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The classification of simple separable KK-contractible C^*-algebras with finite nuclear dimension

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ABSTRACT

The class of simple separable KK-contractible (KK-equivalent to $\{0\}$) C^*-algebra s which have finite nuclear dimension is shown to be classified by the Elliott invariant. In particular, the class of C^*-algebra s $A \otimes \mathcal{W}$ is classifiable, where A is a simple separable C^*-algebra with finite nuclear dimension and $\mathcal{W}$ is the simple inductive limit of Razak algebras with unique trace, which is bounded (see Razak (2002) and Jacelon (2013)).

1. Introduction

The classification of unital simple separable C^*-algebras with finite nuclear dimension which satisfy the UCT has been completed (see, for example, [16,24,29,40], and [50]). As is well known, the case that there exists a non-zero projection in the stabilization of the algebra follows. In the remaining case, that the algebra A is stably projectionless (i.e., if the algebra is finite, the case $\mathrm{K}_0(A)_+ = \{0\}$), a number of classification results are known (see [41,43,53]).

In this paper we consider the general (axiomatically determined) case assuming trivial K-theory. Recall that a C^*-algebra A is said to be KK-contractible if it is KK-equivalent to $\{0\}$. In the presence of the UCT, it is equivalent to say that $\mathrm{K}_i(A) = \{0\}$, $i = 0, 1$. From the order structure of the K_0-group, one sees that the case of stably projectionless simple C^*-algebras is very different from the unital case. In particular, the proofs in this paper do not depend on the unital results—and require rather different techniques.

We obtain the following classification theorem:

Theorem (*Theorem* 7.5)**.** *The class of KK-contractible stably projectionless simple separable C^*-algebras with finite nuclear dimension is classified by the invariant $(\widetilde{\mathrm{T}}(A), \Sigma_A)$. Any C^*-algebra A in this class is a simple inductive limit of Razak algebras.*

Here, $\widetilde{\mathrm{T}}(A)$ is the cone of lower semicontinuous traces finite on the Pedersen ideal $\mathrm{Ped}(A)$ of A, with the topology of pointwise convergence (on $\mathrm{Ped}(A)$), and Σ_A is the norm function (the lower semicontinuous extended positive real-valued function on $\widetilde{\mathrm{T}}(A)$ defined by $\Sigma_A(\tau) = \sup\{\tau(a) : a \in \mathrm{Ped}(A)_+, \|a\| \le 1\}$).

Consider the C^*-algebra $\mathcal{W}$, the (unique) simple inductive limit of Razak algebras with a unique trace (up to a multiple), which is furthermore bounded (see [41] and [28]; $\mathcal{W}$ is also sometime called the Razak–Jacelon algebra). We will show

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that $\mathcal{W}$ is the unique separable simple C*-algebra with a unique tracial state and with finite nuclear dimension which is KK-contractible. Hence $A \otimes \mathcal{W}$ is KK-contractible for any amenable C*-algebra A (see Lemma 3.17). Thus, if A has finite nuclear dimension, so that the C*-algebra $A\otimes\mathcal{W}$ has finite nuclear dimension as well (see Proposition 2.3(ii) of [57]), then $A \otimes \mathcal{W}$ is classifiable (whether it is finite – Theorem 7.5 – or infinite—in which case by [40] it must be $\mathcal{O}_2 \otimes \mathcal{K}$).

Corollary (Corollary 6.7)**.** *Let A be a simple separable C*-algebra with finite nuclear dimension. Then the C*-algebra $A \otimes \mathcal{W}$ is classifiable. In particular, $\mathcal{W} \otimes \mathcal{W} \cong \mathcal{W}$.*

2. The reduction class $\mathcal{R}$, the tracially approximate point-line class $\mathcal{D}$, and model algebras

Let A be a C*-algebra. Denote by $\mathrm{Ped}(A)$ the Pedersen ideal. Denote by $\widetilde{\mathrm{T}}(A)$ the topological cone of lower semicontinuous positive traces defined (i.e., finite) on $\mathrm{Ped}(A)$, with the topology of pointwise convergence (on the elements of $\mathrm{Ped}(A)$). Denote by $\mathrm{T}(A)$ the set of all tracial states of A. Denote by $\overline{\mathrm{T}(A)}^{\mathrm{w}}$ the weak* closure of $\mathrm{T}(A)$ in the space of all positive linear functionals on A. Let X be a topological convex set, or a topological cone. Denote by $\mathrm{Aff}_+(X)$ the cone of all continuous positive real-valued affine functions f on X which vanish at zero and only at that point, together with zero function. Following [43], let us denote by $\mathrm{LAff}_+(X)$ the cone of all lower semicontinuous affine functions with values in $[0,\infty]$ on X which are limits of increasing sequences of functions in $\mathrm{Aff}_+(X)$. We are mostly interested in the case that $X = \widetilde{\mathrm{T}}(A)$. Let $\Sigma_A \in \mathrm{LAff}_+(\widetilde{\mathrm{T}}(A))$ denote the (possibly infinite) norm function: $\Sigma_A(\tau) = \sup\{\tau(a) : a \in \mathrm{Ped}(A)_+,\ \|a\| \le 1\}$. We shall refer to Σ_A as the scale of A.

For $\varepsilon > 0$, let $f_\varepsilon \in C_0((0,\infty))_+$ (throughout the paper) such that $f(t) = 0$ if $t \in (0, \varepsilon/2)$, $f(t) = 1$ if $t \in [\varepsilon, \infty)$ and linear in $[\varepsilon/2, \varepsilon)$.

Let $a \in A_+$, for each $\tau \in \widetilde{\mathrm{T}}(A)$, define $\mathrm{d}_\tau(a) = \lim_{\varepsilon\to 0} \tau(f_\varepsilon(a))$. If $e \in A_+$ is a strictly positive element, then $\Sigma_A(\tau) = \mathrm{d}_\tau(e)$ for all $\tau \in \widetilde{\mathrm{T}}(A)$ (independent of the choice of e). If $S \subseteq \widetilde{\mathrm{T}}(A) \setminus \{0\}$ is a convex subset, denote by $\mathrm{LAff}_{0+}(S)$ the cone $\{f|_S : f \in \mathrm{LAff}_+(\widetilde{\mathrm{T}}(A))\}$ of restrictions to S of the functions in $\widetilde{\mathrm{T}}(A)\setminus\{0\}$. If S is bounded, denote by $\mathrm{LAff}_{b,0+}(S)$ the subset of $\mathrm{LAff}_{0+}(S)$ consisting of those functions bounded on S. In the case that $\mathrm{T}(A)$ is compact, let us denote the cone $\mathrm{LAff}_{0+}(\mathrm{T}(A))$ just by $\mathrm{LAff}_+(\mathrm{T}(A))$.

Definition 2.1. A simple C*-algebra A will be said to be in the reduction class, denoted by $\mathcal{R}$, if A is separable, has continuous scale ([32] and [17]), and $\mathrm{T}(A) \neq \emptyset$. For any non-zero exact Jiang–Su stable separable simple C*-algebra A, by Lemma 6.5 of [19] (combined with Theorem 1.2 of [44]; see Remark 5.2 of [17]), there is a non-zero hereditary sub-C*-algebra $A_0 \subseteq A$ such that A_0 has continuous scale—and so, if $\mathrm{T}(A) \neq \emptyset$, belongs to the class $\mathcal{R}$. In particular, as A is separable and simple, it follows from Brown's theorem [7] that $A\otimes\mathcal{K} \cong A_0\otimes\mathcal{K}$. We will use the fact that $\mathrm{T}(A)$ is a compact base for $\widetilde{\mathrm{T}}(A)$ when A belongs to the class $\mathcal{R}$ (see Theorem 5.3 of [17] and Theorem 3.3 of [32]). (By Theorem 5.3 of [17], when A is as above, with $\mathrm{T}(A) \neq \emptyset$, and $A = \mathrm{Ped}(A)$, these two properties are in fact equivalent.)

Definition 2.2 (*[15]*)**.** Let E and F be finite dimensional C*-algebras, and let $\phi_0, \phi_1 : E \to F$ be homomorphisms (not necessarily unital). The C*-algebra

$$A(E, F, \phi_0, \phi_1) = \{(e, f) \in E \oplus C([0,1], F) : f(0) = \phi_0(e),\ f(1) = \phi_1(e)\}$$

will be called an Elliott–Thomsen algebra or a point-line algebra. (See [15]. These algebras are the one-dimensional case of the non-commutative CW-complexes studied in [13].) The class of point-line algebras will be denoted by $\mathcal{C}$.

Definition 2.3 (*[41]*)**.** Let $k, n \in \mathbb{N}$. Consider the homomorphisms $\phi_0, \phi_1 : \mathrm{M}_k(\mathbb{C}) \to \mathrm{M}_{k(n+1)}(\mathbb{C})$ defined by

$$\phi_0(a) = a \otimes \mathrm{diag}(1_n, 0_k) = \mathrm{diag}(\underbrace{a, \dots, a}_{n}, 0_k) \quad \text{and} \quad \phi_1(a) = a \otimes 1_{n+1} = \mathrm{diag}(\underbrace{a, \dots, a}_{n+1}).$$

The C*-algebra

$$R(k, n) = A(\mathrm{M}_k(\mathbb{C}), \mathrm{M}_{k(n+1)}(\mathbb{C}), \phi_0, \phi_1) \in \mathcal{C} \tag{2.1}$$

will be called a Razak algebra. Let $e \in R(k,n)$ be a strictly positive element. (It is easy to check that

$$\lambda_s(R(k,n)) = \inf\{d_\tau(e) : \tau \in \mathrm{T}(R(k,n))\} = \frac{n}{n+1} \quad --- \text{ see } 5.3). \tag{2.2}$$

Let us also call a direct sum of such C*-algebras a Razak algebra, and denote this class of C*-algebras by $\mathcal{R}_{\mathrm{az}}$.

Definition 2.4. Denote by $\mathcal{C}_0$ the class of all C*-algebras A in $\mathcal{C}$ which satisfy the following conditions: (1) $\mathrm{K}_1(A) = \{0\}$, (2) $\mathrm{K}_0(A)_+ = \{0\}$, and (3) $0 \notin \overline{\mathrm{T}(A)}^{\mathrm{w}}$. (What (2) says is that the C*-algebras in $\mathcal{C}_0$ are stably projectionless. What (3) says is that the spectrum of A is compact.)

Denote by $\mathcal{C}_0^0$ the subclass of C*-algebras in $\mathcal{C}_0$ with $\mathrm{K}_0(A) = \{0\}$. Then every Razak algebra is in $\mathcal{C}_0^0$. Let $\mathcal{C}_0'$ denote the class of all full hereditary sub-C*-algebras of C*-algebras in $\mathcal{C}_0$ and let $\mathcal{C}_0^{0'}$ denote the class of all full hereditary sub-C*-algebras of C*-algebras in $\mathcal{C}_0^0$.

In what follows, for $r > 0$, we will use f_r to denote the continuous positive function defined on $[0, \infty)$ by $f_r(t) = 0$ if $t \in [0, r/2]$, $f_r(t) = 1$ if $t \in [r, 1]$, and f_r is linear on $(r/2, r)$.

Definition 2.5 (*see 8.1 and 8.11 of [17]*)**.** Recall that a simple C*-algebra is said to be in the class $\mathcal{D}$ (or $\mathcal{D}_0$), if the following conditions hold: There are a strictly positive element $e \in A$ with $\|e\| \le 1$ and a real number $1 > \mathfrak{f}_e > 0$, such that for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subseteq A$, and any $a \in A_+ \setminus \{0\}$, there are $\mathcal{F}$-ε-multiplicative completely positive contractive maps $\phi : A \to A_0$ and $\psi : A \to D$ for orthogonal sub-C*-algebras $A_0,\ D \subseteq A$ with $D \in \mathcal{C}_0$ (or $\mathcal{C}_0^0$), satisfying

$$\|x - (\phi(x) + \psi(x))\| < \varepsilon \ \text{ for all } \ x \in \mathcal{F}, \tag{2.3}$$

$$\phi(e) \lesssim a, \quad \text{and} \tag{2.4}$$

$$\tau(f_{1/4}(\psi(e))) \ge \mathfrak{f}_e \ \text{ for all } \ \tau \in \mathrm{T}(D). \tag{2.5}$$

In fact $\mathfrak{f}_e$ can be chosen to be $\inf\{\tau(f_{1/4}(e)) : \tau \in \mathrm{T}(A)\}/2$ (see 9.2 of [17]). Note that, if $A \in \mathcal{D}$ is a separable C*-algebra and B is a hereditary sub-C*-algebra of A, then $B \in \mathcal{D}$ (see 8.6 of [17]). We refer to [17] for a detailed discussion of the definition of the class $\mathcal{D}$.

Definition 2.6. Let us denote by $\mathcal{M}_0$ the class of simple separable C*-algebras which are inductive limits of sequences of C*-algebras in $\mathcal{C}_0^0$, with respect to maps which are injective and take strictly positive elements to strictly positive elements. This class is closed under tensoring with full matrix algebras (as the class of Razak algebras is), and hence is closed under tensoring with any unital simple AF algebra (as the tensor product of a map between two Razak algebras and a unital map between two finite-dimensional algebras is injective and preserves strictly positive elements if both maps have these properties).

Definition 2.7. Recall that the C*-algebra $\mathcal{W}$ is the simple inductive limit of a sequence of Razak algebras with injective connecting maps ((2.1)—see [41,53], and [28]) with a unique trace, which is bounded. By Theorem 1.1 of [41], it is the unique such C*-algebra. (Unique meaning in the Razak limit class, with unique trace which is bounded; in particular it follows that this algebra – indeed any simple such limit of Razak algebras – is isomorphic to its tensor product with a full matrix algebra.) The C*-algebra $\mathcal{W}$ belongs to the class $\mathcal{M}_0$ by Lemma 3.3 of [28].

Furthermore, $\mathcal{W}$ has continuous scale (and so belongs to the class $\mathcal{R}$), by the first part of Proposition 5.4 of [17] (the required property of strict comparison holds by Theorem 4.6 of [51]). Hence by Theorem 3.3 of [32], $\mathcal{W}$ is algebraically simple. Hence by the remark in Definition 9.5 of [17], $\mathcal{W} \in \mathcal{D}_0$.

Theorem 2.8. *For any non-empty metrizable Choquet simplex Δ, there exists a non-unital simple C*-algebra $A \in \mathcal{R}$ (Definition 2.1) such that $A = \lim_{n\to\infty}(B_n, \imath_n)$ where each B_n is a finite direct sum of copies of $\mathcal{W}$ and each $\imath_n$ preserves strictly positive elements (takes strictly positive elements into strictly positive elements), every trace of A is bounded, and*

$$(\mathrm{K}_0(A), \mathrm{K}_1(A), \mathrm{T}(A)) = (\{0\}, \{0\}, \Delta).$$

Moreover, A may be chosen so that $A \in \mathcal{M}_0$ (Definition 2.6), and $A \in \mathcal{D}_0$ (Definition 2.5).

Proof. By 3.10 of [3], there exists a unital simple AF algebra D with $\mathrm{T}(D) = \Delta$. As we shall now show, the C*-algebra $A = D \otimes \mathcal{W}$ has the desired properties. By 2.7, $A \in \mathcal{M}_0$. Since $\mathrm{M}_n(\mathcal{W}) \cong \mathcal{W}$ (see 2.7), it follows easily that $A = \lim_{n\to\infty}(B_n, \iota_n)$, where each B_n is a finite direct sum of copies of $\mathcal{W}$.

Let $e \in \mathcal{W}$ be a strictly positive element. Since W is algebraically simple (see 2.7), $e \in \mathrm{Ped}(\mathcal{W})$. By the definition of the Pedersen ideal, $1 \otimes e \in \mathrm{Ped}(A)$. It follows from Proposition 5.6.2 of [38] that $\mathrm{Ped}(A) = A$ as the hereditary sub-C*-algebra generated by $1 \otimes e$ is A itself. In other words, A is algebraically simple. By (the end of) Definition 9.5 of [17], $A \in \mathcal{D}_0$. Consequently, all traces of A are bounded, and by Theorem 9.4 of [17], A has strict comparison for positive elements. It is clear that $\mathrm{K}_0(A) = \mathrm{K}_1(A) = \{0\}$. Note that the natural affine map from the simplex $\Delta = \mathrm{T}(D)$ to $\mathrm{T}(A)$, consisting of tensoring with the unique tracial state of $\mathcal{W}$, is weak* continuous and bijective and therefore a homeomorphism. It remains to show that A has continuous scale (and so belongs to the class $\mathcal{R}$). This follows from the established facts that A is algebraically simple and $\mathrm{T}(A)$ is compact and Theorem 5.3 of [17]. □

Corollary 2.9 (*[41,53]*)**.** *Let $\widetilde{T}$ be a non-zero topological cone with a compact base which is a metrizable Choquet simplex and let $\gamma : \widetilde{T} \to (0, \infty]$ be a lower semicontinuous affine function, zero at $0 \in \widetilde{T}$, but only there. There exists a simple C*-algebra A which is an inductive limit of Razak algebras such that*

$$(\widetilde{\mathrm{T}}(A), \Sigma_A) = (\widetilde{T}, \gamma).$$

Moreover, A may be chosen to be an inductive limit of finite direct sums of copies of $\mathcal{W}$.

Proof. By Theorem 5.1 of [53], there is a simple C*-algebra B which is an inductive limit of Razak algebras such that $\widetilde{\mathrm{T}}(B) = \widetilde{T}$ and the lower semicontinuous function $\omega(\tau) = \|\tau\|$ (allow values in $[0, \infty]$) is equal to γ. Let $e \in B_+$ be a strictly positive element of B with $\|e\| = 1$. Then $d_\tau(e) = \|\tau\|$ for each $\tau \in \widetilde{\mathrm{T}}(A)$. Thus, $(\widetilde{\mathrm{T}}(B), \Sigma_B) = (\widetilde{T}, \gamma)$.

To see the last part of the theorem, let $b \in \mathrm{Ped}(B)_+ \setminus \{0\}$ with $\|b\| = 1$. Define $\Delta = \{\tau \in \widetilde{\mathrm{T}}(B) : \tau(b) = 1\}$. Since B is separable, by Proposition 2.6 of [49], Δ is a base for $\widetilde{T}$ and it is a metrizable Choquet simplex. Moreover $0 \notin \Delta$. Choose a unital simple AF algebra C with $\mathrm{T}(C) = \Delta$ [3]. Since Δ is compact, $\inf\{\gamma(\tau) : \tau \in \Delta\} > 0$. It follows from Corollary I.1.4 of [1] that there is an increasing sequence of continuous affine functions f_n converging to γ on Δ. By a compactness argument, we may assume that each $f_n \in \mathrm{Aff}_+(\Delta)$. (In other words, $\gamma \in \mathrm{LAff}_+(\widetilde{T})$.) Since $\rho_C(\mathrm{K}_0(C))$ is dense in $\mathrm{Aff}(\Delta)$ (see III.3.4 of [5]), there is an element $a \in (C \otimes \mathcal{K})_+$ such that $\mathrm{d}_\tau(a) = \gamma(\tau)$ for all $\tau \in \Delta = \mathrm{T}(C)$ (see, for example, Theorem 15.2 of [25] and also the proof of III 3.3 of [5]). Set $\overline{a(C \otimes \mathcal{K})a} = C_1$; the hereditary sub-C*-algebra C_1 is also AF. Note that the topological cone $\tilde{\mathrm{T}}(A)$, being completely determined by the compact base Δ (which does not contain zero), is isomorphic to the cone $\tilde{\mathrm{T}}$ which also has Δ as a base. The tensor product $A = C_1 \otimes \mathcal{W}$ has the desired properties. □

3. A stable uniqueness theorem

The following lemma, concerning extensions with non-unital quotient, is a consequence of, and in fact equivalent to, the second part of Corollary 16 of [18] and Theorem 2.1 of [20], in the case of a trivial extension (which is all that we need – this restriction can easily be removed, in the nuclear setting, by working with Choi–Effros liftings). The analogous, purely unital setting – both quotient and extension unital—is dealt with in Theorem 6 of [18]. As pointed out in [20], the mixed case, unital quotient but non-unital extension, while discussed in Section 16 of [18], is not correctly dealt with there, and a corrected statement of the first part of Corollary 16 of [18] was given in Theorem 2.3 of [20]. Closely related earlier results are contained in [11] and [33].

Lemma 3.1. *Let A and B be C*-algebras with B stable and A separable and non-unital. Let $\pi : A \to \mathrm{M}(B)$ be a faithful homomorphism such that the composition with the quotient map to $\mathrm{M}(B)/B$ is also faithful and the induced (trivial) extension is purely large (in the sense of [18]). Then, for any nuclear homomorphism $\sigma : A \to \mathrm{M}(B)$, there is a sequence (u_n) in $\mathrm{M}(\mathrm{M}_2(B))$ with $u_n^* u_n = 1_{\mathrm{M}(B)} \otimes e_{11}$ and $u_n u_n^* = 1_{\mathrm{M}(\mathrm{M}_2(B))}$ such that*

(1) *$\pi(a) - u_n^*(\sigma(a) \oplus \pi(a))u_n \in B$, $n = 1, 2, \ldots$, $a \in A$, and*
(2) *$\lim_{n\to\infty}(\pi(a) - u_n^*(\sigma(a) \oplus \pi(a))u_n) = 0$, $a \in A$.*

Proof. This follows immediately from the second part of Corollary 16 of [18] and Theorem 2.1 of [20], in the case of a trivial extension, with the ideal of that theorem taken to be the C*-algebra direct sum of a countable infinity of copies of the present ideal, B, and the (trivial) extension to be that induced by the infinite repetition of the map π into the Cartesian product of copies of the multiplier algebra $\mathrm{M}(B)$. (This ostensibly special case of 2.1 of [20] is interesting in that it is in fact a stronger result, in the case of trivial extensions – this observation is also valid in the case of a general (non-trivial) extension, in the nuclear setting – again, on considering Choi–Effros liftings.) □

Let A and B be C*-algebras, let $\gamma : A \to B$ be a homomorphism, and consider the ampliated homomorphism

$$\gamma_\infty := \gamma \oplus \gamma \oplus \cdots : A \to \mathrm{M}(\mathcal{K} \otimes B),$$

where $\mathcal{K}$ is the algebra of compact operators on a separable infinite-dimensional Hilbert space, and $\mathrm{M}(\mathcal{K} \otimes B)$ is the multiplier algebra.

Lemma 3.2. *With A and B and γ and γ_∞ as above, assume that A and B are separable and γ is faithful, and that A is not unital. If $\gamma : A \to B$ is full, i.e., if $\overline{B\gamma(a)B} = B$, $a \in A \setminus \{0\}$, then for any nuclear homomorphism $\sigma : A \to \mathrm{M}(\mathcal{K} \otimes B)$, there is a sequence (u_n) in $\mathrm{M}(\mathrm{M}_2(\mathcal{K} \otimes B))$ with $u_n^* u_n = 1_{\mathrm{M}(\mathcal{K}\otimes B)} \otimes e_{11}$ and $u_n u_n^* = 1_{\mathrm{M}_2(\mathrm{M}(\mathcal{K}\otimes B))}$ such that*

(1) *$\gamma_\infty(a) - u_n^*(\sigma(a) \oplus \gamma_\infty(a))u_n \in \mathcal{K} \otimes B$, $n = 1, 2, \ldots$, $a \in A$, and*
(2) *$\lim_{n\to\infty}(\gamma_\infty(a) - u_n^*(\sigma(a) \oplus \gamma_\infty(a))u_n) = 0$, $a \in A$.*

Proof. The lemma follows from Lemma 3.1 immediately once one checks that the extension γ_∞ is purely large.

To see γ_∞ is purely large, let $B_s = B \otimes \mathcal{K}$. Let $c \in \gamma_\infty(A) + B_s$ be a non-zero element which is not in B_s. One may write $c = \gamma_\infty(a) + b$ for some $a \neq 0$ and $b \in B_s$. Let us consider $\overline{cBc^*}$. Replacing c by cc^*, one may assume that $c \geq 0$. Therefore one may assume that $a \geq 0$. It is clear that $\overline{\gamma_\infty(a)B_s\gamma_\infty(a)} \cong \overline{\gamma(a)B\gamma(a)} \otimes \mathcal{K}$. Since $\overline{B\gamma(a)B} = B$, $\overline{B\overline{\gamma_\infty(a)B_s\gamma_\infty(a)}B} = B$. Thus $B \subset \overline{\gamma_\infty(a)B_s\gamma_\infty(a)}$. It follows $\overline{\gamma_\infty(a)B_s\gamma_\infty(a)} = B_s$. In other words, $\overline{\gamma_\infty(a)B_s\gamma_\infty(a)}$ is full in B_s. In what follows, we assume $\|c\| \leq 1$ and $\|a\| \leq 1$.

We now follows the proof of Theorem 17 (iii) of [18]. Since there are some typos there, we will add some details concerning the current situation.

We first show that $\overline{cB_s c}$ is full. Put $c_1 = \gamma_\infty(a) = \gamma(a) \otimes 1$ and choose $u_n = 1 \otimes v_n$, where $v_n \in M(\mathcal{K})$ and (v_n) is a sequence of unitaries corresponding to some permutations of an orthonormal basis such that $\lim_{n\to\infty} \|b_1 u_n b_2\| = 0$ for any $b_1, b_2 \in B_s$. Note that $u_n c_1^{1/2} = c_1^{1/2} u_n$ as $c_1 = \gamma_\infty(a)$, and $u_n c u_n^* \to c_1$ strictly in $M(B_s)$ exactly as on the page 405 of [18]. Hence

$$u_n(c(u_n^*b'u_n)c)u_n^* = (u_ncu_n^*)b'(u_ncu_n^*) \to c_1b'c_1 \ \text{ for all } \ b' \in B$$

(converges in norm). Since $\overline{c_1B_sc_1} = \overline{\gamma_\infty(a)B_s\gamma_\infty(a)}$ is full in B_s, It follows that $\overline{cB_sc}$ is full in B_s.

Put $c' = c^{1/2}c_1c^{1/2}$ and $c'' = c_1^{1/2}cc_1^{1/2}$. Put $x = c^{1/2}c_1^{1/2}$. Then $xx^* = c'$ and $x^*x = c''$. Let $C_1 := \overline{c'B_sc'}$ and $C_2 := \overline{c''B_sc''}$. (Note, since $c'' = \gamma_\infty(a^2)+b'$ for some $b' \in B_s$, C_2 is also full in B_s.) We will show that $C_2 := \overline{c''B_sc''}$ is stable. Since $C_1 \cong C_2$, C_1 is then also stable. (Note also, since the (closed) ideal generated by $\overline{xx^*B_sxx^*}$ contains that of $\overline{x^*xB_sx^*x} = C_2$, C_1 is also full.) Since $0 \le c' \le c^{1/2}$, this implies that $\overline{cB_sc}$ contains a stable sub-C*-algebra C_1. In other words, γ_∞ is purely large.

To show that C_2 is stable, we write $c'' = c_1^2+b_1$ for some $b_1 \in B_s$. We will verify condition (b) of Proposition 2.2 of [26] which by Proposition 2.2 and Theorem 2.1 of [26] is equivalent to the stability of a σ-unital C*-algebra. Fix an element $a \in C_2$ with $0 \le a_1 \le 1$ and $\varepsilon > 0$. Since $C_2 \subset \overline{c_1B_sc_1}$, one may choose an integer $k \ge 4$ such that

$$\|(c'')^{1/2k}a_1^{1/2} - a_1^{1/2}\| < \varepsilon/8 \ \text{ and } \ \|c_1^{1/k}a_1^{1/2} - a_1^{1/2}\| < \varepsilon/8. \tag{3.1}$$

Put $d = c_1^{1/k}$ and $d_1 = (c'')^{1/2k}$. Then $d - d_1 \in B_s$. Since $d = \gamma(a)^{1/k} \otimes 1$, $u_nd = du_n$. Recall that $\lim_{n\to\infty} \|b_1u_nb_2\| = 0$ for any $b_1, b_2 \in B_s$. Hence, there is an integer $n_1 \ge 1$ such that, for all $n \ge n_1$,

$$du_na_1^{1/2} \approx_{\varepsilon/8} d_1u_na_1^{1/2} = u_nd_1a_1^{1/2} \approx_{\varepsilon/8} u_na_1^{1/2}, \ \text{ and} \tag{3.2}$$

$$a_1^{1/2}du_na_1^{1/2} \approx_{\varepsilon/4} 0. \tag{3.3}$$

Put $y_n = du_na_1^{1/2} \in C_2$. Then (see (3.2) and (3.3))

$$y_n^*y_n = a_1^{1/2}u_n^*ddu_na_1^{1/2} \approx_{\varepsilon/4} (a_1^{1/2}u_nd)a_1^{1/2} \approx_{\varepsilon/4} a \ \text{ and} \tag{3.4}$$

$$(y_n^*y_n)(y_ny_n^*) = y_n^*(du_na_1^{1/2}du_na_1^{1/2})y_n = (y_ndu_n)(a_1^{1/2}du_na_1^{1/2})y_n \approx_{\varepsilon/4} 0. \tag{3.5}$$

By 2.2 (b) of [26], C_2 is stable. As mentioned above, it follows that C_1 is stable and is full in B_s. This shows that the extension γ_∞ is purely large. □

Remark 3.3. One may prove directly that the map γ_∞ absorbs any σ as stated in Lemma 3.2 without using the notion of purely large.

Theorem 3.4 (*Theorem 4.2 of [10]*)**.** *Let A be a separable C*-algebra without unit, and let B be a separable C*-algebra. Let $\gamma : A \to B$ be a full homomorphism.*

Let $\phi, \psi : A \to B$ be nuclear homomorphisms with $[\phi] = [\psi]$ in $\mathrm{KK_{nuc}}(\widetilde{A}, B)$. Then for any finite set $\mathcal{F} \subseteq A$ and $\varepsilon > 0$, there exist an integer n and a unitary $u \in \mathrm{M}_{n+1}(B)$ such that

$$\|u^*(\phi(a)) \oplus (\underbrace{\gamma(a) \oplus \cdots \oplus \gamma(a)}_{n})u - \psi(a) \oplus (\underbrace{\gamma(a) \oplus \cdots \oplus \gamma(a)}_{n})\| < \varepsilon, \quad a \in \mathcal{F}.$$

Proof. Since $[\phi] = [\psi]$ in $\mathrm{KK_{nuc}}(A, B)$, one has that $[\phi, \psi, 1] = 0$ in $\mathrm{KK_{nuc}}(A, B)$ in the sense of [10]. Set $\mathrm{M}(\mathcal{K}(H)\otimes B) = D$, $\mathrm{M}(\mathcal{K}(\mathbb{C} \oplus H) \otimes B) = D_1$, $\mathrm{M}(\mathcal{K}(H \oplus H) \otimes B) = D_2$, where $H = l^2$.

Consider the projection $e_n = f_n \otimes 1_{\widetilde{B}} \in \mathrm{M}(\mathcal{K}(H) \otimes B)$, $n = 1, 2, \ldots$, where f_n is the projection onto the first n basis elements of H.

Consider the unital maps $\Phi^\sim, \Psi^\sim : \widetilde{A} \to \mathrm{M}(\mathcal{K}(H) \otimes B)$ defined by

$$\Phi^\sim(a) = \phi(a) \oplus \gamma'_\infty(a) \quad \text{and} \quad \Psi^\sim(a) = \psi(a) \oplus \gamma'_\infty(a) \ \text{ for all } \ a \in A, \tag{3.6}$$

where $\gamma'_\infty(a)$ is considered to be a map from A to $(1_D - e_1)\mathrm{M}(\mathcal{K}(H)\otimes B)(1_D - e_1)$. One checks $[\Phi^\sim, \Psi^\sim, 1] = 0$ in $\mathrm{KK_{nuc}}(\widetilde{A}, B)$. By Proposition 3.6 of [10], there are a unital strictly nuclear representation $\sigma : \widetilde{A} \to \mathrm{M}(\mathcal{K}(H) \otimes B)$ and a continuous path of unitaries $u : [0, \infty) \to \mathrm{U}(\mathcal{K}(H\oplus H) \otimes B + \mathbb{C}1_{D_2})$ such that for any $a \in A$,

$$\lim_{t\to\infty} \|u_t(\Phi^\sim(a) \oplus \sigma(a))u_t^* - (\Psi^\sim(a) \oplus \sigma(a))\| = 0, \ \text{ and}$$

$$u_t(\Phi^\sim(a) \oplus \sigma(a))u_t^* - (\Psi^\sim(a) \oplus \sigma(a)) \in \mathcal{K}(H \oplus H) \otimes B.$$

In particular, there is a sequence of unitaries (u_n) in $\mathrm{U}(\mathcal{K}(H \oplus H) \otimes B + \mathbb{C}1_{D_2})$ such that

$$\lim_{n\to\infty} \|u_n(\phi(a) \oplus \gamma'_\infty(a) \oplus \sigma(a))u_n^* - (\psi(a) \oplus \gamma'_\infty(a) \oplus \sigma(a))\| = 0, \quad a \in \widetilde{A}. \tag{3.7}$$

Since γ is full, by Lemma 3.2, the map γ_∞ is (non-unital) nuclearly absorbing. Therefore $\gamma'_\infty \oplus \sigma \sim \gamma_\infty$; that is, there is a sequence of isometries (v_n) in $\mathrm{M}(\mathcal{K}(H_1 \oplus H) \otimes B)$, with $v_n v_n^* = 1_D$, such that, for any $a \in A$,

$$\lim_{n\to\infty} \|\gamma'_\infty(a) \oplus \sigma(a) - v_n^* \gamma_\infty(a) v_n\| = 0, \quad \text{and}$$

$$\gamma'_\infty(a) \oplus \sigma(a) - v_n^* \gamma_\infty(a) v_n \in \mathcal{K}(H_1 \oplus H) \otimes B,$$

where $H_1 = (1_D - e_1)H$.

Consider the unitaries $w_n = (e_1 \oplus v_n)u_n(e_1 \oplus v_n^*)$ in $\mathrm{M}(\mathcal{K}(\mathbb{C}\oplus H)\otimes B)$, in fact in $\mathcal{K}(\mathbb{C}\oplus H)\otimes B + \mathbb{C}1_{D_1}$. For any contraction $a \in A$,

$$\begin{aligned}
&\|w_n(\phi(a) \oplus \gamma_\infty(a))w_n^* - \psi(a) \oplus \gamma_\infty(a)\| \\
=\ &\|(e_1 \oplus v_n)u_n(e_1 \oplus v_n^*)(\phi(a) \oplus \gamma_\infty(a))(e_1 \oplus v_n)u_n^*(e_1 \oplus v_n^*) - \psi(a) \oplus \gamma_\infty(a)\| \\
\approx\ &\|(e_1 \oplus v_n)u_n(\phi(a) \oplus (\gamma'_\infty(a) \oplus \sigma(a)))u_n^*(e_1 \oplus v_n^*) - \psi(a) \oplus \gamma_\infty(a)\| \\
\approx\ &\|(e_1 \oplus v_n)(\psi(a) \oplus (\gamma'_\infty(a) \oplus \sigma(a)))(e_1 \oplus v_n^*) - \psi(a) \oplus \gamma_\infty(a)\| \\
\approx\ &\|\psi(a) \oplus \gamma_\infty(a) - \psi(a) \oplus \gamma_\infty(a)\| = 0.
\end{aligned}$$

That is, there is a sequence of unitaries (w_k) in $\mathrm{U}(\mathcal{K}(\mathbb{C} \oplus H) \otimes B + \mathbb{C}1_{D_1})$ such that

$$\lim_{k\to\infty} \|w_k(\phi(a) \oplus \gamma_\infty(a))w_k^* - (\psi(a) \oplus \gamma_\infty(a))\| = 0, \quad a \in A.$$

Since $w_k \in \mathcal{K}(\mathbb{C} \oplus H) \otimes B + \mathbb{C}1_{D_1}$, one has that $[w_k, e_n] \to 0$, as $n \to \infty$. Then, for sufficiently large k, and then sufficiently large n, the element $e_n w_k e_n$ of $\mathrm{M}_n(B) + \mathbb{C}1_n$ can be perturbed to a unitary u verifying the conclusion of the theorem. □

Remark 3.5. The unital version of 3.4 can be found in 4.2 of [10] (see an earlier version in [33]). A different approach could also be found in an earlier version of this paper (see [22]).

Proposition 3.6 (*Proposition 2.1 of [2]*)**.** *Let A be a separable C*-algebra (with or without unit). Then there is a countable subset S of A such that if J is any ideal of A, then $S \cap J$ is dense in J.*

Lemma 3.7. *Let D be a C*-algebra. Let $A \subseteq D$ be a separable sub-C*-algebra such that*

$$\overline{DaD} = D \ \text{ for all } \ a \in A \setminus \{0\},$$

and let $B \subseteq D$ be another separable sub-C-algebra. Then, there is a separable sub-C*-algebra C of D such that*

$$A, B \subseteq C \ \text{ and } \ \overline{CaC} = C \ \text{ for all } \ a \in A \setminus \{0\} \tag{3.8}$$

(i.e., such that the inclusion map $A \to C$ is full).

Proof. The proof follows an idea of Blackadar. Applying Proposition 3.6, one obtains a countable set

$$\{a_0, a_1, a_2, \dots\} \subseteq A$$

such that $\{a_0, a_1, a_2, \dots\} \cap J$ is dense in J for any ideal J of A. We may assume that

$$a_0 = 0, \quad \text{so that} \quad a_j \neq 0, \quad j = 1, 2, \dots\ .$$

Set

$$C_1 = \mathrm{C}^*(A \cup B) \subseteq D.$$

It is clear that C_1 is separable. Pick a dense set $\{c_1, c_2, \dots\}$ in C_1. Since $\overline{Da_jD} = D$, $j = 1, 2, \dots$, for any $\varepsilon > 0$ and any c_i, there are finitely non-zero sequences $x_{c_i,a_j,\varepsilon,1}, x_{c_i,a_j,\varepsilon,2}, \dots$ and $y_{c_i,a_j,\varepsilon,1}, y_{c_i,a_j,\varepsilon,2}, \dots$ in D such that

$$\|c_i - (x_{c_i,a_j,\varepsilon,1} a_j y_{c_i,a_j,\varepsilon,1} + x_{c_i,a_j,\varepsilon,2} a_j y_{c_i,a_j,\varepsilon,2} + \cdots)\| < \varepsilon.$$

Set

$$C_2 = \mathrm{C}^*(C_1, x_{c_i,a_j,\frac{1}{n},k}, y_{c_i,a_j,\frac{1}{n},k} : i, j, n, k = 1, 2, \dots).$$

Then

$$\overline{C_2 a_j C_2} \supseteq C_1, \quad j = 1, 2, \dots\ .$$

Repeating the construction above, one obtains a sequence of separable C*-algebras

$$C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n \subseteq \cdots \subseteq D$$

such that

$$\overline{C_{n+1}a_jC_{n+1}} \supseteq C_n, \quad j=1,2,\ldots, \quad n=1,2,\ldots .$$

Setting $\overline{\bigcup_{n=1}^{\infty} C_n} = C$, one has

$$\overline{Ca_jC} = C, \quad j=1,2,\ldots .$$

Then the separable sub-C*-algebra C satisfies the requirements of the lemma. Indeed, let $a \in A \setminus \{0\}$. Consider the ideal $J := \overline{CaC} \cap A$. Since $a \in J$, one has $J \neq \{0\}$. By Proposition 3.6, one has that $\{a_0, a_1, a_2, \ldots\} \cap J$ is dense in J, and in particular, the ideal J contains some $a_j \neq 0$. Since $C = \overline{Ca_jC} \subseteq \overline{CJC} = \overline{CaC}$, one has $\overline{CaC} = C$, as desired. □

Remark 3.8. If A is simple, then, in the proof above, one only needs to pick one non-zero element of A and does not need Proposition 3.6.

Lemma 3.9. *Let B be a σ-unital C*-algebra and let A be a separable amenable C*-algebra which is a sub-C*-algebra of B. Let $h_1, h_2 : A \to B$ be homomorphisms such that $[h_1] = [h_2]$ in $\mathrm{KK}(A,B)$ (which we regard as $\mathrm{KK}^1(A, SB)$). There exists a separable sub-C*-algebra $C \subseteq B$ such that $A, h_1(A), h_2(A) \subseteq C$ and $[h_1] = [h_2]$ in $\mathrm{KK}(A, C)$. If the inclusion of A in B is full (in other words, $\overline{BaB} = B$ for any $0 \neq a \in A$), then C may be chosen such that the inclusion of A in C is full.*

Proof. Consider the extensions $\tau_1, \tau_2 : A \to \mathrm{M}(SB)/SB$ given by the mapping tori

$$M_{h_i} = \{(f, a) \in \mathrm{C}([0,1], B) \oplus A : f(0) = a \text{ and } f(1) = h_i(a)\}, \quad i = 1, 2. \tag{3.9}$$

Let $H_i : A \to \mathrm{M}(SB)$ be a completely positive contractive lifting of τ_i, $i = 1, 2$. There are a monomorphism $\phi_0 : A \to \mathrm{M}(SB \otimes \mathcal{K})$ and a unitary $w \in \mathrm{M}(SB \otimes \mathcal{K})$ such that

$$w^*(H_1(a) \oplus \phi_0(a))w - (H_2(a) \oplus \phi_0(a)) \in SB \otimes \mathcal{K} \;\text{ for all }\; a \in A.$$

Let C_{000} denote the (separable) sub-C*-algebra of B generated by A, $h_1(A)$, and $h_2(A)$.

Choose a system of matrix units $(e_{i,j})$ for $\mathcal{K}$, and choose a dense sequence (t_n) in $(0, 1)$. Choose an increasing approximate unit (E_n) for $SB \otimes \mathcal{K}$ such that $E_n \in \mathrm{M}_{k(n)}(SB)$, $n = 1, 2, \ldots$.

Denote by D_0 the (separable) sub-C*-algebra of $SB \otimes \mathcal{K}$ generated by

$$w^*(\mathrm{diag}(H_1(a), \phi_0(a)))w - \mathrm{diag}(H_2(a), \phi_0(a)), \quad a \in A. \tag{3.10}$$

Denote by D_{00} the sub-C*-algebra of $SB \otimes \mathcal{K}$ generated by

$$\{E_n, wE_n, E_nw, E_n\phi_0(a), \phi_0(a)E_n : a \in A, n \in \mathbb{N}\}.$$

Let D_{000} denote the (separable) sub-C*-algebra of $SB \otimes \mathcal{K}$ generated by D_{00} and D_0. Denote by $\pi_t : SB \otimes \mathcal{K} \to B \otimes \mathcal{K}$ the point evaluation at $t \in (0, 1)$, and by C_{00} the sub-C*-algebra of $B \otimes \mathcal{K}$ generated by

$$\{\pi_{t_n}(D_{000}) + C_{000} \otimes e_{1,1} : n = 1, 2, \ldots\}.$$

Denote by $C_{0,n} \subseteq B \otimes \mathcal{K}$ the sub-C*-algebra generated by $\{(1 \otimes e_{1,i})C_{00}(1 \otimes e_{j,1}) : 1 \leq i, j \leq n\}$, $n = 1, 2, \ldots$. Let C' denote the (separable) sub-C*-algebra of B generated by $\bigcup_{n=1}^{\infty} C_{0,n}$. Choose a separable sub-C*-algebra C of B containing A and C'. By Lemma 3.7, if the inclusion $A \to B$ is full, then we may choose C such that the inclusion $A \to C$ is full. Note that (as $C_{000} \subseteq C_{0,1}$), $h_1(A), h_2(A) \subseteq C' \subseteq C$. Consider the sub-C*-algebra $C_1 = C \otimes \mathcal{K}$ of $B \otimes \mathcal{K}$. Fix

$$b \in \{E_n, wE_n, E_nw, E_n\phi_0(a), \phi_0(a)E_n : a \in A, n \in \mathbb{N}\} \subseteq SB \otimes \mathcal{K}. \tag{3.11}$$

Keep in mind that $E_n \in SB \otimes \mathcal{K} = \mathrm{C}_0((0,1), B \otimes \mathcal{K})$, in particular, $E_n(0) = E_n(1) = 0$, $n \in \mathbb{N}$. Then, for each t_n, $n = 1, 2, \ldots$,

$$\pi_{t_n}(b) \in \pi_{t_n}(SC_1 \otimes \mathcal{K}).$$

It follows that $b \in SC_1 \otimes \mathcal{K}$. To see this, fix $\varepsilon > 0$, and choose a finite sequence $t_{n_i} \in (t_n)$, $i = 1, 2, \ldots, k$, such that

$$0 = t_{n_0} < t_{n_1} < t_{n_2} < \cdots < t_{n_k} < t_{n_{k+1}} = 1$$

and

$$\|b(t) - b(t_{n_i})\| < \varepsilon/4 \;\text{ for all }\; t \in (t_{n_i}, t_{n_{i+1}}), \;\; i = 0, 1, \ldots, k.$$

Set

$$c(t) = \begin{cases} (\dfrac{t}{t_{n_1}})b(t_{n_1}), & t \in (0, t_{n_1}) \\ (\dfrac{t_{n_{i+1}} - t}{t_{n_{i+1}} - t_{n_i}})b(t_{n_i}) + (\dfrac{t - t_{n_i}}{t_{n_{i+1}} - t_{n_i}})b(t_{n_{i+1}}), & t \in [t_{n_i}, t_{n_{i+1}}), i = 1, 2, \ldots, k, \\ (\dfrac{1-t}{1 - t_{n_k}})b(t_{n_k}), & t \in (t_{n_k}, 1). \end{cases}$$

Then $c \in SC_1 \otimes \mathcal{K}$. On the other hand,

$$\|b(t) - c(t)\| < \varepsilon \text{ for all } t \in (0, 1).$$

Since $\varepsilon > 0$ is arbitrary, this implies that $b \in SC_1 \otimes \mathcal{K}$.

In particular, $E_n \in SC_1 \otimes \mathcal{K}$ and (E_n) is an approximate unit for $SC_1 \otimes \mathcal{K}$, and so $\mathrm{M}(SC_1 \otimes \mathcal{K}) \subseteq \mathrm{M}(SB \otimes \mathcal{K})$. Since also $wE_n, E_n w \in SC_1 \otimes \mathcal{K}$, and $w \in \mathrm{M}(SB \otimes \mathcal{K})$, it follows that $w \in \mathrm{M}(SC_1 \otimes \mathcal{K})$. Similarly, since $\phi_0(a)E_n, E_n\phi_0(a) \in SC_1 \otimes \mathcal{K}$ for all $a \in A$ and $\phi_0(w) \in \mathrm{M}(SB \otimes \mathcal{K})$, we may view ϕ_0 as a monomorphism from A to $\mathrm{M}(SC_1 \otimes \mathcal{K}) \subseteq \mathrm{M}(SB \otimes \mathcal{K})$.

A similar argument shows that $D_0 \subseteq SC_1 \otimes \mathcal{K}$.

We now have

$$w, H_1(a) \oplus \phi_0(a), H_2(a) \oplus \phi_0(a) \in \mathrm{M}(SC_1 \otimes \mathcal{K}),$$

$$w^*(H_1(a) \oplus \phi_0(a))w - (H_2(a) \oplus \phi_0(a)) \in SC_1 \otimes \mathcal{K}$$

for all $a \in A$. This implies that $[h_1] = [h_2]$ in $\mathrm{KK}(A, C)$. $\square$

In a similar way (using Lemma 3.7), one also has the following result:

Lemma 3.10. *Let A, D be C*-algebras, with A separable. Let $\phi, \psi, \sigma : A \to D$ be homomorphisms such that*
$[\phi] = [\psi]$ in $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(D))$, and
$\overline{D\sigma(a)D} = D$, $0 \neq a \in A$.
Then there is a separable sub-C-algebra $C \subseteq D$ such that*
$\phi(A), \psi(A), \sigma(A) \subseteq C$,
$[\phi] = [\psi]$ in $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(C))$, and
$\overline{C\sigma(a)C} = C$, $0 \neq a \in A$.

Proof. The proof is in the same spirit as that of 3.9. We sketch it below. Since A is separable, it is easy to find a separable C*-algebra $B_1 \subseteq D$ such that $\phi(A), \psi(A) \subseteq B_1$ and $\phi_{*i} = \psi_{*i}$ $(i = 0, 1)$ viewing ϕ and ψ as maps from A to B_1. For each $m \geq 2$, let $C_m \cong \mathrm{C}_0(X_m)$ for some locally compact and σ-compact metric space X_m such that $\mathrm{K}_0(C_m) = \mathbb{Z}/m\mathbb{Z}$ and $\mathrm{K}_1(C_m) = \{0\}$. Denote by Y_m the one-point compactification of X_m with the point ξ_0 as the additional point. Note Y_m is separable.

Let $\phi^{(m)}, \psi^{(m)} : C_m \otimes A \to C_m \otimes D$ be the natural extensions of ϕ and ψ. Suppose that p and q are two projections in $\mathrm{M}_l((C_m \otimes D)^\sim)$ for some $l \geq 1$ such that there exists $v \in \mathrm{M}_{l+k}((C_m \otimes D)^\sim)$ with $v^*v = p \oplus 1_k$ and $vv^* = q \oplus 1_k$. We now view p, q, v as functions in $C(Y_m, \mathrm{M}_{l+k}(\widetilde{D}))$. Let (y_n) be a dense sequence of Y_m such that $y_1 = \xi_0$. Consider the sub-C*-algebra $B''_{m,0}$ of $\mathrm{M}_{l+k}(\widetilde{D})$ which is generated by $p(y_n)$, $q(y_n)$, and $v(y_n)$ for all $n \geq 1$. Then $B''_{m,0}$ is separable. One then easily constructs a separable sub-C*-algebra $B'_{m,0}$ of D such that p, q, v are in $\mathrm{M}_{l+k}((C_m \otimes B'_{m,0})^\sim)$. Similarly, if u, w are unitaries in $\mathrm{M}_l((C_m \otimes D)^\sim)$ which are connected by a continuous path of unitaries, then one may also construct a separable sub-C*-algebra $B'_{m,1}$ of D such that u, w are in $\mathrm{M}_l((C_m \otimes D)^\sim)$ and are connected by a continuous path of unitaries in $\mathrm{M}_l((C_m \otimes D)^\sim)$.

From this, one concludes that there is a separable sub-C*-algebra $B_m \subseteq D$ such that $\phi^{(m)}(C_m \otimes A), \psi^{(m)}(C_m \otimes A) \subseteq C_m \otimes B_m$ and $\phi^{(m)}_{*i} = \psi^{(m)}_{*i}$ $(i = 0, 1)$ viewing $\phi^{(m)}$ and $\psi^{(m)}$ as maps from $C_m \otimes A$ to $C_m \otimes B_m$, $m = 2, 3, \ldots$ Let D_1 be the sub-C*-algebra generated by B_m, $m = 1, 2, \ldots$. Then D_1 is separable. By 3.9, there is a separable sub-C*-algebra $C \supseteq D_1, \sigma(A)$ such that $\overline{C\sigma(a)C} = C$ for all $a \in A \setminus \{0\}$. Note now that $[\phi] = [\psi]$ in $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(C))$ as $\phi^{(m)}_{*i} = \psi^{(m)}_{*i}$ with $\phi^{(m)}$ and $\psi^{(m)}$ viewed as maps from A into C. $\square$

Definition 3.11. Let $M : (A_+ \setminus \{0\}) \times (0, 1) \to (0, +\infty)$ and $N : (A_+ \setminus \{0\}) \times (0, 1) \to \mathbb{N}$ be maps. A positive map $\phi : A \to B$ will be said to be (N, M)-full if for any $1 > \varepsilon > 0$, any $a \in A_+ \setminus \{0\}$, and any $b \in B^+$ with $\|b\| \leq 1$, there are $b_1, b_2, \ldots, b_{N(a,\varepsilon)} \in B$ with $\|b_i\| \leq M(a, \varepsilon)$, $i = 1, 2, \ldots, N(a, \varepsilon)$, such that

$$\|b - (b_1^*\phi(a)b_1 + b_2^*\phi(a)b_2 + \cdots + b_{N(a,\varepsilon)}^*\phi(a)b_{N(a,\varepsilon)})\| \leq \varepsilon.$$

Write $F := (N, M) : (A_+ \setminus \{0\}) \times (0, 1) \to \mathbb{N} \times \mathbb{R}_+$, and let $\mathcal{H} \subseteq A_+ \setminus \{0\}$. A positive map $L : A \to B$ will be said to be F-$\mathcal{H}$-full if, for any $a \in \mathcal{H}$, any $b \in B_+$ with $\|b\| \leq 1$, and any $\varepsilon > 0$, there are $x_1, x_2, \ldots, x_m \in B$ with $m \leq N(a, \varepsilon)$ and $\|x_i\| \leq M(a, \varepsilon)$ such that

$$\|\sum_{i=1}^m x_i^* L(a) x_i - b\| \leq \varepsilon. \tag{3.12}$$

The map L will be said to be *uniformly* (N, M)-full if N and M are independent of ε, (i.e., $N : A_+ \setminus \{0\} \to \mathbb{N}$ and $M : A_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$) and to be *strongly uniformly* (N, M)-full, if, in addition, ε can be replaced by zero. The map L will be said to be uniformly F-$\mathcal{H}$-full, if F is independent of ε.

Let B be any C*-algebra and $D \subset B$ be a σ-unital sub-C*-algebra. Let $F = (N, M) : (A_+ \setminus \{0\}) \times (0, 1) \to \mathbb{N} \times \mathbb{R}_+$ be a map described above. We would like to make the following remark: If $L : A \to D$ is $(F, \mathcal{H})$-full, then $j \circ L : A \to \overline{DBD}$

is $(F^{(1/2)}, \mathcal{H})$-full, where $F^{(1/2)}((a,\varepsilon)) = F((a,\varepsilon/2))$ and $j : D \to \overline{DBD}$ is the embedding. In fact, for any $\varepsilon > 0$, given any $b \in \overline{DBD}_+$ with $\|b\| \le 1$, there is $d \in D_+$ with $\|d\| = 1$ such that $\|b^{1/2}db^{1/2} - b\| < \varepsilon/2$. Fix $a \in \mathcal{H} \subseteq A_+ \setminus \{0\}$. There are $x_1, x_2, \ldots, x_m$ with $m \le N(a,\varepsilon/2)$ and $\|x_i\| \le M(a,\varepsilon/2)$ such that

$$\|\sum_{i=1}^{m} x_i^* L(a) x_i - d\| \le \varepsilon/2.$$

It follows that, for $a \in \mathcal{H}$,

$$\|\sum_{i=1}^{m} b^{1/2} x_i^* L(a) x_i b^{1/2} - b\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Note that $\|x_i b^{1/2}\| \le \|x_i\|$. So $j \circ L$ is $(F^{(1/2)}, \mathcal{H})$-full. Note also, if $F(a,t) = F(a,t')$ for all $t, t' \in (0,1)$, (uniformly full), then $F^{(1/2)} = F$, whence $j \circ L$ is still $(F, \mathcal{H})$-full.

Let A and B be C*-algebras and $d : A \to B$ a map. For each integer $n \ge 1$, denote by $d_n : A \to \mathrm{M}_n(B)$ the map $d_n : a \mapsto \underbrace{d(a) \oplus d(a) \oplus \cdots \oplus d(a)}_{n}$ (for $a \in A$).

Theorem 3.12 (*cf. Theorem 3.9 of [34]*). *Let A be a separable amenable C*-algebra and let B be a σ-unital C*-algebra. Let $h_1, h_2 : A \to B$ be homomorphisms such that*

$$[h_1] = [h_2] \text{ in } \mathrm{KL}(A,B).$$

Suppose that there is an embedding $d : A \to B$ which is (N,M)-full for some $N : A_+ \setminus \{0\} \times (0,1) \to \mathbb{N}$ and $M : A_+ \setminus \{0\} \times (0,1) \to \mathbb{R}_+ \setminus \{0\}$.

Then, for any $\varepsilon > 0$ and finite subset $\mathcal{F} \subseteq A$, there are an integer $n \ge 1$ and a unitary $u \in \widetilde{\mathrm{M}_{n+1}(B)}$ such that

$$\|u^* \mathrm{diag}(h_1(a), d_n(a)) u - \mathrm{diag}(h_2(a), d_n(a))\| < \varepsilon \text{ for all } a \in \mathcal{F}. \tag{3.13}$$

Proof. Write $C = \prod_{k=1}^{\infty} B$, $C_0 = \bigoplus_{k=1}^{\infty} B$, and let $\pi : C \to C/C_0$ denote the quotient map. Let $H_i = (h_i) : A \to C$ be defined by $H_i(a) = (h_i(a))$ for all $a \in A$, $i = 1, 2$. Define $H_0 : A \to C$ by $H_0(a) = (d(a))$ for all $a \in A$. It follows from 3.5 of [34] that

$$[\pi \circ H_1] = [\pi \circ H_2] \text{ in } \mathrm{KK}(A, C/C_0). \tag{3.14}$$

Since $d : A \to B$ is (N,M)-full, for any $a \in A_+ \setminus \{0\}$, let $M(a,\varepsilon)$ and $N(a,\varepsilon)$ be as in Definition 3.11. Let $(b_n) \in (\prod_{n=1}^{\infty} B)_+$ with $\|(b_n)\| \le 1$. Then, for any $\varepsilon > 0$, there are $b_{1,n}, b_{2,n}, \ldots, b_{N(a,\varepsilon),n} \in B$ with $\|b_{i,n}\| \le M(a,\varepsilon)$ such that

$$\|\sum_{i=1}^{N(a,\varepsilon)} b_{i,n}^* d(a) b_{i,n} - b_n\| < \varepsilon.$$

Set $(b_{i,n}) = z_i$, $i = 1, 2, \ldots, N(a,\varepsilon)$. Then $\|z_i\| = \sup\{\|b_{i,n}\| : n \in \mathbb{N}\} \le M(a,\varepsilon)$, $i = 1, 2, \ldots, N(a,\varepsilon)$. Therefore, $z_i \in \prod_{n=1}^{\infty} B$. We have

$$\|\sum_{i=1}^{N(a,\varepsilon)} z_i^* H_0(a) z_i - (b_n)\| < \varepsilon.$$

This implies that the map $H_0 : A \to \prod_{n=1}^{\infty} B$ is full.

It follows that the embedding $\pi \circ H_0 : A \to C/C_0$ is full. Combining this with (3.14), and applying Lemma 3.9, we obtain a separable sub-C*-algebra $D \subseteq C/C_0$ such that $\pi \circ H_0(A), \pi \circ H_1(A), \pi \circ H_2(A) \subseteq D$, the map $\pi \circ H_0 : A \to D$ is full, and

$$[\pi \circ H_1] = [\pi \circ H_2] \text{ in } \mathrm{KK}(A, D).$$

By Theorem 3.4 there exist an integer $n \ge 1$ and a unitary $U \in \mathrm{M}_{n+1}(D)^\sim$ such that

$$\|U^* \mathrm{diag}(\pi \circ H_1(a), d_n(\pi \circ H_0(a))) U - \mathrm{diag}(\pi \circ H_2(a), d_n(\pi \circ H_0(a)))\| < \varepsilon, \quad a \in \mathcal{F}.$$

Note that $U \in \mathrm{M}_{n+1}(C/C_0)^\sim$. Therefore (by stable relations) there is a unitary $V = (v_k) \in M_{n+1}(C)^\sim$ such that $\pi(V) = U$. Then, for all sufficiently large k,

$$\|v_k^* \mathrm{diag}(h_1(a), d_n(a)) v_k - \mathrm{diag}(h_2(a), d_n(a))\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Thus, the unitary $u = v_k$ with k sufficiently large satisfies the conclusion of the theorem. $\square$

Definition 3.13 (*Definition 2.1 of [23]*). Fix a map $r_0 : \mathbb{N} \to \mathbb{Z}_+$, a map $r_1 : \mathbb{N} \to \mathbb{Z}_+$, a map $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and integers $s \geq 1$ and $R \geq 1$. We shall say a C*-algebra A belongs to the class $\boldsymbol{C}_{(r_0,r_1,T,s,R)}$ if

(a) for any integer $n \geq 1$ and any pair of projections $p,\ q \in \mathrm{M}_n(\widetilde{A})$ with $[p] = [q]$ in $\mathrm{K}_0(A)$, $p \oplus 1_{\mathrm{M}_{r_0(n)}(\widetilde{A})}$ and $q \oplus 1_{\mathrm{M}_{r_0(n)}(\widetilde{A})}$ are Murray–von Neumann equivalent, and moreover, if $p \in \mathrm{M}_n(\widetilde{A})$ and $q \in \mathrm{M}_m(\widetilde{A})$ and $[p] - [q] \geq 0$, then there exists $p' \in \mathrm{M}_{n+r_0(n)}(\widetilde{A})$ such that $p' \leq p \oplus 1_{M_{r_0(n)}}$ and p' is equivalent to $q \oplus 1_{M_{r_0(n)}}$;

(b) if $k \geq 1$, and $x \in K_0(A)$ such that $-n[1_{\widetilde{A}}] \leq kx \leq n[1_{\widetilde{A}}]$ for some integer $n \geq 1$, then

$$-T(n,k)[1_{\widetilde{A}}] \leq x \leq T(n,k)[1_{\widetilde{A}}];$$

(c) the canonical map $\mathrm{U}(\mathrm{M}_s(\widetilde{A}))/\mathrm{U}_0(\mathrm{M}_s(\widetilde{A})) \to \mathrm{K}_1(A)$ is surjective;

(d) if $u \in \mathrm{U}(\mathrm{M}_n(\widetilde{A}))$ and $[u] = 0$ in $\mathrm{K}_1(\widetilde{A})$, then $u \oplus 1_{\mathrm{M}_{r_1(n)}} \in \mathrm{U}_0(\mathrm{M}_{n+r_1(n)}(\widetilde{A}))$;

(f) $\mathrm{cer}(\mathrm{M}_m(\widetilde{A})) \leq R$ for all $m \geq 1$ (see 2.15 of [24], for example).

If A has stable rank one, and (a) to (f) hold, then they hold with $r_0 = r_1 = 0$.

Let A be a unital C*-algebra and let $x \in A$. Suppose that $\|x^*x - 1\| < 1/2$ and $\|xx^* - 1\| < 1/2$. Then x is invertible and $x|x|^{-1}$ is a unitary. Let us use $\lceil x \rceil$ to denote $x|x|^{-1}$. We will use this notation in the next statement (see (3.15)).

Theorem 3.14 (*cf. 5.3 of [33], Theorem 3.1 of [23], Theorem 4.15 of [11], 5.9 of [34], and Theorem 7.1 of [36]*). *Let A be a non-unital separable amenable C*-algebra which satisfies the UCT, let $r_0, r_1 : \mathbb{N} \to \mathbb{Z}_+$, $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be three maps, let $s, R \geq 1$ be integers, and let $F : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ and $\boldsymbol{L} : \mathrm{U}(\mathrm{M}_\infty(\widetilde{A})) \to \mathbb{R}_+$ be two additional maps. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subseteq A$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subseteq A$, a finite subset $\mathcal{P} \subseteq \underline{\mathrm{K}}(A)$, a finite subset $\mathcal{U} \subseteq \mathrm{U}(\mathrm{M}_\infty(\widetilde{A}))$, a finite subset $\mathcal{H} \subseteq A_+ \setminus \{0\}$, and an integer $K \geq 1$ satisfying the following condition: For any two $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $\phi, \psi : A \to B$, where $B \in \boldsymbol{C}_{r_0,r_1,T,s,R}$, and any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $\sigma : A \to \mathrm{M}_l(B)$ (for any integer $l \geq 1$) which is (uniformly) T-$\mathcal{H}$-full and such that*

$$\mathrm{cel}(\lceil \phi(u) \rceil \lceil \psi(u^*) \rceil) \leq \boldsymbol{L}(u) \ \text{ for all } \ u \in \mathcal{U}, \ \text{ and } \tag{3.15}$$

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \tag{3.16}$$

(see 1.1 of [23] and [42] for the definition of cel) there exists a unitary $U \in \widetilde{\mathrm{M}_{1+Kl}(B)}$ such that

$$\|\mathrm{Ad}\, U \circ (\phi \oplus \sigma_K)(a) - (\psi \oplus \sigma_K)(a)\| < \varepsilon \ \text{ for all } \ a \in \mathcal{F}, \tag{3.17}$$

where

$$\sigma_K := \overbrace{\sigma \oplus \sigma \oplus \cdots \oplus \sigma}^{K} : A \to \mathrm{M}_{Kl}(B).$$

Proof. Let us also use ϕ and ψ for $\phi \otimes \mathrm{id}_{\mathrm{M}_m}$ and $\psi \otimes \mathrm{id}_{\mathrm{M}_m}$, respectively. Fix A, r_0, r_1, T, s, R, F, and $\boldsymbol{L}$ as described above. Suppose that the conclusion of the theorem is false for these data. Then there exist $\varepsilon_0 > 0$ and a finite subset $\mathcal{F} \subseteq A$ such that there are a sequence of positive numbers (δ_n) with $\delta_n \searrow 0$, an increasing sequence $(\mathcal{G}_n)$ of finite subsets of A such that $\bigcup_n \mathcal{G}_n$ is dense in A, an increasing sequence $(\mathcal{P}_n)$ of finite subsets of $\underline{\mathrm{K}}(A)$ such that $\bigcup_n \mathcal{P}_n = \underline{\mathrm{K}}(A)$, an increasing sequence $(\mathcal{U}_n)$ of finite subsets of $\mathrm{U}(\mathrm{M}_\infty(\widetilde{A}))$ such that $\bigcup_n \mathcal{U}_n \cap \mathrm{U}(\mathrm{M}_m(\widetilde{A}))$ is dense in $\mathrm{U}(\mathrm{M}_m(\widetilde{A}))$ for each integer $m \geq 1$, an increasing sequence $(\mathcal{H}_n)$ of finite subsets of $A_+^{\mathbf{1}} \setminus \{0\}$ such that, if $a \in \mathcal{H}_n$ and $f_{1/2}(a) \neq 0$, then $f_{1/2}(a) \in \mathcal{H}_{n+1}$, and $\bigcup_n \mathcal{H}_n$ is dense in $A^{\mathbf{1}}$, and (use 3.6) has dense intersection with the unit ball of each closed two-sided ideal of A, a sequence of integers $(k(n))$ with $\lim_{n\to\infty} k(n) = +\infty$, a sequence of unital C*-algebras $B_n \in \boldsymbol{C}_{r_0,r_1,T,s,R}$, two sequences of $\mathcal{G}_n$-δ_n-multiplicative completely positive contractive maps $\phi_n, \psi_n : A \to B_n$ such that

$$[\phi_n]|_{\mathcal{P}_n} = [\psi_n]|_{\mathcal{P}_n} \quad \text{and} \quad \mathrm{cel}(\lceil \phi_n(u) \rceil \lceil \psi_n(u^*) \rceil) \leq \boldsymbol{L}(u), \quad \text{for all } u \in \mathcal{U}_n, \tag{3.18}$$

a sequence of $\mathcal{G}_n$-δ_n-multiplicative completely positive contractive linear maps $\sigma_n : A \to \mathrm{M}_{l(n)}(B_n)$ which are F-$\mathcal{H}_n$-full and satisfy, for each $n = 1, 2, \ldots$,

$$\inf\{\sup \|v_n^*(\phi_n(a) \oplus (\sigma_n)_{k(n)}(a))v_n - (\psi_n(a) \oplus (\sigma_n)_{k(n)}(a))\| : a \in \mathcal{F}\} \geq \varepsilon_0, \tag{3.19}$$

where the infimum is taken among all unitaries $v_n \in \widetilde{\mathrm{M}_{k(n)l(n)+1}(B_n)}$ and $(\sigma_n)_{k(n)} : A \to \mathrm{M}_{k(n)l(n)}(B_n)$ is as above.

Set $\mathrm{M}_{l(n)}(B_n) = B_n'$, $\bigoplus_{n=1}^\infty B_n' = C_0$, $\prod_{n=1}^\infty B_n' = C$, and $C/C_0 = Q(C)$, and denote by $\pi : C \to Q(C)$ the quotient map. Consider the maps $\Phi, \Psi, S : A \to C$ defined by $\Phi(a) = (\phi_n(a))_{n\geq 1}$, $\Psi(a) = (\psi_n(a))_{n\geq 1}$, and $S(a) = (\sigma_n(a))_{n\geq 1}$, $a \in A$. Note that $\pi \circ \Phi$, $\pi \circ \Psi$ and $\pi \circ S$ are homomorphisms. Consider also the truncations $\Phi^{(m)}, \Psi^{(m)}, S^{(m)} : A \to \prod_{n\geq m} B_n'$ defined by $\Phi^{(m)}(a) = (\phi_n(a))_{n\geq m}$, $\Psi^{(m)}(a) = (\psi_n(a))_{n\geq m}$, and $S^{(m)}(a) = (\sigma_n(a))_{n\geq m}$, $a \in A$.

For each $u \in \mathcal{U}_m$, we have $u \in \mathrm{M}_{L(m)}(\widetilde{A})$ for some integer $L(m) \geq 1$. When $n \geq m$, by hypothesis, there exists a continuous path of unitaries $\{u_n(t) : t \in [0, 1]\} \subseteq M_{L(m)}(\widetilde{B_n'})$ such that

$$u_n(0) = \lceil \phi_n(u) \rceil,\ u_n(1) = \lceil \psi_n(u) \rceil \text{ and } \mathrm{cel}(\{u_n(t)\}) \leq \boldsymbol{L}(u).$$

It follows from Lemma 1.1 of [23] that, for all $n \geq m$, there exists a continuous path $\{U(t) : t \in [0, 1]\} \subseteq \mathrm{U}_0(\prod_{n\geq m} \widetilde{B}'_n)$ such that $U(0) = (\lceil \phi_n(u) \rceil)_{n\geq m}$ and $U(1) = (\lceil \psi_n(u) \rceil)_{n\geq m}$. This in particular implies that

$$\lceil \Phi^{(m)}(u) \rceil \lceil \Psi^{(m)}(u^*) \rceil \in \mathrm{U}_0(\mathrm{M}_{L(m)}(\prod_{n\geq m} \widetilde{B}'_n)) \quad \text{and} \quad [\pi \circ \Phi]_{*1} = [\pi \circ \Psi]_{*1}. \tag{3.20}$$

By (3.18), for all $n \geq m$,

$$[\phi_n]|_{\mathcal{P}_m} = [\psi_n]|_{\mathcal{P}_m}. \tag{3.21}$$

By hypothesis and by [23], $\mathrm{K}_0(C) = \prod_b \mathrm{K}_0(B'_n)$, it follows that

$$[\Phi^{(m)}]|_{\mathrm{K}_0(A)\cap \mathcal{P}_m} = [\Psi^{(m)}]|_{\mathrm{K}_0(A)\cap \mathcal{P}_m}, \quad m = 1, 2, \ldots. \tag{3.22}$$

In particular,

$$[\pi \circ \Phi]_{*0} = [\pi \circ \Psi]_{*0}. \tag{3.23}$$

Now let $x_0 \in \mathcal{P}_m \cap \mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z})$ for some $k \geq 2$. Denote by $\tilde{x}_0 \in \mathrm{K}_1(A)$ the image of x_0 under the map $\mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z}) \to \mathrm{K}_1(A)$. We may assume that $\tilde{x}_0 \in \mathcal{P}_{m_0}$ for some $m_0 \geq m$. By (3.20), $[\Phi^{(m_0)}](\tilde{x}_0) = [\Psi^{(m_0)}](\tilde{x}_0)$. Set $y_0 = [\Phi^{(m_0)}](x_0) - [\Psi^{(m_0)}](x_0)$. Then $y_0 \in \mathrm{K}_0((\prod_{n\geq m_0} B'_n), \mathbb{Z}/k\mathbb{Z})$ must be in the image of $\mathrm{K}_0(\prod_{n\geq m_0} B'_n)$, which may be identified with $\mathrm{K}_0(\prod_{n\geq m_0} B'_n)/k\mathrm{K}_0(\prod_{n\geq m_0} B'_n)$ (see [23]). However, by (3.21),

$$y_0 \in \ker \psi_0^{(k)},$$

where $\psi_0^{(k)} : \mathrm{K}_0(\prod_{n\geq m_0} B'_n, \mathbb{Z}/k\mathbb{Z}) \to \prod_{n\geq m_0} \mathrm{K}_0(B'_n, \mathbb{Z}/k\mathbb{Z})$ is as in 4.1.4 of [31]. By [23], $y_0 = 0$. In other words,

$$[\Phi^{(m_0)}](x_0) = [\Psi^{(m_0)}](x_0),$$

which implies that

$$[\pi \circ \Phi]|_{\mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})} = [\pi \circ \Psi]|_{\mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})}, \quad k = 2, 3, \ldots. \tag{3.24}$$

Now let $x_1 \in \mathrm{K}_1(A, \mathbb{Z}/k\mathbb{Z})$. Then $x_1 \in \mathcal{P}_m$ for some $m \geq 1$. Denote by $\tilde{x}_1 \in \mathrm{K}_0(A)$ the image of x_1 under the map $\mathrm{K}_1(A, \mathbb{Z}/k\mathbb{Z}) \to \mathrm{K}_0(A)$. There is $m_1 \geq m$ such that $\tilde{x}_1 \in \mathcal{P}_{m_1}$. By (3.22), $[\Phi^{(m_1)}](\tilde{x}_1) = [\Psi^{(m_1)}](\tilde{x}_1)$. Put $y_1 = [\Phi^{(m_1)}](x_1) - [\Psi^{(m_1)}](x_1)$. Then $y_1 \in \mathrm{K}_1(\prod_{n=m_1} B'_n)/k\mathrm{K}_1(\prod_{n=m_1} B'_n)$ (see [23]). However, by (3.20), $y_1 \in \ker\psi_1^{(k)}$ (see 4.1.4 of [31]). It follows from [23] that $y_1 = 0$. In other words,

$$[\Phi^{(m_1)}](x_1) = [\Psi^{(m_1)}](x_1).$$

Thus,

$$[\pi \circ \Phi]|_{\mathrm{K}_1(A,\mathbb{Z}/k\mathbb{Z})} = [\pi \circ \Psi]|_{\mathrm{K}_1(A,\mathbb{Z}/k\mathbb{Z})}. \tag{3.25}$$

Combining (3.20), (3.23), (3.24), and (3.25), we have

$$[\pi \circ \Phi] = [\pi \circ \Psi] \ \text{in} \ \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(Q(C))),$$

where $C = \prod_{n\geq m} B'_n$. For each $a \in \mathcal{H}_m \subseteq A_+^{\mathbf{1}} \setminus \{0\}$, any $(b_n) \in C_+^{\mathbf{1}}$, and any $\eta > 0$, since σ_n is F-$\mathcal{H}_n$-full, for all $n \geq m$, there are $x_{i,n}(a) \in B'_n$ with $\|x_{i,n}\| \leq M(a)$, $i = 1, 2, \ldots, N(a)$, where $F(a) = M(a) \times N(a)$, such that

$$\| \sum_{i=1}^{N(a)} x_{i,n}(a)^* \sigma_n(a) x_{i,n}(a) - b_n \| < \eta.$$

Define $x(i, a) = (x_{i,n}(a))$. Then $x(i, a) \in C$. It follows that

$$\| \sum_{i=1}^{N(a)} x(i, a)^* S^{(m)}(a) x(i, a) - (b_n)_{n\geq m} \| < \eta.$$

This shows that $\pi \circ S(a)$ is a full element of $Q(C)$ for any $0 \neq a \in \bigcup_{n=1}^{\infty} \mathcal{H}_n$. Let I be an ideal of $Q(C)$ and consider the pre-image

$$J = \{a \in A : \pi \circ S(a) \in I\}.$$

By the choice of $(\mathcal{H}_n)$, $J = \{0\}$. It follows that the map $\pi \circ S : A \to Q(C)$ is full.

By Lemma 3.10, there exists a separable sub-C*-algebra $D \subseteq Q(C)$ such that $\pi \circ S(A), \pi \circ \Phi(A), \pi \circ \Psi(A) \subseteq D$, the map $\pi \circ S : A \to D$ is full, and

$$[\pi \circ \Phi] = [\pi \circ \Psi] \ \text{in} \ \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(D)).$$

Since A satisfies the UCT, by [12], $[\pi\circ\Phi]=[\pi\circ\Psi]$ in $KL(A,D)$. Then, by Theorem 3.12 (as at the end of the proof of Theorem 3.12), there exist an integer $K\geq 1$ and a unitary $V\in \widetilde{M_{K+1}(Q(C))}$ such that

$$\|V^*(\pi\circ\Phi(a)\oplus\Sigma(a))V-(\pi\circ\Psi(a)\oplus\Sigma(a))\|<\varepsilon_0/4,\quad a\in\mathcal{F}$$

where, as above,

$$\Sigma(a)=\overbrace{\pi\circ S(a)\oplus\pi\circ S(a)\oplus\cdots\oplus\pi\circ S(a)}^{K},\quad a\in A.$$

Therefore, there exists a sequence of unitaries $(v_n)\subseteq \widetilde{M_{K+1}(C)}$ and an integer N_1 such that $k(n)\geq K$ for all $n\geq N_1$ and

$$\|v_n^*(\phi_n(a)\oplus(\sigma_n)_K(a))v_n-(\psi_n(a)\oplus(\sigma_n)_K(a))\|<\varepsilon_0/2,\quad a\in\mathcal{F},$$

where

$$(\sigma_n)_K(a)=\overbrace{\sigma_n(a)\oplus\sigma_n(a)\oplus\cdots\oplus\sigma_n(a)}^{K},\quad a\in A.$$

This contradicts (3.19). □

Remark 3.15. Suppose that $K_1(A)\cap\mathcal{P}=\{z_1,z_2,\ldots,z_m\}$. Then, by choosing sufficiently large $\mathcal{P}$, we can always choose $\mathcal{U}=\{w_1,w_2,\ldots,w_m\}$ so that $[w_i]=z_i$, $i=1,2,\ldots,m$. In other words, we do not need to consider unitaries in $U_0(M_\infty(\widetilde{A}))$. In particular, if $K_1(A)=\{0\}$, then we can omit the condition (3.15). Moreover, if B is restricted in the class of C*-algebras of real rank zero, then one can choose $\boldsymbol{L}\equiv 2\pi+1$ and (3.15) always holds if $\mathcal{P}$ is sufficiently large. In other words, in this case, condition (3.15) can also be dropped.

Let B_0 be a C*-algebra with a strictly positive element e_0 and $B=\overline{e_b(M_{K+1}(B_0))e_b}$, where $e_b\in M_{K+1}(B_0)_+$ and $e_b\geq(\sum_{i=1}^K(e_0\otimes e_{ii})$ and $\{e_{i,j}:0\leq i,j\leq K\}$ is a matrix unit for M_{K+1}. Let $B_1=\overline{e_1Be_1}$, where $e_1\in\overline{(e_0\otimes e_{0,0})M_{K+1}(B_0)(e_0\otimes e_{0,0})}_+$. We may view $B_1\subset B_0$. Suppose $B_0\in\boldsymbol{C}_{r_0,r_1,T,s,R}$. Suppose that $\phi,\psi:A\to B_1\subset B$ and $\sigma:A\to B_0\subset B$ are as in Theorem 3.14 (ϕ and ψ are $\mathcal{G}$-δ-multiplicative, and σ is T-$\mathcal{H}$-full in B_0), and that $\mathrm{cel}(\lceil\phi(u)\rceil\lceil\psi(u^*)\rceil)\leq\boldsymbol{L}(u)$ for all $u\in\mathcal{U}$ (viewing ϕ and ψ as maps to B_0 instead of B_1) and (3.16) holds. Then there exists $u\in\widetilde{B}$ such that

$$\|u^*\mathrm{diag}(\phi(a),\sigma_K(a))u-\mathrm{diag}(\psi(a),\sigma_K(a))\|<\varepsilon\ \text{ for all }\ a\in\mathcal{F},\tag{3.26}$$

where $\sigma_K(a)=\mathrm{diag}(\sigma(a),\ldots,\sigma(a))$, where $\sigma(a)$ repeats K times (see also below).

Corollary 3.16. *Let A be a non-unital separable amenable C*-algebra which is KK-contractible and let $T:A_+\setminus\{0\}\to\mathbb{N}\times\mathbb{R}_+\setminus\{0\}$ be a map. For any $\varepsilon>0$ and any finite subset $\mathcal{F}\subseteq A$, there exist $\delta>0$, a finite subset $\mathcal{G}\subseteq A$, a finite subset $\mathcal{H}\subset A_+\setminus\{0\}$ and an integer $K\geq 1$ satisfying the following:*

Let B_0 be any C-algebra with a strictly positive element e_0 and $B=\overline{e_bM_{K+1}(B_0)e_b}$, where $e_b\in M_{K+1}(B_0)$, $e_b\geq\sum_{i=1}^K(e_0\otimes e_{ii})$ and $\{e_{i,j}:0\leq i,j\leq K\}$ is a matrix unit for M_{K+1}. Let $B_1=\overline{e_1Be_1}$, where $e_1\in\overline{(e_0\otimes e_{0,0})M_{K+1}(B_0)(e_0\otimes e_{0,0})}_+$. For any two $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $\phi,\psi:A\to B_1$, and any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $\sigma:A\to B_0$ which is also T-$\mathcal{H}$-full in B_0, there exists a unitary $U\in\widetilde{B}$ such that*

$$\|\mathrm{Ad}\,U\circ(\phi\oplus\sigma_K)(a)-(\psi\oplus\sigma_K)(a)\|<\varepsilon\ \text{ for all }\ a\in\mathcal{F},\tag{3.27}$$

where, as earlier,

$$\sigma_K=\overbrace{\sigma\oplus\sigma\oplus\cdots\oplus\sigma}^{K}:A\to M_K(B_0)\subset B.$$

Proof. In Theorem 3.14, the only reason that the restriction has to be placed on B is for the computation of the K-theory of the maps ϕ and ψ. More precisely, the restriction is used to obtain

$$[\pi\circ\Phi]=[\pi\circ\Psi]\ \text{ in }\ \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(Q(C)))$$

in the proof of 3.14. Since A is KK-contractible, $\underline{K}(A)=\{0\}$. Hence $[\pi\circ\Phi]=[\pi\circ\Psi]=0$. Note that, since $KK(A,A)=0$, A satisfies the UCT. □

Lemma 3.17. *If a separable C*-algebra B is KK-contractible, then $A\otimes B$ is KK-contractible for any separable amenable C*-algebra A.*

Proof. Since B is KK-contractible, i.e., $\mathrm{id}_B\sim_{KK}0_B$, there is a continuous path (in the strict topology) of pairs (ϕ_t^+,ϕ_t^-), $t\in[0,1]$, where

$$\phi_t^{\pm}:B\to M(B\otimes\mathcal{K}),\quad t\in[0,1],$$

are homomorphisms such that

$$\phi_t^+(a) - \phi_t^-(a) \in B \otimes \mathcal{K}, \quad t \in [0,1], \ a \in B,$$

$$(\phi_0^+, \phi_0^-) = (\mathrm{id}_B, 0) \quad \text{and} \quad (\phi_1^+, \phi_1^-) = (0,0).$$

Let A be a separable amenable C*-algebra. Consider the two families of elements

$$\Phi_t^{\pm}(a \otimes b) = a \otimes \phi_t^{\pm}(b) \in A \otimes \mathrm{M}(B \otimes \mathcal{K}) \subseteq \mathrm{M}(A \otimes B \otimes \mathcal{K}), \quad a \in A, \ b \in B, \ t \in [0,1].$$

(Nuclearity of A implies that the two tensor products are unambiguous.) Then $\Phi_t^{\pm}(a \otimes b)$, $t \in [0,1]$, are continuous paths (in the strict topology) in $\mathrm{M}(A \otimes B \otimes \mathcal{K})$, and

$$\Phi_t^+(a \otimes b) - \Phi_t^-(a \otimes b) = a \otimes (\phi_t^+(b) - \phi_t^-(b)) \in A \otimes B \otimes \mathcal{K}.$$

Moreover, $(\Phi_0^+, \Phi_0^-) = (\mathrm{id}_{A\otimes B}, 0)$ and $(\Phi_1^+, \Phi_1^-) = (0,0)$. Therefore, $\mathrm{id}_{A\otimes B} \sim_{\mathrm{KK}} 0$, i.e., $A \otimes B$ is KK-contractible, as asserted. □

4. An isomorphism theorem

Recall that a non-unital C*-algebra A is said to have almost stable rank one if the closure of the set of invertible elements in $\widetilde{A}$ contains A, and if this holds also for each hereditary sub-C*-algebra of A in place of A (see [44]).

Recall also that if $A \in \mathcal{D}$ is a separable simple C*-algebra, then A has (Blackadar) strict comparison for positive elements, A has stable rank one, and the map from $\mathrm{Cu}(A)$ to $\mathrm{LAff}_{0+}(\overline{\mathrm{T}(\overline{aAa})}^{\mathrm{w}})$ is an isomorphism of ordered semigroups (for any non-zero element $a \in \mathrm{Ped}(A)$) (see 11.8 and 11.3 of [17])

In what follows, if A is a C*-algebra, we use $A^{\mathbf{1}}$ for the unit ball of A. We will use the following reformulation of Definition 2.5 given by 11.10 of [17] when $\mathrm{K}_0(A) = \{0\}$.

Proposition 4.1 (*11.10 and 10.8 of [17]*)**.** *Let A be a separable C*-algebra in $\mathcal{D}$ with $\mathrm{K}_0(A) = \{0\}$. Let the strictly positive element $e \in A$ with $\|e\| \le 1$ and the number $1 > \mathfrak{f}_e > 0$ be as in 2.5. There is a map $T : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ with the following property: For any finite subset $\mathcal{F}_0 \subseteq A_+ \setminus \{0\}$, any $\varepsilon > 0$, any finite subset $\mathcal{F} \subseteq A$, any $b \in A_+ \setminus \{0\}$, and any integer $n \ge 1$, there are $\mathcal{F}$-ε-multiplicative completely positive contractive maps $\phi : A \to A$ and $\psi : A \to D$ for some sub-C*-algebra $D = D \otimes e_{11} \subseteq \mathrm{M}_n(D) \subseteq A$ such that $\psi(e)$ is strictly positive in D and T-$\mathcal{F}_0 \cup \{f_{1/4}(e)\}$-full as a map $A \to D$,*

$$\|x - (\phi(x) \oplus \overbrace{\psi(x) \oplus \psi(x) \oplus \cdots \oplus \psi(x)}^{n})\| < \varepsilon \ \ \text{for all} \ \ x \in \mathcal{F} \cup \{e\}, \tag{4.1}$$
$$D \in \mathcal{C}_0, \ \ \phi(e) \lesssim b, \ \ \phi(A) \perp \mathrm{M}_n(D), \tag{4.2}$$

$$\phi(e) \lesssim \psi(e) \ \ \text{and} \ \ t \circ f_{1/4}(\psi(e)) > \mathfrak{f}_e \ \ \text{for all} \ \ t \in T(D). \tag{4.3}$$

Definition 4.2. Let A be a C*-algebra with $\mathrm{T}(A) \neq \emptyset$ such that $0 \notin \overline{\mathrm{T}(A)}^{\mathrm{w}}$. There is an affine map $r_{\mathrm{Aff}} : A_{\mathrm{s.a.}} \to \mathrm{Aff}(\overline{\mathrm{T}(A)}^{\mathrm{w}})$ defined by

$$r_{\mathrm{Aff}}(a)(\tau) = \hat{a}(\tau) = \tau(a), \quad \tau \in \overline{\mathrm{T}(A)}^{\mathrm{w}}, \ a \in A_{\mathrm{s.a.}}.$$

Denote by $A^{\mathfrak{q}}$ the space $r_{\mathrm{Aff}}(A_{\mathrm{s.a.}})$, $A_+^{\mathfrak{q}} = r_{\mathrm{Aff}}(A_+)$ and $A_+^{\mathbf{1},\mathfrak{q}} = r_{\mathrm{Aff}}(A_+^{\mathbf{1}})$.

Theorem 4.3. *Let A and B be two separable simple amenable C*-algebras in the class $\mathcal{D}$ with continuous scale. Suppose that both A and B are KK-contractible. Then $A \cong B$ if and only if there is an affine homeomorphism $\gamma : \mathrm{T}(B) \to \mathrm{T}(A)$. Moreover, the isomorphism $\phi : A \to B$ can be chosen such that $\phi_{\mathrm{T}} = \gamma$, where ϕ_{T} is the map from $\mathrm{T}(B)$ to $\mathrm{T}(A)$ induced by ϕ.*

Proof. By Theorem 2.8, there exists a simple C*-algebra $C = \lim_{n\to\infty}(C_n, \imath_n)$, where each C_n is a finite direct sum of copies of $\mathcal{W}$ and $\imath_n$ maps strictly positive elements to strictly positive elements, which has continuous scale, and is such that

$$\mathrm{T}(A) \cong \mathrm{T}(C).$$

It suffices to show that $A \cong C$. (By symmetry, then also $B \cong C$.) We will use $\Gamma : \mathrm{T}(C) \to \mathrm{T}(A)$ for the affine homeomorphism given above. We will use the approximate intertwining argument of Elliott [14]. We would like recall that $\mathcal{W}$ is an inductive limit of Razak algebras with injective connecting maps and the fact that A has stable rank one (see 11.5 of [17]). Fix two sequences, $\{x_1, x_2, \ldots, x_n, \ldots\}$ of A and $\{y_1, y_2, \ldots, y_n, \ldots\}$ of C, which are dense in the unit ball of A and B, respectively.

Step 1: Construction of L_1.

Fix a finite subset $\mathcal{F}_1 \subseteq A$ and $\varepsilon > 0$. Without loss of generality, we may assume that $x_1 \in \mathcal{F}_1 \subseteq A^{\mathbf{1}}$.

Since A has continuous scale, $A = \mathrm{Ped}(A)$ (3.3 of [32]). Choose a strictly positive element $a_0 \in A_+$ with $\|a_0\| = 1$ and $\mathfrak{f}_{a_0} > 0$ as in Definition 2.5. We may assume, without loss of generality, that

$$a_0 y = y a_0 = y,\ \ a_0 \ge y^* y \ \text{ and } \ a_0 \ge yy^* \ \text{ for all } \ y \in \mathcal{F}_1. \tag{4.4}$$

Let $T : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ with $T(a) = (N(a), M(a))$ $(a \in A_+ \setminus \{0\})$ be as given by Proposition 4.1 (11.10 and 10.8 of [17]).

Let $\delta_1 > 0$ (in place of δ), let $\mathcal{G}_1 \subseteq A$ (in place of $\mathcal{G}$) be a finite subset, let $\mathcal{H}_{1,0} \subseteq A_+ \setminus \{0\}$ (in place of $\mathcal{H}$) be a finite subset, and let $K_1 \ge 1$ (in place of K) be an integer as given by 3.16 for the above T, $\varepsilon/16$ (in place of ε), and $\mathcal{F}_1$. We may assume that $\delta_1 < \varepsilon$.

Without loss of generality, we may assume that $\mathcal{F}_1 \cup \mathcal{H}_{1,0} \subseteq \mathcal{G}_1 \subseteq A^{\mathbf{1}}$.

Choose $b_0 \in A_+ \setminus \{0\}$ with $d_\tau(b_0) < 1/8(K_1 + 1)$.

It follows from Proposition 4.1 that there are $\mathcal{G}_1$-$\delta_1/64$-multiplicative completely positive contractive maps $\phi_0 : A \to A$ and $\psi_0 : A \to D$ for some $D = D \otimes e_{1,1} \subseteq D \otimes \mathrm{M}_{2K_1+1} \subseteq A$ with $D \in \mathcal{C}_0'$ such that $(D \otimes \mathrm{M}_{2K_1+1})\phi_0(A) = 0$ and

$$\|x - (\phi_0 \oplus \overbrace{\psi_0 \oplus \psi_0 \oplus \cdots \oplus \psi_0}^{2K_1+1})(x)\| < \min\{\varepsilon/128, \delta_1/128\} \ \text{ for all } \ x \in \mathcal{G}_1, \tag{4.5}$$

$$\phi_0(a_0) \lesssim b_0,\ \ \phi_0(a_0) \lesssim \psi_0(a_0), \tag{4.6}$$

$\psi_0(a_0)$ is strictly positive in D, and, moreover, ψ_0 is T-$\mathcal{H}_{1,0} \cup \{f_{1/4}(a_0)\}$-full as a map from A to D.

By (4.6), replacing ϕ_0 by $f_\eta(\phi_0(a_0))\phi_0 f_\eta(\phi_0(a_0))$ for some sufficiently small η, applying a result of Rørdam (see also Lemma 3.2 of [17]), as A has stable rank one (see 11.5 of [17]), one may assume that there is a unitary $w_0 \in \widetilde{A}$ such that

$$w_0^* \phi_0(a) w_0 \in \overline{DAD}. \tag{4.7}$$

Define $\phi_0' : A \to A$ by $\phi_0'(a) = \mathrm{diag}(\phi_0(a), \psi_0(a))$ for all $a \in A$. Let $D_{1,1} = \mathrm{M}_{2K_1}(D)$ and $D_{1,1}' = \mathrm{M}_{2K_1+1}(D)$. Let $j_1 : D \to \mathrm{M}_{2K_1}(D)$ be defined by

$$j_1(d) = \overbrace{d \oplus d \oplus \cdots \oplus d}^{2K_1} \ \text{ for all } \ d \in D.$$

Let

$$d_{00}' = \overbrace{\psi_0(a_0) \oplus \psi_0(a_0) \oplus \cdots \oplus \psi_0(a_0)}^{1+2K_1} \in D_{1,1}'.$$

Let $\imath_1 : D_{1,1}' \to A$ denote the embedding map, and let $\mathrm{Cu}^\sim(\imath_1) : \mathrm{Cu}^\sim(D_{1,1}') \to \mathrm{Cu}^\sim(A)$ denote the induced map. By 6.2.3 of [43], $\mathrm{Cu}^\sim(A) = \mathrm{LAff}_+^\sim(\mathrm{T}(A))$ (see also 7.3 and 11.8 of [17]). This also holds with C in place of A. Let $\Gamma^\sim : \mathrm{Cu}^\sim(A) \to \mathrm{Cu}^\sim(C)$ be the isomorphism given by $\Gamma^\sim(f)(\tau) = f(\Gamma(\tau))$ for all $f \in \mathrm{LAff}_+^\sim(\mathrm{T}(A))$ and $\tau \in \mathrm{T}(A)$ (see 7.3 of [17]). By Theorem 1.0.1 of [43], there is a homomorphism $h_1' : D_{1,1}' \to C$ such that

$$\mathrm{Cu}^\sim(h_1') = \Gamma^\sim \circ \mathrm{Cu}^\sim(\imath_1),\ \text{ in particular, } \langle h_1'(d_{00}')\rangle = \Gamma^\sim \circ \mathrm{Cu}^\sim(\imath_1)(\langle d_{00}'\rangle). \tag{4.8}$$

Write $h_1 = (h_1')|_{D_{1,1}}$, and $C' = \{c \in C : ch_1(d) = h_1(d)c = 0 \ \text{ for all } \ d \in D_{1,1}\}$. Note that

$$h_1'(\psi_0(a) \oplus \overbrace{0 \oplus 0 \oplus \cdots \oplus 0}^{2K_1}) \in C' \ \text{ for all } \ a \in A.$$

Define $h_0' : A \to C'$ by

$$h_0'(a) = h_1'(\psi_0(a) \oplus \overbrace{0 \oplus 0 \oplus \cdots \oplus 0}^{2K_1}) \ \text{ for all } \ a \in A.$$

Define $L_1 : A \to C$ by

$$L_1(a) = h_0'(a) \oplus h_1(\overbrace{\psi_0(a) \oplus \psi_0(a) \oplus \cdots \oplus \psi_0(a)}^{2K_1}) \ \text{ for all } \ a \in A. \tag{4.9}$$

Note that L_1 is $\mathcal{G}_1$-$\delta_1/64$-multiplicative (see (4.5)).

Step 2: Construct H_1 and the first approximate commutative diagram.

It follows from Theorem 1.0.1 of [43], as A has stable rank one (by 11.5 of [17]), that there is a homomorphism $H : C \to A$ such that

$$\mathrm{Cu}^\sim(H) = (\Gamma^\sim)^{-1}. \tag{4.10}$$

Note that (by (4.10), (4.6) and the definition of h_0')

$$\langle H \circ h_0'(a_0)\rangle \le \langle \psi_0(a_0)\rangle \tag{4.11}$$

in the C*-algebra A. Choose $\delta_1/4 > \eta_0 > 0$ such that

$$\|f_{\eta_0}(H \circ h_0'(a_0))x - x\|,\ \ \|x - xf_{\eta_0}(H \circ h_0'(a_0))\| < \min\{\varepsilon/128, \delta_1/128\} \tag{4.12}$$

for all $x \in H \circ h_0'(\mathcal{G}_1)$. Again, since A has stable rank one (11.5 of [17]), by a result of Rørdam (see also 3.2 of [17]), there is a unitary $u_0 \in \widetilde{A}$ such that

$$u_0^* f_{\eta_0}(H \circ h_0'(a_0))u_0 \in \overline{\psi_{00}(a_0)A\psi_{00}(a_0)} = \overline{DAD}, \tag{4.13}$$

where

$$\psi_{00}(a) = \psi_0(a) \oplus \overbrace{0 \oplus 0 \oplus \cdots \oplus 0}^{2K_1} \in \mathrm{M}_{1+2K_1}(D) \subseteq A \ \text{ for all } \ a \in A.$$

Set $A_{0,1}' = \overline{u_0^* f_{\eta_0}(H \circ h_0'(a_0))u_0 A u_0^* f_{\eta_0}(H \circ h_0'(a_0))u_0}$. Define $H' : A \to A_{0,1}' \subset \overline{DAD}$ by

$$H'(a) = u_0^*(f_{\eta_0}(H \circ h_0'(a_0)))H \circ h_0'(a)(f_{\eta_0}(H \circ h_0'(a_0)))u_0 \ \text{ for all } \ a \in A.$$

Note that H' is a $\mathcal{G}_1$-$\delta_1/32$-multiplicative completely positive contractive map. Moreover, by (4.12),

$$\|\ \mathrm{Ad}\, u_0 \circ H \circ h_0'(a) - H'(a)\| < \min\{\varepsilon/128, \delta_1/128\} \ \text{ for all } \ a \in \mathcal{G}_1. \tag{4.14}$$

Consider the homomorphisms $\mathrm{Ad}\, u_0 \circ H \circ h_1 \circ j_1$ and $\imath_1 \circ j_1$ (or rather $\imath_1|_{D_{1,1}} \circ j_1$). Then, by (4.8) and (4.10),

$$\mathrm{Cu}^{\sim}(\mathrm{Ad}\, u_0 \circ H \circ h_1 \circ j_1) = \mathrm{Cu}^{\sim}(\imath_1 \circ j_1). \tag{4.15}$$

Put $A' = \{a \in A : a \perp A_{0,1}'\}$. Then A' is a hereditary sub-C*-algebra of A. Thus $A' \in \mathcal{D}$ and $K_0(A') = 0$. Note that we may view both $\imath_1 \circ j_1$ and $\mathrm{Ad}\, u_0 \circ H \circ h_1 \circ j_1$ as maps into A' (recall $h_0'(A) \perp h_1(D_{1,1})$). By Theorem 3.3.1 of [43] (as any hereditary sub-C*-algebra of A has stable rank one) and by (4.15), there exists a unitary $u_1 \in \widetilde{A'}$ such that

$$\|u_1^*(\mathrm{Ad}\, u_0 \circ H \circ h_1 \circ j_1(x))u_1 - \imath_1 \circ j_1(x)\| < \min\{\varepsilon/16, \delta_1/16\} \ \text{ for all } \ x \in \psi_0(\mathcal{G}_1). \tag{4.16}$$

Writing $u_1 = \lambda + z$ with $z \in A'$, we may view u_1 is a unitary in $\widetilde{A}$. Note that, for any $b \in A_{0,1}'$, $u_1^* b u_1 = b$. In particular, for any $a \in A$,

$$\mathrm{Ad}\, u_1 \circ H'(a) = H'(a) \ \text{ for all } \ a \in A. \tag{4.17}$$

Note that the map $\imath' \circ \psi_0 : A \to \overline{DAD}$ is T-$\mathcal{H}_{1,0} \cup \{f_{1/4}(a_0)\}$-full (see the last remark of 3.11), where $\imath' : D \to \overline{DAD}$ is the embedding. By Corollary 3.16, there is $u_2 \in \widetilde{A}$ (see (4.7)) such that

$$\|\mathrm{Ad}\, u_2 \circ (H'(a) \oplus \imath_1 \circ j_1 \circ \psi_0(a)) - (\phi_0'(a) \oplus \imath_1 \circ j_1 \circ \psi_0(a))\| < \varepsilon/16 \tag{4.18}$$

for all $a \in \mathcal{F}_1$. Recall that $H \circ L_1(a) = H \circ h_0'(a) \oplus H \circ h_1 \circ j_1 \circ \psi_0(a)$ for $a \in A$ (see (4.9)). Combining with (4.14), (4.16), and (4.17), we have

$$\|\mathrm{Ad}\,(u_0u_1u_2) \circ H \circ L_1(a) - \mathrm{Ad}\, u_2 \circ (H'(a) \oplus \imath_1 \circ j_1 \circ \psi_0(a))\| < \varepsilon/128 + \varepsilon/16 \tag{4.19}$$

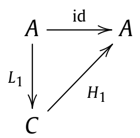
for all $a \in \mathcal{F}_1$. On the other hand, by (4.5),

$$\|\mathrm{id}_A(a) - (\phi_0'(a) \oplus \imath_1 \circ j_1 \circ \psi_0(a))\| < \varepsilon/16 \ \text{ for all } \ a \in \mathcal{F}_1. \tag{4.20}$$

Put $U_1 = u_0u_1u_2$. By (4.20), (4.18), and (4.19), we conclude that

$$\|\mathrm{id}_A(a) - \mathrm{Ad}\, U_1 \circ H \circ L_1(a)\| < \varepsilon \ \text{ for all } \ a \in \mathcal{F}_1. \tag{4.21}$$

Put $H_1 = \mathrm{Ad}\, U_1 \circ H$ (note that H_1 is a homomorphism). Then we have the diagram



which is approximately commutative on the subset $\mathcal{F}_1$ to within ε.

Step 3: Construct L_2 and the second approximately commutative diagram.

We first return to C. Define $\Delta : C_+^{\mathbf{1},q} \setminus \{0\} \to (0, 1)$ by

$$\Delta(\hat{a}) = (1/2)\inf\{\tau(a) : \tau \in T(C)\} \tag{4.22}$$

(Recall that T(C) is compact, by 5.3 of [17] since C has continuous scale.)

Fix any $\eta_1 > 0$ and a finite subset $\mathcal{S}_1 \subseteq C$. We may assume that $y_1 \in \mathcal{S}_1 \subseteq C^{\mathbf{1}}$ and $L_1(\mathcal{F}_1) \subseteq \mathcal{S}_1$.

Let $\mathcal{G}_{2,C} \subseteq C$ (in place of $\mathcal{G}$), $\mathcal{H}_{1,1} \subseteq C_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$), and $\mathcal{H}_{1,2} \subseteq C_{s.a.}$ (in place of $\mathcal{H}_2$) be finite subsets, and $\delta_2 > 0$ (in place of δ) and $\gamma_1 > 0$ (in place of γ) be real numbers as provided by 7.8 of [17] for C, $\eta_1/16$ (in place of ε), and $\mathcal{S}_1$ (in place of $\mathcal{F}$), as well as Δ above.

Without loss of generality, we may assume that $\mathcal{S}_1 \cup \mathcal{H}_{1,2} \subseteq \mathcal{G}_{2,C} \subseteq C^{\mathbf{1}}$.

Fix $\varepsilon_2 > 0$ (with $\varepsilon_2 < \varepsilon/2$) and a finite subset $\mathcal{F}_2$ such that $\{x_1, x_2\} \cup H_1(\mathcal{S}_1) \cup \mathcal{F}_1 \subseteq \mathcal{F}_2$. We may assume that $\mathcal{F}_2 \subseteq A^{\mathbf{1}}$. Let

$$\gamma_0 = \min\{\gamma_1, \inf\{\Delta(\hat{a}) : a \in \mathcal{H}_{1,1} \cup \mathcal{H}_{1,2}\}\}.$$

Fix a strictly positive element a_1 of A with $\|a_1\| = 1$. We may assume, without loss of generality, that

$$a_1 y = y a_1 = y, \ \ a_1 \geq y^* y \ \text{ and } a_1 \geq y y^* \ \text{ for all } \ y \in \mathcal{F}_2. \tag{4.23}$$

Let the map $T : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ with $T(a) = (N(a), M(a))$ $(a \in A_+ \setminus \{0\})$, be as in 4.1 (see 11.10 and 10.8 of [17]) as mentioned in **Step 1**.

Let $\delta_2' > 0$ (in place of δ), let $\mathcal{G}_2 \subseteq A$ (in place of $\mathcal{G}$) be a finite subset, let $\mathcal{H}_{2,0} \subseteq A_+ \setminus \{0\}$ (in place of $\mathcal{H}$) be a finite subset, and let $K_2' \geq 1$ (in place of K) be an integer as given by 3.16 for the above T, $\varepsilon_1/16$ (in place of ε), and $\mathcal{F}_2$.

Without loss of generality, we may assume that $H_1(\mathcal{G}_{2,C}), H_1(\mathcal{H}_{1,1} \cup \mathcal{H}_{1,2}), \mathcal{H}_{2,0} \subseteq \mathcal{G}_2 \subseteq A^{\mathbf{1}}$ and $\delta_2' < \min\{\delta_2, \gamma_0, \delta_1/2\}$. Choose $K_2 \geq K_2'$ such that $1/K_2 < \gamma_0/8$. Choose $b_{2,0} \in A_+ \setminus \{0\}$ with

$$\mathrm{d}_\tau(b_{2,0}) < 1/8(K_2 + 1). \tag{4.24}$$

It follows from Proposition 4.1 (11.10 and 10.7 of [17]) that there are $\mathcal{G}_2$-$\delta_2'/64$-multiplicative completely positive contractive maps $\phi_{2,0} : A \to A$ and $\psi_{2,0} : A \to D_2$ for some $D_2 = D_2 \otimes e_{11} \subseteq D_2 \otimes \mathrm{M}_{2K_2+1} \subseteq A$ with $D_2 \in \mathcal{C}_0$ such that $(D_2 \otimes \mathrm{M}_{2K_2+1})\phi_{2,0}(A) = 0$,

$$\|x - (\phi_{2,0}(x) \oplus \overbrace{\psi_{2,0}(x) \oplus \psi_{2,0}(x) \oplus \cdots \oplus \psi_{2,0}(x)}^{2K_2+1})\| < \min\{\varepsilon_2/128, \delta_2'/128\}, \quad x \in \mathcal{G}_2, \tag{4.25}$$

$$\phi_{2,0}(a_1) \lesssim b_{2,0}, \ \ \phi_{2,0}(a_1) \lesssim \psi_{2,0}(a_1), \tag{4.26}$$

and $\psi_{2,0}(a_1)$ is strictly positive in D_2, and, moreover $\psi_{2,0}$ is T-$\mathcal{H}_{2,0} \cup \{f_{1/4}(a_1)\}$-full in D_2. As in **Step 1**, we may assume that there is a unitary $w_1 \in \widetilde{A}$ such that

$$w_1^* \phi_{2,0}(a_0) w_1 \in \overline{D_2 A D_2} \quad \text{(see (4.7))}. \tag{4.27}$$

Define $\phi_{2,0}' : A \to A$ by $\phi_{2,0}'(a) = \phi_{2,0}(a) \oplus \psi_{2,0}(a)$ for all $a \in A$. Let $D_{2,1} = \mathrm{M}_{2K_2}(D_2)$ and $D_{2,1}' = \mathrm{M}_{2K_2+1}(D_2)$. Let $j_2 : D_2 \to \mathrm{M}_{2K_2}(D_2)$ be defined by

$$j_2(d) = \mathrm{diag}(\overbrace{d, d, \ldots, d}^{2K_2}) \ \text{ for all } \ d \in D_2.$$

Set

$$d_{2,00}' = \overbrace{\psi_{2,0}(a_1) \oplus \psi_{2,0}(a_1) \oplus \cdots \oplus \psi_{2,0}(a_1)}^{2K_2+1} \in D_{2,1}'.$$

With $\imath_2 : D_{2,1}' \to A$ the inclusion map, consider the induced map $\mathrm{Cu}^\sim(\imath_2) : \mathrm{Cu}^\sim(D_{2,1}') \to \mathrm{Cu}^\sim(A)$. It follows from Theorem 1.0.1 of [43] (as C has stable rank one) that there is a homomorphism $h_2' : D_{2,1}' \to C$ such that

$$\mathrm{Cu}^\sim(h_2') = \Gamma^\sim \circ \mathrm{Cu}^\sim(\imath_2), \ \text{ in particular, } \ \langle h_2'(d_{2,00}') \rangle = \Gamma^\sim \circ \mathrm{Cu}^\sim(\imath_2)(\langle d_{2,00}' \rangle). \tag{4.28}$$

Let $h_2 = (h_2')|_{D_{2,1}}$. Denote by $C'' = \{c \in C : ch_2(d) = h_2(d)c = 0, \ \text{ for all } \ d \in D_{2,1}\}$. Note that

$$h_2'(\psi_{2,0}(a) \oplus \overbrace{0 \oplus 0 \oplus \cdots \oplus 0}^{2K_2}) \in C'', \ \text{ for all } \ a \in A.$$

Define $h_{2,0}' : A \to C''$ by

$$h_{2,0}'(a) = h_2'(\psi_{2,0}(a) \oplus \overbrace{0 \oplus 0 \oplus \cdots \oplus 0}^{2K_2}), \ \text{ for all } \ a \in A.$$

Define $L_2' : A \to C$ by, for all $a \in A$,

$$L_2'(a) = h_{2,0}'(a) \oplus h_2(\overbrace{\psi_{2,0}(a) \oplus \psi_{2,0}(a) \oplus \cdots \oplus \psi_{2,0}(a)}^{2K_2}) = h_{2,0}'(a) \oplus h_2 \circ j_2(\psi_{2,0}). \tag{4.29}$$

By (4.28), (4.25), (4.26), and $1/K_2 < \gamma_0/8$, we have, for all $a \in \mathcal{G}_2$,

$$|\tau(h_2 \circ j_2(\psi_{2,0}(a))) - \Gamma(\tau)(a)| = |\Gamma(\tau)(j_2(\psi_{2,0}(a))) - \Gamma(\tau)(a)| < \gamma_0/128 + \gamma_0/8. \tag{4.30}$$

It follows (see (4.29)) that

$$|\tau(L_2'(a)) - \Gamma(\tau)(a)| < \gamma_0/128 + \gamma_0/8 + \gamma_0/8, \quad \text{for all } a \in \mathcal{G}_2 \text{ and } \text{ for all } \tau \in T(C). \tag{4.31}$$

Since $\mathrm{Cu}^{\sim}(H_1) = \mathrm{Cu}^{\sim}(H) = (\Gamma^{\sim})^{-1}$, $\Gamma(\tau)(H_1(x)) = \tau(x)$ for all $x \in C$ and $\tau \in T(C)$. Thus

$$\sup\{|\tau \circ L_2' \circ H_1(x) - \tau(x)| : \tau \in T(C)\} < \gamma_0 \le \gamma_1, \quad \text{for all } x \in \mathcal{H}_{1,1} \cup \mathcal{H}_{1,2}. \tag{4.32}$$

This implies that, in particular,

$$\tau(L_2' \circ H_1(b)) \ge \Delta(\hat{b}), \quad b \in \mathcal{H}_{1,1}. \tag{4.33}$$

Note also that, by construction of C, $\mathrm{K}_0(C) = \mathrm{K}_1(C) = \{0\}$, and so we may apply 7.8 of [17]. In this way, by (4.32) and (4.33), we obtain a unitary $V_1 \in \widetilde{C}$ such that

$$\|\mathrm{Ad}\, V_1 \circ L_2' \circ H_1(a) - \mathrm{id}_C(a)\| < \eta_1/2, \quad \text{for all } a \in \mathcal{S}_1. \tag{4.34}$$

Set $L_2 = \mathrm{Ad}\, V_1 \circ L_2'$. We have the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & A \\ {\scriptstyle L_1}\downarrow & \nearrow{\scriptstyle H_1} & \downarrow{\scriptstyle L_2} \\ C & \xrightarrow[\mathrm{id}]{} & C, \end{array}$$

with the upper triangle approximately commuting on $\mathcal{F}_1$ to within ε and the lower triangle approximately commuting on $\mathcal{S}_1$ to within η_1. Also note that L_2 is $\mathcal{G}_2$-$\delta_2'/64$-multiplicative.

Step 4: Show that the process continues.

We will repeat the argument of **Step 2**.

Recall

$$\mathrm{Cu}^{\sim}(H) = (\Gamma^{\sim})^{-1}. \tag{4.35}$$

Thus

$$\langle H \circ h_{2,0}''(a_1)\rangle \le \langle \psi_{2,0}(a_1)\rangle \tag{4.36}$$

in the C*-algebra A, where $h_{2,0}'' = \mathrm{Ad}\, V_1 \circ h_{2,0}'$. Put $h_2^{\sim} = \mathrm{Ad}\, V_1 \circ h_2$.

Choose $\delta_2/4 > \eta_1 > 0$ such that

$$\|f_{\eta_1}(H \circ h_{2,0}''(a_1))x - x\|, \ \|x - xf_{\eta_1}(H \circ h_{2,0}''(a_1))\| < \min\{\varepsilon_2/128, \delta_2'/128\} \tag{4.37}$$

for all $x \in H \circ h_{2,0}'(\mathcal{G}_2)$. Since A has stable rank one, by a result of Rørdam (see also 3.2 of [17]), there is a unitary $u_{2,0} \in \widetilde{A}$ such that

$$u_{2,0}^* f_{\eta_1}(H \circ h_{2,0}''(a_1))u_{2,0} \in \overline{\psi_{2,00}(a_1)A\psi_{2,00}(a_1)} = \overline{D_2AD_2}, \tag{4.38}$$

where

$$\psi_{2,00}(a_1) = (\psi_{2,0}(a_1) \oplus \overbrace{0 \oplus 0 \oplus \cdots \oplus 0}^{2K_2}).$$

Set $A_{2,0}' = \overline{u_{2,0}^* f_{\eta_1}(H \circ h_{2,0}''(a_1))u_{2,0} A u_{2,0}^* f_{\eta_1}(H \circ h_{2,0}''(a_1))u_{2,0}}$. Note that $A_{2,0}'$ is a hereditary sub-C*-algebra of A. Define $H'' : A \to A_{2,0}' \subset \overline{D_2AD_2}$ by

$$H''(a) = u_{2,0}^*(f_{\eta_1}(H \circ h_{2,0}''(a_1)))H \circ h_{2,0}''(a)(f_{\eta_1}(H \circ h_{2,0}''(a_1)))u_{2,0} \quad \text{for all } a \in A.$$

Note that H'' is a $\mathcal{G}_2$-$\delta_2'/32$-multiplicative completely positive contractive map. Moreover, by (4.37),

$$\|\mathrm{Ad}\, u_{2,0} \circ H \circ h_{2,0}''(a) - H''(a)\| < \min\{\varepsilon_2/128, \delta_2'/128\} \quad \text{for all } a \in \mathcal{G}_2. \tag{4.39}$$

Consider the two homomorphisms $\mathrm{Ad}\, u_{2,0} \circ H \circ h_2^{\sim} \circ j_2$ and $\imath_2 \circ j_2$. Then, by (4.28) and (4.32),

$$\mathrm{Cu}^{\sim}(\mathrm{Ad}\, u_{2,0} \circ H \circ h_2^{\sim} \circ j_2) = \mathrm{Cu}^{\sim}(\imath_2 \circ j_2). \tag{4.40}$$

Put $A'' = \{a \in A : a \perp A_{2,0}'\}$. Note that we may view both $\imath_2 \circ j_2$ and $\mathrm{Ad}\, u_{2,0} \circ H \circ h_2^{\sim} \circ j_2$ as maps into A''. It follows from Theorem 3.3.1 of [43], as A'', a hereditary subalgebra, has stable rank one, that there exists a unitary $u_{2,1} \in \widetilde{A}''$ such that

$$\|u_{2,1}^*(\mathrm{Ad}\, u_{2,0} \circ H \circ h_2^{\sim} \circ j_2(x))u_{2,1} - \imath_2 \circ j_2(x)\| < \min\{\varepsilon_2/16, \delta_2'/16\} \quad \text{for all } x \in \psi_{2,0}(\mathcal{G}_2). \tag{4.41}$$

Writing $u_{2,1} = \lambda + z'$ for some $z' \in A''$. Therefore we may view $u_{2,1}$ as a unitary in $\widetilde{A}$. Note that, for any $b \in \overline{D_2AD_2}$, $u_{2,1}^* b u_{2,1} = b$. In particular, for any $a \in A$,

$$\mathrm{Ad}\, u_{2,1} \circ H''(a) = H''(a) \ \text{ for all } \ a \in A. \tag{4.42}$$

Note that the map $\imath'' \circ \psi_{2,0} : A \to \overline{D_2AD_2}$ is T-$\mathcal{H}_{1,0} \cup \{f_{1/4}(a_1)\}$-full (see the last remark of 3.11), where $\imath'' : D_2 \to \overline{D_2AD_2}$ is the embedding. By Corollary 3.16, there is a unitary $u_{2,2} \in \widetilde{A}$ (see (4.27)) such that

$$\|\mathrm{Ad}\, u_{2,2} \circ (H''(a) \oplus \imath_2 \circ j_2 \circ \psi_{2,0}(a)) - (\phi'_{2,0}(a) \oplus j_2 \circ \psi_{2,0}(a))\| < \varepsilon_2/16 \tag{4.43}$$

for all $a \in \mathcal{F}_2$. Recall that $H \circ L_2(a) = H \circ h''_{2,0}(a) \oplus H \circ h_2^{\sim} \circ j_1 \circ \psi_0(a)$ for $a \in A$ (see (4.29)) and the line after (4.36). Combining with (4.39), (4.41), and (4.42), we have

$$\|\mathrm{Ad}\,(u_{2,0}u_{2,1}u_{2,2}) \circ H \circ L_2(a) - \mathrm{Ad}\, u_{2,2} \circ (H''(a) \oplus \imath_2 \circ j_2 \circ \psi_{2,0}(a))\| < \varepsilon_2/128 + \varepsilon_2/16, \tag{4.44}$$

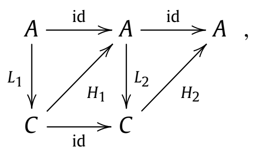
for all $a \in \mathcal{F}_2$. On the other hand, by (4.25),

$$\|\mathrm{id}_A(a) - (\phi'_{2,0}(a) \oplus j_2 \circ \psi_{2,0}(a))\| < \varepsilon_2/16 \ \text{ for all } \ a \in \mathcal{F}_2. \tag{4.45}$$

Set $U_2 = u_{2,0}u_{2,1}u_{2,2}$. By (4.45), (4.43), and (4.44), we conclude that

$$\|\mathrm{id}_A(a) - \mathrm{Ad}\, U_2 \circ H \circ L_2(a)\| < \varepsilon_2 \ \text{ for all } \ a \in \mathcal{F}_2. \tag{4.46}$$

Thus, we have expanded the diagram above to the diagram



where $H_2 := \mathrm{Ad}U_2 \circ H$ (which is a homomorphism), with the last triangle approximately commuting on $\mathcal{F}_2$ to within $\varepsilon_2 (< \varepsilon/2)$.

After continuing in this way (to construct L_3 and so on), the Elliott approximate intertwining argument (see [14], Theorem 2.1) shows that A and C are isomorphic. □

Corollary 4.4. *Let A be a non-unital simple separable amenable C^*-algebra with continuous scale and satisfying the UCT. Suppose that $A \in \mathcal{D}$ and $\mathrm{K}_0(A) = \ker \rho_A$, where ρ_A is the canonical map $\mathrm{K}_0(A) \to \mathrm{Aff}(\mathrm{T}(A))$. Suppose that $B \in \mathcal{D}_0$ satisfies the UCT, has continuous scale and satisfies $\mathrm{K}_0(B) = \mathrm{K}_1(B) = \{0\}$, and suppose that there is an affine homeomorphism $\gamma : \mathrm{T}(B) \to \mathrm{T}(A)$. Then there is an embedding $\phi : A \to B$ such that $\phi_T = \gamma$.*

Proof. Since $\ker \rho_A = \mathrm{K}_0(A)$, then, in the previous proof, Γ (extended to be zero on $\mathrm{K}_0(A)$) now gives a homomorphism from $\mathrm{Cu}^{\sim}(A)$, which is equal to $\mathrm{K}_0(A) \sqcup \mathrm{LAff}_+^{\sim}(\mathrm{T}(A))$, by 6.2.3 of [43] and 7.3 of [17], to $\mathrm{Cu}^{\sim}(C)$, where C is a simple inductive limit of Razak algebras with continuous scale such that $\mathrm{T}(A) \cong \mathrm{T}(C)$. Note that it follows from Theorem 4.3 that $C \cong B$. We simply omit the construction of H_1 and keep Step 1 and Step 3 (in the (new) first step now we ignore anything related to Step 2). A one-sided Elliott intertwining yields a homomorphism from A to C. □

5. Tracial approximation and non-unital versions of some results of Winter

Lemma 5.1 (*Prop. 2.1 of [55]*). *Let A be a simple C^*-algebra (with or without unit) belonging to the reduction class $\mathcal{R}$, and assume that A has strict comparison.*

Let F be a finite dimensional C^*-algebra, and let

$$\phi : F \to A \ \text{ and } \ \phi_i : F \to A \ \text{ for all } \ i \in \mathbb{N} \tag{5.1}$$

be c.p.c. order-zero maps such that for each $c \in F_+$ and $f \in \mathrm{C}_0^+((0,1])$,

$$\lim_{i\to\infty} \sup_{\tau \in \mathrm{T}(A)} |\tau(f(\phi)(c) - f(\phi_i)(c))| = 0 \ \text{ and } \tag{5.2}$$

$$\limsup_{i\to\infty} \|f(\phi_i)(c)\| \le \|f(\phi)(c)\|. \tag{5.3}$$

It follows that there are contractions

$$s_i \in \mathrm{M}_4 \otimes A \ \text{ for all } \ i \in \mathbb{N}$$

such that

$$\lim_{i\to\infty} \|s_i(1_4 \otimes \phi(c)) - (e_{1,1} \otimes \phi_i(c))s_i\| = 0 \ \text{for all} \ c \in F_+ \ \text{and} \tag{5.4}$$

$$\lim_{i\to\infty} \|(e_{1,1} \otimes \phi_i(c))s_is_i^* - e_{1,1} \otimes \phi_i(c)\| = 0. \tag{5.5}$$

(See 4.2 of [56] for the definition of $f(\psi)$ where ψ is an order zero map.)

Proof. The proof is the same as for Proposition 2.1 of [55] (the argument does not require the C*-algebra to be unital; the hypothesis of strict comparison is sufficient for the argument to proceed). □

The following lemma is a slight modification of 4.2 of [54].

Lemma 5.2. *Let A be a separable C*-algebra with nuclear dimension at most m. Let (e_n) be an increasing approximate unit for A. Then there is a sequence of $(m+1)$-decomposable completely positive approximations*

$$\widetilde{A} \xrightarrow{\tilde{\psi}_j} F_j^{(0)} \oplus F_j^{(1)} \oplus \cdots \oplus F_j^{(m)} \oplus \mathbb{C} \xrightarrow{\tilde{\phi}_j} \widetilde{A}, \quad j = 1, 2, \ldots$$

(i.e., each $\tilde{\phi}_j|_{F_j^{(l)}}$ is of order zero) such that, for each $j = 1, 2, \ldots,$

$$\tilde{\phi}_j(F_j^{(l)}) \subseteq A, \quad l = 0, 1, \ldots, m, \tag{5.6}$$

$$\tilde{\phi}_j|_{\mathbb{C}}(1_{\mathbb{C}}) = 1_{\widetilde{A}} - e_{n_j}, \quad \text{for some } e_{n_j} \text{ in the approximate unit } (e_n), \text{ and} \tag{5.7}$$

$$\lim_{j\to\infty} \|\tilde{\phi}_j\tilde{\psi}_j(a) - a\| = 0, \ \lim_{j\to\infty} \|\tilde{\phi}_j^{(l)}\tilde{\psi}_j^{(l)}(1_{\widetilde{A}})a - \tilde{\phi}_j^{(l)}\tilde{\psi}_j^{(l)}(a)\| = 0, \quad l = 0, 1, \ldots, m, \ a \in \widetilde{A}, \tag{5.8}$$

where $\tilde{\phi}_j^{(l)}$ and $\tilde{\psi}_j^{(l)}$ are the restriction of $\tilde{\phi}_j$ to $F_j^{(l)}$ and the projection of $\tilde{\psi}_j$ to $F_j^{(l)}$, respectively.

Proof. Let $\mathcal{F} \subseteq \widetilde{A}$ be a finite set of positive elements with norm one, and let $\varepsilon > 0$ be arbitrary. Each element $a \in \mathcal{F}$ may be written as $\pi(a) \cdot 1_{\widetilde{A}} + x(a)$, where $\pi : \widetilde{A} \to \mathbb{C}$ is the canonical quotient map and $x(a) \in A$. Let $\{e_n\}$ be an approximate identity of A with $e_{n+1}e_n = e_ne_{n+1}$, $n = 1, 2, \ldots$. Choose N such that

$$\|e_Nx(a)e_N - x(a)\| < \varepsilon/4 \ \text{and} \ \|x(a)\| \le 2 \ \text{for all} \ a \in \mathcal{F}. \tag{5.9}$$

Set $e = e_{N+1}$ and, for $a \in \mathcal{F}$, $a' = \pi(a) \cdot 1_{\widetilde{A}} + e_Nx(a)e_N$. It follows that $a - a' \in A$. Moreover,

$$\|a - a'\| < \varepsilon/4, \quad a'e = ea', \quad \text{and} \quad (a' - \pi(a') \cdot 1_{\widetilde{A}})(1 - e) = 0 \tag{5.10}$$

where $\pi : \widetilde{A} \to \mathbb{C}$ is the canonical quotient map. Denote by $\mathcal{F}'$ the set of such a'. Let $\mathcal{F}_1 = \{e^{\frac{1}{2}}a'e^{\frac{1}{2}}, e^{\frac{1}{2}}(a - a')e^{\frac{1}{2}} : a \in \mathcal{F}, a' \in \mathcal{F}'\}$. Then choose a factorization

$$A \xrightarrow{\psi} F^{(0)} \oplus F^{(1)} \oplus \cdots \oplus F^{(m)} \xrightarrow{\phi} A$$

such that

$$\|\phi(\psi(x)) - x\| < \varepsilon/4 \ \text{for all} \ x \in \mathcal{F}_1, \tag{5.11}$$

and the restriction of ϕ to each direct summand $F^{(l)}$, $l = 0, 1, \ldots, m$, is of order zero.

Then, define maps

$$\tilde{\psi} : \widetilde{A} \ni a \mapsto \psi(e^{\frac{1}{2}}ae^{\frac{1}{2}}) \oplus \pi(a) \in (F^{(0)} \oplus F^{(1)} \oplus \cdots \oplus F^{(m)}) \oplus \mathbb{C}, \ \text{and} \tag{5.12}$$

$$\tilde{\phi} : (F^{(0)} \oplus F^{(1)} \oplus \cdots \oplus F^{(m)}) \oplus \mathbb{C} \ni (a, \lambda) \mapsto \phi(a) + \lambda(1 - e). \tag{5.13}$$

For any $a \in \mathcal{F}$, one has,

$$\begin{aligned}
\|\tilde{\phi}(\tilde{\psi}(a)) - a\| &= \|\tilde{\phi}(\tilde{\psi}(a')) + \tilde{\phi}(\tilde{\psi}(a - a')) - a' - (a - a')\| \\
&< \|\tilde{\phi}(\tilde{\psi}(a')) - a'\| + \|\tilde{\phi}(\tilde{\psi}(a - a')) - (a - a')\| \\
&= \|\tilde{\phi}(\tilde{\psi}(a')) - a'\| + \|\phi(\psi(a - a')) - (a - a')\| \qquad (\text{recall } a - a' \in A) \\
&< \|\tilde{\phi}(\tilde{\psi}(a')) - a'\| + \varepsilon/4 \qquad (\text{see } (5.11)) \\
&= \|\phi(\psi(e^{\frac{1}{2}}a'e^{\frac{1}{2}})) + \pi(a')(1 - e) - a'\| + \varepsilon/4 \\
&< \|e^{\frac{1}{2}}a'e^{\frac{1}{2}} + \pi(a')(1 - e) - a'\| + \varepsilon/2 < \varepsilon \qquad (\text{see } (5.10)).
\end{aligned}$$

It is clear that the restriction of $\tilde{\phi}$ to each direct summand $F^{(l)}$ of $F^{(0)} \oplus F^{(1)} \oplus \cdots \oplus F^{(m)} \oplus \mathbb{C}$, $l = 0, 1, \ldots, m$, has order zero.

Since $\mathcal{F}$ and ε are arbitrary, one obtains the $(m+1)$-decomposable completely positive approximations $(\tilde{\psi}_j, \tilde{\phi}_j)$, $j = 1, 2, \ldots$, which satisfy (5.6) and (5.7) of the lemma.

In the same way as in the proof of Proposition 4.2 of [54], $\tilde{\psi}_j$ and $\tilde{\phi}_j$ can be modified to satisfy (5.8). Indeed, consider the maps

$$\hat{\psi}_j : \widetilde{A} \ni a \mapsto \tilde{\psi}_j(1_{\widetilde{A}})^{-\frac{1}{2}}\tilde{\psi}_j(a)\tilde{\psi}_j(1_{\widetilde{A}})^{-\frac{1}{2}} \in (F_j^{(0)} \oplus F_j^{(1)} \oplus \cdots \oplus F_j^{(m)}) \oplus \mathbb{C},$$

where the inverse is taken in the hereditary sub-C*-algebra generated by $\tilde{\psi}(1_{\widetilde{A}})$, and

$$\hat{\phi}_j : (F_j^{(0)} \oplus F_j^{(1)} \oplus \cdots \oplus F_j^{(m)}) \oplus \mathbb{C} \ni a \mapsto \tilde{\phi}_j(\tilde{\psi}_j(1_{\widetilde{A}})^{\frac{1}{2}} a \tilde{\psi}_j(1_{\widetilde{A}})^{\frac{1}{2}}) \in \widetilde{A}.$$

Then the proof of Proposition 4.2 of [54] shows that

$$\lim_{j\to\infty} \|\hat{\phi}_j^{(l)}\hat{\psi}_j^{(l)}(a) - \hat{\phi}_j^{(l)}\hat{\psi}_j^{(l)}(1_{\widetilde{A}})\tilde{\phi}_j\tilde{\psi}_j(a)\| = 0, \quad a \in A, \quad l = 0, 1, \ldots, m.$$

Note that $\pi(1_{\widetilde{A}}) = 1_{\mathbb{C}}$. One has that $\pi(1_{\widetilde{A}})\mathbb{C}\pi(1_{\widetilde{A}}) = \mathbb{C}$, and the restriction of $\hat{\phi}$ to $\mathbb{C}$ is the map $\lambda \mapsto \lambda(1-e)$. It follows that the decompositions $(\hat{\psi}_j, \hat{\phi}_j)$ satisfy the requirements of the lemma. □

Definition 5.3. In the next statement, denote by $\mathcal{S}$ a fixed class of non-unital separable amenable C*-algebras C such that $\mathrm{T}(C) \neq \emptyset$ and $0 \notin \overline{\mathrm{T}(C)}^{\mathrm{w}}$. If $C \in \mathcal{S}$ and $e_C \in C$ is a strictly positive element, define $\lambda_s(C) = \inf\{\mathrm{d}_\tau(e_C) : \tau \in \overline{\mathrm{T}(C)}^{\mathrm{w}}\}$, where $\mathrm{d}_\tau(e_C) := \lim_{\varepsilon\to 0}\tau(f_\varepsilon(e_C))$.

Suppose that $C = \overline{\cup_{n=1}^{\infty} C_n}$ is a simple C*-algebra such that $C_n \subset C_{n+1}$ and $C_n \in \mathcal{S}$, $n \in \mathbb{N}$. Suppose that C has continuous scale. In the following statement, we assume that there are $e_n \in C_{n+}$ with $\|e_n\| = 1$ satisfy

(1) $\{e_n\}$ forms an approximate identity for C and $d_t(e_n) > 1 - 1/n$ for all $t \in T(C_m)$ for all $m \geq n$.

This, in fact, is always the case when $C_n \in \mathcal{S}$ and C has continuous scale. Let $c_n \in C_n$ be a strictly positive element with $\|c_n\| = 1$. Then $c = \sum_{n=1}^{\infty} c_n/2^{n+1}$ is a strictly positive element of C. Thus $\{c^{1/k}\}$ forms an approximate identity for C. Since C has continuous scale, $\tau(c^{1/k}) \nearrow 1$ uniformly on $T(C)$. Put $d_n = \sum_{j=1}^{n} c_j/2^{j+1}$. Then $d_n^{1/k} \leq d_m^{1/k_1}$ if $n \leq m$ and $k < k_1$. Note that $d_n^{1/k} \in C_n$. It follows that a choice of subsequence of the form $\{d_n^{1/k}\}$ forms an approximate identity. So, passing to a subsequence, we relabel it as $c_n \in C_n$. Note that $\tau(c_n) \to 1$ uniformly to 1 on $T(C)$. We may assume that $\tau(c_n) > 1 - 1/2n$ for all $\tau \in T(C)$. One then shows that, for each fixed n, there is $N(n) \geq n$ such that $\tau(c_n) > 1 - 1/n$ for all $\tau \in T(C_m)$ for all $m \geq N(n)$, using a weak* compactness argument. This, by passing to another subsequence, implies (1) holds.

Note also condition (1) implies that $\lambda_s(C_n) \geq 1 - 1/n$.

The following is a non-unital version of 2.2 of [55].

Theorem 5.4. *Let A be a stably projectionless separable simple C*-algebra in $\mathcal{R}$ with $\dim_{\mathrm{nuc}} A = m < \infty$.*

Fix a positive element $e \in A_+$ with $0 \leq e \leq 1$ such that $\tau(e)$, $\tau(f_{1/2}(e)) \geq r_0 > 0$ for all $\tau \in \mathrm{T}(A)$. Let $C = \overline{\bigcup_{n=1}^{\infty} C_n}$ be a non-unital simple C-algebra with continuous scale, where $C_n \subseteq C_{n+1}$ and $C_n \in \mathcal{S}$ which also satisfies condition (1) in 5.3. Suppose that there is an affine homeomorphism $\Gamma : \mathrm{T}(C) \to \mathrm{T}(A)$ and suppose that there are sequences of completely positive contractive maps $\sigma_n : A \to C$ and homomorphisms $\rho_n : C \to A$ such that*

$$\lim_{n\to\infty} \|\sigma_n(ab) - \sigma_n(a)\sigma_n(b)\| = 0, \quad a, b \in A, \tag{5.14}$$

$$\lim_{n\to\infty} \sup\{|t \circ \sigma_n(a) - \Gamma(t)(a)| : t \in \mathrm{T}(C)\} = 0, \quad a \in A, \tag{5.15}$$

$$\lim_{n\to\infty} \sup\{|\tau(\rho_n \circ \sigma_n(a)) - \tau(a)| : \tau \in \mathrm{T}(A)\} = 0, \quad a \in A, \ \ \textit{and} \tag{5.16}$$

$\sigma_n(e)$ is strictly positive in C for all $n \in \mathbb{N}$.

Then A has the following property: For any finite set $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there are a projection $p \in \mathrm{M}_{4(m+2)}(\widetilde{A})$, a sub-C-algebra $S \subseteq p\mathrm{M}_{4(m+2)}(A)p$ with $S \in \mathcal{S}$, and an $\mathcal{F}$-ε-multiplicative completely positive contractive map $L : A \to S$ such that*

(1) $\|[p, 1_{4(m+2)} \otimes a]\| < \varepsilon$, $a \in \mathcal{F}$,
(2) $p(1_{4(m+2)} \otimes a)p \in_\varepsilon S$, $a \in \mathcal{F}$,
(3) $\|L(a) - p(1_{4(m+2)} \otimes a)p\| < \varepsilon$, $a \in \mathcal{F}$,
(4) $p \sim e_{11}$ *in* $\mathrm{M}_{4(m+2)}(\widetilde{A})$,
(5) $\tau(L(e))$, $\tau(f_{1/2}(L(e))) > 7r_0/32(m+2)$ *for all* $\tau \in \mathrm{T}(\mathrm{M}_{4(m+2)}(A))$,
(6) $(1_{4(m+2)} - p)\mathrm{M}_{4(m+2)}(A)(1_{4(m+2)} - p) \in \mathcal{R}$, *and*
(7) $t(f_{1/4}(L(e))) \geq (3r_0/8)\lambda_s(C_1)$ *for all* $t \in \mathrm{T}(S)$.

Proof. Since A has finite nuclear dimension, one has that $A \cong A \otimes \mathcal{Z}$ [54] for the unital case and [49] for the non-unital case). Therefore, A has strict comparison for positive elements (Corollary 4.7 of [47]).

The proof is essentially the same as that of Theorem 2.2 of [55]. We give the proof in the present very much analogous situation for the convenience of the reader. Let $e \in A_+$ with $\|e\| = 1$, $\tau(e) > r_0$, and $\tau(f_{1/2}(e)) > r_0$ for all $\tau \in \mathrm{T}(A)$.

Let (e_n) be an (increasing) approximate unit for A. Since $A \in \mathcal{R}$, and since A is also assumed to be projectionless, one may assume that $\mathrm{sp}(e_n) = [0, 1]$. Since $\dim_{\mathrm{nuc}}(A) \leq m$, by Lemma 5.2, there is a system of $(m+1)$-decomposable completely positive approximations

$$\widetilde{A} \xrightarrow{\psi_j} F_j^{(0)} \oplus F_j^{(1)} \oplus \cdots \oplus F_j^{(m)} \oplus \mathbb{C} \xrightarrow{\phi_j} \widetilde{A}, \quad j = 1, 2, \ldots$$

such that

$$\phi_j(F_j^{(l)}) \subseteq A, \quad l = 0, 1, \ldots, m, \quad \text{and} \tag{5.17}$$

$$\phi_j|_{\mathbb{C}}(1_{\mathbb{C}}) = 1_{\widetilde{A}} - e_j, \tag{5.18}$$

where e_j is an element of (e_n).

Write

$$\phi_j^{(l)} = \phi_j|_{F_j^{(l)}} \quad \text{and} \quad \phi_j^{(m+1)} = \phi_j|_{\mathbb{C}}, \quad l = 0, 1, \ldots, m.$$

As in Lemma 5.2, one may assume that

$$\lim_{j\to\infty} \|\phi_j^{(l)}\psi_j^{(l)}(1_{\widetilde{A}})a - \phi_j^{(l)}\psi_j^{(l)}(a)\| = 0, \quad l = 0, 1, \ldots, m, \ a \in A. \tag{5.19}$$

Note that $\phi_j^{(l)} : F_j^{(l)} \to A$ is of order zero, and the relation for an order zero map is weakly stable (see $(\mathcal{P})$ and $(\mathcal{P}1)$ of 2.5 of [30]). On the other hand, if i is large enough, then $\sigma_i \circ \phi_j^{(l)}$ satisfies the relation for order zero to within an arbitrarily small tolerance, since σ_i will be sufficiently multiplicative. It follows that there are order zero maps

$$\tilde{\phi}_{j,i}^{(l)} : F_j^{(l)} \to C$$

such that

$$\lim_{i\to\infty} \|\tilde{\phi}_{j,i}^{(l)}(c) - \sigma_i(\phi_j^{(l)}(c))\| = 0, \quad c \in F_j^{(l)}.$$

We will identify C with $S_i = \rho_i(C) \subseteq A$, $\sigma_i : A \to C$ with $\rho_i \circ \sigma_i : A \to S_i \subseteq A$, and $\tilde{\phi}_{j,i}^{(l)}$ with $\rho_i \circ \tilde{\phi}_{j,i}^{(l)}$. There is a positive linear map (automatically order zero)

$$\tilde{\phi}_{j,i}^{(m+1)} : \mathbb{C} \ni 1 \mapsto 1_{\widetilde{A}} - \sigma_i(e_j) \in \tilde{S}_i = C^*(S_i, 1_{\widetilde{A}}) \subseteq \widetilde{A}, \quad i \in \mathbb{N}.$$

Note that

$$\tilde{\phi}_{j,i}^{(m+1)}(\lambda) = \sigma_i(\phi_j^{(m+1)}(\lambda)), \quad \lambda \in F_j^{(m+1)} = \mathbb{C}, \tag{5.20}$$

where one still uses σ_i to denote the induced map $\widetilde{A} \to \widetilde{S_i}$.

Note that for each $l = 0, 1, \ldots, m$,

$$\lim_{i\to\infty} \|f(\tilde{\phi}_{j,i}^{(l)})(c) - \sigma_i(f(\phi_j^{(l)})(c))\| = 0, \quad c \in (F_j^{(l)})_+, \ f \in C_0((0,1])_+,$$

(see the comment before the proof of 5.1 for the notation $f(\tilde{\phi}_{j,i}^{(l)})$ and $f(\phi_j^{(l)})$) and hence, from (5.16),

$$\lim_{i\to\infty} \sup_{\tau\in\mathrm{T}(A)} |\tau(f(\tilde{\phi}_{j,i}^{(l)})(c) - f(\phi_j^{(l)})(c))| = 0, \quad c \in (F_j^{(l)})_+, \ f \in C_0((0,1])_+.$$

Also note that

$$\limsup_{i\to\infty} \|f(\tilde{\phi}_{j,i}^{(l)})(c)\| \leq \|f(\phi_j^{(l)})(c)\|, \quad c \in (F_j^{(l)})_+, \ f \in C_0((0,1])_+.$$

Applying Lemma 5.1 to $(\tilde{\phi}_{j,i}^{(l)})_{i\in\mathbb{N}}$ and $\phi_j^{(l)}$ for each $l = 0, 1, \ldots, m$, we obtain contractions

$$s_{j,i}^{(l)} \in M_4(A) \subseteq M_4(\widetilde{A}), \quad i \in \mathbb{N},$$

such that

$$\lim_{i\to\infty} \|s_{j,i}^{(l)}(1_4 \otimes \phi_j^{(l)}(c)) - (e_{1,1} \otimes \tilde{\phi}_{j,i}^{(l)}(c))s_{j,i}^{(l)}\| = 0, \quad c \in F_j^{(l)}, \quad \text{and} \tag{5.21}$$

$$\lim_{i\to\infty} \|(e_{1,1} \otimes \tilde{\phi}_{j,i}^{(l)}(c))s_{j,i}^{(l)}(s_{j,i}^{(l)})^* - e_{1,1} \otimes \tilde{\phi}_{j,i}^{(l)}(c)\| = 0. \tag{5.22}$$

Note that $\mathrm{sp}(e_j) = [0, 1]$. Put $C_0 = C_0((0, 1])$. Define

$$\Delta_j(\hat{f}) = \inf\{\tau(f(e_j)) : \tau \in \mathrm{T}(A)\} \quad \text{for all} \quad f \in (C_0)_+ \setminus \{0\}. \tag{5.23}$$

Since A is assumed to have continuous scale, $\mathrm{T}(A)$ is compact and $\Delta_j(\hat{f}) > 0$ for all $f \in (C_0)_+ \setminus \{0\}$. For $l = m+1$, since $\mathrm{sp}(e_j) = [0, 1]$, by considering Δ_j for each j, since i is chosen after j is fixed, by applying A.16 in the Appendix, one obtains unitaries

$$s_{j,i}^{(m+1)} \in \widetilde{A}, \quad i \in \mathbb{N},$$

such that

$$\lim_{i\to\infty} \|s_{j,i}^{(m+1)} e_j - \sigma_i(e_j) s_{j,i}^{(m+1)}\| = 0,$$

and hence

$$\lim_{i\to\infty} \|s_{j,i}^{(m+1)} (1_{\widetilde{A}} - e_j) - (1_{\widetilde{A}} - \sigma_i(e_j)) s_{j,i}^{(m+1)}\| = 0.$$

By (5.18) and (5.20), one has

$$\lim_{i\to\infty} \|s_{j,i}^{(m+1)} \phi_j^{(m+1)}(c) - \tilde{\phi}_{j,i}^{(m+1)}(c) s_{j,i}^{(m+1)}\| = 0, \quad c \in F_j^{(m+1)} = \mathbb{C}.$$

Considering the element $e_{1,1} \otimes s_{j,i}^{(m+1)} \in \mathrm{M}_4 \otimes \widetilde{A}$, and still denoting it by $s_{j,i}^{(m+1)}$, we have

$$\lim_{i\to\infty} \|s_{j,i}^{(m+1)} (1_4 \otimes \phi_j^{(m+1)}(c)) - (e_{1,1} \otimes \tilde{\phi}_{j,i}^{(m+1)}(c)) s_{j,i}^{(m+1)}\| = 0, \quad c \in F_j^{(l)}$$

and

$$(e_{1,1} \otimes \tilde{\phi}_{j,i}^{(m+1)}(c)) s_{j,i}^{(m+1)} (s_{j,i}^{(m+1)})^* = e_{1,1} \otimes \tilde{\phi}_{j,i}^{(m+1)}(c).$$

Therefore,

$$\lim_{i\to\infty} \|s_{j,i}^{(l)} (1_4 \otimes \phi_j^{(l)}(c)) - (e_{1,1} \otimes \tilde{\phi}_{j,i}^{(l)}(c)) s_{j,i}^{(l)}\| = 0, \quad c \in F_j^{(l)}, \; l = 0, 1, \ldots, m+1. \tag{5.24}$$

$$\lim_{i\to\infty} \|(e_{1,1} \otimes \tilde{\phi}_{j,i}(c)) s_{j,i}^{(l)} (s_{j,i}^{(l)})^* - \tilde{\phi}_{j,i}(c)\| = 0, \quad c \in F_j^{(l)}, \; l = 0, 1, \ldots, m+1. \tag{5.25}$$

Let $\tilde{\sigma}_i : \widetilde{A} \to \widetilde{C}$ and $\tilde{\rho}_i : \widetilde{C} \to \widetilde{A}$ denote the unital maps induced by $\sigma_i : A \to C$ and $\rho_i : C \to A$, respectively. Consider the contractions

$$s_j^{(l)} := (s_{j,i}^{(l)})_{i\in\mathbb{N}} \in (\mathrm{M}_4 \otimes \widetilde{A})_\infty, \quad l = 0, 1, \ldots, m+1, \quad j = 1, 2, \ldots.$$

By (5.24) and (5.25), these satisfy

$$s_j^{(l)} (1_4 \otimes \bar{\iota}(\phi_j^{(l)}(c))) = (e_{1,1} \otimes \bar{\rho}\bar{\sigma}(\phi_j^{(l)}(c))) s_j^{(l)} \text{ and}$$

$$(e_{1,1} \otimes \bar{\rho} \circ \bar{\sigma}(\phi_j^{(l)}(c))) s_j^{(l)} (s_j^{(l)})^* = (e_{1,1} \otimes \bar{\rho} \circ \bar{\sigma}(\phi_j^{(l)}(c))),$$

where

$$\bar{\sigma} : \widetilde{A}_\infty \to \prod \widetilde{C} / \bigoplus \widetilde{C} \quad \text{and} \quad \bar{\rho} : \prod \widetilde{C} / \bigoplus \widetilde{C} \to \widetilde{A}_\infty$$

are the homomorphisms induced by $\tilde{\sigma}_i$ and $\tilde{\rho}_i$, and the map

$$\bar{\iota} : (\widetilde{A})_\infty \to ((\widetilde{A})_\infty)_\infty$$

is the embedding induced by the canonical embedding $\iota : \widetilde{A} \to (\widetilde{A})_\infty$.

Let

$$\bar{\gamma} : \widetilde{A}_\infty \to ((\widetilde{A})_\infty)_\infty$$

denote the homomorphism induced by the composed map

$$\bar{\rho}\bar{\sigma} : \widetilde{A}_\infty \to (\widetilde{A})_\infty,$$

For each $l = 0, 1, \ldots, m+1$, let

$$\bar{\phi}^{(l)} : \prod_j F_j^{(l)} / \bigoplus_j F_j^{(l)} \to A_\infty \quad \text{and} \tag{5.26}$$

$$\bar{\psi}^{(l)} : A \to \prod_j F_j^{(l)} / \bigoplus_j F_j^{(l)} \tag{5.27}$$

denote the maps induced by $\phi_j^{(l)}$ and $\psi_j^{(l)}$.

Consider the contraction

$$\bar{s}^{(l)} = (s_j^{(l)}) \in (\mathrm{M}_4 \otimes \widetilde{A}_\infty)_\infty.$$

Then

$$\bar{s}^{(l)}(1_4 \otimes \bar{\iota}\bar{\phi}^{(l)}\bar{\psi}^{(l)}(a)) = (e_{1,1} \otimes \bar{\gamma}\bar{\phi}^{(l)}\bar{\psi}^{(l)}(a))\bar{s}^{(l)}, \quad a \in \widetilde{A}, \text{ and}$$

$$(e_{1,1} \otimes \bar{\gamma}\bar{\phi}^{(l)}\bar{\psi}^{(l)}(a))\bar{s}^{(l)}(\bar{s}^{(l)})^* = (e_{1,1} \otimes \bar{\gamma}\bar{\phi}^{(l)}\bar{\psi}^{(l)}(a)).$$

By (5.19), one has

$$\bar{\phi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}})\iota(a) = \bar{\phi}^{(l)}\bar{\psi}^{(l)}(a), \quad a \in A.$$

In particular,

$$(\bar{\phi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{\frac{1}{2}}\iota(a) \in C^*(\bar{\phi}^{(l)}\bar{\psi}^{(l)}(A)),$$

and hence

$$\begin{aligned}
\bar{s}^{(l)}(1_4 \otimes (\bar{\iota}\bar{\phi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{\frac{1}{2}})(1_4 \otimes \bar{\iota}\iota(a)) &= \bar{s}^{(l)}(1_4 \otimes \bar{\iota}\bar{\phi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}})^{\frac{1}{2}}\iota(a)) \\
&= (e_{1,1} \otimes \bar{\gamma}(\bar{\phi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{\frac{1}{2}}\iota(a))\bar{s}^{(l)} \\
&= (e_{1,1} \otimes \bar{\gamma}\iota(a)(\bar{\phi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{\frac{1}{2}})\bar{s}^{(l)} \\
&= (e_{1,1} \otimes \bar{\gamma}(\iota(a)))(e_{1,1} \otimes \bar{\gamma}(\bar{\phi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{\frac{1}{2}})\bar{s}^{(l)}.
\end{aligned} \tag{5.28}$$

Set

$$\begin{aligned}
\bar{v} &= \sum_{l=0}^{m+1} e_{1,l} \otimes ((e_{1,1} \otimes \bar{\gamma}\bar{\phi}^{l}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{\frac{1}{2}}\bar{s}^{(l)}) \\
&= \sum_{l=0}^{m+1} e_{1,l} \otimes (\bar{s}^{(l)}(1_4 \otimes \bar{\iota}\bar{\phi}^{l}\bar{\psi}^{(l)}(1_{\widetilde{A}}))^{\frac{1}{2}}) \in M_{m+2}(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes (\widetilde{A}_\infty)_\infty.
\end{aligned}$$

Then

$$\bar{v}\bar{v}^* = \sum_{l=0}^{m+1} e_{1,1} \otimes (e_{1,1} \otimes \bar{\gamma}\bar{\phi}^{l}\bar{\psi}^{(l)}(1_{\widetilde{A}})) = e_{1,1} \otimes e_{1,1} \otimes \bar{\gamma}(1_{\widetilde{A}}).$$

Thus, $\bar{v}$ is an partial isometry. Moreover, for any $a \in \widetilde{A}$,

$$\begin{aligned}
\bar{v}(1_{(m+2)} \otimes 1_4 \otimes \bar{\iota}\iota(a)) &= \sum_{l=0}^{m+1} e_{1,l} \otimes (\bar{s}^{(l)}(1_4 \otimes \bar{\iota}\bar{\phi}^{l}\bar{\psi}^{(l)}(1_{\widetilde{A}})^{\frac{1}{2}})(1_4 \otimes \bar{\iota}\iota(a))) \\
&= \sum_{l=0}^{m+1} e_{1,l} \otimes (e_{1,1} \otimes \bar{\gamma}(\iota(a)))(e_{1,1} \otimes \bar{\gamma}(\bar{\phi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}})^{\frac{1}{2}}\bar{s}^{(l)})) \qquad \text{(by (5.28))} \\
&= (e_{1,1} \otimes e_{1,1} \otimes \bar{\gamma}(\iota(a)))\sum_{l=0}^{m+1} e_{1,l} \otimes e_{1,1} \otimes \bar{\gamma}(\bar{\phi}^{(l)}\bar{\psi}^{(l)}(1_{\widetilde{A}})^{\frac{1}{2}}\bar{s}^{(l)}) \\
&= (e_{1,1} \otimes e_{1,1} \otimes \bar{\gamma}(\iota(a)))\bar{v}.
\end{aligned}$$

Hence

$$\bar{v}^*\bar{v}(1_{m+2} \otimes 1_4 \otimes \bar{\iota}\iota(a)) = \bar{v}^*(e_{1,1} \otimes e_{1,1} \otimes \bar{\gamma}\iota(a))\bar{v} = (1_{m+2} \otimes 1_4 \otimes \bar{\iota}\iota(a))\bar{v}^*\bar{v}, \quad a \in \widetilde{A}.$$

Then, for any finite set $\mathcal{G} \subseteq \widetilde{A}$ and any $\delta > 0$, there are $i \in \mathbb{N}$ and $v_i \in M_{m+2}(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes \widetilde{A}$ such that

$$v_i v_i^* = e_{1,1} \otimes e_{1,1} \otimes \tilde{\rho}_i(1_{\tilde{S}_i}) = e_{1,1} \otimes e_{1,1} \otimes 1_{\widetilde{A}}, \tag{5.29}$$

$$\|[v_i^* v_i, 1_{m+2} \otimes 1_4 \otimes a]\| < \delta \text{ for all } a \in \mathcal{G}, \tag{5.30}$$

$$\|v_i^* v_i(1_{m+2} \otimes 1_4 \otimes a) - v_i{}^*(e_{1,1} \otimes e_{1,1} \otimes \tilde{\rho}_i\tilde{\sigma}_i(a))v_i\| < \delta \text{ for all } a \in \mathcal{G} \text{ and} \tag{5.31}$$

$$\tau(\rho_i \circ \sigma_i(e)),\ \tau(f_{1/2}(\rho_i \circ \sigma_i(e))) \geq 15r_0/16 \text{ for all } \tau \in T(A). \tag{5.32}$$

Define $\kappa_i : \widetilde{S_i} \to M_{m+2} \otimes M_4 \otimes \widetilde{A}$ by

$$\kappa_i(s) = v_i{}^*(e_{1,1} \otimes e_{1,1} \otimes \rho_i(s))v_i.$$

Note that

$$\kappa_i(S_i) \subseteq M_{m+2} \otimes M_4 \otimes A.$$

Then κ_i is an embedding; and on setting $p_i = 1_{\kappa_i(\tilde{S}_i)} = v_i^* v_i$, one has

(i) $p_i \sim e_{1,1} \otimes e_{1,1} \otimes 1_{\tilde{A}}$,
(ii) $\|[p_i, 1_{m+2} \otimes 1_4 \otimes a]\| < \delta$, $a \in \mathcal{G}$,
(iii) $p_i(1_{m+2} \otimes 1_4 \otimes a)p_i \in_\delta \kappa_i(\tilde{S}_i)$, $a \in \mathcal{G}$.

Note that A is $\mathcal{Z}$-stable (by [54]) and hence has strict comparison (by [47]). Let $e \in (1_{4(m+2)} - p_i)\mathrm{M}_{4(m+2)}(A)(1_{4(m+2)} - p_i)$ be a strictly positive element. By (i), $\mathrm{d}_\tau(e) = \tau(1_{4(m+2)} - p_i) = \tau(1_{4(m+2)} - e_{1,1} \otimes e_{1,1} \otimes 1_{\tilde{A}})$ for all $\tau \in T(A)$, where τ is naturally extended to $\tilde{A}$. Since A and $M_{4(m+2)}(A)$ have continuous scale, $\tau \mapsto \mathrm{d}_\tau$ is continuous on $T(A)$. Hence $(1_{4(m+2)} - p_i)\mathrm{M}_{4(m+2)}(A)(1_{4(m+2)} - p_i)$ also has continuous scale (see 5.4 of [17]) and is still in the reduction class $\mathcal{R}$ (so condition (6) holds).

Define $L_i : A \to \kappa_i(S_i)$ by $L_i(a) = v_i{}^*(e_{1,1} \otimes e_{1,1} \otimes \rho_i(\sigma_i(a)))v_i$ for all $a \in A$. Then

(iv) $\|L_i(a) - p_i(1_{4(m+2)} \otimes a)p_i\| < \delta$ for all $a \in \mathcal{G}$ and
(v) $\tau(L_i(e)),\ \tau(f_{1/2}(L_i(e))) \geq \dfrac{15r_0}{64(m+2)}$ for all $\tau \in \mathrm{T}(\mathrm{M}_{4(m+2)}(A))$.

Let $\tau_i \in \mathrm{T}(\kappa_i(S_i))$. Then $\tau_i \circ L_i$ is a positive linear functional. Let $\bar{t}$ be a weak *-limit of $\{\tau_i \circ L_i\}$. Note that, for any $1/2 > \varepsilon > 0$, since A has continuous scale, there is $e_A \in A$ with $\|e_A\| = 1$ such that $\tau(e_A) > 1 - \varepsilon/2$ for all $\tau \in T(A)$. By (5.16) (see also (5.29)), we may assume that $\tau_i \circ L_i(e_A) > 1 - \varepsilon$ for all large i. It follows that $\bar{t}(e_A) \geq 1 - \varepsilon$. Hence $\|\bar{t}\| \geq 1 - \varepsilon$ for any $1/2 > \varepsilon > 0$. It follows that $\bar{t}$ is a state of A. Then, by (5.14) and (5.16), $\bar{t}$ is a tracial state of A. Therefore, with sufficiently small δ and large $\mathcal{G}$ (and sufficiently large i), by also (5.32), we may assume that

$$t(f_{1/4}(L_i(e))) \geq 7r_0/8 \text{ for all } t \in \mathrm{T}(\kappa_i(S_i)). \tag{5.33}$$

Since $\kappa_i(S_i) \cong C$, we may write $\kappa_i(S_i) = \overline{\bigcup_{n=1}^{\infty} S_{i,n}}$, where each $S_{i,n} \cong C_n$ and, by condition (1) of 5.3, there exists a positive element $e_C \in S_{i,1} \subset S_{i,n}$ with $\|e_C\| = 1$ such that $t(e_C) > \lambda_s(C_1)/2$ for all $t \in T(S_{i,n})$ for all $n \geq 1$. Since each $S_{i,n}$ is amenable, there exist completely positive contractive maps $\Phi_n : \kappa_i(S_i) \to S_{i,n}$ such that

$$\lim_{n\to\infty} \|\Phi_n(s) - s\| = 0 \text{ for all } s \in \kappa_i(S_i) \text{ and } \|\Phi_n(e_C) - e_C\| < 1/2^{n+1}. \tag{5.34}$$

We assert that, for all sufficiently large n,

$$t(f_{1/4}(\Phi_n \circ L_i(e))) > (3r_0/8)\lambda_s(C_1) \text{ for all } t \in \mathrm{T}(S_{i,n}). \tag{5.35}$$

Otherwise, there exists a sequence $(n(k))$ and $t_k \in \mathrm{T}(S_{i,n(k)})$ such that

$$t_k(f_{1/4}(\Phi_{n(k)} \circ L_i(e))) < (3r_0/8)\lambda_s(C_1). \tag{5.36}$$

Note that, since $e_C \in S_{i,n(k)}$, $t_{k+1}|_{S_{i,n(k)}} \in T(S_{i,n(k)})$. Let t_0 be a weak* limit of $\{t_k \circ \Phi_{n(k)}\}$. Then, by (5.36),

$$t_0(f_{1/4}(L_i(e))) \leq (3r_0/8)\lambda_s(C_1). \tag{5.37}$$

Note that $t_k(e_C) \geq \lambda_s(C_1)/2$ for all k. Thus, by (5.34), one computes that $t_0(e_C) \geq \lambda_s(C_1)/2$. It follows that t_0 is a trace of S_i with $\|t_0\| \geq \lambda_s(C_1)/2$. Then, by (5.33),

$$t_0(f_{1/4}(L_i(e))) \geq (7r_0/8)(\lambda(C_1)/2). \tag{5.38}$$

This contradicts (5.37) and so the assertion (5.35) holds. We then define $L = \Phi_n \circ L_i$ for some sufficiently large n (and i). The conclusion of the theorem follows from (i),(ii), (iii), (iv), (v), and (5.35). □

Lemma 5.5. *Let A be a stably projectionless simple separable C^*-algebra with almost stable rank one (recall that by definition this includes hereditary sub-C^*-algebras). Suppose that A has continuous scale and has strict comparison for positive elements. Suppose also that the map $\imath : \mathrm{W}_+(A) \to \mathrm{LAff}_{b,+}(\mathrm{T}(A))$ is surjective. Suppose that there are $1 > \eta > 0$ and $1 > \lambda > 0$ such that every hereditary sub-C^*-algebra B with continuous scale has the following property:*

Let $r_0 > 0$ and let $a_0 \in B_+$ be a positive element with $\|a_0\| = 1$ with $\tau(a_0) \geq r_0$ and $\tau(f_{1/2}(a_0)) \geq r_0 > 0$ for all $\tau \in \mathrm{T}(B)$. Suppose that, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subseteq B$, there are $\mathcal{F}$-ε-multiplicative completely positive contractive maps $\phi : B \to B'$, where B' is a hereditary sub-C^*-algebra of B, and $\psi : B \to D$ for some sub-C^*-algebra $D \subseteq B$, and $D \perp B'$, such that

$$\|x - (\phi(x) + \psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F} \cup \{a_0\}, \tag{5.39}$$

$$d_\tau(\phi(a_0)) < 1 - \eta \text{ for all } \tau \in \mathrm{T}(A), \tag{5.40}$$

$$\tau'(\phi(a_0)), \tau'(f_{1/2}(\phi(a_0))) \geq r_0 - \varepsilon \text{ for all } \tau' \in T(B'), \tag{5.41}$$

$$D \in \mathcal{C}_0'(\in \mathcal{C}_0^{0'}), \tag{5.42}$$

$$\tau(\psi(a_0)) \geq r_0\eta \text{ for all } \tau \in \mathrm{T}(B), \tag{5.43}$$

$$t(f_{1/4}(\psi(a_0))) \geq r_0\lambda \text{ for all } t \in T(D). \tag{5.44}$$

Then $A \in \mathcal{D}$ (or $\mathcal{D}_0$).

Proof. Let $b_0 \in A_+ \setminus \{0\}$ with $\|b_0\| = 1$. Choose $k \geq 1$ such that

$$(1-\eta/2)^k < \inf\{d_\tau(b_0) : \tau \in T(A)\}. \tag{5.45}$$

Choose a strictly positive element $a_0 \in A$ with $\|a_0\| = 1$ such that $\tau(a_0)$, $\tau(f_{1/2}(a_0)) \geq 1 - 1/64$ for all $\tau \in T(A)$. Put $r_0 = 1 - 1/64$ and put $\mathfrak{f}_a = (r_0/2)\lambda$.

Fix $1 > \varepsilon > 0$. Put $\varepsilon_1 = \min\{r_0\varepsilon/2(k+1), r_0\eta/4(k+1)\}$. We choose $\delta_1 > 0$ small enough such that

$$\|f_{\sigma'}(a') - f_{\sigma'}(b')\| < \varepsilon_1, \tag{5.46}$$

whenever $\|a' - b'\| < \delta_1$ for any $0 \leq a', b' \leq 1$ in any C*-algebra, where $\sigma' \in \{1/2, 1/4\}$.

Fix a finite subset $\mathcal{F} \subseteq A^{\mathbf{1}}$. Let $\delta_2 = \min\{\delta_1/2(k+1), \varepsilon_1/2(k+1)\}$. Choose some $g \in C_0((0,1])$ with $0 \leq g \leq 1$ and let $a_1 = g(a_0)$ such that $a_1 \geq a_0$ and

$$\|a_1 x a_1 - x\| < \delta_2/64 \text{ for all } x \in \mathcal{F} \cup \{a_0\}. \tag{5.47}$$

Let $\mathcal{F}_1$ be a finite subset containing $\mathcal{F} \cup \{a_i, f_{1/4}(a_i), f_{1/2}(a_i) : i = 0, 1\}$.

By hypothesis, there are $\mathcal{F}_1$-$\delta_2/64$-multiplicative completely positive contractive maps $\phi_1' : A \to B'$, where B' is a hereditary sub-C*-algebra of A, and $\psi_1 : A \to D_1$ for some sub-C*-algebra $D_1 \subseteq A$ such that $D_1 \in \mathcal{C}_0'$ (or $\in \mathcal{C}_0^{0'}$), $D_1 \perp \phi_1'(A)$, and

$$\|x - (\phi_1'(x) + \psi_1(x))\| < \delta_2/16 \text{ for all } x \in \mathcal{F}_1, \tag{5.48}$$
$$d_\tau(\phi_1'(a_0)) < 1 - \eta \text{ for all } \tau \in T(A), \tag{5.49}$$
$$\tau'(\phi_1'(a_0)), \tau'(f_{1/2}(\phi_1'(a_0))) \geq r_0 - \delta_2/16 \text{ for all } \tau' \in T(B') \tag{5.50}$$
$$\tau(\psi_1(a_0)), \tau(f_{1/2}(\psi_1(a_0))) \geq r_0\eta \text{ for all } \tau \in T(A), \tag{5.51}$$
$$t(f_{1/4}(\psi_1(a_0))) \geq r_0\lambda \text{ for all } t \in T(D_1). \tag{5.52}$$

We have, by (5.47),

$$\|\phi_1'(a_1)\phi_1'(x)\phi_1'(a_1) - \phi_1'(x)\| < \delta_2/8 \text{ for all } x \in \mathcal{F}_1. \tag{5.53}$$

Therefore, for some $\sigma > 0$,

$$\|f_\sigma(\phi_1'(a_1))\phi_1'(x)f_\sigma(\phi_1'(a_1)) - \phi_1'(x)\| < \delta_2/4 \text{ for all } x \in \mathcal{F}_1. \tag{5.54}$$

By 7.2 of [17], there exists $0 \leq e \leq 1$ such that

$$f_\sigma(\phi_1'(a_1)) \leq e \leq f_{2\sigma'}(\phi_1'(a_1)) \tag{5.55}$$

and $d_\tau(e)$ is continuous on $\overline{T(A)}^w$, where $0 < \sigma' < \sigma/4$. Define $\phi_1 : A \to A$ by

$$\phi_1(a) = e^{1/2}\phi_1'(a)e^{1/2} \text{ for all } a \in A. \tag{5.56}$$

We also have

$$e^{1/2}((\phi_1'(a_1) - \sigma'/2)_+)e^{1/2} \leq e^{1/2}\phi_1'(a_1)e^{1/2} \leq e. \tag{5.57}$$

But

$$e = e^{1/2}f_{\sigma'}(\phi_1'(a_1))e^{1/2} \leq e^{1/2}((2/\sigma')(\phi_1'(a_1) - \sigma'/2)_+)e^{1/2} \tag{5.58}$$
$$= (2/\sigma')(e^{1/2}((\phi_1'(a_1) - \sigma'/2)_+)e^{1/2}). \tag{5.59}$$

Combining these two inequalities, we conclude that $d_\tau(\phi_1(a_1)) = d_\tau(e)$ for all $\tau \in T(A)$. In particular, $d_\tau(\phi_1(a_1))$ is continuous on $T(A)$. By (5.54), we have

$$\|\phi_1'(a) - \phi_1(a)\| < \delta_2/4 \text{ for all } a \in \mathcal{F}_1. \tag{5.60}$$

By the choice of δ_1, we have

$$\|f_{1/2}(\phi_1(a_0)) - f_{1/2}(\phi_1'(a_0))\| < \varepsilon_1. \tag{5.61}$$

It follows that

$$\tau'(f_{1/2}(\phi_1(a_0))) \geq r_0 - \delta_2/16 - \varepsilon_1 \text{ for all } \tau' \in T(B'). \tag{5.62}$$

Put $B_1 = \overline{\phi_1(a_1)A\phi_1(a_1)}$. Then by 5.4 of [17] B_1 has continuous scale. Note ϕ_1 maps A into B_1. We also have

$$\|x - (\phi_1(x) + \psi_1(x))\| < \delta_2/2 \text{ for all } x \in \mathcal{F}_1. \tag{5.63}$$

Moreover, since $B_1 \subset B'$, we have $B_1 \perp D_1$, and

$$\tau'(\phi_1(a_0)), \tau'(f_{1/2}(\phi_1(a_0))) \geq r_0 - \delta_2/16 - \varepsilon_1 \text{ for all } \tau' \in \mathrm{T}(B_1). \tag{5.64}$$

Since $B_1 \in \mathcal{D}$ (or in $\mathcal{D}_0$), we can repeat the process above for B_1. Therefore we may now apply the hypothesis to B_1 in place of A, and continue the process and stop at stage k.

In this way, we obtain hereditary sub-C*-algebras $B_1, B_2, \ldots, B_k$, and sub-C*-algebras $D_1, D_2, \ldots, D_k$ such that $B_{i+1} \subseteq B_i$, $B_i \perp D_i$, $D_{i+1} \subseteq B_i$, $D_i \in \mathcal{C}_0'$ (or $\mathcal{C}_0^{0'}$), $\mathcal{F}_i$-$\delta_2/16 \cdot 2^{i+1}$-multiplicative completely positive contractive maps $\phi_{i+1} : B_i \to B_{i+1}$ and $\psi_{i+1} : B_i \to D_{i+1}$ such that

$$\mathcal{F}_{i+1} = \{\phi_i(x); x \in \mathcal{F}_i, c_j, f_{1/2}(c_j), f_{1/4}(c_j), j = 0, 1\},$$

where $c_j = \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1(a_j)$, $j = 0, 1$, $i = 1, 2, \ldots, k-1$,

$$\|x - (\phi_{i+1}(x) \oplus \psi_{i+1}(x))\| < \delta_2/2^{i+1} \text{ for all } x \in \mathcal{F}_{i+1} \quad \text{(as in (5.63))}, \tag{5.65}$$

$$d_\tau(\phi_{i+1}(c_{i,0})) < (1-\eta)^{i+1} \text{ for all } \tau \in \mathrm{T}(A), \quad \text{(as in (5.49), see also (5.56))}, \tag{5.66}$$

$$\tau'(\phi_{i+1}(c_{i,0})), \tau'(f_{1/2}(\phi_{i+1}(c_{i,0}))) \geq (r_0 - (i+1)(\delta_2/16 + \varepsilon_1)) \tag{5.67}$$

$$\text{for all } \tau' \in \mathrm{T}(B_{i+1}) \quad \text{(as in (5.62))},$$

$$\tau(\psi_{i+1}(c_{i,0})) \geq (r_0 - (i+1)(\delta_2/16 + \varepsilon_1))\eta \text{ for all } \tau \in \mathrm{T}(B_i) \quad \text{(as in (5.51))}, \tag{5.68}$$

$$t(f_{1/4}(\psi_{i+1}(c_{i,0}))) \geq (r_0 - (i+1)(\delta_2/16 + \varepsilon_1))\lambda \text{ for all } t \in T(D_{i+1}) \quad \text{(as in (5.52))}, \tag{5.69}$$

and B_{i+1} has continuous scale, $i = 1, 2, \ldots, k-1$. Note that $(r_0 - k(\delta_2/16+\varepsilon_1)) \geq r_0/2$. Let $D = \bigoplus_{i=1}^k D_i$ and let $\Psi : A \to D$ be defined by

$$\Psi(a) = (\psi_1(a)) \oplus \psi_2(\phi_1((a)) \oplus \cdots \oplus \psi_k(\phi_{k-1} \circ \cdots \circ \phi_1(a))) \text{ for all } a \in A.$$

By (5.65), with $\Phi = \phi_k \circ \phi_{k-1} \circ \cdots \phi_1 : A \to B_k$,

$$\|x - (\Phi(x) \oplus \Psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F}, \tag{5.70}$$

$$t(f_{1/4}(\Psi(a_0))) \geq (r_0/2)\lambda = \mathfrak{f}_a \text{ for all } t \in \mathrm{T}(D), \tag{5.71}$$

We also have $D \in \mathcal{C}_0'$, or $D \in \mathcal{C}_0^{0'}$.

Moreover,

$$d_\tau(\Phi(a_0)) \leq (1-\eta)^k \text{ for all } \tau \in \mathrm{T}(A).$$

This implies, by (5.45), that

$$\Phi(a_0) \lesssim b_0, \tag{5.72}$$

since A is assumed to have strict comparison for positive elements. By (5.70), (5.71), and (5.72), we conclude that A is in $\mathcal{D}$ or in $\mathcal{D}_0$. □

Definition 5.6 (*10.1 of [17]*)**.** Let A be a non-unital and σ-unital simple C*-algebra. A is said to be tracially approximately divisible in the non-unital sense if the following property holds:

For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subseteq A$, any $b \in A_+ \setminus \{0\}$, and any integer $n \geq 1$, there are σ-unital sub-C*-algebras A_0, A_1 of A such that

$$\mathrm{dist}(x, B_d) < \varepsilon \text{ for all } x \in \mathcal{F},$$

where $B_d \subseteq B := A_0 \oplus \mathrm{M}_n(A_1) \subseteq A$, $A_0 \perp \mathrm{M}_n(A_1)$,

$$B_d = \{(x_0, \overbrace{x_1, x_1, \ldots, x_1}^{n}) : x_0 \in A_0, x_1 \in A_1\} \tag{5.73}$$

and $a_0 \lesssim b$, where a_0 is a strictly positive element of A_0.

Theorem 5.7. *Let A be a stably projectionless separable simple C*-algebra in the class $\mathcal{R}$ with $\dim_{\mathrm{nuc}} A = m < \infty$.*

Suppose that every hereditary sub-C-algebra B of A with continuous scale has the following properties: Let $e_B \in B$ be a strictly positive element with $\|e_B\| = 1$ and $\tau(e_B) > 1 - 1/64$ for all $\tau \in \mathrm{T}(B)$. With C the unique non-unital simple C*-algebra C in $\mathcal{M}_0 \cap \mathcal{R}$ such that $\mathrm{T}(C) \cong \mathrm{T}(B)$, for each affine homeomorphism $\gamma : \mathrm{T}(B) \to \mathrm{T}(C)$, there exist sequences of completely positive contractive maps $\sigma_n : B \to C$ and homomorphisms $\rho_n : C \to B$ such that*

$$\lim_{n\to\infty} \|\sigma_n(ab) - \sigma_n(a)\sigma_n(b)\| = 0 \text{ for all } a, b \in B, \tag{5.74}$$

$$\lim_{n\to\infty} \sup\{|t \circ \sigma_n(a) - \gamma^{-1}(t)(a)| : t \in T(C)\} = 0 \text{ for all } a \in A, \tag{5.75}$$

$$\lim_{n\to\infty} \sup\{|\tau(\rho_n \circ \sigma_n(b)) - \tau(b)| : \tau \in \mathrm{T}(B)\} = 0. \tag{5.76}$$

Suppose also that every hereditary sub-C-algebra A is tracially approximately divisible. Then $A \in \mathcal{D}_0$.*

Proof. By [49] (see also [54]), $A'\otimes\mathcal{Z}\cong A'$ for every hereditary sub-C*-algebra A' of A. It follows from [44] that A has almost stable rank one. Let B be a hereditary sub-C*-algebra with continuous scale. Then B has finite nuclear dimension (see [57]). By [49] again, B is $\mathcal{Z}$-stable. It follows from 6.6 of [19] that the map from $\mathrm{Cu}(A)$ to $\mathrm{LAff}_+(\tilde{\mathrm{T}}(B))$ is surjective. Note that the map from $\mathrm{W}(A)_+$ to $\mathrm{LAff}_{b,+}(\mathrm{T}(A))$ is surjective. We will apply Theorem 5.4 and Lemma 5.5.

Fix a strictly positive element $e\in B$ with $\|e\|=1$. Since B has continuous scale, we may assume there is $e'\in B_+$ with $\|e'\|=1$ such that $f_{1/2}(e)e'=e'=f_{1/2}(e)e'$ and $d_\tau(f_{1/2}(e'))>1-1/64(m+2)$ for all $\tau\in\mathrm{T}(B)$. Let $1>\varepsilon>0$, $\mathcal{F}\subseteq B$ be a finite subset. and let $b\in B_+\setminus\{0\}$. Choose $b_0\in B_+\setminus\{0\}$ and $64(m+2)\langle b_0\rangle\le\langle b\rangle$ in $\mathrm{Cu}(A)$. Since we assume that A is tracially approximately divisible (see (5.6)), there are $e_0\in B_+$ and a hereditary sub-C*-algebra A_0 of B such that $e_0\perp M_{4(m+2)}(A_0)$, $e_0\lesssim b_0$ and

$$\mathrm{dist}(x,B_{1,d})<\varepsilon/64(m+2)\ \text{ for all }\ x\in\mathcal{F}\cup\{e\},$$

where $B_{1,d}\subseteq B_s:=\overline{e_0Be_0}\oplus \mathrm{M}_{4(m+2)}(A_0)\subseteq B$ and

$$B_{1,d}=\{x_0\oplus(\overbrace{x_1\oplus x_1\oplus\cdots\oplus x_1}^{4(m+2)}):x_0\in\overline{e_0Be_0},\ x_1\in A_0\}. \tag{5.77}$$

Without loss of generality, we may further assume that $\mathcal{F}\cup\{e'\}\subseteq B_{1,d}$. Let $P:B_s\to \mathrm{M}_{4(m+2)}(A_0)$ be the projection map and $P^{(1)}:M_{4(m+2)}(A_0)\to A_0=A_0\otimes e_{11}$ be defined by $P^{(1)}(a)=(1_{A_0}\otimes e_{11})a(1_{A_0}\otimes e_{11})$, where $\{e_{ij}\}_{4(m+2)\times4(m+2)}$ is a system of matrix unit. Therefore, we may assume, without loss of generality, that $\|e_0x-xe_0\|<\varepsilon/64(m+2)$, and there is $e_1\in M_{4(m+2)}(A_0)$ with $0\le e_1\le1$ such that $\|e_1x-xe_1\|<\varepsilon/64(m+2)$ and $\|e_1P(x)-P(x)\|<\varepsilon/64(m+2)$ for all $x\in\mathcal{F}\cup\{e,e',f_{1/2}(e),f_{1/4}(e),f_{1/2}(e')\}$. Moreover, since the map from $W(A)_+$ to $\mathrm{LAff}_{b,+}(\mathrm{T}(A))$ is surjective, as in the proof of 5.5 (when 7.2 of [17] is applied), without loss of generality, we may assume that A_0 has continuous scale. Write

$$x=x_0+\overbrace{x_1\oplus x_1\oplus\cdots\oplus x_1}^{4(m+2)}.$$

Let $\mathcal{F}_1=\{x_1:x\in\mathcal{F}\cup\{e',f_{1/2}(e')\}\}$. Note that we may write $\overbrace{x_1\oplus x_1\oplus\cdots\oplus x_1}^{4(m+2)}=x_1\otimes 1_{4(m+2)}$. Then $\dim_{\mathrm{nuc}}A_0=m$ (see [57]). Also, A_0 is a non-unital separable simple C*-algebra which has continuous scale. We may then apply 5.4 to A_0 with $\mathcal{S}=\mathcal{R}_{\mathrm{az}}$. By 2.8, in 5.4, we may choose $C=\overline{\bigcup_{n=1}^\infty W_n}$, where each W_n is a finite direct sum of $\mathcal{W}$'s, $W_n\subset W_{n+1}$ and strictly positive elements of W_n are strictly positive elements of W_{n+1} for all n. Since (see 9.6 of [17]) $\mathcal{W}=\overline{\bigcup_{k=1}^\infty R_k}$, where $R_m\subset R_{m+1}$, strictly positive elements of R_m are strictly positive elements of R_{k+1}, and each R_k is Razak algebra (as in 2.3), where $\lambda_s(R_k)\to1$, as $k\to\infty$ (see for λ_s in 5.3, and also (2.2)), we may write $C=\overline{\bigcup_{n=1}^\infty C_n}$, where $C_n\subset C_{n+1}$, strictly positive elements of C_n are strictly positive elements of C_{n+1}. Moreover, $\lambda_s(C_n)\ge1/2$ for all n. Put $r_0=(1-1/64(m+2))$. Choose $\eta_0=7/32(m+2)$ and $\lambda=3/16$. Thus, by applying 5.4, we have, with $\phi_1(b)=(E-p)b(E-p)$ for all $b\in M_{4(m+2)}(A_0)$, where $E=1_{M_{4(m+2)}(\widetilde{A_0})}$, and $p\in M_{4(m+2)}(\widetilde{A_0})$ is a projection given by 5.4, and $L:A_0\to D_1$ is an $\mathcal{F}_1$-ε-multiplicative completely positive contractive map

$$\|x\otimes1_{4(m+2)}-(\phi_1(x\otimes1_{4(m+2)})+L(x))\|<\varepsilon/4\ \text{ for all }\ x\in\mathcal{F}_1, \tag{5.78}$$

$$d_\tau(\phi_1(e))\le1-1/4(m+2)\ \text{ for all }\ \tau\in\mathrm{T}(A_0), \tag{5.79}$$

$$\tau'(\phi_1(e)),\tau'(f_{1/2}(\phi_1(e)))\ge r_0-\varepsilon/4\ \text{ for all }\ \tau'\in\mathrm{T}((1-p)M_{4(m+1)}(A_0)(1-p)), \tag{5.80}$$

$$D_1\in\mathcal{C}_0^0,\ D_1\subseteq pM_{4(m+2)}(A_0)p, \tag{5.81}$$

$$\tau(L(P^{(1)}(e)))\ge r_0\eta_0\ \text{ for all }\ \tau\in\mathrm{T}(M_{4(m+2)}(A_0))\ \text{ and} \tag{5.82}$$

$$t(f_{1/4}(L(P^{(1)}(e))))\ge r_0\lambda\ \text{ for all }\ t\in\mathrm{T}(D_1). \tag{5.83}$$

Let $B_1=(1-p)M_{4(m+1)}(A_0))(1-p)\oplus\overline{e_0Be_0}$ and $\phi:B\to B_1$ be defined by $\phi(b)=\phi_1(e_1be_1)+e_0be_0$ for $b\in B$. Define $L_1:B\to D_1$ by $L_1(b)=L(P^{(1)}(e_1^{1/2}be_1^{1/2}))$. Then both L_1 and ϕ is $\mathcal{F}$-ε-multiplicative. Put $\eta=\eta_0/2<\frac{\eta_0}{1+\varepsilon/64(m+2)}$. Then, in addition to (5.83) and (5.81),

$$\|x-(\phi(x)+L_1(x))\|<\varepsilon\ \text{ for all }\ x\in\mathcal{F},$$
$$d_\tau(\phi(e))\le1-\eta\ \text{ for all }\ \tau\in\mathrm{T}(B),$$
$$\tau'(\phi_1(e)),\tau'(f_{1/2}(\phi_1(e)))\ge r-\varepsilon\ \text{ for all }\ \tau'\in\mathrm{T}(B_1),$$
$$\tau(L(e))\ge r_0\eta\ \text{ for all }\ \tau\in\mathrm{T}(B).$$

Note this holds for every such B. Thus, the hypotheses of 5.5 are satisfied. We then apply 5.5. □

6. The C*-algebra $\mathcal{W}$ and UHF-stability

Definition 6.1 (*12.1 of [17]*). Let A be a non-unital separable C*-algebra. Suppose that $\tau\in\mathrm{T}(A)$. Recall that τ was said to be a $\mathcal{W}$-trace in [17] if there exists a sequence of completely positive contractive maps (ϕ_n) from A into $\mathcal{W}$ such that

$$\lim_{n\to\infty}\|\phi_n(ab)-\phi_n(a)\phi_n(b)\|=0\ \text{ for all }\ a,\ b\in A,\ \text{ and}$$

$$\tau(a) = \lim_{n\to\infty} \tau_{\mathcal{W}}(\phi_n(a)) \text{ for all } a \in A, \tag{6.1}$$

where $\tau_{\mathcal{W}}$ is the unique tracial state on $\mathcal{W}$.

The following two statements (6.2, and 6.3) are taken from [17] (and the proofs are straightforward).

Proposition 6.2 (*12.4 of [17]*). *Let A be a separable simple C^*-algebra with a $\mathcal{W}$-tracial state $\tau \in \mathrm{T}(A)$. Let $0 \le a_0 \le 1$ be a strictly positive element of A. Then there exists a sequence of completely positive contractive maps $\phi_n : A \to \mathcal{W}$ such that $\phi_n(a_0)$ is a strictly positive element,*

$$\lim_{n\to\infty} \|\phi_n(a)\phi_n(b) - \phi_n(ab)\| = 0 \text{ for all } a, b \in A \text{ and}$$

$$\tau(a) = \lim_{n\to\infty} \tau_{\mathcal{W}} \circ \phi_n(a) \text{ for all } a \in A, \tag{6.2}$$

where $\tau_{\mathcal{W}}$ is the unique tracial state of $\mathcal{W}$.

Theorem 6.3 (*12.2 of [17]*). *Let A be a separable simple C^*-algebra with $A = \mathrm{Ped}(A)$. If every tracial state $\tau \in \mathrm{T}(A)$ is a $\mathcal{W}$-trace, then $\mathrm{K}_0(A) = \ker\rho_A$.*

Proposition 6.4. *Let A be a separable C^*-algebra with $A = \mathrm{Ped}(A)$ such that every tracial state τ of A is quasidiagonal. Let $Y \in \mathcal{D}_0$ be a simple C^*-algebra which is an inductive limit of C^*-algebras in $\mathcal{C}_0'$ such that $\mathrm{K}_0(Y) = \ker\rho_Y$, and Y has a unique trace, which is bounded. Then all tracial states of $A \otimes Y$ are $\mathcal{W}$-tracial states. In particular, all tracial states of $A \otimes \mathcal{W}$ are $\mathcal{W}$-tracial states.*

Proof. Let $\tau \in \mathrm{T}(A)$. Denote by t the unique tracial state of Y. We will show $\tau \otimes t$ is a $\mathcal{W}$-trace on $A \otimes Y$.

By 8.12 of [17], Y is an inductive limit of 1-dimensional non-commutative CW complexes (C^*-algebras in $\mathcal{C}_0$) with $K_1(Y) = \{0\}$. For each n, there is a homomorphism $h_n : \mathrm{M}_n(Y) \to \mathcal{W}$ (by Theorem 1.0.1 of [43]) such that h_n maps a strictly positive element of $\mathrm{M}_n(Y)$ to a strictly positive element of $\mathcal{W}$. Consider $\tau_{\mathcal{W}} \in \mathrm{T}(\mathcal{W})$. Then $\tau_{\mathcal{W}} \circ h_n$ is a tracial state of Y. Therefore $t \otimes \mathrm{tr_n}(a) = \tau_{\mathcal{W}} \circ h_n(a)$ for all $a \in \mathrm{M}_n(Y)$. Moreover, for any $a \in \mathrm{M}_n$ and $b \in Y$,

$$\mathrm{tr}_n(a)t(b) = \tau_{\mathcal{W}} \circ h_n(a \otimes b),$$

where tr_n is the normalized trace on M_n, $n = 1, 2, \ldots$.

Since τ is quasidiagonal, there is a sequence $\psi_n : A \to \mathrm{M}_{k(n)}$ of completely positive contractive maps such that

$$\lim_{n\to\infty} \|\psi_n(ab) - \psi_n(a)\psi_n(b)\| = 0 \text{ for all } a, b \in A \text{ and} \tag{6.3}$$

$$\tau(a) = \lim_{n\to\infty} \mathrm{tr}_{k(n)} \circ \psi_n(a) \text{ for all } a \in A. \tag{6.4}$$

Define $\phi_n : A \otimes Y \to \mathcal{W}$ by $\phi_n(a \otimes b) = h_{k(n)}(\psi_n(a) \otimes b)$ for all $a \in A$ and $b \in Y$. Then ϕ_n is completely positive contractive map and, for any $a \in A$ and $b \in Y$,

$$(\tau \otimes t)(a \otimes b) = \lim_{n\to\infty} \mathrm{tr}_{k(n)}(\psi_n(a))t(b) \tag{6.5}$$

$$= \lim_{n\to\infty} \tau_{\mathcal{W}} \circ h_{k(n)}(\psi_n(a) \otimes b) = \lim_{n\to\infty} \tau_{\mathcal{W}}(\phi_n(a \otimes b)). \tag{6.6}$$

Therefore $\tau \otimes t$ is a $\mathcal{W}$-trace. □

Theorem 6.5. *Let A be a simple separable C^*-algebra with finite nuclear dimension which has bounded scale and is such that $\mathrm{K}_0(A) = \ker \rho_A$ and every tracial state is a $\mathcal{W}$-trace. Suppose that every hereditary sub-C^*-algebra of A with continuous scale is tracially approximately divisible. Then $A \in \mathcal{D}_0$. (In particular, $A \otimes U \in \mathcal{D}_0$ for any UHF-algebra U.)*

Proof. By [49], A is $\mathcal{Z}$-stable. By Remark 5.2 of [17], A has a non-zero hereditary sub-C^*-algebra A_0 with continuous scale. Then $\mathrm{M}_k(A_0)$ also has continuous scale for every integer $k \ge 1$. Since A has bounded scale, it is isomorphic to a hereditary sub-C^*-algebra of $\mathrm{M}_k(A_0)$ for some possibly large k. Since $\mathrm{M}_k(A_0)$ has the same properties as assumed for A, it then follows from 8.6 of [17] that, to prove that A is in $\mathcal{D}_0$, we may assume that A has continuous scale.

It follows from Theorem 2.8 that there is a simple C^*-algebra $B = \lim_{n\to\infty}(B_n, \imath_n)$, where each B_n is a finite direct sum of copies of $\mathcal{W}$ and $\imath_{n,\infty}$ maps strictly positive elements to strictly positive elements, each B_n has bounded scale, and $\mathrm{T}(B) \cong \mathrm{T}(A)$. Denote by $\gamma : \mathrm{T}(A) \to \mathrm{T}(B)$ the affine homeomorphism. By [53], we may assume that $B = \lim_{n\to\infty}(R_n, \imath_n)$, where each R_n is a Razak algebra and $\imath_n$ is injective. It follows from Corollary A.27 of the Appendix that there exists a homomorphism $\rho : B \to A$ which induces γ, i.e.,

$$\tau(\rho(b)) = \gamma(\tau)(b) \text{ for all } b \in B \text{ and } \tau \in T(A). \tag{6.7}$$

Let $(\imath_{n,\infty})_{\mathrm{T}} : \mathrm{T}(B) \to \mathrm{T}(B_n)$ be the continuous affine map such that, for $t \in \mathrm{T}(B)$,

$$t \circ \imath_{n,\infty}(b) = (\imath_{n,\infty})_T(t)(b)$$

for all $b \in B_n$, $n = 1, 2, \ldots$. Recall that $\mathrm{T}(\mathcal{W}) = \{\tau_{\mathcal{W}}\}$, where $\tau_{\mathcal{W}}$ is the unique tracial state of $\mathcal{W}$.

Fix a strictly positive element $a_0 \in A$. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subseteq A$, since $B = \lim_{n\to\infty}(B_n, \imath_n)$ and $\mathrm{T}(B_n)$ has finitely many extremal traces, then, as is standard and easy to see, there are an integer $n_1 \geq 1$ and a continuous affine map $\kappa : \mathrm{T}(B_{n_1}) \to \mathrm{T}(A)$ such that, for $\tau \in \mathrm{T}(B)$,

$$\sup_{\tau\in \mathrm{T}(B)} |\kappa \circ (\imath_{n_1,\infty})_{\mathrm{T}}(\tau)(f) - \gamma^{-1}(\tau)(f)| < \varepsilon/3 \ \text{ for all } \ f \in \mathcal{F}. \tag{6.8}$$

Write $B_{n_1} = W_1 \oplus W_2 \oplus \cdots \oplus W_m$, where each $W_i \cong \mathcal{W}$. Denote by $\tau_{\mathrm{W}_1}, \tau_{\mathrm{W}_2}, \ldots, \tau_{\mathrm{W}_m}$ the unique tracial states on W_i, and $\theta_i = \kappa(\tau_{\mathrm{W}_i})$, $i = 1, 2, \ldots, m$. By the assumption, there exists, for each i, a sequence of completely positive contractive maps $\phi_{n,i} : A \to W_i$ such that

$$\lim_{n\to\infty} \|\phi_{n,i}(a)\phi_{n,i}(b) - \phi_{n,i}(ab)\| = 0 \ \text{ for all } \ a, b \in A, \text{ and} \tag{6.9}$$

$$\theta_i(a) = \lim_{n\to\infty} \tau_{\mathrm{W}_i} \circ \phi_{n,i}(a) \ \text{ for all } \ a \in A. \tag{6.10}$$

Moreover, by 6.2, we may assume that $\phi_{n,i}(a_0)$ is strictly positive. Define $\phi_n : A \to B_{n_1}$ by

$$\phi_n(a) = \phi_{n,1}(a) \oplus \phi_{n,2}(a) \oplus \cdots \oplus \phi_{n,m}(a), \quad a \in A. \tag{6.11}$$

Then

$$\lim_{n\to\infty} \sup_{\tau\in \mathrm{T}(B_{n_1})} \{|\tau(\phi_n(a)) - \kappa(\tau)(a)|\} = 0 \ \text{ for all } \ a \in A. \tag{6.12}$$

Define $\sigma_n : A \to B$ by

$$\sigma_n(a) = \imath_{n_1,\infty} \circ \phi_n(a), \quad a \in A. \tag{6.13}$$

Note that $\sigma_n(a_0)$ is a strictly positive element. We also have that

$$\lim_{n\to\infty} \|\sigma_n(ab) - \sigma_n(a)\sigma_n(b)\| = 0, \quad a,\ b \in A. \tag{6.14}$$

Moreover, for any $\tau \in \mathrm{T}(B)$ and any $f \in \mathcal{F}$,

$$|\gamma^{-1}(\tau)(f) - \tau \circ \sigma_n(f)| \leq |\gamma^{-1}(\tau)(f) - \kappa \circ (\imath_{n_1,\infty})_T(\tau)(f)| \tag{6.15}$$

$$+ |\kappa \circ (\imath_{n_1,\infty})_{\mathrm{T}}(\tau)(f) - \tau \circ \sigma_n(f)| \tag{6.16}$$

$$< \varepsilon/3 + |\kappa \circ (\imath_{n_1,\infty})_{\mathrm{T}}(\tau)(f) - \tau \circ \imath_{n_1,\infty} \circ \phi_n(f)| \tag{6.17}$$

$$\leq \varepsilon/3 + \sup_{t\in \mathrm{T}(B_{n_1})} \{|\tau(\phi_n(f)) - \kappa(\tau)(f)|\}. \tag{6.18}$$

By (6.12), there exists $N \geq 1$ such that, for all $n \geq N$,

$$\sup_{\tau\in \mathrm{T}(B)} \{|\gamma^{-1}(\tau)(f) - \tau \circ \sigma_n(f)|\} < 2\varepsilon/3 \ \text{ for all } \ f \in \mathcal{F}. \tag{6.19}$$

Thus the map σ_n satisfies (5.74) and (5.75). By (6.7) and (6.19), for all $n \geq N$,

$$\sup_{\tau\in \mathrm{T}(A)} \{|\tau(f) - \tau(\rho \circ \sigma_n(f))|\} = \sup_{\tau\in \mathrm{T}(A)} \{|\tau(f) - \gamma(\tau)(\sigma_n(f))|\} < \varepsilon \ \text{ for all } \ f \in \mathcal{F}.$$

Thus (5.76) also holds (with $\rho = \rho_n$). Therefore, by 5.7, $A \in \mathcal{D}_0$. □

Theorem 6.6. *Let A be a non-unital separable simple C^*-algebra with finite nuclear dimension and with $A = \mathrm{Ped}(A)$. Suppose that $\mathrm{T}(A) \neq \emptyset$. Then $A \otimes \mathcal{W} \in \mathcal{D}_0$.*

Proof. By Lemma 3.17, $A\otimes\mathcal{W}$ is KK-contractible. Therefore $A\otimes\mathcal{W}$ satisfies the UCT. Since $\mathcal{W}$ has finite nuclear dimension, so also does $A \otimes \mathcal{W}$. Hence by [50], every tracial state is quasi-diagonal. It follows by 6.4 that every tracial state of $A \otimes \mathcal{W}$ is a $\mathcal{W}$-trace. We also have $K_0(A \otimes \mathcal{W}) = \{0\}$. Let $b \in (A \otimes \mathcal{W})_+$. Since $\mathcal{W}$ has a unique tracial state, by 11.8 of [17], $W(A \otimes \mathcal{W}) = \mathrm{LAff}_{b,0+}(\overline{\mathrm{T}(A)}^w)$. Therefore, there are $a \in A$ and $b_1 \in M_2(\mathcal{W})_+$ such that $b \sim a \otimes b_1$. Put $B = \overline{b(A \otimes \mathcal{W})b}$ and $B_1 = \overline{(a \otimes b_1)(A \otimes \mathcal{W})(a \otimes b_1)}$. Then $B \cong B_1$. Note that $\overline{b_1\mathcal{W}b_1} \cong \mathcal{W}$. It follows that $B_1 \cong \overline{aAa} \otimes \mathcal{W}$. But $\mathcal{W} \otimes Q \cong \mathcal{W}$. This implies that B_1 is tracially approximately divisible. Therefore B is tracially approximately divisible. Then 6.5 applies. □

Added in proof: The condition of finite nuclear dimension in 6.6 can be much weakened to the condition that A is amenable. Since $A \otimes \mathcal{W}$ is $\mathcal{Z}$-stable, by a recent preprint of J. Castillejos and S. Evington, arXiv:1901.11441, it has finite nuclear dimension, as kindly pointed out by the referee.

Corollary 6.7. *Let A be a simple separable finite C^*-algebra such that $A \otimes \mathcal{Z}$ has finite nuclear dimension. Then the C^*-algebra $A\otimes\mathcal{W}$ belongs to the class $\mathcal{M}_0$, and so $A\otimes\mathcal{W}$ is isomorphic to an inductive limit of C^*-algebras in $\mathcal{R}_{\mathrm{az}}$. in particular, $\mathcal{W}\otimes\mathcal{W} \cong \mathcal{W}$.*

Proof. By 6.6, $A \otimes \mathcal{W} \in \mathcal{D}_0$ and, by 3.17, $A \otimes \mathcal{W}$ is KK-contractible. Then 4.3 applies. □

Lemma 6.8. *Let A be a separable simple C*-algebra in $\mathcal{R}$ with finite nuclear dimension which is KK-contractible and assume that all tracial states of A are $\mathcal{W}$-traces. Let $\mathrm{M}_{\mathfrak{p}}$ and $\mathrm{M}_{\mathfrak{q}}$ be two UHF algebras, where $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime supernatural numbers. Then, there exist an isomorphism $\phi : A\otimes\mathrm{M}_{\mathfrak{p}} \to A\otimes\mathrm{M}_{\mathfrak{p}}\otimes\mathrm{M}_{\mathfrak{q}}$ and a continuous path of unitaries $u_t \in \mathrm{M}(A\otimes\mathrm{M}_{\mathfrak{p}}\otimes\mathrm{M}_{\mathfrak{q}})$, $1 \le t < \infty$, such that $u_1 = 1$ and*

$$\lim_{t\to\infty} u_t^*(a \otimes r \otimes 1_{\mathfrak{q}})u_t = \phi(a \otimes r), \quad a \in A, \ r \in \mathrm{M}_{\mathfrak{p}}.$$

Proof. Note that every hereditary sub-C*-algebra B of $A\otimes\mathrm{M}_{\mathfrak{p}}$ and $A\otimes Q$ is tracial approximately divisible, since $M_{\mathfrak{p}}$ and Q are strongly self-absorbing. By the assumption and 6.5, $A\otimes\mathrm{M}_{\mathfrak{p}}$ and $A\otimes Q$ are in $\mathcal{D}_0$. It follows from 4.3 that $A\otimes\mathrm{M}_{\mathfrak{p}} \cong A\otimes Q$. Let $\xi : A \otimes \mathrm{M}_{\mathfrak{p}} \to A \otimes Q$ be an isomorphism. It is well known that any isomorphism $\psi : Q \to Q \otimes \mathrm{M}_{\mathfrak{q}}$ is asymptotically unitarily equivalent to the embedding $Q \to Q \otimes \mathrm{M}_{\mathfrak{q}}$ given by $r \to r \otimes 1_{\mathfrak{q}}$ for all $r \in Q$ (see, for instance, [31]). Therefore there exists a continuous path of unitaries $v_t \in Q \otimes \mathrm{M}_{\mathfrak{q}}$ such that $v_1 = 1$ and

$$\lim_{t\to\infty} v_t^*(r \otimes 1_{\mathfrak{q}})v_t = \psi(r) \ \text{ for all } \ r \in Q.$$

Define $\phi_1 : A \otimes Q \to A \otimes Q \otimes M_{\mathfrak{q}}$ by $\phi_1(a \otimes r) = a \otimes \psi(r)$ for all $a \in A$ and $r \in Q$. Therefore

$$\lim_{t\to\infty} (1_A \otimes v_t^*)(b \otimes 1_{\mathfrak{q}})(1_A \otimes v_t) = \phi_1(b) \ \text{ for all } \ b \in A \otimes Q. \tag{6.20}$$

Define $\phi : A \otimes \mathrm{M}_{\mathfrak{p}} \to A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}}$ by $\phi = (\xi^{-1} \otimes \mathrm{id}_{M_{\mathfrak{q}}}) \circ \phi_1 \circ \xi$ and let $u_t = \tilde{\xi}^{-1}(1_A \otimes v_t)$, where $\tilde{\xi} : \mathrm{M}(A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}}) \to \mathrm{M}(A \otimes Q \otimes \mathrm{M}_{\mathfrak{q}})$ is the extension of $\xi \otimes \mathrm{id}_{M_{\mathfrak{q}}} : A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}} \to A \otimes Q \otimes \mathrm{M}_{\mathfrak{q}}$. Note that $u_1 = 1$, since $v_1 = 1$ and $\{u_t\}$ is a continuous path of unitaries in $\mathrm{M}(A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}})$. Suppose that $a \in A$ and $r \in \mathrm{M}_{\mathfrak{p}}$. So $\xi(a \otimes r) \in A \otimes Q$. Then we have

$$\begin{aligned}
&\lim_{t\to\infty} u_t^*(a \otimes r \otimes 1_{\mathfrak{q}})u_t = \lim_{t\to\infty} \tilde{\xi}^{-1}(1_A \otimes v_t^*)\left((\xi^{-1} \otimes \mathrm{id}_{\mathrm{M}_{\mathfrak{q}}})(\xi(a \otimes r) \otimes 1_{\mathfrak{q}})\right) \tilde{\xi}^{-1}(1_A \otimes v_t)\\
&= (\xi^{-1} \otimes \mathrm{id}_{\mathrm{M}_{\mathfrak{q}}})(\lim_{t\to\infty} \left((1_A \otimes u_t^*)(\xi(a \otimes r) \otimes 1_{\mathfrak{q}})(1_A \otimes u_t)\right))\\
&=^{\text{see (6.20)}} \xi^{-1} \otimes \mathrm{id}_{\mathrm{M}_{\mathfrak{q}}}(\phi_1(\xi(a \otimes r))) = \phi(a \otimes r)
\end{aligned}$$

as desired. □

Theorem 6.9. *Let A be a non-unital separable simple C*-algebra in $\mathcal{R}$ with finite nuclear dimension which is KK-contractible and such that every trace is a $\mathcal{W}$-trace. Then $A \cong A \otimes Q$.*

Proof. It follows from [49] that $A \cong A \otimes \mathcal{Z}$. Decompose $A \otimes \mathcal{Z}$ as an inductive limit of copies of $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, where $\mathfrak{p}$, $\mathfrak{q}$ are two relatively prime supernatural numbers such that $\mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}} = Q$. By Corollary 3.4 of [52], in order to show that A is Q-stable, it is enough to show that $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ is Q-stable. Note that

$$A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} = \{f \in \mathrm{C}([0, 1], A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}}) : f(0) \in A \otimes \mathrm{M}_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}, \ f(1) \in A \otimes 1_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}}\}.$$

Applying Lemma 6.8 to both endpoints, one obtains isomorphisms

$$\phi_0 : A \otimes \mathrm{M}_{\mathfrak{p}} \to A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}}, \quad \phi_1 : A \otimes \mathrm{M}_{\mathfrak{q}} \to A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}},$$

together with a continuous path of unitaries $u_t \in \mathcal{M}(A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}})$, $0 < t < 1$, such that $u_{\frac{1}{2}} = 1$,

$$\lim_{t\to 0} u_t^*(a \otimes r \otimes 1_{\mathfrak{q}})u_t = \phi_0(a \otimes r), \quad a \in A, \ r \in \mathrm{M}_{\mathfrak{p}},$$

and

$$\lim_{t\to 1} u_t^*(a \otimes 1_{\mathfrak{p}} \otimes r)u_t = \phi_1(a \otimes r), \quad a \in A, \ r \in \mathrm{M}_{\mathfrak{q}}.$$

Define the continuous field map $\Phi : A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to \mathrm{C}([0, 1], A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}})$ by

$$\Phi(f)(t) = u_t^* f(t)u_t, \quad t \in [0, 1],$$

where $\Phi(f)(0)$ and $\Phi(f)(1)$ are understood as $\phi_0(f(0))$ and $\phi_1(f(1))$, respectively. Then the map Φ is an isomorphism (the inverse is $\Phi^{-1}(g)(t) = u_t g(t)u_t^*$, $t \in (0, 1)$, $\Phi^{-1}(g)(0) = \phi_0^{-1}(g(0))$, and $\Phi^{-1}(g)(1) = \phi_0^{-1}(g(1))$), and hence $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \cong \mathrm{C}([0, 1], A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}})$. Since the trivial field $\mathrm{C}([0, 1], A \otimes \mathrm{M}_{\mathfrak{p}} \otimes \mathrm{M}_{\mathfrak{q}})$ is Q-stable, one has that $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ is Q-stable, as desired. □

7. The case of finite nuclear dimension

Let A be a non-unital separable C*-algebra. Since $\widetilde{A} \otimes Q$ is unital, we may view $\widetilde{A \otimes Q}$ as a sub-C*-algebra of $\widetilde{A} \otimes Q$ with the unit $1_{\widetilde{A\otimes Q}}$. In the following corollary we use $\imath$ for the embedding from $A \otimes Q$ to $\widetilde{A} \otimes Q$ as well as from $\widetilde{A \otimes Q}$

to $\widetilde{A}\otimes Q$. Since $\mathrm{K}_1(Q)=\{0\}$, from the six-term exact sequence in K-theory, one concludes that the homomorphism $\imath_{*0}:\mathrm{K}_0(A\otimes Q)\to \mathrm{K}_0(\widetilde{A}\otimes Q)$ is injective.

We will use this fact and identify x with $\imath_{*0}(x)$ for all $x\in \mathrm{K}_0(A\otimes Q)$ in the following corollary.

Lemma 7.1. *Let A be a non-unital separable C^*-algebra and let (ψ_n) be a sequence of approximately multiplicative completely positive contractive maps from $\widetilde{A}\otimes Q$ to Q. Then $\phi_n=\psi_n\circ\imath$ is a sequence of approximately multiplicative completely positive contractive maps from A into Q, where $\imath_0:A\to\widetilde{A}\otimes Q$ is the embedding defined by $a\mapsto a\otimes 1$ for all $a\in A$.*

Conversely, if (ϕ_n) is a sequence of approximately multiplicative completely positive contractive maps from A to Q, then, there exists a sequence of approximately multiplicative completely positive contractive maps $(\psi_n):\widetilde{A}\otimes Q\to Q$ such that

$$\lim_{n\to\infty}\|\phi_n(a)-\psi_n\circ\imath_0(a)\|=0 \text{ for all } a\in A.$$

Moreover, if $\limsup\|\phi_n(a)\|\neq 0$ for some $a\in A$ and if $\{e_n\}$ is an approximate unit, then, we can choose ψ_n such that

$$\mathrm{tr}(\psi_n(1))=\mathrm{d}_{\mathrm{tr}}(\phi_n(e_n)) \text{ for all } n.$$

Proof. We prove only the second part. Write $Q=\overline{\bigcup_{n=1}^{\infty}\mathrm{M}_{n!}}$ with the embedding $j_n:B_n:=\mathrm{M}_{n!}\to\mathrm{M}_{n!}\otimes\mathrm{M}_{n+1}=\mathrm{M}_{(n+1)!}$, $n=1,2,\ldots$. Without loss of generality, we may assume that ϕ_n maps A into B_n, $n=1,2,\ldots$. Consider $\phi_n'(a)=\phi_n(e_n^{1/2}ae_n^{1/2})$, $n=1,2,\ldots$. Choose p_n to be the range projection of $\phi_n(e_n)$ in B_n. Define $\psi_n':\widetilde{A}\otimes Q\to Q\otimes Q$ by $\psi_n'(a\otimes 1_Q)=\phi_n'(a)\otimes 1_Q$ for all $a\in A$, $\psi_n'(1\otimes r)=p_n\otimes r$ for all $r\in Q$. Then

$$\lim_{n\to\infty}\|\psi_n'(a\otimes 1)-\phi_n(a)\otimes 1\|=0 \text{ for all } a\in A.$$

Moreover, $\mathrm{tr}(\psi_n'(1))=\mathrm{d}_{\mathrm{tr}}(\phi_n(e_n))$ for all n. There is an isomorphism $h:Q\otimes Q\to Q$ such that $h\circ\imath_Q$ is approximately unitarily equivalent to id_Q, where $\imath_Q:a\mapsto a\otimes 1_Q$ is the embedding. By choosing some unitaries $u_n\in Q$, we can choose $\psi=\mathrm{Ad}\,u_n\circ h\circ\psi_n$, $n=1,2,\ldots$. $\square$

The following is a non-unital version of Lemma 4.2 of [16].

Lemma 7.2. *Let A be a non-unital simple separable amenable C^*-algebra with $\mathrm{T}(A)\neq\emptyset$ which has bounded scale and which satisfies the UCT. Fix a strictly positive element $a\in A_+$ with $\|a\|=1$ such that*

$$\tau(f_{1/2}(a))\geq d \text{ for all } \tau\in\overline{\mathrm{T}(A)}^{\mathrm{w}}. \tag{7.1}$$

For any $\varepsilon>0$ and any finite subset $\mathcal{F}$ of A, there exist $\delta>0$, a finite subset $\mathcal{G}$ of A, and a finite subset $\mathcal{P}$ of $K_0(A)$ with the following property. Let $\psi,\phi:A\to Q$ be two $\mathcal{G}$-δ-multiplicative completely positive contractive maps such that

$$[\psi]|_{\mathcal{P}}=[\phi]|_{\mathcal{P}} \text{ and} \tag{7.2}$$

$$\mathrm{tr}(f_{1/2}(\psi(a)))\geq d/2 \text{ and } \mathrm{tr}(f_{1/2}(\phi(a)))\geq d/2, \tag{7.3}$$

where tr is the unique tracial state of Q. Then there are a unitary $u\in Q$ and an $\mathcal{F}$-ε-multiplicative completely positive contractive map $L:A\to \mathrm{C}([0,1],Q)$ such that

$$\pi_0\circ L=\psi,\quad \pi_1\circ L=\mathrm{Ad}u\circ\phi. \tag{7.4}$$

Moreover, if

$$|\mathrm{tr}\circ\psi(h)-\mathrm{tr}\circ\phi(h)|<\varepsilon'/2 \text{ for all } h\in\mathcal{H}, \tag{7.5}$$

for a finite set $\mathcal{H}\subseteq A$ and $\varepsilon'>0$, then L may be chosen such that

$$|\mathrm{tr}\circ\pi_t\circ L(h)-\mathrm{tr}\circ\pi_0\circ L(h)|<\varepsilon' \text{ for all } h\in\mathcal{H} \text{ and } t\in[0,1]. \tag{7.6}$$

Here, $\pi_t:\mathrm{C}([0,1],Q)\to Q$ is the point evaluation at $t\in[0,1]$.

Proof. Let $T:A_+\setminus\{0\}\to\mathbb{N}\times\mathbb{R}_+\setminus\{0\}$ be given by 5.7 of [17] (with above d and a). In the notation in 3.13, $Q\in\boldsymbol{C}_{0,0,t,1,2}$, where $t:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is defined to be $t(n,k)=n/k$ for all $n,k\geq 1$. Now $\boldsymbol{C}_{0,0,t,1,2}$ is fixed. We are going to apply Theorem 3.14 together with Remark 3.15 (note that Q has real rank zero and $\mathrm{K}_1(Q)=\{0\}$).

Let $\mathcal{F}\subseteq A$ be a finite subset and let $\varepsilon>0$ be given. We may assume that $a\in\mathcal{F}$ and every element of $\mathcal{F}$ has norm at most one. Write $\mathcal{F}_1=\{ab:a,b\in\mathcal{F}\}\cup\mathcal{F}$.

Let $\delta_1>0$ (in place δ), $\mathcal{G}_1$ (in place of $\mathcal{G}$) and $\mathcal{H}_1$(in place of $\mathcal{H}$), $\mathcal{P}$, and K be as assured by Theorem 3.14 for $\mathcal{F}_1$ and $\varepsilon/4$ as well as T (in place of F). (As stated earlier we will also use Remark 3.15 so that we drop $\boldsymbol{L}$ and condition (3.15).) Since $\mathrm{K}_1(Q)=\{0\}$ and $\mathrm{K}_0(Q)=\mathbb{Q}$, we may choose $\mathcal{P}\subseteq\mathrm{K}_0(A)$.

We may also assume that $\mathcal{F}_1\cup\mathcal{H}_1\subseteq\mathcal{G}_1$ and $K\geq 2$.

Now, let $\mathcal{G}_2\subseteq A$ (in place of $\mathcal{G}$) be a finite subset and let $\delta_2>0$ (in place of δ_1) given by 5.7 of [17] for the above $\mathcal{H}_1$ and T.

Let $\delta = \min\{\varepsilon/4, \delta_1/2, \delta_2/2\}$ and $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$. Without loss of generality, we may assume that $\mathcal{G} \subseteq A^{\mathbf{1}}$.

Since $Q \cong Q \otimes Q$, we may assume, without loss of generality, that $\phi(a), \psi(a) \in Q \otimes 1$ for all $a \in A$. Pick mutually equivalent projections $e_0, e_1, e_2, \dots, e_{2K} \in Q$ satisfying $\sum_{i=0}^{2K} e_i = 1_Q$. Then, consider the maps $\phi_i, \psi_i : A \to Q \otimes e_iQe_i$, $i = 0, 1, \dots, 2K$, which are defined by

$$\phi_i(a) = \phi(a) \otimes e_i \quad \text{and} \quad \psi_i(a) = \psi(a) \otimes e_i, \quad a \in A,$$

and consider the maps

$$\Phi_{K+1} := \phi = \phi_0 \oplus \phi_1 \oplus \cdots \oplus \phi_{2K}, \quad \Phi_0 := \psi = \psi_0 \oplus \psi_1 \oplus \cdots \oplus \psi_{2K}$$

and

$$\Phi_i := \phi_0 \oplus \cdots \oplus \phi_{i-1} \oplus \psi_i \oplus \cdots \oplus \psi_{2K}, \quad i = 1, 2, \dots, 2K.$$

Since e_i is unitarily equivalent to e_0 for all i, one has

$$[\phi_i]|_{\mathcal{P}} = [\psi_j]|_{\mathcal{P}}, \quad 0 \le i, j \le 2K.$$

and in particular,

$$[\phi_i]|_{\mathcal{P}} = [\psi_i]|_{\mathcal{P}}, \quad i = 0, 1, .., 2K. \tag{7.7}$$

Note that, for each $i = 0, 1, \dots, n$, Φ_i is unitarily equivalent to

$$\psi_i \oplus (\phi_0 \oplus \phi_1 \oplus \cdots \oplus \phi_{i-1} \oplus \psi_{i+1} \oplus \psi_{i+2} \oplus \cdots \oplus \psi_{2K}),$$

and Φ_{i+1} is unitarily equivalent to

$$\phi_i \oplus (\phi_0 \oplus \phi_1 \oplus \cdots \oplus \phi_{i-1} \oplus \psi_{i+1} \oplus \psi_{i+2} \oplus \cdots \oplus \psi_{2K}).$$

Using (7.3), on applying 5.7 of [17], we obtain that maps ϕ_i and ψ_i are T-$\mathcal{H}_1$-full in e_iQe_i, $i = 0, 1, 2, \dots, 2K$.

In view of this, and (7.7), applying Theorem 3.14 (and its remarks), we obtain unitaries $u_i \in Q$, $i = 0, 1, \dots, 2K$, such that

$$\|\tilde{\Phi}_{i+1}(a) - \tilde{\Phi}_i(a)\| < \varepsilon/4, \quad a \in \mathcal{F}_1, \quad \text{where} \tag{7.8}$$

$$\tilde{\Phi}_0 := \Phi_0 = \psi \quad \text{and} \quad \tilde{\Phi}_{i+1} := \mathrm{Ad}\, u_i \circ \cdots \circ \mathrm{Ad}\, u_1 \circ \mathrm{Ad}\, u_0 \circ \Phi_{i+1}, \quad i = 0, 1, \dots, 2K.$$

Put $t_i = i/(2K+1)$, $i = 0, 1, \dots, 2K+1$, and define $L : A \to \mathrm{C}([0,1], Q)$ by

$$\pi_t \circ L = (2K+1)(t_{i+1} - t)\tilde{\Phi}_i + (2K+1)(t - t_i)\tilde{\Phi}_{i+1}, \quad t \in [t_i, t_{i+1}], \ i = 0, 1, \dots, 2K.$$

By construction,

$$\pi_0 \circ L = \tilde{\Phi}_0 = \psi \quad \text{and} \quad \pi_1 \circ L = \tilde{\Phi}_{n+1} = \mathrm{Ad}\, u_n \circ \cdots \circ \mathrm{Ad}\, u_1 \circ \mathrm{Ad}\, u_0 \circ \phi. \tag{7.9}$$

Since $\tilde{\Phi}_i$, $i = 0, 1, \dots, 2K$, are $\mathcal{G}$-δ-multiplicative (in particular $\mathcal{F}$-$\varepsilon/4$-multiplicative), it follows from (7.8) that L is $\mathcal{F}$-ε-multiplicative. By (7.9), L satisfies (7.4) with $u = u_{2K} \cdots u_1 u_0$.

Moreover, if there is a finite set $\mathcal{H}$ such that (7.5) holds, it is then also straightforward to verify that L satisfies (7.6), as desired. □

Remark 7.3. If A is KK-contractible, then the assumption that A satisfies the UCT can of course be dropped.

Theorem 7.4. *Let A be a non-unital simple separable amenable C^*-algebra with $\mathrm{K}_0(A) = \mathrm{Tor}(\mathrm{K}_0(A))$ which satisfies the UCT. Suppose that $A = \mathrm{Ped}(A)$. Then every trace in $\overline{\mathrm{T}(A)}^{\mathrm{w}}$ is a $\mathcal{W}$-trace.*

Proof. It suffices to show that every tracial state of A is a $\mathcal{W}$-trace. It follows from [50] that every trace is quasidiagonal. For a fixed $\tau \in \mathrm{T}(A)$, there exists a sequence of approximately multiplicative completely positive contractive maps (ϕ_n) from A into Q such that

$$\lim_{n\to\infty} \mathrm{tr} \circ \phi_n(a) = \tau(a) \ \text{for all} \ a \in A.$$

By Lemma 7.1, we may assume that $\phi_n = \psi_n \circ \imath$, where $\imath : A \to A \otimes Q$ is the embedding defined by $\imath(a) = a \otimes 1_Q$ for all $a \in A$ and $\psi_n : A \otimes Q \to Q$ is a sequence of approximate multiplicative completely positive contractive maps.

Therefore it suffices to show that every tracial state of $A \otimes Q$ is a $\mathcal{W}$-trace. Set $A_1 = A \otimes Q$. Then $\mathrm{K}_0(A_1) = \{0\}$.

Fix $1 > \varepsilon > 0$, $1 > \varepsilon' > 0$, a finite subset $\mathcal{F} \subseteq A_1$ and a finite $\mathcal{H} \subseteq A_1$. Put $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{H}$. Without loss of generality, we may assume that $\mathcal{F}_1 \subseteq A_1^{\mathbf{1}}$. Note that A is non-unital. Choose a strictly positive element $a \in A_+$ with $\|a\| = 1$. We may also assume that

$$\tau(f_{1/2}(a)) \ge d > 0 \ \text{for all} \ \tau \in \overline{\mathrm{T}(A)}^{\mathrm{w}} \ (\text{for some} \ d > 0).$$

Let $1 > \delta > 0$, $\mathcal{G} \subseteq A_1$ be a finite subset as provided by 7.2 for A_1 (in place of A), $d/2$ (in place of d), $\varepsilon/16$ (in place of ε), and $\mathcal{F}_1$. (Note since $\mathrm{K}_0(A_1) = \{0\}$, the required set $\mathcal{P}$ in 7.2 does not appear here.)

Let $\mathcal{G}_1 = \mathcal{G} \cup \mathcal{F}_1$ and let $\varepsilon_1 = \varepsilon \cdot \varepsilon' \cdot \delta/2$. Let $\tau \in \mathrm{T}(A_1)$. Since τ is quasidiagonal, there exists a $\mathcal{G}_1$-ε_1-multiplicative completely positive contractive map $\psi : A_1 \to Q$ such that

$$|\tau(b) - \mathrm{tr} \circ \psi(b)| < \varepsilon'/16 \text{ for all } b \in \mathcal{G} \cup \mathcal{F}, \text{ and} \tag{7.10}$$

$$\mathrm{tr}(f_{1/2}(\psi(a))) > 2d/3. \tag{7.11}$$

Choose an integer $m \geq 3$ such that

$$1/m < \min\{\varepsilon_1/64, d/8\}.$$

Let $e_1, e_2, \ldots, e_{m+1} \in Q$ be a set of mutually orthogonal and mutually equivalent projections such that

$$\sum_{i=1}^{m+1} e_i = 1_Q \quad \text{and} \quad \mathrm{tr}(e_i) = \frac{1}{m+1}, \quad i = 1, 2, \ldots, m+1.$$

Let $\psi_i : A_1 \to (1 \otimes e_i)(Q \otimes Q)(1 \otimes e_i)$ be defined by $\psi_i(b) = \psi(b) \otimes e_i$, $i = 1, 2, \ldots, m+1$. Set

$$\sum_{i=1}^{m} \psi_i = \Psi_0 \quad \text{and} \quad \sum_{i=1}^{m+1} \psi_i = \Psi_1.$$

Identify $Q \otimes Q$ with Q. Note that

$$\mathrm{tr}(f_{1/2}(\Psi_i(a))) \geq d/2, \quad i = 0, 1. \tag{7.12}$$

Moreover,

$$|\tau \circ \Psi_0(b) - \tau \circ \Psi_1(b)| < \frac{1}{m+1} < \min\{\varepsilon_1/64, d/8\} \text{ for all } b \in A_1.$$

Again, keep in mind that $K_0(A_1) = \{0\}$. Applying 7.2, we obtain a unitary $u \in Q$ and a $\mathcal{F}_1$-$\varepsilon/16$-multiplicative completely positive contractive map $L : A \to \mathrm{C}([0, 3/4], Q)$ such that

$$\pi_0 \circ L = \Psi_0, \quad \pi_{3/4} \circ L = \mathrm{Ad}u \circ \Psi_1. \tag{7.13}$$

Moreover,

$$|\mathrm{tr} \circ \pi_t \circ L(h) - \mathrm{tr} \circ \pi_0 \circ L(h)| < 1/m < \varepsilon'/64, \quad h \in \mathcal{F}_1, \ t \in [0, 3/4]. \tag{7.14}$$

Here, $\pi_t : \mathrm{C}([0, 3/4], Q) \to Q$ is the point evaluation at $t \in [0, 3/4]$. There is a continuous path of unitaries $\{u(t) : t \in [3/4, 1]\}$ such that $u(3/4) = u$ and $u(1) = 1_Q$. Define $L_1 : A_1 \to \mathrm{C}([0, 1], Q)$ by $\pi_t \circ L_1 = \pi_t \circ L$ for $t \in [0, 3/4]$ and $\pi_t \circ L_1 = \mathrm{Ad}u_t \circ \Psi_1$ for $t \in (3/4, 1]$. L_1 is a $\mathcal{F}_1$-$\varepsilon/16$-multiplicative completely positive contractive map from A_1 into $\mathrm{C}([0, 1], Q)$. Note now

$$\pi_0 \circ L_1 = \Psi_0 \quad \text{and} \quad \pi_1 \circ L_1 = \Psi_1 \text{ and} \tag{7.15}$$

$$|tr \circ \pi_t \circ L(h) - \mathrm{tr} \circ \Psi_1(h)| < \varepsilon'/64 \text{ for all } h \in \mathcal{H}. \tag{7.16}$$

Fix an integer $k \geq 2$. Let $\kappa_i : \mathrm{M}_k \to \mathrm{M}_{k(m+1)}$ $(i = 0, 1)$ be defined by

$$\kappa_0(c) = (\overbrace{c \oplus c \oplus \cdots \oplus c}^{m} \oplus 0), \quad \text{and} \quad \kappa_1(c) = (\overbrace{c \oplus c \oplus \cdots \oplus c}^{m+1}) \tag{7.17}$$

for all $c \in \mathrm{M}_k$. Define

$$C_0 = \{(f, c) : \mathrm{C}([0, 1], \mathrm{M}_{k(m+1)}) \oplus \mathrm{M}_k : f(0) = \kappa_0(c) \text{ and } f(1) = \kappa_1(c)\}.$$

and set

$$C_0 \otimes Q = C_1.$$

Note that $C_0 \in \mathcal{C}_0^0$ and C_1 is an inductive limit of Razak algebras $C_0 \otimes \mathrm{M}_{n!}$. Moreover $\mathrm{K}_0(C_1) = \mathrm{K}_1(C_1) = \{0\}$. Put $p_0 = \sum_{i=1}^m 1_Q \otimes e_i$. Define $\bar{\kappa}_0 : Q \to p_0(Q \otimes Q)p_0$ to be the unital homomorphism defined by $\bar{\kappa}_0(a) = a \otimes \sum_{i=1}^m e_i$ and $\bar{\kappa}_1(a) = a \otimes 1_Q$ for all $a \in Q$.

Then one may write

$$C_1 = \{(f, c) \in \mathrm{C}([0, 1], Q) \oplus Q : f(0) = \bar{\kappa}_0(c) \text{ and } f(1) = \bar{\kappa}_1(c)\}.$$

Note that $\bar{\kappa}_0 \circ \psi(b) = \Psi_0(b)$ for all $b \in A_1$ and $\bar{\kappa}_1 \circ \psi(b) = \Psi_1(b)$ for all $b \in A_1$. Thus one can define $\Phi' : A_1 \to C_1$ by $\Phi'(b) = (L_1(b), \psi(b))$ for all $b \in A_1$.

Then Φ' is a $\mathcal{F}_1$-$\varepsilon/16$-multiplicative completely positive contractive map such that

$$|\mathrm{tr}(\pi_t \circ \Phi'(h)) - \mathrm{tr} \circ \psi(h)| < \varepsilon'/4 \text{ for all } h \in \mathcal{H}. \tag{7.18}$$

Let μ denote Lebesgue measure on [0, 1]. There is a homomorphism $\Gamma : \mathrm{Cu}^{\sim}(C_1) \to \mathrm{Cu}^{\sim}(\mathcal{W})$ (see [43]) such that $\Gamma(f)(\tau_{\mathcal{W}}) = (\mu \otimes \mathrm{tr})(f)$ for all $f \in \mathrm{Aff}(\mathrm{T}(C_1))$, where $\tau_{\mathcal{W}}$ is the unique tracial state of $\mathcal{W}$. By [43], there exists a homomorphism $\lambda : C_1 \to \mathcal{W}$ such that

$$\tau_{\mathcal{W}} \circ \lambda((f, c)) = \int_0^1 \mathrm{tr}(f(t))dt \text{ for all } (f, c) \in C_1. \tag{7.19}$$

Finally, let $\Phi = \lambda \circ \Phi'$. Then Φ is a $\mathcal{F}_1$-ε-multiplicative completely positive contractive map from A_1 into $\mathcal{W}$. Moreover, one computes that

$$|\tau_{\mathcal{W}} \circ \Phi(h) - \tau(h)| < \varepsilon' \text{ for all } h \in \mathcal{H}, \tag{7.20}$$

as desired. □

Theorem 7.5. *Let A and B be non-unital separable simple (finite) C*-algebras with finite nuclear dimension and with non-zero traces. Suppose that both A and B are KK-contractible. Then $A \cong B$ if and only if there is an isomorphism (scale preserving affine homeomorphism) $\Gamma : (\widetilde{\mathrm{T}}(B), \Sigma_B) \cong (\widetilde{\mathrm{T}}(A), \Sigma_A)$.*

Moreover, there is an isomorphism $\phi : A \to B$ such that ϕ induces Γ.

Proof. Let $\Gamma : (\widetilde{\mathrm{T}}(B), \Sigma_B) \to (\widetilde{\mathrm{T}}(A), \Sigma_A)$ be an isomorphism. By 2.9, we may assume that $B \in \mathcal{M}_0$ (an inductive limit of Razak algebras). Recall that A is $\mathcal{Z}$-stable (by [49]). Let $a \in \mathrm{Ped}(A)_+$ with $\|a\| = 1$ such that $A_0 = \overline{aAa}$ has continuous scale (see 5.2 of [17]). Then $\mathrm{T}(A_0)$ is a metrizable Choquet simplex and is a base for the cone $\widetilde{\mathrm{T}}(A)$. Let $b \in B_+$ be such that

$$\mathrm{d}_{\Gamma(\tau)}(b) = d_\tau(a) \text{ for all } \tau \in \widetilde{\mathrm{T}}(A).$$

Set $\overline{bBb} = B_0$. Then Γ gives an affine homeomorphism from $\mathrm{T}(A_0)$ onto $\mathrm{T}(B_0)$. It follows from 7.4 that every tracial state of A_0 or B_0 is a $\mathcal{W}$-trace. By 6.9, A_0 and B_0 are tracially approximately divisible. It follows from 6.5 that $A_0, B_0 \in \mathcal{D}_0$. Then, by 4.3, there is an isomorphism $\phi : A_0 \to B_0$ such that ϕ_T gives $\Gamma|_{T(B_0)}$ and by By [7], this induces an isomorphism $\tilde{\phi} : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$. Fix a strictly positive element $a_0 \in A$ with $\|a_0\| = 1$ such that

$$d_\tau(a_0) = \Sigma_A(\tau), \quad \tau \in \widetilde{\mathrm{T}}(A).$$

Set $\tilde{\phi}(a_0) = b_0$. Then $\tilde{\phi}$ gives an isomorphism from A to $B_1 := \overline{b_0(B \otimes \mathcal{K})b_0}$. Let $b_1 \in B$ be a strictly positive element, so that

$$d_\tau(b_1) = \Sigma_B(\tau), \quad \tau \in \widetilde{\mathrm{T}}(B).$$

Then

$$d_\tau(b_1) = d_\tau(b_0), \quad \tau \in \widetilde{\mathrm{T}}(B).$$

Since B is a separable simple C*-algebra with stable rank one, this implies there exists an isomorphism $\phi_1 : B_1 \to B$ such that $(\phi_1)_{\mathrm{T}} = \mathrm{id}_{\widetilde{\mathrm{T}}(A)}$ (see Theorem 3 of [9]; this also follows from [41] as B is an inductive limit of Razak algebras—see also [43]). Then the composition $\phi_1 \circ \tilde{\phi}|_A$ gives the required isomorphism. □

Corollary 7.6. *Let A, B be simple separable KK-contractible finite C*-algebras with finite nuclear dimension. If there is a homomorphism $\xi : (\widetilde{\mathrm{T}}(B), \Sigma_B) \to (\widetilde{\mathrm{T}}(A), \Sigma_A)$, then there is a C*-algebra homomorphism $\phi : A \to B$ such that $\phi_* = \xi$.*

Proof. In view of 7.5 and 2.9, this follows from the classification of limits of Razak algebras [41]. □

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Appendix

This section, mainly contributed by Huaxin Lin, removes the necessity of assuming that A has stable rank one in Theorem 7.5 and Corollary 7.6.

The main purpose of this appendix is to prove Corollary A.27. The existence of a map as stated in A.27 was proved in [43] under the additional assumption that A has stable rank one. It is needed in the proof of 6.5.

Appendix A.1 of the appendix also contains some results of independent interest. In particular, Corollary A.8, together with the construction of models in [15], establishes the range of the Elliott invariant for Jiang–Su stable separable simple exact C*-algebras.

A.1. Strict comparison in $\widetilde{A}$

Definition A.1. Let B be a C*-algebra with $\mathrm{T}(B) \neq \emptyset$ and let $S \subseteq \mathrm{T}(B)$ be a subset. Suppose that $a \in (B \otimes \mathcal{K})_+$ is such that $d_\tau(a) < +\infty$ for all $\tau \in \mathrm{T}(B)$. Define

$$\omega_S(a) = \inf\{\sup\{d_\tau(a) - \tau(c) : \tau \in S\} : c \in \overline{a(B \otimes \mathcal{K})a},\ \ 0 \leq c \leq 1\}. \tag{A.1}$$

Let us note that, when S is compact, $\omega_S(a) = 0$ if and only if $d_\tau(a)$ is continuous on S. Also note that if $a, b \in (B \otimes \mathcal{K})_+$, $0 \leq a, b \leq 1$ and $a \lesssim b$, then there exists a sequence $x_n \in B \otimes \mathcal{K}$ such that $x_n^* x_n \to a$ and $x_n x_n^* \in \overline{b(B \otimes \mathcal{K})b}$. It follows, for any $1 > \delta > 0$, $f_\delta(x_n^* x_n) \to f_\delta(a)$. Note that $\tau(f_\delta(x_n^* x_n)) = \tau(f_\delta(x_n x_n^*))$ for any $\tau \in \mathrm{T}(B)$. We conclude that, if $a \lesssim b$, there exists a sequence $\{c_k\}$ in $\overline{b(B \otimes \mathcal{K})b}_+$ with $0 \leq c_k \leq 1$ such that

$$\lim_{k \to \infty} \sup\{|\tau(c_k) - \tau(f_{1/k}(a))| : \tau \in \mathrm{T}(B)\} = 0.$$

Consequently, if we further assume $d_\tau(a) = d_\tau(b)$, then $\omega_S(a) \geq \omega_S(b)$. Note that, if $a \sim b$, then $d_\tau(a) = d_\tau(b)$. Hence, when $a \sim b$, we have $\omega_S(a) = \omega_S(b)$.

Now let A be a C*-algebra with $\mathrm{T}(A) \neq \emptyset$ and with compact $\mathrm{T}(A)$. Let $a \in (\widetilde{A} \otimes \mathcal{K})_+$ be such that $d_\tau(a) < +\infty$ for all $\tau \in \mathrm{T}(A)$. We will write $\omega(a)$ for $\omega_{\mathrm{T}(A)}(a)$, namely,

$$\omega(a) = \inf\{\sup\{d_\tau(a) - \tau(c) : \tau \in \mathrm{T}(A)\} : c \in \overline{a(\widetilde{A} \otimes \mathcal{K})a},\ 0 \leq c \leq 1\}. \tag{A.2}$$

As mentioned above, if $b \in (\widetilde{A} \otimes \mathcal{K})_+$, $0 \leq b \leq 1$ and $a \sim b$, in $\widetilde{A} \otimes \mathcal{K}$, then $\omega(a) = \omega(b)$.

Lemma A.2. *Let A be a separable stably projectionless simple C*-algebra such that $\mathrm{M}_r(A)$ almost has stable rank one for every integer $r \geq 1$ and $\mathrm{QT}(A) = \mathrm{T}(A)$ which has strict comparison for positive elements and has continuous scale. Suppose that $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$. Suppose also that $a \in \mathrm{M}_r(\widetilde{A})$ with $0 \leq a \leq 1$ for some integer $r \geq 1$ and $0 < \langle \pi(a) \rangle$, where $\pi : \widetilde{A} \to \mathbb{C}$ is the quotient map. Suppose further that*

$$\inf\{d_\tau(a) : \tau \in \mathrm{T}(A)\} > 4\omega(a). \tag{A.3}$$

Then, for any $d > 2\omega(a)$ and $\omega(a)/2 > \varepsilon_0 > 0$, there is $b \in \mathrm{M}_r(A)_+$ with $b \leq a$ such that

$$2\omega(a) < d_\tau(b) < d \ \text{ for all } \ \tau \in \mathrm{T}(A) \tag{A.4}$$

and, for any $0 < \varepsilon < \inf\{d_\tau(b) : \tau \in \mathrm{T}(A)\}$, there is also $a_1 \in \mathrm{M}_r(\widetilde{A})_+$ such that

$$\pi(a_1) = \pi(a'), \quad b \oplus a_1 \leq a',$$

with $\langle a' \rangle = \langle a \rangle$, $d_\tau(a_1) > \underline{d_\tau(a) - d}$ for all $\tau \in \mathrm{T}(A)$, and a_1 also has the following property: if $\{c_n\} \in \mathrm{M}_r(\widetilde{A})_+$ is an increasing sequence such that $c_n \in a_1(\widetilde{A} \otimes \mathcal{K})a_1$ and $\tau(c_n) \nearrow d_\tau(a_1)$, then, for some $n_0 \geq 1$,

$$d_\tau(a_1) - \tau(c_n) < \omega(a) + \varepsilon_0 + \varepsilon \ \text{ for all } \ \tau \in \mathrm{T}(A) \ \text{ and for all } \ n \geq n_0. \tag{A.5}$$

Proof. We first consider the case that $\langle a \rangle$ is not represented by a projection. There exists an invertible matrix $y \in \mathrm{M}_r(\mathbb{C})_+$ such that $y^{1/2}\pi(a)y^{1/2} = p$ is a projection. Let $Y \in \mathrm{M}_r(\mathbb{C} \cdot 1_{\widetilde{A}})_+$ be the same invertible scalar matrix. Then $\pi(Y^{1/2}aY^{1/2}) = p$. It is clear that $\langle a \rangle = \langle Y^{1/2}aY^{1/2} \rangle$ and we may replace a by $Y^{1/2}aY^{1/2}$. So we assume that $\pi(a) = p$.

Choose $\eta_0 > 0$ such that, for $0 < \eta < \eta_0$,

$$d_\tau(a) - \tau(f_\eta(a)) < \omega(a) + \varepsilon_0 \ \text{ for all } \ \tau \in \mathrm{T}(A). \tag{A.6}$$

Let (e_n) be an approximate identity for $\overline{a\mathrm{M}_r(A)a}$ such that $e_{n+1}e_n = e_n$, $n = 1, 2, \ldots$.

There exists $n_0 \geq 1$ such that

$$d_\tau(a) - \tau(e_n) < \omega(a) + \varepsilon_0 \ \text{ for all } \ \tau \in \mathrm{T}(A) \ \text{ and } \ \text{ for all } \ n \geq n_0. \tag{A.7}$$

By a standard compactness argument, for a fixed $n_0 + 1$, there exists $\eta_0 > \eta_1 > 0$ such that

$$\tau(f_{\eta_1}(a)) > \tau(e_{n_0+3}) \ \text{ for all } \ \tau \in \mathrm{T}(A). \tag{A.8}$$

Let $(e_{1,n})$ be an approximate identity for $\overline{f_{\eta_1}(a)\mathrm{M}_r(A)f_{\eta_1}(a)}$ with $e_{1,n}e_{1,n+1} = e_{1,n}$, $n = 1, 2, \ldots$. By the same compactness argument, we have $e_{1,n}$ for some n such that

$$\tau(e_{1,n}) > \tau(e_{n_0+3}) \text{ for all } \tau \in \mathrm{T}(A). \tag{A.9}$$

It follows that $e_{n_0+2} \lesssim e_{1,n+1}$. Since A has almost stable rank one, one has

$$w^* e_{n_0+2} w \le f_{\eta_1/2}(a) \tag{A.10}$$

for some unitary $w \in \mathrm{M}_r(A)^\sim$ (see the last part of Lemma 3.2 of [17]). Choose a strictly positive function $g_{1,\eta_1} \in C_0((0, 1])_+$ such that $g_{1,\eta_1}(t) = 1$, if $t \ge \eta_1/4$, and $g_{1,\eta_1}(t)$ is linear on $[0, \eta_1/4)$. In particular, $f_{\eta_1/2}g_{1,\eta_1} = f_{\eta_1/2}$. Put $a' = g_{1,\eta_1}(a)$. Note $\pi(a') = \pi(a)$ and $\langle a'\rangle = \langle a\rangle$. Note also that

$$0 \le w^* e_{n_0} w \le w^* e_{n_0+1} w \le w^* e_{n_0+2} w \le f_{\eta_1/2}(a) \le a' \text{ and} \tag{A.11}$$

$$d_\tau(a') - \tau(w^* e_{n_0} w) < \omega(a) + \varepsilon_0 \text{ for all } \tau \in \mathrm{T}(A). \tag{A.12}$$

In particular, $d_\tau(w^* e_{n_0} w) > 5\omega/2$ for all $\tau \in \mathrm{T}(A)$. There exists $b_0 \in \mathrm{M}_r(A)_+$ with

$$d_\tau(b_0) = 2\omega(a) + \min\{(3/4)(d - 2\omega(a)), (3/4)(\tau(w^* e_{n_0} w) - 2\omega(a))\}$$

for all $\tau \in \mathrm{T}(A)$. Note that $d_\tau(b_0) \in \mathrm{Aff}(\mathrm{T}(A))$. We have

$$d_\tau(b_0) < d_\tau(w^* e_{n_0} w) \text{ for all } \tau \in \mathrm{T}(A). \tag{A.13}$$

Since $\mathrm{M}_r(A)$ almost has stable rank one, by 3.2 of [17], one concludes that there exists $b' \in \overline{w^* e_{n_0} w \mathrm{M}_r(A) w^* e_{n_0} w}$ such that $d_\tau(b') = d_\tau(b_0)$ for all $\tau \in \mathrm{T}(A)$. Note that $b'a' = b'$. Let $\varepsilon > 0$. Since $d_\tau(b_0)$ is continuous on $\mathrm{T}(A)$ and $\mathrm{T}(A)$ is compact, there exists $\delta_0 > 0$ such that

$$\tau(f_\delta(b_0)) > d_\tau(b_0) - \min\{(d - 2\omega(a))/2, \varepsilon/4\} \text{ for all } \tau \in \mathrm{T}(A) \tag{A.14}$$

and $0 < \delta \le \delta_0$.

Put $b = f_{\delta_0}(b')$, $b_1 = f_{\delta/2}(b')$ and $b_2 = f_{\delta/4}(b')$. Note that $b \le b_1 \le b_2 \le w^* e_{n_0+1} w$. Note also that

$$2\omega(a) < d_\tau(b) < d \text{ and } 0 < d_\tau(b_2) - \tau(b) < \varepsilon/4 \text{ for all } \tau \in \mathrm{T}(A). \tag{A.15}$$

So (A.4) holds. Put $a_1 = a' - b_1$. Note that $a'b = b$. So $a_1 \oplus b \lesssim a_1$. Since $\pi(a_1) = \pi(a')$, $\langle \pi(a_1)\rangle = \langle \pi(a)\rangle$. Let p_a be the open projection corresponding to a, p_{a_1} the open projection corresponding to a_1 and $p_{b'}$ be the open projection corresponding to b' in $\mathrm{M}_r(\widetilde{A})^{**}$. Note that p_a is the same as the open projection corresponding to a'. Then $p_a \ge p_{a_1} \ge p_a - p_{b'}$,

$$d_\tau(a_1) = \tau(p_{a_1}) \ge \tau(p_{a'} - p_{b'}) = \tau(p_{a'}) - \tau(p_{b'}) \tag{A.16}$$

$$= \tau(p_{a'}) - d_\tau(b') > d_\tau(a) - d \text{ and} \tag{A.17}$$

$$d_\tau(a_1) = \tau(p_{a_1}) < \tau(p_a - b) < \tau(p_a - p_{b'}) + \varepsilon/4 \tag{A.18}$$

$$< \tau(p_a) - \tau(p_{b'}) + \varepsilon/4 = d_\tau(a) - d_\tau(b') + \varepsilon/4 \text{ for all } \tau \in \mathrm{T}(A). \tag{A.19}$$

If c_n is as stated, then $\tau(c_n) \nearrow \tau(p_{a_1})$. Therefore, on $\mathrm{T}(A)$, which is compact, by a standard compactness argument, there is $n_1 \ge 1$ such that

$$\tau(w^* e_{n_0+1} w) - d_\tau(b') < \tau(c_n) \le \tau(p_{a_1}) = d_\tau(a_1) \tag{A.20}$$

for all $\tau \in \mathrm{T}(A)$ and for all $n \ge n_1$. It follows from (A.19), (A.12) and (A.20) that

$$d_\tau(a_1) - \tau(c_n) < d_\tau(a') - \tau(c_n) \tag{A.21}$$

$$< ((\tau(w^* e_{n_0+1} w) + \omega(a) + \varepsilon_0) - d_\tau(b') + \varepsilon/4) - (\tau(w^* e_{n_0+1} w) - d_\tau(b')) \tag{A.22}$$

$$= \omega(a) + \varepsilon_0 + \varepsilon/4 \text{ for all } \tau \in \mathrm{T}(A). \tag{A.23}$$

Now we consider the case that $\langle a\rangle$ is represented by a projection $p \in M_m(\widetilde{A})$. We may write $p = a_1 + b_1$, where $a_1 \in M_m(A)$ and $b_1 \in M_m(\mathbb{C} \cdot 1_{\widetilde{A}})$ is a scalar matrix. In particular, $d_\tau(p)$ is continuous on $\mathrm{T}(A)$. Therefore $\omega(p) = 0$. Let $d > 0$. We may assume that $d < 1/2$. Since $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$, choose an element $b_0 \le A$ such that $d_\tau(b_0) = d/4$ for all $\tau \in \mathrm{T}(A)$. Note that $pb_0 = b_0$ and $d_\tau(b_0)$ is continuous. Now with $\omega(p) = 0$, with $a = p$, and with this new b_0, the rest of the proof above (beginning with b_0 as constructed) applies. □

Lemma A.3. *Let A be a separable stably projectionless simple C^*-algebra such that $\mathrm{M}_n(A)$ has almost stable rank one for all integers $n \ge 1$ and $\mathrm{QT}(A) = \mathrm{T}(A)$ which has strict comparison for positive elements and has continuous scale. Suppose also that $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$. Let $a, b \in \mathrm{M}_r(\widetilde{A})_+$. Suppose that $\langle \pi(a)\rangle \le \langle \pi(b)\rangle (< \infty)$, where $\pi : \widetilde{A} \to \mathbb{C}$ is the quotient map, and*

$$d_\tau(a) + 4\omega(b) < d_\tau(b) \text{ for all } \tau \in \mathrm{T}(A). \tag{A.24}$$

Then $a \lesssim b$.

Proof. If $\langle a\rangle$ is represented by a projection, then $d_\tau(a)$ is continuous. So

$$\inf\{d_\tau(b)-d_\tau(a):\tau\in \mathrm{T}(A)\}>4\omega(b). \tag{A.25}$$

Otherwise, fix $1/2>\eta_1>0$. By applying 7.1 of [17], there exist $\eta_1>\eta_2>0$ and a continuous function $f:\mathrm{T}(A)\to\mathbb{R}^+$ such that

$$d_\tau((a-\eta_1)_+)<f(\tau)<d_\tau((a-\eta_2)_+)<d_\tau(b)\ \text{ for all }\ \tau\in\mathrm{T}(A). \tag{A.26}$$

Then

$$\inf\{d_\tau(b)-d_\tau(a):\tau\in\mathrm{T}(A)\}\ge\inf\{d_\tau(a)-f(\tau):\tau\in\mathrm{T}(A)\}>0. \tag{A.27}$$

Thus, in both cases,

$$d=\inf\{d_\tau(b)-d_\tau(a):\tau\in\mathrm{T}(A)\}>4\omega(b). \tag{A.28}$$

By applying A.2, one obtains non-zero and mutually orthogonal elements $b_0\in \mathrm{M}_r(A)_+$ and $b_1, b'\in \mathrm{M}_r(\widetilde{A})_+$ such that

$$b_0+b_1\le b',\quad \langle b'\rangle=\langle b\rangle, \pi(b_1)=\pi(b'), \tag{A.29}$$

$$2\omega(a)<d_\tau(b_0)<d/2,\ \ d_\tau(b_1)>d_\tau(b)-d/2\ \text{ for all }\ \tau\in\mathrm{T}(A). \tag{A.30}$$

and, for any $c_n\in\mathrm{M}_r(A)_+$ with $c_n\in\overline{b_1(\widetilde{A}\otimes\mathcal{K})b_1}$ and $d_\tau(c_n)\nearrow d_\tau(b_1)$ on $\mathrm{T}(A)$, there exists $n_0\ge1$ such that

$$d_\tau(b_1)-d_\tau(c_n)<\omega(b)+(1/64)\inf\{\tau(b_0):\tau\in\mathrm{T}(A)\}\ \text{ for all }\ \tau\in\mathrm{T}(A). \tag{A.31}$$

Moreover, $\langle\pi(b_1)\rangle=\langle\pi(b)\rangle$. Replacing b by b', without loss of generality, we may assume that $b_0+b_1\le b$.

Put $d_0=\inf\{\tau(b_0):\tau\in\mathrm{T}(A)\}$.

There exists an invertible matrix $y_1\in\mathrm{M}_r(\mathbb{C})_+$ such that $y_1^{1/2}\pi(b_1)y_1^{1/2}=p_1$ is a projection. Let $Y_1\in\mathrm{M}_r(\widetilde{A})$ denote the scalar matrix such that $\pi(Y_1)=y_1$. Note that $\langle Y_1^{1/2}b_1Y^{1/2}\rangle=\langle b_1\rangle$, $Y_1^{1/2}c_nY^{1/2}\le Y_1^{1/2}b_1Y^{1/2}$ and $d_\tau(Y^{1/2}c_nY^{1/2})=d_\tau(c_n)$. So, replacing b_1 by $Y_1^{1/2}b_1Y^{1/2}$, we may assume that $\pi(b_1)=p_1$. Similarly, we may assume that $\pi(a)=p_2$ is also a projection. There is a scalar matrix $U\in\mathrm{M}_r(\widetilde{A})$ such that $\pi(U^*aU)\le p_2$. Without loss of generality, we may assume that $p_2\le p_1$.

We may further assume that there are integers $0\le m_2\le m_1$ such that

$$p_i=\mathrm{diag}(\overbrace{1,1,\ldots,1}^{m_i},0,\ldots,0),\quad i=1,2. \tag{A.32}$$

Let $P_i=\mathrm{diag}(\overbrace{1_{\widetilde{A}},1_{\widetilde{A}},\ldots,1_{\widetilde{A}}}^{m_i},0,\ldots,0)\in\mathrm{M}_r(\widetilde{A})$ so that $\pi(P_i)=p_i$, $i=1,2$.

Note $(b_1-1/n)_+\le b_1$ and $d_\tau((b_1-1/n)_+)\nearrow d_\tau(b_1)$, so by (A.31), for some $\delta_1>0$,

$$d_\tau(b_1)-d_\tau(f_\delta(b_1))<\omega(b)+d_0/64\ \text{ for all }\ \tau\in\mathrm{T}(A) \tag{A.33}$$

and all $0<\delta<\delta_1$.

Let (e_n) be an approximate identity for A such that $e_ne_{n+1}=e_{n+1}e_n=e_n$, $n=1,2,\ldots$. Put

$$E_n=\mathrm{diag}(e_n,e_n,\ldots,e_n)\in\mathrm{M}_r(A),\quad n=1,2,\ldots. \tag{A.34}$$

Then (E_n) is an approximate identity for $\mathrm{M}_r(A)$ and $P_iE_n=E_nP_i$, $i=1,2$, and $n=1,2,\ldots$.

We have $b_1^{1/2}E_n^2b_1^{1/2}\nearrow b_1$ (in the strict topology). Let $c_n=E_nb_1E_n$, $n=1,2,\ldots$. It follows that $d_\tau(c_n)\nearrow d_\tau(b_1)$ on $\mathrm{T}(A)$. By the construction of b_1, there exists $n_0\ge1$ such that

$$d_\tau(b_1)-d_\tau(b_1^{1/2}E_n^2b_1^{1/2})=d_\tau(b_1)-d_\tau(c_n)<\omega(b)+d_0/64 \tag{A.35}$$

for all $\tau\in\mathrm{T}(A)$ and for all $n\ge n_0$.

One then computes, by (A.30) and (A.35), that, for $n\ge n_0$,

$$d_\tau(a)<d_\tau(c_n)\ \text{ for all }\ \tau\in\mathrm{T}(A). \tag{A.36}$$

On the other hand, since $\pi(b_1)=\pi(P_1)$ and $\pi(a)=\pi(P_2)$,

$$\lim_{k\to\infty}\|(E_kb_1E_k+(1-E_k)P_1(1-E_k))-b_1\|=0\ \text{ and} \tag{A.37}$$

$$\lim_{k\to\infty}\|(E_kaE_k+(1-E_k)P_2(1-E_k))-a\|=0. \tag{A.38}$$

Put $x_k=E_kb_1E_k+(1-E_k)P_1(1-E_k)$ and $y_k=E_kaE_k+(1-E_k)P_2(1-E_k)$, $k=1,2,\ldots$. Since

$$\lim_{k\to\infty}\|f_{\delta_1/2}(x_k)-f_{\delta_1/2}(b_1)\|=0, \tag{A.39}$$

we may assume, without loss of generality, for all $k \geq 1$, that

$$\tau(f_{\delta_1/2}(x_k)) \geq \tau(f_{\delta_1/2}(b_1)) - d_0/64 \text{ for all } \tau \in \mathrm{T}(A). \tag{A.40}$$

It follows by (A.33) (with $\delta = \delta_1/2$) that

$$\tau(f_{\delta_1/2}(x_k)) > d_\tau(b_1) - \omega(b) - 3d_0/64 \text{ for all } \tau \in \mathrm{T}(A). \tag{A.41}$$

Since A has continuous scale, there is $k_0 \geq n_0$ such that

$$\tau(1 - e_n) < d_0/64 \text{ for all } \tau \in \mathrm{T}(A) \text{ and for all } n \geq k_0. \tag{A.42}$$

It follows that, for $k \geq k_0$,

$$\tau(f_{\delta_1/2}(x_k)) \leq d_\tau(x_k) \leq d_\tau(c_k) + d_0/64 \tag{A.43}$$
$$= d_\tau(b_1^{1/2}E^2b_1^{1/2}) + d_0/64 \leq d_\tau(b_1) + d_0/64 \text{ for all } \tau \in \mathrm{T}(A). \tag{A.44}$$

Let $g_{\delta_1} \in C_0((0,1])_+$ with $1 \geq g(t) > 0$ for all $t \in (0, \delta_1/4)$, $g_{\delta_1}(t) \geq t$ for $t \in (0, \delta_1/16)$, $g_{\delta_1}(t) = 1$ for $t \in (\delta_1/16, \delta_1/8)$ and $g_{\delta_1}(t) = 0$ if $t \geq \delta_1/4$.

Since $g_{\delta_1}(x_k)f_{\delta_1/2}(x_k) = 0$, by (A.43), we conclude that, for $k \geq k_0$,

$$d_\tau(g_{\delta_1}(x_k)) + \tau(f_{\delta_1/2}(x_k)) \leq d_\tau(x_k) \leq d_\tau(b_1) + d_0/64 \text{ for all } \tau \in \mathrm{T}(A). \tag{A.45}$$

Then, by (A.41),

$$d_\tau(g_{\delta_1}(x_k)) \leq (d_\tau(b_1) - \tau(f_{\delta_1/2}(x_k))) + d_0/64 \tag{A.46}$$
$$\leq \omega(b) + 3d_0/64 + d_0/64 = \omega(b) + d_0/16 \tag{A.47}$$

for all $\tau \in \mathrm{T}(A)$ and for all $k \geq k_0$. Moreover, since $\pi(x_k) = \pi((1-E_n)P_1(1-E_n)) = p_1$ for all n,

$$g_{\delta_1}(x_k) \in \mathrm{M}_r(A). \tag{A.48}$$

It should be noted and will be used later that, for any $0 \leq x \leq 1$,

$$x \leq f_\delta(x) + g_{\delta_1}(x) \text{ for all } 0 < \delta < \delta_1/8. \tag{A.49}$$

Fix an $\eta > 0$. Then there exists $k_1 \geq k_0 + 2$ such that, since $\lim_{k\to\infty} \|y_k - a\| = 0$,

$$(a-\eta)_+ \lesssim y_k = E_kaE_k + (1-E_k)P_2(1-E_k). \tag{A.50}$$

Note that this holds regardless of whether $\langle a \rangle$ is represented by a projection or not. Fix any $n \geq k_0 \geq n_0$,. By (A.36),

$$d_\tau(E_kaE_k) = d_\tau(a^{1/2}E_k^2a^{1/2}) \leq d_\tau(a) < d_\tau(c_n) \text{ for all } \tau \in \mathrm{T}(A) \tag{A.51}$$

and for any k. Since A has strict comparison,

$$E_kaE_k \lesssim c_n \tag{A.52}$$

for any $n \geq k_0$ and any k. Choose $k \geq \max\{k_1, n\} + 2$. In particular, E_n and $(1-E_k)$ are mutually orthogonal. Then

$$(a-\eta)_+ \lesssim y_k \lesssim E_kaE_k + (1-E_k)P_2(1-E_k) \tag{A.53}$$
$$\lesssim c_n + (1-E_k)P_2(1-E_k) \leq c_n + (1-E_k)P_1(1-E_k) \tag{A.54}$$
$$= c_n + P_1(1-E_k)^2P_1 \leq c_n + P_1(1-E_n)^2P_1 \tag{A.55}$$
$$= c_n + (1-E_n)P_1(1-E_n) = x_n. \tag{A.56}$$

In other words,

$$(a-\eta)_+ \leq x_n \text{ for all } n \geq k_0. \tag{A.57}$$

Choose $n \geq k_0$ such that (note that, by (A.37), $x_n \to b_1$ as $n \to \infty$) $f_{\delta_1/2}(x_n) \lesssim b_1$. By (A.49),

$$\langle x_n \rangle \leq \langle f_{\delta_1/2}(x_n) + g_{\delta_1}(x_n) \rangle \leq \langle f_{\delta_1/2}(x_n) \rangle + \langle g_{\delta_1}(x_n) \rangle \tag{A.58}$$
$$\leq \langle b_1 \rangle + \langle g_{\delta_1}(x_n) \rangle. \tag{A.59}$$

By (A.47) and (A.30) and the strict comparison of A,

$$\langle g_{\delta_1}(x_n) \rangle \leq b_0. \tag{A.60}$$

Combining (A.57), (A.58) and (A.60)

$$\langle (a-\eta)_+ \rangle \leq \langle b_1 \rangle + \langle b_0 \rangle = \langle b_1 + b_0 \rangle \leq \langle b \rangle. \tag{A.61}$$

Since this holds for any $\eta > 0$, we conclude that

$$a \lesssim b. \quad \square \tag{A.62}$$

Corollary A.4. *Let A and a be as in A.3. Suppose that $b \in \mathrm{M}_r(\widetilde{A})$ is such that $d_\tau(b)$ is continuous on $\mathrm{T}(A)$ and suppose that $\langle \pi(a) \rangle \le \langle \pi(b) \rangle$ and*

$$d_\tau(a) < d_\tau(b) \ \ for \ \ all \ \ \tau \in \mathrm{T}(A). \tag{A.63}$$

Then $a \lesssim b$.

Definition A.5. Let A be a separable simple stably projectionless C*-algebra with continuous scale and with strict comparison. Suppose also that $M_m(A)$ has almost stable rank one for all $m \ge 1$, $\mathrm{QT}(A) = \mathrm{T}(A)$ and $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$. In what follows we will continue to denote by π the quotient map from $\widetilde{A} \to \mathbb{C}$ and its extension from $M_m(\widetilde{A}) \to M_m$ for all $m \ge 1$, as well as from $\widetilde{A} \otimes \mathcal{K} \to \mathcal{K}$. Let $S(\widetilde{A})$ be the sub-semigroup of $\mathrm{Cu}(\widetilde{A})$ generated by $\langle a \rangle \in \mathrm{Cu}(A)$ and those $x \in \mathrm{Cu}(\widetilde{A})$ which is equal to the supremum of an increasing sequence $(\langle a_n \rangle)$, where $d_\tau(a_n) \in \mathrm{Aff}_+(\mathrm{T}(A))$ and $\langle \pi(a_n) \rangle < +\infty$, and $\langle x \rangle$ is not represented by a projection. If $\langle a \rangle \in \mathrm{Cu}(\widetilde{A})$, we will write $\langle a \rangle\hat{}$ for the function $d_\tau(a)$ on $\mathrm{T}(A)$.

For each $\langle a \rangle \in S(\widetilde{A})$, note that $\langle \pi(a) \rangle = j(a)$ is either an integer or ∞. Let

$$L(\widetilde{A}) = \{(f, n) : f \in \mathrm{LAff}_+(\mathrm{T}(A)), \ \ n \in \mathbb{N} \cup \{0\} \cup \{\infty\}\}.$$

We also define $(f, n) \le (g, m)$ if $f \le g$ and $n \le m$. Define $\Gamma_0(\langle a \rangle) : S(\widetilde{A}) \to L(\widetilde{A})$ by $\Gamma_0(\langle a \rangle) = (\langle a \rangle\hat{}, j(a))$.

For any C*-algebra B, as a tradition, we use $V(B)$ for the semigroup of Murray–von Neumann equivalence classes of projections in $B \otimes \mathcal{K}$.

Theorem A.6. *Let A be a stably projectionless simple C*-algebra such that $\mathrm{M}_r(A)$ has almost stable rank one for all $r \ge 1$, $\mathrm{QT}(A) = \mathrm{T}(A)$, A has continuous scale, and $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$. Then $\Gamma_0 : S(\widetilde{A}) \to L(\widetilde{A})$ is an ordered semigroup isomorphism.*

For any $x = \langle a \rangle, y = \langle b \rangle \in V(\widetilde{A}) \sqcup S(\widetilde{A})$, if $\hat{x} < \hat{y}$ for all $\tau \in \mathrm{T}(A)$ and $\langle \pi(a) \rangle \le \langle \pi(b) \rangle$, then $x \le y$. Moreover, if x is not represented by a projection, then $\hat{x} \le \hat{y}$ and $\langle \pi(a) \rangle \le \langle \pi(b) \rangle$ imply that $x \le y$.

Furthermore, if $\langle a \rangle\hat{} \le \langle b \rangle\hat{}$ and $\langle \pi(a) \rangle \le \langle \pi(b) \rangle$, and if $\langle b \rangle \in S(\widetilde{A})$ and $\langle a \rangle \in \mathrm{Cu}(\widetilde{A})$ is any element which is not represented by a projection, then $x \le y$.

Proof. We will leave the additive part to the reader. We first note that $\Gamma_0|_{\mathrm{Cu}(A)}$ is an ordered semigroup isomorphism to $\{(f, 0) : f \in \mathrm{LAff}_+(\mathrm{T}(A))\}$ (Note that we also use the fact that A is stably projectionless). It is then also clear that Γ_0 is order preserving.

Claim 1. *If $\langle a \rangle \in \mathrm{Cu}(\widetilde{A})$, $\langle b \rangle \in S(\widetilde{A})$ and $\langle b \rangle\hat{} \in \mathrm{Aff}_+(\mathrm{T}(A))$ (i.e., $\langle b \rangle\hat{}$ is continuous), and if $\Gamma_0(\langle a \rangle) \le \Gamma_0(\langle b \rangle)$, then $\langle a \rangle \le \langle b \rangle$, provided that $\langle a \rangle$ is not represented by a projection.*

If $\langle a \rangle \in \mathrm{Cu}(\widetilde{A})$, then, since A is stably projectionless, for any $\varepsilon > 0$,

$$\langle (a - \varepsilon)_+ \rangle\hat{} < \langle b \rangle\hat{}. \tag{A.64}$$

Note that $\langle \pi(b) \rangle < \infty$. Note also $\pi((a - \varepsilon)_+) \le \pi(a)$. It follows from A.3 (and A.4) that

$$(a - \varepsilon)_+ \lesssim b. \tag{A.65}$$

Therefore $a \lesssim b$. This proves Claim 1.

Claim 2. *If $\langle a \rangle, \langle b \rangle \in S(\widetilde{A})$, $\Gamma_0(\langle a \rangle) = \Gamma_0(\langle b \rangle)$ and $\langle b \rangle\hat{} \in \mathrm{Aff}_+(\mathrm{T}(A))$, Then $\langle a \rangle = \langle b \rangle$.*

Note that $\langle a \rangle\hat{} = \langle b \rangle\hat{}$ (so both continuous). If $j(a) = j(b) = 0$, then this follows from the fact that $\Gamma_0|_{\mathrm{Cu}(A)}$ is an isomorphism. So we assume $j(a) = j(b) \neq 0$. By Claim 1, $\langle a \rangle \le \langle b \rangle \le \langle a \rangle$. So $\langle a \rangle = \langle b \rangle$.

Now assume that $a \in (\widetilde{A} \otimes \mathcal{K})_+$, $\langle a \rangle$ is not represented by a projection, $\langle b \rangle \in S(\widetilde{A})$ and

$$\langle a \rangle\hat{} \le \langle b \rangle\hat{} \text{ and } \langle \pi(a) \rangle \le \langle \pi(b) \rangle. \tag{A.66}$$

Write $\langle b_n \rangle \le \langle b_{n+1} \rangle$ and $b = \sup\{\langle b_n \rangle\}$, where $\langle b_n \rangle\hat{}$ are continuous and $\langle \pi(b_n) \rangle < \infty$. Then

$$\langle (a - \varepsilon)_+ \rangle\hat{} \le \langle b \rangle\hat{} \text{ and } \langle \pi((a - \varepsilon)_+) \rangle < \infty \tag{A.67}$$

(for any $\varepsilon > 0$). Since $\langle a \rangle$ is not represented by projections, for any sufficiently small $\varepsilon > 0$,

$$\langle (a - \varepsilon)_+ \rangle\hat{} < \langle (a - \varepsilon/2)_+ \rangle\hat{} < \langle b \rangle\hat{}. \tag{A.68}$$

On the compact set $\mathrm{T}(A)$, one finds an integer $k \ge 1$ such that

$$\langle (a - \varepsilon)_+ \rangle\hat{} < \langle b_k \rangle\hat{} \text{ and } \langle \pi((a - \varepsilon)_+) \rangle \le \langle \pi(b_k) \rangle. \tag{A.69}$$

It then follows from A.3 that

$$\langle (a - \varepsilon)_+ \rangle < \langle b_k \rangle \le \langle b \rangle. \tag{A.70}$$

Therefore

$$\langle a\rangle \le \langle b\rangle. \tag{A.71}$$

This also implies that if $\Gamma_0(a)=\Gamma_0(b)$ then $\langle a\rangle=\langle b\rangle$. In particular, Γ_0 is the injective and the inverse restricted to the image is also order preserving.

To complete the proof of the first part the statement, it remains to show that the map is surjective. Note that $\mathrm{Cu}(A)=\mathrm{LAff}_+(\mathrm{T}(A))$. Therefore elements with the form $(f,0)$ are in the image of Γ_0.

Let $f\in \mathrm{Aff}_+(\mathrm{T}(A))$ and $m\in\mathbb{N}$. Choose $m_0\ge 1$ such that $f(t)-m_0<0$ for all $t\in \mathrm{T}(A)$. Put $\gamma=m_0-f(t)\in \mathrm{Aff}_+(\mathrm{T}(A))$. We then borrow the proof of surjectivity in 6.2.3 of [43] but we also use A.3 with possibly nonzero $\omega(b)$.

Choose $a_1\in M_{m_1}(A)$ such that $a_1=2\gamma$. For each large $n\ge 2$, $\gamma\ll(1+1/n)\gamma$. Thus there exists $a_n\in \mathrm{M}_{m_n}(A)_+$ such that, for some $\delta_n>0$,

$$\gamma<\langle (a_n-\delta_n)_+\rangle\hat{}<\langle a_n\rangle\hat{}<(1+1/n)\gamma. \tag{A.72}$$

Note that A has strict comparison as $\mathrm{Cu}(A)=\mathrm{LAff}_+(\mathrm{T}(A))$. Therefore we may assume that $a_n\le a_1$ ($\langle a_1\rangle\hat{}=\gamma$). In particular, we may assume that $m_n=m_1$, $n=1,2,\ldots$. We may also assume that $m_1\ge m_0$. We may further assume that $\|a_n\|=1$, $n=1,2,\ldots$

Since $a_n\in M_{m_1}(A)_+$ and A is stably projectionless, we may assume that $sp(a_n)=[0,1]$. Consider the commutative sub-C*-algebra generated by a_n and $1_{M_{m_1}}$. Then it is isomorphic to $C([0,1])$. Denote by c_{δ_n} a function in the sub-C*-algebra which is zero at 1, strictly positive on $[0,\delta_n/2)$, zero elsewhere and $\|c_{\delta_n}\|=1$. Note $c_{\delta_n}\in M_{m_1}(\widetilde{A})$ and $\pi(c_{\delta_n})=1_{M_{m_1}}$ (in $M_{m_1}(\mathbb{C})$). Let g_n be also in the sub-C*-algebra which is given by a non-zero positive continuous function with support in $(\delta_n/2,\delta_n)$. Note that $g_n\ne 0$. We may assume that $\|g_n\|\le 1$.

Then

$$\langle c_{\delta_n}\rangle+\langle g_n\rangle+\langle (a-\delta_n)_+\rangle\le m_1\langle 1_{\widetilde{A}}\rangle\le\langle c_{\delta_n}\rangle+\langle a_n\rangle. \tag{A.73}$$

We compute that

$$-(1+1/n)\gamma<(\langle c_{\delta_n}\rangle-m_1\langle 1_{\widetilde{A}}\rangle)\hat{}<(\langle c_{\delta_n}\rangle-m_1\langle 1_{\widetilde{A}}\rangle)\hat{}+\langle g_n\rangle\hat{}<-\gamma. \tag{A.74}$$

Therefore

$$m_1\langle 1_{\widetilde{A}}\rangle\hat{}-(1+1/n)\gamma<\langle c_{\delta_n}\rangle\hat{}<\langle c_{\delta_n}\rangle\hat{}+\langle g_n\rangle\hat{}<m_1\langle 1_{\widetilde{A}}\rangle\hat{}-\gamma. \tag{A.75}$$

Note that, for each n, $\omega(c_{\delta_n})\le\gamma/n$, since both γ and $\langle 1_{\widetilde{A}}\rangle$ are continuous. For each n_k there exists $n_{k+1}>n_k$ such that $7\gamma/n_{k+1}<\langle g_{n_k}\rangle\hat{}$. Hence

$$\langle c_{\delta_{n_k}}\rangle<m_1\langle 1_{\widetilde{A}}\rangle\hat{}-\gamma-7\gamma/n_{k+1}. \tag{A.76}$$

Therefore, there exists a subsequence $\{n_k\}$ such that

$$\langle c_{\delta_{n_k}}\rangle\hat{}+6\omega(c_{\delta_{n_{k+1}}})<\langle c_{\delta_{n_k}}\rangle\hat{}+6\gamma/n_{k+1} \tag{A.77}$$

$$<m_1\langle 1_{\widetilde{A}}\rangle\hat{}-\gamma-\gamma/n_{k+1}<\langle c_{\delta_{n_{k+1}}}\rangle\hat{},\qquad k=1,2,\ldots. \tag{A.78}$$

It follows from A.3 that $\langle c_{\delta_{n_k}}\rangle\le\langle c_{\delta_{n_{k+1}}}\rangle$, $k=1,2,\ldots$. Let $c\in\mathrm{Cu}(\widetilde{A})$ such that $c=\sup\{c_{\delta_{n_k}}\}$. Since $c_{\delta_{n_k}}\in M_{m_1}(\widetilde{A})$ and $\pi(c_{\delta_{n_k}})=1_{M_{m_1}}$, we conclude that $c\le 1_{M_{m_1}}$ and $\langle\pi(c)\rangle=m_1$. We also have

$$\langle c\rangle\hat{}=m_1\langle 1_{\widetilde{A}}\rangle\hat{}-\gamma=(m_1-m_0)\langle 1_{\widetilde{A}}\rangle\hat{}+f. \tag{A.79}$$

Note that

$$\Gamma_0(\langle c\rangle)=((m_1-m_0)\langle 1_{\widetilde{A}}\rangle\hat{}+f,m_1). \tag{A.80}$$

If $m_0-m>0$, then there exists $a_{00}\in M_l(A)_+$ for some $l\ge 1$ such that $\langle a_{00}\rangle\hat{}=(m_0-m)\langle 1_{\widetilde{A}}\rangle\hat{}$. Put $c_1=c_0\oplus a_{00}$. If $m=m_0$, keep $c=c_1$. Then

$$\Gamma_0(\langle c_1\rangle)=(f+(m_1-m)\langle 1_{\widetilde{A}}\rangle\hat{},m_1). \tag{A.81}$$

If $m_1=m$, then $\Gamma_0(\langle c_1\rangle)=(f,m)$. If $m_1-m>0$, we have

$$(m_1-m)\langle 1_{\widetilde{A}}\rangle\hat{}<(m_1-m)\langle 1_{\widetilde{A}}\rangle\hat{}+f. \tag{A.82}$$

Since $(m_1-m)\langle 1_{\widetilde{A}}\rangle\hat{}+f\in\mathrm{Aff}_+(\mathrm{T}(A))$, by A.4, we conclude that

$$(m_1-m)\langle 1_{\widetilde{A}}\rangle\le\langle c_1\rangle. \tag{A.83}$$

Since $(m_1-m)\langle 1_{\widetilde{A}}\rangle$ is represented by a projection, one has $c_2\in M_{m_1}(\widetilde{A})_+$ such that

$$(m_1-m)\langle 1_{\widetilde{A}}\rangle+\langle c_2\rangle=\langle c_1\rangle. \tag{A.84}$$

It follows that $\langle\pi(c_2)\rangle = m$. Note that $\Gamma_0(\langle c_2\rangle) = (f, m)$. To see that we can choose c_2 so that it is not represented by a projection, choose an integer $k \geq 1$ such that $f > 1/k$ on $\mathrm{T}(A)$. Choose c_2' so that $\langle c_2'\rangle = (f - 1/k, m)$ and $c_{2,0} \in (A \otimes \mathcal{K})_+$ such that $d_\tau(c_{2,0}) = 1/k$. Now $c_2 = c_2' \oplus c_2''$ cannot be represented by a projection but $\Gamma_0(c_2) = (f, m)$.

Now let $f \in \mathrm{LAff}_+(\mathrm{T}(A))$ and $m \in \mathbb{N} \cup \{\infty\}$. Choose a sequence (f_n) in $\mathrm{LAff}_+(\mathrm{T}(A))$ with $f_n \nearrow f$ and $m_n \nearrow m$, where $m_n < \infty$, $n = 1, 2, \ldots$. As in the previous paragraph, choose $x_n \in S(\widetilde{A})$ with $\Gamma_0(x_n) = (f_n, m_n)$ such that they are not represented by projections. By what has been proved, $x_n \leq x_{n+1}$, $n = 1, 2, \ldots$. Put $x = \sup\{x_n\}$. Then it is easy to check that $\Gamma_0(x) = (f, m)$.

This shows that Γ_0 is surjective.

For the last part of the statement, let $\langle\pi(a)\rangle \leq \langle\pi(b)\rangle$. Suppose that $y = \langle b\rangle$ is represented by a projection p and $x = \langle a\rangle$ is not represented by a projection, and $\langle a\rangle\hat{} \leq \langle b\rangle\hat{}$. Then, for any $\varepsilon > 0$,

$$\langle(a-\varepsilon)_+\rangle\hat{} < \hat{y} \tag{A.85}$$

Then since $\hat{y}$ is now continuous, by A.4,

$$\langle(a-\varepsilon)_+\rangle \leq y. \tag{A.86}$$

It follows that $x \leq y$.

Now suppose that x is represented by a projection and $\hat{x} < \hat{y}$. If y is also represented by a projection, then by A.4, $x \leq y$.

It remains to check the case that $\langle a\rangle$ is represented by a projection and y is not, and $\langle a\rangle\hat{} < \hat{y}$, as well as $\langle\pi(a)\rangle \leq \mathrm{Cu}(\pi)(y)$. Note that since $\langle a\rangle$ is represented by a projection, $\langle\pi(a)\rangle < \infty$. In this case, there exists an increasing sequence $(\langle b_n\rangle)$ in $\mathrm{Aff}_+(\mathrm{T}(A))$ such that $y = \sup\{\langle b_n\rangle\}$. Since $\langle a\rangle\hat{}$ is continuous, one finds b_n such that $\langle a\rangle\hat{} < \langle b_n\rangle\hat{}$ for some large n. We may also assume that $\langle\pi(b_n)\rangle \geq \langle a\rangle\hat{}$. Now $\langle b_n\rangle\hat{} \in \mathrm{Aff}_+(\mathrm{T}(A))$. From what has been proved, $\langle a\rangle \leq y$. This completes the proof. □

Corollary A.7. *Let A be a stably projectionless simple C*-algebra such that $\mathrm{M}_r(A)$ almost has stable rank one for all $r \geq 1$, $\mathrm{QT}(A) = \mathrm{T}(A)$, A has strict comparison for positive elements and has continuous scale, and $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$. Then $K_0(A)$ has the following property: for any $x \in K_0(A)$, there exists $\tau \in \mathrm{T}(A)$ such that $\rho_A(x)(\tau) = 0$.*

Proof. Since A is stably projectionless, by [4], A is stably finite (see also Theorem 1.2 of [37]). It follows from [6] that $\mathrm{T}(A) \neq \emptyset$. Let $x = [p] - [q]$, where $p, q \in \mathrm{M}_r(\widetilde{A})$ are projections such that $[\pi(p)] = [\pi(q)]$, where $\pi : \mathrm{M}_r(\widetilde{A}) \to \mathrm{M}_r(\mathbb{C})$ is the quotient map. Suppose that $\rho_A(x)(\tau) > 0$ for all $\tau \in \mathrm{T}(A)$. Then,

$$\tau(p) > \tau(q) \ \text{ for all } \ \tau \in \mathrm{T}(A). \tag{A.87}$$

By Theorem A.6,

$$q \lesssim p. \tag{A.88}$$

Thus, there is a projection $p' \leq p$ such that $[p'] = [q]$. Put $P = p - p'$. Then P is a non-zero projection in $M_r(\widetilde{A})$, as $\tau(P) > 0$ for all $\tau \in \mathrm{T}(A)$. Since $\pi(p') \leq \pi(p)$ and $[\pi(p')] = [\pi(q)] = [\pi(p)]$, without loss of generality, we may assume that

$$\pi(P) = 0. \tag{A.89}$$

This implies that $P \in M_r(A)$, which is impossible. By considering $-x$, we conclude that it is also impossible to have $\rho_A(x)(\tau) < 0$ for $\tau \in T(A)$.

If there were no τ such that $\rho_A(x)(\tau) = 0$, then there would be $\tau_1, \tau_2 \in \mathrm{T}(A)$ such that $\rho_A(x)(\tau_1) = t_1 > 0$, $\rho_A(x)(\tau_2) = t_2 < 0$. Then $0 < \alpha := t_2/(t_2 - t_1) < 1$. Put $\tau = \alpha\tau_1 + (1-\alpha)\tau_2 \in \mathrm{T}(A)$. Then $\rho_A(x)(\tau) = 0$. This implies there is $\tau \in \mathrm{T}(A)$ such that $\rho_A(x)(\tau) = 0$. □

Corollary A.8. *Let A be a separable, exact, $\mathcal{Z}$-stable simple C*-algebra, where $\mathcal{Z}$ is the Jiang–Su algebra. Suppose that $x \in K_0(A)$ is such that $\tau(x) > 0$ for all non-zero traces τ of A. Then x is represented by a projection $p \in A \otimes \mathcal{K}$.*

Proof. Since A is assumed to be exact, $\mathrm{QT}(A) = \mathrm{T}(A)$. Also, since A is $\mathcal{Z}$-stable, by Lemma 6.5 of [19], $\mathrm{Cu}(A) = \mathrm{LAff}_+(\tilde{T}(A))$. It follows from [44] that $\mathrm{M}_n(A)$ almost has stable rank one as $\mathrm{M}_n(A)$ is $\mathcal{Z}$-stable. Moreover, there is a non-zero $a \in \mathrm{Ped}(A)_+$ (see 5.2 of [17]) such that $C = \overline{aAa}$ has continuous scale. By Brown's theorem [7], $C \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. It follows from A.7 that we may assume that $A \otimes \mathcal{K}$ has a non-zero projection e. Then, by Brown's theorem [7] again, $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where B is the hereditary sub-C*-algebra generated by e. Now since B is unital and $B \otimes \mathcal{K}$ is $\mathcal{Z}$-stable, by [21] (see also 4.6 of [47]), $K_0(B)$ is weakly unperforated. Thus $x > 0$ and it is represented by a projection. □

Corollary A.9. *Let A be a stably projectionless simple C*-algebra such that $\mathrm{M}_r(A)$ has almost stable rank one for all $r \geq 1$, $\mathrm{QT}(A) = \mathrm{T}(A)$, A has finitely many extremal traces, $\mathrm{Ped}(A) = A$, and $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$. Then $\mathrm{Cu}(\widetilde{A}) = V(\widetilde{A}) \sqcup L(\widetilde{A})$.*

Proof. Since $\mathrm{Ped}(A) = A$ and A has finitely many extremal traces, $d_\tau(e_A)$ is continuous on $\mathrm{T}(A)$ for any strictly positive element $e_A \in A$. Since A has strict comparison, A has continuous scale (see the proof 5.4 of [17], for example). Since $\mathrm{T}(A)$ has only finitely many extreme points, any finite affine function on $\mathrm{T}(A)$ is continuous, and so, for any integer $r \geq 1$, and any $a \in \mathrm{M}_r(\widetilde{A})$, $\langle a\rangle\hat{} \in \mathrm{Aff}_+(\mathrm{T}(A))$. Therefore, if $x \in \mathrm{Cu}(\widetilde{A})$ is not represented by a projection, then $x \in S(\widetilde{A})$. In other words, $\mathrm{Cu}(\widetilde{A}) = V(\widetilde{A}) \sqcup S(\widetilde{A})$. Thus, the corollary follows from A.6. $\square$

Corollary A.10. *Let A be a stably projectionless simple C^*-algebra with continuous scale and with stable rank one such that $\mathrm{QT}(A) = \mathrm{T}(A)$, and $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$. Then $\mathrm{Cu}(\widetilde{A})_+ = S(\widetilde{A})$ and Γ_0 is an ordered semigroup isomorphism from $\mathrm{Cu}_+(\widetilde{A})$ onto $L(\widetilde{A})$. Moreover, $\mathrm{Cu}(\widetilde{A}) = V(\widetilde{A}) \sqcup L(\widetilde{A})$.*

Remark A.11. Let A be a stably projectionless simple C*-algebra which is $\mathcal{Z}$-stable such that $\mathrm{QT}(A) = \mathrm{T}(A)$. Then, by [44], $\mathrm{M}_r(A)$ (for all $r \geq 1$) almost has stable rank one. A combination of [47,48], and [19] shows that A also satisfies the rest of the conditions of A.6.

There are several other immediate consequences of A.6 and related facts about $\mathrm{Cu}^\sim$ (see [43]). Let A be as A.6.

(i) Then the canonical map $\iota_{0,A} : \mathrm{Cu}(A) \to \mathrm{Cu}^\sim(A)$ is injective. To see this, let $\langle a\rangle, \langle b\rangle \in \mathrm{Cu}(A)$. If $\langle a\rangle + k[1_{\tilde{A}}] = \langle b\rangle + k[1_{\tilde{A}}]$, then $\langle a\rangle\hat{} = \langle b\rangle\hat{}$. Since $\mathrm{Cu}(A) = \mathrm{Laff}_+(\mathrm{T}(A))$, $\langle a\rangle = \langle b\rangle$.

(ii) Let $x_n \in S(\widetilde{A})$ with $x_n \leq x_{n+1}$, $n = 1, 2, \ldots$. Then $\sup_n x_n \in S(\widetilde{A})$. This follows from the definition immediately.

(iii) As indicated in the proof of A.6, if $\langle p\rangle \in V(\widetilde{A})$ and $x \in S(\widetilde{A}) \setminus \{0\}$, then $\langle p\rangle + x \in S(\widetilde{A})$.

(iv) Denote by $S^\sim(A) = \{\langle a\rangle - \langle\pi(a)\rangle \cdot [1_{\tilde{A}}] : \langle a\rangle \in S(\widetilde{A}),\ \langle\pi(a)\rangle < \infty\}$ as a sub-semigroup of $\mathrm{Cu}^\sim(A)$ (see [43]). Then, by A.6, $\mathrm{Cu}(A) \subset S^\sim(A)$.

Let $x = \langle a\rangle, y = \langle b\rangle \in S(\widetilde{A})$ such that $\langle\pi(a)\rangle = n$ and $\langle\pi(b)\rangle = m$, where n, m are nonnegative integers. Suppose that $\hat{x} - n = \hat{y} - m$. Then $\hat{x} + m = \hat{y} + n$ and $\langle\pi(a)\rangle + m = n + m = \langle\pi(y)\rangle + n$. If $x = 0$, by (iii), $y = 0$. Let us assume neither are zero. It follows from (iii) that $x + m\langle 1\rangle$ and $y + n\langle 1\rangle$ are not represented by projections. By A.6, $x + m\langle 1\rangle = y + n\langle 1\rangle$. It follows that $x - n\langle 1\rangle = y - m\langle 1\rangle$ in $\mathrm{Cu}^\sim(A)$. Therefore we may write $S^\sim(A) = \mathrm{LAff}_+^\sim(\mathrm{T}(A)) = \{f - g : f \in \mathrm{LAff}_+(\mathrm{T}(A)), g \in \mathrm{Aff}_+(\mathrm{T}(A))\}$ (see [43] for the notation).

(v) Let $\iota_{\tilde{A}}^\sim : \mathrm{Cu}^\sim(A) \to \mathrm{Cu}^\sim(\widetilde{A})$ be the natural map. Then, by A.6, $\iota_{\tilde{A}}^\sim$ is injective on $S^\sim(A)$. In fact, let $x - n\langle 1\rangle, y - m\langle 1\rangle \in S^\sim(A)$, such that, for some integer $k \geq 0$,

$$x + m\langle 1\rangle + k\langle 1\rangle = y + n\langle 1\rangle + k\langle 1\rangle \in S(\widetilde{A}) \subset \mathrm{Cu}(\widetilde{A}). \tag{A.90}$$

Then

$$\hat{x} + m + k = \hat{y} + n + k \ \text{ and } \ \langle\pi(a)\rangle + m + k = \langle\pi(b)\rangle + n + k. \tag{A.91}$$

It follows that

$$\hat{x} + m = \hat{y} + n \ \text{ and } \ \langle\pi(a)\rangle + m = \langle\pi(b)\rangle + n. \tag{A.92}$$

As (iv), we may assume neither x nor y are zero. Since $x + m\langle 1\rangle$ and $y + n\langle 1\rangle$ are not represented by projections, by A.6, $x + n\langle 1\rangle = y + m\langle 1\rangle$. Thus $x - n\langle 1\rangle = y - m\langle 1\rangle$ in $S^\sim(A) \subset \mathrm{Cu}^\sim(\widetilde{A})$.

(vi) Exactly the same argument shows that $S(\widetilde{A})$ maps to $\mathrm{Cu}^\sim(\widetilde{A})$ injectively. Let us denote this map by $\iota_S^\sim$. Let us also set $S^\sim(\widetilde{A}) = \{x - n\langle 1_{\tilde{A}}\rangle : x \in S(\widetilde{A}), n \in \mathbb{N} \cup \{0\}\} \subset \mathrm{Cu}^\sim(\widetilde{A})$. So $S(\widetilde{A}) \subset S^\sim(\widetilde{A})$.

(vii) Note that $V(\widetilde{A}) \sqcup S(\widetilde{A})$ maps into $K_0(\widetilde{A}) \sqcup S^\sim(\widetilde{A})$. Let us identify $\mathbb{N} \subset V(\widetilde{A})$ with $\mathbb{N} \cdot \langle 1_{\tilde{A}}.\rangle$ The map above maps $\mathbb{N} \sqcup S(\widetilde{A})$ into $\mathbb{Z} \sqcup S^\sim(\widetilde{A})$ injectively.

A.2. An existence theorem and some uniqueness theorems

The following is a variation of a result of Pedersen and Rørdam [39].

Lemma A.12. *Let A be a non-unital C^*-algebra and let $x \in A$ and $1 > \delta > \beta > \gamma > 0$. Suppose that there exists $y \in GL(\widetilde{A})$ such that $\|x - y\| < \gamma$. Then there is a unitary $u \in \widetilde{A}$ with the form $u = 1 + z$, where $z \in A$, such that*

$$uf_\delta(|x|) = vf_\delta(|x|), \tag{A.93}$$

where $x = v|x|$ is the polar decomposition of x in A^{**}.

Proof. This is a modification of the proof of Pedersen in [39]. We will follow the proof and keep the notation of [39] and point out where to make the changes.

In Lemma 1 of [39], write $A = \lambda + A'$, where $A' \in \mathfrak{A}$ for some $\lambda \neq 0$. Let $\pi : \tilde{\mathfrak{A}} \to \mathbb{C}$ denote the quotient map with $\ker\pi = \mathfrak{A}$. If $T \in \mathfrak{A}$ and $\|T - A\| < \gamma$, then $\|\pi(A)\| < \gamma$. It follows that $|\lambda| < \gamma$. There is a continuous path $\{g_1(t) : t \in [|\lambda|, \gamma]\}$ such that $g_1(|\lambda|) = \lambda^{*-1}$, $g_1(\gamma) = \gamma^{-1}$ and $|g_1(t)| \geq \gamma^{-1}$. We define a complex valued function

$g' \in C([0, \|A\|])$ as follows:

$$g'(t) = \begin{cases} \lambda^{*-1} & \text{if } t \in [0, |\lambda|]; \\ g_1(t) & \text{if } t \in (\lambda, \gamma]; \\ t^{-1} & \text{if } t \in (\gamma, \infty). \end{cases} \tag{A.94}$$

This g' will replace the function g in the proof of Lemma 1 of [39]. Put

$$B = (g')^{-1}(|T^*|)A^{*-1}(1-f)(|T|) + Vf(|T|) \tag{A.95}$$

with f as described in [39]. Note that $fg'(t) = 0$ if $t \in [0, \gamma]$, $fg'(t) = t^{-1}$ if $t \in [\beta, \infty)$, and $tf(t)g'(t) = f(t)$ if $t \in [0, \infty)$. Exactly as in the proof of [39], one has $BE_\beta = VE_\beta$. Set $C = f(|T|) - A^*Vg'(|T|)f(|T|)$. We still have $f(|T|) = |T|V^*Vg'(|T|)f(|T|)$. Therefore

$$C = f(|T|) - A^*Vg'(|T|)f(|T|) = (T^* - A^*)V(fg')(|T|). \tag{A.96}$$

The same estimate yields

$$\|C\| \le \|T^* - A^*\| \|fg'\|_\infty \le \|T - A\|\gamma^{-1} < 1. \tag{A.97}$$

As in the proof of Lemma 1 of [39], this implies that B defined above is invertible and $BE\beta = VE_\beta$. Note that $\pi(B) = \lambda^*\lambda^{*-1} = 1$. In other words $B = 1 + z'$ for some $z \in \mathfrak{A}$. As in [39], B also satisfies the conclusion of Lemma 2 of [39], i.e., $F_\beta B^{*-1} = F_\beta V$.

Define $h(t) = (t - \beta) \vee 0$. Then $Bh(|T|)) = BE_\beta h(|T|) = VE_\beta h(|T|) = Vh(|T|)$. Let A_0 be as in Lemma 3 of [39] with B defined above. Then the conclusion of Lemma 3 of [39] holds.

Then, as in Lemma 4 of [39], one obtains $B_0 \in \tilde{\mathfrak{A}}$ with $\pi(B_0) = 1$ defined as B defined with g_0' instead of g_0 as we demonstrated above. The same computation provides

$$B_0 - Vf_0(|T|) = (g_0')^{-1}(|T^*|)A_0^{*-1}(1-f_0)(|T|) \tag{A.98}$$

$$= (g_0')^{-1}(|T^*|)B^{*-1}(h+\varepsilon)^{-1}(|T|)(1-f_0)(|T|). \tag{A.99}$$

Exactly as in the proof of Lemma 4 of [39], we have $B_0E_\delta = F_\delta B_0 = F_\delta V = VE_\delta$. Since B_0 is invertible, we have the polar decomposition $B_0 = U|B_0|$ in $\tilde{\mathfrak{A}}$. Note that $\pi(U) = 1$ since $\pi(B_0) = 1$. Hence $U = 1 + z$ for some $z \in \mathfrak{A}$. As in Theorem 5 of [39], $UE_\delta = VE_\delta$. Then $Uf_\delta(|T|) = Vf_\delta(|T|)$. The lemma follows. □

Corollary A.13. *Let A be a non-unital C^*-algebra which almost has stable rank one. Then, for any $x \in A$ and any $\varepsilon > 0$, there is a unitary $u \in \widetilde{A}$ with form $u = 1 + y$ for some $y \in A$ such that*

$$\|u|x| - x\| < \varepsilon. \tag{A.100}$$

Proof. We have $\|f_{\varepsilon/8}(|x|)x - x\| < \varepsilon/4$. Since A almost has stable rank one, by A.12, there exists a unitary $u \in \widetilde{A}$ with the form $u = 1 + z$ for some $z \in A$ such that

$$\|uf_{\varepsilon/8}(|x|) - vf_{\varepsilon/8}(|x|)\| < \varepsilon/4, \tag{A.101}$$

where $x = v|x|$ is the polar decomposition in A^{**}. It follows that

$$\|u|x| - x\| < \|u|x| - uf_{\varepsilon/8}(|x|)|x|\| + \|uf_{\varepsilon/8}(|x|)|x| - vf_{\varepsilon/8}(|x|)|x|\| \tag{A.102}$$

$$+ \|vf_{\varepsilon/8}(|x|)|x| - v|x|\| < \varepsilon. \quad \square \tag{A.103}$$

Lemma A.14 (*Theorem 3.3.1 of [43]*)**.** *Let B be a simple C^*-algebra which has almost stable rank one. Then, for any finite subset $\mathcal{F}$ and $\varepsilon > 0$, there exists a finite subset $\mathcal{G} \subseteq \mathrm{Cu}(C)$ such that, for any two homomorphisms $\phi_1, \phi_2 : C := C_0((0, 1]) \to B$, if*

$$\mathrm{Cu}(\phi_1)(f) \le \mathrm{Cu}(\phi_2)(g) \text{ and } \mathrm{Cu}(\phi_2)(f) \le \mathrm{Cu}(\phi_1)(g) \text{ for all } f, g \in \mathcal{G} \text{ with } f \ll g, \tag{A.104}$$

there exists a unitary $u \in \widetilde{B}$ such that

$$\|u^*\phi_2(f)u - \phi_1(f)\| < \varepsilon \text{ for all } f \in \mathcal{F}. \tag{A.105}$$

Proof. The lemma is based on the fact that ϕ and ψ are approximately unitarily equivalent if $\mathrm{Cu}(\phi) = \mathrm{Cu}(\psi)$ (see the proof of "(iii) implies (i)" in "Proof of Theorem 1.3" in [44]).

The actual proof is almost the same as that of 3.3.1 of [43]. Let us present the details.

Let us point out what is the difference. In the proof of 3.3.1 of [43], consider $(b_G) \in \prod_G B_G$ and let $\varepsilon > 0$. Since each B_G almost has stable rank one, by A.13, there is $u_G \in \widetilde{B}_G$ such that $u_G = 1 + z_G$ for some $z_G \in B_G$ with $\|z_G\| \le 2$ and

$$\|u_G|b_G| - b_G\| < \varepsilon. \tag{A.106}$$

Note that $(u_G) \in 1 + \prod_G B_G$. Since elements with polar decomposition, in the sense of being the (non-unique) product of a unitary and a positive element, are in the closure of the invertible elements, this implies that (with notation as in the proof of 3.3.1 of [43]) both $\prod_G B_G$ and B almost have stable rank one. The rest of the proof then can proceed just as in the proof of 3.3.1 of [43] (note that we only compute $\mathrm{Cu}(\phi)$ and $\mathrm{Cu}(\psi)$ which is easier). □

Remark A.15. A direct proof of the lemma above could also be obtained using [39] directly.

Corollary A.16. *Let $C = C_0((0, 1])$ and let $\Delta : C^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ be an order preserving map. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subseteq C$, there exist a finite subset $\mathcal{H}_1 \subseteq C_+^{\mathbf{1}} \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subseteq C_{s.a.}$, and $\gamma > 0$ satisfying the following condition: for any two homomorphisms $\phi_1, \phi_2 : C \to A$ for some A which is separable, simple, exact, stably projectionless, and has continuous scale, almost stable rank one, and the property that the map $\mathrm{Cu}(A) \to \mathrm{LAff}_+(\mathrm{T}(A))$ is an ordered semigroup isomorphism such that*

$$\tau(\phi_i)(a) \geq \Delta(\hat{a}) \text{ for all } a \in \mathcal{H}_1 \text{ and } \text{ for all } \tau \in \mathrm{T}(A) \text{ and } \tag{A.107}$$

$$|\tau(\phi_1(b)) - \tau(\phi_2(b))| < \gamma \text{ for all } b \in \mathcal{H}_2 \text{ and } \text{ for all } \tau \in \mathrm{T}(A), \tag{A.108}$$

there exists a unitary $u \in \widetilde{A}$ such that

$$\|u^*\phi_2(f)u - \phi_1(f)\| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

Proof. The proof of this is the combination of A.14 and the proof of 7.8 of [17]. □

Remark A.17. Let $\alpha, \beta : \mathrm{Cu}(C_0((0, 1])) \to \mathrm{Cu}(A)$ be two morphisms in **Cu**. Recall the pseudo-metric d_w introduced in [8]:

$$\mathrm{d}_w(\alpha, \beta) = \inf\{r \in \mathbb{R}^+ : \alpha(\langle e_{t+r}\rangle) \leq \beta(\langle e_t\rangle) \text{ and } \beta(\langle e_{t+r}\rangle) \leq \alpha(\langle e_t\rangle),\ t \in \mathbb{R}^+\}, \tag{A.109}$$

where $e_t(x) = (x - t)_+$ is a function on $(0, 1]$.

If $\phi, \psi : C_0((0, 1]) \to A$ are two homomorphisms, define $\mathrm{d}_w(\phi, \psi) = \mathrm{d}_w(\mathrm{Cu}(\phi), \mathrm{Cu}(\psi))$. Let $J \subseteq (0, 1]$ be any relatively open interval $(\alpha, \beta) \cap (0, 1]$. Define, for each $r > 0$, $J_r = \{t \in (0, 1] : \mathrm{dist}(J, t) < r\}$. For each J fix a positive function e_J which is strictly positive on J and zero elsewhere. To be more symmetric than the definition of d_w, one can also define the following metric:

$$D_w(\phi, \psi) = \inf\{r \in \mathbb{R}^+ : \phi(e_J) \lesssim \psi(e_{J_r}),\ \ \psi(e_J) \lesssim \phi(e_{J_r}),\ J \subseteq (0, 1]\}. \tag{A.110}$$

(see some related discussion in [27]). Then D_w is a metric (see the proof of Proposition 2 of [45]). If $\mathrm{Cu}(A)$ has the weak cancellation, d_w is a metric (Proposition 2 of [45]), and d_w and D_w are equivalent.

Another version of A.14 can be stated as follows:

(**A**): For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subseteq C$, there exists $\delta > 0$ with the following property: if $D_w(\phi, \psi) < \delta$, then there exists a unitary $u \in \widetilde{A}$ such that

$$\|u^*\phi(f)u - \psi(f)\| < \varepsilon \text{ for all } f \in \mathcal{F}, \tag{A.111}$$

and, if, furthermore, $\mathrm{Cu}(A)$ has weak cancellation, $D_w(\phi, \psi) < \delta$ can be replaced by $d_w(\phi, \psi) < \delta$ (with possibly a different δ).

Suppose that A is a stably projectionless simple C*-algebra with $\mathrm{T}(A) \neq \emptyset$. Consider any $x+z \ll y+z$ for $x, y, z \in \mathrm{Cu}(A)$, where $x \neq y$. Suppose that $b \in (A\otimes\mathcal{K})_+$ is such that $\langle b\rangle = y+z$ and $0 \leq b \leq 1$. Then, for any $1/2 > \delta > 0, f_{\delta/2}(b) - f_\delta(b) > 0$. Therefore $d_\tau(x + z) < d_\tau(y + z)$ for all $\tau \in \mathrm{QT}(A)$. Thus $d_\tau(x) < d_\tau(y)$ for all $\tau \in \mathrm{QT}(A)$. If A is also assumed to have strict comparison, then $x \leq y$. This implies that $\mathrm{Cu}(A)$ has weak cancellation. As shown in Proposition 2 of [45], d_w is then a metric.

Proposition A.18. *Let $C = C_0((0, 1])$. Then, for any $\varepsilon > 0$, any $\sigma > 0$, and any finite subset $\mathcal{F} \subseteq C$, there exists $\delta > 0$ satisfying the following condition: Suppose that A is a stably projectionless simple C*-algebra with continuous scale which almost has stable rank one and suppose that $\phi, \psi : C \to \widetilde{A}$ are homomorphisms. If $\mathrm{d}_w(\phi, \psi) < \delta$, then there exists $\psi' : C \to \widetilde{A}$ such that $\pi \circ \psi' = \pi \circ \phi$,*

$$\|\psi'(f) - \psi(f)\| < \varepsilon \text{ and } \mathrm{d}_w(\phi, \psi') < \sigma, \tag{A.112}$$

where $\pi : \widetilde{A} \to \mathbb{C}$ is the quotient map.

Proof. Let $\iota : (0, 1] \to (0, 1]$ denote the identity map which we view as a generating element of $C_0((0, 1])$. Fix $0 < \eta < \varepsilon/2$ and a finite subset $\mathcal{G} \subseteq C_0((0, 1])$. There exists $\delta' > 0$ such that, if $|t - t'| < \delta'$, then

$$\|g(t) - g(t')\| < \eta \text{ for all } g \in \mathcal{G}. \tag{A.113}$$

If $d_w(\phi, \psi) < \delta'$, then it is easy to see that $\|\pi(\phi(\iota)) - \pi(\psi(\iota))\| < \delta'$. Let $\lambda_1, \lambda_2 \in (0, 1]$ such that $\pi(\phi(\iota)) = \lambda_1$ and $\pi(\psi(\iota)) = \lambda_2$. Then $|\lambda_1 - \lambda_2| < \delta'$. There exists a continuous map $j : (0, 1] \to (0, 1]$ such that

$$|j(t) - t| < \delta' \text{ and } j(\lambda_2) = \lambda_1.$$

Define $\psi' : C_0((0, 1]) \to \widetilde{A}$ by $\psi'(f) = \psi(f \circ j)$. Then $\pi(\psi'(\iota)) = \lambda_1 = \pi(\phi(\iota))$. Moreover

$$\|\psi(g) - \psi'(g)\| = \|\psi(g - g \circ j)\| < \eta \text{ for all } g \in \mathcal{G}. \tag{A.114}$$

If $\sigma > 0$ is given one can choose large $\mathcal{G}$ and sufficiently small η so that

$$d_w(\psi(f), \psi'(f)) < \sigma/2. \tag{A.115}$$

We also can choose $\delta = \min\{\delta', \sigma/2\}$. $\square$

Remark A.19. Let $I = (\alpha, \beta]$ (or $I = [\alpha, \beta)$). Let A be a stably projectionless simple C*-algebra which almost has stable rank one. Fix a homeomorphism $h_I : (\alpha, \beta] \to (0, 1]$ given by $h(t) = \frac{t-\alpha}{\beta-\alpha}$ for $t \in (\alpha, \beta]$, or ($h_I(t) = \frac{\beta-t}{\beta-\alpha}$ for $t \in [\alpha, \beta)$.) If $\phi : C_0(I) \to A$ is a homomorphism, denote by $\phi \circ h_I^* : C_0((0, 1]) \to A$ the homomorphism defined by $\phi \circ h_I^*(f) = \phi(f \circ h)$ for all $f \in C_0((0, 1])$.

Suppose now there are two homomorphisms $\phi, \psi : C_0(I) \to A$. Define

$$D_{w,I}(\phi, \psi) = D_w(\phi \circ h_I^*, \psi \circ h_I^*) \text{ and } d_{w,I}(\phi, \psi) = d_w(\phi \circ h_I^*, \psi \circ h_I^*). \tag{A.116}$$

Put $\iota_I'(t) = t - \alpha$, if $I = (\alpha, \beta]$ and $\iota_I'(t) = \beta - t$, if $I = [\alpha, \beta)$. Now assume that $0 < \beta - \alpha \le 1$. Put $f_I(t) = (\beta - \alpha)t \in C_0((0, 1])$. Then $\iota_I' = f_I \circ h$. Let $\mathcal{F} \subset C_0(0, 1]$. Then $g \circ \iota_I' = g \circ f_I \circ h$ for each $g \in \mathcal{F}$. Let $\lambda \in (0, 1]$. Define $f_\lambda(t) = \lambda t$ for $t \in (0, 1]$.

For any $\varepsilon > 0$, there is a finite subset $K \in (0, 1]$ such that, for any $0 < \beta - \alpha \le 1$, with $I = (\alpha, \beta]$ or $I = [\alpha, \beta)$, for each $g \in \mathcal{F}$, $\|g \circ \iota_I' - g \circ f_\lambda \circ h\| < \varepsilon/2$ for some $\lambda \in K$. Let $\mathcal{G}_{\mathcal{F}}, \varepsilon = \{g \circ f_\lambda : g \in \mathcal{F},\ \lambda \in K\}$. Then, by (**A**) of A.17, we have the following:

(**B**): Let $\varepsilon > 0$, let $\mathcal{F} \subset C_0((0, 1])$ be a finite subset, Let $\delta > 0$ be given by (**A**) in A.17 for $\varepsilon/2$ and $\mathcal{G}_{\mathcal{F}}, \varepsilon$. Suppose $I = (\alpha, \beta]$ or $I = [\alpha, \beta)$ with $0 < \beta - \alpha < 1$. Then, for any homomorphisms $\phi, \psi : C_0(I) \to A$ such that $D_{w,I}(\phi, \psi) < \delta$, there exists a unitary $u \in \widetilde{A}$ such that

$$\|u^*\psi(f)u - \phi(f)\| < \varepsilon \text{ for all } f \in \{\iota_I', g \circ \iota_I' : g \in \mathcal{F}\}. \tag{A.117}$$

Furthermore, if Cu(A) has weak cancellation, $D_{w,I}$ above could be replaced by $d_{w,I}$.

Lemma A.20. *Let A be a stably projectionless simple C^*-algebra with continuous scale which almost has stable rank one. Suppose also that A has strict comparison for positive elements and that* QT(A) = $\underset{\sim}{\mathrm{T}}$($A$)*. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subseteq C$, there exists $\delta > 0$ satisfying the following condition: If $\phi, \psi : C_0((0, 1]) \to \widetilde{A}$ are two homomorphisms such that*

$$d_w(\phi, \psi) < \delta, \tag{A.118}$$

then there exists a unitary $u \in \widetilde{A}$ such that

$$\|u^*\psi(f)u - \phi(f)\| < \varepsilon \text{ for all } f \in \mathcal{F}. \tag{A.119}$$

Moreover, (A.119) *also holds, without assuming A has strict comparison, on replacing* (A.118) *by*

$$D_w(\phi, \psi) < \delta. \tag{A.120}$$

Proof. By A.18, without loss of generality, we may assume that $\pi \circ \phi = \pi \circ \psi$, where $\pi : \widetilde{A} \to \mathbb{C}$ is the quotient map. Let $\iota : (0, 1] \to (0, 1]$ denote the identity map which we view as a generator of $C_0((0, 1])$. Fix $\varepsilon > 0$. Put $a = \phi(\iota)$ and $b = \psi(\iota)$. It suffices to establish the case that $\mathcal{F} = \{\iota\}$.

Choose $\sigma_0 = \varepsilon/256$. We will prove the last part of the statement first. The first part will follow, since, by A.17, with the assumption that A has strict comparison, Cu(A) has weak cancellation. It follows that d_w and D_w are equivalent. Let $\mathcal{F}_0' = \{\iota, f_{\sigma_0}\}$.

By (**B**) of A.19, we obtain $1/2 > \delta_0 > 0$ with the following property: for any interval $I = [\alpha, \beta)$ or $I = (\alpha, \beta]$ with $0 < \beta - \alpha \le 1$, and if $\phi', \psi' : C_0(I) \to A$ are two homomorphisms such that $D_w(\phi', \psi') < \delta_0$, then there exists a unitary $u' \in \widetilde{A}$ such that

$$\|(u')^*\psi'(f)u' - \phi'(f)\| < \varepsilon/64 \text{ for all } f \in \mathcal{F}_0, \tag{A.121}$$

where $\mathcal{F}_0 = \{\iota_I', f_{\sigma_0/2}(\iota_I')\}$ and where ι_I' is the function defined in A.19,

Put $\delta = (\varepsilon/4)\delta_0 > 0$. Let $\phi, \psi : C_0((0, 1]) \to \widetilde{A}$ be two homomorphisms which satisfy (A.118) for δ.

Now suppose that $\pi(a) = \pi(b) = \lambda$ for some $\lambda \in (0, 1]$. Let $x = a - \lambda \cdot 1_{\widetilde{A}}$ and $y = b - \lambda \cdot 1_{\widetilde{A}}$. Note that $sp(x), sp(y) \subset [-\lambda, 1 - \lambda]$. If $f \in C_0([-\lambda, 0)) \oplus C_0((0, 1 - \lambda]) \subset C([-\lambda, 1 - \lambda])$, then $f(0) = 0$. Therefore

$\pi(f(x)) = \pi(f(y)) = 0$. Define two homomorphisms $\phi_1, \psi_1 : C_0([-\lambda, 0)) \oplus C_0((0, 1-\lambda]) \to A$ by $\phi_1(f) = f(x)$ and $\psi_1(f) = f(y)$ for all $f \in C_0([-\lambda, 0)) \oplus C_0((0, 1-\lambda])$.

Define $c_- \in C_0([-\lambda, 0)) \oplus C_0((0, 1-\lambda])$ by $c_-(t) = \max(-t, 0)$ ($c_-(t) = -t$ in $[-\lambda, 0)$ and $c_-(t) = 0$ in $(0, 1-\lambda]$), and $c_+(t) = \max(t, 0)$. Then $\phi_1(c_+(x)) = (a-\lambda)_+$, $\phi_1(c_-(x)) = (a-\lambda)_-$, and $\psi_1(c_+) = (b-\lambda)_+$ and $\psi_1(c_-(x)) = (b-\lambda)_-$. Let $\phi_{1+} = \phi_1|_{C_0((0,1-\lambda])}$, $\psi_{1,+} = \psi_1|_{C_0((0,1-\lambda])}$, $\phi_{1-} = \phi_1|_{C_0([-\lambda,0))}$, and $\psi_{1,-} = \psi_1|_{C_0([-\lambda,0))}$.

Note that

$$\|f_\sigma(c_-)c_- - c_-\| < \varepsilon/64, \ \ \|c_-f_\sigma(c_-) - c_-\| < \varepsilon/64, \ \text{ and} \tag{A.122}$$

$$\|f_\sigma(c_+)c_+ - c_+\| < \varepsilon/64, \ \ \|c_+f_\sigma(c_+) - c_+\| < \varepsilon/64 \tag{A.123}$$

for all $0 < \sigma \le \sigma_0$. Let us also assume that, for all $0 < \sigma \le \sigma_0$,

$$\|f_\sigma(c_-)^{1/2}c_- - c_-\| < \varepsilon/64, \ \ \|c_-f_\sigma(c_-)^{1/2} - c_-\| < \varepsilon/64, \ \text{ and} \tag{A.124}$$

$$\|f_\sigma(c_+)^{1/2}c_+ - c_+\| < \varepsilon/64, \ \ \|c_+f_\sigma(c_+)^{1/2} - c_+\| < \varepsilon/64. \tag{A.125}$$

Let us consider the case $\lambda \ge \varepsilon/2$ and $\lambda < 1 - \varepsilon/2$ first. Let $I_+ = (0, 1-\lambda]$ and $I_- = [-\lambda, 0)$. Recall that $h_{I_+}(t) = \frac{t}{1-\lambda}$ and $h_{I_-}(t) = \frac{-t}{\lambda}$ (see A.19). Therefore the condition that $\mathrm{D}_w(\phi, \psi) < \delta = (\varepsilon/4)\delta_0$ implies (see A.19) that

$$\mathrm{D}_{w,I_+}(\phi_{1,+}, \psi_{1,+}) = D_w(\phi_+ \circ h_{I_+}^*, \phi_+ \circ h_{I_+}^*) < \delta_0. \tag{A.126}$$

Note this holds in $\mathrm{Cu}(A)$. We also have that

$$\mathrm{D}_{w,I_-}(\phi_{1,-}, \psi_{1,-}) < \delta_0. \tag{A.127}$$

Put $\mathcal{F}_1' = \{f_{\sigma_0/2}(c_-), c_-\}$ and $\mathcal{F}_2' = \{f_{\sigma_0/2}(c_+), c_+\}$. By the choice of δ_0, there are unitaries $u_1, u_2 \in \widetilde{A}$ such that

$$\|u_i^*\phi_1(f)u_i - \psi_1(f)\| < \varepsilon/64 \ \text{ for all } \ f \in \mathcal{F}_i', \quad i = 1, 2. \tag{A.128}$$

By replacing u_i by $\alpha_i u_i$ for some $\alpha_i \in \mathbb{T}$, we may assume that $\pi(u_i) = 1$, $i = 1, 2$. Put $z = \phi_1(f_{\sigma_0/2}(c_-)^{1/2})u_1\psi_1(f_{\sigma_0}(c_-)^{1/2}) + \phi_1(f_{\sigma_0/2}(c_+)^{1/2})u_2\psi_1(f_{\sigma_0}(c_+)^{1/2}) \in A$. Keep in mind that $\phi_1(c_+)\phi_1(c_-) = 0$ and $\psi_1(c_+)\psi_1(c_-) = 0$. Then we have

$$\|z^*\phi_1(f)z - \psi_1(f)\| < \varepsilon/16 \ \text{ for all } \ f \in \mathcal{F}_0'. \tag{A.129}$$

We also have, by (A.128)

$$\psi_1(f_{\sigma_0}(c_-)^{1/2})u_1^*\phi_1(f_{\sigma_0/2}(c_-))u_1\psi_1(f_{\sigma_0}(c_-)^{1/2}) \tag{A.130}$$

$$\approx_{\varepsilon/64} \psi_1(f_{\sigma_0}(c_-)^{1/2})\psi_1(f_{\sigma_0/2}(c_-))\psi_1(f_{\sigma_0}(c_-)^{1/2}) = \psi_1(f_{\sigma_0}(c_-)). \tag{A.131}$$

Similarly,

$$\psi_1(f_{\sigma_0}(c_+)^{1/2})u_1^*\phi_1(f_{\sigma_0/2}(c_+))u_1\psi_1(f_{\sigma_0}(c_+)^{1/2}) \approx_{\varepsilon/64} \psi_1(f_{\sigma_0}(c_+)). \tag{A.132}$$

It follows from Lemma 5 of [39] (see also A.13 here for convenience) that there exists a unitary $u \in \widetilde{A}$ such that

$$\|u|z| - z\| < \varepsilon/64. \tag{A.133}$$

Combining this with (A.129), (A.130) and (A.132), we estimate that

$$\|u^*\phi_1(f)u - \psi_1(f)\| < \varepsilon \ \text{ for all } \ f \in \mathcal{F}. \tag{A.134}$$

Thus, there exists a unitary $w \in \widetilde{A}$ such that $\|w^*xw - y\| < \varepsilon$. Then

$$w^*aw = w^*(x + \lambda 1_{\widetilde{A}})w \approx_\varepsilon y + w^*(\lambda 1_{\widetilde{A}})w = b. \tag{A.135}$$

For the case $\lambda > 1 - \varepsilon/2$, note that we have reduced the general case (of this) to the case $\mathcal{F} = \{\iota\}$. Choose a 1–1 continuous function $h : (0, 1] \to (0, \lambda]$ such that

$$\|h - \iota\| < \varepsilon/2 \ \text{ and } \ \|\iota - h \circ \iota\| < \varepsilon/2. \tag{A.136}$$

Consider the composed maps $\phi_1 = \phi \circ h$ and $\psi_2 = \psi \circ h$. This reduces the problem to the case $\lambda = 1$. So there exists only one interval $[-1, 0)$. In this case we can choose δ depending only on $\varepsilon/2$ not on λ.

In the case $\lambda < \varepsilon/2$, one has a unitary $u \in \widetilde{A}$ such that $\|u^*\phi_1(c_+)u - \psi_1(c_+)\| < \varepsilon/64$. Then

$$u^*\phi(\iota)u \approx_{\varepsilon/2} u^*\phi_1(c_+)u \approx_{\varepsilon/64} \psi_1(c_+) \approx_{\varepsilon/2} \psi(\iota). \quad \square \tag{A.137}$$

Corollary A.21. *Let $C = C_0([0, 1)) \otimes \mathcal{K}$ and let A be a stably projectionless simple C^*-algebra such that $M_m(A)$ almost has stable rank one for each $m \ge 1$ and suppose that A has strict comparison for positive elements and that* $\mathrm{QT}(A) = \mathrm{T}(A)$. *Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subseteq C$ there exists $\delta > 0$ satisfying the following:*

Let $\phi, \psi\colon C \to \widetilde{A}\otimes\mathcal{K}$ be two homomorphisms such that $\mathrm{d}_w(\phi,\psi)<\delta$. Let $C_0=C_0([0,1))=C_0([0,1))\otimes e_{11}\subseteq C$ denote the 1–1 corner. If $\phi(C_0), \psi(C_0)\subseteq \widetilde{A}\otimes e_{11}$, then for any $\varepsilon>0$ and any finite subset $\mathcal{F}\subseteq C$, there exists a unitary $U\in (A\otimes\mathcal{K})^{\sim}$ such that

$$\|U^*\phi(c)U-\psi(c)\|<\varepsilon \text{ for all } c\in\mathcal{F}, \tag{A.138}$$

where $U=\mathrm{diag}(\overbrace{u,u,\dots,u}^{n},1,1,\dots)$, where $u\in U(\widetilde{A})$ for some $n\ge 1$.

Proof. We will write $\mathrm{M}_n(C_0([0,1)))$ as a sub-C*-algebra of C, $\mathrm{M}_n(\widetilde{A})$ as a sub-C*-algebra of $\widetilde{A}\otimes\mathcal{K}$ and $\mathrm{M}_n(A)$ as a sub-C*-algebra of $A\otimes\mathcal{K}$ for all integers $n\ge 1$. Let $\varepsilon>0$ and $\mathcal{F}\subseteq C$ be a finite subset. Without loss of generality, we may assume that $\mathcal{F}\subseteq \mathrm{M}_n(C)$. Furthermore we may write $\mathcal{F}=\{(c_{i,j})_{n\times n}: c_{i,j}\in\mathcal{G}\}$, where $\mathcal{G}\subseteq C_0$ is a finite subset. We will apply A.20 with ε/n^2 in place of ε and $\mathcal{G}$ in place of $\mathcal{F}$. Choose δ as provided for ε/n^2 and $\mathcal{G}$ (in place of $\mathcal{F}$) in A.20. Suppose that $\mathrm{d}_w(\phi,\psi)<\delta$.

Define $\phi_1=\phi|_{C_0}$ and $\psi_1=\psi|_{C_0}$. By A.20, there exists a unitary $u\in\widetilde{A}$ such that

$$\|u^*\phi_1(a)u-\psi_1(a)\|<\varepsilon/n^2 \text{ for all } a\in\mathcal{G}. \tag{A.139}$$

We may assume that $\pi(u)=1$, where $\pi:\widetilde{A}\to\mathbb{C}$ is the quotient map. Define

$$U=\mathrm{diag}(\overbrace{u,u,\dots,u}^{n},1,1,\dots).$$

Then $U\in(A\otimes\mathcal{K})^{\sim}$. Moreover,

$$\|U^*\phi(c)U-\psi(c)\|<\varepsilon \text{ for all } c\in\mathcal{F}.\quad\square \tag{A.140}$$

Corollary A.22. *Let $C=C_0((0,1])$ with a strictly positive element e_c and A be a stably projectionless simple C*-algebra with continuous scale such that $M_m(A)$ almost has stable rank one ($m\ge 1$). Suppose also that A has strict comparison for positive elements and that $\mathrm{QT}(A)=\mathrm{T}(A)$.*

*(a) Then, for any $\gamma:\mathrm{Cu}(C)\to\mathrm{Cu}(\widetilde{A})$ which is an ordered semigroup homomorphism in **Cu** such that $\langle e_c\rangle\le\langle 1_{\widetilde{A}}\rangle$, there exists a homomorphism $\phi: C\to\widetilde{A}$ such that $\mathrm{Cu}(\phi)=\gamma$.*

(b) Let $\phi,\psi: C\to\widetilde{A}$ be two unital homomorphisms such that $\mathrm{Cu}(\phi)=\mathrm{Cu}(\psi)$. Then $\mathrm{Cu}^{\sim}(\phi)=\mathrm{Cu}^{\sim}(\psi)$.

Proof. For part (a), for any integer $n\ge 1$, by Theorem 4 of [45], there exists homomorphism $\psi_n: C\to\widetilde{A}$ such that $d_w(\mathrm{Cu}(\psi_n),\gamma)<1/2^n$. By Corollary A.21, one obtains a sequence of homomorphisms $\phi_k: C\to\widetilde{A}$ such that $d_w(\mathrm{Cu}(\phi_k),\gamma)\to 0$ and $(\phi_k(c))_{k=1}^{\infty}$ is a Cauchy sequence for all $c\in C$. Let ϕ be the limit homomorphism. Then $\mathrm{Cu}(\phi)=\gamma$.

For part (b) follows from Corollary A.21 immediately. $\square$

Definition A.23. Let $R=R_{1,n}$ be the Razak algebra as below:

$$R=R_{1,n}=\{f\in\mathrm{M}_n(C([0,1])): f(0)=\alpha\cdot 1_{M_{n-1}} \text{ and } f(1)=\alpha\cdot 1_{M_n},\ \alpha\in\mathbb{C}\}. \tag{A.141}$$

Put

$$D=\{f\in\mathrm{M}_n(C_0([0,1))): f(0)=\begin{pmatrix}0_{n-1} & 0\\ 0 & \alpha\end{pmatrix}, \alpha\in\mathbb{C}\}.$$

Then $\widetilde{R}$ is the unitization of D. Denote by $\phi^{\sim},\psi^{\sim}:\widetilde{D}\to\widetilde{A}$ the unital extension of ϕ and ψ. Consider

$$C_0=\{f\in\mathrm{M}_n(C_0([0,1))): f(t)=\begin{pmatrix}0_{n-1} & 0\\ 0 & a(t)\end{pmatrix},\ a(t)\in C_0([0,1))\}. \tag{A.142}$$

Then $C_0\cong C_0([0,1))$. Note also that $C_0\subset D$ is a full hereditary sub-C*-algebra Let $j_0: C_0([0,1))\to C_0\subset D$ be the embedding. By Brown's theorem (see [7]), there is an isomorphism $s: C_0\otimes\mathcal{K}\cong D\otimes\mathcal{K}$. Note, from the construction in [7], the isomorphism s (given by partial isometry in $M_2(M(D\otimes\mathcal{K}))$ has the property that $\mathrm{Cu}(s)=\mathrm{Cu}(j_0)$. This was discussed in 4.3 of [43]. Let e_C be a strictly positive element of $s(C_0([0,1))\otimes e_{1,1})$ and e_{C_0} be a strictly positive element of $C_0\otimes e_{1,1}\subset D\otimes e_{1,1}\subset D\otimes\mathcal{K}$. Then $\langle e_C\rangle=\langle e_{C_0}\rangle$ in $D\otimes\mathcal{K}$. Since $D\otimes\mathcal{K}$ has stable rank one, by [9] (see also 1.7 of [35]), there is a partial isometry $w\in(D\otimes\mathcal{K})^{**}$ such that $wa, aw^*\in D\otimes\mathcal{K}$ for all $a\in s(C_0([0,1))\otimes e_{1,1})$ and $w^*aw\in C_0\otimes e_{1,1}$. Denote by $s(e_{1,1})$ the range projection of $s(C_0([0,1))\otimes e_{1,1})$. Then $s(e_{1,1})\in M(s(C_0([0,1)))\otimes\mathcal{K})$. Also $w^*s(e_{11})w=\bar{e}_{1,1}$, where $\bar{e}_{1,1}$ is the range projection of $C_0\subset D$. Clearly $\bar{e}_{1,1}\in M(D)\subset M(D\otimes\mathcal{K})$. Denote by p_D the range projection of $D\otimes e_{1,1}$. Let $P=1-p_D$ in $M(D\otimes\mathcal{K})$. Then we may write $1-\bar{e}_{11}=((1_D-\bar{e}_{1,1})\otimes 1)\oplus(\bar{e}_{1,1}\otimes P)$ which is Murray-Von Neumann equivalent to $((1_D-\bar{e}_{1,1})\otimes 1)\oplus(\bar{e}_{1,1}\otimes 1)=1_D\otimes 1$ in $M(D\otimes\mathcal{K})$. Note also $1-s(e_{1,1})$ is also Murray–Von Neumann equivalent to 1, as $s(C_0\otimes\mathcal{K})=D\otimes\mathcal{K}$. It follows that there is a partial isometry $W_1\in M(D\otimes\mathcal{K})$ such that $W_1^*W_1=(1-s(e_{1,1}))$ and $W_1W_1^*=1-\bar{e}_{1,1}$ (see also Lemma 2.5 of [7]). Define $W=W_1\oplus w$. Then $W\in M(D\otimes\mathcal{K})$ is a unitary. Set $j=\mathrm{Ad}\,W\circ s$. Note $\mathrm{Cu}(j)=\mathrm{Cu}(\mathrm{id}_{C_0})$. The additional feature is that $j(C_0([0,1)))\subset D$.

For any homomorphism $\phi : \widetilde{D} \to B$ (for some C*-algebra B), denote by ϕ again for the extension from $\widetilde{D}\otimes\mathcal{K} \to B\otimes\mathcal{K}$. Define $\phi_{C_0} = \phi\circ j : C_0([0,1))\otimes\mathcal{K} \to B\otimes\mathcal{K}$. If $\psi : R \to A$ (for any C*-algebra) is a homomorphism let $\psi^{\sim} : \widetilde{D} = \widetilde{R} \to \widetilde{A}$ be the extension. We will use $\psi_{C_0} := \psi^{\sim}\circ j : C_0([0,1))\otimes\mathcal{K} \to \widetilde{A}\otimes\mathcal{K}$.

With the definition above, we present the following lemma:

Lemma A.24. *Let D be as in A.23 and let A be a stably projectionless simple C*-algebra such that $M_m(A)$ almost has stable rank one for every $m \geq 1$, has strict comparison and $\mathrm{QT}(A) = \mathrm{T}(A)$. Then, for any $\eta > 0$ and any finite subset $\mathcal{S} \subseteq \widetilde{D}$, there exists $\delta_0 > 0$ satisfying the following condition:*

For any two unital homomorphisms $\phi, \psi : \tilde{D} \to \widetilde{A}$, if

$$\mathrm{d}_w(\phi_{C_0}, \psi_{C_0}) < \delta_0, \tag{A.143}$$

then there exists a unitary $u \in \widetilde{A}$ such that

$$\|u^*\psi(f)u - \phi(f)\| < \eta \text{ for all } f \in \mathcal{S}. \tag{A.144}$$

Proof. Fix $\eta > 0$ and finite subset $\mathcal{S} \subset \widetilde{D}$. We may assume that $\mathcal{S} = \{g + r\cdot 1_{\widetilde{D}} : g \in \mathcal{G}, r \in K\}$, where $\mathcal{G} \subset D$ is a finite subset and K is a finite subset of $\mathbb{C}$. Let $j : C_0([0,1))\otimes\mathcal{K} \to D\otimes\mathcal{K}$ be the isomorphism defined in A.23. Let $\mathcal{F} = j^{-1}(\mathcal{G})$. Note $j(C_0([0,1))\otimes e_{11}) \subset D$. Let $\delta_0 > 0$ be given for η (in place of ε) and $\mathcal{F}$ by A.21. Consider $\phi\circ j, \psi\circ j : C_0([0,1))\otimes\mathcal{K} \to \widetilde{A}\otimes\mathcal{K}$. By applying A.21, there exist a unitary $u \in \widetilde{A}$ and an integer $n \geq 1$ such that

$$\|U^*\phi\circ j(f)U - \psi\circ j(f)\| < \eta \text{ for all } f \in \mathcal{F}, \tag{A.145}$$

where $U = \mathrm{diag}(\overbrace{u, u, \ldots, u}^{n}, 1, 1, \ldots)$. This implies that

$$\|U^*\phi(g)U - \psi(g)\| < \eta \text{ for all } g \in \mathcal{G}. \tag{A.146}$$

Since $\phi(g), \phi(g) \in \widetilde{A}$ for all $g \in \mathcal{G}$, we actually have

$$\|u^*\phi(g)u - \psi(g)\| < \eta \text{ for all } g \in \mathcal{G}. \tag{A.147}$$

Since both ϕ and ψ are unital and $u^*r\cdot 1_{\widetilde{A}}u = r\cdot 1_{\widetilde{A}}$, we finally conclude that (A.144) holds. □

Theorem A.25. *Let R be a Razak algebra and let A be a separable stably projectionless simple C*-algebra with continuous scale such that $\mathrm{M}_n(A)$ almost has stable rank one for every $n \geq 1$. Suppose that A also has strict comparison for positive elements and $\mathrm{QT}(A) = \mathrm{T}(A)$. Let $\gamma : \mathrm{Cu}(\widetilde{R}) \to \mathbb{N}\sqcup S(\widetilde{A}) \subset \mathrm{Cu}(\widetilde{A})$ (see A.5 and (vii) of A.11) be an ordered semigroup homomorphism in* **Cu** *with $\gamma(\langle 1_{\widetilde{R}}\rangle) = \langle 1_{\widetilde{A}}\rangle$ such that $\gamma|_{\mathrm{Cu}(R)} \subset \mathrm{Cu}(A)$ and $\gamma(\langle a\rangle) \neq 0$ for all $\langle a\rangle \neq 0$ in $\mathrm{Cu}(\widetilde{R})$. We also assume that γ maps elements which cannot be represented by projections to the sub-semigroup $S(\widetilde{A})$. Then there exists a homomorphism $\phi : R \to A$ such that $\mathrm{Cu}(\phi) = \gamma|_{\mathrm{Cu}(R)}$.*

Proof. We will keep notation in A.23. In what follows denote by $\pi : \widetilde{A} \to \mathbb{C}$ as well as $\pi : \widetilde{D} \to \mathbb{C}$ for the quotient maps. By the assumption above, we $\mathrm{Cu}(\pi)\circ\gamma|_{\mathrm{Cu}(R)} = 0$.

Since $\widetilde{R} = \widetilde{D}$, D is a hereditary sub-C*-algebra of $\widetilde{R}$. It follows that $\mathrm{Cu}(D)$ is an ordered sub-semigroup of $\mathrm{Cu}(\widetilde{R})$. Note also $D\otimes\mathcal{K} \cong C_0([0,1))\otimes\mathcal{K}$. We specify a strictly positive element $e_D(t) = \begin{pmatrix} g(t)\cdot 1_{n-1} & 0 \\ 0 & (1-t)\end{pmatrix}$ (for $t \in [0,1]$), where $g(t) = 2t$ if $t \in [0, 1/2]$ and $g(t) = 2(1-t)$ for $t \in (1/2, 1]$. Note $\gamma(\langle e_D\rangle) \leq \langle 1_{\mathrm{A}}\rangle$. By part (a) of A.22, there is a homomorphism $\phi_C : C_0([0,1))\otimes\mathcal{K} \to \mathrm{A}\otimes\mathcal{K}$ such that $\mathrm{Cu}(\phi_C) = \gamma\circ\mathrm{Cu}(j)$. Then $\psi := \phi_C\circ j^{-1} : D\otimes\mathcal{K} \to \widetilde{A}\otimes\mathcal{K}$ is a homomorphism such that $\mathrm{Cu}(\phi_C\circ j^{-1}) = \gamma|_{\mathrm{Cu}(D)}$.

Define $e_n = \psi(f_{1/2^n}(e_D))$, $n = 1, 2, \ldots$. Note $\gamma(\langle e_D\rangle) \leq \gamma(\langle 1_{\widetilde{D}}\rangle) = \langle 1_{\widetilde{A}}\rangle$. It follows from Proposition 2.4 of [46] that there exists $x_n \in \widetilde{A}\otimes\mathcal{K}$ such that

$$e_n = x_n^* 1_{\widetilde{A}} x_n, \quad n = 1, 2, \ldots. \tag{A.148}$$

Put $y_n = 1_{\widetilde{A}}x_n$. Let $y_n = v_n|y_n|$ be the polar decomposition of y_n in $(\widetilde{A})^{**}$. Then $v_n a \in \widetilde{A}\otimes\mathcal{K}$ for all $a \in \overline{e_n(\widetilde{A}\otimes\mathcal{K})e_n}$ and $v_n^* a v_n \in \widetilde{A}$ for all $a \in e_n(\widetilde{A}\otimes\mathcal{K})e_n$, $n = 1, 2, \ldots$ Define $\psi_n : D \to \widetilde{A}$ by $\psi_n(d) = v_n^* e_n\psi(d)e_n v_n$ for all $d \in D$. Since D is semiprojective, there exists, for each large n, a homomorphism $h_n : D \to \widetilde{A}$ such that

$$\lim_{n\to\infty} \|h_n(d) - \psi_n(d)\| = 0 \text{ for all } d \in D. \tag{A.149}$$

Define $h_n^{\sim} : \widetilde{D} \to \widetilde{A}$ by defining $h_n(1_{\widetilde{D}} + d) = 1_{\widetilde{A}} + h_n(d)$. It is a unital homomorphism.

Since $\lim_{n\to\infty}\|e_n^*\psi(d)e_n - \psi(d)\| = 0$, it is easy to compute that

$$\lim_{n\to\infty} d_w(\mathrm{Cu}((h_n^{\sim})_{C_0}), \mathrm{Cu}(\phi_{C_0})) = 0 \tag{A.150}$$

Let $\mathcal{F}_n \subset \widetilde{D}$ be finite subsets such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\cup_{n=1}^{\infty}\mathcal{F}_n$ is dense in $\widetilde{D}$. It follows from A.24 that there exist a subsequence $\{n_k\}$ and a sequence of unitaries $u_k \in \widetilde{A}$ such that,

$$\|\mathrm{Ad}\, u_{k+1} \circ h_{n_{k+1}}(d) - \mathrm{Ad}\, u_k \circ h_{n_k}(d)\| < 1/2^k \ \text{ for all } \ d \in \mathcal{F}_k,\ \ k = 1, 2, \ldots. \tag{A.151}$$

It follows that $(\mathrm{Ad}\, u_k \circ h_{n_k}(a))_{k=1}^{\infty}$ a Cauchy sequence for each $d \in \widetilde{D}$. Let $H(a)$ be the limit. Then H defines a unital homomorphism from $\widetilde{D}$ to $\widetilde{A}$. By (A.150), $\mathrm{Cu}(H_{C_0}) = \mathrm{Cu}(\phi_C)$. Since $D\otimes\mathcal{K} \cong C_0([0, 1))\otimes\mathcal{K}$, we then have $\mathrm{Cu}(H|_D) = \gamma|_{\mathrm{Cu}(D)}$.

Since $\mathrm{Cu}(\pi) \circ \gamma|_{\mathrm{Cu}(R)}= 0$, if $\mathrm{Cu}(H) = \gamma$, then $H|_R \subset A$. Therefore it remains to show $\mathrm{Cu}(H) = \gamma$. To show that, we will apply part (b) of A.22.

It follows by [53] that there is a separable simple C*-algebra B which is an inductive limit of Razak algebras with continuous scale such that $\mathrm{T}(B) = \mathrm{T}(A)$. Note that B has stable rank one and $K_0(B) = \{0\}$. By 6.2.3 of [43] (see also 7.3 of [17]), $\mathrm{Cu}(\widetilde{B}) = \mathbb{N} \sqcup L(\widetilde{B})$. It follows by A.6 there is an ordered semigroup isomorphism $\gamma_b : L(\widetilde{B}) \to S(\widetilde{A})$. This extends to an ordered semigroup isomorphism $\gamma_B : \mathrm{Cu}(\widetilde{B}) \to \mathbb{N}\sqcup S(\widetilde{A})$. It then extends an ordered semigroup isomorphism $\gamma_B^{\sim} : \mathrm{Cu}^{\sim}(\widetilde{B}) \to \mathbb{Z} \cup S^{\sim}(\widetilde{A})$ defined by $\gamma_B(m) = m$ and $\gamma_B^{\sim}(x - k\langle 1_{\tilde{B}}\rangle) = \gamma_b(x) - k\langle 1_{\tilde{A}}\rangle$. (Recall that $\widetilde{B}$ has stable rank one and, by 3.16 of [43], $\mathrm{Cu}(\widetilde{B})$ embedded into $\mathrm{Cu}^{\sim}(\widetilde{B})$ and, by (vi) of A.11, $S(\widetilde{A})$ embedded into $S^{\sim}(\widetilde{A})$.) Moreover it induces an isomorphism $\gamma_{B^{\sim}} : \mathrm{Cu}^{\sim}(B) \to S^{\sim}(A)$. Let γ_b^{-1}, γ_B^{-1} and $\gamma_{B^{\sim}}^{-1}$ be the inverse maps of γ_b, γ_B and $\gamma_{B^{\sim}}$.

Define $\gamma_{\sim} : \mathrm{Cu}^{\sim}(D) \to S^{\sim}(A)$ by $\gamma_{\sim}(\langle a\rangle - n\langle 1_{\tilde{D}}\rangle) = \gamma(\langle a\rangle) - n\langle 1_{\tilde{A}}\rangle$ for all $a \in \mathrm{Cu}(\widetilde{D})$ which are not represented by projections (recall also $V(\widetilde{D}) = \mathbb{N}$ and $K_0(D) = K_0(C_0([0, 1))) = \{0\}$.) and $n = \langle\pi(a)\rangle < \infty$. This extends $\gamma|_{\mathrm{Cu}(D)}$. We can also define $\gamma^{\sim} : \mathrm{Cu}^{\sim}(\widetilde{D}) \to \mathbb{Z} \sqcup S^{\sim}(\widetilde{A})$ (see A.11) by $\gamma^{\sim}(m\langle 1_{\tilde{D}}\rangle) = m\langle 1_{\tilde{A}}\rangle$ and $\gamma^{\sim}(\langle a\rangle - k\langle 1_{\tilde{D}}\rangle) = \gamma(\langle a\rangle) - k\langle 1_{\tilde{D}}\rangle$ for all $\langle a\rangle \in \widetilde{A}$ which are not represented by projections. Note, in fact, since both $\widetilde{D}$ and $\widetilde{A}$ are unital, $\gamma^{\sim}$ is uniquely determined by γ (see 3.1 of [43]). Note that $\gamma^{\sim}|_{\mathrm{Cu}^{\sim}(D)}= \gamma_{\sim}$, $\gamma^{\sim}|_{\mathrm{Cu}(\tilde{D})}= \gamma$ and $\gamma_{\sim}|_{\mathrm{Cu}(D)}= \gamma|_{\mathrm{Cu}(D)}$. Recall that $\mathbb{N}\sqcup S(\widetilde{A}) \cong \mathrm{Cu}(\widetilde{B})$, Note that, by the assumption, $\gamma_{\sim}$, $\gamma^{\sim}$ are ordered semigroup homomorphisms in **Cu**.

Note $\widetilde{D}$ is a 1-dimensional NCCW and $\widetilde{B}$ has stable rank one. By Theorem 1.0.1 of [43], there is a unital homomorphism $\Psi_{d,b} : \widetilde{D} \to \widetilde{B}$ such that $\mathrm{Cu}^{\sim}(\Psi_{d,b}) = (\gamma_B^{\sim})^{-1} \circ \gamma^{\sim}$.

Let $D_1 = H(D)$ and let $\imath_{D_1} : D_1 \to \widetilde{A}$ be the embedding. Denote also $\imath_{D_1} : \widetilde{D}_1 \to \widetilde{A}$ the unital extension. Since γ is strictly positive, $\imath_{D_1} \circ H|_D$ is an isomorphism. Then there exists a homomorphism $\Psi_{D_1} : D_1 \to \widetilde{B}$ such that $\mathrm{Cu}(\Psi_{D_1}) = \gamma_B^{-1} \circ \mathrm{Cu}(\imath_{D_1})$. Let $\Psi_{D_1}^{-1} : \Psi|_{D_1}(D_1) \to D_1$ be the inverse of Ψ_{D_1}. Then $\mathrm{Cu}(\Psi_{D_1} \circ H|_D) = \gamma_B^{-1} \circ\gamma|_{\mathrm{Cu}(D)}$. In particular, $\mathrm{Cu}(\Psi_{d,b}\circ j) = \gamma_B^{-1}\circ\gamma\circ\mathrm{Cu}(j)$. It follows from part (b) of A.22 that $\mathrm{Cu}^{\sim}(\Psi_{d,b} \circ j) = \mathrm{Cu}^{\sim}(\Psi_{C_1} \circ H \circ j)$. Note that j is also an isomorphism from $C_0([0, 1))\otimes\mathcal{K}$ onto $D \otimes \mathcal{K}$. It follows that $\mathrm{Cu}^{\sim}(\Psi_{D_1} \circ H|_D) = \mathrm{Cu}^{\sim}(\Psi_{d,b}|_D)$. In other words, $\mathrm{Cu}^{\sim}(\Psi_{D_1} \circ H|_D) = (\gamma_B^{\sim})^{-1} \circ \gamma^{\sim}|_{\mathrm{Cu}^{\sim}(D)}$. Since $\Psi_{D_1}^{-1} \circ \Psi_{D_1} \circ H = H$, it follows that

$$\mathrm{Cu}^{\sim}(H)|_{\mathrm{Cu}^{\sim}(D)}= \gamma^{\sim}|_{\mathrm{Cu}^{\sim}(D)}= \gamma_{\sim}, \tag{A.152}$$

where $\mathrm{Cu}^{\sim}(H) : \mathrm{Cu}^{\sim}(\widetilde{D}) \to \mathrm{Cu}^{\sim}(\widetilde{A})$ is the map induced by H.

Now let $x \in \mathrm{Cu}(\widetilde{D})$ with $\langle\pi(x)\rangle = n < \infty$. Then

$$\mathrm{Cu}(H)(x) = \mathrm{Cu}^{\sim}(H)(x - n\langle 1_{\tilde{D}}\rangle) + \mathrm{Cu}^{\sim}(H)(n\langle 1_{\tilde{D}}\rangle) \tag{A.153}$$

$$= \gamma_{\sim}(x - n\langle 1_{\tilde{D}}\rangle) + n\langle 1_{\tilde{A}}\rangle = \gamma^{\sim}(x - n\langle 1_{\tilde{D}}\rangle) + \gamma^{\sim}(n\langle 1_{\tilde{D}}\rangle) \tag{A.154}$$

$$= \gamma^{\sim}(x) = \gamma(x). \tag{A.155}$$

It follows that $\mathrm{Cu}(H) = \gamma$. □

Theorem A.26. *Let A be a separable simple stably projectionless C^*-algebra with continuous scale such that $M_m(A)$ has almost stable rank one for all $m \geq 1$. Suppose that $\mathrm{QT}(A) = \mathrm{T}(A)$ and $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$. Suppose also that B is a simple C^*-algebra which is an inductive limit of Razak algebras with injective connecting maps, with continuous scale and with $\mathrm{T}(A) = \mathrm{T}(B)$. Then there exists a homomorphism $\phi : B \to A$ which maps strictly positive elements to strictly positive elements and which induces the identification $\mathrm{T}(A) = \mathrm{T}(B)$.*

Proof. Let us construct the required homomorphism ϕ.

Note $\mathrm{Cu}(A) = \mathrm{LAff}_+(\mathrm{T}(A))$ and $\mathrm{Cu}(B) = \mathrm{LAff}_+(\mathrm{T}(B))$. Denote by $\Lambda : \mathrm{T}(A) \to \mathrm{T}(B)$ the affine homeomorphism. Then Λ induces an ordered semigroup isomorphism $\lambda_0 : \mathrm{Cu}(B) \to \mathrm{Cu}(A)$ in **Cu**.

Fix strictly positive elements e_B of B and e_A of A, respectively. Then $\lambda_0(\langle e_B\rangle) \leq \langle e_A\rangle$. Consider the sub-semigroup $S = S(\widetilde{A}) \subseteq \mathrm{Cu}(\widetilde{A})$ defined in A.5. Note that, by A.10, $\mathrm{Cu}(\widetilde{B}) = \mathbb{N} \sqcup L(\widetilde{B})$. By A.6, this induces an ordered semigroup isomorphism $\lambda_1 : \mathrm{Cu}(\widetilde{B}) \to \mathbb{N} \sqcup S(\widetilde{A}) \subseteq \mathrm{Cu}(\widetilde{A})$ with $\lambda_1(\langle 1_{\tilde{B}}\rangle) = \langle 1_{\tilde{A}}\rangle$ (see also (vii) of A.11). Write $B = \lim_{n\to\infty}(R_n, \imath_n)$ (see [53]), where each $\imath_n$ is injective. Let $\gamma_n : \mathrm{Cu}(\widetilde{R}_n) \to \mathrm{Cu}(\widetilde{A})$ be given by $\lambda_1 \circ \mathrm{Cu}(\imath_{n,\infty})$. Note that $\gamma_{n+1} \circ \mathrm{Cu}(\imath_n) = \gamma_n$ and $\gamma_n(\langle 1_{\tilde{R}_n}\rangle) = \langle 1_{\tilde{A}}\rangle$. It follows from A.25 that there is a unital homomorphism $\phi_n : \widetilde{R}_n \to \widetilde{A}$ such that $\mathrm{Cu}(\phi_n) = \gamma_n$ and $\phi_n|_{R_n} \subset A$, $n = 1, 2, \ldots$. Note also $\mathrm{Cu}(\phi_{n+1} \circ \imath_n) = \mathrm{Cu}(\phi_n)$, $n = 1, 2, \ldots$. We also have $\gamma_n(\mathrm{Cu}(R_n)) \subset \mathrm{Cu}(A)$. If $x \in \mathrm{Cu}(\widetilde{R}_n)$ is not represented by a projection, neither $\mathrm{Cu}(\imath_{n,\infty})(x)$. It follows that $\gamma_n(x) \subset S(\widetilde{A})$, $n = 1, 2, \ldots$.

Let (ε_n) be a decreasing sequence of positive numbers with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Let $\mathcal{F}_n \subseteq R_n$ be finite subsets such that $\imath_n(\mathcal{F}_n) \subseteq \mathcal{F}_{n+1}$, $n = 1, 2, \ldots$, and we assume that $\bigcup_{n=1}^{\infty} \imath_{n,\infty}(\mathcal{F}_n)$ is dense in B. By A.24, there exists a sequence of unitaries

$u_n \in \widetilde{A}$ such that

$$\|\mathrm{Ad}\, u_n \circ \phi_{n+1}(b) - \mathrm{Ad} \circ \phi_n(b)\| < 1/2^n \quad \text{for all} \quad b \in \mathcal{F}_n, \tag{A.156}$$

$n = 1, 2, \ldots$. Then $(\mathrm{Ad}\, u_n \circ \phi_n(b))_{n=1}^{\infty}$ is a Cauchy sequence in A for each $b \in B$. Let $\phi(b)$ be the limit (for each $b \in B$). Then ϕ is a homomorphism from B to A such that $\mathrm{Cu}(\phi) = \lambda_1$. From the definition of λ_1, we see that ϕ meets the requirement. □

Corollary A.27. *Let A be a separable simple C^*-algebra which has finite nuclear dimension and continuous scale. Then there exist a simple C^*-algebra B which is an inductive limit of Razak algebras with injective connecting maps and with $\mathrm{T}(A) = \mathrm{T}(B)$ and a homomorphism $\phi : B \to A$ which maps strictly positive elements to strictly positive elements and which induces the identification $\mathrm{T}(A) = \mathrm{T}(B)$.*

Proof. First, the existence of such a C^*-algebra B with $\mathrm{T}(B) = \mathrm{T}(A)$ is given by 2.8. It follows from [54] that A is $\mathcal{Z}$-stable and has strict comparison for positive elements. Moreover, by [44], $M_r(A)$ (for every integer $r \geq 1$) and $A \otimes \mathcal{K}$ have almost stable rank one. By Lemma 6.5 of [19], $\mathrm{Cu}(A) = \mathrm{LAff}_+(\widetilde{\mathrm{T}}(A))$. Thus A.26 applies. □

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