory is closely aligned with the ergodic theory, the dynamics of discrete groups, probability, and descriptive set theory. Further reading can be found in [**7**] and [**12**] and their references. Further background on descriptive set theory can be found in [**11**].

This article is organized as follows. After reviewing some background material and fixing some terminology, we will present the definitions of amenable and hyperfinite equivalence relations in Section 3. This section will culminate with a theorem connecting these two apparently different notions. Section 4 will present several examples of nonamenable equivalence relations. Section 5 will discuss the analogs of the closure properties of amenable groups in the setting of equivalence relations. These played an important role in the isolation of the group in [**16**].

This article does not contain any new results, although Theorem 4.8 below is cast in a more abstract way than in [**9**]. (It is also to the author's knowledge, the first account of this proof in English.) The article's goal is to encourage the reader to pursue further reading in, e.g., [**7**] and [**12**] which contains a much more complete treatment of the subject matter presented here.

2. Preliminaries

Before proceeding, we will fix some terminology. In a few places we will refer to the continuity of extended real valued functions. In this context, the neighborhoods of infinity are the co-bounded sets. Recall that a *Polish space* is a topological space which is separable and completely metrizable. The σ-algebra of Borel sets in a Polish space is said to be a *standard Borel space*. A function f between standard Borel spaces X and Y is *Borel* if preimages of Borel sets are Borel. This is equivalent to the graph of f being a Borel subset of $X \times Y$. It is well known that any two uncountable standard Borel spaces are isomorphic in the sense that there is a bijection f between them such that f and f^{-1} are Borel.

A *Borel measure* on a standard Borel space is a countably additive σ-finite measure defined on its Borel sets. Such a measure extends uniquely to the σ-algebra generated by the Borel sets and the subsets of measure 0 Borel sets. We will generally not distinguish between these measures but note here that *measurable* will always refer to the larger σ-algebra.

In this article we will write (X, μ) is a *measured Polish space* to mean that X is a Polish space and that μ is a Borel measure on X. If in addition the topology on X is generated by the open sets of finite measure, then we say that (X, μ) is *locally finite*. If Γ is a topological group, then we will say that Γ acts continuously on a measured Polish space (X, μ) if:

- the map $(g, x) \mapsto g \cdot x$ is continuous and
- the maps $g \mapsto \mu(g \cdot E)$ are continuous for each measurable $E \subseteq X$.

Notice that this is stronger than the assertion that Γ acts continuously on the metric space X.

We note some useful facts about measured Polish spaces.

FACT 2.1. If (X, μ) is a measured Polish space and $E \subseteq X$ is measurable, then $\mu(E)$ is the supremum of all $\mu(F)$ where F is a closed subset of E.

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FACT 2.2. Suppose that (X, μ) is a locally finite measured Polish space. If $E \subseteq X$ is measurable and has positive measure, then for every $\epsilon > 0$ there is an open set $U \subseteq X$ such that $0 < \mu(U) < \infty$ and $\mu(E \cap U) > (1 - \epsilon)\mu(U)$.

We will also need the following proposition.

PROPOSITION 2.3. *Suppose that (X, μ) is a locally finite measured Polish space and Γ is a metrizable group acting continuously on (X, μ). If $E \subseteq X$ is a measurable set of positive measure, then there is an open neighborhood V of the identity of Γ and an $\epsilon > 0$ such that if g is in V, then $\mu((g \cdot E) \cap E) > \epsilon$.*

PROOF. Let $E \subseteq X$ be given and let $U \subseteq X$ be an open set with

$$0 < \frac{3}{4}\mu(U) < \mu(E \cap U) < \mu(U) < \infty.$$

Set $\epsilon = \frac{1}{4}\mu(U)$. Observe by our continuity assumption on the action, we have that for every x in U there is an open $W_x \subseteq U$ containing x and a $\delta_x > 0$ such that if the distance from g to the identity is less than δ_x, then $g \cdot W_x \subseteq U$. Find a $\delta > 0$ such that

$$\mu(\{x \in U : \delta_x \geq \delta\}) > \frac{3}{4}\mu(U)$$

and define $W = \bigcup\{W_x : \delta_x \geq \delta\}$, observing that $\mu(W) > \frac{3}{4}\mu(U)$. In particular, $\mu(E \cap W) > \frac{1}{2}\mu(U)$. By our assumption that Γ acts continuously on (X, μ), there is an open set V containing the identity such that every element of V has distance less than δ to the identity and $\mu(g \cdot (E \cap W)) > \frac{1}{2}\mu(U)$ whenever g is in V. Since $\mu(E \cap U) > \frac{3}{4}\mu(U)$ and since $g \cdot (E \cap W) \subseteq (g \cdot E) \cap U$, it follows that $\mu(E \cap (g \cdot E) \cap U) > \frac{1}{4}\mu(U) = \epsilon$. □

If X is a standard Borel space, an equivalence relation E on X is Borel if it is Borel as a subset of X^2. A Borel equivalence relation is said to be *countable* if every equivalence class is countable. Notice that while this meaning conflicts with the literal interpretation of "countable," there is never a cause for confusion since for an equivalence relation to be countable as a set, it must have a countable underlying set and in the present context one is generally only interested in uncountable standard Borel spaces.

The principal example of a countable Borel equivalence relation is as follows: if G is a countable discrete group acting by Borel automorphisms on a standard Borel space X, then the orbit equivalence relation E_X^G is a countable Borel equivalence relation. That is, (x, y) is in E_X^G if and only if there is a g in G such that $g \cdot x = y$. In fact all countable Borel equivalence relations arise in this way:

THEOREM 2.4. [**6**] *If E is a countable Borel equivalence relation on a standard Borel space, then there is a countable group G and a Borel action of G on X such that $E = E_X^G$.*

The advantage of working with equivalence relations is in part that the notion of a countable Borel equivalence relation is much more flexible than that of a group. For instance while subgroups give rise to subequivalence relations, the converse is not generally true. A more sophisticated use of this generality occurs in the proof of Theorem 1.1: orbit equivalence relations are used to transfer the nonamenability of $\mathrm{PSL}_2(A)$ to the group $H(A)$ even though these groups are quite unrelated from an algebraic perspective.

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3. Amenable and hyperfinite equivalence relations

Any action of a countable group on a standard Borel space gives rise to a countable Borel equivalence relation and, conversely, any countable Borel equivalence relation can be generated as the orbit equivalence relation of some group action. The fundamental problem of this subject is to understand the extent to which properties of the group which generated a countable Borel equivalence relation are reflected in properties of the equivalence relation and vice versa.

Our focus in this article will be to develop the properties of equivalence relations which are analogs of the group-theoretic property of *amenability*. Roughly speaking, the notion of an amenable equivalence relation has the property that every action of an amenable group gives rise to an amenable equivalence relation and a group is amenable only when every orbit equivalence relation is amenable.

Now to be more precise. Suppose that (X, μ) is Borel measure on a standard Borel space and E is a countable Borel equivalence relation on X. We say that E is *μ-amenable* if there is a μ-measurable assignment $x \mapsto \nu_x$ such that:

- each ν_x is a finitely additive probability measure on X satisfying that $\nu_x([x]_E) = 1$.
- if $(x, y) \in E$, then $\nu_x = \nu_y$.

By *measurable* we mean that if A is any measurable subset of $X \times X$, then

$$x \mapsto \nu_x(\{y \in X : (x, y) \in A\})$$

is μ-measurable. While we will generally quantify *amenable* with a measure, a Borel equivalence relation is said to be *amenable* if it is μ-amenable with respect to every Borel measure on the underlying standard Borel space.

While it is not apparent from the definition, it is true that every orbit equivalence relation of a countable amenable group acting on standard Borel space is necessarily μ-amenable with respect to any Borel measure μ (this does not require any invariance of μ with respect to the group action). This will follow from Theorem 3.5 below.

Next we turn to a seemingly unrelated notion. A countable Borel equivalence relation E on a standard Borel space X is *hyperfinite* if E is an increasing union of Borel equivalence relations with finite equivalence classes. A good example to keep in mind is that of *eventual equality* on infinite binary sequences: define $x =^* y$ if $x(k) = y(k)$ for all but finitely many k. Notice that this is the union of the equivalence relations $=^n$ defined by $x =^n y$ if $x(k) = y(k)$ for all $k \geq n$. The following theorem gives a powerful criterion for verifying hyperfiniteness.

THEOREM 3.1. [**4**] *Suppose that X is a standard Borel space and $f : X \to X$ is a Borel function which such that $f^{-1}(x)$ is at most countable for each x. The smallest equivalence relation E satisfying that, for each $x \in X$, $(x, f(x)) \in E$ is hyperfinite.*

EXAMPLE 3.2. [**4**] Define an equivalence relation E all infinite binary sequences by xEy if for some m and n, $x(m + i) = y(n + i)$ for all $i > 0$. This equivalence relation is called *tail equivalence* and is generated by the shift map $f : 2^\omega \to 2^\omega$ given by $f(x)(i) = x(i + 1)$.

EXAMPLE 3.3. [**2**] Recall that the real projective line $P^1(\mathbf{R})$ is the collection of all lines in $\mathbf{R}^2$ passing through the origin. Identify an element of $P^1(\mathbf{R})$ with the x-coordinate of its intersection with the line $y = 1$, adopting the convention that

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the line $y = 0$ becomes identified with ∞. This identification gives $P^1(\mathbf{R})$ a natural compact metric topology — it is in fact homeomorphic to a circle. $\mathrm{PSL}_2(\mathbf{R})$ is the group of fractional linear transformations $t \mapsto \frac{at+b}{ct+d}$ of $P^1(\mathbf{R})$, interpreted as the extended real line. Define a map $\Phi : 2^\omega \to P^1(\mathbf{R})$ by

$$\Phi(\mathtt{1}x) = \phi(x) \qquad \Phi(\mathtt{0}x) = -\phi(\sim x)$$

($\sim x$ denotes the bitwise complement of x) where ϕ is the unique order preserving map from 2^ω to $[0, \infty]$ which satisfies the equation:

$$\phi(\mathtt{1}x) = 1 + \phi(x) \qquad \phi(\mathtt{0}x) = \frac{1}{1 + \frac{1}{\phi(x)}}.$$

Informally, the first digit of x determines the sign of $\Phi(x)$ and the remaining digits are used to determine the continued fraction expansion of $|\Phi(x)|$ so that the n^{th} "digit" in this expansion is the length of the n^{th} block of consecutive equal digits in x (after the first digit). Formally, the existence and uniqueness of ϕ can be seen as follows. First, the implicit definition gives a recursive formula for ϕ on on those binary sequences which are eventually 0, noting that $\phi(\mathtt{0})$ is forced to be 0. Moreover ϕ maps these eventually 0 sequences in an order preserving manner to the nonnegative rationals. It is then readily checked that ϕ extends to 2^ω so as to still preserve order and satisfy the implicit equation. The action of $\mathrm{PSL}_2(\mathbf{Z}) = \langle t \mapsto t+1, t \mapsto -1/t \rangle$ on $P^1(\mathbf{R})$ naturally lifts to an action on 2^ω. Specifically, if $a(t) = t + 1$ and $b(t) = -1/t$, then the action of these generators is given by

$$\begin{aligned} a \cdot (\mathtt{00}x) &= \mathtt{0}x & b \cdot (\mathtt{0}x) &= \mathtt{1}x \\ a \cdot (\mathtt{01}x) &= \mathtt{10}x & b \cdot (\mathtt{1}x) &= \mathtt{0}x \\ a \cdot (\mathtt{1}x) &= \mathtt{11}x \end{aligned}$$

It can be verified that the corresponding orbit equivalence relation on 2^ω is tail equivalence. In particular, this orbit equivalence relation of $\mathrm{PSL}_2(\mathbf{Z})$'s action on $P^1(\mathbf{R})$ is hyperfinite.

The following gives an important characterization of the hyperfinite equivalence relations:

THEOREM 3.4. [**21**] [**22**] *Every Borel action of* $\mathbf{Z}$ *on a standard Borel space generates a hyperfinite orbit equivalence relation. Conversely, every hyperfinite Borel equivalence relation is the orbit equivalence relation of a Borel action of* $\mathbf{Z}$.

While it is not obvious, it turns out that every hyperfinite Borel equivalence relation is in fact μ-amenable with respect to any Borel measure on the underlying space. In fact, a natural weakening captures the notation of μ-amenability exactly. If μ is a Borel measure on X, then we say that E is *μ-hyperfinite* if there is a μ-measure 0 set $Y \subseteq X$ such that the restriction of E to $X \setminus Y$ is hyperfinite.

The following theorem is an amalgamation of several results stated in modern language (see also [**12**]).

THEOREM 3.5. [**2**] [**5**] [**13**] [**19**] [**24**] *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel measure on X. The following are equivalent:*

(1) *E is μ-amenable;*
(2) *E is μ-hyperfinite;*

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(3) $E = E_X^G$ *for some* μ*-measurable action of an amenable group* G *on* X*;*
(4) $E = E_X^{\mathbf{Z}}$ *for some* μ*-measurable action of* $\mathbf{Z}$ *on* X*.*

It is worth noting that the extent to which the measure 0 sets can be omitted in the previous theorem is a major open problem in descriptive set theory. For instance:

PROBLEM 3.6. [**4**] If E_n $(n < \infty)$ is an increasing sequence of hyperfinite Borel equivalence relations on a standard Borel space, is $\bigcup_{n=0}^{\infty} E_n$ hyperfinite?

PROBLEM 3.7. [**22**] Is every orbit equivalence relation of a Borel action of a countable amenable group acting on a standard Borel space hyperfinite?

In fact it was only relatively recently that a positive solution to Problem 3.7 was proved for the class of abelian groups [**8**]; the strongest result at the time of this writing is [**20**]. Problem 3.7 remains open for the Baumslag-Solitar group $BS(1,2)$ generated by the affine transformations $t \mapsto t+1$ and $t \mapsto 2t$. While not directly related, Marks has recently demonstrated differences between the so-called *Borel context* and *measure-theoretic context* [**17**].

4. Examples

In this section we will consider a number of examples. Perhaps the easiest way to generate nonamenable equivalence relations is through actions of groups which preserve a probability measure.

THEOREM 4.1. [**19**] *Suppose that* G *is a countable group,* (X, μ) *is a standard Borel space equipped with a Borel probability measure, and* E *is the orbit equivalence relation of a measure preserving action of* G *which is free* μ*-a.e.. The equivalence relation* E *is* μ*-amenable if and only if* G *is amenable.*

The following are two typical — but quite different — examples of such actions.

EXAMPLE 4.2. If G is any countably infinite group and (X, μ) is any probability space, then G acts by shift on X^G as follows: $(g \cdot x)(h) = x(g^{-1}h)$. This action preserves the product measure and, unless μ is a point-mass, is free almost everywhere with respect to the product measure.

EXAMPLE 4.3. The action of $\mathrm{SL}_2(\mathbf{Z})$ on the torus $\mathbf{T}^2$ equipped with Lebesgue measure is measure preserving and free λ-a.e.. Since $\mathrm{SL}_2(\mathbf{Z})$ contains the free group on two generators, this orbit equivalence relation is not λ-amenable.

Example 4.3 is in contract with Example 3.3 in which the group $\mathrm{PSL}_2(\mathbf{Z})$ acts on $P^1(\mathbf{R})$, which is homeomorphic to the circle. It is well known that $\mathrm{PSL}_2(\mathbf{Z})$ contains a free subgroup (even one of finite index) and hence is nonamenable. On the other hand, we have seen above that the orbit equivalence relation induced on $P^1(\mathbf{R})$ is just tail equivalence on 2^ω in disguise; in particular it is hyperfinite and hence amenable. Notice that, unlike the action of $\mathrm{SL}_2(\mathbf{Z})$ on the torus, there is no probability measure on $P^1(\mathbf{R})$ which is preserved by the action of $\mathrm{PSL}_2(\mathbf{Z})$.

It turns out, however, that *dense* subgroups of $\mathrm{PSL}_2(\mathbf{R})$ do produce a nonamenable orbit equivalence relation when they act on $P^1(\mathbf{R})$.

THEOREM 4.4. [**9**] *Every nondiscrete subgroup of* $\mathrm{PSL}_2(\mathbf{R})$ *is either solvable or else contains a nondiscrete free subgroup on two generators.*

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THEOREM 4.5. [**9**] *If Γ is a rank 2 free subgroup of a finite dimensional Lie group G and Γ is nondiscrete, then the orbit equivalence relation of Γ's action on G is nonamenable with respect to Haar measure.*

In the case of $\Gamma = \mathrm{PSL}_2(\mathbf{Z}[1/2])$, it is not difficult to exhibit a nondiscrete free subgroup explicitly.

EXAMPLE 4.6. The matrices

$$\alpha = \begin{pmatrix} 1/2 & -4 \\ 1/4 & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1/2 & -1/4 \\ 4 & 0 \end{pmatrix}$$

generate a nondiscrete free subgroup of $\mathrm{PSL}_2(\mathbf{Z}[1/2])$. In order to see this, first observe that the traces of these matrices are $1/2$ and hence both matrices describe *elliptic* transformations of the real projective line (i.e. there are no fixed points). Since any elliptic element of $\mathrm{PSL}_2(\mathbf{R})$ of infinite order generates a nondiscrete subgroup, it suffices to show that the above matrices generate a free group.

Define X to be the set of all rational numbers in $P^1(\mathbf{R}) = \mathbf{R} \cup \{\infty\}$ which can be represented by a fraction with an odd denominator and let Y denote the remaining rational numbers in $P^1(\mathbf{R})$. By the *Ping-Pong Lemma* (see, e.g., [**3**]), it suffices to show that if $n \neq 0$ is an integer, then $\alpha^n Y \subseteq X$ and $\beta^n X \subseteq Y$. Let X_0 consist of those elements of X which can be represented by a fraction of the form $(4p+2)/q$ where q is odd. Notice that $\alpha(X_0 \cup Y) \subseteq X_0$ and that X_0 and Y are disjoint. It follows that $\alpha^n Y \subseteq X$ whenever n is a nonzero integer. Similarly, $\beta^n X \subseteq Y$.

Theorem 4.4 was generalized considerably by the following result.

THEOREM 4.7. [**1**] *If Γ is a dense subgroup of a connected semi-simple real Lie group, then Γ contains a dense free subgroup of rank 2.*

The following theorem is a generalization of Theorem 4.5, although the argument closely follows that of [**9**].

THEOREM 4.8. *Suppose that (X, μ) is a locally finite measured Polish space. If $\Gamma = \langle a, b \rangle$ is a free nondiscrete metrizable group which is acting freely and continuously on (X, μ), then the orbit equivalence relation is not μ-amenable.*

PROOF. Suppose for contradiction that E_X^Γ is μ-amenable and fix a μ-measurable assignment $x \mapsto \nu_x$ such that:

- for each x, ν_x is a finitely additive probability measure supported on the orbit of x;
- if x and y are in the same orbit, then $\nu_x = \nu_y$.

For $u \in \{a, b\}$, define Γ_u to be all those elements of Γ which are representable by a reduced word beginning with u and ending with u^{-1}. Observe that if g is a nonidentity element of Γ, then there is a h in $\{a, ab, ab^{-1}\}$ such that hgh^{-1} is in Γ_a. Thus there is an $h \in \{e, a, ab, ab^{-1}\}$ and a $\Gamma' \subseteq \Gamma$ which accumulates to the identity such that $h\Gamma' h^{-1} \subseteq \Gamma_a$. Since conjugation is continuous, it follows that Γ_a also accumulates to the identity. Furthermore, $b\Gamma_a b^{-1} \subseteq \Gamma_b$ and thus Γ_b accumulates to the identity as well.

Since the action of Γ on X is free, for each $x, y \in X$ which lie in the same orbit, there is a unique $\gamma = \gamma(x, y)$ in Γ such that $x = \gamma \cdot y$. Notice that $\gamma(g \cdot x, y) = g\gamma(x, y)$. Define $\phi : X \to [0, 1]$ by letting $\phi(x) = \nu_x(A_x)$ where A_x is the set of

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those y in the orbit of x such that the reduced word representing $\gamma(x,y)$ begins with a or a^{-1}. Observe that for any x in X and g in Γ_b, $A_x \cap A_{g\cdot x} = \emptyset$. Hence if x is in X and $\phi(x) > 1/2$, then $\phi(g\cdot x) < 1/2$ whenever g is in Γ_b. Similarly, if $\phi(x) < 1/2$ and g is in Γ_a, then $\phi(g\cdot x) > 1/2$. Furthermore, for all x and $g \in \Gamma_a$, the sets A_x, $A_{bgb^{-1}\cdot x}$ and $A_{b^2gb^{-2}\cdot x}$ are pairwise disjoint and consequently $0 \leq \phi(x) + \phi(bgb^{-1}\cdot x) + \phi(b^2gb^{-2}\cdot x) \leq 1$.

I next claim that $Y = \{x \in X : \phi(x) \neq 1/2\}$ has positive measure with respect to μ. Suppose not. Using our assumption that Γ acts continuously on (X,μ), find an open neighborhood V of the identity such that if g is in V, then Y, $bg^{-1}b^{-1}\cdot Y$, and $b^2g^{-1}b^{-2}\cdot Y$ have total measure less than that of X. Now if x is outside these sets and $g \in V \cap \Gamma_a$, we have that $\phi(x)$, $\phi(bgb^{-1}\cdot x)$, and $\phi(b^2gb^{-2}\cdot x)$ are each $1/2$, contradicting that there sum is at most 1. Thus Y must have positive measure.

Let $Y_a = \{y \in Y : \phi(y) > 1/2\}$ and $Y_b = \{y \in Y : \phi(y) < 1/2\}$. Since Y has positive measure, either Y_a or Y_b have positive measure. If Y_a has positive measure, then by Proposition 2.3 there is a g in Γ_b such that $(g\cdot Y_a)\cap Y_a$ has positive measure and in particular is nonempty. This contradicts our observation that if $\phi(y) > 1/2$ and g is in Γ_b, then $\phi(g\cdot y) < 1/2$. Similarly, if Y_b has positive measure, one obtains a contradiction by finding a g in Γ_a such that $g\cdot Y_b$ intersects Y_b and recalling our observation that if $\phi(y) < 1/2$ and g is in Γ_a, then $\phi(g\cdot y) > 1/2$. It must be, therefore, that the orbit equivalence relation is nonamenable. □

We finish this section with a simple but powerful observation of Monod.

EXAMPLE 4.9. [**18**] If A is a countable dense subring of $\mathbf{R}$, let $H(A)$ denote the group consisting of all orientation preserving homeomorphisms of $P^1(\mathbf{R})$ which fix the point at infinity and which are piecewise $\mathrm{PSL}_2(A)$. Suppose that α is in $\mathrm{PSL}_2(A)$ and that α does not fix ∞. As a fractional linear transformation, the graph of α is a hyperbola. If $r \in \mathbf{R}$ is sufficiently large in magnitude, then $\alpha(t) = t + r$ has two solutions $a < b$; set

$$\alpha_r(t) = \begin{cases} \alpha(t) & \text{if } a < t < b \\ t + r & \text{otherwise.} \end{cases}$$

If r is moreover an integer, then α_r is in $H(A)$. It follows that the restriction of the orbit equivalence relation of $\mathrm{PSL}_2(A)$ to $\mathbf{R}$ coincides with the corresponding restriction of the orbit equivalence relation of $H(A)$'s action on $\mathbf{R}$. By Theorem 3.5 and the results of [**9**] mentioned above, $H(A)$ is nonamenable whenever A is a dense subring of $\mathbf{R}$.

EXAMPLE 4.10. [**16**] Let $P(\mathbf{Z})$ denote the subgroup of $H(\mathbf{Z})$ consisting of those elements which have a continuous derivative. By unpublished work of William Thurston, $P(\mathbf{Z})$ is isomorphic to Richard Thompson's group F. In fact

$$\alpha(t) = t + 1 \qquad \beta(t) = \begin{cases} t & \text{if } t \leq 0 \\ \frac{t}{1-t} & \text{if } 0 \leq t \leq \frac{1}{2} \\ 3 - \frac{1}{t} & \text{if } \frac{1}{2} \leq t \leq 1 \\ t + 1 & \text{if } 1 \leq t \end{cases}$$

is the standard set of generators with respect to the usual finite presentation of F (see [**16**]). It is not difficult to see that the orbit equivalence relation of $P(\mathbf{Z})$'s action on $P^1(\mathbf{R})$ coincides with that of $\mathrm{PSL}_2(\mathbf{Z})$ except for the point at infinity.

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Since $\mathrm{PSL}_2(\mathbf{Z}) \cup \{t \mapsto t+1/2\}$ generates $\mathrm{PSL}_2(\mathbf{Z}[1/2])$, it follows that $\langle t \mapsto t/2, \beta\rangle$ is nonamenable.

The previous example is less relevant to the amenability problem for F, however, than it might initially appear. For instance, Lodha has shown that if Γ is any subgroup of $H(\mathbf{R})$ which is isomorphic to F, then the orbit equivalence relation of Γ's action on $\mathbf{R}$ is λ-amenable where λ is Lebesgue measure [**14**].

5. Closure properties of amenable equivalence relations

One of the most basic facts about amenable groups is that they are closed under taking subgroups, extensions, and directed unions. These operations have their analogs in the setting of countable Borel equivalence relations as well. Notice that if $H \leq G$, then $E_X^H \subseteq E_X^G$ whenever G acts on a standard Borel space. Also, if G is an increasing union of a sequence of subgroups G_n $(n < \infty)$, then $E_X^G = \bigcup_n E_X^{G_n}$. Since the property of being μ-hyperfinite is clearly inherited to subequivalence relations, we have the following corollary of Theorem 3.5.

PROPOSITION 5.1. *If E is a subequivalence relation of a μ-amenable equivalence relation is μ-amenable.*

While there is no natural notion of extension in the setting of equivalence relations, it is easy to formulate what is meant by a product of equivalence relations. It is straightforward to verify the following analog of the closure of the class of amenable groups under taking products.

PROPOSITION 5.2. *Products of μ-amenable equivalence relations are amenable with respect to the corresponding product measure.*

The following is the corresponding analog of the amenability of increasing unions of amenable groups.

THEOREM 5.3. [**5**] [**13**] *Suppose that E_n $(n < \infty)$ is an increasing sequence of countable Borel equivalence relations on a standard Borel space X. If μ is a standard probability measure on X and each E_n is μ-hyperfinite, then $\bigcup_n E_n$ is μ-hyperfinite.*

The power of the closure properties mentioned in this section is that they afford some flexibility which has no analog in the algebraic setting. For instance, while the equivalence relations E_n in the previous theorem are required to be nested, they need not come from a nested sequence of groups. It is also sometimes fruitful to generate equivalence relations with partial homeomorphisms rather than full automorphisms of an underlying space.

EXAMPLE 5.4. Consider the following homeomorphisms of $P^1(\mathbf{R}) = \mathbf{R} \cup \{\infty\}$: $\alpha(t) = t + 1/2$ and $\beta(t) = -1/t$. For $0 < r < \infty$, define α_r to be the restriction of α to $[-r, -1/r]$. Let E be the equivalence relation generated by α and β and E_r be the equivalence relation generated by β and α_r (i.e. E_r is the smallest equivalence relation such that for all t, $(t, t+1/2) \in E_r$ and if additionally $-r \leq t \leq -1/r$, then $(t, -1/t) \in E_r$). Notice that if $0 < r < s < \infty$, then $E_r \subseteq E_s \subseteq E$ and that $E = \bigcup_{r>0} E_r$. Thus there is an r such that $0 < r < \infty$ such that E_r is nonamenable. In fact a more careful analysis reveals that $E_4 = E$, although this is not relevant for the point we wish to illustrate here.

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EXAMPLE 5.5. Suppose that α and β are homeomorphisms such that the action of $\langle \alpha, \beta \rangle$ on $\mathbf{R}$ generates a nonamenable equivalence relation. Suppose further that α_n $(n < \infty)$ and β_n $(n < \infty)$ are sequences of homeomorphisms such that for all but countably many t, $\alpha_n(t) = \alpha(t)$ and $\beta_n(t) = \beta(t)$ holds for all but finitely many n. It follows that there exists an n such that the action of $\langle \alpha_n, \beta_n \rangle$ on $\mathbf{R}$ generates a nonamenable equivalence relation. To see this, let X_n denote the set of all t in $\mathbf{R}$ such that for all $k \geq n$, $\alpha_k(t) = \alpha(t)$ and $\beta_k(t) = \beta(t)$. Define E_n to be the equivalence relation generated by the restrictions of α_n and β_n to X_n. It follows that E_n $(n < \infty)$ is an increasing sequence of countable Borel equivalence relations which, off a countable subset of $\mathbf{R}$, unions to the equivalence relation generated by α and β. The claim now follows from Theorem 5.3.

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