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On classification of non-unital amenable simple C*-algebras, II

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ABSTRACT

We present a classification theorem for separable amenable simple stably projectionless C^*-algebras with finite nuclear dimension whose K_0 vanish on traces which satisfy the Universal Coefficient Theorem. One of C^*-algebras in the class is denoted by $\mathcal{Z}_0$ which has a unique tracial state, $K_0(\mathcal{Z}_0)=\mathbb{Z}$ and $K_1(\mathcal{Z}_0)=\{0\}$. Let A and B be two separable amenable simple C^*-algebras satisfying the UCT. We show that $A\otimes\mathcal{Z}_0\cong B\otimes\mathcal{Z}_0$ if and only if $\mathrm{Ell}(A\otimes\mathcal{Z}_0)=\mathrm{Ell}(B\otimes\mathcal{Z}_0)$. A class of simple separable C^*-algebras which are approximately sub-homogeneous whose spectra having bounded dimension is shown to exhaust all possible Elliott invariant for C^*-algebras of the form $A\otimes\mathcal{Z}_0$, where A is any finite separable simple amenable C^*-algebras.

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1. Introduction

Recently some sweeping progresses have been made in the Elliott program [11], the program of classification of separable amenable C^*-algebras by the Elliott invariant (a K-theoretical set of invariant) (see [20,58] and [14]). These are the results of decades of work by many mathematicians (see also [20,58] and [14] for the historical discussion there). These progresses could be summarized briefly as the following: Two unital finite separable simple C^*-algebras A and B with finite nuclear dimension which satisfy the UCT are isomorphic if and only if their Elliott invariant $\mathrm{Ell}(A)$ and $\mathrm{Ell}(B)$ are isomorphic. Moreover, all weakly unperforated Elliott invariant can be achieved by a finite separable simple C^*-algebras in UCT class with finite nuclear dimension (In fact these can be constructed as so-called ASH-algebras—see [20]). Combining with the previous classification of purely infinite simple C^*-algebras, results of Kirchberg and Phillips [46] and [26], now all unital separable simple C^*-algebras in the UCT class with finite nuclear dimension are classified by the Elliott invariant.

This research studies the non-unital cases.

Suppose that A is a separable simple C^*-algebra. In the case that $K_0(A)_+\neq\{0\}$, then $A\otimes\mathcal{K}$ has a non-zero projection, say p. Then $p(A\otimes K)p$ is unital. Therefore if A is in the UCT class and has finite nuclear dimension, then $p(A\otimes\mathcal{K})p$ falls into the class of C^*-algebras which has been classified. Therefore isomorphism theorem for these C^*-algebras is an immediate consequence of that in [20] (see section 8.4 of [39]) using the stable isomorphism theorem of [4].

Therefore this paper considers the case that $K_0(A)_+=\{0\}$. Simple C^*-algebras with $K_0(A)_+=\{0\}$ are stably projectionless in the sense that not only A has no non-zero projections but $M_n(A)$ also has no non-zero projections for every integer $n\geq 1$. However, as one may see in this paper, $K_0(A)$ could still exhaust any countable abelian groups as well as any possible $K_1(A)$. In particular, the results in [20] cannot be applied in the stably projectionless case. It is entirely

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new situation. If one views C^*-algebra theory as the study of non-commutative topological spaces, then unital C^*-algebras correspond to the compact spaces and non-unital ones correspond to non-compact spaces. However, stably projectionless simple C^*-algebras may be viewed as non-commutative topological spaces which are not even locally compact. This causes great difficulties. Different methods have to be developed. In fact, the current paper is mostly independent of [20].

In [15], we introduce a class of stably projectionless simple C^*-algebras $\mathcal{D}$ (see 3.9). We also introduced the notion of generalized tracial rank one for stably projectionless simple C^*-algebras. These are separable stably projectionless simple C^*-algebras which are stably isomorphic to C^*-algebras in $\mathcal{D}$ (see 3.9). If A is stably isomorphic to one in $\mathcal{D}$, we will write $gTR(A) \le 1$. Some study of the structure of these C^*-algebras were also presented in [15]. For example, among other things, we show that C^*-algebras have stable rank one. Let A and B be two stably projectionless simple amenable C^*-algebras satisfy the UCT. Suppose that $K_0(A) = K_1(A) = K_0(B) = K_1(B) = \{0\}$. In the first part of this research (see [16]), we show that $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ (see [16]). In this case the Elliott invariant is reduced to $\mathrm{Ell}(A) = (\tilde{T}(A), \Sigma_A)$ (see 2.10). Combining the above mentioned results, this also gives a classification for separable stably finite projectionless simple C^*-algebras with finite nuclear dimension in the UCT class with trivial K_i-theory.

In the current paper, we study the general case that K-theory of C^*-algebras are non-trivial. We give the following theorem:

Theorem 1.1 (*See* 13.2)**.** *Let A and B be two separable simple amenable C^*-algebras which satisfy the UCT. Suppose that $gTR(A) \le 1$ and $gTR(B) \le 1$ and $K_0(A) = \ker \rho_A$ and $K_0(B) = \ker \rho_B$. Then $A \cong B$ if and only if*

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B). \tag{e1.1}$$

Among all stably projectionless separable simple C^*-algebras, one particularly interesting one is $\mathcal{W}$, a separable C^*-algebra with only one tracial state such that $K_0(\mathcal{W}) = K_1(\mathcal{W}) = \{0\}$. $\mathcal{W}$ is also an inductive limit of sub-homogeneous C^*-algebras (see [47]). It was shown in the first part ([15] and [16]) of this research that if A is a separable simple C^*-algebra in the UCT class, with finite nuclear dimension, with a unique tracial state and zero $K_i(A)$, then $A \cong \mathcal{W}$.

In this part of the research, another stably projectionless simple C^*-algebra $\mathcal{Z}_0$ with a unique tracial state plays a prominent role. This C^*-algebra has the property that $K_0(\mathcal{Z}_0) = \mathbb{Z}$ and $K_1(\mathcal{Z}_0) = \{0\}$. As abelian groups, $K_i(\mathcal{Z}_0) = K_i(\mathbb{C})$, $i = 0, 1$. Therefore, by the Künneth Formula, for any separable C^*-algebra A, $K_i(A \otimes \mathcal{Z}_0) = K_i(A)$, as abelian group, $i = 0, 1$. Moreover, if the tracial state space of A is not empty, then $T(A \otimes \mathcal{Z}_0) = T(A)$, since $\mathcal{Z}_0$ has only one tracial state. As consequence of our main results, $\mathcal{Z}_0 \otimes \mathcal{Z}_0 \cong \mathcal{Z}_0$. Moreover, we show that if A is a separable simple C^*-algebra in the UCT class, with finite nuclear dimension, unique tracial state, $K_1(A) = \{0\}$ and $K_0(A) = \ker \rho_A \cong \mathbb{Z}$, then $A \cong \mathcal{Z}_0$ (see 15.7). Therefore we are particularly interested in $\mathcal{Z}_0$-stable C^*-algebras, i.e., those C^*-algebras with the property that $A \otimes \mathcal{Z}_0 \cong A$.

It should be noted that the condition that $K_0(A) = \ker \rho_A$ ensures that A is stably projectionless. There are cases that $K_0(A)_+ = \{0\}$ but $K_0(A) \ne \ker \rho_A$ which will be dealt in a subsequent paper. However, we prove the following theorem:

Theorem 1.2 (*See* 15.8)**.** *Let A and B be two separable simple C^*-algebras with finite nuclear dimension which satisfies the UCT. Then $A \otimes \mathcal{Z}_0 \cong B \otimes \mathcal{Z}_0$ if and only if*

$$\mathrm{Ell}(A \otimes \mathcal{Z}_0) \cong \mathrm{Ell}(B \otimes \mathcal{Z}_0). \tag{e1.2}$$

(Added in September, 2020: the condition that A and B have finite nuclear dimension could be replaced by the condition that A and B are amneable, as $A \otimes \mathcal{Z}_0$ and $B \otimes \mathcal{Z}_0$ are both $\mathcal{Z}$-stable and hence both have finite nucler dimension by a result of J. Castillejos and S. Evington, arXiv:1901.11441)

When A and B are infinite, then both $A \otimes \mathcal{Z}_0$ and $B \otimes \mathcal{Z}_0$ are purely infinite simple. This case is covered by Kirchberg–Phillips classification theorem (see [26] and [46]).

We also present models for C^*-algebras stably isomorphic to C^*-algebras in $\mathcal{D}$. These model C^*-algebras are locally approximated by sub-homogeneous C^*-algebras whose spectra have dimension no more than 3. We show that these C^*-algebras exhaust all possible Elliott invariant for separable $\mathcal{Z}_0$-stable C^*-algebras as stated as follows (see also 7.12):

Theorem 1.3 (*See* 8.4)**.** *Let A be a finite separable simple amenable C^*-algebra. Then there exists a stably projectionless simple C^*-algebra B which is locally approximated by sub-homogeneous C^*-algebras and which is stably isomorphic to a C^*-algebra in $\mathcal{D}$ such that*

$$\mathrm{Ell}(A \otimes \mathcal{Z}_0) = \mathrm{Ell}(B). \tag{e1.3}$$

Finally, let us point out, if A is a separable simple C^*-algebra with $\mathrm{ke}\rho_A = K_0(A)$ in the UCT class, then A has finite nuclear dimension implies that $gTR(A) \le 1$ (see 15.5) (the converse also holds by the classification theorem). Therefore the conditions $gTR(A) \le 1$ and $gTR(B) \le 1$ in Theorem 1.1 can be replaced by finite nuclear dimension when traces vanish on $K_0(A)$ and $K_0(B)$. In fact, we have the following:

Theorem 1.4 (*See* 15.6)**.** *Let A and B be two finite separable simple C^*-algebras with finite nuclear dimension which satisfy the UCT. Suppose that $K_0(A) = \ker \rho_A$ and $K_0(B) = \ker \rho_B$. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.*

The paper also includes an Appendix which shows every separable and amenable C^*-algebra in $\mathcal{D}$ is $\mathcal{Z}$-stable which is based on [42]. This research is also benefited from previous results related to the classification of simple projectionless C^*-algebras (such as [47,49,50,55], and [57], as well as many others).

2. Preliminaries

Definition 2.1. Let A be a unital C^*-algebra and let $x \in A$. Suppose that $\|xx^* - 1\| < 1$ and $\|x^*x - 1\| < 1$. Then $x|x|^{-1}$ is a unitary. Let us use $\lceil x \rceil$ to denote $x|x|^{-1}$.

Denote by $U(A)$ the unitary group of A and denote by $U_0(A)$ the normal subgroup of $U(A)$ consisting of those unitaries which are path connected with 1_A. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$.

If $u \in A$ is a unitary, then $\bar{u}$ is the image of u in $U(A)/CU(A)$, and if $\mathcal{U} \subset U(A)$ is a subset, then $\overline{\mathcal{U}} = \{\bar{u} : u \in \mathcal{U}\}$.

Definition 2.2. Let A be a C^*-algebra. Denote by $A^{\mathbf{1}}$ the unit ball of A.

Let B be another C^*-algebra and let $\varphi : A \to B$ be a completely positive linear map. Suppose that $r \geq 1$ be an integer. This map induces a completely positive linear map $\varphi \otimes \mathrm{id}_{M_r} : A \otimes M_r \to B \otimes M_r$. *Throughout this paper, we will use notation φ instead of $\varphi \otimes \mathrm{id}_{M_r}$ whenever it is convenient.*

Let A be a non-unital C^*-algebra and let $\varphi : A \to B$ (for some C^*-algebra B) be a linear map. *Sometime in the paper, we will continue to use φ for the unital extension from $\tilde{A}$ to $\tilde{B}$, whenever it is convenient.*

Definition 2.3. Let A be a C^*-algebra. Denote by $T(A)$ the tracial state of A (which could be an empty set). Let $\mathrm{Aff}(T(A))$ be the space of all real valued affine continuous functions on $T(A)$. Let $\tilde{T}(A)$ be the cone of densely defined, positive lower semi-continuous traces on A equipped with the topology of point-wise convergence on elements of the Pedersen ideal $\mathrm{Ped}(A)$ of A. Let B be another C^*-algebra with $T(B) \neq \emptyset$ and let $\varphi : A \to B$ be a homomorphism. We will use then $\varphi_T : T(B) \to T(A)$ for the induced continuous affine map.

Let $r \geq 1$ be an integer and $\tau \in \tilde{T}(A)$. We will continue to use τ on $A \otimes M_r$ for $\tau \otimes \mathrm{Tr}$, where Tr is the standard trace on M_r. Let $S \subset \tilde{T}(A)$ be a convex subset. Define (see [49])

$$\mathrm{Aff}(S)_+ = \{f : C(S, \mathbb{R})_+ : f \text{ affine}, f(\tau) \geq 0\}, \tag{e2.1}$$
$$\mathrm{Aff}_+(S) = \{f : C(S, \mathbb{R})_+ : f \text{ affine}, f(\tau) > 0 \text{ for } \tau \neq 0\} \cup \{0\}, \tag{e2.2}$$
$$\mathrm{LAff}_f(S)_+ = \{f : S \to [0, \infty) : \exists \{f_n\}, f_n \nearrow f,\ f_n \in \mathrm{Aff}(S)_+\}, \tag{e2.3}$$
$$\mathrm{LAff}_{f,+}(S) = \{f : S \to [0, \infty) : \exists \{f_n\}, f_n \nearrow f,\ f_n \in \mathrm{Aff}_+(S)\}, \tag{e2.4}$$
$$\mathrm{LAff}(S)_+ = \{f : S \to [0, \infty] : \exists \{f_n\}, f_n \nearrow f,\ f_n \in \mathrm{Aff}(S)_+\}, \tag{e2.5}$$
$$\mathrm{LAff}_+(S) = \{f : S \to [0, \infty] : \exists \{f_n\}, f_n \nearrow f,\ f_n \in \mathrm{Aff}_+(S)\} \text{ and} \tag{e2.6}$$
$$\mathrm{LAff}^{\sim}(S) = \{f_1 - f_2 : f_1 \in \mathrm{LAff}_+(S) \text{ and } f_2 \in \mathrm{Aff}_+(S)\}. \tag{e2.7}$$

For most part of this paper, $S = \tilde{T}(A)$ or $S = T(A)$ in the above definition will be used. Moreover, $\mathrm{LAff}_{b,+}(\tilde{T}(A))$ is the subset of those bounded functions in $\mathrm{LAff}_{f,+}(\tilde{T}(A))$.

Definition 2.4. Let A be a C^*-algebra with $T(A) \neq \emptyset$. Let $\pi_A : \tilde{A} \to \mathbb{C}$ be the quotient map and $s : \mathbb{C} \to \tilde{A}$ be the homomorphism such that $\pi \circ s = \mathrm{id}_{\mathbb{C}}$. Recall that we also use π_A for the induced homomorphism $\pi_A \otimes \mathrm{id}_{M_r} : M_r(\tilde{A}) \to M_r$ and use s for the induced homomorphism $s \otimes \mathrm{id}_{M_r} : M_r \to M_r(\tilde{A})$ for all integer $r \geq 1$. Let $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ be the order preserving homomorphism defined by $\rho([p] - [s \circ \pi_A(p)])(\tau) = \tau(p - s \circ \pi_A(p))$ for any projections in $M_r(\tilde{A})$ for all integer $r \geq 1$.

Suppose that A is non-unital and separable, and $\tilde{T}(A) \neq \{0\}$. Suppose that there exists $a \in \mathrm{Ped}(A)_+$ which is full. Let $A_a = \overline{aAa}$. Then $T(A_a) \neq \emptyset$. We define

$$\ker \rho_A = \{x \in K_0(A_a) : \rho_A(x) = 0\} \tag{e2.8}$$

Here we also identify $K_0(A_a)$ with $K_0(A)$ using the Brown's stable isomorphism theorem [4].

Suppose that A is unital and has stable rank one. Then we have (by [56] and [21]) the following splitting short exact sequence (we will fix one such J_c)

$$0 \longrightarrow \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \longrightarrow U(A)/CU(A) \underset{J_c}{\overset{\kappa_1^A}{\rightleftarrows}} K_1(A) \longrightarrow 0. \tag{e2.9}$$

If $u \in U_0(A)$ and $\{u(t) : t \in [0, 1]\}$ is a piece-wise smooth and continuous path of unitaries in A such that $u(0) = u$ and $u(1) = 1$. Then, for each $\tau \in T(A)$,

$$D_A(u)(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt \tag{e2.10}$$

modulo $\overline{\rho_A(K_0(A))}$ induces (independent of the path) an isomorphism (denote by $\bar{D}_A$) from $U_0(A)/CU(A)$ onto $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ as mentioned above (see also 2.15 of [20]).

Now suppose that A is a non-unital separable C^*-algebra and $\mathrm{Ped}(A) = A$ with $T(A) \neq \emptyset$. Suppose that $\ker \rho_A = K_0(A)$. Then

$$\mathrm{Aff}(T(\tilde{A}))/\overline{\rho_A(K_0(\tilde{A}))} = \mathrm{Aff}(T(\tilde{A}))/\mathbb{Z}. \tag{e2.11}$$

Definition 2.5. Let A be a non-unital C^*-algebra. We say that A has almost stable rank one (see [50] and [15]) if, for each n, the invertible elements in any nonzero hereditary C^*-subalgebra $\tilde{B}$ of $M_n(\tilde{A})$ is dense in B, i.e., for any $b \in B$ and any $\varepsilon > 0$, there exists an invertible element $x \in \tilde{B}$ such that $\|b - x\| < \varepsilon$.

Proposition 2.6 (*cf. Theorem 3 of [8]; See also 1.5 of [36]*). *Let A be a σ-unital C^*-algebra which has almost stable rank one and let $a, b \in A_+ \setminus \{0\}$ such that $a \sim b$ in Cuntz semigroup. Then there is a partial isometry $w \in A$ such that $w^*x, xw \in A$ and $ww^*x = xww^* = x$ for all $x \in \overline{aAa}$, $wy, yw^* \in A$ for all $y \in \overline{bAb}$ and w^*aw is a strictly positive element of $\overline{bAb}$.*

Proof. Let $H_1 = \overline{aA}$ and $H_2 = \overline{bA}$ be Hilbert A-modules. By 3.3 of [50], there is a Hilbert A-module isomorphism $\varphi : H_1 \to H_2$. Since $a^{1/2} \in \overline{aA}$, $\varphi(a^{1/2})\varphi(a^{1/2})^*$ is a strictly positive element of $\overline{bAb}$ and $\varphi(a^{1/2})^*\varphi(a^{1/2}) = \langle\varphi(a^{1/2}), \varphi(a^{1/2})\rangle_{H_2} = \langle a^{1/2}, a^{1/2}\rangle_{H_1} = a$. Consider $H = H_1 \oplus H_2$, $a_1 = \mathrm{diag}(a, 0)$ and $b_1 = \mathrm{diag}(0, b) \in M_2(A)$. Let $\{e_{i,j}\}_{1\le i,j\le 2}$ be a matrix unit for M_2. Set $B = \overline{(a_1 + b_1)M_2(A)(a_1 + b_1)}$. There is $T_1 \in LM(K(H)) = LM(B)$ (see Theorem 1.5 of [27]) such that $T_1(x_1 \oplus x_2) = 0 \oplus \varphi(x_1)$ for all $(x_1, x_2) \in H$. Put $T = e_{1,2}T$. Then $Tx = \varphi(x)$ for all $x \in H_1$ and $a^{1/2}T^*Ta^{1/2} = \varphi(a^{1/2})^*\varphi(a^{1/2}) = a$. Moreover $TaT^* = \varphi(a^{1/2})\varphi(a^{1/2})^*$ is a strictly positive element in $\overline{bAb}$ and $Ta^{1/2} \in A$. Write $Ta^{1/2} = v|a^{1/2}T^Ta^{1/2}| = v|a|$ as polar decomposition in A^{**}. One then checks that $w := v^*$ satisfies the requirement. $\square$

Definition 2.7. Let A be a unital separable amenable C^*-algebra. For any finite subset $\mathcal{U} \subset U(A)$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: If B is another unital C^*-algebra and if $L : A \to B$ is a $\mathcal{G}$-δ- multiplicative completely positive contractive linear map, then $\overline{\lceil L(u)\rceil}$ is a well defined element in $U(B)/CU(B)$ for all $u \in \mathcal{U}$. We may assume that $[L]|_{\mathcal{S}}$ is well defined, where $\mathcal{S}$ is the image of $\mathcal{U}$ in $K_1(A)$ (see, for example, 2.12 of [20]). Let $G(\mathcal{U})$ be the subgroup generated by $\mathcal{U}$. Suppose that $1/2 > \varepsilon > 0$ is given. By Appendix in [38], we may assume that there is a homomorphism $L^\dagger : G(\mathcal{U}) \to U(B)/CU(B)$ such that

$$\mathrm{dist}(L^\dagger(\bar{u}), \overline{\lceil L(u)\rceil}) < \varepsilon \ \text{ for all } \ u \in \mathcal{U}. \tag{e2.12}$$

Moreover, as in Definition 2.17 of [20], we may also assume that

$$L^\ddagger((G(\mathcal{U}) \cap U_0(A))/CU(A)) \subset U_0(B)/CU(B). \tag{e2.13}$$

It follows that $\kappa_1^B \circ L^\ddagger(\overline{u}) = [L] \circ \kappa_1^A([u])$ for all $u \in G(\mathcal{U})$, where κ_1^A and κ_1^B are defined as in (e2.9) (see Definition 2.17 of [20]). In what follows, when $1/2 > \varepsilon > 0$ is given, whenever we write $L^\dagger$, we mean that δ is small enough and $\mathcal{G}$ is large enough so that $L^\dagger$ is defined, (e2.12) and (e2.13) hold (see 2.17 of [20]). Moreover, for an integer $k \geq 1$, we will also use $L^\dagger$ for the map on some given subgroup of $U(M_k(A))/CU(M_k(A))$ induced by $L \otimes \mathrm{id}_{M_k}$. In particular, when L is a unital homomorphism, the map $L^\dagger$ is well defined on $U(M_k(A))/CU(M_k(A))$.

If A is not unital, $L^\dagger$ is defined to be $\tilde{L}^\dagger$, where $\tilde{L} : \tilde{A} \to \tilde{B}$ is the unital extension of L.

Definition 2.8. Let $1 > \varepsilon > 0$. Define

$$f_\varepsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \varepsilon/2]; \\ \frac{t-\varepsilon/2}{\varepsilon/2}, & \text{if } t \in (\varepsilon/2, \varepsilon]; \\ 1 & \text{if } t \in (\varepsilon, \infty). \end{cases} \tag{e2.14}$$

Definition 2.9. Let A be a C^*-algebra and let $a \in A_+$. Suppose that $\tilde{T}(A) \neq \{0\}$. Recall that

$$d_\tau(a) = \lim_{\varepsilon\to 0} \tau(f_\varepsilon(a))$$

with possible infinite value. Note that $f_\varepsilon(a) \in \mathrm{Ped}(A)_+$ for any $\varepsilon > 0$. Therefore $\tau \mapsto d_\tau(a)$ is a lower semi-continuous affine function on $\tilde{T}(A)$ (to $[0, \infty]$). Suppose that A is non-unital. Let $a \in A_+$ be a strictly positive element. Define

$$\Sigma_A(\tau) = d_\tau(a) \ \text{ for all } \ \tau \in \tilde{T}(A).$$

It is standard and routine to check that Σ_A is independent of the choice of a. The lower semi-continuous affine function Σ_A is called the scale function of A. (see 2.3 of [15]).

Definition 2.10. Let C_1 and C_2 be two cones. A cone map $\gamma : C_1 \to C_2$ is an additive map such that $\gamma(0) = 0$, $\gamma(rc) = r\gamma(c)$ for all $r \in \mathbb{R}_+$.

Let A be a stably projectionless simple C^*-algebras such that $K_0(A) = \ker \rho_A$. Then the Elliott invariant is defined as follows:

$$\mathrm{Ell}(A) = (K_0(A), K_1(A), \tilde{T}(A), \Sigma_A).$$

Suppose that B is another stably projectionless simple C^*-algebras such that $K_0(B) = \ker \rho_B$. Then we write

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B),$$

if there are group isomorphisms $\kappa_i : K_i(A) \to K_i(B)$, $i = 0, 1$, a cone homeomorphism $\kappa_T : \tilde{T}(A) \to \tilde{T}(B)$, i.e., κ_T is $1-1$ and onto, κ_T and κ^{-1} are both cone maps which are continuous (regarding topology of point-wise convergence on elements in $\mathrm{Ped}(A)$), and $\Sigma_A(\tau) = \Sigma_B(\kappa_T(\tau))$ for all $\tau \in \tilde{T}(A)$. In the case that A has continuous scale, then one can simplify $\mathrm{Ell}(A)$ to

$$\mathrm{Ell}(A) = (K_0(A), K_1(A), T(A)).$$

Definition 2.11. Let A and B be C^*-algebras with $T(A) \neq \emptyset$ and $T(B) \neq \emptyset$ and both have stable rank one. Let $\kappa \in KL(A, B)$, $\kappa_T : T(B) \to T(A)$ be an affine continuous map and $\kappa_u : U(\tilde{A})/CU(\tilde{A}) \to U(\tilde{B})/CU(\tilde{B})$ be a continuous homomorphism. We say $(\kappa, \kappa_T, \kappa_u)$ is compatible, if $\rho_B(\kappa(x))(t) = \rho_A(x)(\kappa_T(t))$ for all $x \in K_0(A)$ and $t \in T(B)$, $\kappa(\kappa_1^A(\bar{w})) = \kappa_1^B(\kappa_u(\bar{w}))$ for all $\bar{w} \in U(\tilde{A})/CU(\tilde{A})$ and $D_{\tilde{B}}(z)(t) = D_{\tilde{A}}(w)(\kappa_T(t))$ for all $t \in T(B)$, where $w \in U_0(A)$, $z \in U_0(B)$ such that $\bar{z} = \kappa_u(\bar{w})$ for all $w \in U_0(A)$, where κ_1^A (and κ_1^B) are as in (e2.9).

Definition 2.12. Let A and B be two separable C^*-algebras and let $\varphi_n : A \to B$ be a sequence of linear maps. We say that $\{\varphi_n\}$ is approximately multiplicative, if

$$\lim_{n\to\infty} \|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| = 0 \ \text{ for all } \ a, b \in A. \tag{e2.15}$$

Recall that τ is said to be a $\mathcal{W}$-trace in [15] if there exists a sequence of approximately multiplicative completely positive contractive linear maps $\{\varphi_n\}$ from A into $\mathcal{W}$ such that

$$\tau(a) = \lim_{n\to\infty} \tau_{\mathcal{W}}(\varphi_n(a)) \ \text{ for all } \ a \in A,$$

where $\tau_{\mathcal{W}}$ is the unique tracial state on $\mathcal{W}$.

Definition 2.13. Throughout this paper, Q will be the universal UHF-algebra with $K_0(Q) = \mathbb{Q}$ and $[1_Q] = 1$.

Definition 2.14. Let $\mathcal{B}$ be a class of C^*-algebras and let A be a separable C^*-algebra. We say A is *locally approximated by* C^*-algebras in $\mathcal{B}$, if, for $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a C^*-subalgebra $B \in \mathcal{B}$ such that $\mathrm{dist}(a, B) < \varepsilon$ for all $a \in \mathcal{F}$.

Definition 2.15. Let A be a C^*-algebra with $T(A) \neq \emptyset$. Suppose that A has a strictly positive element $e_A \in \mathrm{Ped}(A)_+$ with $\|e_A\| = 1$. Then $0 \notin \overline{T(A)}^w$, the closure of $T(A)$ in $\tilde{T}(A)$ (see section 5 of [15]). Define

$$\lambda_s(A) = \inf\{d_\tau(e_A) : \tau \in A\}.$$

Let A be a C^*-algebra with $T(A) \neq \{0\}$ such that $0 \notin \overline{T(A)}^w$. There is an affine map $r_{\mathrm{aff}} : A_{s.a.} \to \mathrm{Aff}(\overline{T(A)}^w)$ defined by

$$r_{\mathrm{aff}}(a)(\tau) = \hat{a}(\tau) = \tau(a) \ \text{ for all } \ \tau \in \overline{T(A)}^w$$

and for all $a \in A_{s.a.}$. Denote by $A^q_{s.a.}$ the space $r_{\mathrm{aff}}(A_{s.a.})$ and $A^q_+ = r_{\mathrm{aff}}(A_+)$.

Definition 2.16 (*See 2.5 of [28]*)**.** Let A be a σ-unital, nonunital, non-elementary, simple C^*-algebra and $\{e_n\}$ be an approximate identity such that $e_{n+1}e_n = e_n$ for all n. We say A has continuous scale if, for any $a \in A_+ \setminus \{0\}$, there exists $n_0 \geq 1$ such that $e_m - e_n \lesssim a$ for all $m \geq n \geq n_0$.

Definition 2.17 (*5.5 of [15]*)**.** Let A be a separable C^*-algebra, let B be a non-unital C^*-algebra and let $L : A \to B$ be a positive linear map. Let $F : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$. Suppose that $\mathcal{H} \subset A_+ \setminus \{0\}$ is a subset. We shall say that L is F-$\mathcal{H}$-full, if, for any $a \in \mathcal{H}$, for any $b \in B_+$ with $\|b\| \leq 1$, any $\varepsilon > 0$, there are $x_1, x_2, \ldots, x_m \in B$ such that $m \leq N(a)$ and $\|x_i\| \leq M(a)$, where $(N(a), M(a)) = F(a)$, and

$$\|\sum_{i=1}^m x_i^* L(a) x_i - b\| < \varepsilon. \tag{e2.16}$$

This term is consistent with the uniformly F-$\mathcal{H}$-fullness (3.11 of [16]) since F does not depend on ε.

3. Non-commutative 1-dimensional complices, revisited

Definition 3.1 (*See [17] and [12]*)**.** Let F_1 and F_2 be two finite dimensional C^*-algebras. Suppose that there are two (not necessary unital) homomorphisms $\varphi_0, \varphi_1 : F_1 \to F_2$. Denote the mapping torus M_{φ_1,φ_2} by

$$A = A(F_1, F_2, \varphi_0, \varphi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \varphi_0(g) \ \text{ and } \ f(1) = \varphi_1(g)\}.$$

Denote by $\mathcal{C}$ the class of all C^*-algebras of the form $A = A(F_1, F_2, \varphi_0, \varphi_1)$ and all finite dimensional C^*-algebras. These C^*-algebras are called Elliott–Thomsen building blocks as well as one dimensional non-commutative CW complexes.

Recall that $\mathcal{C}_0$ is the class of all $A \in \mathcal{C}$ with $K_0(A)_+ = \{0\}$ such that $K_1(A) = 0$, and $\mathcal{C}_0^{(0)}$ the class of all $A \in \mathcal{C}_0$ such that $K_0(A) = 0$. Denote by $\mathcal{C}'$, $\mathcal{C}_0'$ and $\mathcal{C}_0^{0'}$ the class of all full hereditary C^*-subalgebras of C^*-algebras in $\mathcal{C}$, $\mathcal{C}_0$ and $\mathcal{C}_0^0$, respectively.

Recall that $\mathcal{R}$ denotes the class of finite direct sums of Razak algebras and $\mathcal{M}_0$ denotes the class of all simple inductive limits of C^*-algebras in $\mathcal{R}$ (with injective connecting maps) (see 6.1 and 9.5 of [15] and also 10.1, 16.2 and 16.5 of [18]).

3.2. Let $F_1 = M_{R_1}(\mathbb{C}) \oplus M_{R_2}(\mathbb{C}) \oplus \cdots \oplus M_{R_l}(\mathbb{C})$, let $F_2 = M_{r_1}(\mathbb{C}) \oplus M_{r_2}(\mathbb{C}) \oplus \cdots \oplus M_{r_k}(\mathbb{C})$ and let $\varphi_0,\ \varphi_1 :\ F_1 \to F_2$ be (not necessary unital) homomorphisms, where R_j and r_i are positive integers. Then φ_0 and φ_1 induce homomorphisms

$$\varphi_{0*}, \varphi_{1*} :\ K_0(F_1) = \mathbb{Z}^l \longrightarrow K_0(F_2) = \mathbb{Z}^k$$

by matrices $(a_{ij})_{k\times l}$ and $(b_{ij})_{k\times l}$, respectively, and $\sum_{j=1}^{l} a_{ij}R_j \le r_i$ for $i = 1, 2, \ldots, k$. We may write $C([0,1], F_2) = \oplus_{j=1}^{k} C([0,1]_j, M_{r_j})$, where $[0,1]_j$ denotes the jth interval.

Theorem 3.3. *Let A be a full hereditary C^*-subalgebra of a C^*-algebra in $\mathcal{C}$. Then $\mathrm{cer}(u) \le 2 + \varepsilon$ if $u \in U_0(\tilde{A})$. Moreover, if $u \in CU(\tilde{A})$ then, for any $\varepsilon > 0$, there exists a continuous path $\{u(t) : t \in [0,1]\} \subset CU(\tilde{A})$ with $u(0) = u$, $u(1) = 1_{\tilde{A}}$ and $\mathrm{length}(\{u(t) : t \in [0,1]\}) \le 4\pi + \varepsilon$. In particular, $\mathrm{cel}(u) \le 4\pi$.*

Proof. Let $e \in B := A(F_1, F_2, \varphi_0, \varphi_1)$ with $\|e\| = 1$ and $A = \overline{eBe}$. Let $u \in U_0(\tilde{A})$ and let $\varepsilon > 0$. Without loss of generality, we may assume that $\varepsilon < \frac{1}{4\max\{R(i)r_j : 1\le i\le l, 1\le j\le k\}}$.

It follows 8.8 of [15] that e is approximately unitarily equivalent (in $\tilde{B}$) to another positive element e' which has the following form $e' = (g, a) \in B$ such that

$$g_j := g|_{[0,1]_j} = \sum_{i=1}^{r_j} \lambda_{i,j} p_{i,j}, \ \ j = 1, 2, \ldots, k, \tag{e3.1}$$

where $\lambda_{1,j}, \lambda_{2,j}, \ldots, \lambda_{r_j,j} \in C([0,1])$ and $p_{1,j}, p_{2,j}, \ldots, p_{r_j,j} \in C([0,1], M_{r_j})$ are mutually orthogonal rank one projections.

It follows that $\langle e' \rangle = \langle e \rangle$ in the Cuntz semi-group. Since B has stable rank one, by [7], A is isomorphic to $C := \overline{e'Be'}$. Therefore, without loss of generality, we may assume that $u \in \tilde{C}$. Note that, for any $f \in C([0,1])_+$,

$$f(e')|_{[0,1]_j} = \sum_{i=1}^{r_j} f(\lambda_{i,j}) p_{i,j}, \ \ j = 1, 2, \ldots, k. \tag{e3.2}$$

Write $u = \prod_{i=1}^{m} \exp(\sqrt{-1}a_i)$, where each $a_i = \alpha_i \cdot 1_{\tilde{A}} + x_i$ with $\alpha_i \in \mathbb{R}$ and $x_i \in C_{s.a.}$, $i = 1, 2, \ldots, m$. Let $\delta > 0$. There is $1/2 > \eta > 0$ such that $\|f_\eta(e')x_i f_\eta(e') - x_i\| < \delta$, $i = 1, 2, \ldots, m$. By choosing δ small enough, we have that

$$\|u - \prod_{i=1}^{m} \exp(\sqrt{-1}\alpha_i \cdot 1_{\tilde{C}} + f_\eta(e')x_i f_\eta(e'))\| < \varepsilon/4. \tag{e3.3}$$

To simplify notation, without loss of generality, we may further assume that $f_\eta(e')x_i f_\eta(e') = x_i$, $i = 1, 2, \ldots, m$. Let $\delta_1 > 0$. It follows from 8.9 of [15] that there is $e'' \le f_\eta(e')$ such that

$$\|e'' - f_\eta(e')\| < \delta_1 \tag{e3.4}$$

and $\overline{e''Ce''} \in \mathcal{C}$. With sufficiently small δ_1, we may assume that

$$\|u - \prod_{j=1}^{m} \exp(i\alpha_j \cdot 1_{\tilde{C}} + e''x_j e'')\| < \varepsilon/3. \tag{e3.5}$$

Put $v = \prod_{j=1}^{m} \exp(\sqrt{-1}\alpha_j \cdot 1_{\tilde{C}} + e''x_j e'')$. We may now view $v \in \tilde{D}$, where $D = \overline{e''Ce''}$. Since $\tilde{D} \in \mathcal{C}$ (see 6.2 of [15]), it follows from 5.19 of [39] that there are $b_1, b_2 \in \tilde{D}_{s.a.}$ such that $\|v - \exp(ib_1)\exp(ib_2)\| < \varepsilon/3$. Note that, if we view $v \in U_0(\tilde{A})$, b_1, b_2 may be viewed as elements in $\tilde{C}_{s.a.}$ since $e'' \le f_\eta(e')$. This follows that $\mathrm{cer}(A) \le 2 + \varepsilon$.

Now suppose that $u \in CU(\tilde{A})$. There exists $v \in CU(\tilde{A})$ such that $\|u - v\| < \varepsilon/4$, $v = \prod_{s=1}^{m_1} v_s$, and $v_s = v_{s,1}v_{s,2}\cdots v_{s,r(s)}v_{s,1}^* v_{s,2}^* \cdots v_{s,r(s)}^*$, where each $v_{s,i} \in U(\tilde{A})$, $s = 1, 2, \ldots, m_1$. Write $v_{s,i} = \beta_{s,i} \cdot 1_{\tilde{A}} + z_{s,i}$, where $\beta_{s,i} \in \mathbb{C}$ with $|\beta_{s,i}| = 1$ and $z_{s,i} \in A$. For any $\delta_2 > 0$, with sufficiently small $\eta > 0$, we may assume that

$$\|z_{s,i} - f_\eta(e')z_{s,i}f_\eta(e')\| < \delta_2/16m_1(\sum_{i=1}^{m_1} r(s)), \ \ 1 \le i \le r(s), \ \ 1 \le s \le m_1. \tag{e3.6}$$

So we may assume that

$$\|z_{s,i} - e''z_{s,i}e''\| < \delta_2/8m_1(\sum_{i=1}^{m_1} r(s)), \ \ 1 \le i \le r(s), \ \ 1 \le s \le m_1. \tag{e3.7}$$

It follows that there is a unitary in $w_{s,i} \in \mathbb{C}\cdot 1_{\tilde{A}} + \overline{e''Ae''}$ such that

$$\|v_{s,i} - w_{s,i}\| < \delta_2/4m_1(\sum_{i=1}^{m_1} r(s)), \quad 1 \le i \le r(s), \quad 1 \le s \le m_1. \tag{e3.8}$$

Put $w_s = w_{s,1}w_{s,2}\cdots w_{s,r(s)}w_{s,1}^*w_{s,2}^*\cdots w_{s,r(s)}^*$ and $w = \prod_{s=1}^{m_1} w_s$. With sufficiently small δ_2, we may assume that

$$\|w - v\| < \varepsilon/4. \tag{e3.9}$$

Now $v \in CU(\mathbb{C}\cdot 1_{\tilde{A}} + \overline{e''Ae''})$. As mentioned above, $\mathbb{C}\cdot 1_{\tilde{A}} + \overline{e''Ae''} \in \mathcal{C}$. By 3.16 of [20], in $\mathbb{C}\cdot 1_{\tilde{A}} + \overline{e''Ae''}$, there is a continuous path $\{u(t) : t \in [1/2, 1]\} \subset CU(\mathbb{C}\cdot 1_{\tilde{A}} + \overline{e''Ae''})$ such that $u(1/2) = w$ and $u(1) = 1_{\tilde{A}}$ which has the length no more than $4\pi + \varepsilon/16\pi$. Note $v \in CU(\tilde{A})$ and

$$\|w - u\| < \varepsilon/2, \quad \text{or} \quad \|uw^* - 1\| < \varepsilon/2. \tag{e3.10}$$

Write $uw^* = \exp(\sqrt{-1}d)$ for some $d \in \tilde{A}_{s.a.}$. Then $\|d\| < 2\arcsin(\varepsilon/4)$. Note that $uw^* \in CU(\tilde{A})$. Therefore, for each irreducible representation π of $\tilde{A}_{s.a.}$, $\mathrm{Tr}_\pi(d) = 2m'\pi$ for some integer m', where Tr_π is the standard trace on $\pi(\tilde{A})$. Since we choose $\varepsilon < \frac{1}{4\max\{R(i)r_j : i,j\}}$, $\mathrm{Tr}_\pi(d) = 0$. It follows that $\tau(d) = 0$ for all $\tau \in T(\tilde{A})$. Define $u(t) = \exp(\sqrt{-1}(1-2t)d)w$ for $t \in [0, 1/2]$. Note that $u(t)$ is in $CU(\tilde{A})$ for all $t \in [0, 1]$ with $u(0) = u$, $u(1) = 1$ and total length no more than $4\pi + \varepsilon$. □

3.4. Let $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$, where $F_1 = M_{R_1}(\mathbb{C}) \oplus M_{R_2}(\mathbb{C}) \oplus \cdots \oplus M_{R_l}(\mathbb{C})$, $F_2 = M_{r_1}(\mathbb{C}) \oplus M_{r_2}(\mathbb{C}) \oplus \cdots \oplus M_{r_k}(\mathbb{C})$. Recall that the irreducible representations of A, are given by

$$\coprod_{i=1}^{k}(0,1)_i \cup \{\rho_1, \rho_2, \ldots, \rho_l\} = \mathrm{Irr}(A),$$

where $(0, 1)_i$ is the same open interval $(0, 1)$. Any trace $\tau \in T(A)$ is corresponding to $(\mu_1, \mu_2, \ldots, \mu_k, s_1, s_2, \ldots, s_l)$, where μ_i are nonnegative measures on $(0, 1)_i$ and $s_j \in \mathbb{R}_+$ and we have

$$\|\tau\| = \sum_{i=1}^{k}\int_0^1 d\mu_i \; + \; \sum_{j=1}^{l} s_j.$$

Let $t \in (0, 1)_i$ and δ_t be the canonical point measure at point t with measure 1, then

$$\lim_{t\to 0}\delta_t = (\mu_1, \mu_2, \ldots, \mu_k, s_1, s_2, \ldots, s_l) \quad \text{and} \quad \lim_{t\to 1}\delta_t = (\mu_1, \mu_2, \ldots, \mu_k, s_1', s_2', \ldots, s_l')$$

with $\mu_j = 0$, $s_j = a_{ij}\cdot\frac{R_j}{r_i}$ and $s_j' = b_{ij}\cdot\frac{R_j}{r_i}$, where $(a_{ij})_{k\times l} = \varphi_{0*}$ and $(b_{ij})_{k\times l} = \varphi_{1*}$ as in 3.2. Let

$$\lambda = \min_i\{\frac{\sum_{j=1}^{l} a_{ij}R_j}{r_i}, \frac{\sum_{j=1}^{l} b_{ij}R_j}{r_i}\}.$$

A direct calculation shows that if $\tau_n \in T(A)$ converge to τ in weak* topology, then $\|\tau\| \ge \lambda\cdot\limsup\|\tau_n\|$. In notation of 2.15, we have

$$\lambda_s(A) = \lambda. \tag{e3.11}$$

Evidently, the number λ above is the largest positive number satisfying the following conditions

$$\varphi_{0*}([\mathbf{1}_{F_1}]) \ge \lambda\cdot[\mathbf{1}_{F_2}], \quad \varphi_{1*}([\mathbf{1}_{F_1}]) \ge \lambda\cdot[\mathbf{1}_{F_2}] \;\text{ in }\; K_0(F_2).$$

In the notation of 2.3, both affine spaces $\mathrm{Aff}(\tilde{T}(A))$ and $\mathrm{Aff}(\mathrm{T}(A))$ can be identified with the subset of

$$\bigoplus_{j=1}^{k} C([0,1]_j, \mathbb{R}) \oplus \mathbb{R}^l = \bigoplus_{j=1}^{k} C([0,1]_j, \mathbb{R}) \oplus \underbrace{(\mathbb{R}\oplus\mathbb{R}\oplus\cdots\oplus\mathbb{R})}_{l\ \text{copies}}$$

consisting of $(f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_l)$ satisfying the condition

$$f_i(0) = \frac{1}{r_i}\sum_{j=1}^{l} a_{ij}g_j\cdot R_j \qquad \text{and} \qquad f_i(1) = \frac{1}{r_i}\sum_{j=1}^{l} b_{ij}g_j\cdot R_j.$$

The positive cone $\mathrm{Aff}(\tilde{T}(A))_+$ is the subset of $\mathrm{Aff}(\tilde{T}(A))$ consisting all elements of those elements $(f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_l)$ with $f_i(t) \ge 0$ and $g_j \ge 0$ for all i, j, t. Set $\mathbb{R}^\sim = \mathbb{R}\cup\{\infty\}$, $\mathbb{R}_+^\sim = \mathbb{R}_+\cup\{\infty\}$. Then $\mathrm{LAff}(\tilde{T}(A))_+$ ($\mathrm{LAff}^\sim(\tilde{T}(A))$, respectively) is identified with the subset of

$$\bigoplus_{j=1}^{k} LSC([0,1]_j, \mathbb{R}_+^\sim) \oplus (\mathbb{R}_+^\sim)^l \;\; (\text{or } \bigoplus_{j=1}^{k} LSC([0,1]_j, \mathbb{R}^\sim) \oplus (\mathbb{R}^\sim)^l) \tag{e3.12}$$

consisting of $(f_1, f_2, \dots, f_k, g_1, g_2, \dots, g_l)$ satisfying the same condition

$$f_i(0) = \frac{1}{r_i}\sum_{j=1}^{l} a_{ij} g_j \cdot R_j \qquad \text{and} \qquad f_i(1) = \frac{1}{r_i}\sum_{j=1}^{l} b_{ij} g_j \cdot R_j.$$

3.5. Suppose that $A = A(F_1, F_2, \varphi_0, \varphi_1)$ is not unital. Let $e_{i,F_2} = (e_{i,1}, e_{i,2}, \dots, e_{i,k}) \in F_2$ be a projection such that $\mathbf{1}_{F_2} - \varphi_i(\mathbf{1}_{F_1}) = e_{i,F_2}$, $i = 0, 1$. Put $F_{2,i} = e_{i,F_2} F_2 e_{i,F_2}$, $i = 0, 1$. Define $\varphi_i' : \mathbb{C} \to F_{2,i}$ by $\varphi_i'(\lambda) = \lambda e_{i,F_2}$, $i = 1, 2$. Define $F_1^{\sim} = F_1 \oplus \mathbb{C}$ and $\varphi_i^{\sim} : F_1^{\sim} \to F_2$ by $\varphi_i^{\sim}(a \oplus \lambda) = \varphi_i(a) \oplus \lambda e_{i,F_2}$, $i = 0, 1$. Then $\tilde{A} = A(F_1^{\sim}, F_2, \varphi_0^{\sim}, \varphi_1^{\sim})$.

In what follows, we will use notations $\mathbb{Z}^{\sim} = \mathbb{Z} \cup \{\infty\}$, and $\mathbb{Z}_+^{\sim} = \mathbb{Z}_+ \cup \{\infty\}$. Let $B = A(F_1, F_2, \varphi_0, \varphi_1)$. Let $a \in B_+$, define $r_a \in \mathrm{LAff}(\tilde{T}(A))_+$ by $r_a(\tau) = d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$. When one identifies $\mathrm{LAff}(\tilde{T}(A))_+$ with the subspace of $\bigoplus_{j=1}^k LSC([0,1]_j, \mathbb{R}_+^{\sim}) \oplus (\mathbb{R}_+^{\sim})^l$ as in 3.4, $r_a \in \bigoplus_{j=1}^k LSC([0,1]_j, \frac{1}{r_j}\mathbb{Z}_+^{\sim}) \oplus \bigoplus_{i=1}^l (\frac{1}{R_i}\mathbb{Z}_+^{\sim})$. (Recall that map $\varphi_{i,*} : K_0(F_1) = \mathbb{Z}^l \to K_0(F_2) = \mathbb{Z}^k$ $(i = 0, 1)$, induced by $\varphi_i : F_1 \to F_2$ is given by the matrix $(a_{ij})_{k\times l}$ and $(b_{ij})_{k\times l}$ with nonnegative integer entries, which can be extended to maps (still denoted by $\varphi_{i,*}$) from $(\mathbb{Z}^{\sim})^l$ to $(\mathbb{Z}^{\sim})^k$.) If we identify each $\frac{1}{r_j}\mathbb{Z}$ (or $\frac{1}{R_i}\mathbb{Z}$ respectively) with $\mathbb{Z}$ by identifying $\frac{1}{r_j}$ with $1 \in \mathbb{Z}$ (or by identifying $\frac{1}{R_i}$ with $1 \in \mathbb{Z}$), r_a is identified with

$$\big((f_1, f_2, \dots, f_k), (j_1, j_2, \dots, j_l)\big) \in \bigoplus_{j=1}^k LSC([0,1]_j, \mathbb{Z}_+^{\sim}) \oplus (\mathbb{Z}_+^{\sim})^l$$

which satisfy

$$(f_1(0), f_2(0), \dots, f_k(0)) = \varphi_{0,*}(j_1, j_2, \dots, j_l) \ \text{ and } \ (f_1(1), f_2(1), \dots, f_k(1)) = \varphi_{1,*}(j_1, j_2, \dots, j_l).$$

Let $LSC([0,1], \mathbb{R}^{\sim})$ be the set of lower-semicontinuous functions from $[0,1]$ to $\mathbb{R}^{\sim}$. We will use the notation $LSC([0,1], (\mathbb{R}^{\sim})^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{R}^{\sim})^l$ to denote the subset of $LSC([0,1], (\mathbb{R}^{\sim})^k) \bigoplus (\mathbb{R}^{\sim})^l$ consisting of elements $\big((f_1, f_2, \dots, f_k), (j_1, j_2, \dots, j_l)\big) \in LSC([0,1], (\mathbb{R}^{\sim})^k) \bigoplus (\mathbb{R}^{\sim})^l$ satisfying

$$(f_1(0), f_2(0), \dots, f_k(0)) = \varphi_{0,*}(j_1, j_2, \dots, j_l) \ \text{ and } \ (f_1(1), f_2(1), \dots, f_k(1)) = \varphi_{1,*}(j_1, j_2, \dots, j_l).$$

Let $LSC([0,1], (\mathbb{R}_+^{\sim})^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{R}_+^{\sim})^l$ ($LSC([0,1], (\mathbb{Z}^{\sim})^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{Z}^{\sim})^l$, or $LSC([0,1], (\mathbb{Z}_+^{\sim})^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{Z}_+^{\sim})^l$ respectively) be the subset of $LSC([0,1], (\mathbb{R}^{\sim})^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{R}^{\sim})^l$ consisting of the above elements with $f_i(t)$ and $j_i \in \mathbb{R}_+^{\sim}$ ($\in \mathbb{Z}^{\sim}$ or $\in Z_+^{\sim}$ respectively). If we insist not take the value $+\infty$, then we will use the notation LSC_f instead of LSC. So the sets $LSC_f([0,1], (\mathbb{R}_+)^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{R}_+)^l$ and $LSC_f([0,1], (\mathbb{Z}_+)^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{Z}_+)^l$ can also be defined similarly.

Now let $B \in \mathcal{C}_0$. Let C be a full hereditary subalgebra of B. Using the rank function in 3.17 of [20] and applying 3.18 of [20], the map $r : \langle a \rangle \mapsto r_a$ gives an injective semi-group homomorphism from $W(C)$ to $LSC_f([0,1], (\mathbb{Z}_+)^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{Z}_+)^l$ (see also 3.18 of [20]) which extends to an order injective semi-group homomorphism from $Cu(C)$ to $LSC([0,1], (\mathbb{Z}_+^{\sim})^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{Z}_+^{\sim})^l$. Note $\tilde{C} \in \mathcal{C}$. Also note that $Cu^{\sim}(C)$ (see [49]) is the semigroup of the formal differences $f - n[1_{\tilde{C}}]$, with $n \in \mathbb{Z}_+$ and $f \in Cu(\tilde{C})$ such that $Cu(\pi_C)(f) = [n]$, where $Cu(\pi_C)$ is the map induced by the quotient map $\pi_C : \tilde{C} \to \mathbb{C}$. With the help of discussion of 8.8 of [15], it is straight forward to check the following:

Proposition 3.6. *Let $C \in \mathcal{C}_0'$. Then*

$$W(C) = LSC_f([0,1], (\mathbb{Z}_+)^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{Z}_+)^l \ \text{ and} \tag{e3.13}$$

$$Cu(C) = LSC([0,1], (\mathbb{Z}_+^{\sim})^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{Z}_+^{\sim})^l. \tag{e3.14}$$

Moreover (see [49] for the definition of $Cu^{\sim}$)

$$Cu^{\sim}(C) = K_0(C) \sqcup LSC([0,1], (\mathbb{Z}^{\sim})^k) \bigoplus_{(\varphi_{0,*},\varphi_{1,*})} (\mathbb{Z}^{\sim})^l. \tag{e3.15}$$

Since C is stably projectionless, it follows that the order $Cu^{\sim}(C)$ is determined by $Cu(C)$.

Definition 3.7. Fix an integer $a_1 \geq 1$. Let $\alpha = \frac{a_1}{a_1+1}$. For each $r \in \mathbb{Q}_+ \setminus \{0\}$, let $e_r \in Q$ (see 2.13) be a projection with $\mathrm{tr}(e_r) = r$. Let $\bar{Q}_r := (1 \otimes e_r)(Q \otimes Q)(1 \otimes e_r)$. Define $q_r : Q \to \bar{Q}_r$ by $a \mapsto a \otimes e_r$ for $a \in Q$. We will also use q_r to denote any homomorphism from B to $B \otimes e_r Q e_r$ (or to $B \otimes Q$) defined by sending $b \in B$ to $b \otimes e_r \in B \otimes e_r Q e_r \subset B \otimes Q$.

For $r = \alpha = \frac{a_1}{a_1+1}$, one can identify Q with $Q \otimes M_{a_1+1}$, then the projection e_α is identified with $\mathbf{1}_Q \otimes \mathrm{diag}(\underbrace{1, \dots, 1}_{a_1}, 0)$.

Let

$$R(\alpha, 1) = \{(f, a) \in C([0,1], Q \otimes Q) \oplus Q : f(0) = q_\alpha(a) \ \text{ and } \ f(1) = a \otimes 1_Q\}.$$

Note that an element (f, a) is full in $R(\alpha, 1)$ if and only if $a \neq 0$ and $f(t) \neq 0$ for all $t \in (0, 1)$. Let $a_\alpha = (f, 1)$ be defined as follows. Let

$$f(t) = (1-t)(1 \otimes e_\alpha) + t(1 \otimes 1) \ \text{for all} \ t \in (0, 1). \tag{e3.16}$$

Note that a_α is a *strictly positive element* of $R(\alpha, 1)$, moreover, for any $1/2 > \eta > 0$, $f_\eta(a_\alpha)$ is full. *C^*-algebra $R(\alpha, 1)$ and a_α will appear frequently in this paper.*

Let $LSC([0, 1], \mathbb{R}^\sim) \oplus_\alpha \mathbb{R}^\sim$ (or $LSC_f([0, 1], \mathbb{R}_+) \oplus_\alpha \mathbb{R}_+$ respectively) be the subset of $LSC([0, 1], \mathbb{R}^\sim) \oplus \mathbb{R}^\sim$ (or $LSC_f([0, 1], \mathbb{R}_+) \oplus \mathbb{R}_+$ respectively) consisting of elements (f, x) such that $f(0) = \alpha x$ and $f(1) = x$. The rank function $r : \langle a \rangle \mapsto r(a) = d_\tau(a)$ gives maps from $W(R(\alpha, 1))$ to $LSC_f([0, 1], \mathbb{R}_+) \oplus_\alpha \mathbb{R}_+$ and from $Cu(R(\alpha, 1))$ to $LSC([0, 1], \mathbb{R}_+^\sim) \oplus_\alpha \mathbb{R}_+^\sim$ which are order semi-group homomorphisms. But these maps are only surjective not injective.

Recall that $W(Q)$ and $Cu(Q)$ can be identified with the semi-groups $\mathbb{R}_+ \setminus \{0\} \sqcup \mathbb{Q}_+$ and $\mathbb{R}_+^\sim \setminus \{0\} \sqcup \mathbb{Q}_+$, where the second copy of $\mathbb{Q}$ is identified with $K_0(Q)$ and $\mathbb{R}_+^\sim \setminus \{0\}$ identified with the rank functions of non-projection and non-zero positive elements. If $s \in \mathbb{Q} \subset \mathbb{R}$, we will use $[s]$ for the corresponding element in $K_0(Q)$. With the order in $Cu(Q)$, in $\mathbb{R}^\sim \sqcup \mathbb{Q}$, $t < [t]$ for $t \in \mathbb{Q} \subset \mathbb{R}$ and $[t] \in K_0(Q) = \mathbb{Q}$. But $s > [t]$ if $s > t$ as in $\mathbb{R}^\sim$. The addition on $\mathbb{R} \sqcup \mathbb{Q}$ is defined by $s + [r] = s + r$ and $[s] + [r] = [s + r]$.

A function $f : [0, 1] \to \mathbb{R}^\sim \sqcup \mathbb{Q}$ is called lower-semicontinuous if, for each $t_0 \in [0, 1]$, and if $f(t_0) = [r] \in K_0(Q)$, there exists $\delta > 0$ such that $f(t) \geq f(t_0)$ for all $t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1]$, or, if $f(t_0) = r \in \mathbb{R}^\sim$, for any non zero $\varepsilon \in \mathbb{R}_+^\sim \setminus \{0\} \sqcup \mathbb{Q}_+$, there exists $\delta > 0$ such that

$$f(t) + \varepsilon \geq f(t_0) \ \text{for all} \ \in [0, 1] \cap (t_0 - \delta, t_0 + \delta) \setminus \{t_0\},$$

where the order is in $\mathbb{R}^\sim \sqcup \mathbb{Q}$ mentioned above.

Let $LSC([0, 1], \mathbb{R}^\sim \sqcup \mathbb{Q})$ be the set of all lower-semicontinuous functions. Let $LSC([0, 1], \mathbb{R}^\sim \sqcup \mathbb{Q}) \oplus_\alpha \mathbb{R}^\sim \sqcup \mathbb{Q}$ be the subset of $LSC([0, 1], \mathbb{R}^\sim \sqcup \mathbb{Q}) \oplus \mathbb{R}^\sim \sqcup \mathbb{Q}$ consisting of elements (f, x) such that $f(0) = \alpha x$ and $f(1) = x$. (Here we define $\alpha[r] = [\alpha r]$. Note that α is rational.) The sets $LSC([0, 1], (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+$ and $LSC_f([0, 1], (\mathbb{R} \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R} \setminus \{0\} \sqcup \mathbb{Q})_+$ can be defined similarly. Then we have the following fact.

Corollary 3.8. *Let $A = R(\alpha, 1)$ for some $1 > \alpha > 0$. Then*

$$W(A) = LSC_f([0, 1], (\mathbb{R} \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R} \setminus \{0\} \sqcup \mathbb{Q})_+, \tag{e3.17}$$

$$Cu(A) = LSC([0, 1], (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+ \ \text{and} \tag{e3.18}$$

$$Cu^\sim(A) = LSC([0, 1], \mathbb{R}^\sim \sqcup \mathbb{Q}) \oplus_\alpha \mathbb{R}^\sim \sqcup \mathbb{Q}. \tag{e3.19}$$

Note, with (e3.19), map r can be extended to an order semi-group homomorphism from $Cu^\sim(A)$ to $LSC([0, 1], \mathbb{R}^\sim) \oplus_\alpha \mathbb{R}^\sim$ defined by $r(f(s), a) = (r(f(s)), r(a))$, where $r(t) = t$ for all $t \in \mathbb{R}^\sim$ and $r([t]) = t$ for all $t \in \mathbb{Q}$.

Definition 3.9 (*cf. 8.1 and 8.2 of [15]*)**.** Recall the definition of class $\mathcal{D}$ and $\mathcal{D}_0$.

Let A be a non-unital simple C^*-algebra with a strictly positive element $a \in A$ with $\|a\| = 1$. Suppose that there exists $1 > \mathfrak{f}_a > 0$, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $b \in A_+ \setminus \{0\}$, there are $\mathcal{F}$-ε-multiplicative completely positive contractive linear maps $\varphi : A \to A$ and $\psi : A \to D$ for some C^*-subalgebra $D \subset A$ with $D \in \mathcal{C}_0^{0'}$ (or $\mathcal{C}_0'$), $D \perp \varphi(A)$, and

$$\|x - (\varphi(x) + \psi(x))\| < \varepsilon \ \text{for all} \ x \in \mathcal{F} \cup \{a\}, \tag{e3.20}$$

$$c \lesssim b, \tag{e3.21}$$

$$t(f_{1/4}(\psi(a))) \geq \mathfrak{f}_a \ \text{for all} \ t \in T(D), \tag{e3.22}$$

where c is a strictly positive element of $\overline{\varphi(A)A\varphi(A)}$. Then we say $A \in \mathcal{D}_0$ (or $\mathcal{D}$).

Note, by Remark 8.11 of [15], D can *always* be chosen to be in $\mathcal{C}_0$ (or $\mathcal{C}_0^0$).

When $A \in \mathcal{D}$ and is separable, then $A = \mathrm{Ped}(A)$ (see 11.3 of [15]). Let $a \in A_+$ with $\|a\| = 1$ be a strict positive element. Put

$$d = \inf\{\tau(f_{1/4}(a)) : \tau \in T(A)\}. \tag{e3.23}$$

Then, for any $0 < \eta < d$, $\mathfrak{f}_a$ can be chosen to be $d - \eta$ (see Remark 9.8 of [15]). One may also assume that $f_{1/4}(\psi(a))$ is full in D. Furthermore, there exists a map: $T : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}$ ($a \mapsto (N(a), M(a))$ for all $a \in A_+ \setminus \{0\}$) which is independent of $\mathcal{F}$ and ε such that, for any finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$, we can further require that ψ is T-$\mathcal{H}$-full (see 8.3 and 9.2 of [15]). For any $n \geq 1$, one can choose a strictly positive element $b \in A$ with $\|b\| = 1$ such that $f_{1/4}(b) \geq f_{1/n}(a)$. Therefore, if A has continuous scale, d can be chosen to be 1, if the strictly positive element is chosen accordingly.

Let A be a separable stably projectionless simple C^*-algebra. Recall that A has generalized tracial rank at most one and write $gTR(A) \leq 1$, if there exists $e \in \mathrm{Ped}(A)_+$ with $\|e\| = 1$ such that $\overline{eAe} \in \mathcal{D}$ (see 11.6 of [15]).

Definition 3.10. Let $A \in \mathcal{D}$ as defined 3.9. If, in addition, for any integer n, $D = M_n(D_1)$ for some $D_1 \in \mathcal{C}_0$ such that

$$\psi(x) = \mathrm{diag}(\overbrace{\psi_1(x), \psi_1(x), \ldots, \psi_1(x)}^{n}) \ \text{for all} \ x \in \mathcal{F}, \tag{e3.24}$$

where $\psi_1 : A \to D_1$ is an $\mathcal{F}$-ε-multiplicative completely positive contractive linear map, then we say $A \in \mathcal{D}^d$.

Note that here, as in 8.3 and 9.2 of [15], the map T mentioned in 3.9 is also assumed to exist and $\mathfrak{f}_a$ can be also chosen as $d-\eta$ for any $\eta>0$ with d as in (e3.23) for a certain strictly positive element a.

Remark 3.11. It follows from 10.4 and 10.7 of [15] that, if $A\in\mathcal{D}_0$, then $A\in\mathcal{D}^d$. Moreover, D_1 can be chosen in $\mathcal{C}_0^{(0)}$, and if $A\in\mathcal{D}$, then D_1 can be chosen in $\mathcal{C}_0$. If A is a separable simple C^*-algebra in $\mathcal{D}$ and A is tracially approximate divisible (in the sense of 10.1 of [15]), then $A\in\mathcal{D}^d$.

Proposition 3.12. *Let A be a non-unital simple C^*-algebra which is tracially approximate divisible. Then every hereditary C^*-subalgebra is also tracially approximate divisible. Consequently, if $A\in\mathcal{D}^d$, then every hereditary C^*-subalgebra is in $\mathcal{D}^d$.*

Proof. Let $B\subset A$ be a hereditary C^*-subalgebra. Fix $\varepsilon>0$, a finite subset $\mathcal{F}\subset B$, a nonzero element $b\in B_+$ and an integer $n\geq 1$. By choosing a member b_e in an approximate identity of B, without loss of generality (with an error within, say $\varepsilon/2$), we may assume that $xb_e=b_ex=x$ for all $x\in\mathcal{F}$.

Since A is tracially approximate divisible, there are C^*-subalgebras A_0 and A_1 of A such that

$$\mathrm{dist}(x, C_d)<\varepsilon \ \text{ for all } \ x\in\mathcal{F}, \tag{e3.25}$$

where $C_d\subset C\subset A$, $C=A_0\oplus M_n(A_1)$,

$$C_d=\{(y_0, \mathrm{diag}(\overbrace{y_1, y_1, \dots, y_1}^{n})) : y_0\in A_0, y_1\in A_1\},$$

and where $a_0\lesssim b$, where a_0 is a strictly positive element of A_0.

Let B_0 be the C^*-subalgebra generated by b_eab_e for all $a\in A_0$ and let B_1 be the C^*-subalgebra generated by b_ecb_e for all $c\in A_1$. Then B_0 and B_1 are C^*-subalgebras of B. Since $B_0\subset\overline{b_ea_0b_eAb_ea_0b_e}$, $b_ea_0b_e$ is a strictly positive element of B_0. Moreover, $b_ea_0b_e\lesssim a_0\lesssim b$. Put

$$B_1^d=\{(x_0, \overbrace{x_1, x_1, \dots, x_1}^{n}) : x_0\in B_0, x_1\in B_1\}, \tag{e3.26}$$

$B_1^d\subset B_3$, where $B_3=B_0\oplus M_n(B_1)$. For each $x\in\mathcal{F}$, let $y_x=(y_{0,x}, y_{1,x}, \dots, y_{1,x})\in C_d$ such that $\|x-y_x\|<\varepsilon/2$. Then

$$\|x-b_ey_xb_e\|<\varepsilon \ \text{ for all } \ x\in\mathcal{F}. \tag{e3.27}$$

Note that $b_ey_xb_e\in B_1^d$. This proves the first part of the statement. If $A\in\mathcal{D}^d$, then, $B\in\mathcal{D}$ for any hereditary C^*-subalgebra B, by 8.6 of [15]. By the first part of the statement, B is tracially approximately divisible. Therefore $B\in\mathcal{D}^d$. $\Box$

Proposition 3.13. *Let $A\in\mathcal{D}$ be with continuous scale and let $e\in A_+$ with $\|e\|=1$ be a strictly positive element, and $1>\mathfrak{f}_e>0$ be as in 3.9. Then, for any finite subset $\mathcal{F}\subset A$, any $\varepsilon>0$, any $b\in A_+\setminus\{0\}$ and any integer $n\geq 1$, there are $\mathcal{F}$-ε-multiplicative completely positive contractive linear maps $\varphi: A\to A$ and $\psi: A\to M_n(D)$ for some C^*-subalgebra $D\in\mathcal{C}_0$ with $M_n(D)\subset A$ and $\varphi(A)\perp M_n(D)$ such that*

$$\|x-(\varphi(x)\oplus\psi(x))\|<\varepsilon \ \textit{ for all } \ x\in\mathcal{F}\cup\{e\}, \tag{e3.28}$$

$$\varphi(e)\lesssim b, \tag{e3.29}$$

$$t(f_{1/4}(\psi(e)))\geq\mathfrak{f}_e/2 \ \textit{ for all } \ t\in T(D). \tag{e3.30}$$

Proof. Fix $\varepsilon>0$, b and $\mathcal{F}$ as described in the statement. Let $\eta=\inf\{\tau(b):\tau\in\overline{T(A)}^w\}>0$. Choose $e_0\in A_+$ with $\|e_0\|=1$ such that $\|e_0ee_0-e\|<\varepsilon/16$. Without loss of generality, we may also assume that $e_0f=fe_0=f$ for all $f\in\mathcal{F}$. It follows from 11.8 of [15] that the map from $Cu(A)$ to $\mathrm{LAff}_{b+}(\overline{T(A)}^w)$ is an isomorphism. Therefore there is $e_{0,1}\in A_+$ such that $n\langle e_{0,1}\rangle=\langle e_0\rangle$ and $\langle e_0\rangle=\langle b\rangle$, where $b=\mathrm{diag}(e_{0,1}, e_{0,1}, \dots, e_{0,1})$ ($e_{0,1}$ repeated n times) in $M_n(A)_+$. By 11.5 of [15], A has stable rank one. It follows that $\overline{e_0Ae_0}$ and $\overline{bM_n(A)b}$ are isomorphic. In particular, $\overline{e_0Ae_0}\cong M_n(\overline{e_{0,1}Ae_{0,1}})$. Therefore, without loss of generality, (replacing e_0 by another strictly positive element in $\overline{e_0Ae_0}$), we may also write that $e_0=\sum_{i=1}^n e_{0,i}$, where $\{e_{0,1}, e_{0,2}, \dots, e_{0,n}\}$ are mutually orthogonal and there exists $w_i\in A$ such that $w_i^*w_i=e_{0,1}$ and $w_iw_i^*=e_{0,i}$, $i=1,2,\dots,n$.

Since A is stably projectionless, without loss of generality, we may assume that $sp(e_0)=[0,1]$. Then elements $e_{0,i}$ and w_i generate a C^*-subalgebra C which is isomorphic to $C_0((0,1])\otimes M_n$ which is semi-projective. Let $\mathcal{G}_1=\{e_{0,i}, w_i : 1\leq i\leq n\}$.

Put $\delta_0=\min\{\varepsilon/16(n+1), \eta/2(n+1), \mathfrak{f}_e/4(n+1)\}$. Choose $\delta_1>0$ such that for any $\mathcal{G}_1$-δ_1-multiplicative completely positive contractive linear map L from C to a C^*-algebra B, there is a homomorphism $\varphi': C\to B$ such that

$$\|\varphi'(g)-L(g)\|<\delta_0/4 \ \text{ for all } \ g\in\mathcal{G}_1. \tag{e3.31}$$

Put $\mathcal{F}_1=\mathcal{F}\cup\mathcal{G}_1\cup\{ab : a, b\in\mathcal{F}\cup\mathcal{G}_1\}$.

Fix a positive number $\varepsilon_1 < \min\{\delta_0, \delta_1/2\}/(4(n+1))$. Since $A \in \mathcal{D}$, there are $\mathcal{F}_2$-ε_1-multiplicative completely positive contractive linear maps $\varphi : A \to A$ and $\psi_0 : A \to B$ for some C^*-subalgebra $B \subset A$ with $B \in \mathcal{C}_0$ such that $\varphi(e) \lesssim b$, $\varphi(A) \perp B$,

$$\|x - (\varphi(x) \oplus \psi_0(x))\| < \varepsilon_1 \text{ for all } x \in \mathcal{F}_1 \cup \{e, e_0\}, \tag{e3.32}$$

$$t(f_{1/4}(\psi(e))) \geq \mathfrak{f}_e \text{ for all } t \in T(B). \tag{e3.33}$$

By the choice of $\mathcal{G}_1$ and δ_1, we obtain a homomorphism $h : C \to B$ such that

$$\|h(g) - \psi_0(g)\| < \delta_0/4 \text{ for all } g \in \mathcal{G}_1. \tag{e3.34}$$

Let $e_i' = h(e_i)$ and $v_i = h(w_i)$, $i = 1, 2, \ldots, n$. Let $B' = h(e_0)Bh(e_0)$. Since h is a homomorphism and $e', v_i \in B'$, $B' \cong M_n(\overline{e_1'Be_1'})$. Set $D = \overline{e_1'Be_1'}$. Define $\psi : A \to B'$ by $\psi(a) = h(e_0)\psi(a)h(e_0)$. One checks

$$\tau(\psi(e)) \geq \mathfrak{f}_a/2 \text{ for all } \tau \in T(B') \tag{e3.35}$$

and ψ is $\mathcal{F}$-ε-multiplicative. Moreover,

$$\|x - (\varphi(x) \oplus \psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F}. \quad \Box \tag{e3.36}$$

4. The unitary group

Lemma 4.1. *Let A be a non-unital C^*-algebra and let $e_1, e_2 \in A_+$ with $\|e_i\| = 1$ ($i = 1, 2$) such that*

$$e_1e_2 = e_2e_1 = 0$$

and there is a unitary $u \in \tilde{A}$ such that $u^*e_1u = e_2$. Suppose that $w = 1_{\tilde{A}_0} + x_0 \in \tilde{A}_0$ is a unitary with $x_0 \in A_0$, where $A_0 = \overline{e_1Ae_1}$. Then $w_1 = 1 + x_0 + u^*x_0^*u \in CU(\tilde{A})$, $\mathrm{cel}(w_1) \leq \pi$ and $\mathrm{cer}(w_1) \leq 1 + \varepsilon$.

Proof. Let B be the C^*-subalgebra of A generated by A_0 and ue_2. Note that $u^*A_0u = \overline{e_2Ae_2}$. One can define a map from $M_2(\overline{e_1Ae_1}) = M_2(\overline{e_1^2Ae_1^2})$ to B by

$$M_2(\overline{e_1Ae_1}) \ni \begin{pmatrix} e_1^2a_{11}e_1^2 & e_1^2a_{12}e_1^2 \\ e_1^2a_{21}e_1^2 & e_1^2a_{22}e_1^2 \end{pmatrix} \mapsto e_1^2a_{11}e_1^2 + e_1^2a_{12}e_1ue_2 + e_2u^*e_1a_{21}e_1^2 + e_2u^*e_1a_{22}e_1ue_2.$$

It is easy to verify that this is an isomorphism by using $e_1e_2 = e_2e_1 = 0$ and $u^*e_1u = e_2$. Therefore $B \cong M_2(A_0) \in B$. Consider $M_2(\tilde{A}_0)$. Put $p_{1,1} = 1_{\tilde{A}_0}$. We view $p_{1,1}$ as the open projection associated to A_0. Let $p_{2,2} = u^*p_{1,1}u$. Since $1_{\tilde{A}_0} + x_0$ is a unitary, we have

$$(p_{11} + x_0^*)(p_{11} + x_0) = (p_{11} + x_0)(p_{11} + x_0^*) = p_{11}.$$

Define, for $t \in [0, 1]$,

$$X(t) = ((\cos(t\pi/2))p_{1,1} + (\sin(t\pi/2))p_{1,1}u + (\sin(t\pi/2))u^*p_{1,1} + (\cos(t\pi/2))p_{2,2}) + ((1_{\tilde{A}}) - p_{1,1} - p_{2,2}).$$

Define

$$W(t) = (1 + x_0)X(t)(1 + x_0^*)X(t)^* \text{ for all } t \in [0, 1].$$

Let $X'(t) = X(t) - ((1_{\tilde{A}}) - p_{1,1} - p_{2,2}) \in M_2(\tilde{A}_0)$ (by identifying $p_{11}u$ with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, u^*p_{11} with $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and p_{22} with $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$). Set

$$W'(t) = (p_{1,1} + p_{2,2} + x_0)X'(t)(p_{1,1} + p_{2,2} + x_0^*)X'(t)^* \in M_2(\tilde{A}_0).$$

We have

$$X'(0) = p_{1,1} + p_{2,2} \text{ and } X'(1) = p_{1,1}u + u^*p_{1,1}.$$

Then

$$W'(0) = p_{1,1} + p_{2,2} \text{ and } W'(1) = (p_{1,1} + x_0) + (p_{2,2} + u^*x_0^*u).$$

Let $\pi : M_2(\tilde{A}_0) \to M_2$ be the quotient map. Then $\pi(W'(t)) = 1_{M_2}$ for all $t \in [0, 1]$. This implies that $W'(t) \in \widetilde{M_2(A_0)}$ for all $t \in [0, 1]$. It follows that $W(t) \in U(\tilde{A})$ for all $t \in [0, 1]$. Note that $W(0) = 1_{\tilde{A}}$ and $W(1) = 1 + x_0 + u^*x_0^*u$. Moreover, one computes that (since each $W(t) \in U_0(\tilde{A})$),

$$\mathrm{cel}(\{W(t)\}) \leq \pi.$$

It follows that $\mathrm{cer}(W(1)) \le 1+\varepsilon$. Moreover

$$1+x_0+u^*x_0^*u = (1+x_0)u^*(1+x_0^*)u.$$

It follows that $1+x_0+u^*x_0u \in CU(\tilde{A})$. □

The following is a variation of a lemma of N. C. Phillips

Lemma 4.2 (*Lemma 3.1 of [37]*). *Let $H > 0$ be a positive number and let $N \ge 2$ be an integer. Then, for any non-unital C^*-algebra which has almost stable rank one, any positive element $e_0 \in A_+$ with $\|e_0\| = 1$, and $u = \lambda \cdot 1_{\tilde{A}_0} + x_0' \in \tilde{A}_0$ (where $x_0' \in A_0$ and $|\lambda| = 1$) such that $\mathrm{cel}_{\tilde{A}_0}(u) \le H$, where $A_0 = \overline{e_0Ae_0}$. Suppose that there are mutually orthogonal positive elements $e_1, e_2, \dots, e_{2N} \in A_0^{\perp}$ such that $e_0 \sim e_i$, $i = 1, 2, \dots, 2N$. Then there exists $z \in CU(\tilde{A})$ with $\mathrm{cel}(z) \le 2\pi$ and $\mathrm{cer}(z) \le 2+\varepsilon$ such that*

$$\|u' - \lambda \cdot z\| < 2H/N,$$

where $u' = \lambda \cdot 1_{\tilde{A}} + x_0'$.

Proof. Since $\mathrm{cel}_{\tilde{A}_0}(u) \le H$, there are $u_0, u_1, \dots, u_N \in \tilde{A}_0$ such that

$$u_0 = u,\ \ u_N = 1_{\tilde{A}_0} \ \text{and}\ \|u_i - u_{i-1}\| < H/N,\ \ i = 1, 2, \dots, N. \tag{e4.1}$$

Write $u_i = \lambda_i \cdot 1_{\tilde{A}_0} + x_i'$, where $x_i' \in A_0$, $i = 1, 2, \dots, N$. In particular, $x_N' = 0$. It follows from (e4.1) that ($\lambda_0 = \lambda$)

$$|\lambda_i - \lambda_{i-1}| < H/N,\ \ i = 1, 2, \dots, N.$$

Let $v = v_0 = \bar{\lambda}u = 1_{\tilde{A}_0} + \bar{\lambda}x_0'$ and $v_i = \bar{\lambda}_iu_i = 1_{\tilde{A}_0} + \bar{\lambda}_ix_i'$, $i = 1, 2, \dots, N$. Put $x_i = \bar{\lambda}_ix_i'$, $i = 0, 1, \dots, N$. We have $x_N = 0$ and $v_N = 1_{\tilde{A}_0}$. Now

$$\|v_i - v_{i-1}\| = \|\bar{\lambda}_iu_i - \bar{\lambda}_{i-1}u_{i-1}\| < 2H/N,\ \ i = 1, 2, \dots, N. \tag{e4.2}$$

Let

$$\varepsilon_0 = 2H/N - \sup\{\|v_i - v_{i-1}\| :, i = 1, 2, \dots, N\}.$$

Choose $1 > \delta > 0$ such that

$$\|x_i - f_\delta(e_0)x_if_\delta(e_0)\| < \varepsilon_0/16N,\ \ i = 0, 1, 2, \dots., N.$$

Put $B_0 = \overline{f_\delta(e_0)Af_\delta(e_0)}$. There is a unitary $w_i \in 1_{\tilde{A}_0} + B_0$ such that

$$\|v_i - w_i\| < \varepsilon_0/4N,\ \ i = 0, 1, \dots, N.$$

Write $w_i = 1_{\tilde{A}_0} + y_i$, where $y_i \in B_0$ and $y_N = 0$. Since A has almost stable rank one, there are unitaries $U_i \in \tilde{A}$ such that

$$U_i^*f_{\delta/2}(e_0)U_i \in \overline{e_iAe_i},\ \ i = 1, 2, \dots, 2N.$$

Let

$$X_1 = 1_{\tilde{A}} + y_0 + \sum_{i=1}^N U_{2i-1}^*y_i^*U_{2i-1} + \sum_{i=1}^N U_{2i}^*y_iU_{2i} \tag{e4.3}$$

$$X_2 = 1_{\tilde{A}} + y_0 + \sum_{i=1}^N U_{2i-1}^*y_{i-1}^*U_{2i-1} + \sum_{i=1}^N U_{2i}^*y_iU_{2i} \ \text{and} \tag{e4.4}$$

$$X_3 = 1_{\tilde{A}} + \sum_{i=1}^N U_{2i-1}^*y_iU_{2i-1} + \sum_{i=1}^N U_{2i}^*y_i^*U_{2i}. \tag{e4.5}$$

Note that $X_1 \in U(\tilde{A})$. Since $y_N = 0$, as in 4.1, for $i = 2, 3$, we have

$$X_i \in CU(\tilde{A}),\ \ \mathrm{cel}(X_i) \le \pi \ \text{and}\ \mathrm{cer}(X_i) \le 1+\varepsilon. \tag{e4.6}$$

Moreover

$$\|X_1 - X_2\| \le \sup\{\|y_i^* - y_{i-1}^*\| :, i = 1, 2, \dots, N\} \tag{e4.7}$$

$$< \varepsilon_0/4N + \sup\{\|v_i - v_{i-1}\| :, i = 1, 2, \dots, N\} \tag{e4.8}$$

Furthermore,

$$1_{\tilde{A}} + y_0 = X_1X_3. \tag{e4.9}$$

Put $z = X_2X_3$. Then, by (e4.6),

$$z \in CU(\tilde{A}), \ \ \mathrm{cel}(z) \le 2\pi \ \text{ and } \ \mathrm{cer}(z) \le 2+\varepsilon.$$

Moreover,

$$\|\bar{\lambda}\cdot u' - z\| \le \|(1_{\tilde{A}} - 1_{\tilde{A}_0}) + v_0 - (1_{\tilde{A}} + y_0)\| + \|(1_{\tilde{A}} + y_0) - z\| \tag{e4.10}$$

$$< \varepsilon_0/8N + \varepsilon_0/4N + \sup\{\|v_i - v_{i-1}\| :, i = 1, 2, \ldots, N\} < 2H/N. \quad \square \tag{e4.11}$$

Theorem 4.3 (*cf. Theorem 6.5 of [33]*). *Let A be a non-unital separable simple C^*-algebra in $\mathcal{D}$ and let $u \in U_0(\tilde{A})$ with $u = \lambda\cdot 1 + x_0$, where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $x_0 \in A$. Then, for any $\varepsilon > 0$, there exists a unitary $u_1, u_2 \in \tilde{A}$ such that u_1 has exponential length no more than 2π, u_2 has exponential rank 3 and*

$$\|u - u_1u_2\| < \varepsilon.$$

Moreover, $\mathrm{cer}(A) \le 5+\varepsilon$.

Proof. Let $1/2 > \varepsilon > 0$. Let $u' = \bar{\lambda}\cdot u$. Let $v_0, v_1, \ldots, v_n \in U_0(\tilde{A})$ such that

$$v_0 = u', \ \ v_n = 1 \ \text{ and } \ \|v_i - v_{i-1}\| < \varepsilon/32, \ \ i = 0, 1, \ldots, n-1.$$

Write $v_i = \lambda_i\cdot 1 + x_i$, where $|\lambda_i| = 1$ and $x_i \in A$, $i = 1, \ldots, n-1$, and $v_0 = 1 + \tilde{x}_0$, where $\tilde{x}_0 = \bar{\lambda}x_0$. Note that $x_n = 0$.

As demonstrated in the proof of 4.2, we may assume that there is a strictly positive element $e \in A_+$ such that $\|e\| = 1$ such that

$$f_\eta(e)x_i = x_if_\eta(e) = x_i, \ \ i = 0, 1, 2, \ldots, n, \tag{e4.12}$$

for some $\eta > 0$. Let

$$\mathcal{G}_1 = \{e, f_\eta(e), f_{\eta/2}(e), \tilde{x}_0, x_i, \ 0 \le i \le n\}.$$

Put

$$d = \inf\{d_\tau(e) : \tau \in \overline{T(A)}^w\} > 0.$$

Without loss of generality, we may assume that $\tau(f_{1/2}(e)) \ge d/2$ for all $\tau \in \overline{T(A)}^w$.

Note that we may assume that A is infinite dimensional. Hence we may choose mutually orthogonal positive non-zero elements $c_0, c_1, \ldots, c_{n+1}$ such that $c_0 \sim c_i$ $(1 \le i \le n+1)$ and

$$d_\tau(c_0) < d/5(n+1) \ \text{ for all } \ \tau \in T(A). \tag{e4.13}$$

Let $\delta > 0$ and let $\mathcal{G} \supset \mathcal{G}_1$ be a finite subset of A. Since $A \in \mathcal{D}$, there are A_0 and $D \subset A$ with $D \in \mathcal{C}_0'$ and $A_0 \perp D$, $\mathcal{G}$-δ-multiplicative completely positive contractive linear maps $\varphi_0 : A \to A_0$ and $\varphi_1 : A \to D$, such that

$$\|x - (\varphi_0(x) \oplus \varphi_1(x))\| < \delta \ \text{ for all } \ x \in \mathcal{G} \tag{e4.14}$$

$$\varphi_0(e) \lesssim c_0, \tag{e4.15}$$

$$\tau(f_{1/4}(\varphi_1(e))) \ge d/4 \ \text{ for all } \ \tau \in \overline{T(A)}^w. \tag{e4.16}$$

By choosing smaller δ and larger $\mathcal{G}$, we may assume the following: there are $y_i \in \overline{\varphi_0(f_{\eta/2}(e))A\varphi_0(f_{\eta/2}(e))}$ such that $1 + y_0$, $\lambda_i\cdot 1 + y_i$ are unitaries with $y_n = 0$ such that $\|\varphi_0(x_i) - y_i\| < \varepsilon/32$, $i = 1, 2\ldots, n$, and $\|\varphi_0(\tilde{x}_0) - y_0\| < \varepsilon/32$. Consequently,

$$\|y_i - y_{i+1}\| < \varepsilon/16, \ \ \|(\lambda_i\cdot 1 + y_i) - (\lambda_i\cdot 1 + y_{i-1})\| < \varepsilon/16, \tag{e4.17}$$

$i = 0, 1, \ldots, n$. Moreover, there is $z_1 \in \overline{f_{\eta/2}(\varphi_1(e))Df_{\eta/2}(\varphi_1(e))}$ such that $1_{\tilde{A}} + z_1$ is a unitary and

$$\|v_0 - (1_{\tilde{A}} + y_0 + z_1)\| < \varepsilon/16. \tag{e4.18}$$

Put $u_1' = 1 + y_0$, $u_2' = 1 + z_1$ and $u_2 = \lambda u_2'$. Then

$$\|u - u_1'\cdot u_2\| < \varepsilon/4.$$

Put $B_0 = \overline{\varphi_0(f_{\eta/2}(e))A\varphi_0(f_{\eta/2}(e))}$. Let $w_i = \lambda_i\cdot 1_{\tilde{B}_0} + y_i$, $i = 0, 1, \ldots, n$. Then $w_n = 1_{\tilde{B}_0}$, $w_0 = 1\cdot 1_{\tilde{B}_0} + y_0$ and

$$\|w_i - w_{i-1}\| < \varepsilon/16, \ \ i = 1, 2, \ldots, n.$$

This implies that $w_0 \in U_0(\tilde{B}_0)$ and $H := \mathrm{cel}(w_0) \le n\pi\varepsilon/8$. By (e4.13) and (e4.16), there are mutually orthogonal elements $c_i' \in A_0^\perp$, with $c_i' \sim c_0$, $i = 0, 1, \ldots, n+1$. Then, by (e4.15) and by Lemma 4.2, $\mathrm{cel}(u_1') \le 2\pi + 2H/n < 2\pi + \pi\varepsilon/8$. On the other hand, by 3.3, $\mathrm{cer}(u_2) \le 2+\varepsilon$. Lemma then follows. $\square$

Theorem 4.4. *Let A be a separable simple C^*-algebra in $\mathcal{D}$ and let $u \in CU(\tilde{A})$. Then $u \in U_0(\tilde{A})$ and $\mathrm{cel}(u) \le 6\pi$.*

Proof. Let $\pi : \tilde{A} \to \mathbb{C}$ be the quotient map. Since $u \in CU(\tilde{A})$, $\pi(u) = 1$. So we write $u = 1 + x_0$, where $x_0 \in A$.

Let $1/2 > \varepsilon > 0$. There are $v_1, v_2, \dots, v_k \in U(\tilde{A})$ such that

$$\|u - v_1 v_2 \cdots v_k\| < \varepsilon/32,$$

and $v_i = a_i b_i a_i^* b_i^*$, $a_i, b_i \in U(\tilde{A})$. It is standard that $v_1 v_2 \cdots v_k \oplus 1_{M_{4k}} \in U_0(M_{4k+1}(\tilde{A}))$. Since $\tilde{A}$ has stable rank one (see 11.5 of [15] and 15.5 of [18]), by [48], $v_1 v_2 \cdots v_k \in U_0(\tilde{A})$. It follows that $u \in U_0(\tilde{A})$. Put $u_0 = v_1 v_2 \cdots v_k$. Let $H = \mathrm{cel}(u_0)$.

Write $a_i = \lambda_i + x_i$ and $b_i = \mu_i + y_i$, where $|\lambda_i| = |\mu_i| = 1$ and $x_i, y_i \in A$, $i = 1, 2, \dots, k$.

The rest of the proof is similar to that of 4.3. We will repeat some of the argument. we may assume that there is a strictly positive element $e \in A_+$ such that $\|e\| = 1$ and

$$f_\eta(e)x_i = x_i f_\eta(e) = x_i, f_\eta(e)y_i = y_i f_\eta(e) = y_i, \quad i = 0, 1, 2, \dots, k, \tag{e4.19}$$

for some $\eta > 0$. Let

$$\mathcal{G}_1 = \{e, f_\eta(e), f_{\eta/2}(e), x_i, y_i, 0 \le i \le k\}.$$

Put

$$d = \inf\{d_\tau(e) : \tau \in \overline{T(A)}^w\} > 0.$$

Without loss of generality, we may assume that $\tau(f_{1/2}(e)) \ge d/2$ for all $\tau \in \overline{T(A)}^w$.

Choose $n \ge 1$ such that

$$4H/n < \varepsilon/64k.$$

There are mutually orthogonal elements $c_0, c_1, \dots, c_{n+1}$ in A such that $c_0 \sim c_i$ and $d_\tau(c_0) < \varepsilon d/n64k$ for all $\tau \in T(A)$. Let $\delta > 0$ and let $\mathcal{G} \supset \mathcal{G}_1$ be a finite subset of A.

Since $A \in \mathcal{D}$, there are A_0 and $D \subset A$ with $D \in \mathcal{C}_0'$ and $A_0 \perp D$, $\mathcal{G}$-δ-multiplicative completely positive contractive linear maps $\varphi_0 : A \to A_0$ and $\varphi_1 : A \to D$, such that

$$\|x - (\varphi_0(x) \oplus \varphi_1(x))\| < \delta \ \text{ for all } \ x \in \mathcal{G} \tag{e4.20}$$

$$\varphi_0(e) \lesssim c_0, \tag{e4.21}$$

$$\tau(f_{1/4}(\varphi_1(e))) \ge d/4 \ \text{ for all } \ \tau \in \overline{T(A)}^w. \tag{e4.22}$$

By choosing smaller δ and larger $\mathcal{G}$, we may assume the following: there is $x_0' \in \overline{\varphi_0(f_{\eta/2}(e))A\varphi_0(f_{\eta/2}(e))}$ such that $1 + x_0'$ is a unitary, $\mathrm{cel}(p + x_0') \le 2H$, where p is the unit of unitization of $\tilde{B}$, where $B = \overline{\varphi_0(f_{\eta/2}(e))A\varphi_0(f_{\eta/2}(e))}$, and there are $z, z_i, x_i', y_i' \in \overline{f_{\eta/2}(\varphi_1(e))Df_{\eta/2}(\varphi_1(e))}$ such that $\lambda_i + a_i'$ and $\mu_i + b_i'$ are unitaries, and

$$\|(1+z) - (1+z_1)(1+z_2)\cdots(1+z_k)\| < \varepsilon/16 \ \text{ and } \ \|u_0 - (1 + x_0' + z)\| < \varepsilon/16, \tag{e4.23}$$

where

$$1 + z_i = (\lambda_i \cdot 1 + x_i')(\mu_i \cdot 1 + y_i')(\lambda_i \cdot 1 + x_i')^*(\mu_i + y_i')^*, \quad i = 1, 2, \dots, k.$$

In particular, $(1+z_1)(1+z_2)\cdots(1+z_k) \in CU(\tilde{C})$, where $C = \overline{f_{\eta/2}(\varphi_1(e))Df_{\eta/2}(\varphi_1(e))}$. It follows from 3.3 that

$$\mathrm{cel}((1+z_1)(1+z_2)\cdots(1+z_k)) \le 4\pi.$$

As in the proof of 4.3, using (e4.21), by applying 4.2, we have

$$\mathrm{cel}(1 + x_0') \le 4H/n + 2\pi + \varepsilon < 2\pi + 2\varepsilon.$$

It follows that

$$\mathrm{cel}(u) < 6\pi. \quad \square$$

Proposition 4.5 (*cf. Theorem 4.6 of [21]*)**.** *Let A be a separable simple C^*-algebra with continuous scale and let $e \in A_+ \setminus \{0\}$. Then the map $\imath_e : U_0(\widetilde{eAe})/CU(\widetilde{eAe}) \to U_0(\tilde{A})/CU(\tilde{A})$ is surjective. If, in addition, A has stable rank one, then the map is also injective.*

Proof. The proof is almost identical to that of the unital case (see Theorem 4.6 of [21]).

First, we claim that, for any $h \in A_{s.a.}$, there exists $h' \in (\overline{eAe})_{s.a.}$ such that $\tau(h') = \tau(h)$ for all $\tau \in T(A)$. Let $h = h_+ - h_-$.

Put $A_0 = \overline{eAe}$. By Proposition 5.6 of [15], there are $x_i, y_j \in A$ ($1 \le i \le n$ and $1 \le j \le m$) such that

$$\sum_{i=1}^n x_i^* e x_i = h_+ \ \text{ and } \ \sum_{j=1}^m y_j^* e y_j = h_-. \tag{e4.24}$$

Then

$$h' := \sum_{i=1}^{n} e^{1/2}x_i^*x_ie^{1/2} - \sum_{j=1}^{m} e^{1/2}y_j^*y_je^{1/2} \in A_0. \tag{e4.25}$$

Moreover, $\tau(h') = \tau(h)$ for all $\tau \in T(A)$. This proves the claim.

To show ι_e is surjective, let $u \in U_0(\tilde{A})$ with $u = \prod_{j=1}^{l} \exp(i2\pi h_j)$ with $h_j \in \tilde{A}_{s.a.}$. Write $h_j = \alpha_i \cdot 1_{\tilde{A}} + h_j'$, where $\alpha_j \in \mathbb{R}$ with $|\alpha_j| = 1$ and $h_j' \in A_{s.a.}$. By the claim that there exists $h_{0,j}' \in (A_0)_{s.a.}$ such that $\tau(h_{0,j}') = \tau(h_j')$ for all $\tau \in T(A)$. Let $h_{0,j} = \alpha_j \cdot 1_{\tilde{A}_0} + h_{0,j}'$, $j = 1, 2, \ldots, l$. Put $w = \prod_{j=1}^{l} \exp(ih_{0,j})$. Then $w \in U_0(\tilde{A}_0)$. Put $v = \prod_{j=1}^{l} \exp(i\tilde{h}_{0,j})$, where $\tilde{h}_{0,j} = \alpha_j \cdot 1_{\tilde{A}} + h_{0,j}'$, $j = 1, 2, \ldots, l$. Then $v \in U_0(\tilde{A})$. Moreover, $\iota_e(\bar{w}) = \bar{v}$. Since

$$D_{\tilde{A}}(v)(\tau) = \sum_{j=1}^{l} \tau(\tilde{h}_{0,j}) = \sum_{j=1}^{l} \tau(\tilde{h}_j) = D_{\tilde{A}}(u)(\tau) \tag{e4.26}$$

for all $\tau \in T(\tilde{A})$, by Lemma 3.1 of [56], $\iota_e(\bar{w}) = \bar{u}$. This proves that ι_e is surjective.

To see it is injective, let $e_A \in A$ be a strictly positive element of A with $\|e_A\| = 1$. Since A has continuous scale, by (the proof of) Proposition 5.6 of [15], there exists an integer $K \geq 1$ such that

$$K\langle a_0 \rangle > \langle e_A \rangle \tag{e4.27}$$

(in Cuntz semi-group). Since A has stable rank one, without loss of generality, we may write $A \subset M_K(A_0)$. Put $E_0 = 1_{\tilde{A}_0}$. Let $u \in \tilde{A}_0$ with $u = \lambda \cdot E_0 + x$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $x \in (A_0)_{s.a.}$. Write $w = \lambda \cdot 1_{\tilde{A}} + x$. Then $\iota_e(\bar{u}) = \bar{w}$. Suppose that $w \in CU(\tilde{A})$. Write $E = 1_{M_K(\tilde{A}_0)}$ and $w' = \lambda \cdot E + x$. Then $w' \in CU(M_K(\tilde{A}_0))$. However, since $\tilde{A}_0$ has stable rank one, it follows from Theorem 4.6 of [21] that $\bar{u} \in CU(\tilde{A}_0)$. This shows that ι_e is injective. □

Lemma 4.6. *Let A be a non-unital and σ-unital simple C^*-algebra of stable rank one with continuous scale. Suppose that there is $H > 0$ such that, for any hereditary C^*-subalgebra B of A, $\mathrm{cel}(z) \leq H$ for any $z \in CU(\tilde{B})$. Suppose that there are two mutually orthogonal σ-unital hereditary C^*-subalgebras A_0 and A_1 (of A) with strictly positive elements a_0 and a_1 with $\|a_0\| = 1$ and $\|a_1\| = 1$, respectively. Suppose that $x \in A_0$ and suppose that for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, $w = \lambda + x \in U_0(\tilde{A})$. Suppose also that there is an integer $K \geq 1$ such that*

$$Kd_\tau(a_0) \geq 1 \ \text{ for all } \ \tau \in T(A). \tag{e4.28}$$

Let $u = \lambda \cdot 1_{\tilde{A}_0} + x$. Suppose that, for some $\eta \in (0, 2]$,

$$\mathrm{dist}(\bar{w}, \bar{1}) \leq \eta \ \text{ in } \ U_0(\tilde{A})/CU(\tilde{A}).$$

Then, if $\eta < 2$, one has

$$\mathrm{cel}_{\tilde{A}_0}(u) < (\frac{K\pi}{2} + 1/16)\eta + H \ \text{ and } \ \mathrm{dist}(\bar{u}, \bar{1}_{\tilde{A}_0}) < (K + 1/8)\eta,$$

and if $\eta = 2$, one has

$$\mathrm{cel}_{\tilde{A}_0}(u) < \frac{K\pi}{2}\mathrm{cel}(w) + 1/16 + H.$$

Proof. Let $L = \mathrm{cel}(w)$. Since A is simple and has stable rank one, $u \in U_0(\tilde{A}_0)$.

First consider the case that $\eta < 2$. Let $c \in CU(\tilde{A})$ such that

$$\|c - w\| \leq \eta.$$

Choose $\frac{\eta}{32K(K+1)\pi} > \varepsilon > 0$ such that $\varepsilon + \eta < 2$. Choose $h \in \tilde{A}_{s.a.}$ such that with $\|h\| \leq 2\arcsin(\frac{\varepsilon+\eta}{2})$ such that

$$w\exp(ih) = c. \tag{e4.29}$$

Thus

$$\overline{D_{\tilde{A}}}(w\exp(ih)) = \bar{0} \ \ (\text{in} \ \ \mathrm{Aff}(T(\tilde{A}))/\overline{\rho_{\tilde{A}}(K_0(\tilde{A}))}). \tag{e4.30}$$

It follows that

$$|\overline{D_{\tilde{A}}}(w)(\tau)| \leq 2\arcsin(\frac{\varepsilon + \eta}{2}). \tag{e4.31}$$

Put $h = \alpha \cdot 1_{\tilde{A}} + h_0$, where $\alpha \in \mathbb{R}$ with $|\alpha| \leq 2\arcsin(\frac{\varepsilon+\eta}{2})$ and $h_0 \in A_{s.a.}$. As in the proof of surjectivity of ι_e in 4.5, there is $h_0' \in (A_0)_{s.a}$ such that $\tau(h_0') = \tau(h_0)$ for all $\tau \in T(A)$. Put $h_0'' = \alpha \cdot 1_{\tilde{A}} + h_0'$. Moreover, $\tau(h_0'') = \tau(h)$ for all $\tau \in T(\tilde{A})$.

Therefore

$$\overline{D_{\tilde{A}}}(w\exp(ih_0''))(\tau)=\bar{0}. \tag{e4.32}$$

It follows from 4.5 that

$$D_{\tilde{A}_0}(u\exp(ih_{00}))=\bar{0}\ \text{ (in }\ \mathrm{Aff}(T(\tilde{A}_0))/\overline{\rho_{\tilde{A}_0}(K_0(\tilde{A}_0))}), \tag{e4.33}$$

where $h_{00}=\alpha\cdot 1_{\tilde{A}_0}+h_0'$. By (e4.28), $\|\tau|_{A_0}\|\geq 1/K$. Then, by (e4.31), in $\tilde{A}_0$, one computes

$$|\overline{D_{\tilde{A}_0}}(u)|\leq K2\arcsin(\frac{\varepsilon+\eta}{2}). \tag{e4.34}$$

Thus there is $v\in CU(\tilde{A}_0)$ and $h_1\in\tilde{A}_{s.a.}$ such that

$$u=v\exp(2\pi ih_1)\ \text{ and }\ \|h_1\|\leq K2\arcsin(\frac{\varepsilon+\eta}{2}). \tag{e4.35}$$

Therefore

$$\mathrm{cel}(u)\leq H+K2\arcsin(\frac{\varepsilon+\eta}{2})\leq H+K(\varepsilon+\eta)\frac{\pi}{2} \tag{e4.36}$$

$$\leq H+(K\frac{\pi}{2}+\frac{1}{64(K+1)})\eta. \tag{e4.37}$$

One can also compute that

$$\mathrm{dist}(\bar{u},\bar{1}_{\tilde{A}_0})\leq K(\varepsilon+\eta)\leq K\eta+\frac{\eta}{32(K+1)\pi}.$$

This proves the case that $\eta<2$.

Now suppose that $\eta=2$. Define $R=[\mathrm{cel}(w)+1]$. Note that $\frac{\mathrm{cel}(w)}{R}<1$. Put $w'=\lambda\cdot 1_{M_{R+1}}+x$. It follows from 4.2 that

$$\mathrm{dist}(\overline{w'},\overline{1_{M_{R+1}}})<\frac{\mathrm{cel}(w)}{R+1} \tag{e4.38}$$

Put $K_1=K(R+1)$. To simplify notation, replacing A by $M_{R+1}(A)$, without loss of generality, we may now consider that

$$K_1d_\tau(a_0)\geq 1\ \text{ and }\ \mathrm{dist}(\bar{w},\bar{1})<\frac{\mathrm{cel}(w)}{R+1}. \tag{e4.39}$$

Then we can apply the case that $\eta<2$ with $\eta=\frac{\mathrm{cel}(w)}{R+1}$. □

5. A uniqueness theorem for C^*-algebras in $\mathcal{D}$

Proposition 5.1. *Let A be a separable amenable C^*-algebra. Let $\varepsilon>0$ and $\mathcal{F}\subset A$ be a finite subset. Then there exist $\delta>0$ and a finite subset $\mathcal{G}\subset A$ satisfy the following: Suppose that there are two mutually orthogonal C^*-subalgebras A_0 and A_1 and two $\mathcal{F}$-$\varepsilon/2$-multiplicative completely positive contractive linear maps $\varphi_0:A\to A_0$ and $\varphi_1:A\to A_1$ such that*

$$\|x-(\varphi_0(x)\oplus\varphi_1(x))\|<\varepsilon/2\ \textit{ for all }\ x\in\mathcal{F}$$

and suppose that there is $\psi:A\to B$ (for any C^*-algebra B) which is a $\mathcal{G}$-δ-multiplicative completely positive contractive linear map. Then there exist a pair of mutually orthogonal C^*-subalgebras B_0 and B_1 of B and $\mathcal{F}$-ε-multiplicative completely positive contractive linear maps $\psi_0:A\to B_0\subset B$ and $\psi_1:A\to B_1\subset B$ such that

$$\|\psi_0(x)-\psi\circ\varphi_0(x)\|<\varepsilon\ \text{ and} \tag{e5.1}$$

$$\|\psi_1(x)-\psi\circ\varphi_1(x)\|<\varepsilon\ \textit{ for all }\ x\in\mathcal{F}. \tag{e5.2}$$

Proof. Fix $1/2>\varepsilon>0$ and a finite subset $\mathcal{F}\subset A$. Let $\{B_n\}$ be any sequence of C^*-algebras and let $\varphi_n:A\to B_n$ be any sequence of completely positive contractive linear maps such that

$$\lim_{n\to\infty}\|\varphi_n(a)\varphi_n(b)-\varphi_n(ab)\|=0\ \text{ for all }\ a,b\in A. \tag{e5.3}$$

Let $B_\infty=\prod_{n=1}^\infty B_n$, $B_q=B_\infty/\oplus_{n=1}^\infty B_n$ and $\Pi:B_\infty\to B_q$ be the quotient map. Define $\Phi:A\to B_\infty$ by $\Phi(a)=\{\varphi_n(a)\}$ for all $a\in A$. Then $\Pi\circ\Phi:A\to B_q$ is a homomorphism. Suppose A_0 and A_1 are in the statement of the proposition. Let $a_0\in(A_0)_+$ with $\|a_0\|=1$ and $a_1\in(A_1)_+$ with $\|a_1\|=1$ be strictly positive elements of A_0 and A_1, respectively. Then $a_0a_1=a_1a_0=0$. Therefore there are $b^{(0)},b^{(1)}\in B_\infty$ such that $b^{(0)}b^{(1)}=b^{(1)}b^{(0)}=0$ and such that $\Pi(b^{(i)})=\Pi\circ\Phi(a_i)$, $i=0,1$ (see, for example, 10.1.10 of [41]). Write $b^{(i)}=\{b_n^{(i)}\}$. Let $B_{n,i}=\overline{b_n^{(i)}B_nb_n^{(i)}}$, $i=0,1$. Then $B_{n,0}$ and $B_{n,1}$ are mutually orthogonal. Since A is amenable, there is a completely positive contractive linear map $\Psi : A \to B_\infty$ such that $\Psi = \Pi \circ \Phi$. Define $\psi_n' : A \to B_n$ by

$$\psi_n'(a) = b_n^{(i)}\varphi_n(a)b_n^{(i)} \text{ for all } a \in A. \tag{e5.4}$$

Let $\psi_0 = \psi_n' \circ \varphi_0$ and $\psi_1 = \psi_n' \circ \varphi_0$. If n is sufficiently large, then ψ_0 and ψ_1 can be $\mathcal{F}$-ε-multiplicative. Moreover, if n sufficiently large,

$$\|\psi_0(a) - \varphi_n \circ \varphi_0(a)\| < \varepsilon \text{ for all } a \in \mathcal{F} \text{ and} \tag{e5.5}$$
$$\|\psi_1(a) - \varphi_n \circ \varphi_1(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}. \tag{e5.6}$$

If the proposition fails, then such $\{\varphi_n\}$ could not exist for some choice of $\{B_n\}$, ε and $\mathcal{F}$. This proves the proposition. □

5.2. Fix a map $\mathbf{T}(n,k) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Let $A \in \mathcal{D}$. Denote by $\mathcal{D}_{\mathbf{T}(n,k)}$ the class of C^*-algebras in $\mathcal{D} \cap \mathbf{C}_{(r_0,r_1,T,s,R)}$ with $r_0 = 0$, $r_1 = 0$, $\mathbf{T} = \mathbf{T}(n,k)$, $s = 1$ and $R = 7$, as defined in 3.13 of [16].

Note if $A \in \mathcal{D}$, then A has stable rank one (see 11.5 of [15]) (so $r_0 = 0$ and $r_1 = 0$ in 3.14 of [15]) and by 4.4, $\mathrm{cer}(M_n(\tilde{A})) \le 6 + \varepsilon$ ($R \le 7$) for all n. If A is also $\mathcal{Z}$-stable, then $K_0(A)$ is weakly unperforated. Thus $A \in \mathcal{D}_{\mathbf{T}(n,k)}$ for $\mathbf{T}(n,k) = n$ for all $(n,k) \in \mathbb{N} \times \mathbb{N}$ (see 5.5).

In the Appendix to this paper, we show that every amenable C^*-algebra in $\mathcal{D}$ are $\mathcal{Z}$-stable. In fact, it is shown that $K_0(A)$ is always weakly unperforated in the appendix of [16] for all $A \in \mathcal{D}$. Therefore $A \in \mathcal{D}_{\mathbf{T}(n,k)}$ for the above $\mathbf{T}$.

Theorem 5.3. *Fix $\mathbf{T}(n,k)$. Let A be a non-unital separable simple C^*-algebra in $\mathcal{D}^d$ with continuous scale which satisfies the UCT. Let $T : A_+ \setminus \{0\} \to \mathbb{N} \times (\mathbb{R}_+ \setminus \{0\})$ be a map. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, $\gamma > 0$, $\eta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{P} \subset \underline{K}(A)$, a finite subset $\mathcal{U} = \{v_1, v_2, \dots, v_{m_0}\} \subset U(\tilde{A})$ such that $\{[v_1], [v_2], \dots, [v_{m_0}]\} = \mathcal{P} \cap K_1(A)$, and a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ satisfy the following: Suppose that $\varphi_1, \varphi_2 : A \to B$ are two $\mathcal{G}$-δ-multiplicative completely positive contractive linear maps which are T-$\mathcal{H}_1$-full (see 2.17), where $B \in \mathcal{D}_{\mathbf{T}(n,k)}$ with continuous scale such that*

$$[\varphi_1]|_{\mathcal{P}} = [\varphi_2]|_{\mathcal{P}}, \tag{e5.7}$$
$$|\tau \circ \varphi_1(h) - \tau \circ \varphi_2(h)| < \gamma \text{ for all } h \in \mathcal{H}_2 \text{ and } \tau \in T(B) \text{ and} \tag{e5.8}$$
$$\mathrm{dist}(\overline{\lceil \varphi_1(v_i) \rceil}, \overline{\lceil \varphi_2(v_i) \rceil}) < \eta \text{ for all } v_i \in \mathcal{U} \text{ (recall 2.1 for } \lceil - \rceil). \tag{e5.9}$$

Then there exists a unitary $w \in \tilde{B}$ such that

$$\|\mathrm{Ad}\, w \circ \varphi_1(a) - \varphi_2(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}. \tag{e5.10}$$

Proof. Fix a finite subset $\mathcal{F}$ and $1/4 > \varepsilon > 0$. As pointed out in 5.2, $B \in \mathbf{C}_{(0,0,\mathbf{T}(n,k),1,7)}$ for all $B \in \mathcal{D} = \mathcal{D}_{\mathbf{T}(n,k)}$, where $\mathbf{T}(n,k) = n$ for all (k,n). Without loss of generality, we may assume that $\mathcal{F} \subset A^{\mathbf{1}}$.

Since A has the continuous scale, $T(A)$ is compact (see 5.3 of [15]). Fix a strictly positive element $a_0 \in A_+$ with $\|a_0\| = 1$. We may assume, without loss of generality, that

$$a_0 y = y a_0 = y,\ a_0 \ge y^* y \text{ and } a_0 \ge y y^* \text{ for all } y \in \mathcal{F} \text{ and} \tag{e5.11}$$
$$\tau(f_{1/4}(a_0)) \ge 1 - \varepsilon/2^{12} \text{ for all } \tau \in T(A). \tag{e5.12}$$

Let $T_1 : A_+ \setminus \{0\} \to \mathbb{N} \times (\mathbb{R}_+ \setminus \{0\})$ with $T_1(a) = (N(a), M(a))$ ($a \in A_+ \setminus \{0\}$) be the map described after (e3.23) in 3.10 and 3.9 (see also 8.3 and 10.8 of [15]) (in place of T). Suppose that $T(a) = (N_T(a), M_T(a))$ for $a \in A_+ \setminus \{0\}$.

Define $T_2, T_3 : A_+ \setminus \{0\} \to \mathbb{N} \times (\mathbb{R}_+ \setminus \{0\})$ by $T_2(a) = (N(a), (4/3)M(a))$ and $T_3(a) = (N_T(a)N(a), (8/6)(M_T(a)+1)M(a))$ for all $a \in A_+ \setminus \{0\}$. Define $\mathbf{L}(u) = 8\pi$ for all $u \in U(\tilde{A})$.

Let $\delta_1 > 0$ (in place of δ), let $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be a finite subset, let $\mathcal{H}_{1,0} \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}$) be a finite subset, $\mathcal{P}_1 \subset \underline{K}(A)$ (in place of $\mathcal{P}$) be a finite subset, let $\mathcal{U}_1 \subset U(\tilde{A})$ (in place of $\mathcal{U}$) be a finite subset and let $K_1 \ge 1$ (in place of K) be an integer given by 3.14 and 3.15 of [16], or by 7.9 (together with 7.13 of [18]) for the above T_3 (in place of F), $\varepsilon/16$ (in place of ε) and $\mathcal{F}$ and $\mathbf{L}$ with $\mathbf{L}(u) = 8\pi$. We assume that $a_0, f_{1/16}(a_0), f_{1/8}(a_0)$ and $f_{1/4}(a_0) \in \mathcal{F} \cup \mathcal{H}_{1,0}$ (with $r_0 = r_1 = 0$, $T = T(k,n)$ above, $s = 1$ and $R = 7$).

We may also assume that δ_1 is sufficiently small and $\mathcal{G}_1$ is sufficiently large that $[L_i]|_{\mathcal{P}}$ is well-defined, and

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}},$$

provided that L_i is $\mathcal{G}_1$-$2\delta_1$-multiplicative and

$$\|L_1(x) - L_2(x)\| < \delta_1 \text{ for all } x \in \mathcal{G}_1.$$

Without loss of generality, we may also assume that

$$\mathcal{F} \cup \mathcal{H}_{1,0} \cup \{xy : x, y \in \mathcal{F}\} \subset \mathcal{G}_1 \subset A^{\mathbf{1}}.$$

Choose $b_0 \in A_+ \setminus \{0\}$ with

$$d_\tau(b_0) < 1/2^{10}(2K_1+1) \ \text{for all} \ \tau \in T(A). \tag{e5.13}$$

Choose also a larger finite subset $\mathcal{G}_1'$ of A and a smaller δ_1' so that

$$\|\lceil L(u)\rceil - L(u)\| < \min\{1/4, \varepsilon \cdot \delta_1/2^{10}\}/8\pi \ \text{for all} \ u \in \mathcal{U}_1 \ \text{and} \tag{e5.14}$$

$$\|f_{1/8}(L(a_0)) - L(f_{1/8}(a_0))\| < 1/2^{10}(K_1+1) \tag{e5.15}$$

provided that L is a $\mathcal{G}_1'$-δ_1'-multiplicative completely positive contractive linear map (to any other C^*-algebra).

We may assume that $0 < \delta_1' \le \frac{\varepsilon\cdot\delta_1}{2^{12}(K_1+1)}$. For each $v \in \mathcal{U}_1$, there is $\alpha(v) \in \mathbb{C}$ and $a(v) \in A$ such that

$$v = \alpha(v)\cdot 1_{\tilde{A}} + a(v), \ \ |\alpha(v)| = 1 \ \text{and} \ \|a(v)\| \le 2. \tag{e5.16}$$

Let $\Omega = \{a(v) : v \in \mathcal{U}_1\}$. We may also assume that $\mathcal{G}_1' \supset \mathcal{G}_1 \cup \mathcal{F} \cup \mathcal{H}_{1,0} \cup \{xy : x, y \in \mathcal{G}_1\} \cup \Omega$. It follows from 3.10 and 3.9 (see also 8.3 and 10.8 of [15]) that there are $\mathcal{G}_1'$-$\delta_1'/64$-multiplicative completely positive contractive linear maps $\varphi_0 : A \to A$ and $\psi_0 : A \to D$ for some $M_{2K_1+1}(D) \subset A$ with $D \in \mathcal{C}_0'$ and $A \perp M_{2K_1+1}(D)$ such that

$$\|x - (\varphi_0(x), \mathrm{diag}(\overbrace{\psi_0(x), \psi_0(x), \ldots, \psi_0(x)}^{2K_1+1}))\| < \min\{\varepsilon/K_1 2^{12}, \delta_1'/128K_1\} \ \text{for all} \ x \in \mathcal{G}_1', \tag{e5.17}$$

$$a_{00}' \lesssim b_0 \ \text{and} \ \tau(f_{1/4}(\psi_0(a_0))) \ge 1 - \varepsilon/2^{10} \ \text{for all} \ \tau \in T(D), \tag{e5.18}$$

and $\psi_0(a_0)$ is strictly positive, where a_{00}' is a strictly positive element of $\overline{\varphi_0(a_0)A\varphi_0(a_0)}$. Moreover, ψ_0 is T_1- $\mathcal{H}_{1,0}$-full in $\overline{DAD}$.

We compute that, by (e5.12), (e5.17) and (e5.15),

$$2\tau(f_{1/8}(\psi_0(a_0))) \ge 3/4K_1 \ \text{for all} \ \tau \in T(A). \tag{e5.19}$$

We also compute that (see (e5.12), (e5.13), (e5.17) and (e5.18)), for all $\tau \in T(A)$,

$$\tau(f_{1/4}(\varphi_0(a_0)), \mathrm{diag}(\overbrace{f_{1/4}(\psi_0(a_0)), f_{1/4}(\psi_0(a_0)), \ldots, f_{1/4}(\psi_0(a_0))}^{2K_1+1}))) > 1 - \varepsilon/2^9. \tag{e5.20}$$

Let $A_{00} = \overline{(\varphi_0(a_0), \psi_0(a_0))A(\varphi_0(a_0), \psi_0(a_0))}$ and let $\varphi_{00} : A \to A_{00}$ be defined by

$$\varphi_{00}(x) = \varphi_0(x) \oplus \psi_0(x) \ \text{for all} \ x \in A.$$

Let $a_{00} = a_{00}' \oplus \psi_0(a_0) \in A_{00}$ be a strictly positive element of A_{00}.

By choosing even possibly smaller δ_1' and larger $\mathcal{G}_1'$, if necessary, we may assume that $[\varphi_{00}]|_{\mathcal{P}_1}$ is well defined and denote $\mathcal{P}_2 = [\varphi_{00}](\mathcal{P}_1)$. Moreover, we may also assume, without loss of generality, that

$$[L']|_{\mathcal{P}_2} = [L'']|_{\mathcal{P}_2}, \tag{e5.21}$$

if

$$\|L'(x) - L''(x)\| < \delta_1' \ \text{for all} \ x \in \mathcal{G}_1'$$

and L' and L'' are $\mathcal{G}_1'$-δ_1'-multiplicative completely positive contractive linear maps. We may also assume that

$$\|f_{\delta'}(a_{00})\varphi_{00}(x) - \varphi_{00}(x)\| < \delta_1'/2^{10} \ \text{and} \tag{e5.22}$$

$$\|f_{\delta'}(a_{00})\varphi_{00}(x)f_{\delta'}(a_{00}) - \varphi_{00}(x)\| < \delta_1'/2^{10} \ \text{for all} \ x \in \mathcal{G}_1' \tag{e5.23}$$

for some $1/64 > \delta' > 0$. Furthermore,

$$\|f_{\delta'}(\psi_0(a_0))\psi_0(x) - \psi_0(x)\| < \delta_1'/2^{10} \ \text{and} \tag{e5.24}$$

$$\|f_{\delta'}(\psi_0(a_0))\psi_0(x)f_{\delta'}(\psi_0(a_0)) - \psi_0(x)\| < \delta_1'/2^{10} \ \text{for all} \ x \in \mathcal{G}_1'. \tag{e5.25}$$

It follows from (e5.19) that $a_{00}' \lesssim b_0 \lesssim f_{1/8}(\psi_0(a_0))$ and, by 3.1 of [15], there exists $x_0 \in A$ such that

$$f_{\delta'/256}(a_{00}')(x_0^* f_{1/8}(\psi_0(a_0))x_0) = f_{\delta'/256}(a_{00}'). \tag{e5.26}$$

Let $g \in C_0((0,1])_+$ be such that $\|g\| = 1$, $g(t) = 0$ for all $t \in (0, \delta'/64)$ and $t \in (\delta'/8, 1]$.

Put (keep in mind that A is projectionless and simple)

$$\sigma_0 = \inf\{\tau(g(a_{00})) : \tau \in T(A)\} > 0. \tag{e5.27}$$

Let $\bar{D} = M_{2K_1}(D)$. Let $j_1 : D \to M_{2K_1}(D) = \bar{D}$ be defined by

$$j_1(d) = \mathrm{diag}(d, d, \ldots, d) \ \text{for all} \ d \in D.$$

Let $\imath_1 : \bar{D} \to A$ be the embedding. Let $\varepsilon_1 = \min\{\varepsilon/2^{10}, \delta_1/2^{10}, \delta_1'/2^{10}\}$. Choose a finite subset $\mathcal{G}_2' \subset \bar{D}$ which contains $\bigoplus_{i=1}^{2K_1} \pi_i \circ \psi_0(\mathcal{G}_1')$, where $\pi_i : \bigoplus_{i=1}^{2K_1} D \to D$ is the projection to the ith summand. Let $e_d \in D_+$ with $\|e_d\| = 1$ such that

$$\|f_{1/4}(e_d)y - y\| < \varepsilon_1/16 \quad \text{and} \quad \|yf_{1/4}(e_d) - y\| < \varepsilon_1/16 \tag{e5.28}$$

for all $y \in \psi_0(\mathcal{G}_1')$. Let $\bar{e}_d = \mathrm{diag}(\overbrace{e_d, e_d, \ldots, e_d}^{2K_1})$. Without loss of generality, we may assume that $\bar{e}_d, f_{1/4}(\bar{e}_d) \in \mathcal{G}_2'$.

Define $\Delta : D_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ by, for $d \in D_+^{\mathbf{1}} \setminus \{0\}$,

$$\Delta(\hat{d}) = \min\{\inf\{\tau \circ \imath_1 \circ j_1(d) : \tau \in T(A)\}, \min\{\frac{1}{2^{10}M(d)^2N(d)} : d \in \hat{d}\}\}. \tag{e5.29}$$

For ε_1, choose $\varepsilon_2 > 0$ (in place of δ) associated with $\varepsilon_1/16$ (in place of ε) and $1/16$ (in place of σ) required by Lemma 3.3 of [15]. Without loss of generality, we may assume that $\varepsilon_2 < \varepsilon_1$.

Let $\mathcal{G}_d$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_d \subset K_0(\bar{D})$ (in place of $\mathcal{P}$) be a finite subset, $\mathcal{H}_{1,d} \subset (\bar{D})_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, $\mathcal{H}_{2,d} \subset (\bar{D})_{s.a.}$ (in place of $\mathcal{H}_2$) be a finite subset, $\delta_d > 0$ (in place of δ), $\gamma_d > 0$ (in place of γ) required by Theorem 7.8 of [15] for $C = \bar{D}$, $\varepsilon_2/4$ (in place of ε), $\mathcal{G}_2'$ (in place of $\mathcal{G}$) and Δ above.

By (e5.14), there is a finite subset $\mathcal{U}_2 \subset U(\widetilde{A}_{00})$ such that, for any $w \in \mathcal{U}_1$, there is $w' \in \mathcal{U}_2$ with

$$\|\varphi_{00}(w) - w'\| < \min\{1/4, \varepsilon_1/2^{10}\}/8\pi. \tag{e5.30}$$

For each $w' \in \mathcal{U}_2$, there is $\alpha(w') \in \mathbb{C}$ with $|\alpha(w')| = 1$ and $a(w') \in A_{00}$ with $\|a(w')\| \le 2$ such that

$$w' = \alpha(w') \cdot 1_{\tilde{A}_{00}} + a(w').$$

Define

$$\Omega_0 = \{a(w') : w' \in \mathcal{U}_2\}.$$

Note that by viewing $\widetilde{A}_{00}$ as a C^*-subalgebra of $\tilde{A}$, we may also view $\mathcal{U}_2$ as a subset of $\tilde{A}$.

Let

$$\mathcal{G}_2 = \{a_{00}, f_{\delta'/4}(a_{00}), g(a_{00}), x_0, x_0^*\} \cup \mathcal{G}_1' \cup \varphi_0(\mathcal{G}_1') \cup \psi_0(\mathcal{G}_1') \cup \mathcal{G}_2' \cup \mathcal{G}_d \cup \mathcal{H}_{1,d} \cup \mathcal{H}_{2,d} \cup \Omega_0 \subset A_{00},$$

$$\mathcal{H}_1 = \{a_{00}, f_{\delta'/4}(a_{00}), f_{1/4}(a_0), f_{1/4}(\psi_0(a_0)), g(a_{00})\} \cup \mathcal{H}_{1,0} \cup \psi_0(\mathcal{H}_{1,0}) \cup \mathcal{H}_{1,d},$$

$$\mathcal{H}_2 = \mathcal{H}_1 \cup \mathcal{H}_{2,d}, \quad K_2 = 2^8 \max\{M(a)^2N(a)^2 : a \in \mathcal{H}_1\},$$

$$\sigma_{00} = \frac{\sigma_0}{K_2}, \quad \delta_2 = \frac{\min\{\delta_1/16, \delta_d/4, \gamma/2, \eta/2, \delta'/256, \sigma_{00}/4\}}{4(K_1+1)},$$

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup (j_1)_{*0}(\mathcal{P}_d) \cup \{[w'] : w' \in \mathcal{U}_2\},$$

$$\gamma = \frac{\gamma_d \cdot \delta' \cdot \sigma_{00}}{128(K_1+1)},$$

$\eta = 1/2^{10}(K_1+1)K_2$, and $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_3$

Now let $\mathcal{G}_0$ (in place of $\mathcal{G}$) and δ_0 (in place of δ) be as required by 5.1 for $\mathcal{G}_2$ (in place of $\mathcal{F}$) and δ_2 (in place of ε). Since $\bar{D}$ is weakly semi-projective, we may choose even large $\mathcal{G}_0$ and smaller δ_0 such that there is a homomorphism Φ from $\bar{D}$ such that

$$\|L(x) - \Phi(x)\| < \delta_2/2 \quad \text{for all } x \in \mathcal{G}_2 \cap \bar{D}$$

for any $\mathcal{G}_0$-δ_0-multiplicative completely positive contractive linear map$L$ from $\bar{D}$. We also assume that

$$\|L(f_{\delta'/4}(a_{00})) - f_{\delta'/4}(L(a_{00}))\| < \min\{\delta_2/2, \delta'/32\}, \tag{e5.31}$$

$$\|L(g(a_{00})) - g(L(a_{00}))\| < \min\{\delta_2/2, \delta'/32\}, \tag{e5.32}$$

$$\tau(g(L(a_{00}))) > (1/2)\sigma_{00} \quad \text{for all} \quad \tau \in T(C) \quad \text{and} \tag{e5.33}$$

$$\tau(f_{\delta'/128}(L(a_{00}'))) < 1/16(2K_1+1) \quad \text{for all} \quad \tau \in T(C) \tag{e5.34}$$

$$\tau(f_{1/8}(L(\psi_0(a_0)))) \ge 1/K_2 \quad \text{for all} \quad \tau \in T(C) \quad (\text{since} f_{1/4}(\psi_0(a_0)) \in \mathcal{H}_1) \tag{e5.35}$$

for any $\mathcal{G}_0$-δ_0-multiplicative completely positive contractive linear map L from A to C which is also T-$\mathcal{H}_1$-full (used for (e5.33) and (e5.35)), where C is any C^*-algebra with $T(C) \neq \emptyset$.

Let $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_0$ and $\delta = \min\{\delta_0/2, \delta_2/2\}$.

Now suppose that $\varphi_1, \varphi_2 : A \to B$ satisfy the assumption of the theorem for the above chosen $\mathcal{G}$, δ, γ, $\mathcal{P}$, η, $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{U}$ (for T).

Let $\varphi_{i,0} = \varphi_i \circ \varphi_{00}$, $i = 1, 2$. Let $\psi_{i,1}' : \bar{D} \to B$ be defined by $(\varphi_i)|_{\bar{D}}$. By applying 5.1 without loss of generality, we may assume that there are two pairs of hereditary C^*-subalgebras B_0, B_1 and B_0' and B_1', with $B_0 \perp B_1$ and $B_0' \perp B_1'$ such that

$\varphi_1(A_{00}) \subset B_0$ and $\psi_{1,1}(\bar{D}) \subset B_1$, $\varphi_2(A_{00}) \subset B_0'$ and $\psi_{2,1}(\bar{D}) \subset B_1'$, and $\varphi_i|_{A_{00}}$ is $\mathcal{G}_2$-δ_2-multiplicative, $\psi_{1,1} : \bar{D} \to B_1$ and $\psi_{2,1} : \bar{D} \to B_1'$ are homomorphisms such that

$$\|\psi_{i,1}'(x) - \psi_{i,1}(x)\| < \delta_2/2 \text{ for all } x \in \mathcal{G}_2 \cap \bar{D},\ i = 1, 2. \tag{e5.36}$$

We may further assume, by (e5.26) (and $x_0 \in \mathcal{G}_2$),

$$f_{\delta'/128}(\varphi_1(a_{00}')) \lesssim \psi_{1,1}(f_{1/16}(\psi_0(a_0))). \tag{e5.37}$$

Choose $b_{00} \in B_+$ such that $\tau(b_{00}) \geq 1/2$ for all $\tau \in T(B)$. Since both φ_1, φ_2 are T-$\mathcal{H}_1$-full, $\psi_{1,0}$ and $\psi_{2,0}$ are $(4/3)T$-$(\psi_0(\mathcal{H}_{1,0}) \cup \mathcal{H}_{1,d})$-full. We then compute that

$$\tau(\psi_{i,0}(x)) \geq \Delta(\hat{x}) \text{ for all } x \in \mathcal{H}_{1,d} \text{ and } \text{ for all } \tau \in T(B). \tag{e5.38}$$

Then, by the choice of $\mathcal{P}$, $\mathcal{H}_{2,d}$ and γ, by applying 7.8 of [15] we obtain a unitary $U_1' \in \tilde{B}$ such that

$$\|\mathrm{Ad}\, U_1' \circ \psi_{2,1}(x) - \psi_{1,1}(x)\| < \varepsilon_2/4 \text{ for all } x \in \mathcal{G}_2'. \tag{e5.39}$$

In particular,

$$\|\mathrm{Ad}\, U_1' \circ \psi_{2,1}(\bar{e}_d) - \psi_{1,1}(\bar{e}_d)\| < \varepsilon_2/4. \tag{e5.40}$$

By applying Lemma 3.3 of [15], there is a unitary $U_1'' \in \tilde{B}$ such that

$$\mathrm{Ad}\, U_1'' \circ \mathrm{Ad} U_1' \circ \psi_{2,1}(x) \in \overline{\psi_{1,1}(\bar{e}_d)B\psi_{1,1}(\bar{e}_d)} \text{ for all } x \in \overline{\psi_{2,1}(\bar{e}_d)B\psi_{2,1}(\bar{e}_d)} \text{ and} \tag{e5.41}$$

$$\|(U_1'')^* c U_1'' - c\| < (\varepsilon_1/16)\|c\| \text{ for all } c \in \overline{\psi_{2,1}(\bar{e}_d)B\psi_{2,1}(\bar{e}_d)}. \tag{e5.42}$$

Put $U_1 = U_1'U_1''$. Then we have

$$\mathrm{Ad}\, U_1 \circ \psi_{2,1}(f_{1/4}(\bar{e}_d)x f_{1/4}(\bar{e}_d)) \in \overline{\psi_{1,1}(\bar{e}_d)B\psi_{1,1}(\bar{e}_d)} \text{ for all } x \in A \text{ and} \tag{e5.43}$$

$$\|\mathrm{Ad}\, U_1 \circ \psi_{2,1}(x) - \psi_{1,1}(x)\| < \varepsilon_1/4 \text{ for all } x \in j_1 \circ \psi_0(\mathcal{G}_1'). \tag{e5.44}$$

Let $B' = \overline{(\mathrm{Ad}\, U_1 \circ \psi_{2,1}(f_{1/4}(\bar{e}_d)))B(\mathrm{Ad}\, U_1 \circ \psi_{2,1}(f_{1/4}(\bar{e}_d)))}$ and let

$$B_p = \{b \in B : bx = xb = 0 \text{ for all } x \in B'\}. \tag{e5.45}$$

By the choice of $\mathcal{H}_2$ and the assumption (e5.8), for all $\tau \in T(B)$,

$$|\tau(\varphi_{1,0}(f_{\delta'/4}(a_{00}))) - \tau(\varphi_{2,0}(f_{\delta'/4}(a_{00})))| < \min\{\gamma/2, \delta_2/2\}, \tag{e5.46}$$

With (e5.31) in mind, by the assumption, we have that

$$\|f_{\delta'/4}(\varphi_i(a_{00})) - \varphi_i(f_{\delta'/4}(a_{00}))\| < \min\{\delta_2/2, \delta'/32\} \text{ and} \tag{e5.47}$$

$$\|g(\varphi_i(a_{00})) - \varphi_i(g(a_{00}))\| < \min\{\delta_2/2, \delta'/32\}, \tag{e5.48}$$

$i = 1, 2$. We then compute that, by (e5.31), by the choice of $\mathcal{H}_2$ and γ, and by (e5.33),

$$\begin{aligned}
\tau(f_{\delta'/4}(\varphi_2(a_{00}))) &\leq \min\{\delta_2/2, \delta'/32\} + \tau(\varphi_2(f_{\delta'/4}(a_{00}))) && \text{(e5.49)}\\
&\leq \min\{\delta_2/2, \delta'/32\} + \gamma + \tau(\varphi_1(f_{\delta'/4}(a_{00}))) && \text{(e5.50)}\\
&< \min\{\delta_2/2, \delta'/32\} + \gamma + \min\{\delta_2/2, \delta'/32\} && \text{(e5.51)}\\
&\qquad + \tau(f_{\delta'/4}(\varphi_1((a_{00})))) && \text{(e5.52)}\\
&< \tau(g(\varphi_1(a_{00}))) + \tau(f_{\delta'/4}(\varphi_1((a_{00})))) && \text{(e5.53)}\\
&\leq \tau(f_{\delta'/64}(\varphi_1(a_{00}))) && \text{(e5.54)}
\end{aligned}$$

for all $\tau \in T(B)$. It is important to note that

$$U_1^* f_{\delta'/2}(\varphi_2(a_{00}))U_1,\ f_{\delta'/64}(\varphi_1(a_{00})) \in B_p.$$

Also note that B_p is a hereditary C^*-subalgebra of B. Since B has strictly comparison for positive elements and B has stable rank one, by 3.2 of [15], there is a unitary $U_2' \in \tilde{B}_p$ such that

$$(U_2')^* U_1^* f_{\delta'/2}(\varphi_2(a_{00}))U_1(U_2') \in \overline{f_{\delta'/128}(\varphi_1(a_{00}))Bf_{\delta'/128}(\varphi_1(a_{00}))} := B_{00}. \tag{e5.55}$$

Write $U_2' = \alpha \cdot 1_{\tilde{B}_p} + z$ with $z \in B_p$ and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Put $U_2 = \alpha \cdot 1_{\tilde{B}} + z$. Then (e5.55) still holds by replacing U_2' by U_2. Moreover,

$$U_2^* x U_2 = x \tag{e5.56}$$

for any $x \in B'$. In particular,

$$\|U_2^*(\mathrm{Ad}\, U_1 \circ \psi_{2,1}(x))U_2 - \psi_{1,1}(x)\| < \varepsilon_1/4 + \varepsilon_1/16 = 5\varepsilon_1/16 \tag{e5.57}$$

for all $x \in j_1 \circ \psi_0(\mathcal{G}_1')$.

Put $\varphi_{2,0}' = \mathrm{Ad}\, U_2 \circ \mathrm{Ad}\, U_1 \circ \varphi_{2,0}$ and define $\varphi_{2,0}'' : A \to B_{00}$ by

$$\varphi_{2,0}''(x) = U_2^* U_1^* f_{\delta'/2}(\varphi_2(a_{00}))\varphi_{2,0}(x) f_{\delta'/2}(\varphi_2(a_{00}))U_1 U_2 \ \text{ for all } \ x \in A. \tag{e5.58}$$

By (e5.22), $\psi_{2,1}''$ is $\mathcal{G}_1'$-$\delta_1'/2^4$-multiplicative completely positive contractive linear map. Define $\varphi_{1,0}' : A \to B_{00}$ by

$$\varphi_{1,0}'(x) = f_{\delta'/2}(\varphi_1(a_{00}))\varphi_{1,0}(x) f_{\delta'/2}(\varphi_1(a_{00})) \ \text{ for all } \ x \in A \tag{e5.59}$$

which is also $\mathcal{G}_1'$-$\delta_1'/2^4$-multiplicative completely positive contractive linear map. Now both $\varphi_{1,0}'$ and $\varphi_{2,0}''$ are completely positive contractive linear maps from A into B_{00}. Note that B is separable and simple and has stable rank one. From the assumption, (e5.22) and (e5.21), we have

$$[\varphi_{2,0}'']|_{\mathcal{P}} = [\varphi_{2,0}]|_{\mathcal{P}} = [\varphi_{1,0}]|_{\mathcal{P}} = [\varphi_{1,0}']|_{\mathcal{P}}. \tag{e5.60}$$

It follows from the choice of $\mathcal{U}_2$ and assumption (e5.9) (as well as (e5.22) and (e5.46) among others) that

$$\mathrm{dist}(\overline{\lceil \varphi_{2,0}''(v)\rceil}, \overline{\lceil \varphi_{1,0}'(v)\rceil}) < \eta + \delta_1'/2^4 \ \text{ for all } \ v \in \mathcal{U}_1 \tag{e5.61}$$

as elements in $U(\tilde{B})/CU(\tilde{B})$. It follows from (e5.35) that

$$\tau(f_{\delta'/128}(\varphi_1(a_{00}))) > \tau(f_{\delta'/128}(\varphi_1(\psi_0(a_0)))) \ge 1/K_2 \ \text{ for all } \ \tau \in T(B). \tag{e5.62}$$

It follows from 4.6 that, in $U(\tilde{B}_{00})$, for all $v \in \mathcal{U}_1$,

$$\begin{aligned} \mathrm{cel}_{\tilde{B}_3}(\overline{\lceil \varphi_{2,0}''(v)\rceil \lceil \varphi_{1,0}'(v)\rceil^*}) &< (\frac{K_2\pi}{2} + \frac{1}{16})(\eta + \delta_2) + 6\pi \qquad &\text{(e5.63)}\\ &\le 7\pi < \mathbf{L}(v). \qquad &\text{(e5.64)} \end{aligned}$$

Now let $\tilde{\psi}_d = \psi_{1,1} \circ \mathrm{diag}(\psi_0, \psi_0)$ and $B_2 = \overline{\tilde{\psi}_d(A)B\tilde{\psi}_d(A)}$. Let $b_{00}' \in B_{00}$ be a strictly positive element with $\|b_{00'}\| = 1$ and let $b_2 \in B_2$ be a strictly positive element with $\|b_2\| = 1$. It follows from (e5.37) that

$$b_{00}' \lesssim b_2. \tag{e5.65}$$

Recall that $\psi_{1,1}$ and $\psi_{2,1}$ are assumed to be homomorphisms which are T-$\psi_0(\mathcal{H}_{1,0})$-full. Since ψ_0 is T_1-$\mathcal{H}_{1,0}$-full in D, $\tilde{\psi}_d$ is also T_3-$\mathcal{H}_{1,0}$-full in B_2. Recall that

$$\psi_{1,1}(j_1 \circ \psi_0(x)) = \mathrm{diag}(\overbrace{\tilde{\psi}_d(x), \tilde{\psi}_d(x), \ldots, \tilde{\psi}_d(x)}^{K_1}) \ \text{ for all } \ x \in A. \tag{e5.66}$$

Now we are ready to apply the stable uniqueness theorem 3.14 of [16]. By that theorem, viewing B_2 as a hereditary C^*-subalgebra of B, there exists a unitary $U_3 \in M_{K_1+1}(B_2)$ such that

$$\|U_3^*(\varphi_{2,0}''(x) \oplus \psi_{1,1}(j_1 \circ \psi_0(x)))U_3 - (\varphi_{1,0}'(x) \oplus \psi_{1,1}(j_1 \circ \psi_0(x)))\| < \varepsilon/16 \tag{e5.67}$$

for all $x \in \mathcal{F}$. It follows from (e5.28), (e5.59), (e5.22) and (e5.58) that

$$\begin{aligned} &\|U_3^*(\varphi_{2,0}'(x) \oplus \psi_{1,1}(j_1 \circ \psi_0(x)))U_3 - (\varphi_{1,0}(x) \oplus \psi_{1,1}(j_1 \circ \psi_0(x)))\| \qquad &\text{(e5.68)}\\ &\quad < \varepsilon/16 + \varepsilon_1/4 + \delta_1/2^8 < \varepsilon/8 \qquad &\text{(e5.69)} \end{aligned}$$

for all $x \in \mathcal{F}$. Since both $\varphi_{2,0}'(x) \oplus \psi_{1,1}(j_1 \circ \psi_0(x))$ and $\varphi_{1,0}(x) \oplus \psi_{1,1}(j_1 \circ \psi_0(x))$ are in B, and since B has stable rank one, one easily finds a unitary $U_3' \in \tilde{B}$ such that the above holds using U_3' in stead of U_3 but with $\varepsilon/7$ instead of $\varepsilon/8$.

Put $U_4 = U_1 U_2 U_3'$. It follows from (e5.17), (e5.57), (e5.36) and the above that, for all $x \in \mathcal{F}$,

$$\begin{aligned} &\|\mathrm{Ad}\, U_4 \circ \varphi_2(x) - \varphi_1(x)\| \\ &\quad \le \|U_4^* \varphi_2(\varphi_{00}(x) \oplus j_1 \circ \psi_0(x))U_4 - \varphi_1(\varphi_{00}(x) \oplus j_1 \circ \psi_0(x))\| + 2\min\{\varepsilon/128, \delta_1'/128\} \\ &\quad < \|(U_3')^*(\varphi_{2,0}'(x) \oplus \psi_{1,1}(j_1 \circ \psi_0(x)))U_3' - (\varphi_{1,0}(x) \oplus \psi_{1,1}(j_1 \circ \psi_0(x)))\| \\ &\qquad\qquad + 5\varepsilon_1/16 + \delta_2/2 + \varepsilon/64 \\ &\quad < \varepsilon/7 + +5\varepsilon_1/16 + \delta_2/2 + \varepsilon/64 < \varepsilon. \quad \square \qquad \text{(e5.70)} \end{aligned}$$

Remark 5.4. It is easy to see that, with (e5.8), we may assume that $[v_i] \neq \{0\}$ (see (e2.9)).

The following follows from A6 and A7 of the appendix of [16]. We keep here since the proof is much simpler.

Proposition 5.5. *Let A be a non-unital separable stably projectionless exact simple C^*-algebra with continuous scale which is $\mathcal{Z}$-stable and $T(A) \neq \emptyset$. Then $K_0(\tilde{A})$ is weakly unperforated, i.e., if $x \in K_0(\tilde{A})$ with $kx \in K_0(\tilde{A})_+ \setminus \{0\}$ for some integer $k \geq 1$, then $x \in K_0(\tilde{A})_+$. Furthermore, if $p, q \in M_s(\tilde{A})$ (for some $s \geq 1$) are two projections such that $\tau(q) < \tau(p)$ for all $\tau \in T(\tilde{A})$, then $q \lesssim p$.*

Proof. Put $A_1 = \widetilde{A \otimes \mathcal{Z}}$. Note that, since A is $\mathcal{Z}$-stable, $A_1 = \tilde{A}$. Let $B = \tilde{A} \otimes \mathcal{Z}$ and let $\imath : A_1 \to B$ be the embedding. Then $\imath_{*0} : K_0(A_1) \to K_0(B)$ is an isomorphism. Let $\pi_A : A_1 \to \mathbb{C}$ and $\pi_z : B \to \mathcal{Z}$ be the quotient maps. Note that $\pi_z \circ \imath = \imath_{\mathbb{C},\mathcal{Z}} \circ \pi_A$, where $\imath_{\mathbb{C},\mathcal{Z}}$ is the embedding from $\mathbb{C}$ to $\mathcal{Z}$. Let t_0 be the tracial state of A_1 which vanishes on $A \otimes \mathcal{Z} = A$ and t_z be the tracial state of $\mathcal{Z}$. Note $T(A_1) = T(A) \cup \{t_0\}$ and $T(B) = T(A) \cup \{t_z \circ \pi_z\}$.

Let $x \in K_0(A_1)$ such that $kx > 0$ in $K_0(A_1)$ for some integer $k \geq 1$. Suppose that $p, q \in M_s(A_1)$ are two projections such that $[p]-[q] = x$ in $K_0(A_1)$. Suppose that $k[p]-k[q]$ is realized by a projection $r \in M_n(\tilde{A})$. If $\pi_A(r) = 0$, then $r \in M_n(A)$ which is contradicted with A being stable projectionless. That is r is a full projection in A_1. Hence, for all $\tau \in T(A_1)$, $\tau(r) > 0$. That is $\tau(p) > \tau(q)$ for all $\tau \in T(A_1)$. It follows that $\tau(\imath(p)) > \tau(\imath(q))$ for all $\tau \in T(B)$. Also $pM_s(A \otimes \mathcal{Z})p \neq \{0\}$. Note $p \notin M_s(A \otimes \mathcal{Z})$ since A is stably projectionless. Therefore the ideal generated by p in $M_s(B)$ contains q. Since B is $\mathcal{Z}$-stable, by 4.10 of [52], $q \lesssim p$ in $M_s(B)$. Therefore there is a projection $p_1 \leq p$ in $M_s(B)$ such that $[p_1] = \imath_*(x)$. There is a unitary $w \in \mathcal{Z}$ such that $w^*\pi_z(p_1)w = 1_{M_k}$, where $1_{M_k} \in M_s(\mathbb{C})$ is a scalar matrix of rank $k \leq s$. Since $K_1(\mathcal{Z}) = \{0\}$, there exists a unitary $W \in M_s(B)$ such that $\pi_z(W) = w$. Then $W^*p_1W - 1_{M_k} \in \ker \pi_z = M_s(A \otimes \mathcal{Z})$. Let $e = W^*p_1W$. Then $e \in M_s(A_1)$. We compute that $[e] = x$ in $K_0(A_1)$. This implies that $x > 0$ and $K_0(A_1)$ is weakly unperforated. $\square$

Remark 5.6. In Theorem 5.3, if both φ_1 and φ_2 map strictly positive elements to strictly positive elements, then, by the virtue of 5.7 of [15], the fullness condition can be replaced by $\tau(f_{1/2}(\varphi_1(e)))$, $\tau(f_{1/2}(\varphi_2(e))) \geq d$ for some given $1 > d > 0$ and a strictly positive element $e \in A$. for all $\tau \in T(B)$. If furthermore, φ_1 and φ_2 are assumed to be homomorphisms, then, $\tau \circ \varphi_i$ are tracial states of $T(A)$ for all $\tau \in T(B)$. Therefore, the fullness condition can be dropped.

6. C^*-algebras of the form $B \otimes \mathcal{W}$

The main purpose of this section is to prove Theorem 6.9.

The following is known (in particular, the case that $n = 1$).

Lemma 6.1. *Let B be a C^*-algebra and $n \geq 1$ be an integer. Let $u \in 1_{M_n(\tilde{B})} + M_n(B)$. Suppose that $u \in U_0(M_n(\tilde{B}))$. Then there exists a continuous path $\{u(t) : t \in [0,1]\} \subset U_0(M_n(\tilde{B}))$ such that $u(0) = u$, $u(1) = 1_{M_n(\tilde{B})}$ and $\varphi(u(t)) = 1_{M_n(\tilde{B})}$ for all $t \in [0,1]$, where $\varphi : M_n(\tilde{B}) \to M_n$ is the quotient map.*

Moreover, one may write $u = \prod_{k=1}^m \exp(ih_j)$ for some $h_j \in M_n(\tilde{B})_{s.a.}$ with $\varphi(h_j) = 0$ (and $\varphi(\exp(ih_j)) = 1_{M_n(\tilde{B})}$), $j = 1, 2, \ldots, m$ (for some $m \geq 1$).

Proof. Let $\{w(t) : t \in [0,1]\}$ be a continuous path of unitaries such that $w(0) = u$ and $w(1) = 1_{M_n(\tilde{B})}$. Let $\bar{w}(t) = \varphi(w(t)) \in M_n$. Let $w'(t) \in M_n \subset M_n(\tilde{B})$ be the same scalar matrix as $\bar{w}(t)$ with $\varphi(w'(t)) = \bar{w}(t)$. Note that $w'(0) = 1_{M_n(\tilde{B})} = w'(1)$. Define $u(t) = w(t)(w'(t))^*$. Then $u(0) = u$ and $u(1) = 1_{M_n(\tilde{B})}$. Moreover $\varphi(u(t)) = 1_{M_n(\tilde{B})}$ for all $t \in [0,1]$.

To see the last part, one chooses a partition $0 = t_0 < t_1 < \cdots t_m = 1$ of $[0,1]$ such that $\|u(t_{j-1})u(t_j)^* - 1\| < 1$. Define $h_j = \frac{1}{2\pi i}\log(u(t_{j-1})u(t_j)^*)$, $j = 1, 2, \ldots, m$. Then $u = \prod_{j=1}^m \exp(ih_j)$. Note that $\varphi(u(t_j)) = 1_{M_n(\tilde{B})}$, whence $\varphi(h_j) = 0$, $j = 1, 2, \ldots, m$. $\square$

Lemma 6.2. *Let B be a C^*-algebra and $u = 1_{\tilde{B}} + x \in \tilde{B}$ be a unitary, where $x \in B$. Suppose that $\mathrm{diag}(u, 1_{M_m(\tilde{B})}) \in U_0(M_{m+1}(\tilde{B}))$. Let $v = 1_C + x \otimes 1_Q \in C$, where $C = \widetilde{B \otimes Q}$. Then $v \in U_0(C)$. Moreover, there exists a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset C$ such that $v(0) = v$, $v(1) = 1_C$ and $\pi(v(t)) = 1_C$ for all $t \in [0,1]$, where $\pi : C \to \mathbb{C}$ is the quotient map.*

Proof. By 6.1, there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset U_0(\widetilde{M_{m+1}(B)})$ such that

$$u(0) = \mathrm{diag}(u, 1_{M_m(\tilde{B})}), \quad u(1) = 1_{M_{m+1}(\tilde{B})} \text{ and } \pi(u(t)) = 1_{M_{m+1}}, \tag{e6.1}$$

where $\pi : \tilde{B} \to \mathbb{C}$ is the quotient map. Write $u(t) = 1_{m+1}(\tilde{B}) + x(t)$, where $\{x(t) : t \in [0,1]\} \subset M_{m+1}(B)$ is a continuous path such that $x(t) + x(t)^* + x(t)^*x(t) = 0$, $x(t) + x(t)^* + x(t)x^*(t) = 0$ and $x(1) = 0$.

Let $e_1, e_2, \ldots, e_{m+1} \in Q$ be mutually orthogonal and mutually equivalent projections such that $\sum_{i=1}^{m+1} e_i = 1_Q$. Put

$$E_i = 1_{m+1} \otimes e_i = \mathrm{diag}(\overbrace{e_i, e_i, \ldots, e_i}^{m+1}) \in M_{m+1}(\tilde{B} \otimes Q), \quad i = 1, 2, \ldots, m+1. \tag{e6.2}$$

Define

$$w_i = 1_C + x(0) \otimes e_i = 1_C + x \otimes e_i, \quad i = 1, 2, \ldots, m. \tag{e6.3}$$

Then

$$u = w_1 w_2 \cdots w_{m+1}. \tag{e6.4}$$

Let $X_i \in M_{m+1}(\tilde{B} \otimes Q)$ be a unitary such that

$$X_i E_i X_i^* = 1_{\tilde{B}\otimes Q} = 1_C. \tag{e6.5}$$

Note that $X_i(x(t) \otimes e_i)X_i^* \in B \otimes Q$. Define $w_i(t) = 1_C + X_i(x(t) \otimes e_i)X_i^* \in C$ for $t \in [0, 1]$, $i = 1, 2, \dots, m+1$. Then

$$w_i(t)^* w_i(t) = 1_C + X_i(x(t) \otimes e_i)X_i^* + X_i(x^*(t) \otimes e_i)X_i^* + X_i(x(t)x^*(t) \otimes e_i)X_i^* \tag{e6.6}$$

$$= 1_C + X_i(x(t) + x^*(t) + x(t)x^*(t))X_i^* = 1_C. \tag{e6.7}$$

Similarly,

$$w_i^*(t) w_i(t) = 1_C, \quad i = 1, 2, \dots, m+1. \tag{e6.8}$$

So $\{w_i(t) : t \in [0, 1]\} \subset U_0(C)$ with $w_i(0) = w_i$ and $w_i(1) = 1_C$. Moreover,

$$\pi(w_i(t)) = 1 \text{ for all } t \in [0, 1], \quad i = 1, 2, \dots, m+1. \tag{e6.9}$$

Define $v(t) = w_1(t)w_2(t)\cdots w_{m+1}(t)$ for $t \in [0, 1]$. Then

$$v(t) \in U(C), \quad v(0) = w_1(0)w_2(0)\cdots w_{m+1}(0) = 1_C + x = u \text{ and } v(1) = 1_C. \quad \Box \tag{e6.10}$$

Theorem 6.3. *Let B be a C^*-algebra and $C = \widetilde{B \otimes \mathcal{W}}$. Then $U(M_m(C)) = U_0(M_m(C))$ for all integer $m \geq 1$. Moreover, if $u{=}1_C + x$ is a unitary in C for some $x \in B \otimes \mathcal{W}$, then there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\} \subset C$ such that $\pi(u(t)) = 1$ for all $t \in [0, 1]$, where $\pi : C \to \mathbb{C}$ is the quotient map.*

Proof. Note $K_1(\mathcal{W}) = \{0\}$. Therefore $K_1(B \otimes \mathcal{W}) = \{0\}$.

Let $u \in U(M_m(C))$. Write $u = w + x$, where $w \in M_m$ is a scalar unitary and $x \in M_m(C)$. By considering w^*u, without loss of generality, we may assume $u = 1_{M_m} + x$ for some $x \in M_m(C)$. Hence (by 6.1) there exists a continuous path $\{w(t) : t \in [0, 1]\} \subset 1_{M_{n+m}(C)} + M_{n+m}(B \otimes \mathcal{W})$ such that $w(0) = \mathrm{diag}(u, 1_n)$ and $w(1) = 1_{n+m}$ for some integer $n \geq 1$. Since $\mathcal{W} \cong \mathcal{W} \otimes Q$ and Q is strong self absorbing, without loss of generality, we may assume that $u = 1_{M_m(C)} + x \otimes 1_Q$. Thus 6.2 applies. This proves the second part of the lemma. To see the first part, we let $m = 1$. $\Box$

Lemma 6.4. *Let B be a C^*-algebra and let $q \in M_m(\tilde{C})$ be a projection, where $C = B \otimes \mathcal{W}$, such that $\pi_C(q) = p \in M_m(\mathbb{C})$, a projection matrix, where $\pi_C : \tilde{C} \to \mathbb{C}$ is the quotient map. Then there exist an integer $r_1 \geq r \geq 0$ and a unitary $w \in M_{m+r_1}(\tilde{C})$ such that $w(q \oplus 1_r)w^* = P \oplus 1_r$, where P is the matrix in $M_m(\mathbb{C} \cdot 1_{\tilde{C}})$ which is the same matrix as p. Moreover, $\pi_C(w) = 1_{m+r_1}$.*

Proof. Since $\mathcal{W}$ is KK-contractible, $K_0(B \otimes \mathcal{W}) = \{0\}$. Therefore, for some large $r_1 \geq r \geq 0$, there is $w_1 \in M_{m+r_1}(\tilde{C})$ such that

$$w_1(q \oplus 1_r)w_1^* = P \oplus 1_r. \tag{e6.11}$$

Note $\pi_C(w_1)(P \oplus 1_r)\pi_C(w_1)^* = P \oplus 1_r$, where we identify these elements with matrices with scalar entries. Write $W = \pi_C(w_1)$ as a unitary matrix with scalar entries. Note that $W(P \oplus 1_r)W^* = P \oplus 1_r$. Let $w = W^* w_1$. Then, $\pi_C(w) = 1_{m+r_1}$ and

$$w(1 \oplus 1_r)w^* = P \oplus 1_r. \quad \Box \tag{e6.12}$$

Lemma 6.5. *Let B be a C^*-algebra and $u = 1_{\tilde{B}} + x \in U_0(\tilde{B})$, where $x \in B$. Then, for any $\varepsilon > 0$, there exists $m \geq 1$ such that there exists a continuous path of unitaries $\{y(t) : t \in [0, 1]\} \subset 1_{M_{m+1}} + M_{m+1}(B)$ such that $y(0) = \mathrm{diag}(u, 1_{M_m})$, $y(1) = 1_{M_{m+1}}$ and* $\mathrm{length}(\{y(t) : t \in [0, 1]\}) \leq 4\pi + \varepsilon$.

Moreover, let $C = \widetilde{B \otimes \mathcal{K}}$ and $v = 1_C + z \in U_0(C)$ for some $z \in C$. Then, for any $\varepsilon > 0$, there exists a continuous path of unitaries $\{v(t) : t \in [0, 1]\}$ such that $v(0) = v$, $v(1) = 1_C$, $\Pi(v(t)) = 1$ for all $t \in [0, 1]$ and $\mathrm{cel}(v(t)) \leq 4\pi + \varepsilon$, where $\Pi : C \to \mathbb{C}$ is the quotient map. Consequently, $v \in U_0(C)$ and $\mathrm{cel}(v) \leq 4\pi + \varepsilon$.

Proof. By 6.1, we may assume that there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\} \subset \tilde{B}$ such that $u(t) = 1_{\tilde{B}} + x(t)$, where $x(t) \in B$ such that $u(0) = u$ and $u(1) = 1_{\tilde{B}}$. Note that $\{x(t) : t \in [0, 1]\}$ is continuous, $x(0) = x$ and $x(1) = 0$. Moreover, for all $t \in [0, 1]$,

$$x(t) + x(t)^* + x(t)x^*(t) = 0 \text{ and } x(t) + x(t)^* + x(t)^*x(t) = 0. \tag{e6.13}$$

Fix $1/4 > \varepsilon > 0$. There is a partition $0 = t_0 < t_1 < \cdots t_n = 1$ such that

$$\|x(t) - x(t_i)\| < \varepsilon/2 \text{ for all } t \in [t_i, t_{i+1}], \quad i = 0, 1, 2..., n-1. \tag{e6.14}$$

Put $D = M_{2n+1}(B)$. Then

$$Z_0 = 1_{\tilde{D}} + \mathrm{diag}(x(0), x^*(0), x(t_1), x^*(t_1), \ldots, x(t_{n-1}), x^*(t_{n-1}), 0) \in \tilde{D}. \tag{e6.15}$$

Put $y_i = x(t_i) + x^*(t_i)$, $i = 0, 1, \ldots, n-1$. Then, by (e6.13),

$$\begin{aligned} Z_0Z_0^* &= 1_{\tilde{D}} + \mathrm{diag}(y_0 + x(0)x^*(0), y_0 + x(0)^*x(0), y_1 + x(t_1)x^*(t_1), y_1 + x(t_1)^*x(t_1), \ldots, 0) \\ &= 1_{\tilde{D}}. \end{aligned}$$

Similarly, $Z_0^*Z_0 = 1_{\tilde{D}}$. In other words, Z_0 is a unitary in $\tilde{D}$. Put, for $t \in [0, 1]$,

$$V(t) = \begin{pmatrix} \cos(t\pi/2) & -\sin(t\pi/2) \\ \sin(t\pi/2) & \cos(t\pi/2) \end{pmatrix} \tag{e6.16}$$

Note that $V(t)$ is a unitary in $M_2(\tilde{B})$, $V(0) = 1_{M_2(\tilde{B})}$ and $V(t)^*V(t) = 1_{M_2(\tilde{B})}$. Put $w_i(t) = V(t)\mathrm{diag}(1_{\tilde{B}}+x(t_i), 1_{\tilde{B}})V^*(t)\mathrm{diag}(1_{\tilde{B}}, 1_{\tilde{B}} + x^*(t_i))$, $t \in [0, 1]$ and $i = 0, 1, \ldots, n-1$. Note that

$$w_i(0) = \mathrm{diag}(1_{\tilde{B}} + x(t_i), 1_{\tilde{B}} + x^*(t_i)), \quad \text{and} \quad w_i(1) = \mathrm{diag}(1_{\tilde{B}}, 1_{\tilde{B}}), \quad i = 0, 1, \ldots, n-1. \tag{e6.17}$$

Let $\varphi : M_2(\tilde{B}) \to M_2$ be the quotient map. We also have

$$\varphi(w_i(t)) = \varphi(V(t))\mathrm{diag}(1, 1)\varphi(V^*(t))\mathrm{diag}(1, 1) = \mathrm{diag}(1, 1). \tag{e6.18}$$

Define

$$Z(t) = \mathrm{diag}(w_0(t), w_1(t), \ldots, w_{n-1}(t), 1_{\tilde{B}}) \tag{e6.19}$$

Then $Z(0) = Z_0$, $Z(1) = 1_{\tilde{D}}$. Moreover, by (e6.18),

$$\pi_D(Z(t)) = 1_{\pi_D(\tilde{D})},$$

where $\pi_D : \tilde{D} \to \mathbb{C}$ is the quotient map. Note that $\{Z(t) : t \in [0, 1]\}$ is a continuous path of unitaries in $\tilde{D}$. It is standard to compute that length($\{Z(t) : t \in [0, 1]\}$) $= 2\pi$. Put

$$Z_{-1} = 1_{\tilde{D}} + \mathrm{diag}(x(0), x^*(t_1), x(t_1), x^*(t_2), x(t_2), \ldots, x^*(t_n), x(t_n)). \tag{e6.20}$$

By (e6.14),

$$\|Z_0 - Z_{-1}\| < \varepsilon. \tag{e6.21}$$

There exists a continuous path of unitaries $\{Z_{-1}(t) : t \in [0, 1]\} \subset \tilde{D}$ such that $Z_{-1}(0) = Z_{-1}$, $Z_{-1}(1) = Z_0$ and length($\{Z_{-1}(t) : t \in [0, 1]\}$) $\le 2\arcsin(\varepsilon/2)$. Define

$$\omega_i(t) = V(t)\mathrm{diag}(1_{\tilde{B}} + x^*(t_i), 1_{\tilde{B}})V^*(t)\mathrm{diag}(1_{\tilde{B}} + x(t_i), 1_{\tilde{B}})$$

for $t \in [0, 1]$ and $i = 1, 2, \ldots, n$. Then, similar to some computation above, $\{\omega_i(t) : t \in [0, 1]\} \subset M_2(\tilde{B})$ is a continuous path of unitaries such that $\omega_i(0) = 1_{M_2(\tilde{B})}$, $\omega_i(1) = \mathrm{diag}(1_{\tilde{B}} + x^*(t_i), 1_{\tilde{B}} + x(t_i))$ and

$$\varphi(w_i(t)) = \varphi(V(t))\mathrm{diag}(1, 1)\varphi(V^*(t))\mathrm{diag}(1, 1) = \mathrm{diag}(1, 1) \text{ for all } t \in [0, 1], \tag{e6.22}$$

$i = 1, 2, \ldots, n$. Define

$$Z_{-2}(t) = \mathrm{diag}(1_{\tilde{B}} + x(0), \omega_1(t), \omega_2(t), \ldots, \omega_n(t)) \text{ for all } t \in [0, 1]. \tag{e6.23}$$

Then $\{Z_{-2}(t) : t \in [0, 1]\} \subset U_0(\tilde{D})$ is a continuous path such that

$$Z_{-2}(0) = 1_{\tilde{D}} + x(0) = 1_{\tilde{D}} + x = \mathrm{diag}(u, 1_{2n}) \tag{e6.24}$$

$$Z_{-2}(1) = 1_{\tilde{D}} + \mathrm{diag}(x(0), x^*(t_1), x(t_1), \ldots, x_n^*(t_n), x_n(t_n)) = Z_{-1}. \tag{e6.25}$$

Moreover, length($\{Z_{-2}(t) : t \in [0, 1]\}$) $\le 2\pi$. Now define

$$y(t) = \begin{cases} Z_{-2}(3t) & \text{if } t \in [0, 1/3] \\ Z_{-1}(3(t-1/3)) & \text{if } t \in [1/3, 2/3] \\ Z(3(t-2/3)) & \text{if } t \in [2/3, 1]. \end{cases} \tag{e6.26}$$

Now $\{y(t) : t \in [0, 1]\} \subset U_0(\tilde{D})$, $y(0) = Z_{-2}(0) = \mathrm{diag}(u, 1_{2n})$ and $y(1) = Z(1) = 1_{\tilde{D}}$. Moreover, for any $1/4 > \varepsilon > 0$,

$$\text{length}(\{y(t) : t \in [0, 1]\}) \le 4\pi + 2\arcsin(\varepsilon/2). \tag{e6.27}$$

This proves the first part of the statement. The second part follows the same way. □

Theorem 6.6. *Let $\{B_n\}$ be a sequence of C^*-algebras and $C = \prod_{n=1}^{\infty}(B_n \otimes \mathcal{W} \otimes \mathcal{K})$. Then $K_1(C) = \{0\}$. Moreover $K_1(E) = \{0\}$, where $E = \prod_{n=1}^{\infty}(B_n \otimes \mathcal{W} \otimes \mathcal{K})^{\sim}$.*

Proof. Let $u \in M_m(\tilde{C})$. Let $\pi_C : \tilde{C} \to \mathbb{C}$ be the quotient map and let $z = \pi_C(u) \in M_m(\mathbb{C})$ be a unitary matrix. Denote by Z the same scalar unitary matrix (as z) in $M_m(\mathbb{C}1_{\tilde{C}})$. Replacing u by Z^*u, without loss of generality, we may assume that $u \in 1_{M_m(\tilde{C})} + M_m(C)$. To simplify notation, without loss of generality, we may further assume that $u \in 1_{\tilde{C}} + C$. We write $u = \{u_n\}$, where $u_n = 1_{(B_n\otimes\mathcal{W}\otimes\mathcal{K})_n^{\sim}} + x_n$ and where $x_n \in B_n \otimes \mathcal{W} \otimes \mathcal{K}$, $n = 1, 2, \ldots$.

By applying 6.5, we obtain, for each n, a continuous path of unitaries $v_n(t) \in 1_{(B_n\otimes\mathcal{W}\otimes\mathcal{K})^{\sim}} + B_n \otimes \mathcal{W} \otimes \mathcal{K}$ with $v_n(0) = u_n$ and $v_n(1) = 1_{(B_n\otimes\mathcal{W}\otimes\mathcal{K})^{\sim}}$ and $\mathrm{cel}(v_n(t)) < 5\pi$. Thus, we obtain (see Lemma 1.1 of [19]) a sequence of equi-continuous paths of unitaries $\{w_n(t) : t \in [0, 1]\} \subset 1_{(B_n\otimes\mathcal{W}\otimes\mathcal{K})^{\sim}} + B_n \otimes \mathcal{W} \otimes \mathcal{K}$ with $w_n(0) = u_n$ and $w_n(1) = 1_{(B_n\otimes\mathcal{W}\otimes\mathcal{K})^{\sim}}$, $n = 1, 2, \ldots$. Define

$$u(t) = \{w_n(t)\} \subset 1_{\tilde{C}} + C. \tag{e6.28}$$

Then $\{u(t) : t \in [0, 1]\}$ is a continuous path of unitaries in $1_{\tilde{C}} + C$ such that $u(0) = u$ and $u(1) = 1_{\tilde{C}}$. Thus $K_1(C) = \{0\}$.

If $u \in U(E)$, we write $u = \{u_n\}$, where $u_n \in U((B_n \otimes \mathcal{W} \otimes \mathcal{K})^{\sim})$. Denote by $\pi_n : (B_n \otimes \mathcal{W} \otimes \mathcal{K})^{\sim} \to \mathbb{C}$ the quotient map. Let $\lambda_n = \pi_n(u_n)$. Then $\lambda_n \in \mathbb{T}$. Consider the unitary $Z = \{\lambda_n\} \in U(E)$. Then $Z \in U_0(E)$. Now consider $v = uZ^*$. Then $v \in \tilde{C}$. Thus the above shows that $K_1(E) = \{0\}$. □

Lemma 6.7. *Let B_n be a sequence of C^*-algebras and let $C = \prod_{n=1}^{\infty} B_n \otimes \mathcal{W}$. Then $K_i(C)$ is divisible and the map from $K_i(C)$ to $K_i(C, \mathbb{Z}/k\mathbb{Z})$ is zero, $i = 0, 1$ and $k = 2, 3, \ldots$.*

Proof. Let $\psi : Q \otimes Q \to Q$ be an isomorphism. Since $\mathcal{W} \otimes Q \cong \mathcal{W}$, we may write $C = \prod_{n=1}^{\infty} B_n \otimes \mathcal{W} \otimes Q$. Define $\Phi : C \to C \otimes Q$ by $\Phi(\{a_n\}) = \{a_n\} \otimes 1_Q$ for all $\{a_n\} \in C$. It is an injective homomorphism. Let $\Psi : C \otimes Q \to C$ be a homomorphism defined by $\Psi(\{(b_n \otimes r_n) \otimes r\}) = \{b_n \otimes \psi(r_n \otimes r)\}$ for all $\{b_n \otimes r_n\} \in C$, where $b_n \in B_n \otimes \mathcal{W}$ and $r_n \in Q$. Denote by $\tilde{\Phi} : \tilde{C} \to \widetilde{C \otimes Q}$ and $\tilde{\Psi} : \widetilde{C \otimes Q} \to \tilde{C}$ the extensions.

Fix $z \in K_0(C)$. We will show that, for any integer $k \geq 2$, there exists $y \in K_0(C)$ such that $ky = z$. Without loss of generality, we may assume that $z = [p] - [q]$, where $p \in M_r(\mathbb{C})$ is a matrix with scaler entries and $q = p + x$, where $x \in M_r(C)_{s.a}$, and both p and q are projections. Let D be a separable C^*-subalgebra of C such that $M_r(D)$ contains x. Let $\iota : D \to C$ be the embedding. Consider $\Phi(D)$. Then, $\Phi(D) \subset D_1 := \{\iota(d) \otimes r : d \in D, r \in Q\} \cong D \otimes Q$. Note, for each $\varepsilon > 0$ and each finite subset $\mathcal{F} \subset Q$, there exists a unitary $u \in Q$ such that

$$u^*\psi(y \otimes 1_Q)u \approx_{\varepsilon} y \ \text{ for all } \ y \in \mathcal{F}. \tag{e6.29}$$

Now write $x = \{(x_{i,j}^{(n)})_{r\times r}\}$. For each n, i, j, there are $a_{i,j,k}^{(n)} \in B_n \otimes \mathcal{W}$ and $r_{i,j,k}^{(n)} \in Q$, $k = 1, 2, \ldots, N(i, j, n)$, such that

$$\|x_{i,j}^{(n)} - \sum_k a_{i,j,k}^{(n)} \otimes r_{i,j,k}^{(n)}\| < 1/(4^{n+1}(\|x\| + 1)r^2), \quad 1 \leq i, j \leq r. \tag{e6.30}$$

Let $M_n = \max\{\|a_{i,j,k}^{n}\| : 1 \leq k \leq N(i, j, n)\}$. Therefore there is a unitary $u_n \in Q$ such that

$$\|u_n^*\psi(r_{i,j,k}^{(n)} \otimes 1_Q)u_n - r_{i,j,k}^{(n)}\| < 1/(4^n(\|x\| + 1)N(i, j, n)M_n r^2), \quad 1 \leq i, j \leq r, \ \ n = 1, 2, \ldots. \tag{e6.31}$$

Let $u = \{1_{\tilde{B}_n} \otimes u_n\}$ and $U = \mathrm{diag}(\overbrace{u, u, \ldots, u}^{r})$. Then $u^*\{b_n\}u \in C$ for all $\{b_n\} \in C$. In other words, $\mathrm{Ad}\, u : C \to C$ is an automorphism. Moreover

$$\|U^*(\Psi \circ \Phi(x))U - x\| < 1/4. \tag{e6.32}$$

Let $H_1 = \mathrm{Ad}\, u \circ \Psi : C \otimes Q \to C$ and $H_2 = H_1 \circ \Phi : C \to C$. Put $\imath_1 = \imath \otimes \mathrm{id}_Q : D_1 := D \otimes Q \to C \otimes Q$. Recall that $p, q \in M_r(\tilde{D})$ and $p - q = x \in M_r(D)$. Thus $[p] - [q]$ also defines an element $z' \in K_0(D)$ and $z = \iota_{*0}(z')$ in $K_0(C)$. Since $\Phi(D) \subset D_1 \cong D \otimes Q$, we may regard $\Phi|_D$ as a map from D to D_1— let us denote it by Φ', that is $\Phi' : D \to D_1$ is the same map as $\Phi|_D = \Phi \circ \iota$ but with different codomain algebra D_1 (instead of $C \otimes Q$). Formally, we have $\Phi|_D = \iota_1 \circ \Phi'$. Then

$$\Phi_{*0}(z) = \Phi_{*0}(\iota_{*0}(z')) = (\Phi|_D)_{*0}(z') = (\iota_1 \circ \Phi')_{*0}(z') = (\iota_1)_{*0}(\Phi'_{*0}(z')).$$

By Künneth formula, $K_0(D_1)$ is divisible. Therefore there is $y' \in K_0(D_1)$ such that $\Phi'_{*0}(z') = ky' \in K_0(D_1)$. That is $\Phi_{*0}(z) = (\iota_1)_{*0}(ky') = k(\iota_1)_{*0}(y')$. Hence $(H_2)_{*0}(z) = (H_1)_{*0}(\Phi_{*0}(z)) = ky$, where $y = (H_1)_{*0}((\iota_1)_{*0}(y'))\in K_0(C)$. That is, $(H_2)_{*0}(z)$ divisible by k.

Since $p = (\lambda_{i,j})_{r\times r}$, where $\lambda_{i,j} \in \mathbb{C}$. (by (e6.32)),

$$\tilde{H_2}(p) = U^*pU = p \ \text{ and } \ \|(\tilde{H}_2)(q) - q\| = \|H_2(x) - x\| < 1/4, \tag{e6.33}$$

where we use $\tilde{H}_2 : \tilde{C} \to \tilde{C}$ for the unitization of $H_2 : C \to C$ and its induced map $\tilde{H}_2 : M_r(\tilde{C}) \to M_r(\tilde{C})$. It follows that

$$(H_2)_{*0}(z) = z \ \text{ in } \ K_0(C). \tag{e6.34}$$

Hence z is divisible by k. This shows that $K_0(C)$ is divisible. It follows that $K_0(C)/kK_0(C) = \{0\}$ for all $k \geq 2$.

A similar argument shows that $K_1(C)$ is also divisible. $\square$

Lemma 6.8. *Let A be a separable C^*-algebra and let $\varphi_n : A \to B_n \otimes \mathcal{W}$ be a sequence of approximately multiplicative completely positive contractive linear maps. Then there exists a sequence of integers $\{m(n)\}$ satisfying the following condition: Put $C = \prod_n (M_{m(n)}(B_n \otimes \mathcal{W}))$ and $C_0 = \bigoplus_{n=1}^{\infty} M_{m(n)}(B_n \otimes \mathcal{W})$. Then $[\pi \circ (\{\varphi_n\})] = 0$ in $\mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(C/C_0))$, where $\pi : C \to C/C_0$ is the quotient map and where we view φ_n maps A into $M_{m(n)}(B_n \otimes \mathcal{W})$ (as $\varphi_n(a) = \mathrm{diag}(\varphi_n(a), 0, \ldots, 0)$).*

Proof. Let $K_0(A) = \{x_1, x_2, \ldots, x_n, \ldots\}$. Put $D_n = (B_n \otimes \mathcal{W})^{\sim}$. Suppose that, without loss of generality, that $[\varphi_n(x_i)]$ is well defined, for all $i \le n$. We may write $x_i = [p_i] - [q_i] \in K_0(A)$, where $p_i \in M_{r(i)}(\mathbb{C})$ is a projection and $q_i = p_i + b_i$ and $b_i \in M_{r(i)}(A)_{s.a.}$. By 6.4, for any $i \le n$, there are integers $r(i,n)'$ and $m(n)' = r(i) + r(i,n)'$, and a unitary $u_{i,n} \in M_{m(n)'}(D_n)$ with $\pi_{d,n}(u_{i,n}) = 1_{m(n)'}$, where $\pi_{d,n} : M_{m(n)'}(D_n) \to M_{m(n)'}$ is the quotient map, such that

$$\|u_{i,n}^*(\varphi_n(p_i) \oplus 1_{r(i,n)'})u_{i,n} - (\varphi_n(q_i) \oplus 1_{r(i,n)'})\| < 1/2^{n+1} \tag{e6.35}$$

for all large n. Note that $\{u_{i,n}\} \in (\prod_n M_{m(n)'}(B_n \otimes \mathcal{W}))^{\sim}$. It follows that, for any $i \ge 1$, there exists $k(i) \ge i$ such that

$$[\{\varphi_n(x_i)\}_{n \ge k(i)}] = 0 \ \text{ in } \ K_0(\prod_{n \ge m(k)} M_{m(n)'}(B_n \otimes \mathcal{W})). \tag{e6.36}$$

Thus, for any $x \in K_0(A)$, $[\pi'(\{\varphi_n(x)\})] = 0$, where $\pi' : \prod_n M_{m(n)'}(B_n \otimes \mathcal{W}) \to \prod_n M_{m(n)'}(B_n \otimes \mathcal{W})/\oplus_n M_{m(n)'}(B_n \otimes \mathcal{W})$ is the quotient map.

Let $K_1(A) = \{g_1, g_2, \ldots, g_n, \ldots\}$ and $z_i \in M_{d(i)}$ be a unitary so that $[z_i] = g_i$, $i = 1, 2, \ldots$. Let $G_m = \{g_1, g_2, \ldots, g_m\}$. By the first part of 6.5, there exist $l(i) \ge 1$, $k_1(i) \ge i$ and $m(n)'' = d(i) + l(i,n)$ such that

$$[\{\langle\varphi_n(z_i)\rangle\}_{n \ge k_1(i)}] = 0 \ \text{ in } \ K_1(\prod_{n \ge k_1(i)} M_{m(n)''}(B_n \otimes \mathcal{W})), \tag{e6.37}$$

by viewing φ_n as a map from A into $M_{m(n)''}(B_n \otimes \mathcal{W})$. Let $m(n) = \max\{m(n)', m(n)''\}$, $n = 1, 2, \ldots$. Put $C = \prod_{n=1}^{\infty} M_{m(n)}(B_n \otimes \mathcal{W})$. Then $(\pi \circ \{\varphi_n\})_{*j} = 0$ $(j = 0, 1)$ as we view $\{\varphi_n\}$ as a map from A to C, where $\pi : C \to C/C_0$ is the quotient map. Fix an integer $k \ge 2$ and a finite subset $F' \subset K_0(A, \mathbb{Z}/k\mathbb{Z})$, we may assume that the image of F' is in $G_i = \{g_1, g_2, \ldots, g_i\}$. Then, by the following commutative diagram

$$\begin{tikzcd} K_0(A) \arrow[r] \arrow[d, "{[\{\varphi_n\}_{n \ge k_1(i)}]}"'] & K_0(A, \mathbb{Z}/k\mathbb{Z}) \arrow[r] \arrow[d, "{[\{\varphi_n\}_{n \ge k_1(i)}]}"'] & K_1(A) \arrow[d, "{[\{\varphi_n\}_{n \ge k_1(i)}]}"] \\ K_0(C') \arrow[r] & K_0(C', \mathbb{Z}/k\mathbb{Z}) \arrow[r] & K_1(C')), \end{tikzcd} \tag{e6.38}$$

where $C' = \prod_{n \ge k_1(i)} M_{m(n)}(B_n \otimes \mathcal{W})$, since $[\{\varphi_n\}_{n \ge k_1(i)}]|_{G_i} = 0$, $[\{\varphi_n\}_{n \ge k_1(i)}]|_{F'} \subset K_0(C')/kK_0(C')$. However, by 6.7, $K_0(C')/kK_0(C') = 0$. It follows that $[\pi(\{\varphi_n\})]|_{F'} = 0$. It follows that

$$[\pi(\{\varphi_n\})]|_{K_0(A, \mathbb{Z}/k\mathbb{Z})} = 0, \quad k = 2, 3, \ldots. \tag{e6.39}$$

Exactly the same argument shows that

$$[\pi(\{\varphi_n\})]|_{K_1(A, \mathbb{Z}/k\mathbb{Z})} = 0, \quad k = 2, 3, \ldots. \tag{e6.40}$$

This implies that

$$[\pi \circ (\{\varphi_n\})] = 0 \ \text{ in } \ \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(C/C_0)), \tag{e6.41}$$

where $C_0 = \bigoplus_{n=1}^{\infty} M_{m(n)}(B_n \otimes \mathcal{W})$. $\square$

We would like to recall Definition 2.17 for definition of T-$\mathcal{H}$-fullness (see also 5.5 of [15]).

Theorem 6.9. *Let A be a non-unital separable amenable C^*-algebra which satisfies the UCT and let $T : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ be a map. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ satisfying the following condition: For any two $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $\varphi, \psi : A \to B \otimes \mathcal{W}$, where B is any σ-unital C^*-algebra, and any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $\sigma : A \to M_l(B \otimes \mathcal{W})$ (for any integer $l \ge 1$) which is also T-$\mathcal{H}$-full, there exist integers $N_1, N_2 \ge 1$ and a unitary $U \in M_{1+N_1 l+N_2}(B \otimes \mathcal{W})^{\sim}$ such that*

$$\|\mathrm{Ad}\, U \circ ((\varphi \oplus S_{N_1})(a) \oplus 0_{N_2}) - ((\psi \oplus S_{N_1})(a) \oplus 0_{N_2})\| < \varepsilon \ \text{ for all } \ a \in \mathcal{F}, \tag{e6.42}$$

where

$$S_K(f) = \mathrm{diag}(\overbrace{\sigma(a), \sigma(a), \dots, \sigma(a)}^{K}) \text{ for all } a \in A$$

for integer $K \geq 1$ *and where* $0_{N_2} = \mathrm{diag}(\overbrace{0, 0, \dots, 0}^{N_2})$.

Proof. This follows from the proof of 3.14 of [15]. Suppose that the conclusion of the theorem is false, then there exist $\varepsilon_0 > 0$ and a finite subset $\mathcal{F} \subseteq A$ such that there are a sequence of positive numbers (δ_n) with $\delta_n \searrow 0$, an increasing sequence $(\mathcal{G}_n)$ of finite subsets of A such that $\bigcup_n \mathcal{G}_n$ is dense in A, an increasing sequence $(\mathcal{H}_n)$ of finite subsets of $A_+^{\mathbf{1}} \setminus \{0\}$ such that, if $a \in \mathcal{H}_n$ and $f_{1/2}(a) \neq 0$, then $f_{1/2}(a) \in \mathcal{H}_{n+1}$, and $\bigcup_n \mathcal{H}_n$ is dense in $A^{\mathbf{1}}$, and has dense intersection with the unital ball of each closed two-sided ideal of A, two sequences of $\mathcal{G}_n$-δ_n-multiplicative completely positive contractive maps $\varphi_n, \psi_n : A \to B_n$ a sequence of unital $\mathcal{G}_n$-δ_n-multiplicative completely positive contractive linear maps $\sigma_n : A \to \mathrm{M}_{l(n)}(B_n)$ which are F-$\mathcal{H}_n$-full and satisfy, for each $n = 1, 2, \dots$,

$$\inf\{\sup \|v_n^*(\varphi_n(a) \oplus S_{k_1(n)}(a) \oplus 0_{k_2(n)})v_n - (\psi_n(a) \oplus S_{k_1(n)}(a) \oplus 0_{k_2(n)})\| : a \in \mathcal{F}\} \geq \varepsilon_0, \tag{e6.43}$$

where the infimum is taken among all $k_1(n), k_2(n) \to \infty$, and all unitaries $v_n \in \mathrm{M}_{k_1(n)l(n)+1+k_2(n)}(B_n)$, and $S_{k_1(n)} : A \to \mathrm{M}_{k_1(n)l(n)}(B_n)$ is as above.

Let $\{m(n)\}$ be as in 6.8. Set $\mathrm{M}_{m(n)l(n)}(B_n) = B_n'$, $\bigoplus_{n=1}^{\infty} B_n' = C_0$, $\prod_{n=1}^{\infty} B_n' = C$, and $C/C_0 = Q(C)$, and denote by $\pi : C \to Q(C)$ the quotient map. Consider the maps $\Phi, \Psi, S : A \to C$ defined by $\Phi(a) = (\varphi_n(a))_{n\geq 1}$, $\Psi(a) = (\psi_n(a))_{n\geq 1}$, and $S(a) = (\bar{\sigma}_n(a))_{n\geq 1}$ for all $a \in A$, where

$$\bar{\sigma}_n(a) = \mathrm{diag}(\overbrace{\sigma_n(a), \sigma_n(a), \dots, \sigma_n(a)}^{m(n)}) \text{ for all } a \in A. \tag{e6.44}$$

Note that $\pi \circ \Phi$, $\pi \circ \Psi$ and $\pi \circ S$ are homomorphisms. Consider also the truncations $\Phi^{(m)}, \Psi^{(m)}, S^{(m)} : A \to \prod_{n\geq m} B_n'$ defined by $\Phi^{(m)}(a) = (\varphi_n(a))_{n\geq m}$, $\Psi^{(m)}(a) = (\psi_n(a))_{n\geq m}$, and $S^{(m)}(a) = (\bar{\sigma}_n(a))_{n\geq m}$.

It follows from 6.8 that

$$[\pi \circ \Phi] = [\pi \circ \Psi] \text{ in } \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(C/C_0)). \tag{e6.45}$$

We will show that $\bar{\sigma}_n$ is T-$\mathcal{H}_n$-full in $M_{m(n)l(n)}(B_n \otimes \mathcal{W})$. Let $T(a) = (N(a), M(a)) \in \mathbb{N} \times \mathbb{R} \setminus \{0\}$ for all $a \in A_+ \setminus \{0\}$. Fixed any nonzero element $0 \leq a \leq 1$ in $\mathcal{H}_n$. Let $b \in M_{m(n)l(n)}(B_n \otimes \mathcal{W})_+$ with $\|b\| \leq 1$, and $\varepsilon_1 > 0$. Since B_n is σ-unital, there exists $0 \leq e \leq 1$ in $B_n \otimes \mathcal{W}$ such that

$$\|b - b^{1/2}(1_{m(n)l(n)} \otimes e)b^{1/2}\| < \varepsilon_1/2. \tag{e6.46}$$

Choose $\varepsilon_1/2 > \eta > 0$ such that

$$\|b - b^{1/2}(1_{m(n)l(n)} \otimes (e - \eta)_+)b^{1/2}\| < \varepsilon_1. \tag{e6.47}$$

Since σ_n is T-$\mathcal{H}_n$-full, by also applying 3.1 of [15], there are $x_1, x_2, \dots, x_{N(a)} \in M_{l(n)}(B_n \otimes \mathcal{W})$ with $\|x_i\| \leq M(a)$, $1 \leq i \leq N(a)$ such that $(e - \eta)_+ \otimes 1_{l(n)} = \sum_{i=1}^{N(a)} x_i^* \sigma_n(a) x_i$. Therefore (identifying $\bar{\sigma}_n(a)$ with $1_{m(n)} \otimes \sigma_n(a)$)

$$\|\sum_{i=1}^{N(a)} b^{1/2}(1_{m(n)} \otimes x_i)^* \bar{\sigma}_n(a)(1_{m(n)} \otimes x_i)b^{1/2} - b\| < \varepsilon_1.$$

This shows that $\bar{\sigma}_n$ is T-$\mathcal{H}_n$-full in $M_{m(n)l(n)}((B \otimes \mathcal{W}))$. As in the proof of 3.14 of [15], this implies $\pi \circ \{\bar{\sigma}_n\}$ is full in C/C_0.

Then, by the proof 3.14 of [15], there exists an integer $K \geq 1$ and there exists a unitary $v \in M_{Km(n)l(n)+m(n)l(n)}(C/C_0)$ such that

$$\|v^* \mathrm{diag}(\pi \circ \Phi(a), \Sigma_n(a))v - \mathrm{diag}(\pi \circ \Phi_n(a), \Sigma_n(a))\| < \varepsilon_0/4 \text{ for all } a \in \mathcal{F},$$

where

$$\Sigma_n(a) = \mathrm{diag}(\overbrace{\pi \circ \bar{\sigma}_n(a), \pi \circ \bar{\sigma}_n(a), \dots, \pi \circ \bar{\sigma}_n(a)}^{K}).$$

Lifting this to C, one obtains, for all sufficiently large $n \geq 1$, a unitary $u_n \in M_{Km(n)l(n)+m(n)l(n)}((B_n \otimes \mathcal{W})^{\sim})$ such that

$$\|u_n^* \mathrm{diag}(\varphi_n(a) \oplus 0_{m(n)l(n)-1}, \bar{\sigma}_n(a))u_n - \mathrm{diag}(\psi_n(a) \oplus 0_{m(n)l(n)-1}, \bar{\sigma}_n(a))\| < \varepsilon_0/2 \text{ for all } a \in \mathcal{F}.$$

By replacing u_n by another unitary w_n, if necessary, we have, for all sufficiently large $n \geq 1$,

$$\|u_n^* \mathrm{diag}(\varphi_n(a), \bar{\sigma}_n(a) \oplus 0_{m(n)l(n)-1})u_n - \mathrm{diag}(\psi_n(a), \bar{\sigma}_n(a) \oplus 0_{m(n)l(n)-1})\| < \varepsilon_0/2, \tag{e6.48}$$

for all $a \in \mathcal{F}$.

This contradicts (e6.43). Lemma follows. □

7. Models and range of invariant

Lemma 7.1. *Let A be an AF algebra and $\varphi_1, \varphi_2 : A \to Q$ be two unital homomorphisms with $(\varphi_1)_{*0} = (\varphi_2)_{*0}$. Let n be a positive integer. Define B_i ($i = 1, 2$) to be the C^*-subalgebra of $C([0,1], Q \otimes M_{n+1}) \oplus A$ given by*

$$B_i = \{(f, a) \in C([0,1], Q \otimes M_{n+1}) \oplus A : \begin{array}{l} f(0) = \varphi_i(a) \otimes \mathrm{diag}(\overbrace{1, \ldots, 1}^{n}, 0) \\ f(1) = \varphi_i(a) \otimes \mathrm{diag}(\underbrace{1, \ldots, 1, 1}_{n+1}) \end{array}\} \tag{e7.1}$$

for $i = 1, 2$. Then $B_1 \cong B_2$.

Proof. Since both A and Q are AF algebras and $(\varphi_1)_{*0} = (\varphi_2)_{*0}$, there is a unitary path $\{u(t)\}_{0 \le t < 1}$ such that $\varphi_2(a) = \lim_{t \to 1} u(t)\varphi_1(a)u(t)^*$ (see [34]). Define the isomorphism $\psi : B_1 \to B_2$ by sending $(f, a) \in B_1$ to $(g, a) \in B_2$, where g is given by

$$g(t) = \begin{cases} (u(|2t-1|) \otimes \mathbf{1}_{n+1}) f(t) (u(|2t-1|) \otimes \mathbf{1}_{n+1})^* & \text{if } t \in (0,1), \\ \varphi_2(a) \otimes \mathrm{diag}(\overbrace{1, \ldots, 1}^{n}, 0) & \text{if } t = 0, \\ \varphi_2(a) \otimes \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, 1) & \text{if } t = 1. \end{cases}$$

It is straight forward to verify that g is continuous, that $(g, a) \in B_2$, and that ψ defines a desired isomorphism. $\square$

Definition 7.2. Let G_0 and G_1 be two countable abelian groups. Let A be a unital AH-algebra with $TR(A) = 0$, unique tracial state, $K_1(A) = G_1$ and $K_0(A) = \mathbb{Q} \oplus G_0$ with $\ker \rho_A = G_0$ and $[1_A] = (1, 0)$.

There is a unital homomorphism $s : A \to Q$ such that $s_{*0}(r, g) = r$ for $(r, g) \in \mathbb{Q} \oplus G_0$. Fix a unital embedding $j : Q \to A$ with $j_{*0}(r) = (r, 0)$ for $r \in \mathbb{Q}$. (Note that both $j \circ s$ and $s \circ j$ induce the identity maps on $T(A)$ and $T(Q)$ respectively. Furthermore the homomorphism j and s identify the spaces $T(A)$ and $T(Q)$)

Fix an integer $a_1 \ge 1$. Let $\alpha = \frac{a_1}{a_1+1}$. For each $r \in \mathbb{Q}_+ \setminus \{0\}$, let $e_r \in Q$ be a projection with $\mathrm{tr}(e_r) = r$. Let $\bar{Q}_r := (1 \otimes e_r)(Q \otimes Q)(1 \otimes e_r)$. Define $q_r : Q \to \bar{Q}_r$ by $a \mapsto a \otimes e_r$ for $a \in Q$. We will also use q_r to denote a homomorphism from B to $B \otimes e_r Q e_r$ (or to $B \otimes Q$) defined by sending $b \in B$ to $b \otimes e_r \in B \otimes e_r Q e_r \subset B \otimes Q$.

We fix an isomorphism $Q \otimes Q \to Q$ which will be denoted by ι^Q. Moreover the composition of the maps which first maps a to $a \otimes 1_Q$ and then to Q via ι^Q is approximately inner. In fact every unital endomorphism on Q is approximately inner. If we identify Q with $Q \otimes 1_Q$ in $Q \otimes Q$ then ι^Q is an approximately inner endomorphism.

For each $1 > r > r' > 0$, we assume that $e_r \ge e_{r'}$. Fix $1 > r > 0$, define $\iota_r^Q : \bar{Q}_r \to Q_r := e_r Q e_r$ by $\iota_r^Q = \mathrm{Ad}\, v_r \circ \iota^Q|_{\bar{Q}_r}$, where $v_r^*(\iota^Q(1 \otimes e_r))v_r = e_r$.

Let

$$R(\alpha, r) = \{(f, a) \in C([0,1], Q \otimes Q_r) \oplus Q : f(0) = a \otimes e_{r\alpha} \text{ and } f(1) = a \otimes e_r\}.$$

(Recall that $R(\alpha, 1)$ has been defined in 3.7.)

Let

$$A(W, \alpha) = \{(f, a) \in C([0,1], Q \otimes Q) \oplus A : f(0) = q_\alpha \circ s(a) \text{ and } f(1) = s(a) \otimes 1_Q\}.$$

We also note that (f, a) is full in $A(W, \alpha)$ if and only if $a \ne 0$ and $f(t) \ne 0$ for all $t \in (0, 1)$.

Let $\mathcal{M}_+$ denote the set of nonnegative regular measures on $(0, 1)$. As in 3.4, trace spaces $\tilde{T}(A(W, \alpha))$ and $\tilde{T}(R(\alpha, 1))$ are isomorphic, and each $\tau \in \tilde{T}(R(\alpha, 1)) \cong \tilde{T}(A(W, \alpha))$ corresponds to $(\mu, s) \in \mathcal{M}_+(0, 1) \times \mathbb{R}_+$. Furthermore we have

$$\|\tau\| = \|\mu\| + s = \int_0^1 d\mu \ + \ s.$$

Note that in the weak topology of $\tilde{T}(A(W, \alpha))$ (or $\tilde{T}(R(\alpha, 1))$), under the above identification, one has that

$$\lim_{t \to 0}(\delta_t, 0) = (0, \alpha) \in \mathcal{M}_+(0,1) \times \mathbb{R}_+ \quad \text{and} \quad \lim_{t \to 1}(\delta_t, 0) = (0, 1) \in \mathcal{M}_+(0,1) \times \mathbb{R}_+,$$

where δ_t is the unit measure of the point mass at t.

The affine space $\mathrm{Aff}(\tilde{\mathrm{T}}(A(W, \alpha)))$ and $\mathrm{Aff}(\tilde{\mathrm{T}}(R(\alpha, 1)))$ can be identified with

$$\{(f, x) \in C([0,1], \mathbb{R}) \oplus \mathbb{R} : f(0) = \alpha \cdot x \text{ and } f(1) = x\}, \tag{e7.2}$$

a subspace of $C([0,1], \mathbb{R}) \oplus \mathbb{R}$.

Let

$$A(W, \alpha, r) = \{(f, a) \in C([0,1], Q \otimes Q_r) \oplus A : f(0) = q_{r\alpha} \circ s(a) \text{ and } f(1) = q_r \circ s(a)\}.$$

Define $\varphi_{A,R,\alpha} : A(W, \alpha) \to R(\alpha, 1)$ by

$$\varphi_{A,R,\alpha}((f, a)) = (f, s(a)) \text{ for all } (f, a) \in A(W, \alpha).$$

Define $\tilde{sj} : C([0, 1], Q \otimes Q) \to C([0, 1], Q \otimes Q)$ by

$$\tilde{sj}(f)(t) = ((s \circ j) \otimes \mathrm{id}_Q)(f(t)).$$

Define $\varphi_{R,A,\alpha} : R(\alpha, 1) \to A(W, \alpha, 1)$ by

$$\varphi_{R,A,\alpha}((f, a)) = (\tilde{sj}(f), j(a)) \text{ for all } (f, a) \in R(\alpha, 1).$$

Note that

$$\tilde{sj}(f)(0) = ((s \circ j) \otimes \mathrm{id}_Q)(a \otimes e_\alpha) = s \circ j(a) \otimes e_\alpha \text{ and} \tag{e7.3}$$
$$\tilde{sj}(f)(1) = ((s \circ j) \otimes \mathrm{id}_Q)(a \otimes 1) = s \circ j(a) \otimes 1. \tag{e7.4}$$

Also

$$q_\alpha \circ s \circ j(a) = s \circ j(a) \otimes e_\alpha.$$

In particular, $\varphi_{R,A,\alpha}$ does map $R(\alpha, 1)$ into $A(W, \alpha, 1)$. Moreover $\varphi_{R,A,\alpha}$ is injective and map the strictly positive element a_α to a strictly positive element (with the same form–see 3.7).

With the identification of both $\mathrm{Aff}(\tilde{\mathrm{T}}(A(W, \alpha)))$ and $\mathrm{Aff}(\tilde{\mathrm{T}}(R(\alpha, 1)))$ with the same subspace of $C([0, 1], \mathbb{R}) \oplus \mathbb{R}$, the homomorphism $\varphi_{A,R,\alpha}$ and $\varphi_{R,A,\alpha}$ induce the identity map on that subspace at the level of $\mathrm{Aff}(\tilde{T}(-))$ maps. They also induce the identity maps at level of trace spaces, when we identify the corresponding trace spaces. In particular, $\varphi^*_{A,R,\alpha} : \tilde{\mathrm{T}}(R(\alpha, 1)) \to \tilde{\mathrm{T}}(A(W, \alpha))$ (or $\varphi^*_{R,A,\alpha} : \tilde{\mathrm{T}}(A(W, \alpha)) \to \tilde{\mathrm{T}}(R(\alpha, 1))$, respectively) takes the subset $\mathrm{T}(R(\alpha, 1))$ to the subset $\mathrm{T}(A(W, \alpha))$ (or takes $\mathrm{T}(A(W, \alpha))$ to $\mathrm{T}(R(\alpha, 1))$, respectively)

Fix $\alpha, r \in \mathbb{Q}_+ \setminus \{0\}$. There are unitaries $u_{\alpha,r},\ u_{1,r} \in \bar{Q}_r$ such that

$$u^*_{\alpha,r}(e_\alpha \otimes e_r)u_{\alpha,r} = (\iota^Q_r)^{-1}(e_{r\alpha}) \text{ and } u^*_{1,r}(1 \otimes e_r)u_{1,r} = (\iota^Q_r)^{-1}(e_r) = 1 \otimes e_r.$$

(Note that $u_{1,r}$ can be chosen to be $1_{\bar{Q}_r}$.) There is a continuous path of unitaries $\{u(t) : t \in [0, 1]\}$ in $\bar{Q}_r$ such that $u(0) = u_{\alpha,r}$ and $u(1) = u_{1,r}$.

Let $v(t) = 1 \otimes u(t) \in Q \otimes \bar{Q}_r$ for $t \in [0, 1]$. Note if $f(t) \in Q \otimes Q$, then

$$v(t)^*(f(t) \otimes e_r)v(t) \in Q \otimes \bar{Q}_r \text{ for all } t \in (0, 1).$$

Let $\varphi_{R,r} : R(\alpha, 1) \to R(\alpha, r)$ be defined by

$$\varphi_{R,r}((f, a)) = (\mathrm{id}_Q \otimes \iota^Q_r) \circ \mathrm{Ad}\, v \circ q_r(f, a).$$

Note that, for $t \in (0, 1)$,

$$(\mathrm{id}_Q \otimes \iota^Q_r) \circ \mathrm{Ad}\, v(t) \circ q_r(f)(t) = (\mathrm{id}_Q \otimes \iota^Q_r) \circ \mathrm{Ad}\, v(t)(f(t) \otimes e_r) \tag{e7.5}$$
$$= (\mathrm{id}_Q \otimes \iota^Q_r)(v(t)^*(f(t) \otimes e_r)v(t)) \in Q \otimes Q_r, \tag{e7.6}$$

$$(\mathrm{id}_Q \otimes \iota^Q_r) \circ \mathrm{Ad}\, v(0) \circ q_r(f)(0) = (\mathrm{id}_Q \otimes \iota^Q_r) \circ \mathrm{Ad}\, v(0)(a \otimes e_\alpha \otimes e_r) \tag{e7.7}$$
$$= (\mathrm{id}_Q \otimes \iota^Q_r)(a \otimes (\iota^Q_r)^{-1}(e_{\alpha r})) \tag{e7.8}$$
$$= a \otimes e_{\alpha r} \text{ and} \tag{e7.9}$$

$$(\mathrm{id}_Q \otimes \iota^Q_r) \circ \mathrm{Ad}\, v(1) \circ q_r(f)(1) = (\mathrm{id}_Q \otimes \iota^Q_r) \circ \mathrm{Ad}\, v(1)(a \otimes 1 \otimes e_r) \tag{e7.10}$$
$$= (\mathrm{id}_Q \otimes \iota^Q_r)(a \otimes (\iota^Q_r)^{-1}(e_r)) \tag{e7.11}$$
$$= a \otimes e_r. \tag{e7.12}$$

Evidently, when we identify $\tilde{\mathrm{T}}(R(\alpha, r))$ and $\tilde{\mathrm{T}}(R(\alpha, 1))$ with $\mathcal{M}_+(0, 1) \times \mathbb{R}_+$, the map $\varphi^*_{R,r}$ is the identity map and takes the subset $\mathrm{T}(R(\alpha, r))$ to the subset $\mathrm{T}(R(\alpha, 1))$.

Define $s^{(2,3)} : Q \otimes Q \otimes Q \to Q \otimes Q \otimes Q$ by

$$s^{(2,3)}(x \otimes y \otimes z) = x \otimes z \otimes y$$

for all $x, y, z \in Q$. To make the notation clearer, we will often write the above $x \otimes z \otimes y$ as $(x \otimes z) \otimes y$, later. Define a homomorphism $\tilde{\iota^Q} : R(\alpha, 1) \otimes Q \to R(\alpha, 1)$ by

$$\widetilde{\iota^Q}(f \otimes b, a \otimes b) = (((\iota^Q) \otimes \mathrm{id}_Q) \circ s^{(2,3)}(f \otimes b), \iota^Q(a \otimes b))$$

for $(f, a) \in R(\alpha, 1)$ and $b \in Q$.

Note, at $t = 0$,

$$(\iota^Q \otimes \mathrm{id}_Q) \circ s^{(2,3)}(f \otimes b)(0) = (\iota^Q \otimes \mathrm{id}_Q) \circ s^{(2,3)}(a \otimes e_\alpha \otimes b) \tag{e7.13}$$
$$= (\iota^Q \otimes \mathrm{id}_Q)((a \otimes b) \otimes e_\alpha) \tag{e7.14}$$
$$= \iota^Q(a \otimes b) \otimes e_\alpha; \tag{e7.15}$$

and, at $t = 1$,

$$(\iota^Q \otimes \mathrm{id}_Q) \circ s^{(2,3)}(f \otimes b)(1) = (\iota^Q \otimes \mathrm{id}_Q) \circ s^{(2,3)}(a \otimes 1 \otimes b) \tag{e7.16}$$
$$= (\iota^Q \otimes \mathrm{id}_Q)((a \otimes b) \otimes 1) \tag{e7.17}$$
$$= \iota^Q(a \otimes b) \otimes 1. \tag{e7.18}$$

Let $m \geq 2$ be an integer. Viewing M_m as a unital C^*-subalgebra of Q, Put $\iota^{M_m} = \iota^Q|_{Q \otimes M_m}$. Define $\widetilde{\iota^{M_m}} : R(\alpha, 1) \otimes M_m \to R(\alpha, 1)$ by $\widetilde{\iota^{M_m}} = \widetilde{\iota^Q}|_{R(\alpha,1)\otimes M_m}$. Note also that (recall (e3.16))

$$\widetilde{\iota^Q}(a_\alpha \otimes 1_Q) = a_\alpha \ \text{ and } \ \widetilde{\iota^{M_m}}(a_\alpha \otimes 1_{M_m}) = a_\alpha. \tag{e7.19}$$

We need one more map. Let $\psi_{A_w} : A(W, \alpha) \to C([0, 1], Q) \oplus A$ be defined by

$$\psi_{A_w}(f, a) = (g, a),$$

where $g(t) = s(a)$ for all $t \in [0, 1]$. Define $\psi_{A_w,r} : A(W, \alpha) \to C([0, 1], Q \otimes Q_r) \oplus A$ by

$$\psi_{A_w,r}((f, a)) = (q_r(g), a)$$

with $g(t) = s(a)$ (and $q_r(g) = q_r \circ s(a)$). Note that $\psi_{A_w,r}(a_\alpha) = (1 \otimes e_r, 1)$ is the unit of $C([0, 1], Q \otimes Q_r) \oplus A$. It follows that $\psi_{A_w,r}$ maps strictly positive elements to strictly positive elements.

When we identify $\tilde{\mathrm{T}}(A(W, \alpha))$ with $\mathcal{M}_+(0, 1) \times \mathbb{R}_+$, and $\tilde{\mathrm{T}}(C([0, 1], Q \otimes Q_r) \oplus A)$ with $\mathcal{M}_+[0, 1] \times \mathbb{R}_+$, the map $\psi^*_{A_w,r}$ is given by

$$\psi^*_{A_w,r}(\mu, s) = (0, s + \int_0^1 d\mu),$$

which takes $\mathrm{T}(C([0, 1], Q \otimes Q_r))$ to $\mathrm{T}(A(W, \alpha))$.

Warning: $C([0, 1], Q \otimes Q_r) \oplus A \neq A(W, \alpha)$.

One more notation: define $P_f : (f, a) \to f$ and $P_a : (f, a) = a$.

Now let $\alpha < \beta < 1$. Let us choose x such that $\beta(1/2 + x) = (\alpha/2 + x)$. So

$$x = \frac{(1/2)(\beta - \alpha)}{1 - \beta} > 0.$$

Let

$$y = 1/2 + x = \frac{1}{2} + \frac{(1/2)(\beta - \alpha)}{(1 - \beta)} = \frac{(1 - \alpha)}{2(1 - \beta)}.$$

Let $r_1 = (1/2)(1/y) = \frac{(1-\beta)}{(1-\alpha)}$ and $r_2 = x(1/y) = \frac{(\beta-\alpha)}{(1-\alpha)}$. Then

$$\alpha r_1 + r_2 = (1/y)(1/2 + x) = \beta \ \text{ and } \ r_1 + r_2 = (1/y)(1/2 + x) = 1.$$

Define $\Phi_{A_w,\alpha,\beta} : A(W, \alpha) \to A(W, \beta)$ by

$$P_a(\Phi_{A_w,\alpha,\beta}((f, a))) = a \ \text{ and}$$
$$P_f(\Phi_{A_w,\alpha,\beta}((f, a))) = \mathrm{diag}(P_f \circ \varphi_{R,r_1} \circ \varphi_{A,R,\alpha}((f, a)), P_f \circ \psi_{A_w,r_2}((f, a))).$$

One computes that, for $t \in (0, 1)$,

$$P_f(\varphi_{R,r_1} \circ \varphi_{A,R,\alpha}((f, a)))(t) = (\mathrm{id}_Q \otimes \iota^Q_{r_1}) \circ \mathrm{Ad}\, v(t) \circ q_{r_1}(f)(t) \tag{e7.20}$$
$$= (\mathrm{id}_Q \otimes \iota^Q_{r_1})(v(t)^* f(t) \otimes e_{r_1})v(t) \tag{e7.21}$$
$$\in Q \otimes Q_{r_1} \subset Q \otimes Q \ \text{ and} \tag{e7.22}$$
$$P_f(\psi_{A_w,r_2}((f, a)))(t) = q_{r_2}(s(a)) = s(a) \otimes e_{r_2} \in Q \otimes Q. \tag{e7.23}$$

At $t = 0$,

$$P_f(\varphi_{R,r_1} \circ \varphi_{A,R,\alpha}((f, a))(0)) = (\mathrm{id}_Q \otimes \iota^Q_{r_1}) \circ \mathrm{Ad}\, v(0) \circ q_{r_1}(f)(0) \tag{e7.24}$$
$$= (\mathrm{id}_Q \otimes \iota^Q_{r_1}) \circ \mathrm{Ad}\, v(0)(s(a) \otimes e_\alpha \otimes e_{r_1}) \tag{e7.25}$$

$$= (\mathrm{id}_Q \otimes \iota_{r_1}^Q)(a \otimes (\iota_{r_1}^Q)^{-1}(e_{\alpha r_1})) \tag{e7.26}$$

$$= s(a) \otimes e_{\alpha r_1}. \tag{e7.27}$$

Hence

$$P_f(\Phi_{A_w,\alpha,\beta}((f,a)))(0) = \mathrm{diag}(s(a) \otimes e_{\alpha r_1}, s(a) \otimes e_{r_2}) \tag{e7.28}$$

$$= s(a) \otimes e_{\alpha r_1 + r_2} = s(a) \otimes e_\beta. \tag{e7.29}$$

At $t = 1$,

$$P_f(\varphi_{R,R,r_2} \circ \varphi_{A,R,\alpha}((f,a))(1)) = (\mathrm{id}_Q \otimes \iota_{r_1}^Q) \circ \mathrm{Ad}\, v(1) \circ q_{r_1}(f)(1) \tag{e7.30}$$

$$= (\mathrm{id}_Q \otimes \iota_{r_1}^Q) \circ \mathrm{Ad}\, v(1)(s(a) \otimes 1 \otimes e_{r_1}) \tag{e7.31}$$

$$= (\mathrm{id}_Q \otimes \iota_{r_1}^Q)(s(a) \otimes (\iota_{r_1}^Q)^{-1}(e_{r_1})) \tag{e7.32}$$

$$= s(a) \otimes e_{r_1}. \tag{e7.33}$$

Hence

$$P_f(\Phi_{A_w,\alpha,\beta}((f,a)))(1) = \mathrm{diag}(s(a) \otimes e_{r_1}, s(a) \otimes e_{r_2}) \tag{e7.34}$$

$$= s(a) \otimes e_{r_1+r_2} = s(a) \otimes 1. \tag{e7.35}$$

Therefore, indeed, $\Phi_{A_w,\alpha,\beta}$ defines a homomorphism from $A(W,\alpha)$ to $A(W,\beta)$. It is injective. We also check that $\Phi_{A_w,\alpha,\beta}(a_\alpha)$ is a strictly positive element of $A(W,\beta)$ (recall (e3.16)).

Furthermore $\Phi^*_{A_w,\alpha,\beta} : \tilde{\mathrm{T}}(A(W,\beta))(\cong \mathcal{M}_+(0,1) \times \mathbb{R}_+) \to \tilde{\mathrm{T}}(A(W,\alpha))(\cong \mathcal{M}_+(0,1) \times \mathbb{R}_+)$ is given by

$$\Phi^*_{A_w,\alpha,\beta}(\mu, s) = (r_1\mu, r_2(\int_0^1 d\mu) + s),$$

which takes $\mathrm{T}(A(W,\beta))$ to $\mathrm{T}(A(W,\alpha))$.

Fix any $a \in A_+$ with $\|a\| = 1$. Define $f(t) = (1-t)(s(a) \otimes e_\alpha) + t(s(a) \oplus 1)$. Then $(f,a) \in A(\alpha, 1)$ is a full positive element. Note that $\Phi_{A_w,\alpha,\beta}((f,a))$ is also a full positive element.

Let m, m' be two positive integers such that $m|m'$. Let $\frac{m'}{m} = a+1$. Let $F_2 = M_{m'}(\mathbb{C})$, $F_1 = M_m(\mathbb{C})$, and $\varphi_0, \varphi_1 : F_1 \to F_2$ be defined by

$$\varphi_0(x) = \mathrm{diag}(\underbrace{x, \dots, x}_{a}, 0), \quad \text{and} \quad \varphi_0(x) = \mathrm{diag}(\underbrace{x, \dots, x}_{a+1}).$$

Denote that

$$A(m,m') = A(F_1, F_2, \varphi_0, \varphi_1) = \{(f,x) \in C([0,1], M_m(\mathbb{C}) \otimes M_{a+1}(\mathbb{C})) \oplus M_m(\mathbb{C}) : \\ f(0) = x \otimes \mathrm{diag}(\underbrace{1, \dots, 1}_{a}, 0) \;\; f(1) = x \otimes \mathrm{diag}(\underbrace{1, \dots, 1}_{a+1})\}.$$

Then $A(m,m') \in \mathcal{C}_0^0$ with $\lambda_s(A(m,m')) = \frac{a}{a+1}$.

In [25], the author constructed a simple inductive limit $\mathcal{W} = \lim W_i'' = \lim(A(m_i, (a_i+1)m_i), \omega_{i,j})$ such that $K_0(\mathcal{W}) = 0 = K_1(\mathcal{W})$ and $T(\mathcal{W}) = \{pt\}$. In the construction, one has $a_i + 1 = 2(a_{i-1}+1)$ and $m_i = a_i m_{i-1}$. Consequently $\lim_{i\to\infty} a_i = \infty$. From the construction in [25], the map $\omega_{i,j}$ takes strictly positive elements to strictly positive elements, and $\omega^*_{i,j}$ maps tracial state space $T(W_j'')$ to tracial state space $T(W_i'')$. Furthermore, $A_i \in \mathcal{C}_0^0$ with $\lambda_s(A_i) = \frac{a_i}{a_{i+1}} \to 1$ as $i \to \infty$.

Note that $\mathcal{W} \otimes Q \cong \mathcal{W}$. Identify $Q \otimes M_m$ and $Q \otimes M_{a+1}$ with Q, we can identify $A(m,(a+1)m) \otimes Q$ with $R(\alpha, 1)$ for $\alpha = a/(a+1)$. Moreover, $\mathcal{W} = \lim(W_n' = R(\alpha_n, 1), \iota'_{W,n})$, where $\iota'_{W,n} : R(\alpha_n, 1) \to R(\alpha_{n+1}, 1)$ are injective. Again, we have that $(\iota'_{W,n})^*$ takes $T(R(\alpha_{n+1}, 1))$ to $T(R(\alpha_n, 1))$.

Let C be a unital AF-algebra so that $T(C) = T$. We write $C = \lim_{n\to\infty}(F_n, \iota_{F,n})$, where $\dim(F_n) < \infty$ and $\iota_{F,n} : F_n \to F_{n+1}$ are unital injective homomorphisms.

Let $\mathcal{W}$ be as before. Write

$$W_T = \mathcal{W} \otimes C.$$

Then $T(W_T) = T$ and W_T has continuous scale.

Suppose that

$$F_n = \bigoplus_{i=0}^{k(n)} M_{n_i}.$$

By identifying $R(\alpha_n, 1)$ with $R(\alpha_n, 1) \otimes M_{n_i}$ and $R(\alpha_n, 1) \otimes Q$, we may write that

$$W_T = \lim_{n\to\infty}(W_n, \imath_n),$$

where W_n is a direct sum of $k(n)$ summand of $R(\alpha_n, 1) : W_n = \bigoplus_{i=0}^{k(n)} R(\alpha_n, 1)^{(i)}$, where $\alpha_1 < \alpha_2 < \cdots < 1$. Again, we have that $\imath_n^*$ takes $T(W_{n+1})$ to $T(W_n)$.

We write

$$W_n = R_{0,n} \bigoplus D_n,$$

where $R_{0,n} = R(\alpha_n, 1)^{(0)}$ and

$$D_n = \bigoplus_{i=1}^{k(n)} R(\alpha_n, 1)^{(i)}.$$

In the case that W_n has only one summand, we understand that $W_n = R_{0,n}$ and $D_n = \{0\}$. We also use

$$P_{0,n} : W_n \to R_{0,n} \ \text{ and } \ P_{1,n} : W_n \to D_n$$

for the projection map, i.e., $P_{0,n}(a \oplus b) = a$ and $P_{1,n}(a \oplus b) = b$ for all $a \in R_{0,n}$ and $b \in D_n$.

Consider

$$B_n = W_n \oplus M_{(n!)^2}(A(W, \alpha_n)), \quad n = 1, 2,$$

Let $r_n = \frac{1}{2^{n+1}k(n)}$, $n = 1, 2, \ldots$.

Let us define a homomorphism $\Psi_{n,n+1} : B_n \to B_{n+1}$ as follows.

On $M_{(n!)^2}(A(W, \alpha_n))$ define $\Psi_{n,n+1,A,A} : M_{(n!)^2}(A(W, \alpha_n)) \to M_{((n+1)!)^2}(A(W, \alpha_{n+1}))$ by

$$\Psi_{n,n+1,A,A}(a) = \mathrm{diag}(\Phi_{A_w,\alpha_n,\alpha_{n+1}}(a), \overbrace{0, 0, \ldots, 0}^{((n+1)!)^2-(n!)^2}) \ \text{ for all } \ a \in M_{(n!)^2}(A(W, \alpha_n))$$

and define $\Psi_{n,n+1,A,W} : M_{(n!)^2}(A(W, \alpha_n)) \to R_{0,n+1} \otimes e_{r_n} Q e_{r_n}$ by

$$\Psi_{n,n+1,A,W} = q_{r_n} \circ \imath'_{W,n} \circ \widetilde{\iota^{M_{(n!)^2}}} \circ (\varphi_{A,R,\alpha_n} \otimes \mathrm{id}_{M_{(n!)^2}}).$$

(Recall that $\widetilde{\iota^{Q}} : R(\alpha, 1) \otimes Q \to R(\alpha, 1)$ is an isomorphism and $\widetilde{\iota^{M_m}} : R(\alpha, 1) \otimes M_m \to R(\alpha, 1)$ is defined by $\widetilde{\iota^{M_m}} = \widetilde{\iota^{Q}}|_{R(\alpha,1)\otimes M_m}$.) It is injective.

On W_n define $\Psi_{n,n+1,W,W} : W_n \to R_{0,n+1} \otimes (1 - e_{r_n})Q(1 - e_{r_n}) \oplus D_{n+1} \subset W_{n+1}$ by, for $a \in R_{0,n}$, $b \in D_n$,

$$\begin{aligned}&\Psi_{n,n+1,W,W}((a \oplus b)) = \Psi^0_{n,n+1,W,W}((a \oplus b)) \oplus \Psi^1_{n,n+1,W,W}((a \oplus b)) = \\ &q_{1-r_n}((P_{0,n+1} \circ \imath_{n,n+1}(a)) \oplus (P_{0,n+1} \circ \imath_{n,n+1}(b))) \\ &\qquad \oplus P_{1,n+1} \circ \imath_{n,n+1}(a) \oplus P_{1,n+1} \circ \imath_{n,n+1}(b).\end{aligned} \tag{e7.36}$$

Suppose that $a, b \geq 0$. Then, for any $t \in T(W_{n+1})$,

$$t(\Psi_{n,n+1,W,W}(a \oplus b)) \geq (1 - r_n)t(\imath_{n,n+1}(a \oplus b)). \tag{e7.37}$$

Define $\Psi_{n,n+1,W,A} : R_{0,n} \to M_{((n+1)!)^2}(A(W, \alpha_{n+1}))$ by

$$\Psi_{n,n+1,W,A}(a) = \mathrm{diag}(0, \overbrace{(\varphi_{R,A,\alpha_{n+1}} \circ \imath'_{W,n})(a), \ldots, (\varphi_{R,A,\alpha_{n+1}} \circ \imath'_{W,n})(a)}^{((n+1)!)^2-(n!)^2}).$$

Now if $(a \oplus b) \oplus c \in W_n \oplus A(W, \alpha_n)$ (with $a \in M_{(n!)^2}(R_{0,n})$, $b \in D_n$, and $c \in A(W, \alpha_n)$, define

$$\Phi_{n,n+1}((a \oplus b) \oplus c) = d \oplus c',$$

where

$$d = \widetilde{\iota^{Q}}\big(\Psi_{n,n+1,A,W}(c) \oplus \Psi^0_{n,n+1,W,W}(a \oplus b)\big) \oplus \Psi^1_{n,n+1,W,W}(a \oplus b) \in W_{n+1}$$

($\Psi_{n,n+1,A,W}(c) \in R_{0,n+1} \otimes (e_{r_n} Q e_{r_n})$, $\Psi^0_{n,n+1,W,W}(a \oplus b) \in R_{0,n+1} \otimes (e_{1-r_n} Q e_{1-r_n})$, and $\Psi^1_{n,n+1,W,W}(a \oplus b) \in D_{n+1}$) and

$$c' = \mathrm{diag}(\Psi_{n,n+1,A,A}(c), \Psi_{n,n+1,W,A}(a)) \in M_{((n+1)!)^2}(A(W, \alpha_{n+1})).$$

Since all partial maps of $\Phi_{n,n+1}$ take the strictly positive elements to the strictly positive elements in corresponding corners, $\Phi_{n,n+1}$ itself takes strictly positive elements to strictly positive elements. This also implies that $\Phi^*_{n,n+1}(T(B_{n+1})) \subset T(B_n)$. Note also that $\Phi_{n,n+1}$ maps full elements to full elements and it is injective.

Define

$$B_T = \lim_{n\to\infty}(B_n, \Phi_{n,n+1}).$$

Remark 7.3. In the construction above, C^* algebras A and Q are $\mathcal{Z}$-stable, one can also choose the homomorphism $s: A \to Q$ and $j: Q \to A$ to be of the form $s' \otimes id_{\mathcal{Z}}: A \otimes \mathcal{Z} \to Q \otimes \mathcal{Z}$ and $j' \otimes id_{\mathcal{Z}}: Q \otimes \mathcal{Z} \to A \otimes \mathcal{Z}$ respectively, when one identifies $A \cong A \otimes \mathcal{Z}$ and $Q \cong Q \otimes \mathcal{Z}$. Then $R(\alpha, 1)$, $A(W, \alpha_n)$, W_n, B_n are all $\mathcal{Z}$-stable. One can also make the map $\Phi_{n,n+1}: B_n \otimes \mathcal{Z} \to B_{n+1} \otimes \mathcal{Z}$ to be of form of $\Phi' \otimes \mathrm{id}_{\mathcal{Z}}$. In such a way, we will have that B_T is $\mathcal{Z}$-stable.

By section 4 of [13], one can write $A = \lim_{n\to\infty}(A_n, \varphi_n)$, where each $A_n = M_{k(n)}(C(X_n))$, where each X_n is a finite CW complex with dimension no more than 3. Let $s: A \to Q$ be at the beginning of 7.2. Then, by the proof of 4.7 (and using 2.29) of [13], there exists a sequence of $M_{l(n)} \subset Q$ and homomorphisms $s_n: A_n \to M_{l(n)}$ such that, for each fixed m,

$$\lim_{n\to\infty} s \circ \varphi_{m,\infty}(a) = \lim_{n\to\infty} s_n \circ \varphi_{m,n}(a) \text{ for all } a \in A_m. \tag{e7.38}$$

This also follows from the following. Note $s_{*i}(G_i) = 0$, $i = 0, 1$. Since $K_1(Q) = \{0\}$ and $K_0(Q) = \mathbb{Q}$ which is divisible, by Theorem 3.9 of [23], for each fixed m, there exists a sequence of homomorphisms $\psi_k: A_m \to Q$ such that $\psi_k(A_m)$ has finite dimension and $\lim_{k\to\infty} \psi_k(a) = s \circ \varphi_{m,\infty}(a)$ for all $a \in A_m$. Since finite dimensional C^*-algebras are semiprojective, one also obtains (e7.38). Then for any finite set $\mathcal{F} \subset A(W, \alpha)$ and any $\varepsilon > 0$, there is a C^*-algebra of the form

$$D_n{}' = \Big\{ (f, a) \in C([0,1], M_{l(n)} \otimes M_{l(n)}) \oplus A_n : \ f(0) = s_n(a) \otimes \mathrm{diag}(\underbrace{1, \ldots, 1}_{\alpha l(n)}, 0),$$
$$f(1) = s_n(a) \otimes \mathrm{diag}(\underbrace{1, \ldots, 1}_{l(n)}) \Big\}$$

such that $\mathcal{F} \subset_\varepsilon D_n'$, where $\alpha l(n)$ is an integer. Put $D_n = D_n' \oplus W_n$. Then that D_n is a sub-homogeneous C^*-algebras with 3-dimensional spectrum. Moreover, $D_n \in \overline{\mathcal{D}_2}$ defined in 4.8 of [20].

Hence B_T has the decomposition rank at most three. (In fact, one can prove that B_T is an inductive limit sub-homogeneous C^*-algebras with spectrum having dimension no more than 3, but we do not need this fact.)

Lemma 7.4. *Suppose that $a \in (W_n)_+$. Then, for any integer $k \geq 1$ and any $t \in T(W_{n+k})$,*

$$t(\Psi_{n,n+k,W,W}(a)) \geq (1 - \sum_{j=0}^{k-1} r_{n+j}) t(\imath_{n,n+k}(a)). \tag{e7.39}$$

Proof. Note $\tau \circ \Phi_{n+1,n+2}$ is in $T(W_{n+1})$ for all $\tau \in T(W_{n+2})$. Thus this lemma follows from (e7.37) and induction immediately. □

Lemma 7.5. *Let $n \geq 1$ be an integer. There is a strictly positive element $e_0' \in W_n$ with $\|e_0'\| = 1$ such that $\imath_{n,\infty}(e_0')$ is a strictly positive element. Moreover, for any $a \in (W_n)_+ \setminus \{0\}$, there exist $n_0 \geq n$, $x_1, x_2, \ldots, x_m \in W_{n_0}$ such that*

$$\sum_{i=1}^m x_i^* \imath_{n,n_0}(a) x_i = \imath_{n,n_0}(e_0').$$

Moreover,

$$t(\imath_{n,m}(e_0')) \geq 7/8 \ \textit{for all} \ t \in T(W_m) \ \textit{and} \ \ \textit{for all} \ m \geq n_0,$$

$$\textit{and} \ \ \tau(\imath_{n,\infty}(e_0')) > 15/16 \ \textit{for all} \ \tau \in T(W_T).$$

Proof. To simplify the notation, without loss of generality, we may let $n = 1$. Since W_T is simple, pick a strictly positive element in $e_0' \in (W_1)_+$ with $\|e_0'\| = 1$ so that $e' = \imath_{1,\infty}(e_0')$ is a strictly positive in W_T. By replacing e_0' by $g(e_0')$ for some $g \in C_0((0,1])_+$ we may assume that

$$\tau(e_0') > 15/16 \ \text{for all} \ \tau \in T(W_T).$$

There is an integer $n_0' \geq 1$ such that

$$t(\imath_{1,n}(e_0')) \geq 7/8 \ \text{for all} \ n \geq n_0' \ \text{and} \ t \in T(W_n). \tag{e7.40}$$

Note that this implies that

$$t(\imath_{1,n}(f_\eta(e_0'))) \geq 3/4 \ \text{for all} \ n \geq n_0 \ \text{and} \ t \in T(W_n) \tag{e7.41}$$

whenever $1/16 > \eta > 0$.

Fixed $a \in (W_1)_+ \setminus \{0\}$. Since W_T is simple, there exist $n_0 \geq n_0' \geq 1$ and $x_1', x_2', \ldots, x_{m'}' \in W_{n_0}$ such that

$$\| \sum_{i=1}^{m'} (x_i')^* \imath_{1,n_0}(a) x_i' - \imath_{1,n_0}(e_0') \| < 1/128. \tag{e7.42}$$

It follows from Lemma 2.2 of [51] that there are $y_1', y_2', \ldots, y_m' \in W_{n_0}$ such that

$$\sum_{i=1}^{m'} (y_i')^* \imath_{1,n_0}(a) y_i' = \imath_{1,n_0}(f_\eta(e_0')) \tag{e7.43}$$

for some $1/16 > \eta > 0$. By (e7.41), $\imath_{1,n_0}(f_\eta(e_0'))$ is full in W_{n_0}. Therefore there are $x_1, x_2, \ldots, x_m \in W_{n_0}$ such that

$$\sum_{i=1}^{m} x_i^* \imath_{1,n_0}(a) x_i = \imath_{1,n}(e_0'). \quad \Box \tag{e7.44}$$

Proposition 7.6. *B_T is a simple C^*-algebra.*

Proof. It suffices to show that every element in $(B_T)_+ \setminus \{0\}$ is full in B_T. It suffices to show that every non-zero positive element in $\cup_{n=1}^\infty \Phi_{n,\infty}(B_n)$ is full. Take $b \in \cup_{n=1}^\infty \Phi_{n,\infty}(B_n)$ with $b \geq 0$ and $\|b\| = 1$. To simplify notation , without loss of generality, we may assume that there is $b_0 \in B_1$ such that $\Phi_{1,\infty}(b_0) = b$.

Write $b_0 = b_{00} \oplus b_{0,1}$, where $b_{00} \in (W_1)_+$ and $b_{0,1} \in (A(W, \alpha_1))_+$.

First suppose that $b_{00} \neq 0$.

By applying 7.5, one obtains an integer $n_0 > 1$, $x_1, x_2, \ldots, x_m \in W_{n_0}$ such that

$$\sum_{i=1}^{m} x_i^*(\imath_{1,n_0}(b_{00})) x_i = \imath_{1,n_0}(e_0'). \tag{e7.45}$$

Let $M = \max\{\|x_i\| : 1 \leq i \leq m\}$. The above implies that

$$t(\imath_{1,n_0}(b_{00})) \geq \frac{7}{8mM^2} \quad \text{for all} \quad t \in T(W_{n_0}). \tag{e7.46}$$

Let $P_{W,m} : B_m \to W_m$ and $P_{A,m} : B_m \to M_{(m!)}(A(W, \alpha_m))$ be the projections ($m \geq 1$). Then, by 7.4,

$$t(P_{W,n_0}(\Phi_{1,n_0}(b_{00}))) \geq t(\Psi_{1,n_0,W,W}(b_{00})) \tag{e7.47}$$

$$\geq (1 - \sum_{j=0}^{n_0-1} r_{1+j}) t(\imath_{1,n_0}(b_{00})) \tag{e7.48}$$

$$\geq (1 - \sum_{j=0}^{n_0-1} r_{1+j})(\frac{7}{8mM^2}) \quad \text{for all} \quad t \in T(W_{n_0}). \tag{e7.49}$$

It follows that $P_{W,n_0}(\Phi_{1,n_0}(b_{00}))$ is full in W_{n_0}. Put $b_{00}' = P_{W,n_0}(\Phi_{1,n_0}(b_{00}))$. By applying 7.4 again, one concludes that $P_{W,n_0+1} \circ \Phi_{n_0,n_0+1}(b_{00}')$ is full in W_{n_0+1}.

Since b_{00}' is full in W_{n_0}, $P_{0,n_0}(b_0')$ is full in $R_{0,n_0} = R(\alpha_{n_0}, 1)$. Since $\varphi_{R,A,\alpha_{n+1}} \circ \imath_{W,n}'$ maps full elements of $R_{\alpha_{n_0},1}$ to full elements in $A(W, \alpha_{n_0+1})$, $P_{A,n_0+1} \circ \Phi_{n_0,n_0+1}(b_{00}')$ is full in $M_{(n+1)!}(A(W, \alpha_{n_0+1}))$. It follows that $\Phi_{n_0,n_0+1}(b_{00}')$ is full in B_{n_0+1}.

Note that what has been proved: for any $b' \in (W_n)_+ \setminus \{0\}$, there is $m_0 \geq 1$ such that $\Phi_{n,m_0}(b')$ is full in B_{m_0}. Therefore $\Phi_{n,m}(b')$ is full in B_m for all $m \geq m_0$.

In particular, this shows that $\Phi_{n,\infty}(b_{00})$ is full. Therefore $b \geq \Phi_{n,\infty}(b_{00})$ is full.

Now consider the case that $b_{00} = 0$. Then $b_{1,0} \neq 0$. Since $\Psi_{1,2,A,W}$ is injective, $P_{W,1}(\Phi_{1,2}(b_{1,0})) \neq 0$. Applying what has been proved, $\Phi_{2,\infty}(P_{W,1}(\Phi_{1,2}(b_{1,0})))$ is full in B_T. But

$$\Phi_{1,\infty}(b_{1,0}) \geq \Phi_{2,\infty}(P_{W,1}(\Phi_{1,2}(b_{1,0}))).$$

This shows that, in all cases, b is full in B_T. Therefore B_T is simple. $\Box$

Proposition 7.7. *$B_T \in \mathcal{D}_0$ and $T(B_T) = T$. In particular, B_T has continuous scale. Moreover B_T is locally approximated by sub-homogeneous C^*-algebras with spectrum having dimension no more than 3, has finite nuclear dimension, $\mathcal{Z}$-stable and has stable rank one.*

Proof. Let us first show that $T(B_T) = T$. Recall $\tilde{T}(A)$ is the set of all lower semi-continuous traces on A and $T(A)$ is the set of tracial states on A. In the rest of the proof, for all C^* algebras $A = B_n$ and $A = W_n$, we have that $0 < \alpha_n \leq \inf\{d_\tau(a) : \tau \in \overline{T(A)}^w\}$, and that all traces $\tau \in \tilde{T}(A)$ are bounded trace.

Note the homomorphisms $\Phi_{n,n+1} : B_n \to B_{n+1}$ and $\imath_{n,n+1} : W_n \to W_{n+1}$ induce maps $\Phi_{n,n+1}^* : \tilde{T}(B_{n+1}) \to \tilde{T}(B_n)$ and $\imath_{n,n+1}^* : \tilde{T}(W_{n+1}) \to \tilde{T}(W_n)$. From the construction above, (see also [25]), since $\Phi_{n,n+1}$ and $\imath_{n,n+1}$ map strictly positive elements to strictly positive elements, $\Phi_{n,n+1}^*$ and $\imath_{n,n+1}^*$ take tracial states to tracial states. That is, $\Phi_{n,n+1}^* : T(B_{n+1}) \subset T(B_n)$ and $\imath_{n,n+1}^* : T(W_{n+1}) \subset T(W_n)$. Consequently for any $\tau \in \tilde{T}(B_{n+1})$ (or $\tau \in \tilde{T}(W_{n+1})$), we have $\|\Phi_{n,n+1}^*(\tau)\| = \|\tau\|$ (or $\|\imath_{n,n+1}^*(\tau)\| = \|\tau\|$).

Hence we have the following inverse limit systems of compact convex spaces:

$$\overline{T(B_1)}^w \xleftarrow{\Phi^*_{1,2}} \overline{T(B_2)}^w \xleftarrow{\Phi^*_{2,3}} \overline{T(B_3)}^w \longleftarrow \cdots\cdots \longleftarrow \varprojlim \overline{T(B_n)}^w \quad ,$$

$$\overline{T(W_1)}^w \xleftarrow{\iota^*_{1,2}} \overline{T(W_2)}^w \xleftarrow{\iota^*_{2,3}} \overline{T(W_3)}^w \longleftarrow \cdots\cdots \longleftarrow \varprojlim \overline{T(W_n)}^w \quad .$$

Here we write that

$$\varprojlim \overline{T(B_n)}^w = \{(\tau_1, \tau_2, \ldots, \tau_n, \ldots) \in \Pi_n \overline{T(B_n)}^w : \Phi^*_{n,m}(\tau_m) = \tau_n\},$$

which is a subspace of the product space $\Pi_n \overline{T(B_n)}^w$ with product topology. On the other hand, since all the map $\Phi^*_{n,m}$ are affine map, $\lim_{\leftarrow} \overline{T(B_n)}^w$ has a natural affine structure defined by

$$t(\tau_1, \tau_2, \ldots, \tau_n, \ldots) + (1-t)(\tau'_1, \tau'_2, \ldots, \tau'_n, \ldots) = (t\tau_1 + (1-t)\tau'_1, \tau_2 + (1-t)\tau'_2, \ldots, \tau_n + (1-t)\tau'_n),$$

for any $(\tau_1, \tau_2, \ldots, \tau_n, \ldots), (\tau'_1, \tau'_2, \ldots, \tau'_n, \ldots) \in \lim_{\leftarrow} \overline{T(B_n)}^w$ and $t \in (0, 1)$.

Note that each element in $\lim_{\leftarrow} \overline{T(B_n)}^w$ is given by $(\tau_1, \tau_2, \ldots \tau_n, \ldots,)$ with $\Phi^*_{n,m}(\tau_m) = \tau_n$, for $m > n$. This element decides a unique element $\tau \in \tilde{T}(B)$ defined by $\tau|_{B_n} = \tau_n$. However, since $\|\tau_n\| \geq \alpha_n$ and $\lim_n \alpha_n = 1$, $\tau \in T(B_T)$. On the other hand, each element $\tau \in T(B_T)$ defines a sequence $\{\tau_n = \tau|_{B_n} \in \tilde{T}(B_n)\}_n$. Since $\cup_n B_n$ is dense in B, $\|\tau\| = \lim_{n\to\infty} \|\tau_n\|$. From $\|\Phi^*_{n,n+1}(\tau')\| = \|\tau'\|$ for any $\tau' \in \tilde{T}(B_{n+1})$, we know that $\|\tau_n\| = \|\tau_{n+1}\|$. Consequently $\|\tau_n\| = \|\tau\| = 1$ for all n.

Hence $\tau_n \in T(B_n) \subset \overline{T(B_n)}^w$. Consequently, $T(B_T) = \lim_{\leftarrow} \overline{T(B_n)}^w$. Similarly, $T(W_T) = \lim_{\leftarrow} \overline{T(W_n)}^w$. (Note that the map $T(B_T) \to \overline{T(B_n)}^w$ from the reverse limit is the same as $\Phi^*_{n,\infty} : T(B_T) \to \overline{T(B_n)}^w$, the restrict map. That is, $\tau \in T(B_T)$ corresponds to the sequence

$$(\Phi^*_{1,\infty}(\tau), \Phi^*_{2,\infty}(\tau), \ldots, \Phi^*_{n,\infty}(\tau), \ldots,) = (\tau|_{B_1}, \tau|_{B_2}, \ldots, \tau|_{B_n}, \ldots).$$

In other words, the homeomorphism between $T(B_T)$ and $\lim_{\leftarrow} \overline{T(B_n)}^w$ also preserve the convex structure.)

Similarly, we also have the following inverse limit systems of the topological cones:

$$\tilde{T}(B_1) \xleftarrow{\Phi^*_{1,2}} \tilde{T}(B_2) \xleftarrow{\Phi^*_{2,3}} \tilde{T}(B_3) \longleftarrow \cdots\cdots \longleftarrow \tilde{T}(B_T) \quad ,$$

$$\tilde{T}(W_1) \xleftarrow{\iota^*_{1,2}} \tilde{T}(W_2) \xleftarrow{\iota^*_{2,3}} \tilde{T}(W_3) \longleftarrow \cdots\cdots \longleftarrow \tilde{T}(W_T) \quad .$$

(Again, the reverse limit is taking in the category of topological space in weak* topology, but it automatically preserves cone structure)

Let $\pi_n : B_n = W_n \oplus M_{(n!)^2}(A(W, \alpha_n)) \to W_n$ be the projection and let $\tilde{\Phi}_{n,n+1} = \Phi_{n,n+1}|_{W_n}$, then we have the following (not commutative) diagram:

$$\begin{tikzcd}
B_1 \arrow[r, "\Phi_{1,2}"] \arrow[d, "\pi_1"'] & B_2 \arrow[r, "\Phi_{2,3}"] \arrow[d, "\pi_2"'] & B_3 \arrow[r, "\Phi_{3,4}"] \arrow[d, "\pi_3"'] & {} \\
W_1 \arrow[r, "\iota_{1,2}"] \arrow[ur, "\tilde{\Phi}_{1,2}"] & W_2 \arrow[r, "\iota_{2,3}"] \arrow[ur, "\tilde{\Phi}_{2,3}"] & W_3 \arrow[r, "\iota_{3,4}"] & {} .
\end{tikzcd}$$

Even though the diagram is not commutative, from the construction, it induces an approximate commuting diagram

$$\begin{tikzcd}
\tilde{T}(B_1) & \tilde{T}(B_2) \arrow[l, "\Phi^*_{1,2}"'] \arrow[dl, "\tilde{\Phi}^*_{1,2}"'] & \tilde{T}(B_3) \arrow[l, "\Phi^*_{2,3}"'] \arrow[dl, "\tilde{\Phi}^*_{2,3}"'] & \cdots\cdots \arrow[l] & \tilde{T}(B_T) \arrow[l] \\
\tilde{T}(W_1) \arrow[u, "\pi^*_1"] & \tilde{T}(W_2) \arrow[l, "\iota^*_{1,2}"] \arrow[u, "\pi^*_2"] & \tilde{T}(W_3) \arrow[l, "\iota^*_{2,3}"] \arrow[u, "\pi^*_3"] & \cdots\cdots \arrow[l] & \tilde{T}(W_T) \arrow[l] .
\end{tikzcd}$$

That is

$$|\big(\tilde{\Phi}^*_{n,n+1}(\pi^*_{n+1}(\tau))\big)(g) - \big(\iota^*_{n,n+1}(\tau)\big)(g)| \leq k(n) r_n \|g\| \|\tau\| \quad \text{for all } g \in W_n,\ \tau \in \tilde{T}(W_{n+1}); \text{ and}$$

$$|\big(\pi^*_n(\tilde{\Phi}^*_{n,n+1}(\tau))\big)(f) - \big(\Phi^*_{n,n+1}(\tau)\big)(f)| \leq \Big(\frac{1}{(n+1)^2} + k(n) r_n\Big) \|f\| \|\tau\| \quad \text{for all } f \in B_n,\ \tau \in \tilde{T}(B_{n+1}).$$

(Note that $k(n) r_n = \frac{1}{2^{n+1}}$.)

Note that from the above, for $\tau_{n+1} \in \tilde{T}(W_{n+1})$ if $\tau_n = \iota^*_{n,n+1}(\tau_{n+1})$, then

$$\|\pi^*_{n+1}(\tau_{n+1})\| \ge (1 - \frac{1}{2^{n+1}})\|\tau_n\| \tag{e7.50}$$

So, we have the following fact:
if $(\tau_1, \tau_2, \ldots, \tau_n, \ldots,) \in \Pi_n \tilde{T}(W_{n+1})$ satisfies $\tau_n = \iota^*_{n,n+1}(\tau_{n+1})$, then

$$\lim_{n\to\infty} \|\tau_n\| = \lim_{n\to\infty} \|\pi^*_n(\tau_n)\|.$$

The approximate intertwining induces an affine homeomorphisms $\Pi : \tilde{T}(W_T) \to \tilde{T}(B_T)$ as follows.

For each $\tau \in \tilde{T}(W_T)$, for fixed n, we define a sequence of $\{\sigma_{n,m}\}_{m>n} \subset \tilde{T}(B_n)$ by

$$\sigma_{n,m} = \big(\Phi^*_{n,m} \circ \pi^*_m \circ \iota^*_{m,\infty}\big)(\tau) \in \tilde{T}(B_n).$$

Recall that each element in $\tilde{T}(B_n)$ is a bounded trace, whence it is a positive linear functional of B_n. From the above inequalities for approximately commuting diagram, one concludes that $\{\sigma_{n,m}\}_{m>n}$ is a Cauchy sequence (in norm) in the dual space of B_n.

For each n, let $\tau_n = \lim_{m\to\infty} \sigma_{n,m}$. Evidently, from the inductive system above, $\tilde{\Phi}^*_{n,n+1}(\tau_{n+1}) = \tau_n$. Hence the sequence $(\tau_1, \tau_2, \ldots, \tau_n, \ldots)$ determines an element $\tau' \in \tilde{T}(B_T)$. Let $\Pi(\tau) = \tau'$. From (e7.50) and the above mentioned fact, we know that Π preserves the norm and Π maps $T(W_T)$ to $T(B_T)$ Moreover, it is clear that Π is also an affine map on $T(W_T)$.

We can define $\Pi' : \tilde{T}(B_T) \to \tilde{T}(W_T)$ in exactly same way by replacing $\Phi^*_{n,m}$ by $\iota^*_{n,m}$, replacing π^*_m by $\tilde{\Phi}^*_{m,m+1}$, and $\iota^*_{m,\infty}$ by $\Phi^*_{m+1,\infty}$.

We now show that both Π and Π' are continuous on $T(W_T)$ and $T(B_T)$, respectively. Let $\{s_\lambda\} \subset T(W_T)$ be a net which converges to $s \in T(W_T)$ point-wisely on W_T. Write $s_\lambda = (s_{\lambda,1}, s_{\lambda,2}, \ldots, s_{\lambda,n}, \ldots)$ and $s = (s_1, s_2, \ldots, s_n, \ldots)$. Since $s_{\lambda,n} = \iota^*_{n,n+1}(s_{\lambda,n+1})$ and $s_n = \iota^*_{n,n+1}(s_{n+1})$, for each n, $s_{\lambda,n}$ converges to s_n on W_n. Write $\Pi(s_\lambda) = (\tau_{\lambda,1}, \tau_{\lambda,2}, \ldots, \tau_{\lambda,n}, \ldots)$ and $\Pi(s) = (\tau_1, \tau_2, \ldots, \tau_n, \ldots)$. Then, by the definition,

$$\tau_{\lambda,n} = \lim_{m\to\infty} \sigma_{\lambda,n,m} = \lim_{m\to\infty} \big(\Phi^*_{n,m} \circ \pi^*_m \circ \iota^*_{m,\infty}\big)(s_\lambda) \tag{e7.51}$$
$$= \lim_{m\to\infty} \big(\Phi^*_{n,m} \circ \pi^*_m\big)(s_{\lambda,m}) \text{ and} \tag{e7.52}$$
$$\tau_n = \lim_{m\to\infty} \sigma_{n,m} = \lim_{m\to\infty} \big(\Phi^*_{n,m} \circ \pi^*_m \circ \iota^*_{m,\infty}\big)(s) \tag{e7.53}$$
$$= \lim_{m\to\infty} \big(\Phi^*_{n,m} \circ \pi^*_m\big)(s_m). \tag{e7.54}$$

For $b \in B_n$ and $m > n$,

$$\big(\Phi^*_{n,m} \circ \pi^*_m\big)(s_{\lambda,m})(b) = s_{\lambda,m}(\pi_m \circ \Phi_{n,m}(b)) \text{ and} \tag{e7.55}$$
$$\big(\Phi^*_{n,m} \circ \pi^*_m\big)(s_m)(b) = s_m(\pi_m \circ \Phi_{n,m}(b)). \tag{e7.56}$$

Let $\varepsilon > 0$ and let $\mathcal{F} \subset B_n$ be a finite subset. We may assume that $\mathcal{F}$ is in the unit ball of B_n.

There exists $m_0 \ge 1$ such that, for all $m \ge m_0$,

$$|s_{\lambda,n}(\pi_m \circ \Phi_{n,m}(b)) - \tau_{\lambda,n}(b)| < \varepsilon/3 \text{ and} \tag{e7.57}$$
$$|s_n(\pi_m \circ \Phi_{n,m}(b)) - \tau_n(b)| < \varepsilon/3 \tag{e7.58}$$

for all b in the unit ball of B_n.

Since $s_{\lambda,n} \to s_n$ on B_n point-wisely, There exists λ_0 such that, for all $\lambda > \lambda_0$,

$$|s_{\lambda,n}(\pi_{m_0} \circ \Phi_{n,m_0}(b)) - s_n(\pi_{m_0} \circ \Phi_{n,m_0}(b))| < \varepsilon/3 \tag{e7.59}$$

for all $b \in \mathcal{F}$. It follows that, when $\lambda > \lambda_0$, for all $b \in \mathcal{F}$,

$$|\tau_{\lambda,n}(b) - \tau_n(b)| \le |\tau_{\lambda,n}(b) - s_{\lambda,n}(\pi_{m_0} \circ \Phi_{n,m_0}(b))| \tag{e7.60}$$
$$+ |s_{\lambda,n}(\pi_{m_0} \circ \Phi_{n,m_0}(b)) - s_n(\pi_{m_0} \circ \Phi_{n,m_0}(b))| \tag{e7.61}$$
$$+ |s_n(\pi_{m_0} \circ \Phi_{n,m_0}(b)) - \tau_n(b)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \tag{e7.62}$$

This verifies that $\Pi(s_\lambda)$ converges to $\Pi(s)$ on B_n for each n. Since $\cup_{n=1} B_n$ is dense in B_T, it is easy to see that $\Pi(s_\lambda)$ converges to $\Pi(s)$ point-wisely. It follows that Π is weak*-continuous on $T(W_T)$. A similar argument verifies that Π' is weak*-continuous on $T(B_T)$. From the definition, one can also verify that Π and Π' are inverse each other. Consequently, they induce the homeomorphism between $T(W_T)$ and $T(B_T)$. Hence $T(B_T) = T(W_T) = T$.

From Remark 7.3, we know that B_T is locally approximated by sub-homogeneous C^*-algebras with spectrum having dimension no more than 3, has finite nuclear dimension and $\mathcal{Z}$-stable. It follows from a theorem of Rørdam (see 3.5 of [15]) that B_T has strictly comparison for positive elements. Since T is compact, it follows from 5.3 of [15] that B_T has continuous scale.

It remains to show that $B_T \in \mathcal{D}_0$. Since B_T has continuous scale, to prove $B_T \in \mathcal{D}_0$. let $a \in A_+$ be a strictly positive element with $\|a\| = 1$. Without loss of generality, we may assume that $\tau(f_{1/2}(a)) \geq 15/16$ for all $\tau \in T(B_T)$. We choose a such that $a = (a_a, a_w) \in B_1 = A(W, \alpha_1) \oplus W_1$ such that

$$t(a_w) > 3/4, \quad t(f_{1/2}(a_w)) > 3/4 \text{ for all } t \in T(W_1). \tag{e7.63}$$

Choose $\mathfrak{f}_a = 5/16$. Let $b \in A_+ \setminus \{0\}$ and let $\mathcal{F} \subset B_T$ be a finite set and $\varepsilon > 0$. Let $\delta > 0$. With out lose of generality, we may assume $F \cup \{a, b\} \subset B_n$ for n large enough, and let $\Lambda : B_T \to B_n$ be a completely positive contractive linear map such that

$$\|\Lambda(b) - b\| < \min\{\varepsilon/2, \delta\} \text{ for all } b \in \mathcal{F}. \tag{e7.64}$$

We choose δ so small that

$$\|f_{1/2}(\Lambda(a)) - \Lambda(f_{1/2}(a))\| < 1/16 \text{ and } \|f_{1/2}(\Lambda(a)) - \Lambda(f_{1/2}(a))\| < 1/16. \tag{e7.65}$$

Let $P_A : B_n \to M_{(n!)^2}(A(W, \alpha_n))$ and $P_W : B_n \to W_n$ be the canonical projections. We choose $n \geq 1$ such that

$$\frac{1}{(n+1)^2} < \inf\{\tau(b) : \tau \in T(B_T)\}/2. \tag{e7.66}$$

We will choose the algebra $D \in \mathcal{C}_0^0$ to be $D = \Psi_{n,n+1,W,A}(W_n) \oplus W_{n+1}$ and the map $\varphi : B_T \to B_T$ and $\psi : B_T \to D$ be defined by

$$\varphi = \Phi_{n+1,\infty} \circ \Psi_{n,n+1,A,A} \circ P_A \circ \Lambda \qquad \text{and}$$

$$\psi = \Psi_{n,n+1,W,A} \circ P_W \circ \Lambda \oplus \mathrm{diag}(\Psi_{n,n+1,A,W} \circ P_A \circ \Lambda, \Psi_{n,n+1,W,W} \circ P_W \circ \Lambda).$$

Put

$$\psi' = \Psi_{n,n+1,W,A} \circ P_W \oplus \mathrm{diag}(\Psi_{n,n+1,A,W} \circ P_A, \Psi_{n,n+1,W,W} \circ P_W) \tag{e7.67}$$

from B_n to D. Since $\Psi_{n,n+1,W,A}$ is injective on W_n, $D \in C_0^0$. Since $\Phi_{n,\infty}$ is injective, we will identify D with $\Phi_{n,\infty}(D)$. With this identification, we have

$$\|x - \mathrm{diag}(\varphi(x), \psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F}. \tag{e7.68}$$

It follows from 7.4 that

$$P_W(\Phi_{1,n}(f_{1/2}(a))) \geq \Phi_{1,n}(a_W) \text{ and} \tag{e7.69}$$

$$t(P_W(\Phi_{1.n}(f_{1/2}(a)))) \geq t(\Phi_{1,n}(f1/2(a_W))) \geq (1 - \sum_{j=0}^{n-1} r_{1+j})t(\imath_{1,n}(f_{1/2}(a_W))) \tag{e7.70}$$

for all $t \in T(W_n)$. Since $t \circ \imath_{1,n}$ is a tracial state on W_1 as proved above, by (e7.63),

$$t(P_W(\Phi_{1,n}(f_{1/2}(a)))) \geq (1/2)(3/4) = 3/8 \text{ for all } t \in T(W_n). \tag{e7.71}$$

Since $\Psi_{n,n+1,W,A}$ sends strictly positive elements of W_n to those of $\Psi_{n,n+1,W,A}(W_n)$, any $t' \in T(\Psi_{n,n+1,W,A}(W_n))$ gives a tracial state of W_n, therefore

$$t'(\Psi_{n,n+1,W,A}(P_W(\Phi_{1,n}(f_{1/2}(a))))) \geq 3/8 \text{ for all } \tau' \in T(\Psi_{n,n+1,W,A}(W_n)). \tag{e7.72}$$

For any $t \in T(W_{n+1})$, by applying (e7.63) again,

$$t(\Psi_{n,n+1,W,W}(P_W(f))) \geq (1 - \sum_{j=0}^{n} r_{1+j})t(\imath_{1,n+1}(f_{1/2}(a_W))) \geq (1/2)(3/8) = 3/8. \tag{e7.73}$$

Combining (e7.72) and (e7.73), we have that

$$t(\psi'(\Phi_{1,n}(f_{1/2}(a)))) \geq t(\psi'(P_W(\Phi_{1,n}(f_{1/2}(a))))) \geq 3/8 \text{ for all } t \in T(D). \tag{e7.74}$$

It follows that, for all $t \in T(D)$,

$$t(f_{1/2}(\psi(a))) \geq t(\psi'(\Phi_{1,n}(f_{1/2}(a)))) \geq t(\psi'(P_W(\Phi_{1,n}(f_{1/2}(a))))) - 1/16 \geq 5/16 = \mathfrak{f}_a. \tag{e7.75}$$

On the other hand, from the construction, for any $c \in \Psi_{n,n+1,A,A}(M_{(n!)^2}(A(W, \alpha_n)))_+$ with $\|c\| \leq 1$,

$$\tau(c) \leq \frac{1}{(n+1)^2} \text{ for all } \tau \in T(M_{((n+1)!)^2}(A(W, \alpha_{n+1}))). \tag{e7.76}$$

Therefore, for any integer $m \geq 1$,

$$\tau(\varphi(a)^{1/m}) < \frac{1}{(n+1)^2} \text{ for all } \tau \in T(B_T). \tag{e7.77}$$

Consequently, by (e7.66),

$$d_\tau(\varphi(a)) \le \frac{1}{(n+1)^2} < \inf\{d_\tau(b) : \tau \in T(B_T)\} \tag{e7.78}$$

Since we have proved that B_T has strict comparison for positive elements, (e7.78) implies that

$$\varphi(a) \lesssim b. \tag{e7.79}$$

It follows from 3.9, (e7.68), (e7.79) and (e7.75) that $B_T \in \mathcal{D}_0$. Since $B_T \in \mathcal{D}_0$, it follows from 11.5 of [15] that B_T has stable rank one. This completes the proof of this proposition. $\square$

Proposition 7.8. *$K_0(B_T) = \ker \rho_{B_T} = G_0$ and $K_1(B_T) = G_1$.*

Proof. Let $I = C_0\big((0, 1), Q \otimes Q\big)$ be the canonical ideal of $A(W, \alpha_n)$. Then the short exact sequence

$$0 \to I \to A(W, \alpha_n) \to A \to 0$$

induces six term exact sequence

$$\begin{tikzcd} K_0(I) \arrow[r] & K_0(A(W,\alpha_n)) \arrow[r] & K_0(A) \arrow[d, "\partial"] \\ K_1(A) \arrow[u] & K_1(A(W,\alpha_n)) \arrow[l] & K_1(I). \arrow[l] \end{tikzcd}$$

Note that $K_0(I) = \{0\}$ and $K_1(I) = K_0(Q) = \mathbb{Q}$. Moreover, $K_0(A) = \mathbb{Q} \oplus G_0$ and $K_1(A) = G_1$. The map $\partial : K_0(A) \to K_1(I) \cong K_0(Q \otimes Q)$ is given by $\partial = (1 - \alpha_n)s_{*0}$ (defined by $\partial(x) = (1 - \alpha_n)s_{*0}(x) \in \mathbb{Q}$ for all $x \in K_0(A)$) as the difference of two induced homomorphisms at the end points (recall that $s_{*0}(r, g) = r$ for all $(r, g) \in \mathbb{Q} \oplus G_0$, see the beginning of 7.2). Then one checks $(1 - \alpha_n)s_{*0}$ is surjective as $K_1(I) = \mathbb{Q}$. From the six-term exact sequence above, one computes that $K_0(A(W, \alpha_n)) = \ker \partial = \ker s_{*0} = G_0 = \ker \rho_A$ and $K_1(A(W, \alpha_n)) \cong K_1(A) = G_1$. We also note that ∂ gives an isomorphism on $\mathbb{Q}$. Recall $B_n = W_n \oplus M_{(n!)^2}(A(W, \alpha_n))$. Since $K_*(W) = \{0\}$, one has

$$K_0(B_n) = \ker \rho_{B_n} = \ker \rho_{M_{(n!)^2}(A(W,\alpha_n))} = \ker \rho_A = G_0, \quad \text{and} \tag{e7.80}$$

$$K_1(B_n) = K_1(M_{(n!)^2}(A(W, \alpha_n))) = K_1(A) = G_1. \tag{e7.81}$$

Hence $\Phi_{n,n+1,*} : K_*(B_n) \to K_*(B_{n+1})$ is completely decided by its partial map $\Phi' : M_{(n!)^2}(A(W, \alpha_n)) \to M_{((n+1)!)^2}(A(W, \alpha_{n+1}))$. Also this partial map sends $(f, a) \in M_{(n!)^2}(A(W, \alpha_n))$ to $(g, \mathrm{diag}(a, 0, \dots, 0)) \in M_{((n+1)!)^2}(A(W, \alpha_{n+1}))$, where $g = P_f(\Phi_{A_w,\alpha_n,\alpha_{n+1}}((f, a)))$ is as in Definition 7.2. Therefore Φ' maps $M_{(n!)^2}(I)$ to $M_{((n+1)!)^2}(I)$ and it induces a homomorphism $\Phi'' : M_{(n!)^2}(A) \to M_{((n+1)!)^2}(A)$ which is given by $a \mapsto \mathrm{diag}(a, 0, \dots, 0)$ for all $a \in M_{(n!)^2}(A)$. The latter map induces the identity map id on $K_i(A)$, $i = 0, 1$. Thus we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & K_0(A(W,\alpha_n)) \arrow[rr] \arrow[d, "\Phi'_{*0}"] & & G_0 \oplus \mathbb{Q} \arrow[dl, "\mathrm{id}"'] \arrow[ddd, "\partial"] \\ & 0 \arrow[r] & K_0(A(W,\alpha_{n+1})) \arrow[r] & G_0 \oplus \mathbb{Q} \arrow[d, "\partial"] & \\ & G_1 \arrow[u] & K_1(A(W,\alpha_{n+1})) \arrow[l] & \mathbb{Q} \arrow[l, "0"'] & \\ G_1 \arrow[uuu] \arrow[ur, "\mathrm{id}"] & & K_1(A(W,\alpha_n)) \arrow[ll, "\cong"'] \arrow[u, "\Phi'_{*1}"] & & \mathbb{Q}. \arrow[ll, "0"'] \arrow[ul, "\mathrm{id}"'] \end{tikzcd}$$

This commutative diagram shows that Φ'_{*0} is the identity map on G_0 and Φ'_{*1} is the identity map on G_1. Since $K_i(W) = \{0\}$, this shows that $(\Phi_{n,n+1})_i : K_i(B_n) \to K_i(B_{n+1})$ $(i = 0, 1)$ is the identity map for each n. It follows that $K_0(B_T) = G_0$ and $K_1(B_T) = G_1$. $\square$

Lemma 7.9. *Let G_0 be a torsion free abelian group and let A be the unital AF algebra with*

$$(K_0(A), K_0(A)_+, [1_A]) = \big(\mathbb{Q} \oplus G_0, (\mathbb{Q} \oplus G_0)_+, (1, 0)\big),$$

where $(\mathbb{Q} \oplus G_0)_+ = \{(r, g) : r \in \mathbb{Q}_+ \setminus \{0\}, g \in G_0\} \cup \{(0, 0)\}$. Let $\gamma : K_0(A) \to K_0(Q)$ be given by sending $(r, x) \in \mathbb{Q} \oplus G_0$ to $r \in \mathbb{Q} = K_0(Q)$. Then one can write AF inductive limits $A = \lim_n(A_n, \varphi_{n,m})$ with injective $\varphi_{n,m}$ and $Q = \lim_n(M_{l(n)}(\mathbb{C}), \psi_{n,m})$

such that there are injective homomorphisms $s_n : A_n \to M_{l(n)}(\mathbb{C})$ satisfying the following conditions:

(1) $(s_n)_* : K_0(A_n) \to K_0(M_{l(n)}(\mathbb{C}))$ is surjective;

(2) $s_{n+1} \circ \varphi_{n,n+1} = \psi_{n,n+1} \circ s_n$ and the commutative diagram

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\varphi_{1,2}} & A_2 & \xrightarrow{\varphi_{2,3}} & A_3 & \xrightarrow{\varphi_{3,4}} \cdots\cdots & A \\ \downarrow s_1 & & \downarrow s_2 & & \downarrow s_3 & & \\ M_{l(1)} & \xrightarrow{\psi_{1,2}} & M_{l(2)} & \xrightarrow{\psi_{2,3}} & M_{l(3)} & \xrightarrow{\psi_{3,4}} \cdots\cdots & Q \end{array}$$

induces $s : A \to Q$ satisfy $s_* = \gamma$.

Proof. By the classification theory of AF algebras due to Elliott, there is a one-sided intertwining

$$\begin{array}{ccccccc} F_1 & \xrightarrow{\varphi'_{1,2}} & F_2 & \xrightarrow{\varphi'_{2,3}} & F_3 & \xrightarrow{\varphi'_{3,4}} \cdots\cdots & A \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \\ M_{m(1)} & \xrightarrow{\psi'_{1,2}} & M_{m(2)} & \xrightarrow{\psi'_{2,3}} & M_{m(3)} & \xrightarrow{\psi'_{3,4}} \cdots\cdots & Q, \end{array}$$

which induces a homomorphism $\alpha : A \to Q$ with $\alpha_* = \gamma$, where F_n are finite dimensional C^*-algebras, all homomorphisms α_n, $\varphi'_{n,n+1}$ and $\psi'_{n,n+1}$ are unital and injective. We need to modify the diagram to make the condition (1) holds.

We will define subsequence F_{k_n} and for each n construct a matrix algebra $M_{l(n)}$, unital injective homomorphisms $s_n : F_{k_n} \to M_{l(n)}$, $\xi_n : M_{l(n)} \to M_{m(k_n)}$ and $\beta_{n-1} : M_{m(k_{n-1})} \to M_{l(n)}$ (if $n > 1$) to satisfy the following conditions:

(i): $(s_n)_* : K_0(F_{k_n}) \to K_0(M_{l(n)})$ is surjective;

(ii): $\xi_n \circ s_n = \alpha_{k_n}$ and $\beta_{n-1} \circ \alpha_{k_{n-1}} = s_n \circ \varphi'_{k_{n-1},k_n}$.

Let $k_1 = 1$. By identifying $K_0(M_{m(k_1)})$ with $\mathbb{Z}$, there is a positive integer $j|m(k_1)$ such that $(\alpha_{k_1})_*(K_0(F_{k_1})) = j \cdot \mathbb{Z}$. Let $l(1) = \frac{m(k_1)}{j}$. Choose a homomorphism $s_1 : F_{k_1} \to M_{l(1)}$ to satisfy that $(s_1)_* = \frac{(\alpha_{k_1})_*}{j} : K_0(F_{k_1}) \to K_0(M_{l(1)}) = Z$ (which is surjective). Note that for any finite dimensional C^* algebra F and a matrix algebra M_k, a homomorphism $\beta : F \to M_k$ is injective if and only if $\beta_*(K_0(F)_+ \setminus \{0\}) \subset K_0(M_k)_+ \setminus \{0\}$. Hence the injectivity of α_{k_1} implies the injectivity of s_1. Let $\xi'_1 : M_{l(n)} \to M_{m(k_n)}$ be any unital embedding. Then $(\xi'_1 \circ s_1)_* = (\alpha_{k_1})_*$. There is a unitary $u \in M_{m(k_1)}$ such that $\mathrm{Ad}u \circ \xi'_1 \circ s_1 = \alpha_{k_1}$. Define $\xi_1 = \mathrm{Ad}u \circ \xi'_1$ to finish the initial step $n = 1$ for the induction.

Suppose that we have already carried out the construction until step n. There is a k_{n+1} such that

$$(\psi'_{k_n,k_{n+1}})_*(K_0(M_{m(k_n)})) \subset (\alpha_{k_{n+1}})_*(K_0(F_{k_{n+1}})) \subset K_0(M_{m(k_{n+1})}).$$

Again, there is a positive integer $j|m(k_{n+1})$ such that

$$(\alpha_{k_{n+1}})_*(K_0(F_{k_{n+1}})) = j \cdot \mathbb{Z} \subset \mathbb{Z}(= K_0(M_{m(k_{n+1})})).$$

Let $l(n+1) = \frac{m(k_{n+1})}{j}$. As what we have done in the case for $k_n = k_1$, there are two injective unital homomorphisms $s_{n+1} : F_{k_{n+1}} \to M_{l(n+1)}$ and $\xi_{n+1} : M_{l(n+1)} \to M_{m(k_{n+1})}$ such that $\xi_{n+1} \circ s_{n+1} = \alpha_{k_{n+1}}$. Note that ξ_{n+1} has to be injective as $M_{l(n+1)}$ is simple. Since the map $(\psi'_{k_n,k_{n+1}})_* : K_0(M_{m(k_n)}) \to K_0(M_{m(k_{n+1})})$ factors through $K_0(M_{l(n+1)})$ by $(\xi_{n+1})_*$, one can find a homomorphism $\beta'_n : M_{m(k_n)} \to M_{l(n+1)}$ such that $(\xi_{n+1})_* \circ (\beta'_n)_* = (\psi'_{k_n,k_{n+1}})_*$. Since $(\xi_{n+1})_*$ is injective, we know that $(\beta'_n \circ \alpha_{k_n})_* = (s_{n+1} \circ \varphi'_{k_n,k_{n+1}})_*$. Hence we can choose a unitary $u \in M_{l(n+1)}$ such that $\mathrm{Ad}u \circ \beta'_n \circ \alpha_{k_n} = s_{n+1} \circ \varphi'_{k_n,k_{n+1}}$. In particular, β'_n is injective. Choose $\beta_n = \mathrm{Ad}u \circ \beta'_n$, we conclude that the inductive construction of F_{k_n}, $M_{l(n)}$, $s_n : F_{k_n} \to M_{l(n)}$, $\xi_n : M_{l(n)} \to M_{m(k_n)}$ and $\beta_{n-1} : M_{m(k_{n-1})} \to M_{l(n)}$ to satisfy (i) and (ii) for all n. (Warning: we do not require that $\xi_n \circ \beta_{n-1} = \psi'_{k_{n-1},k_n}$.)

Finally, let $A_n = F_{k_n}$, $\varphi_{n,n+1} = \varphi'_{k_n,k_{n+1}}$ and $\psi_{n,n+1} : M_{l(n)} \to M_{l(n+1)}$ be defined by $\psi_{n,n+1} = \beta_n \circ \xi_n$. Therefore both $\varphi_{n,n+1}$ and $\psi_{n,n+1}$ are injective. Then

$$s_{n+1} \circ \varphi_{n,n+1} = \beta_n \circ \alpha_{k_n} = \beta_n \circ \xi_n \circ s_n = \psi_{n,n+1} \circ s_n.$$

Since $m(k_n)|l(n+1)$, we have $\lim(M_{l(n)}, \psi_{n,m}) = Q$. □

Lemma 7.10. *Let G_0 be torsion free and A be the AF algebra as in 7.9 with $K_0(A) = Q \oplus G_0$. Let a be a positive integer and $\alpha = \frac{a}{a+1}$. Let $A(W, \alpha)$ be defined in 7.2. Then $A(W, \alpha)$ is an inductive limit of a sequence of C^*-algebras $C_n \in \mathcal{C}_0$ with $\lambda_s(C_N) = \alpha$ and with injective connecting maps.*

Proof. Let $s: A \to Q$ be as in 7.9. By Lemma 7.1, $A(W, \alpha)$ is isomorphic to the C^*-subalgebra of $C([0,1], Q \otimes M_{a+1}) \oplus A$ defined by

$$C = \left\{ (f, x) \in C([0,1], Q \otimes M_{a+1}) \oplus A : \begin{array}{l} f(0) = s(x) \otimes \operatorname{diag}(\overbrace{1, \ldots, 1}^{a}, 0), \\ f(1) = s(x) \otimes \operatorname{diag}(\underbrace{1, \ldots, 1}_{a}, 1) \end{array} \right\}.$$

Let $A = \lim_n (A_n, \varphi_{n,m})$ with injective $\varphi_{n,m}$, $Q = \lim_n (M_{l(n)}(\mathbb{C}), \psi_{n,m})$, and $s_n : A_n \to M_{l(n)}(\mathbb{C})$ be described as in 7.9. Evidently C is an inductive limit of

$$C_n = \left\{ (f, x) \in C([0,1], M_{l(n)}(\mathbb{C}) \otimes M_{a+1}) \oplus A_n : \begin{array}{l} f(0) = s_n(x) \otimes \operatorname{diag}(\overbrace{1, \ldots, 1}^{a}, 0), \\ f(1) = s_n(x) \otimes \operatorname{diag}(\underbrace{1, \ldots, 1}_{a}, 1) \end{array} \right\},$$

with connecting homomorphism $\Phi_{n,n+1} : C_n \to C_{n+1}$ given by

$$\Phi_{n,n+1}(f, x) = (g, y) \qquad \text{for} \quad (f, x) \in C_n,$$

where $g(t) = (\psi_{n,n+1} \otimes id_{a+1})(f(t))$ and $y = \varphi_{n,n+1}(x)$. Since both $\varphi_{n,n+1}$ and $\psi_{n,n+1}$ are injective, so is $\Phi_{n,n+1}$. The short exact sequence

$$0 \to C_0\big((0,1), M_{l(n)}(\mathbb{C}) \otimes M_{a+1}\big) \to C_n \to A_n \to 0$$

induces the six term exact sequence of K-theory. Since $(s_n)_{*0} : K_0(A_n) \to K_0(M_{l(n)}(\mathbb{C}))$ is surjective, exactly as the beginning of proof of Proposition 7.8, we have $K_0(C_n) = \ker((s_n)_{*0}) \subset K_0(A_n)$ and $K_1(C_n) = 0$. From a standard calculation (see section 3 of [20]), we know that $K_0(C_n)_+ = \ker(s_n)_{*0} \cap K_0(A_n)_+$. On the other hand, since s_n is injective, $\ker(s_n)_{*0} \cap K_0(A_n)_+ = \{0\}$. In fact, if $x \in \ker(s_n)_{*0} \cap K_0(A_n)_+ \setminus \{0\}$, then there exists a projection $p \in M_r(A_n)$ such that $[p] = x$. However, since s_n is injective, $s_n(p) = q$ is a non-zero projection in $M_r(M_{l(n)})$ which is a non-zero element in $K_0(M_{l(n)})$, whence $x \notin \ker((s_n)_{*0})$. This proves that $K_0(C_n)_+ = \{0\}$. Thus $C_n \in \mathcal{C}_0$. Since s_n are unital, from the very definition (see Definition 3.5), we have $\lambda_s(C_n) = \alpha$. $\square$

Summarize the above, we obtain the following main theorem of this section:

Theorem 7.11. *Let G_0, G_1 be any countable abelian groups and T be any compact metrizable Choquet simplex, then there is a simple C^*-algebra $B \in \mathcal{D}_0$ with continuous scale such that $K_0(B) = \ker(\rho_B) = G_0$, $K_1(B) = G_1$ and $T(B) = T$.*

Furthermore, if, in addition, G_0 is torsion free and $G_1 = 0$, then $B = \lim_{n\to\infty}(C_n, \imath_n)$ with each $C_n \in \mathcal{C}_0$, and $\imath_n$ map strictly positive elements to strictly positive elements. Moreover, B is locally approximated by C^*-algebras in $\mathcal{C}_0$.

Proof. We only need to prove the additional part. But in this case, by Lemma 7.10, we know all B_n in the construction of inductive limit of B in 7.2 are inductive limits of C^*-algebras in $\mathcal{C}_0$ with injective connecting maps. Therefore B is locally approximated by C^*-algebras in $\mathcal{C}_0$ and $B \in \mathcal{D}$. Since the C^*-algebras in $\mathcal{C}_0$ are semi-projective, B itself is an inductive limit of C^*-algebras in $\mathcal{C}_0$. $\square$

Corollary 7.12. *Let G_0, G_1 be any countable abelian groups. Let $\tilde{T}$ be a topological cone with a base T which is a metrizable Choquet simplex and let $\gamma : T \to (0, \infty]$ be a lower semi-continuous function and $\tilde{\gamma} : \tilde{T} \to [0, \infty]$ be the extension of γ defined by $\tilde{\gamma}(s\tau) = s\gamma(\tau)$ for any $s \in \mathbb{R}_+$ and $\tau \in T$. Then there exist a non-unital simple C^*-algebra A, which is stably isomorphic to a C^*-algebra with the form B_T (in 7.7) which is in $\mathcal{D}_0$ such that*

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A, \rho_A) \cong (G_0, G_1, \tilde{T}, \tilde{\gamma}, 0)$$

(Note that $\rho_A = 0$ is equivalent to $K_0(A) = \ker(\rho_A)$.)

Proof. Let B be the C^*-algebra in 7.11 with $K_0(B) = \ker(\rho_B) = G_0$, $K_1(B) = G_1$ and $T(B) = T$. There is a positive element (see 6.2.1 of [49], for example) $a \in B \otimes \mathcal{K}$ such that $d_\tau(a) = \gamma(\tau)$ for all $\tau \in T = T(B)$. Let $A = \overline{a(B \otimes \mathcal{K})a}$. Then A is stably isomorphic to $B \in \mathcal{D}_0$ and

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A, \rho_A) \cong (G_0, G_1, \tilde{T}, \tilde{\gamma}, 0). \quad \square$$

Remark 7.13. We would like to recall the following facts:

Let A be a separable C^*-algebra with $T(A) \neq \emptyset$ and $\mathrm{Ped}(A) = A$. Then $T(A)$ forms a base for the cone $\tilde{T}(A)$. It follows from 3.3 of [43] and 3.1 of [44] that $\tilde{T}(A)$ forms a vector lattice. Therefore, if $T(A)$ is compact, then $T(A)$ is always a metrizable Choquet simplex.

Definition 7.14. In what follows we will use $\mathcal{B}_T$ for the class of C^*-algebras with the form B_T. Note that if $A \in \mathcal{B}_T$ then A is $\mathcal{Z}$-stable with weakly unperforated $K_0(A)$ (see 5.5).

8. C^*-algebras $\mathcal{Z}_0$ and class $\mathcal{D}_0$

Definition 8.1. Let $\mathcal{Z}_0 = B_T$ be as constructed in the previous section with $G_0 = \mathbb{Z}$ and $G_1 = \{0\}$ and with unique tracial state. Note also $\mathcal{Z}_0$ is $\mathcal{Z}$-stable.

From Theorem 7.11 and Corollary 7.12, we have the following fact.

Proposition 8.2. *$\mathcal{Z}_0$ is locally approximated by C^*-algebras in $\mathcal{C}_0$. In fact that $\mathcal{Z}_0 = \lim_{n\to\infty}(C_n, \imath_n)$, where each $C_n \in \mathcal{C}_0$, $\imath_n$ maps strictly positive elements to strictly positive elements.*

(See 15.7 for the uniqueness of $\mathcal{Z}_0$.)

Lemma 8.3. *Let A be a separable exact simple C^*-algebra with continuous scale. Then $A \otimes \mathcal{Z}_0$ also has continuous scale and $A \otimes \mathcal{Z}_0$ is $\mathcal{Z}$-stable.*

Proof. Since $\mathcal{Z}_0$ is $\mathcal{Z}$-stable, so is $A \otimes \mathcal{Z}_0$. Therefore, by [52], $A \otimes \mathcal{Z}_0$ is purely infinite or is stably finite. Since every separable purely infinite simple C^*-algebra has continuous scale [28], we assume that $A \otimes \mathcal{Z}_0$ is stably finite. In particular, $T(A \otimes \mathcal{Z}_0) \neq \emptyset$. Since $\mathcal{Z}$ is unital, it is easy to see that $A \otimes \mathcal{Z}$ has continuous scale. It follows that $T(A)$ is compact. Since $\mathcal{Z}_0$ has a unique tracial state, $T(A \otimes \mathcal{Z}_0)$ is also compact. The lemma follows if we also assume that A is exact by 5.3 of [15]. However, the proof 5.3 of [15] also shows that $T(A \otimes \mathcal{Z}_0)$ is compact.

For general cases, let $B = A \otimes \mathcal{Z}$. We may write $A \otimes \mathcal{Z}_0 = B \otimes \mathcal{Z}_0$. We also note that B has strict comparison (by Theorem 4.5 of [52]).

Let $\{e_n\}$ be an approximate identity for B such that $e_{n+1}e_n = e_ne_{n+1} = e_n$, $n = 1, 2, \ldots$ Let $\{b_n\}$ be an approximate identity for $\mathcal{Z}_0$ such that $b_{n+1}b_n = b_nb_{n+1} = b_n$, $n = 1, 2, \ldots$. It follows that $c_n = e_n \otimes b_n$ is an approximate identity for $B \otimes \mathcal{Z}_0$ such that

$$c_{n+1}c_n = (e_{n+1}e_n) \otimes (b_{n+1}b_n) = e_n \otimes b_n = c_n, \quad n = 1, 2, \ldots. \tag{e8.1}$$

Fix any $d \in B \otimes \mathcal{Z}_0$. Put

$$\sigma = \inf\{d_\tau(d) : \tau \in T(B \otimes \mathcal{Z}_0)\} > 0. \tag{e8.2}$$

Since B has continuous scale, there exists an integer $n_0 \geq 1$ such that

$$\tau(e_n - e_m) < \sigma/4 \ \text{ for all } \ \tau \in T(B) \tag{e8.3}$$

when $n > m \geq n_0$. Let t_Z be the unique tracial state of $\mathcal{Z}_0$. There is $n_1 \geq 1$ such that

$$t_Z(b_n - b_m) < \sigma/4 \ \text{ for all } \ n > m \geq n_1. \tag{e8.4}$$

We have, for $n > m \geq n_0 + n_1$,

$$c_n - c_m = e_n \otimes b_n - e_m \otimes b_m = (e_n - e_m) \otimes b_n + (e_m \otimes b_n - e_m \otimes b_m) \tag{e8.5}$$

$$= (e_n - e_m) \otimes b_n + (e_m \otimes (b_n - b_m)) \tag{e8.6}$$

Therefore, for $n > m \geq n_0 + n_1$,

$$(\tau \otimes t_Z)(c_n - c_m) < \sigma/2 \ \text{ for all } \ \tau \in T(B). \tag{e8.7}$$

By the strict comparison for positive element, the above inequality implies that $c_n - c_m \lesssim d$. It follows that $A \otimes \mathcal{Z}_0$ has continuous scale. □

Now we are ready to state the following theorem which is a variation of 7.12:

Theorem 8.4. *For any separable finite simple amenable C^*-algebra A, there is a C^*-algebra B which is stably isomorphic to a C^*-algebra of the form B_T in $\mathcal{D}_0$ such that* $\mathrm{Ell}(B) \cong \mathrm{Ell}(A \otimes \mathcal{Z}_0)$

Proof. Note that, by 6.2.3 of [49] (see also 7.3 of [15]), one may write

$$Cu^{\sim}(\mathcal{Z}_0) = \mathbb{Z} \sqcup \mathrm{LAff}_+^{\sim}(\tilde{T}(\mathcal{Z}_0)) \ \text{ and } \ Cu^{\sim}(\mathcal{W}) = \mathrm{LAff}_+^{\sim}(\tilde{T}(\mathcal{W})). \tag{e8.8}$$

Since both $\mathcal{Z}_0$ and $\mathcal{W}$ are monotracial, $\mathrm{LAff}_+^{\sim}(\tilde{T}(\mathcal{Z}_0)) = \mathrm{LAff}_+^{\sim}(\tilde{T}(\mathcal{W}))$. Since $K_0(\mathcal{Z}_0) = \ker_{\rho_{\mathcal{Z}_0}}$, one has an ordered semigroup homomorphism $\Lambda : \mathbb{Z} \sqcup \mathrm{LAff}_+^{\sim}(\tilde{T}(\mathcal{Z}_0)) \to \mathrm{LAff}_+^{\sim}(\tilde{T}(\mathcal{W}))$ which maps $\mathbb{Z}$ to zero and identity on $\mathrm{LAff}_+^{\sim}(\tilde{T}(\mathcal{Z}_0)) = \mathbb{R}_+^{\sim}$. In particular, Λ maps 1 to 1. It follows from 8.2 and [49] that there is a homomorphism $\varphi_{z,w} : \mathcal{Z}_0 \to \mathcal{W}$ which maps strictly positive elements to strictly positive elements. Let t_Z and $t_{\mathcal{W}}$ be the unique tracial states of $\mathcal{Z}_0$ and $\mathcal{W}$, respectively. Then $t_{\mathcal{W}} \circ \varphi_{z,w} = t_Z$, since $\mathcal{Z}_0$ has only one tracial state.

Since $\mathcal{Z}\otimes\mathcal{Z}_0\cong\mathcal{Z}_0$, without loss of generality, we may assume that A is $\mathcal{Z}$-stable. Let $a\in \mathrm{Ped}(A)_+$ be such that $\overline{aAa}$ has continuous scale (see 5.2 of [15]). Put $B=\overline{aAa}\otimes\mathcal{Z}_0$. It is easy to verify that B is a hereditary C^*-subalgebra of $A\otimes\mathcal{Z}_0$. Every tracial state of B has the form $\tau\otimes t_Z$, where $\tau\in T(\overline{aAa})$. Fix $\tau\in T(\overline{aAa})$, then

$$(\tau\otimes t_z)(a\otimes z)=\tau(a)t_Z(z)=\tau(a)(t_{\mathcal{W}}\circ\varphi_{z,w}(z))\ \text{ for all }\ a\in A\ \text{ and }\ z\in\mathcal{Z}_0. \tag{e8.9}$$

Let $\psi=\mathrm{id}_A\otimes\varphi_{z,w}:A\otimes\mathcal{Z}_0\to A\otimes\mathcal{W}$ and let $s=\tau\otimes t_z\in T(B)$. Then, by (e8.9), $s=(\tau\otimes t_{\mathcal{W}})\circ\psi$. Since $\mathcal{W}$ satisfies the UCT, by the Künneth formula [54], $K_i(\overline{aAa}\otimes\mathcal{W})=0$, $i=0,1$. Therefore, for any $x\in K_0(B)$, $s(x)=0$. This implies that $\ker\rho_B=K_0(B)$. Since A is separable, simple and B is a hereditary C^*-subalgebra of $A\otimes\mathcal{Z}_0$, by [4], $(A\otimes\mathcal{Z}_0)\otimes\mathcal{K}\cong B\otimes\mathcal{K}$. It follows that $K_0(A\otimes\mathcal{Z}_0)=\ker\rho_{A\otimes\mathcal{Z}_0}$.

Note that (see 7.13) $T(B)$ is a metrizable Choquet simplex. By 7.12, there is a C^*-algebra C which is stably isomorphic to a C^*-algebra of the form B_T in $\mathcal{D}_0$ such that $\mathrm{Ell}(C)\cong\mathrm{Ell}(A\otimes\mathcal{Z}_0)$. □

Theorem 8.5. *Let A be a separable C^*-algebra which is stably isomorphic to a C^*-algebra in $\mathcal{D}_0$. Then $K_0(A)=\ker\rho_A$.*

Proof. Without loss of generality, we may assume that $A\in\mathcal{D}_0$. By 12.3 of [15], it suffices to show that every tracial state of A is a $\mathcal{W}$-trace. By 12.2 of [15], it suffices to produce a sequence of completely positive contractive linear maps $\{\varphi_n\}$ from A into $D_n\in\mathcal{C}_0^{0'}$ such that

$$\lim_{n\to\infty}\|\varphi_n(ab)-\varphi_n(a)\varphi_n(b)\|=0\ \text{ for all }\ a,\ b\in A\ \text{ and}$$
$$\tau(a)=\lim_{n\to\infty}t_n(\varphi_n(a))\ \text{ for all }\ a\in A, \tag{e8.10}$$

where $t_n\in T(D_n)$.

This, of course, follows directly from the definition of $\mathcal{D}_0$. In fact, in the proof of 9.1 of [15] $\varphi_{1,n}$ would work (note, we assume that $A\in\mathcal{D}_0$ instead in $\mathcal{D}$, therefore C^*-algebras $D_n\in\mathcal{C}_0^{0'}$ instead in $\mathcal{C}_0'$). Note also that, 9.1 of [15] shows that $QT(Q)=T(A)$. Thus (e8.10) follows from (e.9.9) of [15]. □

Theorem 8.6. *Let A be a separable simple C^*-algebra in $\mathcal{D}$ with continuous scale. Then the map from $Cu(A)$ to $\mathrm{LAff}_+(T(A))$ is a Cuntz semigroup isomorphism.*

Proof. This follows from 11.8 of [15] (see 15.8 of [18]) immediately (since $\overline{T(A)}^w=T(A)$, as A is assumed to have continuous scale). □

Corollary 8.7. *Let A be a separable simple C^*-algebra in $\mathcal{D}$. Then $Cu^\sim(A)=K_0(A)\sqcup\mathrm{LAff}_+^\sim(\tilde{T}(A))$.*

Proof. Note, by 11.5 of [15], A has stable rank one. This follows from 8.6 (see 7.3 of [15]). □

Theorem 8.8. *Let A be a separable simple C^*-algebra in $\mathcal{D}$ with $\ker\rho_A=K_0(A)$. Then $A\in\mathcal{D}_0$. Moreover, There exists $e_A\in A_+$ with $\|e_A\|=1$ and $0<\sigma_0<1$ which satisfy the following: For any $\varepsilon>0$, $\eta>0$ and any finite subset $\mathcal{F}\subset A$, there are $\mathcal{F}$-ε-multiplicative completely positive contractive linear maps $\varphi:A\to A$ and $\psi:A\to D$ for some C^*-subalgebra $D\in\mathcal{R}$ (see 3.1) with $\varphi(A)\perp D$ such that*

$$\|x-(\varphi(x)\oplus\psi(x))\|<\varepsilon\ \text{ for all }\ x\in\mathcal{F}, \tag{e8.11}$$
$$d_\tau(\varphi(e_A))<\eta\ \text{ for all }\ \tau\in T(A)\ \text{ and} \tag{e8.12}$$
$$t(f_{1/4}(\psi(e_A)))\ge 1-\sigma_0\ \text{ for all }\ t\in T(D). \tag{e8.13}$$

Proof. We may assume, without loss of generality, that A has continuous scale, by considering a hereditary C^*-subalgebra of A (see 11.9 of [15]). We will use the facts that C^*-algebras in $\mathcal{D}$ have stable rank one and strict comparison as well as have the property described in 8.7.

Let $e_A\in A$ be a strictly positive element with $\|e_A\|=1$. Note that $d_\tau(e_A)$ is now assumed to be continuous on $\tilde{T}(A)$. Fix any integer $m\ge 2$, by 8.7, there is a positive element $e_{00}\in A$ such that $d_\tau(e_{00})=(1/m)d_\tau(e_A)$ for all $\tau\in\tilde{T}(A)$. Moreover, as in the proof of 3.13, $A\cong M_m(\overline{e_{00}Ae_{00}})$. Let $\Lambda_{0,m}:\mathrm{Cu}^\sim(A)\to\mathrm{Cu}^\sim(A)$ be defined by $(\Lambda_{0,m})|_{K_0(A)}=\mathrm{id}_{K_0(A)}$ and $(\Lambda_{0,m})|_{\mathrm{LAff}_+^\sim(\tilde{T}(A))}=(1/m)\,\mathrm{id}_{\mathrm{LAff}_+^\sim(\tilde{T}(A))}$. Then, since $K_0(A)=\ker\rho_A$, one sees that $\Lambda_{0,m}$ is a morphism in Cu.

Claim. *For any C^*-subalgebra $D'\subset A$ with $D'\in\mathcal{C}_0$, there is a homomorphism $j_{0,D'}:D'\to A_0:=\overline{e_{00}Ae_{00}}$ such that $\mathrm{Cu}^\sim(j_{0,D'})=\Lambda_{0,m}\circ\mathrm{Cu}^\sim(\iota_{D'})$. To see the claim, note that A_0 has stable rank one (see 11.5 of [15]) and $\Lambda_{0,m}\circ\mathrm{Cu}^\sim(\iota_{D'})$ is a morphism in* **Cu**. *Then, by Theorem 1.0.1. of [49], such homomorphism $j_{0,D'}$ exists.*

Let $1>\sigma_0>0$. We may assume that $\tau(e_A)\ge 1-\sigma_0/64$ for all $\tau\in T(A)$. Suppose also that $\tau(f_{1/2}(e_A))>1-\sigma_0/32$ for all $\tau\in T(A)$. Let $\mathfrak{f}=1-\sigma_0/4$. Let $\varepsilon>0$ and let $\mathcal{F}\subset A$ be a finite subset. Let $\sigma_0/32>\eta>0$. Choose $m\ge 2$ such that $1/m<\min\{\sigma_0/2^{12},\eta/2\}$.

Let $T = T(A)$. Let W_T be the separable simple amenable C^*-algebra with $K_i(W_T) = \{0\}$, $i = 0, 1$, and $T(W_T) = T$ as in 2.8 of [16]. Therefore $\mathrm{LAff}_+^{\sim}(\tilde{T}(W_T)) = \mathrm{LAff}_+^{\sim}(\tilde{T}(A))$. Let $\Gamma : \mathrm{LAff}_+^{\sim}(\tilde{T}(W_T)) \to \mathrm{LAff}_+^{\sim}(\tilde{T}(A))$ be the order semi-group isomorphism. By 8.7, $Cu^{\sim}(A) = K_0(A) \sqcup \mathrm{LAff}_+^{\sim}(\tilde{T}(A))$. Since $K_0(A) = \ker \rho_A$, the map $\Gamma^{-1'} : Cu^{\sim}(A) \to Cu^{\sim}(W_T)$ which maps $K_0(A)$ to zero and $\Gamma^{-1'}|_{\mathrm{LAff}_+^{\sim}(\tilde{T}(A))} = (\frac{m-1}{m})\Gamma^{-1}$ is a morphism in **Cu**.

Fix $0 < \varepsilon_1 < \min\{\varepsilon/4, \sigma/2^{10}\}$ and a finite subset $\mathcal{F}_1 \supset \mathcal{F}$. There are $\mathcal{F}_1$-ε_1-multiplicative completely positive contractive linear maps $\varphi_0 : A \to A$ and $\psi : A \to D$ for some C^*-subalgebra $D \subset A$ such that $\varphi_0(A) \perp D$,

$$\|x - (\varphi_0(x) \oplus \psi(x))\| < \varepsilon_1/4 \text{ for all } x \in \mathcal{F}_1 \cup \{e_A\}, \tag{e8.14}$$

$$D \in \mathcal{C}_0(\text{see } 3.11),\quad d_\tau(\varphi_0(e_A)) < \eta \text{ for all } \tau \in T(A) \text{ and} \tag{e8.15}$$

$$t(f_{1/4}(\psi(e_A))) \geq 1 - \sigma_0/16 \text{ for all } t \in T(D). \tag{e8.16}$$

Let $\imath_D : D \to A$ be the embedding. Consider $\Gamma^{-1'} \circ Cu^{\sim}(\imath_D)$. Then, by [49], there exists a homomorphism $\psi_1 : D \to W_T$ such that $Cu^{\sim}(\psi_1) = \Gamma^{-1'} \circ Cu^{\sim}(\imath_D)$. Let $e_d \in D$ be a strictly positive element of D with $\|e_d\| = 1$ and let $W_1 = \overline{\psi_1(e_d)W_T\psi_1(e_d)}$. By [49] again, there exists a homomorphism $\psi_{w,a} : W_T \to A$ such that $Cu^{\sim}(\psi_{w,a}) = \Gamma$.

Note that $\langle \psi_{w,a} \circ \psi_1(e_d)\rangle \leq (1/n)\langle e_A\rangle$. By the claim above, we may assume that there also exists a homomorphism $j_D : D \to A$ such that $\mathrm{Cu}^{\sim}(j_D) = \Lambda_{0,m} \circ \mathrm{Cu}^{\sim}(\imath_D)$ and $j_D(D) \perp \psi_{w,a} \circ \psi$. Put $\Psi := j_D \oplus \psi_{w,a} \circ \psi_1$. Then $Cu^{\sim}(\imath_D) = Cu^{\sim}(\Psi)$.

By [49], there exists a sequence of unitaries $u_n \in \tilde{A}$ such that

$$\lim_{n\to\infty} \|\imath_D(g) - u_n^*\Psi(g)u_n\| = 0 \text{ for all } g \in D. \tag{e8.17}$$

Let $\delta > 0$, $\mathcal{G} \subset D$ be a finite subset, and $e_n = \Psi(e_d)$. Choose $1/4 > \sigma > 0$ such that

$$\|f_\sigma(e_d)gf_\sigma(e_d) - g\| < \delta/2 \text{ for all } g \in \mathcal{G}. \tag{e8.18}$$

By (e8.17), with sufficiently small δ, by Prop.1 of [8], there is $n_0 \geq 1$ and unitaries $v_n \in \tilde{A}$, for all $n \geq n_0$,

$$\|\imath_D(g) - v_n^* f_\sigma(e_n)\Psi'(g)f_\sigma(e_n)v_n\| < \varepsilon_1 \text{ for all } g \in \mathcal{G} \text{ and } v_n^* f_\sigma(e_n)v_n \in \overline{DAD}. \tag{e8.19}$$

Put $\Phi' : D \to A$ by $\Phi'(c) = v_{n_0}^* \psi_{w,a} \circ \psi_1(c)v_{n_0}$ for all $c \in D$, and $\varphi : A \to A$ by $\varphi(a) = \varphi_0(a) \oplus v_n^* f_\sigma(e_d)j_D(a)f_\sigma(e_d)v_n$ for all $a \in A$. Let $W_0 = v_{n_0}^* \psi_{w,a}(W_1)v_{n_0}$. By the choice of m and by (e8.16), we may also assume that

$$t(f_{1/4}(\Phi'(\psi(e_A)))) > 1 - \sigma_0/8 \text{ for all } t \in T(W_0). \tag{e8.20}$$

Note $W_0 \perp \varphi(A)A\varphi(A)$. Moreover, with sufficiently small δ and large $\mathcal{G}$,

$$\|x - (\varphi(x)) \oplus \Phi'(\psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F}. \tag{e8.21}$$

Note that $W_0 = \overline{\cup_{n=1}^\infty F_n \otimes \mathcal{W}}$, where $F_n \subset F_{n+1}$ are finite dimensional C^*-algebras, and $\mathcal{W} = \overline{\cup_{n=1}^\infty R_n}$, where $R_n \in \mathcal{R}$. Since D is semiprojective, there exists a sequence of homomorphisms $\psi_{0,n} : D \to R_n$ such that

$$\lim_{n\to\infty} \|\psi_{0,n}(g) - \Phi'(g)\| = 0 \text{ for all } g \in D. \tag{e8.22}$$

Then, passing to a subsequence, applying a weak*-compactness argument, if necessarily, we may assume that, for all sufficiently large n,

$$t(f_{1/4}\psi_{0,n}(\psi(e_A))) > 1 - \sigma_0/8 \geq \mathfrak{f} \geq \text{ for all } t \in T(R_n), \tag{e8.23}$$

Moreover,

$$\|x - (\varphi(x)) \oplus \psi_{0,n}(\psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F}.$$

The lemma then follows. $\square$

Proposition 8.9. *Let A be a separable simple C^*-algebra and B be a separable simple C^*-algebra which is tracially approximate divisible (see definition 10.1 of [15]). Suppose that both A and B have continuous scale, and B has strict comparison. Let $C = A \otimes B$ (minimal tensor product) be such that C has continuous scale and also has strict comparison. Then $A \otimes B$ is tracially approximate divisible.*

Proof. Let $\varepsilon > 0$, let $\mathcal{F} \subset A \otimes B$ be a finite subset, let $c \in (A \otimes B)_+ \setminus \{0\}$ and let $n \geq 1$ be an integer. Without loss of generality, we may assume that

$$\mathcal{F} = \{a \otimes b : a \in \mathcal{F}_A \text{ and } b \in \mathcal{F}_B\},$$

where $\mathcal{F}_A \subset A$ and $\mathcal{F}_B$ are finite subsets. We may further assume that $\|a\|, \|b\| \leq 1$ for all $a \in \mathcal{F}_A$ and $b \in \mathcal{F}_B$. Since $A \otimes B$ is simple and $T(C)$ is compact, as C has continuous scale,

$$\inf\{d_\tau(c) : \tau \in T(C)\} = d > 0. \tag{e8.24}$$

Choose $b_0 \in B_+ \setminus \{0\}$ with $\|b_0\| = 1$ such that $d_\tau(b_0) < d/2$ for all $\tau \in T(B)$.

Since B has tracially approximate divisible property, there are C^*-subalgebras B_0 and B_1 of A such that

$$\mathrm{dist}(b, D_d) < \varepsilon/2 \text{ for all } y \in \mathcal{F}_B, \tag{e8.25}$$

where $D_d \subset D \subset B$ which has the form

$$D_d = \{d_0 \oplus \mathrm{diag}(\overbrace{d_1, d_1, \dots, d_1}^{n}) \in B_0 \oplus M_n(B_1) : d_0 \in B_0, d_1 \in B_1\},$$

where $D = B_0 \oplus M_n(B_1)$. Moreover, $b_{e0} \lesssim b_0$, where b_{e0} is a strictly positive element of B_0.

Now let $A_0 = A \otimes B_0$, $A_1 = A \otimes B_1$ and $A_3 = A_0 \oplus M_n(A_1)$. Also let $A_d = A \otimes D_d$. Then

$$\mathrm{dist}(x, A_d) < \varepsilon \text{ for all } x \in \mathcal{F}. \tag{e8.26}$$

We also compute that

$$e_a \otimes b_{e0} \lesssim e_a \otimes b_0 \lesssim c. \tag{e8.27}$$

This implies that C is tracially approximate divisible. □

Proposition 8.10. $B_T \otimes \mathcal{Z}_0 \in \mathcal{D}_0$.

Proof. C^*-algebra B_T has finite nuclear dimension and so does $\mathcal{Z}_0$. By Proposition 2.3 of [60], $B_T \otimes \mathcal{Z}_0$ has finite nuclear dimension. By 8.3, $B_T \otimes \mathcal{Z}_0$ has continuous scale and $\mathcal{Z}$-stable. Therefore, by [52], it has strict comparison. It follows from 8.9 and 3.12 that every hereditary C^*-subalgebra has tracially approximate divisible property. It follows from 6.5 of [16] that every tracial state of $B_T \otimes \mathcal{Z}_0$ is a $\mathcal{W}$-trace. It follows from 6.6 of [16] that $B_T \otimes \mathcal{Z}_0$ is in $\mathcal{D}_0$. □

In the Appendix (A.10), we will show that

Theorem 8.11 (A.10)**.** *Let A be a separable amenable C^*-algebra in $\mathcal{D}$. Then $A \otimes \mathcal{Z} \cong A$.*

Definition 8.12. By [49], there exists a homomorphism $\varphi_{w,z} : \mathcal{W} \to \mathcal{Z}_0$ which maps the strictly positive elements to strictly positive elements, Since $K_0(\mathcal{Z}_0) = \ker \rho_{\mathcal{Z}_0}$, by 8.2 and by [49], there exists also a homomorphism $\varphi_{z,w} : \mathcal{Z}_0 \to \mathcal{W}$ which maps the strictly positive elements to strictly positive elements. Note as in the proof of 8.4 we also have $t_Z = t_{\mathcal{W}} \circ \varphi_{z,w}$ and $t_{\mathcal{W}} = t_Z \circ \varphi_{w,z}$, where t_Z and $t_{\mathcal{W}}$ are the unique tracial states of $\mathcal{Z}_0$ and $\mathcal{W}$ respectively.

There exist also an isomorphism $\varphi_{w21} : M_2(\mathcal{W}) \to \mathcal{W}$ and an isomorphism $\varphi_{z21} : M_2(\mathcal{Z}_0) \to \mathcal{Z}_0$ such that $(\varphi_{z21})_{*0} = \mathrm{id}_{K_0(\mathcal{Z}_0)}$. We will fixed these four homomorphisms.

Definition 8.13. Let $\kappa_0^o : K_0(\mathcal{Z}_0) \to K_0(\mathcal{Z}_0)$ be a homomorphism by sending x to $-x$ for all $x \in K_0(\mathcal{Z}_0) = \ker \rho_{\mathcal{Z}_0}$. Denote also by κ^o the automorphism on $Cu^{\sim}(\mathcal{Z}_0)$ such that $\kappa^o|_{K_0(\mathcal{Z}_0)} = \kappa_0^o$ and identity on $\mathrm{LAff}^{\sim}(\tilde{T}(\mathcal{Z}_0))$ which maps function 1 to function 1. It follows from [49] that there is an endomorphism $j^{\circledast'} : \mathcal{Z}_0 \to \mathcal{Z}_0$ such that $Cu^{\sim}(j^{\circledast'}) = \kappa^o$ and $j^{\circledast'}(a)$ is a strictly positive element of $\mathcal{Z}_0$ for some strictly positive element a. By [49] again, $j^{\circledast'}(\mathcal{Z}_0)$ is isomorphic to $\mathcal{Z}_0$, say, given by $j : j^{\circledast'}(\mathcal{Z}_0) \to \mathcal{Z}_0$. Then $j^{\circledast} = j \circ j^{\circledast'}$ is an automorphism. The automorphism $j^{\circledast}$ will be also used in later sections.

Lemma 8.14. *Define Φ, $\Psi : \mathcal{Z}_0 \to M_2(\mathcal{Z}_0)$ by*

$$\Phi(a) = \mathrm{diag}(a, j^{\circledast}(a)) \text{ and } \Psi(a) = (\varphi_{wz} \otimes \mathrm{id}_{M_2})(\mathrm{diag}(\varphi_{zw}(a), \varphi_{zw}(a))) \text{ for all } a \in \mathcal{Z}_0.$$

Then Φ is approximately unitarily equivalent to Ψ, i.e., there exists a sequence of unitaries $\{u_n\} \subset \widetilde{M_2(\mathcal{Z}_0)}$ such that

$$\lim_{n\to\infty} \mathrm{Ad}\, u_n \circ \Phi(a) = (\varphi_{wz} \otimes \mathrm{id}_{M_2}) \circ \mathrm{diag}(\varphi_{zw}(a), \varphi_{zw}(a)) \text{ for all } a \in \mathcal{Z}_0.$$

In particular, $j^{\circledast}_{*0}(x) = -x$ for $x \in K_0(\mathcal{Z}_0)$.

Moreover $\varphi_{z21} \circ \Phi$ is approximately unitarily equivalent to $\varphi_{z21} \circ \Psi$.

Proof. Using 6.1.1 of [49] (see also 7.3 of [15]), one computes that

$$Cu^{\sim}(\Phi) = Cu^{\sim}(\Psi).$$

It follows from [49] that Φ is approximately unitarily equivalent to Ψ. □

9. $\mathcal{E}(A, B)$

Definition 9.1. Let A be a separable amenable C^*-algebra and let B be another C^*-algebra. We use $B^{\vdash}$ for the C^*-algebra obtained by adding a unit to B (regardless B has a unit or not). We will continue to use the embedding $\varphi_{wz}. : \mathcal{W} \to \mathcal{Z}_0$. Without causing confusion, we will identify $\mathcal{W}$ with $\varphi_{wz}(\mathcal{W})$ from time to time.

An *asymptotic sequential morphism* $\varphi = \{\varphi_n\}$ from A to B is a sequence of approximately multiplicative completely positive contractive linear maps $\varphi_n : A \to B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ which satisfies the following condition: there is $\alpha \in Hom_{\Lambda}(\underline{K}(A), \underline{K}(B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}))$ and there are two sequences of approximately multiplicative completely positive contractive linear maps $h_n,\ h_n' : A \to \mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ such that, for any finite subset $\mathcal{P} \in \underline{K}(A)$, there exists $n_0 \geq 1$ such that

$$[\varphi_n]|_{\mathcal{P}}+[h_n]|_{\mathcal{P}}= \alpha|_{\mathcal{P}}+[h_n']|_{\mathcal{P}} \ \text{ for all } \ n \geq n_0. \tag{e9.1}$$

Let $\varphi = \{\varphi_n\}$ and $\psi = \{\psi_n\}$ be two asymptotic sequential morphisms from A to B. We say φ and ψ are equivalent and write $\varphi \sim \psi$ if there exist two sequences of approximately multiplicative completely positive contractive linear maps $h_n, h_n' : A \to \mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ and a sequence of unitaries $u_n \in B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ such that

$$\lim_{n\to\infty} \|u_n^*\mathrm{diag}(\varphi_n(a), h_n(a))u_n - \mathrm{diag}(\psi_n(a), h_n'(a))\| = 0 \ \text{ for all } \ a \in A.$$

Denote by $\langle\varphi\rangle$ the equivalence class of asymptotic sequential morphisms represented by φ, and by $\mathcal{E}(A, B)$ the set of all equivalence classes of asymptotic sequential morphisms from A to B.

Consider the split short exact sequence

$$0 \to B \otimes \mathcal{Z}_0 \otimes \mathcal{K} \xrightarrow{\iota} B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K} \overset{\pi}{\underset{s}{\rightleftarrows}} \mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K} \to 0.$$

It gives the following split short exact sequence:

$$0 \to KL(A, B \otimes \mathcal{Z}_0) \xrightarrow{[\iota]} KL(A, B^{\vdash} \otimes \mathcal{Z}_0) \overset{[\pi]}{\underset{[s]}{\rightleftarrows}} KL(A, \mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0) \to 0. \tag{e9.2}$$

Define $\lambda_B : \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B^{\vdash} \otimes \mathcal{Z}_0)) \to \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B \otimes \mathcal{Z}_0))$ by

$$\lambda_B(x) = x - [s] \circ [\pi](x) \ \text{ for all } \ x \in KL(A, B^{\vdash} \otimes \mathcal{Z}_0). \tag{e9.3}$$

Note that

$$s \circ \pi \circ g_n = g_n$$

for any completely positive contractive linear map $g_n : A \to \mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K} \subset B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$.

Let $\langle\varphi\rangle \in \mathcal{E}(A, B)$ be represented by $\{\varphi_n\}$ and let α be as in (e9.1). Then, for any fixed finite subset $\mathcal{P} \subset \underline{K}(A)$,

$$\lambda_B \circ ([\varphi_n]|_{\mathcal{P}}+[h_n]|_{\mathcal{P}}-[h_n']|_{\mathcal{P}}) = [\varphi_n]|_{\mathcal{P}}-[s \circ \pi \circ \varphi_n]|_{\mathcal{P}}= \lambda_B \circ \alpha|_{\mathcal{P}} \tag{e9.4}$$

for all $n \geq n_0(\mathcal{P})$ for some integer $n_0(\mathcal{P})$. If $\{\psi_n\}$ is another representation of $\langle\varphi\rangle$, then, there exist two sequences of approximately multiplicative completely positive contractive linear maps $g_n, g_n' : A \to \mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ and a sequence of unitaries $u_n \in B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ such that

$$\lim_{n\to\infty} \|u_n^*\mathrm{diag}(\varphi_n(a), g_n(a))u_n - \mathrm{diag}(\psi_n(a), g_n'(a))\| = 0 \ \text{ for all } \ a \in A.$$

Thus there is an integer $n_1(\mathcal{P}) \geq 1$ such that

$$[\varphi_n]|_{\mathcal{P}}+[g_n]|_{\mathcal{P}} = [\psi_n]|_{\mathcal{P}}+[g_n']|_{\mathcal{P}} \ \text{ and } \tag{e9.5}$$

$$[s \circ \pi \circ \varphi_n]|_{\mathcal{P}}+[g_n]|_{\mathcal{P}} = [s \circ \pi \circ \psi_n] + [g_n']|_{\mathcal{P}} \ \text{ for all } \ n \geq n_1(\mathcal{P}). \tag{e9.6}$$

Therefore

$$([\psi_n]|_{\mathcal{P}}-[s \circ \pi \circ \psi_n]|_{\mathcal{P}}) = ([\varphi_n]|_{\mathcal{P}}+[g_n]|_{\mathcal{P}}-[g_n']|_{\mathcal{P}}) \tag{e9.7}$$

$$-([s \circ \pi \circ \varphi_n]|_{\mathcal{P}}+[s \circ \pi \circ g_n]|_{\mathcal{P}}-[s \circ \pi \circ g_n']|_{\mathcal{P}}) \tag{e9.8}$$

$$= [\varphi_n]|_{\mathcal{P}}-[s \circ \pi \circ \varphi_n]|_{\mathcal{P}}= \lambda_B \circ \alpha|_{\mathcal{P}} \tag{e9.9}$$

for all $n \geq \max\{n_0(\mathcal{P}), n_1(\mathcal{P})\}$. Thus $\beta_A : \mathcal{E}(A, B) \to Hom_{\Lambda}(\underline{K}(A), \underline{K}(B \otimes \mathcal{Z}_0 \otimes \mathcal{K}))$ given by $\beta_A(\langle\varphi\rangle) = \lambda_B \circ \alpha$, is well defined.

If φ and ψ are two asymptotic sequential morphisms from A to B, we define $\varphi \oplus \psi$ by $(\varphi \oplus \psi)(a) = \mathrm{diag}(\varphi(a), \psi(a))$ for all $a \in A$. Here we identify $M_2(\mathcal{K})$ with $\mathcal{K}$ in the usual way. We define $\langle\varphi\rangle + \langle\psi\rangle = \langle\varphi \oplus \psi\rangle$. This clearly defines an addition in $\mathcal{E}(A, B)$. Let $\langle\psi\rangle \in \mathcal{E}(A, B)$ be represented by $\{\psi_n\}$ whose images are in $\mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$. Then, for any $\langle\varphi\rangle \in \mathcal{E}(A, B)$, $\langle\varphi \oplus \{\psi_n\}\rangle = \langle\varphi\rangle$. In other words that $\mathcal{E}(A, B)$ is a semigroup with zero represented by zero asymptotic morphism. Note, if A is unital, then $\mathcal{E}(A, B) = \{0\}$, as there are only zero asymptotic sequential morphisms from A to $B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$.

Definition 9.2. Denote $C = B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$. Let $C_\infty = l^\infty(C)/c_0(C)$. If $\varphi = \{\varphi_n\}$ is an asymptotic sequential morphism, then we may view φ as a homomorphism from A to C_∞. Two asymptotic sequential morphisms φ and ψ are *homotopy* if there is a homomorphism $H : A \to C([0, 1], C_\infty)$ such that $\pi_0 \circ H = \varphi$ and $\pi_1 \circ H = \psi$, where $\pi_t : C([0, 1], C_\infty) \to C_\infty$ is the point-evaluation at $t \in [0, 1]$. Since we assume that A is amenable, there exists a completely positive contractive linear map $L : A \to C([0, 1], l^\infty(C))$ such that $\Pi \circ L = H$, where $\Pi : l^\infty(C) \to C_\infty$ is the quotient map. Denote by $P_n : l^\infty(C) \to C$ the nth coordinate map. Define $\Phi_n' = P_n \circ L$, $n = 1, 2, \ldots$. Define $\varphi_n' = \pi_0 \circ \Phi_n'$ and $\psi_n' = \pi_1 \circ \Phi_n'$. Note that

$$\lim_{n\to\infty} \|\varphi_n(a) - \varphi_n'(a)\| = 0 \ \text{ and } \quad \text{ and } \ \lim_{n\to\infty} \|\psi_n(a) - \psi_n'(a)\| = 0 \ \text{ for all } \ a \in A. \tag{e9.10}$$

Therefore we may assume, without loss of generality, as far as in this section, that φ_n and ψ_n are homotopy for each n. Fix a finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$. There is a partition $0 = t_0 < t_1 < \cdots t_m = 1$ such that

$$\|\pi_t \circ L(a) - \pi_{t_i} \circ L(a)\| < \varepsilon/2 \text{ for all } a \in \{cd, c, d : c, d \in \mathcal{F}\}. \tag{e9.11}$$

Since $\Pi \circ L = H$, there is $n_0 > 1$ such that $\pi_{t_i} \circ P_n \circ L$ is $\mathcal{F}$-$\varepsilon/2$-multiplicative for all $a, b \in \mathcal{F}$ for all $n \geq n_0$, $i = 0, 1, \ldots, m$. It follows from (e9.11) that $\pi_t \circ P_n \circ L$ is $\mathcal{F}$-ε-multiplicative. In other words, φ_n and ψ_n are connected by a path of $\mathcal{F}$-ε-multiplicative completely positive contractive linear maps for all large n.

Definition 9.3. We now fixed a separable amenable C^*-algebra A satisfying the UCT with the following property: There is a map $T : A_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ and a sequence of approximately multiplicative completely positive contractive linear maps $\varphi_n : A \to \mathcal{W}$ such that, for any finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$, there exists an integer $n_0 \geq 1$ such that φ_n is T-$\mathcal{H}$-full (see 5.5 of [15] and 7.8 of [18]) for all $n \geq n_0$.

For the rest of this section, A is as above.

Lemma 9.4. *Let $\{\varphi_n\}$ be an asymptotic sequential morphism from A to $B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ such that the image of φ_n are all contained in $B^{\vdash} \otimes \mathcal{W} \otimes \mathcal{K}$. Then $\langle \{\varphi_n\} \rangle = 0$.*

Proof. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. Let T be given in 9.3. Write $T(a) = (N(a), M(a))$ for all $a \in A_+ \setminus \{0\}$. We will apply 6.9.

Let $\delta > 0$, $\mathcal{G}$ be a finite subset, $\mathcal{H} \subset A_+ \setminus \{0\}$ be a finite subset and let $K \geq 1$ be an integer as required by 6.9 for T.

Suppose that $\varphi_n : A \to B^{\vdash} \otimes \mathcal{W} \otimes \mathcal{K}$ is a $\mathcal{G}$-δ-multiplicative completely positive contractive linear map. We may assume, without loss of generality, that the image of φ_n lies in $M_{k(n)}(B^{\vdash} \otimes \mathcal{W})$. Choose an asymptotic sequential morphism $\{\psi_n\}$ from A to $\mathcal{W}$ given by 9.3. We may assume that ψ_n is $\mathcal{G}$-δ-multiplicative and is T-$\mathcal{H}$-full. Let $\psi_0 : \mathcal{W} \to \mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{W} \otimes \mathcal{K}$ be the homomorphism defined by $\psi_0(a) = 1 \otimes a \otimes e_{1,1}$, where $e_{1,1}$ is a rank one projection of $\mathcal{K}$. By replacing $\{\psi_n\}$ by $\{\psi_0 \circ \psi_n\}$, we assume that the image of ψ_n are in $\mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{W} \otimes \mathcal{K}$. Define $\bar{\psi}_n : A \to M_{k(n)}(\mathcal{W})$ by

$$\bar{\psi}_n(a) = \mathrm{diag}(\overbrace{\psi_n(a), \psi_n(a), \ldots, \psi_n(a)}^{k(n)}) \text{ for all } a \in A.$$

By viewing $\bar{\psi}_n$ as a map from A to $M_{k(n)}((\mathbb{C} \cdot 1_{B^{\vdash}}) \otimes \mathcal{W})$, it is easy to check that $\bar{\psi}_n$ is T-$\mathcal{H}$-full in $M_{k(n)}((\mathbb{C} \cdot 1_{B^{\vdash}}) \otimes \mathcal{W})$ (see the proof of 6.9).

Then, by 6.9, there exist an integer K and a unitary $v \in M_{(K+1)k(n)}(B^{\vdash} \otimes \mathcal{W})^{\sim} \subset M_{(K+1)k(n)}(B^{\vdash} \otimes \mathcal{Z}_0)^{\sim}$ such that

$$\|v^* \mathrm{diag}(\varphi_n(a), \Psi_n(a))v - \mathrm{diag}(0, \Psi_n(a))\| < \varepsilon \text{ for all } a \in \mathcal{F},$$

where $\Psi_n(a) = \bar{\psi}_n(a) \otimes 1_K$. Lemma then follows. □

Proposition 9.5. *$\mathcal{E}(A, B)$ is an abelian group.*

Proof. Define an endomorphism $\iota^{\circledast}$ on $B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ by

$$\iota^{\circledast}(a \otimes b \otimes c) = a \otimes j^{\circledast}(b) \otimes c \text{ for all } a \in B^{\vdash}, b \in \mathcal{Z}_0 \text{ and } c \in \mathcal{K}$$

(see 8.13 for the definition of $j^{\circledast}$). Let $\varphi = \{\varphi_n\}$ be an asymptotic sequential morphism from A to $B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$. Let $\psi_n : A \to B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ be defined by

$$\psi_n(a) = \iota^{\circledast} \circ \varphi_n(a) \text{ for all } a \in A.$$

Define $H : B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K} \to M_2(B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})$ by

$$H(a \otimes b \otimes c) = a \otimes (\varphi_{wz} \otimes \mathrm{id}_{M_2})(\mathrm{diag}(\varphi_{zw}(b), \varphi_{zw}(b))) \otimes c \text{ for all } a \in B^{\vdash}, b \in \mathcal{Z}_0 \text{ and } c \in \mathcal{K}.$$

It follows from 8.14 that there exists a sequence of unitaries $\{u_n\} \subset B^{\vdash} \widetilde{\otimes \mathcal{Z}_0} \otimes \mathcal{K}$ such that

$$\mathrm{Ad}\, u_n \circ H(\varphi_n(x)) = \lim_{n \to \infty} \mathrm{diag}(\varphi_n(x), \psi_n(x)) \text{ for all } x \in A.$$

It follows that $\{\varphi_n \oplus \psi_n\}$ is approximately unitarily equivalent to $\{H \circ \varphi_n\}$. By 9.4, $\langle \varphi_n \oplus \psi_n \rangle = 0$. □

Definition 9.6. Fixed A as in 9.3, we will consider $\mathcal{E}(A, B)$ for separable C^*-algebra B, and denote $\mathcal{E}(A, B)$ by $\mathcal{E}_A(B)$. Suppose that B and C are separable C^*-algebras and $h : B \to C$ is a homomorphism. Define $h_* := \mathcal{E}_A(h) : \mathcal{E}_A(B) \to \mathcal{E}_A(C)$ by $\mathcal{E}_A(h)(\langle \varphi \rangle) = \langle \{h \circ \varphi_n\} \rangle$, where $\{\varphi_n\}$ is a representation of $\langle \varphi \rangle$ and where we also use h for $h^{\sim} \otimes \mathrm{id}_{\mathcal{Z}_0 \otimes \mathcal{K}}$. This gives a homomorphism from the abelian group $\mathcal{E}_A(B)$ to the abelian $\mathcal{E}_A(C)$. Clearly $\mathcal{E}_A(\mathrm{id}_B) = \mathrm{id}_{\mathcal{E}_A(B)}$. If D is another C^*-algebra and $h_1 : C \to D$ is a homomorphism, then one checks that $\mathcal{E}_A(h_1 \circ h) = \mathcal{E}_A(h_1) \circ \mathcal{E}_A(h)$.

Theorem 9.7. *$\mathcal{E}(A, -) = \mathcal{E}_A(-)$ is a covariant functor from separable C^*-algebras to abelian groups which is homotopy invariant and stable, i.e., $\mathcal{E}_A(D) = \mathcal{E}_A(D \otimes \mathcal{K})$.*

Proof. From 9.5 and 9.6, $\mathcal{E}_A(-)$ is a covariant functor from separable C^*-algebras to abelian groups. It is obviously stable. We will show it is homotopy invariant.

Fix a C^*-algebra B. Set $C = B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$. Let φ and ψ be two homotopy asymptotic sequential morphisms from A to C. Let $\delta > 0$ and $\mathcal{G} \subset A$.

Fix a large integer n. As discussed in 9.2, we may assume that there exists $\mathcal{G}$-δ-multiplicative completely positive contractive linear map $L_n : A \to C([0,1], C)$ which is such that $\pi_0 \circ L_n = \varphi_n$ and $\pi_1 \circ L_n = \psi_n$.

Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset.

Let $\mathcal{F}_1$ be a finite subset which contains $\mathcal{F}$. Let $\mathcal{P} : 0 = t_0 < t_1 < \cdots < t_k = 1$ be a partition such that

$$\|\pi_t \circ L_n(g) - \pi_{t_i} \circ L_n(g)\| < \varepsilon/4 \ \text{ for all } \ g \in \mathcal{F}_1 \tag{e9.12}$$

for all $t \in [t_{i-1}, t_{i+1}]$, $i = 1, 2, \ldots, k$. Put $\gamma_i = \pi_{t_i} \circ L_n$, $i = 0, 1, 2, \ldots, k$. Define $\Phi_n, \Psi_n, \Phi_n', \Psi_n' : A \to M_{2k+1}(C)$ as follows.

$$\Phi_n(a) = \mathrm{diag}(\gamma_0(a), \iota^{\circledast} \circ \gamma_1(a), \gamma_1(a), \ldots, \iota^{\circledast} \circ \gamma_k(a), \gamma_k(a)), \tag{e9.13}$$
$$\Phi_n'(a) = \mathrm{diag}(\gamma_0(a), \iota^{\circledast} \circ \gamma_0(a), \gamma_1(a), \ldots, \iota^{\circledast} \circ \gamma_{k-1}(a), \gamma_k(a)), \tag{e9.14}$$
$$\Psi_n'(a) = \mathrm{diag}(\gamma_k(a), \iota^{\circledast} \circ \gamma_0(a), \gamma_0(a), \ldots, \iota^{\circledast} \circ \gamma_{k-1}(a), \gamma_{k-1}(a)), \tag{e9.15}$$
$$\Psi_n(a) = \mathrm{diag}(\gamma_k(a), \iota^{\circledast} \circ \gamma_1(a), \gamma_1(a), \ldots, \iota^{\circledast} \circ \gamma_k(a), \gamma_k(a)) \tag{e9.16}$$

for all $a \in A$. We estimate that, by (e9.12),

$$\|\Phi_n(g) - \Phi_n'(g)\| < \varepsilon/4 \ \text{ and } \ \|\Psi_n(g) - \Psi_n'(g)\| < \varepsilon/4 \ \text{ for all } \ g \in \mathcal{F}_1. \tag{e9.17}$$

There is also a unitary $u \in M_{2k+1}(\tilde{C})$ such that

$$\|\mathrm{Ad}\, u \circ \Phi_n'(g) - \Psi_n'(g)\| < \varepsilon/4 \ \text{ for all } \ g \in \mathcal{F}_1. \tag{e9.18}$$

It follows that

$$\|\mathrm{Ad}\, u \circ \Phi_n(f) - \Psi_n(f)\| < 3\varepsilon/4 \ \text{ for all } \ f \in \mathcal{F}. \tag{e9.19}$$

Define $\Theta : A \to M_{2k}(C)$ by

$$\Theta(a) = \mathrm{diag}(\iota^{\circledast} \circ \gamma_1(a), \gamma_1(a), \ldots, \iota^{\circledast} \circ \gamma_k(a), \gamma_k(a))$$

for all $a \in A$. Then (e9.19) becomes

$$\|\mathrm{Ad}\, u \circ \mathrm{diag}(\varphi_n(g), \Theta(g)) - \mathrm{diag}(\psi_n(g), \Theta(g))\| < 3\varepsilon/4 \ \text{ for all } \ g \in \mathcal{F}_1. \tag{e9.20}$$

On the other hand, by 8.14, there exists a homomorphism $H : B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K} \to B^{\vdash} \otimes \mathcal{W} \otimes \mathcal{K}$ and $\mathcal{G}$-δ-multiplicative completely positive contractive linear map $\Gamma_n : A \to C$ such that

$$\|H \circ \Gamma_n(g) - \Theta(g)\| < \varepsilon/8 \ \text{ for all } \ g \in \mathcal{F}_1 \tag{e9.21}$$

($\Gamma_n = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_k)$). Finally, we obtain that

$$\|\mathrm{Ad}\, u \circ \mathrm{diag}(\varphi_n(f), H \circ \Gamma_n(f)) - \mathrm{diag}(\psi_n(f), H \circ \Gamma_n(f))\| < \varepsilon$$

for all $f \in \mathcal{F}$. Since the image of $H \circ \Gamma_n$ are in $B^{\vdash} \otimes \mathcal{W} \otimes \mathcal{K}$, the above implies that $\langle \varphi \rangle = \langle \psi \rangle$. □

The proof of the following is essentially the same as that in 6.1.4 of [30].

Proposition 9.8. *If*

$$0 \to J \xrightarrow{j} D \xrightarrow{\pi} D/J \to 0 \tag{e9.22}$$

is a split short exact sequence of separable C^*-algebras, then

$$\mathcal{E}(A, J) \xrightarrow{j_*} \mathcal{E}(A, D) \xrightarrow{\pi_*} \mathcal{E}(A, D/J)$$

is exact in the middle.

Proof. Suppose that $\langle \varphi \rangle \in \mathcal{E}(A, J)$ which can be represented by an asymptotic sequential morphism $\{\varphi_n\}$ which maps A to $J^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$. Then $\pi \circ j \circ \varphi_n$ has image in $\mathbb{C} \cdot 1_{(D/J)^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$. It follows from the definition that $\pi_* \circ j_* = 0$.

Now assume that $\langle \varphi \rangle \in \mathcal{E}(A, D)$ which is represented by $\{\varphi_n\}$. Without loss of generality, we may assume that $\mathrm{im}\varphi_n \in M_{k(n)}(D^{\vdash} \otimes \mathcal{Z}_0)$ for some sequence $\{k(n)\}$.

Suppose that $\pi_*(\langle \varphi \rangle) = 0$. Thus we may assume that there exist two asymptotic sequential morphisms $h_n, h_n' : A \to \mathbb{C} \cdot 1_{(D/J)^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ and a sequence of unitaries $u_n \in ((D/J)^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})^{\sim}$ such that

$$\lim_{n \to \infty} \|u_n^* \mathrm{diag}(\pi \circ \varphi_n(a), h_n(a)) u_n - h_n'(a)\| = 0 \ \text{ for all } \ a \in A. \tag{e9.23}$$

By the proof of 9.5, by adding $\iota^{\circledast} \circ h'_n$ (to both terms), we may assume that $[h'_n]|_{\mathcal{P}}= 0$ for all $n \geq n_0$. We also assume that there exists $\alpha \in Hom_{\Lambda}(\underline{K}(A), \underline{K}(D^{\vdash} \otimes \mathcal{Z}_0))$ such that, for any finite subset $\mathcal{P} \subset \underline{K}(A)$ and for all $n \geq n_0$ (for some $n_0 \geq 1$),

$$[\varphi_n]|_{\mathcal{P}}+[h_n]|_{\mathcal{P}}= [J_D] \circ \lambda_D \circ \alpha|_{\mathcal{P}}+[h''_n]|_{\mathcal{P}} \text{ (see (e9.3) for the definition of } \lambda_D), \tag{e9.24}$$

where $\{h''_n\}$ is a sequence approximately multiplicative completely positive contractive linear maps from A to $\mathbb{C} \cdot 1_{(D)^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$, and we also view h_n and h'_n as maps from A to $\mathbb{C} \cdot 1_{(D)^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$, and where $J_D : D \otimes \mathcal{Z}_0 \to D^{\vdash} \otimes \mathcal{Z}_0$ is the embedding. Thus, combining with (e9.24), $[\pi] \circ (\lambda_D(\alpha)) = 0$.

Denote $\Pi_{D/J} : ((D/J)^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})^{\sim} \to \mathbb{C}$ the quotient map. Without loss of generality, we may assume that $\mathrm{im}\,(\varphi_n \oplus h_n), \mathrm{im}\, h'_n \subset M_{K(n)}(\mathbb{C} \cdot 1_{(D/J)^{\vdash}} \otimes \mathcal{Z}_0)$. We may further assume that $K(n) = 2k(n)$. We may view $\mathrm{diag}(u_n, u_n^*) \in ((D/J)^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})^{\sim}$. Replacing u_n by $\mathrm{diag}(u_n, u_n^*)$, we may assume that $u_n \in U_0(((D/J)^{\vdash} \otimes \mathcal{K})^{\sim})$. Therefore, we may assume that there exists a unitary $z_n \in U((D^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})^{\sim})$ such that $\pi(z_n) = u_n$.

By identifying $\mathbb{C} \cdot 1_{(D/J)^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ with $\mathbb{C} \cdot 1_{D^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ and $\mathbb{C} \cdot 1_{J^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$, we may view $h_n, h'_n : A \to \mathbb{C} \cdot 1_{D^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ as well as $h_n, h'_n : A \to \mathbb{C} \cdot 1_{J^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$, whichever it is convenient.

Let $\Pi : l^{\infty}(D^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}) \to l^{\infty}(D^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})/c_0(D^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})$ be the quotient map. Let

$$U = \{z_n\}, \quad Z = \Pi(U), \Phi = \{\varphi_n\}, \quad H = \{h_n\}, H' = \{h'_n\},$$

where we view $\Phi, H, H' : A \to l^{\infty}(D^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})$. Then, by (e9.23)

$$Z^*(\Pi(\Phi(a) \oplus H(a)))Z - \Pi \circ H'(a) \in l^{\infty}(J^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})/c_0(J^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})$$

for all $a \in A$. Since $\Pi \circ H'(a), \ \Pi \circ H(a) \in \mathbb{C} \cdot 1_{J^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$, it follows that

$$Z^*(\Pi(\Phi(a) \oplus 0))Z \in l^{\infty}(J^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})/c_0(J^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K})$$

for all $a \in A$. Since A is amenable, by [6], there exists a completely positive contractive linear map $L = \{l_n\} : A \to J^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ such that $\Pi \circ L = \mathrm{Ad}\, U \circ (\Phi)$. Also

$$\lim_{n\to\infty} \|\mathrm{diag}(l_n(a), h_n(a)) - z_n^*(\mathrm{diag}(\varphi_n(a), h_n(a)))z_n\| = 0 \text{ for all } a \in A. \tag{e9.25}$$

Since (e9.22) is split exact, by Proposition 5.2 of [31], there is a unique $\beta \in \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(J \otimes \mathcal{Z}_0))$ such that $[j] \circ \beta = \lambda_D \circ \alpha$. It follows (by (e9.24) and (e9.25)) that, viewing l_n as maps from A to $J^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$, there exist two sequences of approximately multiplicative completely positive contractive linear maps $H_n, H''_n : A \to \mathbb{C} \cdot 1_{J^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$, for any finite subset $\mathcal{P} \subset \underline{K}(A)$, such that, for all $n \geq n_1$ (for some $n_1 \geq 1$),

$$[l_n]|_{\mathcal{P}}+[H_n]|_{\mathcal{P}}= [J_J] \circ \beta|_{\mathcal{P}}+[H''_n]|_{\mathcal{P}} \quad (J_J : J \otimes \mathcal{Z}_0 \to J^{\vdash} \otimes \mathcal{Z}_0 \text{ is the embedding}).$$

So $\langle\{l_n\}\rangle$ is an asymptotic sequential morphism in $\mathcal{E}(A, J)$ and (by (e9.25)) $j_*\langle\{l_n\}\rangle = \langle\varphi_n\rangle$ which implies that $\langle\varphi\rangle$ is in the $j_*(\mathcal{E}(A, J))$. □

Proposition 9.9. *$\mathcal{E}_A(-)$ is split exact.*

Proof. This is standard from 9.7 and 9.8 (see [22]). Let

$$0 \to J \xrightarrow{j} D \xrightarrow{\pi} D/J \to 0$$

be a short exact sequence of separable C^*-algebras.

Let us first assume that D/J is contractible. Then by 9.7, $\mathcal{E}_A(D/J) = \{0\}$. It follows from 9.8 that j_* gives a surjective map from $\mathcal{E}_A(J)$ onto $\mathcal{E}_A(D)$.

For C^*-algebra C, denote by $S(C) = C_0((0, 1), C)$. Then, by 9.7,

$$\mathcal{E}_A(D/J) = 0 = \mathcal{E}_A(S(D/J))$$

Put

$$S(D, D/J) = \{(a, b) \in D \oplus C_0([0, 1), D/J) : \pi(a) = b(0)\} \text{ and} \tag{e9.26}$$

$$Z(J, D) = \{x \in C([0, 1], D) : x(0) \in J\}. \tag{e9.27}$$

We have the following exact sequence:

$$0 = \mathcal{E}_A(S(D/J)) \longrightarrow \mathcal{E}_A(S(D, D/J)) \longrightarrow \mathcal{E}_A(D). \tag{e9.28}$$

Define $\pi' : Z(J, D) \to C_0([0, 1), D/J)$ by $\pi'(f)(t) = \pi(f)(1-t)$ for $t \in [0, 1)$. Note $\pi'(f)(1) = \pi(f)(0) = 0$ for all $f \in Z(J, D)$. Define $\chi : Z(J, D) \to S(D, D/J)$ by

$$\chi(f) = (f(1), \pi'(f)) \text{ for all } f \in Z(J, D).$$

One obtains the short exact sequence:

$$0 \to C_0([0, 1), J) \to Z(J, D) \to S(D, D/J) \to 0.$$

This gives the following exact sequence:

$$0 = \mathcal{E}_A(C_0([0,1),J)) \longrightarrow \mathcal{E}_A(Z(J,D)) \longrightarrow \mathcal{E}_A(S(D,D/J)). \tag{e9.29}$$

From (e9.28) and (e9.29), it follows that composition map $\mathcal{E}_A(Z(J,D)) \to \mathcal{E}_A(S(D,D/J)) \to \mathcal{E}_A(D)$ is injective.

However, $Z(J,D)$ is homotopically equivalent to J. Moreover, one sees that the composition $J \to Z(J,D) \to S(D,D/J) \to D$ coincides with j. It follows that j_* is injective.

Thus we show that, when D/J is contractible, j_* is an isomorphism from $\mathcal{E}_A(J)$ onto $\mathcal{E}_A(D)$.

In general, let $\imath : J \to S(D,D/J)$ be defined by $\imath(b) = (b,0)$ for $b \in J$. Then $S(D,D/J)/\imath(J) \cong C_0([0,1),D/J)$ which is contractible. So, from what has been proved, $\imath_*$ is an isomorphism.

To see that $\mathcal{E}_A(-)$ is split exact, consider the short exact sequence of separable C^*-algebras:

$$0 \to J \xrightarrow{j} D \underset{s}{\overset{\pi}{\rightleftarrows}} D/J \to 0.$$

By 9.8,

$$\mathcal{E}_A(J) \xrightarrow{j} \mathcal{E}_A(D) \xrightarrow{\pi} \mathcal{E}_A(D/J)$$

is exact in the middle. Since $\pi \circ s = \mathrm{id}_{D/J}$, we check that $\pi_* \circ s_* = (\mathrm{id}_{D/J})_*$.

It remains to show that j_* is injective. Using the exact sequence

$$\mathcal{E}_A(S(D/J)) \to \mathcal{E}_A(S(D,D/J)) \to \mathcal{E}_A(D),$$

and identifying $\mathcal{E}_A(J)$ with $\mathcal{E}_A(S(D,D/J))$, we see that $\ker j_* \subset \mathrm{im}\,(\imath_1)_*$ where $\imath_1 : S(D/J) \to S(D,D/J)$ is the embedding.

Let

$$I = \{(s(b(0)), b) \in S(D,D/J) : b \in C_0([0,1),D/J)\},$$

where s is the split map given above. Since $\pi \circ s = \mathrm{id}_{D/J}$, $I \cong C_0([0,1),D/J)$ which is contractible. On the other hand, $\mathrm{im}\,\imath_1 \subset I$. Therefore $(\imath_1)_* = 0$. Thus $\ker j_* = 0$. In other words, j_* is injective. $\square$

10. An existence theorem

Definition 10.1. Fix A as in 9.3. We assume that A satisfies the UCT. There is a homomorphism β_A^B from $\mathcal{E}_A(B)$ to $KL(A,B)$ defined as follows.

We will identify $KL(A,C)$ with $\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(C))$ for any separable C^*-algebra C (see [10]). Let $\langle\varphi\rangle \in \mathcal{E}_A(B) := \mathcal{E}(A,B)$ be represented by an asymptotic morphism $\{\varphi_n\}$. Recall (see 9.1) that $\beta_A(\langle\varphi_n\rangle) = \lambda_B \circ \alpha$ is well defined which defines a homomorphism $\beta_A^B : \mathcal{E}_A(B) \to KL(A,B)$. Consequently β_A^B is a morphism which maps C^*-algebra B to the abelian group $\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B \otimes \mathcal{Z}_0))$. If B and C are two C^*-algebras and $h : B \to C$ is a homomorphism we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}_A(B) \arrow[r, "\mathcal{E}_A(h)"] \arrow[d, "\beta_A^B"'] & \mathcal{E}_A(C) \arrow[d, "\beta_A^C"] \\
Hom_\Lambda(A, B \otimes \mathcal{Z}_0 \otimes \mathcal{K}) \arrow[r, "{[h]}"'] & Hom_\Lambda(A, C \otimes \mathcal{Z}_0 \otimes \mathcal{K}).
\end{tikzcd}$$

It follows that

$$\beta : \mathcal{E}_A(-) \to Hom_\Lambda(\underline{K}(A), \underline{K}(- \otimes \mathcal{Z}_0))$$

is a natural transformation.

Theorem 10.2. *The transformation β_A maps $\mathcal{E}_A(B)$ onto $Hom_\Lambda(A, B \otimes \mathcal{Z}_0)$ for each separable C^*-algebra B, if A satisfies the UCT.*

Proof. By a theorem of Higson (Theorem 3.7 of [22]), since $\mathcal{E}_A(-)$ is a covariant functor from separable C^*-algebras to abelian groups which is homotopy invariant, stable and split exact (Section 8), there is a unique transformation

$$\alpha : KK(A,-) \to \mathcal{E}_A(-)$$

such that $\alpha_A([\mathrm{id}_A]_{KK}) = \langle \mathrm{id}_A \rangle$. Let $\gamma : KK(A,-) \to KL(A,-)$ be the natural transformation induced by $\Gamma : KK(A,B) \to KL(A,B)$. We have

$$\beta_A \circ \alpha_A([\mathrm{id}_A]) = [\mathrm{id}_A]_{KL},$$

where β is defined in 10.1. Since $\gamma([\mathrm{id}_A]) = [\mathrm{id}_A]$, (the first $[\mathrm{id}_A]$ is in $KK(A,A)$ and the second is in $KL(A,A)$), by the uniqueness of Higson's theorem (3.7 of [22]),

$$\beta \circ \alpha = \gamma.$$

Since $\gamma(KK(A,B)) = Hom_\Lambda(A, B \otimes \mathcal{Z}_0 \otimes \mathcal{K})$, if A satisfies the UCT, $\beta_A : \mathcal{E}_A(B) \to KL(A,B)$ must be surjective for each B. $\square$

Lemma 10.3. *Let B a non-unital and separable simple C^*-algebra with continuous scale. Let $\varphi_0, \varphi_1, \varphi_2 : \mathcal{W} \to M(B)/B$ be homomorphisms with φ_0 nonzero. Then, for any $\varepsilon > 0$, and any finite subset $\mathcal{F} \subset \mathcal{W}$, there exists a unitary $U \in M_2(M(B))$ such that*

$$\|\pi(U)^*\mathrm{diag}(\varphi_1(a), \varphi_0(a))\pi(U) - \mathrm{diag}(\varphi_2(a), \varphi_0(a))\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Proof. It follows from [28] that $M(B)/B$ is simple and purely infinite.

Fix a strictly positive element $a_W \in \mathcal{W}$ with $\|a_W\| = 1$. Let $b_0 = \varphi_0(a_W)$ and let $B_0 = \overline{b_0(M(B)/B)b_0}$.

Since $\mathcal{W}$ and B_0 are both simple, there is a map $T : \mathcal{W}_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ such that $\varphi_0 : \mathcal{W} \to B_0$ is T-$\mathcal{W}_+ \setminus \{0\}$-full. Let $W_0 = \varphi_0(\mathcal{W})$. So $b_0 \in W_0$.

Let $\mathcal{H} \subset \mathcal{W}_+ \setminus \{0\}$ be a finite subset and $K \geq 1$ be an integer as required by Cor. 3.16 of [16] for the above given T, $\varepsilon/2$ (in place of ε) and $\mathcal{F}$.

Note that $\mathcal{W} \otimes Q \cong \mathcal{W}$. Moreover, the map from $\mathcal{W}$ to $\mathcal{W} \otimes 1_Q$ which maps a to $a \otimes 1_Q$ then to $\mathcal{W}$ is approximately inner. To simplify notation, without loss of generality, we may assume that $\varphi_0 : \mathcal{W} \to W_0 \otimes Q$ has the form $\varphi_0(a) \otimes 1_Q$. Let $e_1, e_2, \ldots, e_K \in Q$ be mutually orthogonal and mutually equivalent projections such that $\sum_{i=1}^K e_i = 1_Q$. Define $\varphi_{0,i} : \mathcal{W} \to W_0 \otimes e_i$ by

$$\varphi_{0,i}(a) = \varphi_0(a) \otimes e_i \text{ for all } a \in \mathcal{W}.$$

Put $B_{0,1} = \overline{(b_0 \otimes e_1)(M(B)/B)(b_0 \otimes e_1)}$.

Let $b_1 = \varphi_1(a_W)$, $b_2 = \varphi_2(a_W) \in M(B)/B$. Since $\mathcal{W}$ is projectionless, $\mathrm{sp}(a_W) = [0, 1]$. Thus, since $\mathcal{W}$ is simple, b_1 cannot be invertible in $M(B)/B$. Let $D_1 = \overline{b_1 A b_1}$. By Pedersen's double orthogonal complement theorem (Theorem 15 of [45]), there is a projection $E_1 \in M(B)/B$ such that $1_{M(B)/B} - E_1 \in D^\perp$ is not zero and $E_1 b_1 = b_1 E_1 = b_1$. Similarly, one obtains a projection $E_2 \in M(B)/B$ such that $1_{M(B)/B} - E_2 \neq 0$ and $E_2 b_2 = b_2 E_2 = b_2$. Using the fact that $M(B)/B$ is purely infinite simple again, one obtains a unitary $w_1 \in M(B)/B$ such that

$$w_1^* E_2 w_1 \leq E_1.$$

Thus, without loss of generality, replacing φ_2 by $\mathrm{Ad}\, w_1 \circ \varphi_2$, one may assume that $E_2 \leq E_1$.

Since $M(B)/B$ is purely infinite and simple, $E_1 \lesssim p_0'$ for some projection $p_0' \in B_0$. Thus we obtain a unital hereditary C^*-subalgebra $B_{00} \subset M(B)/B$ such that, we may view that $\varphi_1, \varphi_2 : \mathcal{W} \to B_{00}$ and $\varphi_{0,1} : \mathcal{W} \to B_{00}$ is a T-$\mathcal{W}_+ \setminus \{0\}$-full. Moreover, we view

$$\varphi_0(a) = \mathrm{diag}(\overbrace{\varphi_{0,1}(a), \varphi_{0,1}(a), \ldots, \varphi_{0,1}(a)}^{K}) \text{ for all } a \in \mathcal{W}.$$

Furthermore, $M_{K+1}(B_{00})$ is a unital C^*-subalgebra of $M_2(M(B)/B)$ such that $1_{M_{K+1}(B_{00})}$ is not the unit of $M_2(M(B)/B)$. By applying 3.16 of [16], there is a unitary $u \in M_{K+1}(B_{00}) \subset M_2(M(B)/B)$ such that

$$\|u^*(\mathrm{diag}(\varphi_1(a), \varphi_0(a)))u - \mathrm{diag}(\varphi_2(a), \varphi_0(a))\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Since $1_{M_2} - 1_{M_{K+1}(B_{00})} \neq 0$ and $M_2(M(B)/B)$ is purely infinite and simple, there exists a unitary $v \in (1_{M_2} - 1_{M_{K+1}(B_{00})})(M(B)/B)(1_{M_2} - 1_{M_{K+1}(B_{00})})$ such that $u \oplus v \in U_0(M_2(M(B)/B))$. Thus we may assume that u is a unitary in $U_0(M_2(M(B)/B))$. Hence there is a unitary $U \in M_2(M(B))$ such that $\pi(U) = u$. $\square$

10.4 (*Construction of φ_W*). Let B be a non-unital separable simple C^*-algebra with stable rank one, with $T(B) \neq \emptyset$ and with continuous scale.

Let $\{e_n\} \subset B \otimes \mathcal{Z}_0$ be an approximate identity with

$$e_{n+1}e_n = e_n e_{n+1} = e_n \text{ for all } n \in \mathbb{N}.$$

We may assume that $e_{n+1} - e_n \neq 0$ for all $n \geq 1$. Choose $k(n) \geq 1$ such that

$$\inf\{d_\tau(e_{4n} - e_{4n-1}) : \tau \in T(B \otimes \mathcal{Z}_0)\} > \frac{1}{k(n)}, \quad n = 1, 2, \ldots.$$

Note that $\sum_{n=1}^\infty \frac{1}{k(n)} < 1$. Put $B_n := \overline{(e_{4n} - e_{4n-1})(B \otimes \mathcal{Z}_0)(e_{4n} - e_{4n-1})}$. Fix a strictly positive element $a_w \in \mathcal{W}$ with $\|a_w\| = 1$.

It follows from [49] that there is a homomorphism $\varphi_{0,n} : \mathcal{W} \to B_n$ such that

$$d_\tau(\varphi_{0,n}(a_w)) = \frac{1}{k(n)} \text{ for all } \tau \in T(B).$$

Let $\varphi_{even}, \varphi_{odd}, \varphi_W : \mathcal{W} \to M(B \otimes \mathcal{Z}_0)$ be defined by

$$\varphi_{even} = \sum_{n=1}^{\infty} \varphi_{0,2n}, \quad \varphi_{odd} = \sum_{n=1}^{\infty} \varphi_{0,2n+1} \text{ and} \tag{e10.1}$$

$$\varphi_W = \sum_{n=1}^{\infty} \varphi_{0,n} = \mathrm{diag}(\varphi_{even}, \varphi_{odd}). \tag{e10.2}$$

Proposition 10.5. *Let B be a non-unital separable simple C^*-algebra with stable rank one, with $T(B) \neq \emptyset$ and with continuous scale. Fix an integer $k \geq 1$. Let $j_{w,z} : \mathcal{W} \to M_k(\mathcal{Z}_0)$ be an embedding which maps strictly positive elements to strictly positive elements and $d : \mathcal{Z}_0 \to \mathbb{C} \cdot 1_{M_k(\tilde{B})} \otimes \mathcal{Z}_0 \subset M_k(\tilde{B} \otimes \mathcal{Z}_0) \subset M(M_k(B \otimes \mathcal{Z}_0))$ be the embedding defined by $d(z) = 1 \otimes z$ for all $z \in \mathcal{Z}_0$.*

Let $\varepsilon > 0$ and $\mathcal{F} \subset \mathcal{W}$ be a finite subset. Then there is an integer $K \geq 1$ and a unitary $u \in M_{K+1}(M(M_k(B \otimes \mathcal{Z}_0)))$ such that

$$\|u^* \mathrm{diag}(d_K \circ j_{w,z}(a), 0)u - (d_K \circ j_{w,z}(a) \oplus \varphi_{odd}(a))\| < \varepsilon \text{ for all } a \in \mathcal{F},$$

where

$$d_K(z) = \mathrm{diag}(\overbrace{d(z), d(z), \ldots, d(z)}^{K}) \text{ for all } z \in \mathcal{Z}_0.$$

Proof. The proof has the same spirit as that of 10.3. Keep in mind that B has continuous scale. Therefore $M(M_k(B \otimes \mathcal{Z}_0))$ has only one (closed) ideal $M_k(B \otimes \mathcal{Z}_0)$ (see [28]). Since $\mathcal{W}$ is simple and $d \circ j_{w,z}$ maps a strictly positive element to that of $\mathbb{C} \cdot 1_{M_k(\tilde{B})} \otimes \mathcal{Z}_0$ which is not in $M_k(B \otimes \mathcal{Z}_0)$, $d \circ j_{w,z}(a)$ is full in $M(M_k(B \otimes \mathcal{Z}_0))$ for every $a \in \mathcal{W}_+ \setminus \{0\}$. There is a map $T : \mathcal{W}_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ such that $d \circ j_{w,z}$ is T-$\mathcal{W}_+ \setminus \{0\}$-full in $M(M_k(B \otimes \mathcal{Z}_0))$.

Let $K \geq 1$ be the integer required by Cor. 3.16 of [16] for $\varepsilon/2$ (in place of ε), $\mathcal{F}$ and T. By applying 3.16 of [16] (and considering φ_{odd} and zero map), one obtains (note that $M(M_k(B \otimes \mathcal{Z}_0))$ is unital) a unitary $v \in M_{K+1}(M(M_k(B \otimes \mathcal{Z}_0)))$ such that

$$\|u^* \mathrm{diag}(d_K \circ j_{w,z}(a), 0)u - (d_K \circ j_{w,z}(a) \oplus \varphi_{odd}(a))\| < \varepsilon \text{ for all } a \in \mathcal{F}. \quad \square$$

Lemma 10.6. *For any $\varepsilon > 0$, there is $\delta > 0$ satisfying the following: for any $e \in A_+$ with $\|e\| \leq 1$ and any $a \in A$ with $\|a\| \leq 1$,*

$$\|e^{1/2} a e^{1/2} - ea\| < \varepsilon$$

whenever $\|ea - ae\| < \delta$.

In the following statement and the proof we keep notations in 10.4 and 10.5.

Theorem 10.7. *Let A be a non-unital separable amenable C^*-algebra. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be finite subset.*

There exists $\delta > 0$ with $\delta < \varepsilon/2$, a finite subset $\mathcal{G} \subset A$ with $\mathcal{F} \subset \mathcal{G}$ and an integer $K \geq 1$ satisfying the following: For any $\mathcal{G}$-δ-multiplicative completely positive contractive linear map $\varphi : A \to M_k(\tilde{B} \otimes \mathcal{Z}_0)$ (for any non-unital separable simple C^*-algebra B with continuous scale and any integer $k \geq 1$) such that if there are homomorphisms $\psi_{z,w} : M_k(\mathcal{Z}_0) \to \mathcal{W}$ and $\psi_{w,z} : \mathcal{W} \to M_k(\mathbb{C} \cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0) \cong M_k(\mathcal{Z}_0)$ which map strictly positive elements to strictly positive elements such that

$$\|\pi \circ (\varphi(a)) - \psi_{w,z} \circ \psi_{z,w} \circ \pi \circ (\varphi(a))\| < \delta \text{ for all } a \in \mathcal{G},$$

where $\pi : M_k(\tilde{B} \otimes \mathcal{Z}_0) \to M_k(\mathbb{C} \cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0)$ is the quotient map, then there exist an $\mathcal{F}$-ε-multiplicative completely positive contractive linear map $L_0 : A \to M_{K+2}(M_k(B \otimes \mathcal{Z}_0))$ and an $\mathcal{F}$-ε-multiplicative completely positive contractive linear map $L_1 : A \to M_{K+2}(M_k(\tilde{B} \otimes \mathcal{Z}_0))$ such that

$$\|L_0(a) \oplus L_1(a) - \varphi(a) \oplus d_K \circ s \circ \varphi^{\pi}(a)\| < \varepsilon \text{ for all } a \in \mathcal{F},$$

where $\varphi^{\pi} = \psi_{w,z} \circ \psi_{z,w} \circ \pi \circ \varphi$, $s : M_k(\mathbb{C} \cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0) \to M_k(\tilde{B} \otimes \mathcal{Z}_0)$ is the nature embedding, and furthermore, the following are true:

$$(1) \qquad L_0(a) = p_m^{1/2}(\varphi(a) \oplus d_K \circ s \circ \varphi^{\pi}(a))p_m^{1/2} \text{ for all } a \in A$$

for some $m \geq m_0$, where $\{p_m\}$ is an approximate identity for $M_{K+2}(M_k(B \otimes \mathcal{Z}_0))$ and,

(2) there are $\mathcal{G}$-δ-multiplicative completely positive contractive linear map $L_{00} : A \to \mathcal{W}$ and $L_{0,0}(\mathcal{F})$-$\varepsilon/2$-multiplicative completely positive contractive linear map $L_{w,b} : \mathcal{W} \to M_{K+2}(M_k(\tilde{B} \otimes \mathcal{Z}_0))$ such that $L_1 = L_{w,b} \circ L_{00}$.

Proof. Fix $1/2 > \varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$. We may assume that $\mathcal{F} \subset A^{\mathbf{1}}$.

Let $\mathcal{G} = \{ab : a, b \in \mathcal{F}\} \cup \mathcal{F}$. Let $\{e_n\} \subset M_k(B)$ be an approximate identity as described in 10.4. Let $\delta_1 > 0$ (in place of δ) be in 10.6 for $\varepsilon/64$.

Let $\delta = \min\{\delta_1/2^{12}, \varepsilon/2^{12}\}$. We view $M_k(\tilde{B}\otimes\mathcal{Z}_0)$ as a C^*-subalgebra of $M(M_k(B\otimes\mathcal{Z}_0))$. Suppose that $\varphi : A \to M_k(\tilde{B}\otimes\mathcal{Z}_0)$ is $\mathcal{G}$-δ-multiplicative completely positive contractive linear map. Suppose that there are homomorphisms $\psi_{z,w} : M_k(\mathcal{Z}_0) \to \mathcal{W}$ and $\psi_{w,z} : \mathcal{W} \to M_k(\mathbb{C}\cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0)$ such that

$$\|\pi \circ \varphi(a) - (\psi_{w,z} \circ \psi_{z,w} \circ \pi \circ (\varphi(a)))\| < \delta \text{ for all } a \in \mathcal{G}. \tag{e10.3}$$

Recall that $\varphi^\pi = \psi_{w,z}\circ\psi_{z,w}\circ\pi\circ\varphi$. Put $\varphi^W = \psi_{z,w}\circ\pi\circ\varphi$. Thus $\psi_{w,z}\circ\varphi^W = \varphi^\pi$. Let K be the integer in 10.5 associated with δ (in place of ε) and $\varphi^W(\mathcal{G}) \subset \mathcal{W}$ (in place of $\mathcal{F}$).

By applying 10.3, we obtain a unitary $U_1 \in M_{K+2}(M(M_k(B\otimes\mathcal{Z}_0)))$ such that

$$\|\Pi(U_1)^*\Pi \circ \varphi_W(\varphi^W(a))\Pi(U_1) - \mathrm{diag}(\Pi \circ d_{K+1} \circ \psi_{w,z} \circ \varphi^W(a)), \Pi \circ \varphi_{odd}(\varphi^W(a))\| < \delta \tag{e10.4}$$

for all $a \in \mathcal{G}$, where $\Pi : M_{K+2}(M(M_k(B\otimes\mathcal{Z}_0))) \to M_{K+2}(M((M_k(B\otimes\mathcal{Z}_0))/(M_k(B\otimes\mathcal{Z}_0))))$ is the quotient map.

Let $s : M_k(\mathbb{C}\cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0) \to M_k(\tilde{B}\otimes\mathcal{Z}_0)$ be the embedding such that

$$\pi \circ s(a) = a \text{ for all } a \in M_k(\mathbb{C}\cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0).$$

Consider $L_{1,1} : A \to (M(M_k(B\otimes\mathcal{Z}_0)))$ defined by $L_{1,1} = \varphi_W \circ \varphi^W$ and $L'_{1,0} : A \to M_{K+2}(M(M_k(B\otimes\mathcal{Z}_0)))$ defined by

$$L'_{1,0}(a) = \mathrm{diag}(d'_{K+1} \circ s \circ \psi_{w,z} \circ \varphi^W(a)), \varphi_{odd}(\varphi^W(a)) \text{ for all } a \in A,$$

where notation $d'_m(c)$ means the following:

$$d'_m(c) = \mathrm{diag}(\overbrace{c, c, \ldots, c}^{m}).$$

By 10.5, there is another unitary $U_2 \in M_{K+2}(M(M_k(B\otimes\mathcal{Z}_0)))$ such that

$$\|U_2^* L'_{1,0}(a) U_2 - \mathrm{diag}(d'_{K+1} \circ s \circ \psi_{w,z} \circ \varphi^W(a), 0)\| < \delta \text{ for all } a \in \mathcal{G}. \tag{e10.5}$$

Define a homomorphism $L_{1,0} : A \to M_{K+1}(M_k(\tilde{B}\otimes\mathcal{Z}_0))$ by

$$L_{1,0}(a) = d'_{K+1} \circ s \circ \varphi^\pi(a) \text{ for all } a \in A.$$

Put $\Phi = \varphi \oplus d'_K \circ s \circ \varphi^\pi$ and $U = U_1U_2$. By (e10.4) and (e10.5), for each $a \in \mathcal{G}$, there exist $b(a), b'(a) \in M_{K+2}(M_k(B\otimes\mathcal{Z}_0))$ with $\|b(a)\| \le 1$, $\|b'(a)\| \le 1$ such that

$$\|U^*L_{1,1}(a)U - L_{1,0}(a) + b(a)\| < 2\delta \text{ and} \tag{e10.6}$$

$$\|U^*L_{1,1}(a)U - \Phi(a) + b'(a)\| < 2\delta \text{ for all } a \in \mathcal{G}. \tag{e10.7}$$

Put $\bar{e}_n = \mathrm{diag}(\overbrace{e_n, e_n, \ldots, e_n}^{K+2})$, $n = 1, 2, \ldots$. Let $p_n = U^*\bar{e}_nU$, $n = 1, 2, \ldots$. Then $\{p_n\}$ is an approximate identity for $M_{K+2}(M_k(B\otimes\mathcal{Z}_0))$. Let $S = \mathbb{N} \setminus \{4n-1, 4n : n \in \mathbb{N}\}$. If $m \in S$,

$$(1-p_m)(p_{4n}-p_{4n-1}) = \begin{cases} (p_{4n}-p_{4n-1}) & \text{if } m < 4n-1; \\ 0 & \text{if } m > 4n \end{cases} \text{ and} \tag{e10.8}$$

$$p_m(1-p_m)(p_{4n}-p_{4n-1}) = 0 \text{ for all } m \in \mathcal{S}. \tag{e10.9}$$

There is $N \ge 1$ such that, for any $m \ge N$ and $m \in S$,

$$\|(1-p_m)(U^*L_{1,1}(a)U) - (1-p_m)L_{1,0}(a)\| < 4\delta, \tag{e10.10}$$

$$\|(U^*L_{1,1}(a)U)(1-p_m) - L_{1,0}(a)(1-p_m)\| < 4\delta, \tag{e10.11}$$

$$\|(1-p_m)(U^*L_{1,1}(a)U) - (1-p_m)\Phi(a)\| < 4\delta \text{ and} \tag{e10.12}$$

$$\|(U^*L_{1,1}(a)U)(1-p_m) - \Phi(a)(1-p_m)\| < 4\delta \text{ for all } a \in \mathcal{G}. \tag{e10.13}$$

Note that, by the construction of φ_W and (e10.9), if $m \in S$,

$$(1-p_m)(U^*L_{1,1}(a)U) = (U^*L_{1,1}(a)U)(1-p_m) \tag{e10.14}$$

$$= (1-p_m)(U^*L_{1,1}(a)U)(1-p_m) \text{ for all } a \in A. \tag{e10.15}$$

It follows from (e10.10)–(e10.14), for all $m \ge N$ and $m \in S$,

$$\|p_m\Phi(a) - \Phi(a)p_m\| < 8\delta \text{ and } \|(1-p_m)L_{1,0}(a) - L_{1,0}(a)(1-p_m)\| < 8\delta \text{ for all } a \in \mathcal{G}. \tag{e10.16}$$

By the choice of δ_1 and 10.6, for all $a \in \mathcal{G}$,

$$\|p_m^{1/2}\Phi(a)p_m^{1/2} - p_m\Phi(a)\| < \varepsilon/64 \text{ and} \tag{e10.17}$$

$$\|(1-p_m)^{1/2}L_{1,0}(a)(1-p_m)^{1/2} - (1-p_m)L_{1,0}(a)\| < \varepsilon/64. \tag{e10.18}$$

Moreover, the map $a \mapsto (1-p_m)(U^*L_{1,1}(a)U)$ is $\mathcal{G}$-δ-multiplicative. By (e10.18) and (e10.10), $a \to (1-p_m)^{1/2}L_{1,0}(a)(1-p_m)^{1/2}$ is $\mathcal{F}$-ε-multiplicative. Define

$$L(a) = p_m\Phi(a) + (1-p_m)(U^*L_{1,1}(a)U) \text{ for all } a \in A.$$

Then, by (e10.12),

$$\|L(a) - \Phi(a)\| < 4\delta \text{ for all } a \in \mathcal{G}. \tag{e10.19}$$

Consequently,

$$\|L(ab) - L(a)L(b)\| < 8\delta \text{ for all } a, b \in \mathcal{F}. \tag{e10.20}$$

We compute that

$$L(ab) = p_m\Phi(ab) + (1-p_m)(U^*L_{1,1}(ab)U) \text{ for all } a, b \in A, \tag{e10.21}$$

and, for all $a, b \in \mathcal{G}$, by (e10.9), (e10.15) and (e10.16),

$$\begin{aligned} L(a)L(b) &= (p_m\Phi(a) + (1-p_m)(U^*L_{1,1}(a)U))(p_m\Phi(b) + (1-p_m)(U^*L_{1,1}(b)U)) \\ &= \quad p_m\Phi(a)p_m\Phi(b) + ((1-p_m)((U^*L_{1,1}(a)U))(1-p_m)(U^*L_{1,1}(b)U)) \\ &\approx_{8\delta+\delta} p_m\Phi(a)\Phi(b)p_m + (1-p_m)(U^*L_{1,1}(ab)U). \end{aligned}$$

Combining this with (e10.21), (e10.20)

$$\|p_m\Phi(ab) - p_m\Phi(a)\Phi(b)p_m\| < 8\delta + 8\delta + \delta = 17\delta \text{ for all } a, b \in \mathcal{F}. \tag{e10.22}$$

Therefore (see 10.6)

$$\|p_m^{1/2}\Phi(ab)p_m^{1/2} - p_m^{1/2}\Phi(a)p_m^{1/2}p_m^{1/2}\Phi(b)p_m^{1/2}\| < 17\delta + 3\varepsilon/64 < \varepsilon/16. \tag{e10.23}$$

Define $L_0(a) = p_m^{1/2}\Phi(a)p_m^{1/2}$ and $L_1(a) = (1-p_m)^{1/2}L_{1,0}(a)(1-p_m)^{1/2}$. By (e10.23), L_0 is $\mathcal{F}$-ε-multiplicative. By (e10.19), (e10.10), (e10.12), and the choice of δ_1, we finally have

$$\|(L_0(a) + L_1(a)) - \Phi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Let $L_{00} = \varphi^W : A \to \mathcal{W}$ and $L_{w,b} : \mathcal{W} \to M_{K+2}(M_k(\tilde{B} \otimes \mathcal{Z}_0))$ be defined by $L_{w,b}(b) = (1-p_m)^{1/2}(d_K' \circ s \circ \psi_{w,z}(b))(1-p_m)^{1/2}$ for $b \in \mathcal{W}$. Then $L_1 = L_{00} \circ L_{w,b}$. $\Box$

Theorem 10.8. *Let A be a non-unital separable amenable C^*-algebra which satisfies the UCT and satisfies the condition in 9.3 and let B be a separable simple C^*-algebra with continuous scale. For any $\alpha \in KL(A, B)$, there exists an asymptotic sequential morphism $\{\varphi_n\}$ from A into $B \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ such that*

$$[\{\varphi_n\}] = \alpha.$$

Proof. Let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. We assume that any $\mathcal{F}$-ε-multiplicative completely positive contractive linear map L from A, $[L]|_{\mathcal{P}}$ is well-defined.

If follows from 10.2 that there exist sequences of approximately multiplicative completely positive contractive linear maps $\Phi_n : A \to B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ and $\Psi_n : A \to \mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ such that, for any finite subset $\mathcal{Q} \subset \underline{K}(A)$,

$$[\Phi_n]|_{\mathcal{Q}} = \alpha|_{\mathcal{Q}} + [\Psi_n]|_{\mathcal{Q}}$$

for all sufficiently large n, where $\Psi_n = s \circ \pi \circ \Phi_n$ (without loss of generality) and $\pi : B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K} \to \mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ be the quotient map. Fix a sufficiently large n.

Let $\{e_{i,j}\}$ be a system of matrix unit for $\mathcal{K}$ and let E be the unit of the unitization of $1_{B^{\vdash}} \otimes \mathcal{Z}_0$. By considering maps $a \mapsto (E \otimes \sum_{i=1}^k e_{i,i})\Phi_n(a)(E \otimes \sum_{i=1}^k e_{i,i})$ and maps $a \mapsto (E \otimes \sum_{i=1}^k e_{i,i})\Psi_n(a)(E \otimes \sum_{i=1}^k e_{i,i})$, without loss of generality, we may assume that the image of Φ_n is in $M_k(B^{\vdash} \otimes \mathcal{Z}_0)$ and that of Ψ_n is also in $M_k(\mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0)$ for some sufficiently large k.

Define $\imath^{\circledast} : B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K} \to B^{\vdash} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ by defining $\imath^{\circledast}(b \otimes z \otimes k) = b \otimes j^{\circledast}(z) \otimes k$ for all $b \in B^{\vdash}$, $z \in \mathcal{Z}_0$ and $k \in \mathcal{K}$ (see 8.13 for $j^{\circledast}$). Note that

$$s \circ \pi(\Phi_n \oplus s \circ \pi \circ i^{\circledast} \circ \Phi_n) = \Psi_n \oplus s \circ \pi \circ i^{\circledast} \circ \Phi_n.$$

Let $\delta > 0$ and let $\mathcal{G} \subset A$ be a finite subset.

It follows from virtue of 8.14, replacing Φ_n by $\Phi_n \oplus s \circ \pi \circ i^{\circledast} \circ \Phi_n$ and replacing Ψ_n by $\Psi_n \oplus s \circ \pi \circ i^{\circledast} \circ \Phi_n$, and by implementing a unitary in unitization of $M_k(\mathbb{C} \cdot 1_{B^{\vdash}} \otimes \mathcal{Z}_0)$, we may assume that

$$\|\pi \circ \Phi_n(g) - \varphi_{w,z} \circ \varphi_{z,w} \circ \pi(\Phi_n(a))\| < \delta \text{ for all } g \in \mathcal{G}.$$

and $\Psi_n = s \circ \pi \circ \Phi_n$ approximately factors through $\mathcal{W}$, in particular, $[\Psi_n]|_{\mathcal{P}} = 0$. In other words,

$$[\Phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}. \tag{e10.24}$$

By applying 10.7, we obtain an integer $K \geq 1$, $\mathcal{F}$-ε-multiplicative completely positive contractive linear maps $L_{0,n} : A \to M_k(B \otimes \mathcal{Z}_0)$, $L_{1,n} : A \to M_{(K+2)k}(B^{\vdash} \otimes \mathcal{Z}_0)$ and $L_{2,n} : A \to M_{(K+1)k}(B^{\vdash} \otimes \mathcal{Z}_0)$ such that

$$\|L_{0,n}(a) \oplus L_{1,n}(a) - \Phi_n(a) \oplus L_{2,n}(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}, \tag{e10.25}$$

where $L_{1,n}$ and $L_{2,n}$ factor through $\mathcal{W}$. In particular,

$$[L_{1,n}]|_{\mathcal{P}} = [L_{2,n}]|_{\mathcal{P}} = 0. \tag{e10.26}$$

It follows that, using (e10.24) and (e10.25)

$$[L_{0,n}]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}. \tag{e10.27}$$

Choose $\varphi_n = L_{0,n}$ (for all sufficiently large n). □

11. Existence theorem for determinant maps

Lemma 11.1. *Let A be a stably projectionless simple C^*-algebra such that $Cu(A) = \mathrm{LAff}_+(\tilde{T}(A))$ with strict comparison for positive elements and with continuous scale. Suppose $a, b \in A \otimes \mathcal{K}_+$. Then $\langle a\rangle \ll \langle b\rangle$ ($\langle a\rangle$ is compact contained in $\langle b\rangle$) if and only if, there exists $\delta > 0$, for any $t \in T(A)$, there exists a neighborhood $O(t) \subset T(A)$ such that*

$$d_t(b) > d_\tau(a) + \delta \text{ for all } \tau \in O(t). \tag{e11.1}$$

Proof. The proof of "if" part is a standard compactness argument (see, for example 5.4 of [38]). Recall that $T(A)$ is compact in this case (see [32]). Suppose that (e11.1) holds. Let $f_n \in \mathrm{LAff}_+(\tilde{T}(A))$ such that $f_n \nearrow \sup f_n \geq \langle b\rangle$. Then, for each $t \in T(A)$, there exist n_t such that

$$f_{(n_t)}(t) > d_t(b) - \delta/8. \tag{e11.2}$$

Since each f_{n_t} is lower semi-continuous, there is a neighborhood $U(t) \subset O(t)$ such that

$$f_{(n_t)}(\tau) > d_t(b) - \delta/4 \text{ for all } \tau \in U(t). \tag{e11.3}$$

It follows that

$$f_{n_t}(\tau) > d_t(b) - \delta/4 > d_\tau(a) + \delta/2 \text{ for all } \tau \in U(t). \tag{e11.4}$$

There are finitely many such $U(t_1), U(t_2), \dots, U(t_m)$ covers $T(A)$. Put $n_0 = \max\{n_{t_i} : 1 \leq i \leq m\}$. Then, if $\tau \in U(t_j)$,

$$f_{n_0}(\tau) > f_{n_{t_j}}(\tau) > d_\tau(a) + \delta/2. \tag{e11.5}$$

This implies that $f_{n_0} > \langle a\rangle$ in $\mathrm{LAff}_+(\tilde{T}(A))$, which means $\langle a\rangle \ll \langle b\rangle$.

For the converse, as in Lemma 2.2 of [5] (see 7.2 of [15]), there exists a sequence of continuous $f_n \in \mathrm{Aff}_+(T(A))$ such that $f_n \nearrow b$. Let $g_n = f_n - \frac{1}{n}$. Then $g_n \nearrow b$. The assumption that $\langle a\rangle \ll \langle b\rangle$ implies that, for some $n_0 \geq 1$, $\langle a\rangle < g_{n_0} = f_{n_0} - \frac{1}{n_0}$ in $Cu(A)$. Hence

$$f_{n_0}(\tau) > d_\tau(a) + \frac{1}{n_0} \text{ for all } \tau \in T(A). \tag{e11.6}$$

Since f_{n_0} is continuous, for each $t \in T(A)$, there is a neighborhood $O(t)$ such that

$$f_{n_0}(t) > d_\tau(a) + \frac{1}{2n_0} \text{ for all } \tau \in O(t). \tag{e11.7}$$

Therefore

$$d_t(b) \geq f_{n_0}(t) > d_\tau(a) + \frac{1}{2n_0} \text{ for all } \tau \in O(t). \quad \square \tag{e11.8}$$

Theorem 11.2. *Let A be a stably projectionless simple exact C^*-algebra with strictly comparison for positive elements, with stable rank one and with continuous scale such that $Cu(A) = \mathrm{LAff}_+(\tilde{T}(A))$. Fix $1 > \alpha > 0$ and $1 > \eta \geq 3/4$. Let*

$$h_\eta \in \{f \in C([0,1], \mathbb{R}) : f(0) = \alpha f(1)\}$$

such that h_η is strictly increasing on $[0, \eta]$, $0 \leq h_\eta \leq 1$, $h_\eta(0) = 0 = h_\eta(1)$, and $h_\eta(\eta) = 1$.

Let $c \in A_+$ with $\|c\| = 1$ and $b \in \overline{cAc}_+$ with $\|b\| = 1$. Suppose that there is a non-zero homomorphism $\varphi : R(\alpha, 1) \to \overline{cAc}$. Then, for any $\varepsilon > 0$, there exists a homomorphism $\psi : R(\alpha, 1) \to B := \overline{cAc}$ such that

$$\sup\{|\tau(\psi(h_\eta)) - \tau(b)| : \tau \in T(A)\} < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Since A is stably projectionless, we may assume that $\mathrm{sp}(b) = [0, 1]$.

Note that $(h_\eta)|_{[0,\eta]}: [0, \eta] \to [0, 1]$ is a bijection. Define $h_\eta^{-1} : [0, 1] \to [0, \eta]$ to be the inverse of $(h_\eta)|_{[0,\eta]}$. Note that $h_\eta \circ h_\eta^{-1} = \mathrm{id}_{[0,1]}$. For each $f \in C([0, 1], \mathbb{R})_+$, define $\gamma(f)(\tau) = \tau(f \circ h_\eta^{-1}(b))$ for all $\tau \in T(A)$.

The γ above gives an affine continuous map from $C([0, 1], \mathbb{R}) \to \mathrm{Aff}(T(A))$. Note that $\mathrm{Aff}(\tilde{T}(R(\alpha, 1)))$ and $\mathrm{LAff}(\tilde{T}(R(\alpha, 1)))_+$ are identified with

$$\{(f, s) \in C([0, 1], \mathbb{R}) \oplus \mathbb{R} : f(0) = s\alpha \text{ and } f(1) = s\} = \{f \in C([0, 1], \mathbb{R}) : f(0) = \alpha f(1)\}$$
$$\text{and } LSC([0, 1], \mathbb{R}_+^{\sim}) \oplus_\alpha \mathbb{R}_+^{\sim}$$

(see 3.7), respectively. Let $\gamma_1 = \gamma|_{\mathrm{Aff}(\tilde{T}(R(\alpha,1)))_+}$. Then

$$\gamma_1(h_\eta)(\tau) = \tau(h_\eta \circ h_\eta^{-1}(b)) = \tau(b). \tag{e11.9}$$

It induces an order semi-group homomorphism $\gamma_1 : \mathrm{LAff}(\tilde{T}(R(\alpha, 1)))_+ \to \mathrm{LAff}(\tilde{T}(A))_+$. Note γ_1 takes continuous functions to continuous functions. Let $r : Cu(R(\alpha, 1)) \to \mathrm{LAff}(\tilde{T}(R(\alpha, 1)))_+$ be the rank function defined in 3.7. Define an order semi-group homomorphism $\gamma_2 : Cu(R(\alpha, 1)) \to \mathrm{LAff}_+(\tilde{T}(A))$ by

$$\gamma_2(\langle (f, s)\rangle) = (1 - \varepsilon/4)\gamma_1(r((f, s))) + (\varepsilon/4)Cu(\varphi)((f, s)). \tag{e11.10}$$

We verify that γ_2 is a morphism in **Cu**. Since the rank function r preserves the suprema of increasing sequences, it is easy to check that γ_2 also preserves the suprema of increasing sequences. Suppose that $\langle f\rangle = \langle (f, s_f)\rangle \ll \langle g\rangle = \langle (g, s_g)\rangle$ in $Cu(R(\alpha, 1))$. There is a sequence of $c_n \in Cu(R(\alpha, 1))$ such that $r(c_n)$ is continuous and $r(c_n) \nearrow r(\langle (g, s_g)\rangle)$ (see 3.7 and 3.8). Note that c_n can be identified with an element in $LSC([0, 1], (\mathbb{R}^{\sim} \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R}^{\sim} \setminus \{0\} \sqcup \mathbb{Q})_+$, at each point t, we identify $r(c_n)(t)$ with the corresponding values of $c_n(t)$ in $\mathbb{R}_+^{\sim}$—that is, $[s] \in \mathbb{Q}_+$ is regarded as $s \in \mathbb{R}_+$. Put $c = \sup_n r(c_n)$. Then $c = r(\langle a\rangle)$. For any $\varepsilon_1 > 0$, $(1 + \varepsilon_1)r(c) \geq r(\langle g\rangle)$. Since $\langle f\rangle \ll \langle g\rangle$, there exists $n_0 \geq 1$ such that

$$(1 + \varepsilon_1)r(c_{n_0}) \geq r(\langle f\rangle). \tag{e11.11}$$

This, in particular, implies that $r(\langle f\rangle)$ is a bounded function.

Now let $z_n \in Cu(R(\alpha, 1))$ such that $z_n \nearrow \sup z_n \geq \gamma_2((g, s_g))$. By 11.1, there exists $\delta > 0$ such that, for each $t \in T(A)$, there is a neighborhood $U(t)$ such that

$$d_t(\varphi(g)) > d_\tau(\varphi(f)) + \delta \text{ for all } \tau \in U(t). \tag{e11.12}$$

Choose $0 < \varepsilon_1 < \varepsilon \cdot \delta/16(M + 1)$. Then, for some $n_0 \geq 1$,

$$(1 - \varepsilon/4)(1 + \varepsilon_1)\gamma_1(r(c_{n_0})) > (1 - \varepsilon/4)(1 + \varepsilon_1)\gamma_1(r(\langle f\rangle)). \tag{e11.13}$$

Since $r(c_{n_0})$ is continuous, $\gamma_1(r(c_{n_0}))$ is also continuous. Therefore, for each $t \in T(A)$, there is a neighborhood $O(t)$ such that

$$(1 - \varepsilon/4)\gamma_1(r(c_{n_0}))(t) > (1 - \varepsilon/4)\gamma_1(r(\langle f\rangle))(\tau) - \varepsilon_1 \text{ for all } \tau \in O(t). \tag{e11.14}$$

Put $N(t) = O(t) \cap U(t)$. Then, by (e11.12) and (e11.14) as well as (e11.10),

$$\gamma_2(\langle g\rangle)(t) > \gamma_2(r(\langle f\rangle))(\tau) + \varepsilon\delta/2 \text{ for all } \tau \in N(t). \tag{e11.15}$$

It follows from 11.1 that $\gamma_2(\langle f\rangle) \ll \gamma_2(\langle g\rangle)$. This shows that γ_2 is a morphism in **Cu**. Since $K_0(R(\alpha, 1)) = \{0\}$, it induces a morphism $\gamma_2^{\sim} : Cu^{\sim}(R(\alpha, 1)) \to Cu^{\sim}(A)$ (see 7.3 of [15]).

It follows from [49] that there exists a homomorphism $\psi : R(\alpha, 1) \to B = \overline{cAc}$ such that

$$d_\tau(\psi(g)) = \gamma_2(\langle g\rangle)(\tau) \text{ for all } \tau \in T(A) \tag{e11.16}$$

and for all $g \in R(\alpha, 1)_+$. There is $f \in R(\alpha, 1)_+$ such that $d_\tau(f) = \tau(h_\eta)$ for all $\tau \in T(R(\alpha, 1))$ (see 3.4). Therefore

$$d_\tau(\psi(f)) = \lim_{n\to\infty} \tau(\psi(f^{1/n})) = \lim_{n\to\infty} \tau \circ \psi(f^{1/n}) \tag{e11.17}$$
$$= d_{\tau\circ\psi}(f) = (\tau \circ \psi)(h_\eta) \text{ for all } \tau \in T(A). \tag{e11.18}$$

Then, by (e11.10) and (e11.16),

$$|d_t(\psi(f)) - \gamma_1(r(f))(t)| < \varepsilon/4 \text{ for all } t \in T(R(\alpha, 1)). \tag{e11.19}$$

Since $\gamma_1(r(f)) = \gamma_1(\hat{h}_\eta)$, we estimate that

$$\sup\{|\tau \circ \psi(h_\eta) - \tau(b)| : \tau \in T(A)\} < \varepsilon.$$

The lemma follows. $\square$

11.3. Let A be the AH-algebras of real rank zero with unique tracial state as associated with B_T in Section 6. So $B_T = \lim_{n\to\infty}(B_n, \Phi_n)$. Write

$$B_n = W_n \oplus E_n \ \text{ and } \ E_n = M_{(n!)^2}(A(W, \alpha_n)), \quad n = 1, 2, \ldots .$$

We may write $A = \overline{\cup_{n=1}^{\infty} C_n}$, where $C_n = C_{n,1} \oplus C_{n,2}$, $C_{n,1} \oplus C_{n,2} \subset C_{n+1,1} \oplus C_{n+1,2}$ and $C_{n,1}$ is a circle algebra and $C_{n,2}$ is a homogeneous C^*-algebra with torsion K_1. In fact, $C_{n,2}$ may be written as $M_{r(n)}(C(X_n))$, where X_n is a finite CW complex with dimension no more than 3 and $r(n) \geq 6$ (see [13]). In particular (by [48]), $K_1(C_{n,2}) = U(C_{n,2})/U_0(C_{n,2})$. We use $j_n : C_n \to C_{n+1}$ for the embedding.

Fix a finitely generated subgroup $F_0 \subset K_1(B_T)$. We may assume that $F_0' \subset K_1(B_n)$ such that $(\Phi_{n,\infty})_{*1}(F_0') = F_0$. Write $B_n = E_n \oplus W_n$, where $E_n = M_{(n!)^2}(A(W, \alpha_n))$. We also write

$$C_{k,1} = M_{r(k(1))}(C(\mathbb{T})) \oplus M_{r(k(2))}(C(\mathbb{T})) \oplus \cdots M_{r(k(m_f))}(C(\mathbb{T})).$$

with the identity of each summand being p_j, $j = 1, 2, \ldots, k(m_f) = m_f$,—here we denote m_f by $k(m_f)$ to emphasise that it corresponds to C_k. We choose $n \geq 1$ so that $n \geq m_f$. Put $F_1'' = \pi_{n*1}'(F_0')$, where $\pi_n' : B_n \to A$ defined by $\pi_n'(a \oplus b) = \pi(a)$ for all $a \in M_{(n!)^2}(A(W, \alpha_n))$ and $b \in W_n$, where $\pi_n : M_{(n!)^2}(A(W, \alpha_n)) \to M_{(n!)^2}(A)$ is the quotient map. Note that $\pi_{n*1} : K_1(B_n) \to K_1(A)$ is an isomorphism. We may assume that $F_1'' \subset (j_{k,\infty})_{*1}(K_1(C_k))$. Let $\tilde{F} = \pi_{n*1}^{-1}((j_{k,\infty})_{*1}(K_1(C_k)))$ and $F = (\Phi_{n,\infty})_{*1}(\tilde{F})$. (Here, we identify $K_1(M_{(n!)^2}(A))$ with $K_1(A)$ and $K_1(M_{(n!)^2}(C_k))$ with $K_1(C_k)$.)

The subgroup F may be called *the standard subgroup* of $K_1(B_T)$.

In what follows tr is the unique tracial state on Q. We will define an injective homomorphism $j_{F,u} : F \to U(B_T)/CU(B_T)$. We identify $A(\widetilde{W, \alpha_n})$ with the following C^*-algebra (recall $s : A \to Q$ is defined in the beginning of 7.2):

$$\{(f_\lambda, a) \in C([0,1], Q \otimes Q) \oplus A : f_\lambda(0) = (s(a-\lambda) \otimes e_{\alpha_n}) + \lambda \cdot 1_{Q\otimes Q} \ \text{ and } \ f_\lambda(1) = s(a-\lambda) \otimes 1_Q + \lambda \cdot 1_{Q\otimes Q}\},$$

where $\lambda \in \mathbb{C}$ and $a - \lambda = a - \lambda \cdot 1_A \in A$. Note that $(f, 1_A)$, where $f(t) = 1_Q \otimes 1_Q$, is added to $A(W, \alpha)$.

Write $F = \mathbb{Z}^{k(m_f)} \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \mathbb{Z}/k_{m_t}\mathbb{Z}$. Put $m = k(m_f) + k(m_t)$. Let $x_1, x_2, \ldots, x_{k(m_f)}$ be the free cyclic generators for $\mathbb{Z}^{k(m_f)}$ and $x_{0,j}$ be cyclic generators for each $\mathbb{Z}/k_j\mathbb{Z}$, $j = 1, 2, \ldots, k(m_t)$, respectively.

Fix unitaries $z_1', z_2', \ldots, z_{k(m_f)}', z_{0,1}', z_{0,2}', \ldots, z_{0,k(m_t)}' \in C_k$ such that $[z_i'] = x_i$, $i = 1, 2, \ldots, k(m_f)$ and $[z_{0,j}'] = x_{0,j}$, $j = 1, 2, \ldots, k(m_t) = m_t$. Note that $(z_{0,j}')^{k_j} \in U_0(C_{n,2})$. We may choose $z_{0,j}'$ so that $(z_{0,j}')^{k_j} \in CU(C_{n,2})$. We further assume that $z_j' = \mathrm{diag}(z_j^{(0)}, 1, \ldots, 1)$, where $z_j^{(0)}$ is the standard unitary generator for $C(\mathbb{T})$, $j = 1, 2, \ldots, k(m_f)$.

We write $s(z_j') = \exp(ih_{j,0}')\exp(ih_{j,1}')$, where $h_{j,0}', h_{j,1}' \in s(p_j)Q_{s.a.}s(p_j)$. (Note that here we use the fact that the exponential rank for Q is $1+\varepsilon$ (see [29])). Let $h_{j,0}'', h_{j,1}'' \in \mathbb{R}$ such that $h_{j,l}'' = \mathrm{tr}(h_{j,l}')$, $l = 0, 1$. Put $z_j = z_j'\exp(-2i\pi h_{j,1}'')\exp(-2i\pi h_{j,0}'')$, $j = 1, 2, \ldots, m(k)$. Then $[z_j] = [z_j'] = x_j$. Note that $s(z_j) = \exp(2i\pi h_{j,0})\exp(2i\pi h_{j,1})$ such that $h_{j,0}, h_{j,1} \in (s(p_j)Qs(p_j))_{s.a.}$ and $\mathrm{tr}(h_{j,0}) + \mathrm{tr}(h_{j,1}) = 0$, $j = 1, 2, \ldots, k(m_f)$. We also choose $z_{0,j}$ and $s(z_{0,j}) = \exp(ih_{j,0,0})\exp(ih_{j,0,1})$ such that $\mathrm{tr}(h_{j,0,0}) + \mathrm{tr}(h_{j,0,1}) = 0$, and $[z_{0,j}] = x_{0,j}$.

Define $u_j = (f_j, z_j)$ as follows.

$$f_j(t) = (s(z_j) \otimes e_{\alpha_n}) \oplus ((\exp(i2t\pi h_{j,0})\exp(i2t\pi h_{j,1})) \otimes (1 - e_{\alpha_n})) \ \text{ for all } \ t \in [0,1]. \tag{e11.20}$$

Note that

$$f_j(0) = (s(z_j) \otimes e_{\alpha_n}) \oplus (1 \otimes (1_Q - e_{\alpha_n})) \ \text{ and} \tag{e11.21}$$

$$f_j(1) = (s(z_j) \otimes e_{\alpha_n}) \oplus (\exp(i2\pi h_{j,0})\exp(i2\pi h_{j,1}) \otimes (1 - e_{\alpha_n})) = s(z_j) \otimes 1_Q. \tag{e11.22}$$

In fact

$$f_j(t) = \exp(2i\pi d_{j,0}(t))\exp(2i\pi d_{j,1}(t)), \tag{e11.23}$$

where

$$d_{j,0}(t) = h_{j,0} \otimes e_{\alpha_n} + th_{j,0} \otimes (1_Q - e_{\alpha_n}) \ \text{ and} \tag{e11.24}$$

$$d_{j,1}(t) = h_{j,1} \otimes e_{\alpha_n} + th_{j,1} \otimes (1_Q - e_{\alpha_n}). \tag{e11.25}$$

In particular, $(f_j, z_j) \in A(\widetilde{W, \alpha_n})$ and $u_j \in U(A(\widetilde{W, \alpha_n}))$, $j = 1, 2, \ldots, k(m_f)$.

Write $u_j = \zeta_j + \mu(u_j)$, where $\zeta_j \in A(W, \alpha_n)$ and $\mu(u_j)$ is a scalar. Since $d_{j,0}, d_{j,1} \in A(W, \alpha_n)_{s.a.}$, $\mu(u_j) = 1$. In particular, $(f, z_j) \in A(\widetilde{W, \alpha_n})$ and $u_j \in U(A(\widetilde{W, \alpha_n}))$, $j = 1, 2, \ldots, k(m_f)$.

Let $u_{0,j} = (f_{0,j}, z_{0,j}) \in A(\widetilde{W, \alpha_n})$ be defined as follows:

$$f_{0,j}(t) = \exp(2i\pi d_{j,0,0}(t))\exp(2i\pi d_{j,0,1}(t)), \tag{e11.26}$$

where

$$d_{j,0,0}(t) = h_{j,0,0} \otimes e_{\alpha_n} + th_{j,0,0} \otimes (1_Q - e_{\alpha_n}) \text{ and} \tag{e11.27}$$

$$d_{j,0,1}(t) = h_{j,0,1} \otimes e_{\alpha_n} + th_{j,0,1} \otimes (1_Q - e_{\alpha_n}). \tag{e11.28}$$

One has, for some $\zeta_{0,j} \in A(W, \alpha_n)$,

$$u_{0,j} = \zeta_{0,j} + 1_{\widetilde{A(W,\alpha_n)}}.$$

The map $J_{u,F,n} : \tilde{F} \to U(\widetilde{B_n})/CU(\widetilde{B_n})$ defined by $x_j \mapsto \overline{u_j}$ and $x_{0,j} \mapsto \overline{u_{0,j}}$ is an injective homomorphism and define $J_{u,F} : F \to U(\widetilde{B_T})/CU(\widetilde{B_T})$ by identifying $\bar{u}_j$ with $\overline{\Phi_{n,\infty}(u_j)}$ and $\bar{u}_{0,j}$ with $\overline{\Phi_{n,\infty}(u_{0,j})}$. It should be noted, by our choice, $k_j\overline{u_{0,j}} = 0$.

11.4. We keep notation used in 11.3. Define

$$E_{n,k} = \{(f, a) \in M_{(n!)^2}(C([0, 1], Q \otimes Q)) \oplus M_{(n!)^2}(C_k) : f(0) = s(a)\otimes e_{\alpha_n} \text{ and } f(1) \in s(a) \otimes 1_Q\},$$

$n = 1, 2, \ldots$, where $s : M_{(n!)^2}(C_k) \to M_{(n!)^2}(Q)$ is the restriction of $s : M_{(n!)^2}(A) \to M_{(n!)^2}(Q)$ to $M_{(n!)^2}(C_k) \subset M_{(n!)^2}(A)$. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset B_T$. Without loss of generality, we may assume that $\mathcal{F} \subset B_n$. Denote by $\mathcal{F}^{Aw} = q_{E_n}(\mathcal{F})$, where $q_{E_n} : B_n \to E_n = M_{(n!)^2}(A(W, \alpha_n))$ is the projection map. Let

$$C_{k,1} = \bigoplus_{i=1}^{k(m_f)} M_{r(k(i))}(C(\mathbb{T})).$$

Now write $u_1, u_2, \ldots, u_{k(m_f)} \in \tilde{E}_n$ which represent the free generators of $K_1(E_{n,k})$. We may assume that $\pi_n(u_j) = z_j$, the unitary generator for $M_{r(k(j))}(C(\mathbb{T}))$, $j = 1, 2, \ldots, k(m_f)$, and where $\pi_n : E_n \to M_{(n!)^2}(A)$ is the quotient map. We also assume that z_j and u_j have the form (e11.20).

Fix $\varepsilon/2 > \delta > 0$ and a finite subset $\mathcal{G}' \subset C_k$ with $\mathcal{G}' \supset \pi_{n,k}(\mathcal{F}^{Aw})$, where $\pi_{n,k} : E_{n,k} \to M_{(n!)^2}(C_k)$ is the quotient map. Choose a finite subset $\mathcal{F}_1 \supset \mathcal{F}^{Aw}$ such that $\pi_{n,k}(\mathcal{F}_1) \supset \mathcal{G}'$.

We also assume that there is an $\mathcal{G}'$-δ-multiplicative completely positive contractive linear map $L : A \to C_k$ such that

$$\|L(a) - a\| < \delta/4 \text{ for all } a \in \mathcal{G}', \tag{e11.29}$$

where we also use L to denote $L \otimes \mathrm{id}_{M_{(n!)^2}} : M_{(n!)^2}(A) \to M_{(n!)^2}(C_k)$. Choose $\delta > \delta_0 > 0$ such that, for any $(f, a) \in \mathcal{F}_1$, if $|t - t'| < 2\delta_0$,

$$\|f(t) - f(t')\| < \delta/16 \text{ for all } t, t' \in [0, 1]. \tag{e11.30}$$

Define $\tilde{L} : E_n \to E_{n,k}$ as follows: $\tilde{L}((f, a)) = (g, L(a))$, where

$$g(t) = \begin{cases} ((1 - 2t/\delta_0)s(L(a)) \otimes e_{\alpha_n} + \frac{2t}{\delta_0}s(a) \otimes e_{\alpha_n}) & \text{for all } t \in [0, \delta_0/2], \\ f(\frac{t-\delta_0/2}{1-\delta_0}) & \text{for all } t \in (\delta_0/2, 1 - \delta_0/2], \\ \frac{1-t}{\delta_0/2}s(a)\otimes 1_Q + \frac{t-(1-\delta_0/2)}{\delta_0/2}s(L(a)) \otimes 1_Q & \text{for all } t \in (1 - \delta_0/2, 1]. \end{cases}$$

One verifies that $\tilde{L}$ is an $\mathcal{F}_1$-$\delta/2$-multiplicative completely positive contractive linear map from E_n into $E_{n,k}$.

We now assume that $\alpha_n < \alpha_{n+1}$. Let $r_1 = \frac{1-\alpha_{n+1}}{1-\alpha_n}$ and $r_2 = \frac{\alpha_{n+1}-\alpha_n}{1-\alpha_n}$. Let $1 > \eta > 3/4$ and $\mu_j \geq 0$, $j = 1, 2, \ldots, k(m_f)$. Let $\omega_j = \mu_j/\mathrm{tr}(s(p_j))$, $j = 1, 2, \ldots, k(m_f)$.

Fix a continuous increasing surjective function $g_1 : [0, \eta] \to [0, 1]$ such that $g_1(0) = 0, g_1(\eta) = 1$ and decreasing surjective function $g_2 : [\eta, 1] \to [0, 1]$ such that $g_2(\eta) = 1$, $g_2(1) = 0$. Define $h|_{[0,\eta]} = g_1$ and $h|_{[\eta,1]} = g_2$. In particular, $h = (0) = 0$ and $h(1) = 0$.

Define a homomorphism $\varphi^f_{c,R} : M_{(n!)^2}(C_{k,1}) \to M_{(n!)^2}(C([0, 1], Q) \otimes e_{r_2})$ such that

$$\varphi^f_{c,R}(z_j)(t) = s(z_j)\exp(i2\pi(\omega_j/r_2)h(t))s(p_j) \otimes e_{r_2} \text{ for all } t \in [0, 1]. \tag{e11.31}$$

Define $\varphi_{c,R} = \varphi^f_{c,R}|_{M_{(n!)^2}(C_{k,1})} \oplus (\varphi^t_{c,R})|_{M_{(n!)^2}(C_{k,2})}: M_{(n!)^2}(C_k) \to M_{(n!)^2}(C([0, 1], Q) \otimes e_{r_2})$, where

$$\varphi^t_{c,R}(a)(t) = s(a)\otimes e_{r_2} \text{ for all } t \in [0, 1]. \tag{e11.32}$$

Let $\varphi_{A,R} : E_n \to M_{(n!)^2}(C([0, 1], Q) \otimes e_{r_2})$ be defined by $\varphi_{A,R}((f, a)) = \psi_{c,R} \circ L \circ \pi_A(a)$ for all $a \in E_n$ and where $\pi_A : M_{(n!)^2}(A(W, \alpha_n)) \to M_{(n!)^2}(A)$ is the quotient map.

Now define a completely positive contractive linear map $\Psi : E_n \to M_{(n!)^2}(A(W, \alpha_{n+1}))$ as follows. We will use some of the notation in Section 7. Define (see Section 7 for the notation)

$$P_a(\Psi((f, a))) = L(a) \text{ and} \tag{e11.33}$$

$$P_f(\Psi((f,a))) = \mathrm{diag}(P_f \circ \varphi_{R,r_1} \circ \varphi_{A,R,\alpha_n}(\tilde{L}(f,a)), (\varphi_{A,R}(f,a)))$$
$$= \mathrm{diag}(P_f \circ \varphi_{R,r_1} \circ \varphi_{A,R,\alpha_n}(g, L(a)), (\varphi_{A,R}(f,a))). \tag{e11.34}$$

Note that

$$P_f(\Psi(f,a))(0) = \mathrm{diag}(s(L(a)) \otimes e_{\alpha_n r_1}, s(L(a)) \otimes e_{r_2}) = s(L(a)) \otimes e_{\alpha_{n+1}} \text{ and} \tag{e11.35}$$
$$P_f(\Psi(f,a))(1) = \mathrm{diag}(s(L(a)) \otimes e_{r_1}, s(L(a)) \otimes e_{r_2}) = s(L(a)) \otimes 1_Q. \tag{e11.36}$$

Let

$$W_j(t) = (\exp(i2\pi h_{j,0}) \exp(i2\pi h_{j,1}) \otimes e_{\alpha_n r_1} \oplus (\exp(i2\pi t h_{j,0})) \exp(i2\pi t h_{j,1}) \otimes (e_{r_1} - e_{\alpha_n r_1})$$
$$+ s(z_j) \exp(i2\pi(\omega_j/r_2)h(t)) s(p_j) \otimes e_{r_2}, \qquad j = 1, 2, \ldots, k(m_f).$$

Let $E'_{n+1} := M_{(n!)^2}(A(W, \alpha_{n+1}))$, then in $\tilde{E}'_{n+1}$ (with large $\mathcal{G}'$),

$$\|\Psi(u_j) - (W_j, z_j)\| < \delta, \quad j = 1, 2, \ldots, k(m_f). \tag{e11.37}$$

(Here the unitalization of Ψ is also denoted by Ψ.) Therefore there exists $H_{j,00} \in (E'_{n+1})_{s.a.}$ with $\|H_{j,00}\| \le 2\arcsin(\delta/2)$ such that

$$\lceil \Psi(u_j) \rceil = \exp(i2\pi H_{j,00})(W_j, z_j), \quad j = 1, 2, \ldots, k(m_f). \tag{e11.38}$$

Put

$$H_{j,0}(t) = h_{j,0} \otimes (e_{\alpha_n r_1} \oplus e_{r_2}) \oplus t h_{j,0} \otimes (e_{r_1} - e_{\alpha_n r_1}), \tag{e11.39}$$
$$H_{j,1}(t) = h_{j,1} \otimes (e_{\alpha_n r_1} \oplus e_{r_2}) \oplus t h_{j,1} \otimes (e_{r_1} - e_{\alpha_n r_1}) \text{ and} \tag{e11.40}$$
$$H_{j,2}(t) = (\omega_j/r_2) h(t) s(p_j) \otimes e_{r_2}. \tag{e11.41}$$

Noting $h(0) = 0$ and $h(1) = 0$, we see that $H_{j,l} \in M_{(n!)^2}(R(\alpha_{n+1}, 1))$. Therefore

$$\varphi_{A,R,\alpha_{n+1}}(\lceil \Psi(u_j) \rceil) = \exp(i2\pi H_{j,00}) \exp(i2\pi H_{j,0}) \exp(i2\pi H_{j,1}) \exp(i2\pi H_{j,2}). \tag{e11.42}$$

Note that (recall that $\mathrm{tr}(h_{j,0}) + \mathrm{tr}(h_{j,1}) = 0$, $j = 1, 2, \ldots, k(m_f)$), for all $t \in [0, 1]$,

$$\mathrm{tr}(H_{j,0} + H_{j,1})(t) = 0 \tag{e11.43}$$

We then compute that, for all $t \in [0, 1]$,

$$\mathrm{tr}(H_{j,00} + H_{j,0} + H_{j,1} + H_{j,2})(t) = \mathrm{tr}(H_{j,00}) + (\omega_j/r_2) h(t) \cdot \mathrm{tr}(s(p_j)) \mathrm{tr}(e_{r_2}) \tag{e11.44}$$
$$= \mathrm{tr}(H_{j,00}) + \mu_j h(t). \tag{e11.45}$$

It follows that, in $E''_{n+1} = M_{(n!)^2}(R(\alpha_{n+1}, 1))$, for all $t \in [0, 1]$,

$$|D_{E''_{n+1}}(\varphi_{A,R,\alpha_{n+1}}(\lceil \Psi(u_j) \rceil))(t) - \mu_j h(t)| < \delta. \tag{e11.46}$$

Let

$$W_{0,j}(t) = \exp(i2\pi h_{j,0,0}) \exp(i2\pi h_{j,0,1}) \otimes e_{\alpha_n r_1} \oplus \exp(i2\pi t h_{j,0,0}) \exp(i2\pi t h_{j,0,1}) \otimes (e_{r_1} - e_{\alpha_n r_1})$$
$$+ s(z_j) s(p_j) \otimes e_{r_2}, \qquad j = 1, 2, \ldots, m_t.$$

A similar computation shows that

$$|D_{E''_{n+1}}(\varphi_{A,R,\alpha_{n+1}}(\lceil \Psi(u_{0,j}) \rceil))(t)| < \delta. \tag{e11.47}$$

We will keep notations in 11.3 and 11.4 in the following statement.

Lemma 11.5. *Let C be a non-unital separable simple C^*-algebra in $\mathcal{D}$ with continuous scale such that $\ker \rho_C = K_0(C)$ and let $B = B_T$ be as constructed in 7.2.*

Let $\varepsilon > 0$, $\mathcal{F} \subset B$ be a finite subset, let $\mathcal{P} \subset \underline{K}(B)$ be a finite subset and let $1/2 > \delta_0 > 0$.

For any finitely generated standard subgroup F (see 11.3), any finite subset $S \subset F$, there exists an integer $n \ge 1$ with the following property:
for any finite subset $\mathcal{U} \subset U(\tilde{B}_T)$ such that $\overline{\mathcal{U}} \subset J_{F,u}(F) \subset J_{F,u}((\Phi_{n,\infty})_{*1}(K_1(E_n)))$ (see the end of 11.3) and $\Pi(\overline{\mathcal{U}}) = S$, where $\Pi : U(\tilde{B})/CU(\tilde{B}) \to K_1(B)$ is the quotient map, for any homomorphism
$\gamma : J_{F,u}((\Phi_{n,\infty})_{*1}(K_1(E_n))) \to \mathrm{Aff}(T(\tilde{C}))/\mathbb{Z}$, such that $\gamma|_{\mathrm{Tor} J_{u,F}((\Phi_{n,\infty})_{*1}(K_1(E_n)))} = 0$ and any $c \in C_+$ with $\|c\| = 1$, there exists $\mathcal{F}$-ε-multiplicative completely positive contractive linear map $\Phi : B_T \to \overline{cCc}$ such that

$$[\Phi]|_{\mathcal{P}} = 0 \ \text{ and } \ \mathrm{dist}(\Phi^\dagger(\bar{z}), \gamma(\bar{z})) < \delta_0 \ \text{ for all } \ z \in \mathcal{U}$$
$$(\text{in } U_0(\tilde{C})/CU(\tilde{C}) \cong \mathrm{Aff}(T(\tilde{C}))/\mathbb{Z}). \tag{e11.48}$$

(here we assume $\mathrm{dist}(\Phi^\dagger(\bar{z}), \overline{\lceil \Phi(z) \rceil}) < \delta_0/4$ for all $z \in \mathcal{U}$–see 2.7 for the definition of $\Phi^\dagger$).

Proof. Fix $\varepsilon > 0$, $\mathcal{F}$ and $\mathcal{P}$ as described by this lemma. Fix $\delta_1 > 0$, a finite subset $\mathcal{G} \subset B_T$. We assume that $\mathcal{F} \subset \mathcal{G}$. Choose $n_0 \ge 1$ such that there exists finite subset $\mathcal{G}' \subset B_{n_0}$ such that, for any $b \in \mathcal{G}$, there exists $b' \in \mathcal{G}'$ such that

$$\|b - \Phi_{n,\infty}(b')\| < \delta_1/64. \tag{e11.49}$$

We assume that $\delta_1 < \min\{\delta_0/16, \varepsilon/16\}$.

Choose $k \ge 1$ as in 11.3 and write $F = \mathbb{Z}^{m_f} \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \mathbb{Z}/k_{m_t}\mathbb{Z}$. Fix a set of generator S of F.

To simplify notation, without loss of generality, we may assume that $\mathcal{G} \subset \Phi_{n_0,\infty}(\mathcal{G}')$. We also assume, without loss of generality, that $\mathcal{P} \subset [\Phi_{n_0,\infty}](B_n)$. Let $\mathcal{P}' \subset \underline{K}(B_{n_0})$ be a finite subset such that $\mathcal{P} \subset [\Phi_{n_0,\infty}](\mathcal{P}')$.

Let $\overline{\mathcal{U}} \subset J_{F,u}(F)$ and, let z_j and u_j, $j = 1, 2, \dots, m_f$, and $z_{0,j}$, and $u_{0,j}$, $j = 1, 2, \dots, m_t$, be as described in 11.3. Without loss of generality, we may assume that $\overline{\mathcal{U}} = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{m_f}, \bar{u}_{0,1}, \dots, \bar{u}_{0,m_t}\}$.

We also assume that there exists a completely positive contractive linear map $L : B_T \to B_n$ such that, for all $n \ge n_0$,

$$\|L(\Phi_{n,\infty}(b')) - b'\| < \delta_1/64 \ \text{ for all } \ b' \in \mathcal{G}' \tag{e11.50}$$

We further assume that δ_1 is sufficiently small and $\mathcal{G}$ is sufficiently large so that $[L']|_{\mathcal{P}}$ is well defined for any $\mathcal{G}$-δ_1-multiplicative completely positive contractive linear map from B_T. Moreover, $L'^{\dagger}$ can be defined so that $\mathrm{dist}(L'^{\dagger}(\bar{z}), \overline{\lceil L'(z)\rceil}) < \delta_0/4$ for all $z \in \mathcal{U}$ (see 2.7).

Choose $\delta = \frac{\delta_1}{16(m_f+m_t+2)}$ and choose $n \ge n_0 + m_f + m_t + 2$ as in 11.4 associated with $\delta/64$ (in place ε) and $\mathcal{G}$ (in place of $\mathcal{F}$).

Choose non-zero elements $c_{i,l} \in \overline{cCc}_+$ which are mutually orthogonal, $i = 1, 2, \dots, m_f$, $l = 1, 2$.

Choose $1 > \eta_0 > 0$ such that

$$\eta_0 \le \inf\{d_\tau(c_{j,l}) : \tau \in T(C)\},\ 1 \le j \le m_f \ \text{ and } \ l \in \{1, 2\}.$$

Choose $g_{j,+}, g_{j,-} \in \mathrm{Aff}(T(C))_+$ and $\lambda_{j,+}, \lambda_{j,-} \in \mathbb{R}_+$ such that

$$0 < g_{j+}(\tau) \le \eta_0,\ 0 < g_{j-}(\tau) \le \eta_0 \ \text{ for all } \ \tau \in T(C) \ \text{ and} \tag{e11.51}$$

$$\gamma(\bar{u}_j) = \lambda_{j,+}g_{j+} - \lambda_{j,-}g_{j-},\ \ j = 1, 2, \dots, m_f. \tag{e11.52}$$

Let $P_n : B_n \to E_n$ be the projection map, and let $\mathcal{G}'' \subset E_n$ be a finite subset such that $\mathcal{G}'' \supset P(\mathcal{G}')$.

Define $\varphi'_{j,l} : M_{(n!)^2}(A(W, \alpha_n)) \to M_{(n!)^2}(R(\alpha_{n+1}, 1))$ be as defined (denoted by $\varphi_{A,R,\alpha_{n+1}} \circ \Psi$ there) in 11.4 (with $\mu_j = \lambda_{j,+}$ and $\mu_i = 0$ if $i \ne j$ (for $\varphi'_{j,1}$); and with $\mu_j = \lambda_{j,-}$ and $\mu_i = 0$ if $i \ne j$ (for $\varphi'_{j,2}$)) such that

$$\lceil \varphi'_{j,l}(u_j)\rceil = \exp(\sqrt{-1}2\pi H_{j,00})\exp(\sqrt{-1}2\pi H_{j,0})\exp(\sqrt{-1}2\pi H_{j,1})\exp(\sqrt{-1}2\pi H_{j,2,l}), \tag{e11.53}$$

where $H_{j,00}, H_{j,0}, H_{j,1}, H_{j,2,l} \in M_{(n!)^2}(R(\alpha_{n+1}, 1))$, $l = 1, 2$, such that

$$\mathrm{tr}(H_{j,00}(t) + H_{j,0}(t) + H_{j,1}(t) + H_{j,2,l}(t)) = \mathrm{tr}(H_{j,00}(t)) + \mathrm{tr}(H_{j,2,l})(t),\ \ l = 1, 2, \tag{e11.54}$$

$$\mathrm{tr}(H_{j,2,1})(t) = \lambda_+ h(t),\ \ \mathrm{tr}(H_{j,2,2})(t) = \lambda_- h(t) \ \text{ and} \tag{e11.55}$$

$$|\mathrm{tr}(H_{j,00}(t))| < \delta/4 \tag{e11.56}$$

for all $t \in [0, 1]$, where $h(t)$ is $C([0, 1])_+$ such that $h(0) = 0$, $h(3/4) = 1$, $h(1) = 0$, $h(t)$ is strictly increasing on $[0, 3/4]$ and strictly decreasing on $[3/4, 1]$. Moreover $\varphi'_{j,l}$ is $\mathcal{G}''$-$\delta/8(m_f)$-multiplicative,

$$\lceil \varphi'_{j,l}(u_i)\rceil \ = \ \exp(2\pi\sqrt{-1}H_{i,00})\exp(2\pi\sqrt{-1}H_{i,0})\exp(2\pi\sqrt{-1}H_{i,1}), \text{ if } i \ne j \ \text{ and} \tag{e11.57}$$

$$[\varphi'_{j,l}]|_{\mathcal{Q}} = 0, \tag{e11.58}$$

where $\mathcal{Q} = [P_n \circ \Phi_{n,\infty}](\mathcal{P}')$. (Note that $K_i(R(\alpha_{n+1}, 1)) = \{0\}$, $i = 0, 1$). Note since $C \in \mathcal{D}$, for each j, there exists a non-zero homomorphism $\varphi''_{j,l} : M_{(n!)^2}(R(\alpha_{n+1}, 1)) \to C_{j,l} := \overline{c_{j,l}Cc_{j,l}}$, $j = 1, 2, \dots, m_f$. It follows from 11.2 that there is, for each j and l, a homomorphism $\varphi''_{j,l} : M_{(n!)^2}(R(\alpha_{n+1}, 1)) \to C_{j,l}$ such that

$$\sup\{|\tau \circ \varphi''_{j,1}(h) - g_{j,+}(\tau)| : \tau \in T(C)\} < \delta/2 \ \text{ and} \tag{e11.59}$$

$$\sup\{|\tau \circ \varphi''_{j,2}(h) - g_{j,-}(\tau)| : \tau \in T(C)\} < \delta/2. \tag{e11.60}$$

Let $\varphi_{j,l} = \varphi''_{j,l} \circ \varphi'_{j,l} : E_n \to C_{j,l}$. Recall $\varphi'_{j,l}$ is of the form $\varphi_{A,R,\alpha_{n+1}} \circ \Psi$, we compute that (also using (e11.43) and (e11.46), and, see 2.1 for the notation $\lceil\cdot\rceil$),

$$D_{\tilde{C}}(\sum_{l=1}^{2}\lceil \varphi_{j,l}(u_j)\rceil) \approx_{2\delta_1/16(m_f)} (\lambda_{j+}g_{j,+} - \lambda_{j-}g_{j,-}) \qquad (\text{in } \ \mathrm{Aff}(T(\tilde{C}))/\mathbb{Z}) \tag{e11.61}$$

$$= \qquad \gamma(\bar{u}_j) \qquad\qquad (\text{in } \ \mathrm{Aff}(T(\tilde{C}))/\mathbb{Z}) \tag{e11.62}$$

$$D_{\tilde{C}}(\lceil \varphi_{j,l}(u_i)\rceil) \approx_{2\delta_1/16(m_f)} 0 \qquad\qquad (\text{in } \ \mathrm{Aff}(T(\tilde{C}))/\mathbb{Z}),\ \ i \ne j. \tag{e11.63}$$

Similarly, using (e11.47), we have

$$D_{\tilde{C}}(\sum_{l=1}^{2}(\lceil\varphi_{j,l}(u_{0,i})\rceil)) \approx_{2\delta_1/16(m_f)} 0 \ \text{(in } \mathrm{Aff}(T(\tilde{C}))/\mathbb{Z}). \tag{e11.64}$$

Now define $\Phi' : E_n \to \bigoplus_{j=1}^{m_f}(\bigoplus_{l=1}^{2} C_{j,l})$ by $\Phi' = \sum_{j=1}^{m_f}(\sum_{l=1}^{2}\varphi_{j,l})$. From the above estimates,

$$\mathrm{dist}(\Phi^{\dagger}(\bar{z}), \gamma(\bar{z})) < \delta_0 \ \text{ for all } \ z \in \mathcal{U}. \tag{e11.65}$$

Moreover, since Φ' factors through $M_{(n!)^2}(R(\alpha_{n+1}, 1))$,

$$[\Phi']|_{\mathcal{Q}} = 0. \tag{e11.66}$$

Define $\Phi = \Phi' \circ P_n \circ L$. We check that Φ meets the requirements. □

Lemma 11.6. *Let C be a non-unital separable C^*-algebra. Suppose that $u \in U(M_s(\tilde{C}))$ (for some integer $s \geq 1$) with $[u] \neq 0$ in $K_1(C)$ but $u^k \in CU(M_s(\tilde{C}))$ for some $k \geq 1$. Suppose that $\pi_C(u) = e^{2i\pi\theta}$ for some $\theta \in (M_s)_{s.a.}$, where $\pi_C : M_s(\tilde{C}) \to M_s$ is the quotient map. Then $sk \cdot \mathrm{tr}(\theta) \in \mathbb{Z}$, where tr is the tracial state of M_s.*

Let B be a stably projectionless simple separable C^*-algebra with $\ker\rho_B = K_0(B)$ and with continuous scale. For any $\varepsilon > 0$, there exist $\delta > 0$ and finite subset $\mathcal{G} \subset C$ satisfying the following: If $L_1, L_2 : C \to B$ are two $\mathcal{G}$-δ-multiplicative completely positive contractive linear maps such that $[L_1](u) = [L_2](u)$ in $K_1(B)$, then

$$\mathrm{dist}(\overline{\lceil L_1(u)\rceil}, \overline{\lceil L_2(u)\rceil}) < \varepsilon, \tag{e11.67}$$

where u is as in the first paragraph.

Proof. Write $u = e^{2\sqrt{-1}\pi\theta} + \zeta$, where $\zeta \in M_s(C)$ and $\theta \in (M_s)_{s.a.}$. Therefore, if $u^k \in CU(M_s(\tilde{C}))$, then $sk\mathrm{tr}(\theta) \in \mathbb{Z}$.

Note L_i is originally defined on C and the extension $L_i : M_s(\tilde{C}) \to M_s(\tilde{B})$ has the property that $L_i(u) = e^{2\sqrt{-1}\pi\theta} + L_i(\zeta)$, $i = 1, 2$. To simplify notation, without loss of generality, we may assume that $\lceil L_1(u)\rceil \cdot \lceil L_2(u^*)\rceil \in U_0(M_s(\tilde{B}))$. Note that

$$\pi_B(\lceil L_1(u)\rceil \cdot \lceil L_2(u^*)\rceil) = e^{2\sqrt{-1}\pi\theta}e^{-2\sqrt{-1}\pi\theta} = 1$$

(where $\pi_B : M_s(\tilde{B}) \to M_s$ is the quotient map). By 6.1, we may write

$$\lceil L_1(u)\rceil \cdot \lceil L_2(u^*)\rceil = \prod_{j=1}^{n}\exp(2\sqrt{-1}\pi h_j) \ \text{ for all } \ \text{some } h_1, h_2, \ldots, h_n \in M_s(\tilde{B})_{s.a} \text{ with}$$

$$\pi_B(h_j) = 0 \text{ and } \pi_B(\exp(2\sqrt{-1}\pi h_j)) = 1 \ \text{ for all } j. \tag{e11.68}$$

It follows from 14.5 of [35] that, by choosing small δ and large $\mathcal{G}$ (independent of L_1 and L_2) there is $h_0 \in M_s(\tilde{B})_{s.a.}$ such that $\|h_0\| < \min\{1, \varepsilon\}/4s(k+1)$ and

$$((\exp(2i\pi h_0))(\prod_{j=1}^{n}\exp(2i\pi h_j)))^k \in CU(M_s(\tilde{B})). \tag{e11.69}$$

By (e11.68), $\pi_B(\exp(2ih_0)) \in CU(M_s)$. Then $st_0(h_0) \in \mathbb{Z}$. However, since $\|h_0\| < 1/4s(k+1)$, $t_0(h_0) < 1/4s(k+1)$. This implies that $t_0(h_0) = 0$. Note also $U_0(\tilde{B})/CU(\tilde{B}) = \mathrm{Aff}(T(\tilde{B}))/\mathbb{Z}$. Therefore (by (e11.69)), there is an integer $m \in \mathbb{Z}$ such that, for any $\tau \in T(\tilde{B})$,

$$k\tau(\sum_{j=1}^{n}h_j + h_0/k) = m. \tag{e11.70}$$

For any $\tau_0 \in T(B)$ and any $0 < \alpha < 1$, $t = \alpha\tau_0 + (1-\alpha)t_0$ is a tracial state of $\tilde{B}$. Then (by (e11.68)),

$$kt(\sum_{j=1}^{n}h_j + h_0/k) = k\alpha\tau_0(\sum_{j=1}^{n}h_j) + \alpha\tau_0(h_0/k)) = m. \tag{e11.71}$$

So $k\alpha\tau_0(\sum_{j=1}^{n}h_j + h_0/k) = m$ for any $0 < \alpha < 1$. It follows that

$$\tau_0(\sum_{j=1}^{n}h_j + h_0/k) = 0 \ \text{ for all } \ \tau_0 \in T(\tilde{B}). \tag{e11.72}$$

It follows that

$$|\tau(\sum_{j=1}^{n} h_j)| < \varepsilon/2(k+1) \ \text{ for all } \ \tau \in T(\tilde{B}). \tag{e11.73}$$

Thus (e11.67) holds. □

12. Construction of homomorphism

Proposition 12.1. *Let A be a separable simple C^*-algebra in $\mathcal{D}$. Suppose that $\ker \rho_A = K_0(A)$. Then there exists a sequence of approximately multiplicative completely positive contractive linear maps $\{\varphi_n\}$ from A to $\mathcal{W}$ which maps strictly positive elements to strictly positive elements.*

Proof. Fix $\tau \in T(A)$. Define $\gamma : T(\mathcal{W}) \to T(A)$ by $\gamma(t_{\mathcal{W}}) = \tau$, where $t_{\mathcal{W}}$ is the unique tracial state of $\mathcal{W}$. Then γ induces an order semi-group homomorphism from $\mathrm{LAff}(\tilde{T}(A))$ onto $\mathrm{LAff}(\tilde{T}(\mathcal{W}))$. Since $\ker \rho_A = K_0(A)$ and $K_0(\mathcal{W}) = 0$, this in turn induces a homomorphism $\Gamma : Cu^{\sim}(A) \to Cu^{\sim}(\mathcal{W})$ (see 7.3 of [15]). Fix a strictly positive element $a_0 \in A$ with $\|a\| = 1$. Let $\mathfrak{f}_{a_0} > 0$ be the associated number (see 3.9). There exists a sequence of approximately multiplicative completely positive contractive linear maps $\psi_n : A \to D_n$ such that $\psi_n(a_0)$ is a strictly positive element of D_n, $t(f_{1/4}(a_0)) \geq \mathfrak{f}_{a_0}$ for all $t \in T(D_n)$, (where D_n is the same as constructed in the proof of 9.1 of [15] and ψ_n is the same as $\varphi_{1,n}$). Moreover,

$$\lim_{n\to\infty} \sup\{|\tau(a) - \tau \circ \psi_n(a)| : \tau \in \overline{T(A)}^{w}\} = 0 \ \text{ for all } \ a \in A$$

(see the proof of 9.1 of [15]). In particular, this implies that $\lim_{n\to\infty} \|\psi_n(x)\| = \|x\|$ for all $x \in A$. For each n, let $\iota_n : D_n \to A$ be the embedding.

Let $\lambda_n = \Gamma \circ (Cu^{\sim}(\iota_n))$. It follows from [49] that there is a homomorphism $h_n : D_n \to \mathcal{W}$ such that

$$Cu^{\sim}(h_n) = \lambda_n, \quad n = 1, 2, \ldots.$$

By passing a subsequence if necessary, we may assume that

$$\lim_{n\to\infty} \|h_n \circ \psi_n(ab) - h_n \circ \psi_n(a) h_n \circ \psi_n(b)\| = 0 \ \text{ for all } \ a, b \in A.$$

By using an argument used in the proof of 12.4 of [15], we may also assume that $h_n \circ \psi_n(a_0)$ is a strictly positive element of $\mathcal{W}$. □

Remark 12.2. In the absence of the condition $K_0(A) = \ker \rho_A$, the proof of 12.1 shows that the conclusion of 12.1 holds if the assumption is changed to that A has at least one non-zero $\mathcal{W}$-trace. The proof of Proposition 12.1 also shows that every tracial state of simple C^*-algebras in $\mathcal{D}$ with $K_0(A) = \ker \rho_A$ is a $\mathcal{W}$-trace. Proposition 12.1 can also be obtained from the proof of 8.8.

The following is a number theory lemma which may be known.

Lemma 12.3. *Let $a_1, a_2, \ldots, a_n$ be non-zero integers such that at least one of them is positive and one of them is negative. Then, for any $d \in \mathbb{Z}$, if $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$ has an integer solution, then it must have a positive integer solution.*

Proof. We will prove it by induction. Suppose that $a, b \in \mathbb{Z}$ such that $a > 0$ and $b < 0$. Suppose also there are $x_0, y_0 \in \mathbb{Z}$ such that $ax_0 + by_0 = d$. Then, for any integer $m \in \mathbb{Z}$, and any $x = x_0 + bm$ and $y = y_0 - am$,

$$a(x_0 + bm) + b(y_0 - am) = d. \tag{e12.1}$$

Thus, by choosing negative integer m with large $|m|$, both $x_0 + bm$ and $y_0 - am$ are positive. This proves the case $n = 2$.

Suppose the lemma holds for $n - 1$ with $n \geq 3$. Without lose of generality, let us first assume that a_1 and a_2 have different signs. Suppose $\{x_1^0, x_2^0, \ldots, x_n^0\}$ is an integer solution for $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$, Let $k = a_1x_1^0 + a_2x_2^0 + \cdots + a_{n-1}x_{n-1}^0$. Now we divided it into two cases:

Case 1: k and a_n have opposite signs. By induction assumption there are positive integers $x_1', x_2', \ldots, x_{n-1}'$ such that

$$k = a_1x_1' + a_2x_2' + \cdots + a_{n-1}x_{n-1}', \tag{e12.2}$$

since $a_1a_2 < 0$ and $n \geq 3$. On the other hand, by applying the case $n = 2$, we have integers $x > 0$ and $y > 0$ such that $kx + a_ny = d$.

Let $x_i = xx_i'$ for $i \in \{1, 2 \ldots n - 1\}$ and $x_n = y$ to get desired positive integer solution for

$$\sum_{i=1}^{n} x_i a_i = x \sum_{i=1}^{n-1} x_i' a_i + a_n y = d. \tag{e12.3}$$

Case 2: k and a_n have the same sign.

By the induction assumption there are positive integers: $x_1', x_2', \dots, x_{n-1}'$ such that

$$-k = a_1x_1' + a_2x_2' + \cdots + a_{n-1}x_{n-1}' \tag{e12.4}$$

(recall $a_1a_2 < 0$). On the other hand, applying the case $n = 2$ (note that $-k$ and a_n have opposite signs), we have $x > 0$ and $y > 0$ such that $-kx + a_ny = d$. Finally let $x_i = xx_i'$ for $i \in \{1, 2...n-1\}$ and $x_n = y$ to get the desired positive integer solution. □

12.4. Recall that, from 7.2, $\mathcal{Z}_0$ is an inductive limit of $B_m = W_m \oplus M_{(m!)^2}(A(W, \alpha_m))$ and recall that $K_0(\mathcal{Z}_0) = \mathbb{Z}$ and $K_1(\mathcal{Z}_0) = \{0\}$. Let $E_m = M_{(m!)^2}(A(W, \alpha_m))$ be as in 7.2. For any m, $K_0(E_m) = \mathbb{Z}$ and $K_1(E_m) = \{0\}$. Let id : $K_0(\mathcal{Z}_0) \cong K_0(E_m)$ be the isomorphism. Then it induces a unique element in $KK(\mathcal{Z}_0, E_m)$ and will be denote by id. Let $z_{\mathbb{Z}} = [1] \in \mathbb{Z} = K_0(\mathcal{Z}_0)$ be the generator of $K_0(\mathcal{Z}_0)$. Suppose that C is a separable amenable C^*-algebra satisfies the UCT. Denote by $\kappa_{\mathcal{Z}_0} \in KK(C \otimes \mathcal{Z}_0, C)$ the element such that $(\kappa_{\mathcal{Z}_0})_{*i} : K_i(C \otimes \mathcal{Z}_0) \to K_i(C) \otimes \mathbb{Z} = K_i(C)$ is the isomorphism with $(\kappa_{\mathcal{Z}_0})_{*i}(x \otimes z_{\mathbb{Z}}) = x$ for $x \in K_i(C)$, given by Kunneth's formula, $i = 0, 1$.

Lemma 12.5. *Let C be a separable amenable C^*-algebra which satisfies the condition in 9.3 and which satisfies the UCT. There exists a sequence of approximate multiplicative completely positive contractive linear maps $\varphi_n : C \otimes \mathcal{Z}_0 \to C \otimes M_{k(n)}$ (for some subsequence $\{k(n)\}$) which maps strictly positive elements to strictly positive elements such that*

$$[\varphi_n]|_{\mathcal{P}} = (\kappa_{\mathcal{Z}_0})|_{\mathcal{P}}, \tag{e12.5}$$

where $\kappa_{\mathcal{Z}_0} \in KK(C \otimes \mathcal{Z}_0, C)$ is an invertible element which induces $(\kappa_{\mathcal{Z}_0})_{*i}$, for every finite subset $\mathcal{P} \subset \underline{K}(C)$ and all sufficiently large n.

Proof. Let $\varepsilon > 0$ and let $\mathcal{F} \subset C$ be a finite subset.

Without loss of generality, we may assume that $[L]|_{\mathcal{P}}$ is well-defined for any $\mathcal{F}$-ε-multiplicative completely positive contractive linear map from C. We may also assume that $\mathcal{P}$ generates the subgroup

$$G_{\mathcal{P}} \subset K_0(C) \bigoplus K_1(C) \bigoplus \bigoplus_{i=1,0} \bigoplus_{j=1}^{m} K_i(C, \mathbb{Z}/j\mathbb{Z}) \text{ for some } m \geq 2.$$

Let $\delta > 0$ and $\mathcal{G} \subset A$ be a finite subset. Let A be a unital simple AF-algebra with $K_0(A) = \mathbb{Q} \oplus \mathbb{Z}$ and with $\ker \rho_A = \mathbb{Z}$. Write

$$A = \overline{\cup_{n=1}^{\infty} F_n},$$

where $1_A \in F_n \subset F_{n+1}$ is a sequence of finite dimensional C^*-algebras. Recall that there is an identification of $K_0(\mathcal{Z}_0)$ with $\ker \rho_A \cong \mathbb{Z} \subset K_0(A)$. Therefore there are sequences of pair of projections $p_n, q_n \in F_n$ such that

$$(j_{n,\infty})_{*0}([p_n] - [q_n]) = z_{\mathbb{Z}},$$

where $j_{n,\infty} : F_n \to A$ is the embedding and $z_{\mathbb{Z}}$ is [1] in $\mathbb{Z} \cong \ker \rho_A$. Without loss of generality we may assume that

$$[p_n] \neq [q_n] \in K_0(F_n) \text{ for all } n \geq 1. \tag{e12.6}$$

Write

$$F_n = M_{k_1} \oplus M_{k_2} \oplus \cdots M_{k_l}.$$

Note that $l \geq 3$ (see 7.7.2 of [2]). Let $P_i : F_n \to M_{k_i}$ be the projection map. Let $x_i = [P_i(p_n)] - [P_i(q_n)] \in \mathbb{Z} = K_0(M_{k_i})$, $i = 1, 2, \dots, l$. Then some of $x_i > 0$ and some of $x_i < 0$. Otherwise, we may assume that

$$x_i \geq 0 \text{ for all } i \in \{1, 2, \dots, l\}. \tag{e12.7}$$

Then $[p_n] - [q_n] \geq 0$ for all n. It follows that, for all $k \geq 1$,

$$(j_{n,n+k})_{*0}([p_n] - [q_n]) \geq 0 \text{ and } (j_{n,\infty})_{*0}([p_n] - [q_n]) \geq 0. \tag{e12.8}$$

That is $(j_{n,\infty})_{*0}([p_n] - [q_n]) \in K_0(A)_+$. This contradicts the fact that $(j_{n,\infty})_{*0}([p_n] - [q_n]) = z_{\mathbb{Z}}$.

Note that, as constructed in Section 6, with A above,

$$\mathcal{Z}_0 = \lim_{m\to\infty} (E_m \oplus W_m), \tag{e12.9}$$

where W_m is a single summand of the form $R(\alpha_m, 1)$ for some $0 < \alpha_m < 1$ and $E_m = M_{(m!)^2}(A(W, \alpha_m))$. Note that $K_i(W_m) = \{0\}$, $i = 0, 1$, and $K_0(A(W, \alpha_m)) = \mathbb{Z}$ and $K_1(A(W, \alpha_m)) = \{0\}$. Let id $\in KK(\mathcal{Z}_0, E_m)$ be as described in 12.4. Let $\kappa_{00} \in KK(C \otimes \mathcal{Z}_0, C \otimes E_m)$ be the invertible element given by $[\mathrm{id}_C]$ and id.

By (e12.9), there exists a $\mathcal{G}$-δ-multiplicative completely positive contractive linear map $\Phi : C \otimes \mathcal{Z}_0 \to C \otimes E_m$ (for sufficiently large m) such that

$$[\Phi]|_{\mathcal{P}} = (\kappa_{00})|_{\mathcal{P}} \tag{e12.10}$$

which maps strictly positive elements to strictly positive elements. Consider the short exact sequence

$$0 \to C_0((0,1), Q) \to E_m \to M_{(m!)^2}(A) \to 0.$$

Let $\varphi_{qa} : E_m \to M_{(m!)^2}(A)$ be the quotient map. Note that $(\varphi_{qa})_{*0}$ gives an isomorphism from $\mathbb{Z} = K_0(A(W, \alpha_m))$ onto $\ker \rho_A$. Let $\varphi_q : C \otimes E_m \to C \otimes M_{(m!)}(A)$ be defined by $\mathrm{id}_C \otimes \varphi_{qa}$. Let $\varphi_1 : C \otimes \mathcal{Z}_0 \to C \otimes M_{(m!)^2}(A)$ be defined by $\varphi_1 = \varphi_q \circ \Phi$. For any $\delta_1 > 0$ and finite subset $\mathcal{F}_A \subset M_{(m!)^2}(A)$, there is a unital $\mathcal{F}_A$-δ_1-multiplicative completely positive map, $\Phi_A : M_{(m!)^2}(A) \to M_{(m!)^2}(F_n)$ such that $[\Phi_A]|_{\ker \rho_A}$ is injective. Note that Φ_A maps strictly positive elements of A to strictly positive elements of F_n. Recall

$$F_n = M_{k_1} \oplus M_{k_2} \oplus \cdots M_{k_l}$$

and $x_i = [P_i(p_n)] - [P_i(q_n)] \in \mathbb{Z}$, $i = 1, 2, \ldots, l$. Without loss of generality, we may assume that

$$x_i > 0 \text{ for } i = 1, 2, \ldots, m^+, \; x_i < 0 \text{ for } i = m^+ + 1, \ldots, l', \text{ and } x_i = 0 \text{ for } 1 = l' + 1, \ldots, l. \tag{e12.11}$$

We claim that $x_1, x_2, \ldots, x_{l'}$ are relatively prime. If not, $x_i = Nx_i'$, for some integer x_i', $i = 1, 2, \ldots, l$, for some $N \geq 2$. Then $N(j_{n,\infty})_{*0}((x_1', x_2', \ldots, x_l')) = z_{\mathbb{Z}}$. This is impossible since $K_0(A) = \mathbb{Q} \oplus \mathbb{Z}$ and $z_{\mathbb{Z}} = [1]$ in the summand $\mathbb{Z}$. It follows from 12.3 that there are positive integers $N_1, N_2, \ldots, N_l$ such that

$$\sum_{i=1}^{l} N_i x_i = 1. \tag{e12.12}$$

Let $r = \sum_{i=1} N_i k_i$. Define $\imath : M_{(m!)^2}(F_n) \to M_{(m!)^2}(M_r)$ by

$$\imath((f_1, f_2, \ldots, f_l)) = \bigoplus_{i=1}^{l} \imath_i(f_i), \tag{e12.13}$$

where $\imath_i : M_{k_i} \to M_r$ is defined by

$$\imath_i(f_i) = \mathrm{diag}(\overbrace{f_i, f_i, \ldots, f_i}^{N_i}) \text{ for all } f_i \in M_{k_i}, \quad i = 1, 2, \ldots, l. \tag{e12.14}$$

Let $\kappa_{\mathcal{Z}_0} \in KL(C \otimes \mathcal{Z}_0, C)$ be defined by, for $j = 2, 3, \ldots$,

$$\kappa_{\mathcal{Z}_0}(x \otimes z_{\mathbb{Z}}) = x \text{ for all } x \in K_i(C \otimes \mathcal{Z}_0) \oplus K_i(C \otimes \mathcal{Z}_0, \mathbb{Z}/j\mathbb{Z}), \; i = 0, 1. \tag{e12.15}$$

Note that $(\imath_{*0})([p_n] - [q_n]) = [1] \in \mathbb{Z} = K_0(M_{(m!)^2 r})$. Let $L = (\mathrm{id}_C \otimes \imath) \circ (\mathrm{id}_C \otimes \Phi_A) \circ \varphi_1 : C \otimes \mathcal{Z}_0 \to C \otimes M_{(m!)^2 r}$. By choosing δ and δ_1 sufficiently small and $\mathcal{G}$ and $\mathcal{G}_A$ sufficiently large, L is $\mathcal{F}$-ε-multiplicative. Moreover, we compute that

$$[L]|_{\mathcal{P}} = [\kappa_{\mathcal{Z}_0}]|_{\mathcal{P}}. \quad \square$$

Lemma 12.6. *Let A and B be separable simple C^*-algebras in $\mathcal{D}$ with $K_0(A) = \ker \rho_A$ and $K_0(B) = \ker \rho_B$, respectively, which have continuous scale and satisfy the UCT. Suppose that there is $\kappa \in KL(A, B)$ and an affine continuous map $\kappa_T : T(B) \to T(A)$. Then, there exists a sequence of approximate multiplicative completely positive contractive linear maps $\varphi_n : A \to B$ such that*

$$[\{\varphi_n\}] = \kappa \text{ and} \tag{e12.16}$$

$$\lim_{n \to \infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)|\} = 0 \text{ for all } a \in A_{s.a.}. \tag{e12.17}$$

Proof. Let $\varepsilon > 0$, $\eta > 0$, $\mathcal{F} \subset A$ be a finite subset and $\mathcal{H} \subset A_{s.a}$ be a finite subset.

Fix a finite subset $\mathcal{P} \subset \underline{K}(A)$. We may assume that, for some $m \geq 1$,

$$\mathcal{P} \subset K_0(A) \bigoplus K_1(A) \bigoplus (\bigoplus_{j=1}^{m} K_0(A, \mathbb{Z}/j\mathbb{Z}) \bigoplus K_1(A, \mathbb{Z}/j\mathbb{Z})).$$

Moreover, $m!x = 0$ for all $x \in \mathrm{Tor}(K_0(A)) \cap \mathcal{P}$. Let $G_{0,\mathcal{P}}$ be the subgroup generated by $K_0(A) \cap \mathcal{P}$. We may write $G_{0,\mathcal{P}} := F_0 \oplus G_0$, where F_0 is free and G_0 is torsion. In particular, $m!x = 0$ for all $x \in G_0$.

Choose $\delta > 0$ and finite subset $\mathcal{G} \subset A$ so that $[L]|_{\mathcal{P}}$ is well defined for any $\mathcal{G}$-δ-multiplicative completely positive contractive linear map L from A. We may assume that $\delta < \varepsilon$ and $\mathcal{F} \cup \mathcal{H} \subset \mathcal{G}$. Since both A and B have continuous scale, $T(A)$ and $T(B)$ are compact (5.3 of [15]). Choose $a_0 \in A_+$ such that $\|a_0\| = 1$ and

$$d_\tau(a_0) < \min\{\eta, \delta\}/4 \text{ for all } \tau \in T(A). \tag{e12.18}$$

Let $e_0 \in A$ be a strictly positive element of A with $\|e_0\| = 1$ such that $\tau(e_0) > 15/16$ for all $\tau \in T(A)$.

Since $A \in \mathcal{D}_0$ (see 8.8), by 10.7 of [15], there are $\mathcal{G}$-$\delta/4$-multiplicative completely positive contractive linear maps $\varphi_0 : A \to \overline{\varphi_0(A)A\varphi_0(A)}$ and $\psi_0 : A \to D \subset A$ with $D \in \mathcal{C}_0^{(0)}$ and $M_{m!}(D) \perp \varphi_0(A)$ such that

$$\|x - (\varphi_0(x) \oplus \mathrm{diag}(\overbrace{\psi_0(x), \psi_0(x), \ldots, \psi_0(x)}^{m!}))\| < \delta/16 \ \text{ for all } \ x \in \mathcal{G}, \tag{e12.19}$$
$$\varphi_0(e_0) \lesssim a_0, \tag{e12.20}$$
$$t(f_{1/4}(\psi_0(e_0))) > 1/4 \ \text{ for all } \ t \in T(D). \tag{e12.21}$$

Let $\Psi_0 : A \to M_{m!}(D) \subset A$ be defined by

$$\Psi_0(a) = \mathrm{diag}(\overbrace{\psi_0(x), \psi_0(x), \ldots, \psi_0(x)}^{m!}) \ \text{ for all } \ a \in A. \tag{e12.22}$$

Let $\mathcal{P}_1 = [\varphi_0](\mathcal{P})$ and $\mathcal{P}_2 = [\Psi_0](\mathcal{P})$. Put $\mathcal{P}_3 = \mathcal{P} \cup \mathcal{P}_1 \cup \mathcal{P}_2$. Note that, since $K_i(D) = \{0\}$ $(i = 0, 1)$, $\Psi_0|_{\mathcal{P}\cap K_i(A)} = 0$, $i = 0, 1$. Moreover, by (e12.22),

$$[\Psi_0]|_{\mathcal{P}\cap K_i(\mathbb{Z}/j\mathbb{Z})} = 0, \ \ i = 0, 1, \ j = 2, \ldots, m. \tag{e12.23}$$

Set

$$d = \inf\{d_\tau(\varphi_0(e_0)) : \tau \in T(A)\}. \tag{e12.24}$$

We also have

$$[\varphi_0]|_{F_0} = [\mathrm{id}_A]|_{F_0}. \tag{e12.25}$$

Let $\mathcal{G}_1 = \mathcal{G} \cup \varphi_0(\mathcal{G})$. Choose $0 < \delta_1 < \delta$ and finite subset $\mathcal{G}_1 \subset A$ such that $[L']|_{\mathcal{P}_4}$ is well defined for any $\mathcal{G}_1$-δ_1-multiplicative completely positive contractive linear map from A.

It follows from 10.8, 12.1 and 12.5 that there exists a $\mathcal{G}_1$-$\delta_1/4$-multiplicative completely positive contractive linear map $L : A \to B \otimes M_K$ for some integer K such that

$$[L]|_{\mathcal{P}_3} = \kappa_{\mathcal{Z}_0} \circ (\kappa_{\mathcal{Z}_0}^{-1} \circ \kappa)|_{\mathcal{P}_3} = \kappa|_{\mathcal{P}_3}, \tag{e12.26}$$

where $\kappa_{\mathcal{Z}_0} \in KK(B \otimes \mathcal{Z}_0, B)$ is as in 12.4 with $B = C$. Without loss of generality, we may assume that $\mathcal{G}_1 \subset A^{\mathbf{1}}$.

Let $b_0 \in B$ with $\|b_0\| = 1$ such that

$$\tau(b_0) < \min\{\eta, \delta_1, d\}/16(K+1) \ \text{ for all } \ \tau \in T(B). \tag{e12.27}$$

Let $e_b \in B \otimes M_K$ be a strictly positive element of $B \otimes M_K$ such that

$$\tau(e_b) > 7/8 \ \text{ for all } \ \tau \in T(B \otimes M_K). \tag{e12.28}$$

Let $Q \subset \underline{K}(B)$ be a finite subset which contains $[L](\mathcal{P}_4)$. We assume that

$$\mathcal{Q} \subset K_0(B) \bigoplus K_1(B) \bigoplus \bigoplus_{i=0,1} \bigoplus_{j=1}^{m_1} K_i(B, \mathbb{Z}/j\mathbb{Z}) \tag{e12.29}$$

for some $m_1 \geq 2$. Moreover, we may assume that $m_1 x = 0$ for all $x \in \mathrm{Tor}(G_{0,b})$, where $G_{0,b}$ is the subgroup generated by $\mathcal{Q} \cap K_0(B)$. Without loss of generality, we may assume that $m|m_1$.

Let $\mathcal{G}_b \subset B \otimes M_K$ be a finite subset and $1/2 > \delta_2 > 0$ be such that $[\Phi]|_{\mathcal{Q}}$ is well defined for any $\mathcal{G}_b$-δ_2-multiplicative completely positive contractive linear map Φ from $B \otimes M_K$. Note also, by 8.8, $B \in \mathcal{D}_0$. There are $\mathcal{G}_b$-δ_2-multiplicative completely positive contractive linear maps $\varphi_{0,b} : B \otimes M_K \to \overline{\varphi_{0,b}(B \otimes M_K)(B \otimes M_K)\varphi_{0,b}(B \otimes M_K)}$ and $\psi_{0,b} : B \otimes M_K \to D_b$, $M_{(m_1)!}(D_b) \subset B \otimes M_K$ with $D_b \in \mathcal{C}_0^{(0)}$ such that

$$\|b - (\varphi_{0,b}(b), \mathrm{diag}(\overbrace{\psi_{0,b}(b), \psi_{0,b}(b), \ldots, \psi_{0,b}(b)}^{(m_1)!}))\| < \min\{\delta_2, \varepsilon/16, \eta/16\} \ \text{ for all } \ b \in \mathcal{G}_b \tag{e12.30}$$
$$\text{and } \ \varphi_{0,b}(e_b) \lesssim b_0 \ \text{ and } \ t(\psi_{0,b}) > 3/4 \ \text{ for all } \ t \in T(D_b). \tag{e12.31}$$

Note that $K_1(D_b) = \{0\} = K_0(D_b)$. Moreover, as in (e12.19) and (e12.23), we may also assume that

$$[\psi_{0,b}]|_{\mathrm{Tor}(G_{0,b})} = 0 \ \text{ and } \ [\psi_{0,b}]|_{\mathcal{Q}\cap K_i(B,\mathbb{Z}/j\mathbb{Z})} = 0, \ \ j = 2, 3, \ldots, m_1. \tag{e12.32}$$

Therefore

$$[\varphi_{0,b}]|_{\mathrm{Tor}(G_{0,b})} = [\mathrm{id}_B]|_{\mathrm{Tor}(G_{0,b})}, [\varphi_{0,b}]|_{\mathcal{Q}\cap K_1(B)} = [\mathrm{id}_B]|_{\mathcal{Q}\cap K_1(B)} \ \text{ and} \tag{e12.33}$$
$$[\varphi_{0,b}]|_{\mathcal{Q}\cap K_i(B,\mathbb{Z}/j\mathbb{Z})} = [\mathrm{id}_B]|_{\mathcal{Q}\cap K_i(B,\mathbb{Z}/j\mathbb{Z})}, \ \ j = 2, 3, \ldots, m_1. \tag{e12.34}$$

Let $G_{\mathcal{P}}$ be the subgroup generated by $\mathcal{P}$ and let $\kappa' = \kappa - [\varphi_{0,b}] \circ [L] \circ [\varphi_0]$ be defined on $G_{\mathcal{P}}$.

Then, by (e12.26), (e12.33) and (e12.34), we compute that

$$\kappa'|_{G_{0,\mathcal{P}}} = 0, \quad \kappa'|_{\mathcal{P}\cap K_1(A)} = 0 \text{ and} \tag{e12.35}$$

$$\kappa'|_{\mathcal{P}\cap K_i(A,\mathbb{Z}/j\mathbb{Z})} = 0, \quad j = 2, 3, \ldots, m. \tag{e12.36}$$

Let $\imath : M_{m!}(D) \to A$ be the embedding.

Let $\kappa_T^\sharp : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ be given by κ_T. This induces an order semigroup homomorphism $\tilde{\kappa}^T : \mathrm{LAff}_+(\tilde{T}(A)) \to \mathrm{LAff}_+(\tilde{T}(B))$. By 8.6 and 11.1, one checks easily that $\kappa_T^\sharp$ is a Cuntz semigroup homomorphism and a morphism in **Cu**.

Let $\gamma' : Cu(M_{m!}(D)) \to \mathrm{LAff}_+(\tilde{T}(B))$ be the Cuntz semi-group homomorphism given by $\gamma' = \kappa_T^\sharp \circ Cu(\imath)$. Put $\gamma : Cu(M_{m!}(D)) \to \mathrm{LAff}_+(\tilde{T}(B))$ defined by $\gamma(f) = (1 - \min\{\eta, \eta_0\}/2(m!))\gamma'(f)$ for all $f \in Cu(M_{m!}(D))$.

Let $\gamma_0 : Cu^\sim(M_{m!}(D)) \to Cu^\sim(B)$ be the morphism induced by γ (note $K_0(M_{m!}(D)) = \{0\}$ and see also 7.3 of [15]).

By applying 1.0.1 of [49], one obtains a homomorphism $h_d : M_{m!}(D) \to B$ such that

$$(h_d)_{*0} = \gamma_{00} \text{ and } \tau \circ h_d(c) = \gamma(\widehat{c})(\tau) \text{ for all } \tau \in T(B) \text{ and } c \in (M_{m!}(D))_{s.a.} \tag{e12.37}$$

Define $h : A \to B$ by $h = h_d \circ \Psi_0$. Then

$$[h]|_{\mathcal{P}} = \kappa'|_{\mathcal{P}}, [h]|_{\mathcal{P}\cap K_1(A)} = 0 \text{ and } [h]|_{\mathcal{P}\cap K_i(\mathbb{Z}/j\mathbb{Z})} = 0, \quad i = 2, 3, \ldots, m. \tag{e12.38}$$

Moreover,

$$\tau(h(a)) = \gamma(\widehat{\Psi_0(a)}) \text{ for all } a \in A \text{ and } \tau \in T(B). \tag{e12.39}$$

Let $e_d \in M_{m!}(D)$ be a strictly positive element with $\|e_d\| = 1$. Then, by (e12.24),

$$d_\tau(h_d(e_d)) < 1 - d \text{ for all } \tau \in T(B). \tag{e12.40}$$

It follows from (e12.27) that

$$d_\tau(h(e_d)) + d_\tau(\varphi_{0,b}(e_0)) < 1 \text{ for all } \tau \in T(B). \tag{e12.41}$$

Note that B has stable rank one (see 11.5 of [15] and 15.5 of [18]). By omitting a conjugating unitary in B without loss of generality, we may assume that $\varphi_{0,b} \circ L \oplus h$ maps A into B. Put $\Phi = \varphi_{0,b} \circ L \oplus h$. Then Φ is $\mathcal{G}$-δ-multiplicative. Moreover, we compute that

$$[\Phi]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \text{ and } \sup\{|\tau(\Phi(x)) - \kappa_T(\tau)(x)| : \tau \in T(B_T)\} < \eta \text{ for all } x \in \mathcal{H}. \tag{e12.42}$$

The lemma then follows. □

Lemma 12.7. *Let A be a non-unital simple separable C^*-algebra in $\mathcal{D}$ with $K_0(A) = \ker \rho_A$ and with continuous scale which satisfies the UCT. Let B_T be as in 7.2. Suppose that there is $\kappa \in KL(B_T, A)$, an affine continuous map $\kappa_T : T(A) \to T(B_T)$ and a continuous homomorphism $\kappa_{uc} : U(\tilde{B}_T)/CU(\tilde{B}_T) \to U(\tilde{A})/CU(\tilde{A})$ such that $(\kappa, \kappa_T, \kappa_{uc})$ is compatible. Then there exists a sequence of approximate multiplicative completely positive contractive linear maps $\varphi_n : B_T \to A$ such that*

$$[\{\varphi_n\}] = \kappa \tag{e12.43}$$

$$\lim_{n\to\infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)|\} = 0 \textit{ for all } a \in (B_T)_{s.a.} \textit{ and} \tag{e12.44}$$

$$\lim_{n\to\infty} \mathrm{dist}(\kappa_{uc}(z), \varphi_n^\dagger(z)) = 0 \textit{ for all } z \in U(\tilde{B}_T)/CU(\tilde{B}_T). \tag{e12.45}$$

Proof. Let $\varepsilon > 0$, let $\eta > 0$ and let $\sigma > 0$, let $\mathcal{P} \subset \underline{K}(B_T)$ be a finite subset, let $S_u \subset U(\tilde{B}_T)/CU(\tilde{B}_T)$ be a finite subset, let $\mathcal{H} \subset (B_T)_{s.a.}$ be a finite subset and let $\mathcal{F} \subset B_T$ be a finite subset.

Without loss of generality, we may assume that $\mathcal{F} \subset (B_T)^{\mathbf{1}}$, and, $[L']|_{\mathcal{P}}$ and $(L')^\dagger|_{S_u}$ are well-defined for any $\mathcal{F}$-ε-multiplicative completely positive contractive linear map from B_T.

Let $G_1 \subset K_1(B_T)$ be the subgroup generated by $\mathcal{P} \cap K_1(B_T)$.

Fix $\delta > 0$ and a finite subset $\mathcal{G} \subset B_T$. We assume that $\delta < \min\{\varepsilon/2, \eta/4, \sigma/16\}$. To simplify notation, without loss of generality, we may assume that $G_1 \subset F \subset (\Phi_{n_0,\infty})_{*1}(K_1(B_{n_0}))$ for some $n_0 \geq 1$, where F is a finitely generated standard subgroup of $K_1(B_T)$ (see 11.3). We also choose n_0 larger than that required by 11.5 for δ (in place of ε) $\mathcal{G}$ (in place of $\mathcal{F}$) $\mathcal{P}$ and $\sigma/16$ (in place of δ_0). Without loss of generality, we may write

$$S_u = S_{u,1} \sqcup S_{u,0}, \tag{e12.46}$$

where $S_{u,1} \subset J_{F,u}(F)$ and $S_{u,0} \subset U_0(\tilde{B}_T)/CU(\tilde{B}_T) = \mathrm{Aff}(T(\tilde{B}_T))/\mathbb{Z}$ and both $S_{u,1}$ and $S_{u,0}$ are finite subsets. For $w \in S_{u,0}$, write

$$w = \prod_{j=0}^{l(w)} \exp(\sqrt{-1}2\pi h_{w,j}), \tag{e12.47}$$

where $h_{w,0} \in \mathbb{R}$ and $h_{w,j} \in (B_T)_{s.a.}$, $j = 1, 2, \ldots, l(w)$. Let

$$\mathcal{H}_u = \{h_{w,j} : 1 \le j \le l(w), \ \ w \in S_{u,0}\} \text{ and } M = \max\{\sum_{i=0}^{l(w)} \|h_{w,j}\| : w \in S_{u,0}\}. \tag{e12.48}$$

To simplify notation further, we may assume that $G_1 = F$.

Write $G_1 = \mathbb{Z}^{m_f} \oplus \mathrm{Tor}(G_1)$ and $\mathbb{Z}^{m_f}$ is generated by cyclic and free generators $x_1, x_2, \ldots, x_{m_f}$. Let $\mathrm{Tor}(G)$ be generated by $x_{0,1}, x_{0,2}, \ldots, x_{0,m_t}$. Let $u_1, u_2, \ldots, u_{m_f}, u_{1,0}, u_{2,0}, \ldots, u_{m_t,0} \in U(\tilde{B}_T)$ be unitaries such that $[u_i] = x_i$, $i = 1, 2, \ldots, m_f$, and $[u_{j,0}] = x_{0,j}$, $j = 1, 2, \ldots, m_t$. Let $\pi_u : U(\tilde{B}_T)/CU(\tilde{B}_T) \to K_1(B_T)$ be the quotient map and let G_u be the subgroup generated by $S_{u,1}$. Since $(\kappa, \kappa_T, \kappa_u)$ is compatible, without loss of generality, we may assume that $\pi_u(G_u) = \{x_1, x_2, \ldots, x_{m_f}\} \cup \{x_{0,1}, x_{0,2}, \ldots, x_{0,m_t}\}$ and $S_{u,1} = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{m_f}, \bar{u}_{1,0}, \bar{u}_{2,0}, \ldots, \bar{u}_{m_t,0}\}$ as described in 11.3, in particular, $k_j\bar{u}_{j,0} = 0$ in $U(\tilde{B}_T)/CU(\tilde{B}_T)$, $j = 1, 2, \ldots, m_t$.

Let $\varphi_n : B_T \to A$ be a sequence of approximately multiplicative completely positive contractive linear maps given by 12.6 such that

$$[\{\varphi_n\}] = \kappa \text{ and} \tag{e12.49}$$
$$\lim_{n\to\infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)|\} = 0 \text{ for all } a \in (B_T)_{s.a.}. \tag{e12.50}$$

Fix a strictly positive element $e_b \in B_T$ with $\|e_b\| = 1$ and $\tau(e_b) \ge 15/16$ and $\tau(f_{1/2}(e_b)) \ge 15/16$ for all $\tau \in T(B_T)$.

Let $\mathcal{F}_b \subset B_T$ be a finite subset which contains $\mathcal{F} \cup \mathcal{H} \cup \mathcal{H}_u$. and let $\delta_b > 0$. There are $\mathcal{F}_a$-δ_b-multiplicative completely positive contractive linear maps $\Phi_0 : B_T \to D_b \subset B_T$ with $D_b \in C_0^0$, $\Phi_1 : B_T \to B_T$ and $\Phi_1(B_T) \perp D_b$ such that

$$\|b - (\Phi_0(b) \oplus \Phi_1(b))\| < \delta_b/2 \text{ for all } b \in \mathcal{F}_b \text{ and} \tag{e12.51}$$
$$0 < d_\tau(\Phi_0(e_b)) < \min\{\eta, \sigma/16\}/4(M+1) \text{ for all } \tau \in T(B_T). \tag{e12.52}$$

Note that $K_0(D_b) = K_1(D_b) = \{0\}$. Therefore, for any sufficiently large n,

$$[\varphi_n \circ \Phi_0]|_{\mathcal{P}} = 0, \ \ [\varphi_n \circ \Phi_1]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \text{ and} \tag{e12.53}$$
$$d_\tau(\varphi_n(\Phi_0(e_b))) < \min\{\eta, \sigma/16\}/2(M+1) \text{ for all } \tau \in T(A). \tag{e12.54}$$

Fix a sufficiently large n. Define $\lambda = \kappa|_{G_u} - (\varphi_n \circ \Phi_1)^\dagger|_{G_u} : G_u \to U(\widetilde{A})/CU(\widetilde{A})$. Since $(\kappa, \kappa_T, \kappa_u)$ is compatible, $\pi_u \circ \lambda(\bar{u}_i) = 0$ and $\pi_u \circ \lambda(\bar{u}_{0,j}) = 0$, $i = 1, 2, \ldots, m_f$ and $j = 1, 2, \ldots, m_t$.

Let $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{H}$. It follows from 11.5 that there exists $\mathcal{F}_1$-$\min\{\varepsilon/4, \eta/4\}$-multiplicative completely positive contractive linear map $L : B_T \to \overline{cAc}$, where $c = \varphi_n \circ \Phi_0(e_b)$, such that

$$[L]|_{\mathcal{P}} = 0 \text{ and } \mathrm{dist}(L^\dagger(\bar{u}_j), \lambda(\bar{u}_j)) < \sigma/4, \ \ j = 1, 2, \ldots, m_f. \tag{e12.55}$$

Define $\Psi : B_T \to A$ by

$$\Psi(a) = L(a) \oplus \varphi_n \circ \Phi_1(a) \text{ for all } a \in B_T. \tag{e12.56}$$

Then Ψ is $\mathcal{F}$-ε-multiplicative if n is sufficiently large. By (e12.52), by (e12.50) and by choosing sufficiently large n,

$$\sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)|\} < \min\{\sigma/16, \eta\}/(M+1) \text{ for all } a \in \mathcal{H}_u \text{ and} \tag{e12.57}$$
$$\sup\{|\tau(\Psi(b)) - \kappa_T(\tau)(b)| : \tau \in T(A)\} < \min\{\sigma/16, \eta\} \text{ for all } b \in \mathcal{H}. \tag{e12.58}$$

It follows from (e12.53), (e12.55) and the definition of λ that

$$[\Psi]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \text{ and } \mathrm{dist}(\Psi^\dagger(\bar{u}_j), \kappa_{uc}(\bar{u}_j)) < \sigma/2, \ \ j = 1, 2, \ldots, m_f. \tag{e12.59}$$

By 11.6, we may also have

$$\mathrm{dist}(\Psi^\dagger(\bar{u}_{j,0}), \kappa_{uc}(\bar{u}_{j,0})) < \sigma, \ \ j = 1, 2, \ldots, m_t. \tag{e12.60}$$

By the choice of M and $\mathcal{H}_u$, (e12.52), and (e12.57), and by the assumption that $(\kappa, \kappa_T, \kappa_{uc})$ is compatible,

$$\mathrm{dist}(\Psi^\dagger(\bar{w}), \kappa_{u,c}(\bar{w})) < \sigma \text{ for all } w \in S_{u,0}. \ \ \square \tag{e12.61}$$

Theorem 12.8. *Let A be a separable amenable simple C^*-algebra in $\mathcal{D}_0$ with continuous scale which satisfies the UCT. Let B_T be as in 7.2. Suppose that there is $\kappa \in KL(B_T, A)$, an affine continuous map $\kappa_T : T(A) \to T(B_T)$ and a continuous homomorphism $\kappa_{uc} : U(\tilde{B}_T)/CU(\tilde{B}_T) \to U(\widetilde{A})/CU(\widetilde{A})$ such that $(\kappa, \kappa_T, \kappa_{uc})$ is compatible. Then there exists a homomorphism $\varphi : B_T \to A$ such that*

$$[\varphi] = \kappa, \ \ \tau \circ \varphi(a) = \kappa_T(\tau)(a) \text{ for all } a \in (B_T)_{s.a.} \text{ and } \varphi^\dagger = \kappa_{uc}. \tag{e12.62}$$

Proof. Let $e_b \in B_T$ be a strictly positive element of B_T with $\|e_b\| = 1$. Since A has continuous scale, without loss of generality, we may assume that

$$\min\{\inf\{\tau(e_b) : \tau \in T(B_T)\}, \inf\{\tau(f_{1/2}(e_b)) : \tau \in T(B_T)\}\} > 3/4. \tag{e12.63}$$

Let $T : (B_T)_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ be given by Theorem 5.7 of [15].

By 12.7, there exists a sequence of approximately multiplicative completely positive contractive linear maps $\varphi_n : B_T \to A$ such that

$$[\{\varphi_n\}] = \kappa \tag{e12.64}$$

$$\lim_{n\to\infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a) : \tau \in T(A)|\} = 0 \ \text{ for all } \ a \in (B_T)_{s.a.} \ \text{ and} \tag{e12.65}$$

$$\lim_{n\to\infty} \mathrm{dist}(\kappa_{uc}(z), \varphi_n^{\dagger}(z)) = 0 \ \text{ for all } \ z \in U(\tilde{B}_T)/CU(\tilde{B}_T). \tag{e12.66}$$

Let $\varepsilon > 0$ and $\mathcal{F} \subset B_T$ be a finite subset.

We will apply 5.3. Note that $K_0(\tilde{A})$ is weakly unperforated (see 5.5 and 8.11). $\delta_{1,1} > 0$ (in place of δ), $\gamma_1 > 0$ (in place of γ), $\eta_1 > 0$ (in place of η), let $\mathcal{G}_{1,1} \subset B_T$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{H}_{1,1} \subset (B_T)_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, $\mathcal{P}_1 \subset \underline{K}(B_T)$ (in place of $\mathcal{P}$), $\mathcal{U}_1 \subset U(\tilde{U})$ (in place of $\mathcal{U}$) with $\overline{\mathcal{U}} = \mathcal{P} \cap K_1(B_T)$ and let $\mathcal{H}_{1,2} \subset (B_T)_{s.a.}$ (in place of $\mathcal{H}_2$) be as required by Theorem 5.3 for T, ε and $\mathcal{F}$ (with $T(k, n) = n$, see 5.2).

Without loss of generality, we may assume that $\mathcal{H}_{1,1} \subset (B_T)_+^{\mathbf{1}} \setminus \{0\}$ and $\gamma_1 < 1/64$.

Let $\mathcal{G}_{1,2} \subset B_T$ (in place of $\mathcal{G}$) be a finite subset and let $\delta_{1,2} > 0$ be as required by Theorem 5.7 of [15] for the above $\mathcal{H}_{1,1}$ (in place of $\mathcal{H}_1$). Let $\delta_1 = \min\{\delta_{1,1}, \delta_{1,2}\}$ and $\mathcal{G}_1 = \mathcal{G}_{1,1} \cup \mathcal{G}_{1,2}$.

Choose $n_0 \geq 1$ such that φ_n is $\mathcal{G}_1$-$\delta_1/2$-multiplicative, for all $n \geq n_0$,

$$[\varphi_n]|_{\mathcal{P}_1} = \kappa|_{\mathcal{P}_1}, \tag{e12.67}$$

$$\sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)| : \tau \in T(B_T)\} < \gamma_1/2 \ \text{ for all } \ a \in \mathcal{H}_{1,2}, \tag{e12.68}$$

$$\tau(f_{1/2}(\varphi_n(e_a))) > 3/8 \ \text{ for all } \ \tau \in T(B_T) \ \text{ and} \tag{e12.69}$$

$$\mathrm{dist}(\varphi_n^{\dagger}(\bar{u}), \kappa_{uc}(\bar{u})) < \eta/2 \ \text{ for all } \ u \in \mathcal{U}. \tag{e12.70}$$

By applying 5.7 of [15], φ_n are all T-$\mathcal{H}_{1,1}$-full. By applying Theorem 5.3, we obtain a unitary $u_n \in \tilde{B}_T$ (for each $n \geq n_0$) such that

$$\|u_n^* \varphi_n(a) u_n - \varphi_{n_0}(a)\| < \varepsilon \ \text{ for all } \ a \in \mathcal{F}. \tag{e12.71}$$

Now let $\{\varepsilon_n\}$ be a decreasing sequence of positive elements such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of B_T such that $\cup_{n=1}^{\infty} \mathcal{F}_k$ is dense in B_T.

By what have been proved, we obtain a subsequence $\{n_k\}$ and a sequence of unitaries $\{u_k\} \subset \tilde{B}_T$ such that

$$\|\mathrm{Ad}\, u_{k+1} \circ \varphi_{n_{k+1}}(a) - \mathrm{Ad}\, u_k \circ \varphi_{n_k}(a)\| < \varepsilon_k \ \text{ for all } \ a \in \mathcal{F}_k, \tag{e12.72}$$

$k = 1, 2, \ldots$. Since $\cup_{n=1}^{\infty} \mathcal{F}_k$ is dense in B_T, by (e12.72), $\{\mathrm{Ad}\, u_k \circ \varphi_{n_k}(a)\}$ is a Cauchy sequence. Let

$$\varphi(a) = \lim_{k\to\infty} \mathrm{Ad}\, u_k \circ \varphi_{n_k}(a) \ \text{ for all } \ a \in B_T. \tag{e12.73}$$

Then $\varphi : B_T \to A$ is a homomorphism which satisfies (e12.62). $\square$

Lemma 12.9. *Let A be a non-unital simple separable C^*-algebra in $\mathcal{D}$ with $K_0(A) = \ker \rho_A$ and with continuous scale which satisfies the UCT. Let B_T be as in 7.2. Suppose that there is $\kappa \in KL(A, B_T)$, an affine continuous map $\kappa_T : T(B_T) \to T(A)$ and a continuous homomorphism $\kappa_{uc} : U(\tilde{A})/CU(\tilde{A}) \to U(\widetilde{B_T})/CU(\widetilde{B_T})$ such that $(\kappa, \kappa_T, \kappa_{uc})$ is compatible. Suppose also that $\kappa|_{K_1(A)}$ is injective.*

Then there exists a sequence of approximate multiplicative completely positive contractive linear maps $\varphi_n : A \to B_T$ such that

$$[\{\varphi_n\}] = \kappa, \tag{e12.74}$$

$$\lim_{n\to\infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)|\} = 0 \ \textit{ for all } \ a \in A_{s.a.} \ \textit{ and} \tag{e12.75}$$

$$\lim_{n\to\infty} \mathrm{dist}(\kappa_{uc}(z), \varphi_n^{\dagger}(z)) = 0 \ \textit{ for all } \ z \in U(\tilde{A})/CU(\tilde{A}). \tag{e12.76}$$

Proof. Denote by $\Pi : U(\tilde{A})/CU(\tilde{A}) \to K_1(A)$ the quotient map and fix a splitting map $J_u : K_1(A) \to U(\tilde{A})/CU(\tilde{A})$. Since $(\kappa, \kappa_T, \kappa_{uc})$ is compatible, it suffices to show that there are $\{\varphi_n\}$ which satisfies (e12.74) and (e12.75) and

$$\lim_{n\to\infty} \mathrm{dist}(\kappa_{uc}(J_u(\zeta)), \varphi_n^{\dagger}(J_u(\zeta))) = 0 \ \text{ for all } \ \zeta \in K_1(A). \tag{e12.77}$$

It follows from 12.6 that there exists $\{\varphi_n\}$ which satisfies (e12.74) and (e12.75). Let $G_1 \subset K_1(A)$ be a finitely generated subgroup.

Choose some sufficiently large n, then $\varphi_n^{\dagger}$ induces a homomorphism on the group $J_u(G_1)$. Since $\kappa|_{K_1(A)}$ is injective and $(\kappa, \kappa_T, \kappa_{uc})$ is compatible, $\varphi_n^{\dagger}|_{J_u(G_1)}$ has an inverse γ. Let $G_b = \varphi_n^{\dagger}(J_u(G_1))$ and let $\Pi_b : U(\tilde{B}_T)/CU(\tilde{B}_T) \to K_1(B_T)$ be the quotient map. Again, using the fact that $(\kappa, \kappa_T, \kappa_{uc})$ is compatible, $(\Pi_b)|_{G_b}$ is injective. Let $J_{ub} : K_1(B_T) \to U(\tilde{B}_T)/CU(\tilde{B}_T)$ be a homomorphism such that $\Pi_b \circ J_{uc} = \mathrm{id}_{K_1(B_T)}$.

Put

$$\lambda_0 = ((\kappa_{uc} \circ \gamma) \circ J_{uc} - (\varphi_n)^{\dagger} \circ \gamma \circ J_{uc})|_{\Pi_b(G_b)}. \tag{e12.78}$$

Then, since $(\kappa, \kappa_T, \kappa_{uc})$ is compatible, $\Pi_b \circ \lambda_0 = 0$. Therefore λ_0 maps from $\Pi_b(G_b)$ to $\mathrm{Aff}(T(\tilde{B}_T))/\overline{\rho_{B_T}(K_1(\tilde{B}_T))}$. However, $\mathrm{Aff}(T(\tilde{B}_T))/\overline{\rho_{B_T}(K_1(\tilde{B}_T))}$ is divisible. Therefore there is a homomorphism $\lambda_1 : K_1(B_T) \to \mathrm{Aff}(T(\tilde{B}_T))/\overline{\rho_{B_T}(K_1(\tilde{B}_T))}$ such that

$$(\lambda_1)|_{\Pi_b(G_b)} = \lambda_0. \tag{e12.79}$$

Now defined $\Lambda : U(\tilde{B}_T)/CU(\tilde{B}_T) \to U(\tilde{B}_T)/CU(\tilde{B}_T)$ as follows.

$$\Lambda|_{\mathrm{Aff}(T(\tilde{B}_T))/\overline{\rho_{B_T}(K_1(\tilde{B}_T))}} = \mathrm{id}_{\mathrm{Aff}(T(\tilde{B}_T))/\overline{\rho_{B_T}(K_1(\tilde{B}_T))}}, \tag{e12.80}$$

$$\Lambda|_{J_{ub}(K_1(B_T))} = \lambda_1 \circ \Pi_b + (\mathrm{id}_{B_T})^{\dagger}. \tag{e12.81}$$

Note that $([\mathrm{id}_{B_T}], (\mathrm{id}_{B_T})_T, \Lambda)$ is compatible. It follows from 12.7 that there exists a homomorphism $\psi_n : B_T \to B_T$ such that

$$[\psi_n] = [\mathrm{id}_{B_T}],\ \ (\psi_n)_T = (\mathrm{id}_{B_T})_T \ \text{ and } \ \psi_n^{\dagger} = \Lambda. \tag{e12.82}$$

Now let $\Phi_n = \psi_n \circ \varphi_n$. Then, for $z \in J_u(G_1)$, by (e12.78),

$$\Phi_n^{\dagger}(z) = \psi_n^{\dagger} \circ \varphi_n^{\dagger}(z) = \lambda_1 \circ \Pi_b \circ \varphi_n^{\dagger}(z) + \varphi_n^{\dagger}(z) \tag{e12.83}$$

$$= \lambda_0 \circ \varphi_n^{\dagger}(z) + \varphi_n^{\dagger}(z) = \kappa_{uc}(z). \tag{e12.84}$$

The lemma follows immediately from this construction of Φ_n. □

Lemma 12.10. *Let A be a non-unital simple separable C^*-algebra in $\mathcal{D}_0$ with continuous scale which satisfies the UCT. Let B_T be as in 7.2. Suppose that there is $\kappa \in KL(A, B_T)$, an affine continuous map $\kappa_T : T(B_T) \to T(A)$, and a continuous homomorphism $\kappa_{uc} : U(\tilde{A})/CU(\tilde{A}) \to U(\tilde{B_T})/CU(\tilde{B_T})$ such that $(\kappa, \kappa_T, \kappa_{uc})$ is compatible. Suppose also that $\kappa|_{K_1(A)}$ is injective.*

Then there exists a homomorphism $\varphi : A \to B_T$ such that

$$[\varphi] = \kappa,\ \ \varphi_T = \kappa_T \ \text{ and } \ \varphi^{\dagger} = \kappa_{uc}. \tag{e12.85}$$

Proof. The proof is exactly the same as that of 12.8 but applying 12.9 instead of 12.7. □

13. The isomorphism theorem for $\mathcal{Z}_0$-stable C^*-algebras

Theorem 13.1. *Let A and B be two separable simple amenable C^*-algebras in $\mathcal{D}$ with continuous scale which satisfy the UCT. Suppose that* $\ker \rho_A = K_0(A)$ *and* $\ker \rho_B = K_0(B)$. *Then $A \cong B$ if and only if*

$$(K_0(A), K_1(A), T(A)) \cong (K_0(B), K_1(B), T(B)). \tag{e13.1}$$

Moreover, let $\kappa_i : K_i(A) \to K_i(B)$ be an isomorphism as abelian groups ($i = 0, 1$) and let $\kappa_T : T(B) \to T(A)$ be an affine homeomorphism. Suppose that $\kappa \in KL(A, B)$ which gives κ_i and $\kappa_{cu} : U(\tilde{A})/CU(\tilde{A}) \to U(\tilde{B})/CU(\tilde{B})$ is a continuous affine isomorphism so that $(\kappa, \kappa_T, \kappa_{cu})$ is compatible. Then there is an isomorphism $\varphi : A \to B$ such that

$$[\varphi] = \kappa \ \ (i = 0, 1),\ \ \varphi_T = \kappa_T \ \text{ and } \ \varphi^{\dagger} = \kappa_{cu} \tag{e13.2}$$

Proof. Note it follows from 8.8 that $A, B \in \mathcal{D}_0$. It follows from 7.11 that there is a non-unital simple C^*-algebra B_T constructed in Section 7 such that

$$K_0(B_T) = K_0(B), K_1(B_T) = K_1(B) \ \text{ and } \ T(B_T) = T(B). \tag{e13.3}$$

Let $\kappa \in KL(A, B)$ be an invertible element which gives κ_i ($i = 0, 1$). Let $\kappa_T : T(B) \to T(A)$ be an affine homeomorphism. By the assumption, (κ, κ_T) is always compatible. Choose any κ_{cu} so that $(\kappa, \kappa_T, \kappa_{cu})$ is compatible. Note that there is always at least one: $\kappa_{cu}|_{J_c(K_1(A))} = J_c \circ \kappa|_{K_1(A)} \circ \pi_{cu}$, where $\pi_{cu} : U(\tilde{A})/CU(\tilde{A}) \to K_1(A)$ is the quotient map and $\kappa_{cu}|_{\mathrm{Aff}(T(A))/\mathbb{Z}}$ is induced by κ_T.

Therefore it suffices to show that there is an isomorphism $\varphi : A \to B$ such that (e13.2) holds. We will use the Elliott intertwining argument.

Let $\{\mathcal{F}_{a,n}\}$ be an increasing sequence of finite subsets of A such that $\cup_{n=1}^{\infty} \mathcal{F}_{a,n}$ is dense in A, let $\{\mathcal{F}_{b,n}\}$ be an increasing sequence of finite subsets of B such that $\cup_{n=1}^{\infty} \mathcal{F}_{b,n}$ is dense in B. Let $\{\varepsilon_n\}$ be a sequence of decreasing positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < 1$.

Let $e_a \in A$ and $e_b \in B$ be strictly positive elements of A and B, respectively, with $\|e_a\| = 1$ and with $\|e_b\| = 1$. Note that $d_\tau(e_a) = 1$ for all $\tau \in T(A)$ and $d_\tau(e_b) = 1$ for all $\tau \in T(B)$.

It follows from 12.10 that there is a homomorphism $\varphi_1 : A \to B$ such that

$$[\varphi_1] = \kappa, \quad (\varphi_1)_T = \kappa_T \ \text{ and } \ \varphi_1^\dagger = \kappa_{cu}. \tag{e13.4}$$

Note that $d_\tau(\varphi_1(e_a)) = 1$. Therefore φ_1 maps e_a to a strictly positive element of B. It follows from 12.7 that there is a homomorphism $\psi_1' : B \to A$ such that

$$[\psi_1'] = \kappa^{-1}, \ (\psi_1')_T = \kappa_T^{-1} \ \text{ and } \ (\psi_1')^\dagger = \mathrm{id}_A^\dagger \circ (\varphi_1^\dagger)^{-1}. \tag{e13.5}$$

Thus

$$[\psi_1' \circ \varphi_1] = [\mathrm{id}_A], \ \ (\psi_1' \circ \varphi_1)_T = \mathrm{id}_{T(A)} \ \text{ and } \ (\psi_1' \circ \varphi_1)^\dagger = \mathrm{id}_{U(\tilde{A})/CU(\tilde{A})}. \tag{e13.6}$$

It follows from 5.3 (see also 5.6) that there exists a unitary $u_{1,a} \in \tilde{A}$ such that

$$\mathrm{Ad}\, u_{1,a} \circ \psi_1' \circ \varphi_1 \approx_{\varepsilon_1} \mathrm{id}_A \ \text{ on } \ \mathcal{F}_{a,1}. \tag{e13.7}$$

Put $\psi_1 = \mathrm{Ad}\, u_{1,a} \circ \psi_1'$. Then we obtain the following diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\varphi_1"'] & A \\ B \arrow[ur, "\psi_1"'] & \end{tikzcd}$$

which is approximately commutative on the subset $\mathcal{F}_{a,1}$ within ε_1.

By applying 12.10, there exists a homomorphism $\varphi_2' : A \to B$ such that

$$[\varphi_2'] = \kappa, \ \ (\varphi_2')_T = \kappa_T \ \text{ and } \ (\varphi_2')^\dagger = \mathrm{id}_B^\dagger \circ (\psi_1^\dagger)^{-1} = \kappa_{cu}. \tag{e13.8}$$

Then,

$$[\varphi_2' \circ \psi_1] = [\mathrm{id}_B], \ \ (\varphi_2' \circ \psi_1)_T \ \text{ and } \ (\varphi_2' \circ \psi_1)^\dagger = \mathrm{id}_{U(\tilde{B})/CU(\tilde{B})}. \tag{e13.9}$$

It follows from 5.3 (and 5.6) that there exists a unitary $u_{2,b} \in \tilde{B}$ such that

$$\mathrm{Ad}\, u_{2,b} \circ \varphi_2'' \circ \psi_1 \approx_{\varepsilon_2} \mathrm{id}_B \ \text{ on } \ \mathcal{F}_{b,2} \cup \varphi_1(\mathcal{F}_{a,1}). \tag{e13.10}$$

Put $\varphi_2 = \mathrm{Ad}\, u_{2,b} \circ \varphi_2$. Then we obtain the following diagram:

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\varphi_1"'] & A \arrow[d, "\varphi_2"] \\ B \arrow[ur, "\psi_1"'] \arrow[r, "\mathrm{id}_B"'] & B \end{tikzcd}$$

with the upper triangle approximately commutes on $\mathcal{F}_{a,1}$ within ε_1 and the lower triangle approximately commutes on $\mathcal{F}_{b,2} \cup \varphi_1(\mathcal{F}_{a,1})$ within ε_2. Note also

$$[\varphi_2] = \kappa, \ \ (\varphi_2)_T = \kappa_T \ \text{ and } \ (\varphi_2)^\dagger = \kappa_{cu}. \tag{e13.11}$$

We then continue this process, and, by the induction, we obtain an approximate intertwining:

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\varphi_1"'] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\varphi_2"] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\varphi_3"] & \cdots\cdots A \\ B \arrow[ur, "\psi_1"] \arrow[r, "\mathrm{id}_B"'] & B \arrow[ur, "\psi_2"] \arrow[r, "\mathrm{id}_B"'] & B \arrow[r, "\mathrm{id}_B"'] & \cdots\cdots B \end{tikzcd}$$

By the Elliott approximate intertwining argument, this implies that $A \cong B$ and the isomorphism φ produced by the above diagram meets the requirements of (e13.2). $\square$

The following theorem and its proof gives the proof of Theorem 1.1.

Theorem 13.2. *Let A and B be two stably projectionless separable simple amenable C^*-algebras with $gTR(A) \le 1$ and $gTR(B) \le 1$ and which satisfy the UCT. Suppose that $K_0(A) = \ker \rho_A$ and $K_0(B) = \ker \rho_B$. Then $A \cong B$ if and only if*

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \cong (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B). \tag{e13.12}$$

Proof. Let

$$\Gamma : (K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \to (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B) \tag{e13.13}$$

be an isomorphism. Let $\Gamma_T : \tilde{T}(A) \to \tilde{T}(B)$ be the cone homeomorphism such that

$$\Sigma_B(\Gamma_T(\tau)) = \Sigma_A(\tau) \text{ for all } \tau \in \tilde{T}(A). \tag{e13.14}$$

Let $e_A \in \mathrm{Ped}(A)_+$ such that $\|e_A\| = 1$ such that $A_0 := \overline{e_A A e_A}$ has continuous scale (see 5.3 of [15]). Choose $b_0 \in P(B)_+ \setminus \{0\}$ with $\|b_0\| = 1$ such that $B' := \overline{b_0 B b_0}$ has continuous scale. Then $T(A_0)$ and $T(B')$ are metrizable Choquet simplices. Moreover $T(A_0)$ and $T(B')$ can be identified with

$$T_A = \{\tau \in \tilde{T}(A) : d_\tau(a_A) = 1\} \text{ and } \{s \in \tilde{T}(B') : d_s(b_0) = 1\}, \tag{e13.15}$$

respectively. Let $g(t) = d_{\Gamma^{-1}(t)}(e_A) \in \mathrm{LAff}_f(\tilde{T}(B))$. Since $d_\tau(e_A)$ is continuous and Γ^{-1} is a cone homeomorphism, $g(t)$ is continuous and $g \in \mathrm{Aff}_+(T(B'))$. Since $\mathrm{Aff}_+(T(B'))$ is compact, g is also bounded. By identifying $B' \otimes \mathcal{K}$ with $B \otimes \mathcal{K}$, we find a positive element $b_{00} = \mathrm{diag}(b_0, \dots, b_0) \in B \otimes \mathcal{K}$, where b_0 repeats m times so that $d_s(b_{00}) > g(s)$ on $T(B')$. Then g is continuous on $T(B'')$, where $B'' := \overline{b_{00}(B \otimes \mathcal{K})b_{00}}$. It follows 8.6 that there is $e_B \in B''_+ \subset B \otimes \mathcal{K}$ with $\|e_B\| = 1$ such that $d_s(e_B) = g|_{T(B'')}$. Since B has strictly comparison, $B_0 := \overline{e_B B e_B}$ has continuous scale (see 5.3 of [15]). Let

$$T_B = \{t \in \tilde{T}(B) : d_t(e_B) = 1\}. \tag{e13.16}$$

Then $T(A_0) = T_B$. It follows that Γ induces the following isomorphism

$$(K_0(A_0), K_1(A_0), T(A_0)) \cong (K_0(B_0), K_1(B_0), T(B_0)). \tag{e13.17}$$

It follows from 13.1 that there is an isomorphism $\varphi_0 : A_0 \to B_0$ which induces Γ on $(K_0(A_0), K_1(A_0), T(A_0))$. By [4], φ_0 gives an isomorphism from $A_0 \otimes \mathcal{K}$ onto $B_0 \otimes \mathcal{K}$. Let $a \in A_+$ with $\|a\| = 1$ be a strictly positive element. Then

$$\hat{a}(\tau) = \Sigma_A(\tau) \text{ for all } \tau \in \tilde{T}(A). \tag{e13.18}$$

Let $b \in (B_0 \otimes \mathcal{K})_+$ such that $\varphi(a) = b$. Then

$$d_t(b) = \lim_{n\to\infty} t \circ \varphi(a^{1/n}) \text{ for all } t \in \tilde{T}(B). \tag{e13.19}$$

Note $\Sigma_B(t) = d_t(b)$. Since B is simple and has stable rank one, this implies that $B \cong \overline{b(B_0 \otimes \mathcal{K})b}$. The theorem follows. □

Corollary 13.3. *Let A and B be in $\mathcal{D}_0$ which are amenable and satisfy the UCT. Then $A \cong B$ if and only if*

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B). \tag{e13.20}$$

Proof. Since A and B are in $\mathcal{D}_0$, by 8.5, $K_0(A) = \ker \rho_A$ and $K_0(B) = \ker \rho_B$. Therefore Theorem 13.2 applies. □

Corollary 13.4. *Let A be a stably projectionless simple separable amenable C^*-algebra which satisfies the UCT and $gTR(A) \le 1$. Suppose that $K_0(A) = \ker \rho_A$. Then $A \otimes \mathcal{Z}_0 \cong A$.*

In particular, $\mathcal{Z}_0 \otimes \mathcal{Z}_0 \cong \mathcal{Z}_0$.

Proof. Recall that $K_0(\mathcal{Z}_0) = \mathbb{Z} = \ker \rho_{\mathcal{Z}_0}$, $K_1(\mathcal{Z}_0) = \{0\}$ and $T(\mathcal{Z}_0)$ has exactly one point. Let $A_0 = \overline{eAe}$ for some $e \in A_+ \setminus \{0\}$ such that $A_0 \in \mathcal{D}$. Since $K_0(A) = \ker \rho_A$, $A_0 \in \mathcal{D}_0$, by 8.8. By 12.5 of [15] and 6.6 of [16] (or by 18.5 and 18.6 of [18]), $A_0 \otimes \mathcal{Z}_0 \in \mathcal{D}_0$. Therefore $gTR(A \otimes \mathcal{Z}_0) \le 1$. Moreover, $K_0(A \otimes \mathcal{Z}_0) \cong K_0(A) = \ker \rho_A$, $K_1(A \otimes \mathcal{Z}_0) \cong K_1(A)$, $\tilde{T}(A \otimes \mathcal{Z}_0) = \tilde{T}(A)$ and $\Sigma_A = \Sigma_{A \otimes \mathcal{Z}_0}$. Thus 13.2 applies. □

14. A homotopy lemma

The purpose of this section is to present 14.14 which will be used in next section. The following is known, a proof for the unital case can be found in 12.4 of [20]

Lemma 14.1. *Let C be a separable C^*-algebra, and let $\Delta : C_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. There exists a map $T : C_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$ satisfying the following: For any finite subset $\mathcal{H} \subset C_+^{\mathbf{1}} \setminus \{0\}$ and any σ-unital C^*-algebra A with the strict comparison of positive elements which is quasi-compact, if $\varphi : C \to A$ is a unital contractive completely positive linear map satisfying*

$$\tau \circ \varphi(h) \ge \Delta(\hat{h}) \text{ for all } h \in \mathcal{H} \text{ for all } \tau \in \mathrm{T}(A), \tag{e14.1}$$

then φ is T-$\mathcal{H}$-full.

Recall the class of sub-homogeneous C^*-algebras $\overline{D_r}$ is defined in 4.8 of [20]. The following is a non-unital version of 8.4 of [20] (see 5.2.7 of [39]).

Theorem 14.2. *Let A_0 be a non-unital C^*-algebra such that $A := \tilde{A_0} \in \overline{D_r}$ with finitely generated $K_i(A)$ $(i = 0, 1)$. Let $\mathcal{F} \subset A$ be a finite subset, let $\varepsilon > 0$ be a positive number and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exist a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset \underline{K}(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset J_c(K_1(A))$ (see (e2.9) in 2.4 for the definition of J_c) for which $[\mathcal{U}] \subset \mathcal{P}$ satisfying the following: For any unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $\varphi, \psi : A_0 \to C$ for some $C \in \mathcal{C}_0$ such that*

$$[\varphi^{\sim}]|_{\mathcal{P}} = [\psi^{\sim}]|_{\mathcal{P}}, \tag{e14.2}$$

$$\tau(\varphi^{\sim}(a)) \geq \Delta(\hat{a}), \quad \tau(\psi^{\sim}(a)) \geq \Delta(\hat{a}), \quad \text{for all } \tau \in T(C) \text{ and } a \in \mathcal{H}_1, \tag{e14.3}$$

$$|\tau \circ \varphi^{\sim}(a) - \tau \circ \psi^{\sim}(a)| < \gamma_1 \text{ for all } a \in \mathcal{H}_2, \text{ and} \tag{e14.4}$$

$$\mathrm{dist}((\varphi^{\sim})^{\dagger}(u), (\psi^{\sim})^{\dagger}(u)) < \gamma_2 \text{ for all } u \in \mathcal{U}, \tag{e14.5}$$

there exists a unitary $W \in \tilde{C}$ such that

$$\|W(\varphi^{\sim}(f))W^* - (\psi^{\sim}(f))\| < \varepsilon, \text{ for all } f \in \mathcal{F}, \tag{e14.6}$$

where $\varphi^{\sim}, \psi^{\sim}$ are the unital extension of φ and ψ from A to $\tilde{C}$.

Proof. Without loss of generality, we may assume that A is infinite dimensional.

Since $K_*(A)$ is finitely generated, there is n_0 such that $\kappa \in \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(C))$ is determined by its restriction to $K_*(A, \mathbb{Z}/n\mathbb{Z})$, $n = 0, \ldots, n_0$.

Let $\mathcal{H}_1' \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be a finite subset and let $\mathcal{P}_0 \subset \underline{K}(A)$ (in place of $\mathcal{P}$) be a finite subset required by 4.4.5 of [39] (6.7 of [20]) for $\varepsilon/32$ (in place of ε), $\mathcal{F}$ and Δ. We may assume that $\delta_1 < \varepsilon/32$ and $(2\delta_1, \mathcal{G}_1)$ is a KK-pair (see the end of 2.12 of [20]).

Moreover, we may assume that δ_1 is sufficiently small that if $\|uv - vu\| < 3\delta_1$, then the Exel formula

$$\tau(\mathrm{bott}_1(u, v)) = \frac{1}{2\pi\sqrt{-1}}(\tau(\log(u^*vuv^*)))$$

holds for any pair of unitaries u and v in any unital C^*-algebra C with tracial rank zero and any $\tau \in T(C)$ (see Theorem 3.6 of [34]). Moreover if $\|v_1 - v_2\| < 3\delta_1$, then

$$\mathrm{bott}_1(u, v_1) = \mathrm{bott}_1(u, v_2).$$

Let $g_1, g_2, \ldots, g_{k(A)} \in U(M_{m(A)}(A))$ ($m(A) \geq 1$ is an integer) be a finite subset such that $\{\bar{g_1}, \bar{g_2}, \ldots, \bar{g}_{k(A)}\} \subset J_c(K_1(A))$ and such that $\{[g_1], [g_2], \ldots, [g_{k(A)}]\}$ forms a set of generators for $K_1(A)$. Let $\mathcal{U} = \{\bar{g_1}, \bar{g_2}, \ldots, \bar{g}_{k(A)}\} \subset J_c(K_1(A))$ be a finite subset.

Let $\mathcal{U}_0 \subset A$ be a finite subset such that

$$\{g_1, g_2, \ldots, g_{k(A)}\} \subseteq \{(a_{i,j}) : a_{i,j} \in \mathcal{U}_0\}.$$

Let $\delta_u = \min\{1/256m(A)^2, \delta_1/16m(A)^2\}$, $\mathcal{G}_u = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{U}_0$ and let $\mathcal{P}_u = \mathcal{P}_0 \cup \{[g_1], [g_2], \ldots, [g_{k(A)}]\}$.

Let $\delta_2 > 0$ (in place of δ), $\mathcal{G}_2 \subset A$ (in place of $\mathcal{G}$), $\mathcal{H}_2' \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}$), $N_1 \geq 1$ (in place of N) be the finite subsets and the constants as required by 7.3 of [20] for δ_u (in place of ε), $\mathcal{G}_u$ (in place of $\mathcal{F}$), $\mathcal{P}_u$ (in place of $\mathcal{P}$) and Δ and with $\bar{g}_j$ (in place of g_j), $j = 1, 2, \ldots, k(A)$ (with $k(A) = r$).

Let $\delta_3 > 0$ and let $\mathcal{G}_3 \subset A \otimes C(\mathbb{T})$ be a finite subset satisfying the following: For any $\mathcal{G}_3$-δ_3-multiplicative contractive completely positive linear map $L' : A \otimes C(\mathbb{T}) \to C'$ (for any unital C^*-algebra C' with $T(C') \neq \emptyset$),

$$|\tau([L'](\boldsymbol{\beta}(\bar{g}_j)))| < 1/8N_1, \quad j = 1, 2, \ldots, k(A). \tag{e14.7}$$

Without loss of generality, we may assume that

$$\mathcal{G}_3 = \{g \otimes f : g \in \mathcal{G}_3' \text{ and } f \in \{1, z, z^*\}\},$$

where $\mathcal{G}_3' \subset A$ is a finite subset containing 1_A (by choosing a smaller δ_3 and large $\mathcal{G}_3'$).

Let $\varepsilon_1' = \min\{d/27N_1m(A)^2, \delta_u/2, \delta_2/2m(A)^2, \delta_3/2m(A)^2\}$ and let $\bar{\varepsilon}_1 > 0$ (in place of δ) and $\mathcal{G}_4 \subset A$ (in place of $\mathcal{G}$) be a finite subset as required by 6.4 of [20] for ε_1' (in place of ε) and $\mathcal{G}_u \cup \mathcal{G}_3'$. Put $\varepsilon_1 = \min\{\varepsilon_1', \varepsilon_1'', \bar{\varepsilon}_1\}$. Let $\mathcal{G}_5 = \mathcal{G}_u \cup \mathcal{G}_3' \cup \mathcal{G}_4$.

Let $\mathcal{H}_3' \subseteq A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\delta_4 > 0$ (in place of δ), $\mathcal{G}_6 \subset A$ (in place of $\mathcal{G}$), $\mathcal{H}_4' \subset A_{s.a.}$ (in place of $\mathcal{H}_2$), $\mathcal{P}_1 \subset \underline{K}(A)$ (in place of $\mathcal{P}$) and $\sigma > 0$ be the finite subsets and constants as required by Theorem 5.8 of [20] with respect to $\varepsilon_1/16$ (in place ε) and $\mathcal{G}_5$ (in place of $\mathcal{F}$) and Δ.

Choose $N_2 \geq N_1$ such that $(k(A) + 1)/N_2 < 1/8N_1$. Choose $\mathcal{H}_5' \subset A_+^{\mathbf{1}} \setminus \{0\}$ and $\delta_5 > 0$ and a finite subset $\mathcal{G}_7 \subset A$ such that, for any M_m and unital $\mathcal{G}_7$-δ_5-multiplicative contractive completely positive linear map $L' : A \to M_m$, if $\mathrm{tr} \circ L'(h) > 0$ for all $h \in \mathcal{H}_5'$, then $m \geq 16N_2$.

Put $\delta = \min\{\varepsilon_1/16, \delta_4/4m(A)^2, \delta_5/4m(A)^2\}$, $\mathcal{G} = \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7$, and $\mathcal{P} = \mathcal{P}_u \cup \mathcal{P}_1$. Put

$$\mathcal{H}_1 = \mathcal{H}_1' \cup \mathcal{H}_2' \cup \mathcal{H}_3' \cup \mathcal{H}_4' \cup \mathcal{H}_5'$$

and let $\mathcal{H}_2 = \mathcal{H}_4'$. Let $\gamma_1 = \sigma$ and let $0 < \gamma_2 < \min\{d/16N_2m(A)^2, \delta_u/9m(A)^2, 1/256m(A)^2\}$.

Now suppose that $C \in \mathcal{C}_0$ and $\varphi, \psi : A \to C$ are two unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps satisfying the condition of the theorem for the given Δ, $\mathcal{H}_1$, δ, $\mathcal{G}$, $\mathcal{P}$, $\mathcal{H}_2$, γ_1, γ_2 and $\mathcal{U}$.

We write $C = A(F_1, F_2, h_0, h_1)$, $F_1 = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_{F(1)}}$ and $F_2 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_{F(2)}}$. By the choice of $\mathcal{H}_5'$, one has that

$$n_j \geq 16N_2 \text{ and } m_s \geq 16N_2, \quad 1 \leq j \leq F(2), \quad 1 \leq s \leq F(1). \tag{e14.8}$$

Let $q_{F_1,0} = h_0(1_{F_1})$ and $q_{F_1,1} = h_1(1_{F_1})$. Define $h_0^{\sim} : F_1^{\sim} := F_1 \oplus \mathbb{C} \to F_2$ by $h_0^{\sim}((a, \lambda)) = h_0(a) \oplus \lambda(1 - q_{F_1,0})$ and $h_1^{\sim}((a, \lambda)) = h_1(a) \oplus \lambda(1 - q_{F_1,1})$. Then $\tilde{C} = A(F_1 \oplus \mathbb{C}, F_2, h_0^{\sim}, h_1^{\sim})$. Put $\pi^{C\sim} : \tilde{C} \to \mathbb{C}$. Note that $\pi^{C\sim} \circ \varphi(a) = 0 = \pi^{C\sim} \circ \psi(a)$ for all $a \in A_0 \subset A$, and that $\pi^{C\sim} \circ \varphi(1_{A^\sim}) = 1_{\mathbb{C}} = \pi^{C\sim} \circ \psi(1_{A^\sim})$. Hence

$$\pi^{C\sim} \circ \varphi(a) = \pi^{C\sim} \circ \psi(a) \text{ for all } a \in A. \tag{e14.9}$$

Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of $[0, 1]$ so that

$$\|\pi_t \circ \varphi^{\sim}(g) - \pi_{t'} \circ \varphi^{\sim}(g)\| < \varepsilon_1/16 \text{ and } \|\pi_t \circ \psi^{\sim}(g) - \pi_{t'} \circ \psi^{\sim}(g)\| < \varepsilon_1/16 \tag{e14.10}$$

for all $g \in \mathcal{G}$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

Applying Theorem 5.8 of [20], one obtains a unitary $w_i \in F_2$ if $0 < i < n$, $w_0 \in h_0(F_1)$, such that

$$\|w_i \pi_{t_i} \circ \varphi^{\sim}(g) w_i^* - \pi_{t_i} \circ \psi^{\sim}(g)\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_5, \tag{e14.11}$$

Also there is $w_e' \in F_1$ such that

$$\|(w_e')^* \pi_e \circ \varphi(g) w_e' - \pi_e \circ \psi(g)\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_5. \tag{e14.12}$$

Let $\pi^{F_1^{\sim\prime}} : h_0^{\sim}(F_1^{\sim}) \to \mathbb{C}$ and let $\pi' : h_0(F_1^{\sim}) \to h_0(F_1)$ be the quotient maps. Put $w_0 = h_0(w_e') \oplus (1_{F_2} - q_{F_1,0})$, $w_n = h_1(w_e') \oplus (1_{F_2} - q_{F_1,1})$, $w_0' = h_0(w_e')$ and $w_n' = h_1(w_e')$. Then

$$\|w_i^* \pi_{t_i} \circ \varphi^{\sim}(g) w_i - \pi_{t_i} \circ \psi^{\sim}(g)\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_5, \tag{e14.13}$$

$i = 0$ and $i = n$. Denote $w_e = w_e' \oplus 1_{\mathbb{C}} \in F_1 \oplus \mathbb{C}$. Then $w_0 = h_0^{\sim}(w_e)$, $w_n = h_1^{\sim}(w_e)$.

By (e14.5), there is a unitary $\omega_j' \in M_{m(A)}(\tilde{C})$ such that $\omega_j \in CU(M_{m(A)}(\tilde{C}))$ and

$$\|\lceil(\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rceil\lceil(\psi^{\sim} \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rceil - \omega_j')\| < \gamma_2, \quad j = 1, 2, \ldots, k(A). \tag{e14.14}$$

By (e14.9) and (e14.14), we have $\|(\pi^{C\sim} \otimes \mathrm{id}_{m(A)})(\omega_j') - 1\| < \gamma_2$. Set $\omega_j = \omega_j' \cdot (\pi^{C\sim} \otimes \mathrm{id}_{m(A)})(\omega_j')^*$ (viewing $(\pi^{C\sim} \otimes \mathrm{id}_{m(A)})(\omega_j') \in M_{m(A)}(\mathbb{C}) \subset M_{m(A)}(\tilde{C})$). Consequently, we have

$$\|\lceil\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rceil\lceil(\psi^{\sim} \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rceil - \omega_j\| < 2\gamma_2, \quad j = 1, 2, \ldots, k(A), \tag{e14.15}$$

with an extra condition $\pi^{C\sim} \mathrm{id}_{m(A)}(\omega_j) = 1_{m(A)}(\mathbb{C})$. As mentioned in 2.2, we will use $\pi^{C\sim}$ for $\pi^{C\sim} \otimes \mathrm{id}_{m(A)}$. (*Note that we now have w_i as well as ω_i in the proof.*) Write

$$\omega_j = \prod_{l=1}^{e(j)} \exp(\sqrt{-1} a_j^{(l)})$$

for some self adjoint element $a_j^{(l)} \in M_{m(A)}(\tilde{C})$, $l = 1, 2, \ldots, e(j)$, $j = 1, 2, \ldots, k(A)$. In particular, one can choose $a_j^{(l)}$ such that $\pi^{C\sim}(a_j^{(l)}) = 0 \in M_{m(A)}(\mathbb{C})$ (see 6.1). Write

$$a_j^{(l)} = (a_j^{(l,1)}, a_j^{(l,2)}, \ldots, a_j^{(l,n_{F(2)})}) \text{ and } \omega_j = (\omega_{j,1}, \omega_{j,2}, \ldots, \omega_{j,F(2)})$$

in $C([0, 1], F_2) = C([0, 1], M_{n_1}) \oplus \cdots \oplus C([0, 1], M_{n_{F(2)}})$, where $\omega_{j,s} = \prod_{l=1}^{e(j)} \exp(\sqrt{-1} a_j^{(l,s)})$, $s = 1, 2, \ldots, F(2)$.

Then

$$\sum_{l=1}^{e(j)} \frac{n_s(t_s \otimes \mathrm{Tr}_{m(A)})(a_j^{(l,s)}(t))}{2\pi} \in \mathbb{Z}, \quad t \in (0, 1),$$

where t_s is the normalized trace on M_{n_s}, $s = 1, 2, \ldots, F(2)$. In particular,

$$\sum_{l=1}^{e(j)} n_s(t_s \otimes \mathrm{Tr}_{m(A)})(a_j^{(l,s)}(t)) = \sum_{l=1}^{e(j)} n_s(t_s \otimes \mathrm{Tr}_{m(A)})(a_j^{(l,s)}(t')) \text{ for all } t, t' \in (0, 1). \tag{e14.16}$$

We also have

$$(1/2\pi) \sum_{l=1}^{e(j)} m_s(t_{es} \otimes \mathrm{Tr}_{m(A)})(\pi_e(a_j^{(l)}) \in \mathbb{Z}, \tag{e14.17}$$

where t_{es} is the tracial state on M_{m_s}. Note, for $s = F(1) + 1$, one has $\pi_{e,s}(a_j^{(l)}) = \pi^{C\sim}(a_j^{(l)}) = 0$.

Let $W_i = w_i \otimes \mathrm{id}_{M_{m(A)}}$, $i = 0, 1, \ldots, n$ and $W_e = w_e \otimes \mathrm{id}_{M_{m(A)}(F_1)}$. Then it follows from (e14.11) and (e14.15) that

$$\|(\pi_{t_i}(\lceil\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rceil)W_i(\pi_{t_i}(\lceil\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rceil)W_i^* - \omega_j(t_i)\| \tag{e14.18}$$

$$< 3m(A)^2\varepsilon_1 + 2\gamma_2 < 1/32. \tag{e14.19}$$

We also have (with $\varphi_e = \pi_e \circ \varphi^{\sim}$)

$$\|\lceil(\varphi_e \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rceil(W_e(\lceil\varphi_e \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rceil)W_e^* - \pi_e(\omega_j)\| < 3m(A)^2\varepsilon_1 + 2\gamma_2 < 1/32. \tag{e14.20}$$

It follows from (e14.18) that there exist selfadjoint elements $b_{i,j} \in M_{m(A)}(F_2)$ such that

$$\exp(\sqrt{-1}b_{i,j}) = \omega_j(t_i)^*(\pi_i(\lceil\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rceil)W_i(\pi_i(\lceil\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rceil)W_i^*, \tag{e14.21}$$

and $b_{e,j} \in M_{m(A)}(F_1 \oplus \mathbb{C})$ such that

$$\exp(\sqrt{-1}b_{e,j}) = \pi_e(\omega_j)^*(\pi_e(\lceil\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rceil)W_e(\pi_e(\lceil\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rceil)W_e^*, \tag{e14.22}$$

and

$$\|b_{i,j}\| < 2\arcsin(3m(A)^2\varepsilon_1/2 + \gamma_2), \ \ j = 1, 2, \ldots, k(A), \ i = 0, 1, \ldots, n, e. \tag{e14.23}$$

Write

$$b_{i,j} = (b_{i,j}^{(1)}, b_{i,j}^{(2)}, \ldots, b_{i,j}^{F(2)}) \in M_{m(A)}(F_2) \ \text{ and } \ b_{e,j} = (b_{e,j}^{(1)}, b_{e,j}^{(2)}, \ldots, b_{e,j}^{(F(1))}, b_{e,j}^{F(1)+1}) \in M_{m(A)}(F_1 \oplus \mathbb{C}).$$

From $\pi^{C^{\sim}}(\omega_j) = 1$ and definition of W_e and w_e, we know that $b_{e,j}^{F(1)+1} = 0$. We have that

$$h_0^{\sim}(b_{e,j}) = b_{0,j} \ \text{ and } \ h_1^{\sim}(b_{e,j}) = b_{n,j}. \tag{e14.24}$$

Note that

$$(\pi_{t_i}(\lceil\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rceil))W_i(\pi_{t_i}(\lceil\varphi^{\sim} \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rceil)W_i^* = \pi_{t_i}(\omega_j)\exp(\sqrt{-1}b_{i,j}), \tag{e14.25}$$

$j = 1, 2, \ldots, k(A)$ and $i = 0, 1, \ldots, n, e$. Then,

$$\frac{n_s}{2\pi}(t_s \otimes \mathrm{Tr}_{M_{m(A)}})(b_{i,j}^{(s)}) \in \mathbb{Z}, \tag{e14.26}$$

where t_s is the normalized trace on M_{n_s}, $s = 1, 2, \ldots, F(2)$, $j = 1, 2, \ldots, k(A)$, and $i = 0, 1, \ldots, n$. We also have

$$\frac{m_s}{2\pi}(t_s \otimes \mathrm{Tr}_{M_{m(A)}})(b_{e,j}^{(s)}) \in \mathbb{Z}, \tag{e14.27}$$

where t_s is the normalized trace on M_{m_s}, $s = 1, 2, \ldots, F(1)$, $j = 1, 2, \ldots, k(A)$. Put

$$\lambda_{i,j}^{(s)} = \frac{n_s}{2\pi}(t_s \otimes \mathrm{Tr}_{M_{m(A)}})(b_{i,j}^{(s)}) \in \mathbb{Z},$$

where t_s is the normalized trace on M_{n_s}, $s = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k(A)$ and $i = 0, 1, 2, \ldots, n$.

Put

$$\lambda_{e,j}^{(s)} = \frac{m_s}{2\pi}(t_s \otimes \mathrm{Tr}_{M_{m(A)}})(b_{e,j}^{(s)}) \in \mathbb{Z},$$

where t_s is the normalized trace on M_{m_s}, $s = 1, 2, \ldots, F(1)$ and $j = 1, 2, \ldots, k(A)$. Denote

$$\lambda_{i,j} = (\lambda_{i,j}^{(1)}, \lambda_{i,j}^{(2)}, \ldots, \lambda_{i,j}^{(F(2))}) \in \mathbb{Z}^{F(2)}, \ \text{ and } \ \lambda_{e,j} = (\lambda_{e,j}^{(1)}, \lambda_{e,j}^{(2)}, \ldots, \lambda_{e,j}^{F(1)}, 0) \in \mathbb{Z}^{F(1)+1}.$$

We have, by (e14.23), for $j = 1, 2, \ldots, k(A)$ and $i = 0, 1, 2, \ldots, n$,

$$|\frac{\lambda_{i,j}^{(s)}}{n_s}| < 1/4N_1, \ \ s = 1, 2, \ldots, F(2), \ \ |\frac{\lambda_{e,j}^{(s)}}{m_s}| < 1/4N_1, \ \ s = 1, 2, \ldots, F(1), \tag{e14.28}$$

Define $\alpha_i^{(0,1)} : K_1(A) \to \mathbb{Z}^{F(2)}$ by mapping $[g_j]$ to $\lambda_{i,j}$, $j = 1, 2, \ldots, k(A)$, $i = 0, 1, 2, \ldots, n$, and define $\alpha_e^{(0,1)} : K_1(A) \to \mathbb{Z}^{F(1)} \oplus \mathbb{Z}$ by mapping $[g_j]$ to $(\lambda_{e,j}, 0)$, $j = 1, 2, \ldots, k(A)$. We write $K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \boldsymbol{\beta}(K_1(A))$ (see 2.10 of [35] for the definition of $\boldsymbol{\beta}$). Define $\alpha_i : K_*(A \otimes C(\mathbb{T})) \to K_*(F_2)$ as follows: On $K_0(A \otimes C(\mathbb{T}))$, define

$$\alpha_i|_{K_0(A)} = [\pi_i \circ \varphi]|_{K_0(A)}, \ \ \alpha_i|_{\boldsymbol{\beta}(K_1(A))} = \alpha_i \circ \boldsymbol{\beta}|_{K_1(A)} = \alpha_i^{(0,1)} \tag{e14.29}$$

and on $K_1(A \otimes C(\mathbb{T}))$, define $\alpha_i|_{K_1(A \otimes C(\mathbb{T}))} = 0$, $\ i = 0, 1, 2, \ldots, n$.

Also define $\alpha_e \in \mathrm{Hom}(K_*(A \otimes C(\mathbb{T})), K_*(F_1 \otimes \mathbb{C}))$, by

$$\alpha_e|_{K_0(A)} = [\pi_e \circ \varphi^{\sim}]|_{K_0(A)}, \ \ \alpha_e|_{\boldsymbol{\beta}(K_1(A))} = \alpha_e \circ \boldsymbol{\beta}|_{K_1(A)} = \alpha_e^{(0,1)} \tag{e14.30}$$

on $K_0(A\otimes C(\mathbb{T}))$ and $(\alpha_e)|_{K_1(A\otimes C(\mathbb{T}))}=0$. Note that

$$(h_0^{\sim})_*\circ\alpha_e=\alpha_0 \text{ and } (h_1^{\sim})_*\circ\alpha_e=\alpha_n. \tag{e14.31}$$

Since $A\otimes C(\mathbb{T})$ satisfies the UCT, the map α_e can be lifted to an element of $KK(A\otimes C(\mathbb{T}), F_1\oplus\mathbb{C})$ which is still denoted by α_e. Then define

$$\alpha_0=\alpha_e\times[h_0^{\sim}] \text{ and } \alpha_n=\alpha_e\times[h_1^{\sim}] \tag{e14.32}$$

in $KK(A\otimes C(\mathbb{T}), F_2)$. For $i=1,\dots,n-1$, also pick a lifting of α_i in $KK(A\otimes C(\mathbb{T}), F_2)$, and still denote it by α_i. We estimate that

$$\|(w_i^*w_{i+1})\pi_{t_i}\circ\varphi^{\sim}(g)-\pi_{t_i}\circ\varphi^{\sim}(g)(w_i^*w_{i+1})\|<\varepsilon_1/4 \text{ for all } g\in\mathcal{G}_5, \tag{e14.33}$$

$i=0,1,\dots,n-1$. Let $\Lambda_{i,i+1}: C(\mathbb{T})\otimes A\to F_2$ be a unital contractive completely positive linear map given by the pair $w_i^*w_{i+1}$ and $\pi_{t_i}\circ\varphi$ (by 6.4 of [20], see 2.8 of [35]). Denote $V_{i,j}=\lceil\pi_{t_i}\circ\varphi^{\sim}\otimes\mathrm{id}_{M_{m(A)}}(g_j)\rceil$, $j=1,2,\dots,k(A)$ and $i=0,1,2,\dots,n-1$.

Write

$$V_{i,j}=(V_{i,j,1},V_{i,j,2},\dots,V_{i,j,F(2)})\in M_{m(A)}(F_2),\ \ j=1,2,\dots,k(A),\ \ i=0,1,2,\dots,n.$$

Similarly, write

$$W_i=(W_{i,1},W_{i,2},\dots,W_{i,F(2)})\in M_{m(A)}(F_2),\ \ i=0,1,2,\dots,n. \tag{e14.34}$$

We have

$$\|W_iV_{i,j}^*W_i^*V_{i,j}V_{i,j}^*W_{i+1}V_{i,j}W_{i+1}^*-1\|<1/16 \tag{e14.35}$$

$$\|W_iV_{i,j}^*W_i^*V_{i,j}V_{i+1,j}^*W_{i+1}V_{i+1,j}W_{i+1}^*-1\|<1/16 \tag{e14.36}$$

and there is a continuous path $Z(t)$ of unitaries such that $Z(0)=V_{i,j}$ and $Z(1)=V_{i+1,j}$. Since

$$\|V_{i,j}-V_{i+1,j}\|<\delta_1/12,\ \ j=1,2,\dots,k(A),$$

we may assume that $\|Z(t)-Z(1)\|<\delta_1/6$ for all $t\in[0,1]$. We also write

$$Z(t)=(Z_1(t),Z_2(t),\dots,Z_{F(2)}(t))\in F_2 \text{ and } t\in[0,1].$$

We obtain a continuous path $W_iV_{i,j}^*W_i^*V_{i,j}Z(t)^*W_{i+1}Z(t)W_{i+1}^*$ which is in $CU(M_{nm(A)})$ for all $t\in[0,1]$ and

$$\|W_iV_{i,j}^*W_i^*V_{i,j}Z(t)^*W_{i+1}Z(t)W_{i+1}^*-1\|<1/8 \text{ for all } t\in[0,1].$$

It follows that

$$(1/2\pi\sqrt{-1})(t_s\otimes\mathrm{Tr}_{M_{m(A)}})[\log(W_{i,s}V_{i,j,s}^*W_{i,s}^*V_{i,j,s}Z_s(t)^*W_{i+1,s}Z_s(t)W_{i+1,s}^*)]$$

is a constant integer, where t_s is the normalized trace on M_{n_s}. In particular,

$$(1/2\pi\sqrt{-1})(t_s\otimes\mathrm{Tr}_{M_{m(A)}})(\log(W_{i,s}V_{i,j,s}^*W_{i,s}^*W_{i+1,s}V_{i,j,s}W_{i+1,s}^*)) \tag{e14.37}$$

$$=(1/2\pi\sqrt{-1})(t_s\otimes\mathrm{Tr}_{M_{m(A)}})(\log(W_{i,s}V_{i,j,s}^*W_{i,s}^*V_{i,j}V_{i+1,j,s}^*W_{i+1,s}V_{i,j,s}W_{i+1,s}^*)). \tag{e14.38}$$

One also has

$$W_iV_{i,j}^*W_i^*V_{i,j}V_{i+1,j}^*W_{i+1}V_{i+1,j}W_{i+1}^*=(\omega_j(t_i)\exp(\sqrt{-1}b_{i,j}))^*\omega_j(t_{i+1})\exp(\sqrt{-1}b_{i+1,j}) \tag{e14.39}$$

$$=\exp(-\sqrt{-1}b_{i,j})\omega_j(t_i)^*\omega_j(t_{i+1})\exp(\sqrt{-1}b_{i+1,j}). \tag{e14.40}$$

Note that, by (e14.14) and (e14.10), for $t\in[t_i,t_{i+1}]$,

$$\|\omega_j(t_i)^*\omega_j(t)-1\|<2(m(A)^2)\varepsilon_1/16+2\gamma_2<1/32, \tag{e14.41}$$

$j=1,2,\dots,k(A)$, $i=0,1,\dots,n-1$. By Lemma 3.5 of [40],

$$(t_s\otimes\mathrm{Tr}_{m(A)})(\log(\omega_{j,s}(t_i)^*\omega_{j,s}(t_{i+1})))=0. \tag{e14.42}$$

It follows that (by the Exel formula (see [24]), using (e14.38), (e14.40) and (e14.42))

$$(t\otimes\mathrm{Tr}_{m(A)})(\mathrm{bott}_1(V_{i,j},W_i^*W_{i+1})) \tag{e14.43}$$

$$=(\frac{1}{2\pi\sqrt{-1}})(t\otimes\mathrm{Tr}_{m(A)})(\log(V_{i,j}^*W_i^*W_{i+1}V_{i,j}W_{i+1}^*W_i)) \tag{e14.44}$$

$$=(\frac{1}{2\pi\sqrt{-1}})(t\otimes\mathrm{Tr}_{m(A)})(\log(W_iV_{i,j}^*W_i^*W_{i+1,s}V_{i,j}W_{i+1}^*)) \tag{e14.45}$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t\otimes \mathrm{Tr}_{m(A)})(\log(W_iV_{i,j}^*W_i^*V_{i,j}V_{i+1,j}^*W_{i+1}V_{i+1,j}W_{i+1}^*)) \tag{e14.46}$$

$$= (\frac{1}{2\pi\sqrt{-1}})(t\otimes \mathrm{Tr}_{m(A)})(\log(\exp(-\sqrt{-1}b_{i,j})\omega_j(t_i)^*\omega_j(t_{i+1})\exp(\sqrt{-1}b_{i+1,j}))) \tag{e14.47}$$

$$= (\frac{1}{2\pi\sqrt{-1}})[(t\otimes \mathrm{Tr}_{m(A)})(-\sqrt{-1}b_{i,j}) + (t\otimes \mathrm{Tr}_{m(A)})(\log(\omega_j(t_i)^*\omega_j(t_{i+1}))) \tag{e14.48}$$

$$+(t\otimes \mathrm{Tr}_{m(A)})(\sqrt{-1}b_{i,j})] \tag{e14.49}$$

$$= \frac{1}{2\pi}(t\otimes \mathrm{Tr}_{m(A)})(-b_{i,j}+b_{i+1,j}) \tag{e14.50}$$

for all $t\in T(F_2)$. In other words,

$$\mathrm{bott}_1(V_{i,j}, W_i^*W_{i+1}) = -\lambda_{i,j}+\lambda_{i+1,j} \tag{e14.51}$$

$j=1,2,\ldots,m(A)$, $i=0,1,\ldots,n-1$.

Define $\beta_0=0$, $\beta_1=[\Lambda_{0,1}]-\alpha_1+\alpha_0+\beta_0$,

$$\beta_i = [\Lambda_{i-1,i}]-\alpha_i+\alpha_{i-1}+\beta_{i-1},\quad i=2,3,\ldots,n. \tag{e14.52}$$

Then

$$\beta_1([g_j]) = \Lambda_{0,1}([g_j])-\lambda_{1,j}+\lambda_{0,j}=0,$$
$$\beta_2([g_j]) = \Lambda_{1,2}([g_j])-\lambda_{2,j}-\lambda_{1,j}+\beta_1([g_j])=0 \ \text{ and}$$
$$\beta_i([g_j]) = \lambda_{i-1,i}([g_j])-\lambda_{i,j}-\lambda_{i-1,j}-\beta_{i-1}([g_j])=0,\quad i=3,\ldots,n.$$

It follows 5.2.5 of [39] that there is $\varrho\in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(F_1\otimes\mathbb{C}))$ such that

$$\varrho(\boldsymbol{\beta}(K_1(A)))=0 \ \text{ and}$$
$$\varrho\times([h_1^{\sim}]-[h_0^{\sim}])|_{\boldsymbol{\beta}(\underline{K}(A))}=\beta_n|_{\boldsymbol{\beta}(\underline{K}(A))}.$$

Define $\kappa_0=\alpha_0+\beta_0+\varrho\times[h_0^{\sim}]$, $\kappa_i=\alpha_i+\beta_i+\varrho\times[h_0^{\sim}]$, $i=1,2,\ldots,n$. Note that, on $\boldsymbol{\beta}(\underline{K}(A))$,

$$\kappa_n = \alpha_n+\beta_n+\varrho\times[h_0^{\sim}] = \alpha_n+\varrho\times([h_1^{\sim}]-[h_0^{\sim}])+\varrho\times[h_0^{\sim}] \tag{e14.53}$$

$$= \alpha_n+\varrho\times[h_1^{\sim}] = (\alpha_e+\varrho)\times[h_1^{\sim}], \tag{e14.54}$$

and, by (e14.32), $\kappa_0=\alpha_0+\varrho\times[h_0^{\sim}]=\alpha_e\times[h_0^{\sim}]+\varrho\times[h_0^{\sim}]$. We also have, for each $j=1,2,\ldots,k(A)$,

$$\kappa_i([g_j]) = \lambda_{i,j}+(h_0^{\sim})_{*0}\circ\varrho([g_j])=\lambda_{i,j},\quad i=0,1,\ldots,n \ \text{ and}$$
$$(\varrho+\alpha_e)([g_j]) = \lambda_{e,j}.$$

Applying 7.4 of [20] (using (e14.28), (e14.3)), there are unitaries $z_i\in F_2$, $i=1,2,\ldots,n-1$, and $z_e\in F_1\otimes\mathbb{C}$ with $z_e=z_e'\oplus 1$ such that, for $i=1,2,\ldots,n-1$,

$$\|[z_i,\ \pi_{t_i}\circ\varphi^{\sim}(g)]\|<\delta_u \ \text{ for all } \ g\in\mathcal{G}_u,\ \mathrm{Bott}(z_i,\ \pi_{t_i}\circ\varphi^{\sim})=(\kappa_i)|_{\boldsymbol{\beta}(\underline{K}(A))}, \ \text{ and} \tag{e14.55}$$

$$\|[z_e,\ \pi_e\circ\varphi^{\sim}(g)]\|<\delta_u \ \text{ for all } \ g\in\mathcal{G}_u \ \text{ and } \ \mathrm{Bott}(z_e,\ \pi_e\circ\varphi^{\sim})=(\varrho+\alpha_e)|_{\boldsymbol{\beta}(\underline{K}(A))}. \tag{e14.56}$$

Put

$$z_0=h_0(z_e)\otimes(1_{F_2}-h_0(1_{F_1})) \quad\text{and}\quad z_n=h_1(z_e)\oplus(1_{F_2}-h_1(1_{F_1})).$$

Note that, as above,

$$\mathrm{Bott}(z_0,\ \pi_0\circ\varphi^{\sim})=\kappa_0|_{\boldsymbol{\beta}(\underline{K}(A))} \ \text{ and } \ \mathrm{Bott}(z_n,\ \pi_0\circ\varphi^{\sim})=\kappa_n|_{\boldsymbol{\beta}(\underline{K}(A))}.$$

Let

$$U_i=z_iw_iw_{i+1}^*z_{i+1},\quad i=0,1,\ldots,n-1. \tag{e14.57}$$

Then, by (e14.55), (e14.56) and (e14.33),

$$\|[U_i,\ \pi_{t_i}\circ\varphi^{\sim}(g)]\|<2\delta_u+2\varepsilon_1/4<\delta_1/2 \ \text{ for all } \ g\in\mathcal{G}_u. \tag{e14.58}$$

We also compute that (using the choice of δ_1 and (e14.52))

$$\begin{aligned}\mathrm{Bott}(U_i,\ \pi_{t_i}\circ\varphi^{\sim}) &= \mathrm{Bott}(z_i,\ \pi_{t_i}\circ\varphi^{\sim})+\mathrm{Bott}(w_i^*w_{i+1},\ \pi_{t_i}\circ\varphi^{\sim})\\ &= \mathrm{Bott}(z_{i+1},\ \pi_{t_i}\circ\varphi^{\sim})=\kappa_i+[\Lambda_{i,i+1}]-\kappa_{i+1}\\ &= \alpha_i+\beta_i+\varrho\times[h_0]+[\Lambda_{i,i+1}]-(\alpha_{i+1}+\beta_{i+1}+\varrho\times[h_0])\\ &= \alpha_i+\beta_i+[\Lambda_{i,i+1}]-\alpha_{i+1}-([\Lambda_{i,i+1}]-\alpha_{i+1}+\alpha_i+\beta_i)=0,\end{aligned}$$

$i = 0, 1, \ldots, n-1$. Note that, by the assumption (e14.3),

$$t_s \circ \pi_t \circ \varphi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1', \tag{e14.59}$$

where t_s is the normalized trace on M_{n_s}, $1 \leq s \leq F(2)$. Then, by this, (e14.58), (e14.59) and by applying 6.7 of [20] we obtain a continuous path of unitaries $\{U_i(t) : t \in [t_i, t_{i+1}]\} \subset F_2$ such that $U_i(t_i) = 1_{F_2}$ and $U(t_{i+1}) = z_i(w_i)^* w_{i+1} z_{i+1}^*$ and

$$\|[U_i(t), \pi_t \circ \varphi^{\sim}(f)]\| < \varepsilon/32 \text{ for all } f \in \mathcal{F}, \tag{e14.60}$$

$i = 0, 1, \ldots, n-1$. Now define $W(t) = w_i z_i^* U_i(t)$ for $t \in [t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$. Then $W(t) \in C([0,1], F_2)$ but also

$$W(0) = w_0 z_0^* = h_0^{\sim}(w_e z_e^*) \text{ and } W(1) = w_n z_n^* = h_1^{\sim}(w_e z_e^*).$$

Therefore $W \in \tilde{C}$. One then checks that, by (e14.10), (e14.60), (e14.55) and (e14.11),

$$\begin{aligned}
&\|W(t)(\pi_t \circ \varphi^{\sim})(f)W(t)^* - (\pi_t \circ \psi^{\sim})(f) \otimes 1_{M_N}\| && \text{(e14.61)}\\
< &\|W(t)(\pi_t \circ \varphi^{\sim})(f)W(t)^* - W(t)(\pi_{t_i} \circ \varphi^{\sim})(f)W^*(t)\| && \text{(e14.62)}\\
&+ \|W(t)(\pi_{t_i} \circ \varphi^{\sim})(f)W(t)^* - W(t_i)(\pi_{t_i} \circ \varphi^{\sim})(f)W(t_i)^*\| && \text{(e14.63)}\\
&+ \|W(t_i)(\pi_{t_i} \circ \varphi^{\sim})(f)W(t_i)^* - (w_i \pi_{t_i} \circ \varphi^{\sim})(f)w_i^*\| && \text{(e14.64)}\\
&+ \|w_i(\pi_{t_i} \circ \varphi^{\sim})(f)w_i^* - \pi_{t_i} \circ \psi^{\sim}(f)\| && \text{(e14.65)}\\
&+ \|\pi_{t_i} \circ \psi^{\sim}(f) - \pi_t \circ \varphi^{\sim}(f)\| && \text{(e14.66)}\\
< &\ \varepsilon_1/16 + \varepsilon/32 + \delta_u + \varepsilon_1/16 + \varepsilon_1/16 < \varepsilon && \text{(e14.67)}
\end{aligned}$$

for all $f \in \mathcal{F}$ and for $t \in [t_i, t_{i+1}]$. $\square$

Definition 14.3. Let D be a non-unital C^*-algebra. Denote by $C(\mathbb{T}, \tilde{D})^o$ the C^*-subalgebra of $C(\mathbb{T}, \tilde{D})$ generated by $C_0(\mathbb{T} \setminus \{1\}) \otimes 1_{\tilde{D}}$ and $1_{C(\mathbb{T})} \otimes D$. The unitization of $C(\mathbb{T}, \tilde{D})^o$ is $C(\mathbb{T}, \tilde{D}) = C(\mathbb{T}) \otimes \tilde{D}$. Let C be another non-unital C^*-algebra, $L : C(\mathbb{T}, \tilde{D})^o \to C$ be a completely positive contractive linear map and $L^{\sim} : C(\mathbb{T}) \otimes \tilde{D} \to \tilde{C}$ be the unitization. Denote by z the standard unitary generator of $C(\mathbb{T})$. For any finite subset $\mathcal{F} \subset C(\mathbb{T}) \otimes D$, any finite subset $\mathcal{F}_d \subset \tilde{D}$, and $\varepsilon > 0$, there exist a finite subset $\mathcal{G} \subset D$ and $\delta > 0$ such that, whenever $\varphi : D \to C$ is a $\mathcal{G}$-δ -multiplicative completely positive contractive linear map (for any C^*-algebra C) and $\|[u, \varphi(g)]\| < \delta$ for all $g \in \mathcal{G}$, there exists a $\mathcal{F}$-ε-multiplicative completely positive contractive linear map $L' : C(\mathbb{T}) \otimes \tilde{D} \to \tilde{C}$ such that

$$\|L'(z \otimes 1) - u\| < \varepsilon \text{ and } \|L'(1 \otimes d) - \varphi^{\sim}(d)\| < \varepsilon \text{ for all } d \in \mathcal{F}_d. \tag{e14.68}$$

We will denote such L' by $\Phi_{u,\varphi}$.

Conversely, there exist a finite subset $\mathcal{G}' \subset C(\mathbb{T}, \tilde{D})^o$ and $\delta' > 0$, if $L : C(\mathbb{T}, D)^o \to C$ is $\mathcal{G}'$-δ'-multiplicative completely positive contractive linear map, there is a unitary $u \in \tilde{C}$ such that

$$\|\tilde{L}(z \otimes 1) - u\| < \varepsilon \tag{e14.69}$$

and $\varphi = L^{\sim}|_{1 \otimes D}$ is a completely positive contractive linear map.

In what follows, we use $\mathcal{A}$ for the family of C^*-algebras which can be approximated by C^*-algebras $D \in \overline{\mathcal{D}}_r$ for some integer $r \geq 1$, Note that $B_T \subset \mathcal{A}$.

Lemma 14.4. *Let $A = C(\mathbb{T}) \otimes \tilde{D}$, where $D \in \mathcal{A}$. Let $\mathcal{F} \subset A$ be a finite subset, let $\varepsilon > 0$ be a positive number and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. There exist a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$, and a finite subset $\mathcal{P} \subset \underline{K}(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset J_c(K_1(A))$ for which $[\mathcal{U}] \subset \mathcal{P}$ satisfying the following: For any unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $\Phi_{u,\varphi}, \Phi_{v,\psi} : A \to \tilde{C}$ for some amenable $C \in \mathcal{D}^d$ with continuous scale, where $u, v \in U(\tilde{C})$ and $\varphi, \psi : D \to C$ are two $\mathcal{G}_d$-δ-multiplicative completely positive contractive linear maps ($\mathcal{G}_d = \{g : g \otimes 1 \in \mathcal{G}\}$) such that*

$$[\Phi_{u,\varphi}]|_{\mathcal{P}} = [\Phi_{v,\psi}]|_{\mathcal{P}}, \tag{e14.70}$$

$$\tau(\Phi_{u,\varphi}(a)) \geq \Delta(\hat{a}), \quad \tau(\Phi_{v,\psi}(a)) \geq \Delta(\hat{a}) \text{ for all } \tau \in T(C) \text{ and } a \in \mathcal{H}_1, \tag{e14.71}$$

$$|\tau \circ \Phi_{u,\varphi}(a) - \tau \circ \Phi_{v,\psi}(a)| < \gamma_1 \text{ for all } a \in \mathcal{H}_2 \text{ and} \tag{e14.72}$$

$$\mathrm{dist}(\Phi_{u,\varphi}^{\dagger}(y), \Phi_{v,\psi}^{\dagger}(y)) < \gamma_2 \text{ for all } y \in \mathcal{U}, \tag{e14.73}$$

there exists a unitary $W \in \tilde{C}$ such that

$$\|W(\Phi_{u,\varphi}(f))W^* - (\Psi_{v,\psi}(f))\| < \varepsilon, \text{ for all } f \in \mathcal{F}. \tag{e14.74}$$

Proof. Let us first reduce the general case to the case that $D \in \overline{\mathcal{D}}_r$. Fix any finite subset $\mathcal{F}_d \subset D$ and any $\varepsilon_d > 0$, by 7.3, there is $D_n \in \overline{\mathcal{D}}_r$ such that

$$\mathrm{dist}(x, D_n) < \varepsilon_d \ \text{ for all } \ x \in \mathcal{F}_d. \tag{e14.75}$$

This effectively allows us to assume that $D \in \overline{\mathcal{D}}_r$. It should then be noted that $C(\mathbb{T}, \tilde{D}) \in \overline{\mathcal{D}}_{r+1}$.

Now we assume that $D \in \overline{\mathcal{D}}_r$.

Let $\mathbf{L} = 8\pi$, $r_0 = 0$, $r_1 = 0$, $\mathbf{T}(n, k) = n$ for all (n, k), $s = 1$ and $R = 7$. Let $1/2 > \varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. Let $\Delta_0 = \Delta/2$. Let $F' : A_+ \setminus \{0\} \to \mathbb{R} \times \mathbb{N}$ be given by 14.1 associated with Δ_0.

Put $A_0 = C(\mathbb{T}, \tilde{D})^o$. Let $\mathcal{F}_I \subset A_0$ be a finite subset such that, if $x \in \mathcal{F}$, then $x = \lambda + y$ for some $y \in \mathcal{F}_I$.

Let $\delta_0 > 0$ (in place of δ), $\mathcal{G}_0 \subset A_0$ (in place of $\mathcal{G}$) be finite subset, $\mathcal{P}_0 \subset \underline{K}(A_0)$ (in place of $\mathcal{P}$), $\mathcal{U}_0 \subset U(M_N(A))$ (for some integer $N \ge 1$) $\mathcal{H}_0 \subset (A_0)_+ \setminus \{0\}$ (in place of $\mathcal{H}$) and $K \ge 1$ be an integer required by Theorem 3.14 of [16] for A_0, $\varepsilon/16$ (in place of ε), $\mathcal{F}_I$ (in place of $\mathcal{F}$), $\mathbf{L}$, F', (in place F), as well as r_0, r_1, T, s and R above. As in 3.15 of [16], we can choose $\mathcal{U}_0 = \{g_1, g_2, \dots, g_{k(A)}\}$ so that $K_1(A) \cap \mathcal{P}_0 = \{[g_1], [g_2], \dots, [g_{k(A)}]\}$.

Let $\gamma_1' > 0$, $\gamma_2' > 0$, $\delta' > 0$, $\mathcal{G}' \subset A$, $\mathcal{H}_1' \subset (A)_+^{\mathbf{1}} \setminus \{0\}$, $\mathcal{P}' \subset \underline{K}(A)$, $\overline{\mathcal{U}'} \subset J_c(K_1(A))$ and $\mathcal{H}_2' \subset A_{s.a.}$ be finite subsets required by 14.2 for $\min\{\delta_0/4, \varepsilon/16\}$ (in place of ε) $\mathcal{G}_0$ (in place of $\mathcal{F}$) and for Δ_0 (in place of Δ).

Put $\gamma_1 = \gamma_1'/4$, $\gamma_2 = \frac{1}{2K+1}\min\{\gamma_2'/16, \varepsilon/64\}$, $\delta = \min\{\delta'/16, \delta_0/16, \gamma_1/16, \gamma_2/16, \varepsilon/2^{10}\}$, $\mathcal{H}_1 = \mathcal{H}_1'$, $\mathcal{H}_2 = \mathcal{H}_2$ and $\mathcal{G} = \mathcal{G}'$.

Now suppose that $\Phi_1, \Phi_2 : A \to \tilde{C}$ are two $\mathcal{G}$-δ-multiplicative completely positive contractive linear maps such that $\Phi_1 = \Phi_{u,\varphi}$ and $\Phi_2 = \Phi_{v,\psi}$, where u, v and φ, ψ are as given. Moreover, Φ_1, Φ_2 satisfy the condition (e14.70), (e14.71), (e14.72), (e14.73) and (e14.73) for the above mentioned Δ, $\mathcal{P}$, $\mathcal{H}_1$, $\mathcal{H}_2$, γ_1, γ_2 and $\mathcal{U}$.

Since $C \in \mathcal{D}^d$, there exist a sequence of positive elements $\{b_n\}$ of C, a sequence of C^*-subalgebras $C_{0,n} \in \mathcal{C}_0$, two sequences of completely positive contractive linear maps $\varphi_{0,n} : A \to B_n$ and $\varphi_{1,n} : C \to C_{0,n}$ such that $C_{0,n} \perp B_n$,

$$\lim_{n\to\infty} \|\varphi_{i,n}(ab) - \varphi_{i,n}(a)\varphi_{i,n}(b)\| = 0 \ \text{ for all } \ a, b \in C, \tag{e14.76}$$

$$\lim_{n\to\infty} \|x - (\varphi_{0,n}(x) \oplus \mathrm{diag}(\overbrace{\varphi_{1,n}(x), \varphi_{1,n}(x), \dots, \psi_{1,n}(x)}^{K}))\| = 0 \ \text{ for all } \ x \in C \tag{e14.77}$$

$$\lim_{n\to\infty} \sup_{\tau \in T(C)} d_\tau(b_n) = 0, \quad t(f_{1/4}(\varphi_{1,n}(e_C))) \ge 1/2 \ \text{ for all } \ t \in T(C_{0,n}), \tag{e14.78}$$

$$\text{and } \ \tau(f_{1/4}(\varphi_{1,n}(e_C))) > 1/2 \ \text{ for all } \ \tau \in T(C), \tag{e14.79}$$

where $e_C \in C$ is a strictly positive element with $\|e_C\| = 1$, $B_n = \overline{b_n C b_n}$ (see 9.2 of [15]). Put $C_n = M_K(C_{0,n})$, $n = 1, 2, \dots$ It should be noted that $C_n \perp B_n$, $n = 1, 2, \dots$. We may assume, without loss of generality, for all n,

$$\sup_{\tau \in T(C)} d_\tau(b_n) < \min\{\gamma_1/64K, \gamma_2/64K, \min\{\Delta_0(\hat{h}) : h \in \mathcal{H}_1\}/4(K+2)\}. \tag{e14.80}$$

Let $u_i, v_i \in M_N(\tilde{C})$ $(i = 1, 2, \dots, k(A))$ be two unitaries such that

$$\|(\Phi_1 \otimes \mathrm{id}_{M_N})(g_i) - u_i\| < \min\{\varepsilon/2^8, \gamma_2/8\} \ \text{ and } \ \|(\Phi_2 \otimes \mathrm{id}_{M_N})(g_i) - v_i\| < \min\{\varepsilon/2^8, \gamma_2/8\}.$$

Let $w_i \in CU(\tilde{C})$ be such that

$$\|u_i v_i^* - w_i\| < (5/4)\gamma_2 \ \text{ and } \ w_i = \prod_{j=1}^{m(i)} w_{i,j}, \quad w_{i,j} = w_{1,i,j}^* w_{2,i,j}^* w_{1,i,j} w_{2,i,j}, \tag{e14.81}$$

where $w_{s,i,j} \in U(\tilde{C})$, $s = 1, 2$, $j = 1, 2, \dots, m(i)$ and $i = 1, 2, \dots, k(A)$. Let $m = \max\{m(i) : 1 \le i \le k(A)\}$.

Write $w_{s,i,j} = \alpha_{s,i,j} + c(w_{s,i,j})$, where $\alpha_{s,i,j} \in \mathbb{T} \subset \mathbb{C}$ and $c(w_{s,i,j}) \in C$, $j = 1, 2, \dots, m(i)$. Note that $\|c(w_{s,i,j})\| \le 2$, $j = 1, 2, \dots, m(i)$, $i = 1, 2, \dots, k(A)$.

Define $\psi_{1,n} : A \to C_n$ by $\psi_{1,n}(a) = \mathrm{diag}(\overbrace{\varphi_{1,n}(a), \varphi_{1,n}(a), \dots, \varphi_{1,n}(a)}^{K})$ for all n. Put $\Psi_j = \psi_{1,n} \circ \Phi_j : A \to C_n$, $j = 1, 2$.

Let $\mathcal{G}_2 = \mathcal{G} \cup \{c(w_{s,i,j}) : s = 1, 2, \ 1 \le j \le m(i), 1 \le i \le k(A)\}$. We can choose n large enough so that $\psi_{0,n}$ and $\psi_{1,n}$ are $\mathcal{G}_2$-$\frac{\delta}{2^{12} m N^2}$-multiplicative. In particular, by (e14.73) and the choice of γ_2,

$$\mathrm{dist}(\overline{\lceil \widetilde{\varphi_{0,n}}(u_i) \rceil}, \overline{\lceil \widetilde{\varphi_{0,n}}(v_i) \rceil}) \le \gamma_2'/4 \ \text{ in } \ U(\tilde{B}_n)/CU(\tilde{B}_n) \ \text{ and } \tag{e14.82}$$

$$\mathrm{dist}(\overline{\lceil \widetilde{\psi_{1,n}}(u_i) \rceil}, \overline{\lceil \widetilde{\psi_{1,n}}(v_i) \rceil}) \le \gamma_2'/4 \ \text{ in } \ U(\tilde{C}_n)/CU(\tilde{C}_n). \tag{e14.83}$$

It is standard to check that, by choosing sufficiently large n, we may assume that Ψ_j are $\mathcal{G}$-δ-multiplicative completely positive contractive linear maps satisfying the following:

$$t \circ \Psi_1(h) \ge \Delta_0(\hat{h}), \quad t \circ \Psi_2(h) \ge \Delta_0(\hat{h}) \ \text{ for all } \ h \in \mathcal{H}_1, \tag{e14.84}$$

$$|t \circ \Psi_1(g) - t \circ \Psi_2(g)| < \gamma_2' \ \text{ for all } \ g \in \mathcal{H}_2. \tag{e14.85}$$

Combining these with (e14.83), by applying 14.2, one obtains a unitary $U_1 \in \tilde{C}_n$ such that

$$\|U_1^* \Psi_1(x) U_1 - \Psi_2(x)\| < \min\{\delta_0/4, \varepsilon/4\} \text{ for all } x \in \mathcal{G}_0. \tag{e14.86}$$

Write $U_1 = \lambda \cdot 1_{\tilde{C}_n} + c(U_1)$, where $\lambda \in \mathbb{T} \subset \mathbb{C}$ and $c(U_1) \in C_n$. Define $V_1 = \lambda \cdot 1_{\tilde{C}} + c(U_1)$. Then $V_1 \in U(\tilde{C})$. Note, since $B_n \perp C_n$, $V_1^* b V_1 = b$ for all $b \in B_n$.

Let $E_n = \overline{C_{0,n} C C_{0,n}}$ and e_{E_n} be a strictly positive element with $\|e_{E_n}\| = 1$. Put $\Lambda : A \to C_{1,n} \subset E_n$ by defining $\Lambda(a) = \mathrm{Ad}\, V_1 \circ \varphi_{1,n} \circ \Phi_1(a)$ for all $a \in A_0$, By (e14.84), Λ is F'-$\mathcal{H}_1$-full in $C_{1,n}$. It follows it is F'-$\mathcal{H}_1$-full in E_n. By (e14.80), we may assume that $b_n \lesssim e_{E_n}$.

Let $L_i = \varphi_{0,n} \circ \Phi_i$, $i = 1, 2$. By (e14.77), we assume that L_i is also $\mathcal{G}$-2δ-multiplicative and

$$\|L_i(x) \oplus \Psi_i(x) - \Phi_i(x)\| < \delta \text{ for all } x \in \mathcal{G}. \tag{e14.87}$$

Since $K_i(C_n) = \{0\}$, $i = 0, 1$, we conclude that

$$[L_1]|_{\mathcal{P}} = [\Phi_1]|_{\mathcal{P}} = [\Phi_2]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}. \tag{e14.88}$$

It follows from 4.4 and (e14.82) that, in B_n,

$$\mathrm{cel}(\lceil L_1(z \otimes 1)\rceil \lceil L_2(z \otimes 1)\rceil^*) < 8\pi = \mathbf{L}. \tag{e14.89}$$

It follows from 3.14 of [16] that there exists a unitary $W_1 \in \tilde{B}$ such that

$$\|W_1^*(L_1(a) \oplus S(a))W_1 - (L_2(a) \oplus S(a))\| < \varepsilon/16, \tag{e14.90}$$

where $S(a) = \mathrm{diag}(\overbrace{\Lambda(a), \Lambda(a), \ldots, \Lambda(a)}^{K}) = V_1^* \Psi_1(a) V_1$, for all $a \in \mathcal{F}_I$. Put $W = V_1 W_1$. One then estimates, by (e14.87), (e14.90) and (e14.86),

$$\mathrm{Ad}\, W \circ \Phi_1 \approx_\delta \mathrm{Ad}\, W \circ (L_1 \oplus \mathrm{Ad}\, V_1 \circ \Psi_1) \tag{e14.91}$$

$$\approx_{\varepsilon/16} L_2 \oplus V_1 \circ \Psi_1 \approx_{\varepsilon/4} L_2 \oplus \Psi_2 \approx_\delta \Phi_2 \text{ on } \mathcal{F}_I. \tag{e14.92}$$

Therefore

$$\|\mathrm{Ad}\, W \circ \Phi_1(a) - \Phi_2(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}. \quad \Box \tag{e14.93}$$

Lemma 14.5. *Let A be a non-unital C^*-algebra and $T(A) \neq \emptyset$, let U be an infinite dimensional UHF-algebra and $B \subset A$ be a hereditary C^*-subalgebra of B. Suppose that there exists $e \in A_+$ with $\|e\| = 1$ and $eb = be = b$ for all $b \in B$. Then there is a unitary $w \in \tilde{A} \otimes U$ with the form $w = \exp(i\pi(e \otimes h))$ for some $h \in U_{s.a.}$ with $\tau_U(h) = 0$ (where τ_U is the unique tracial state of U) such that for any unitary $u = \lambda + x \in \tilde{A}$ with $\lambda \in \mathbb{T} \subset \mathbb{C}$ and $x \in B$, one has, for any $b \in B$ and $f \in C(\mathbb{T})$,*

$$\tau(bf((u \otimes 1)w)) = \tau(b)\tau(f(1_A \otimes \exp(ih))) = \tau(b)\int_{\mathbb{T}} f dm \tag{e14.94}$$

and for all $\tau \in T(A \otimes U)$, where m is the normalized Lebesgue measure on $\mathbb{T}$. Moreover, for any $a \in B$ and $\tau \in T(A \otimes U)$, $\tau((a \otimes 1)w^j) = 0$ if $j \neq 0$. Furthermore, if A has continuous scale, then, for any $\varepsilon > 0$, and any $N \geq 1$, one can choose e such that

$$|\tau((u \otimes 1)w)^j| < \varepsilon \text{ for all } 0 < |j| \leq N. \tag{e14.95}$$

Proof. Denote by τ_U the unique trace of U. Then any trace $\tau \in T(A \otimes U)$ is a product trace, i.e.,

$$\tau(a \otimes b) = \tau(a \otimes 1) \otimes \tau_U(b), \quad a \in A, b \in U.$$

Pick a selfadjoint element $h \in U$ such that the spectral measure of the unitary $w_0 = \exp(ih)$ is the Lebesgue measure (a Haar unitary). Moreover, $\mathrm{sp}(h) = [-\pi, \pi]$ and $\tau(h) = 0$.

Then one has, for each $n \in \mathbb{Z}$,

$$\tau_U(w_0^n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Put $w = \exp(i(e \otimes h)) \in \widetilde{A \otimes U}$. Thus $w = \sum_{k=0}^\infty \frac{ie^k \otimes h^k}{k!}$. Hence, for any $\tau \in T(A \otimes U)$, one has, for each $n \in \mathbb{Z}$, and any $b \in B$ (note that $eb = be = b$),

$$\tau(b((u \otimes 1)w)^n) = \tau(b(u^n \otimes 1)(e \otimes 1)(1 \otimes w_0^n)) = \tau(bu^n \otimes 1)\tau_U(w_0^n) = \begin{cases} \tau(b), & \text{if } n = 0, \\ 0, & \text{otherwise;} \end{cases}$$

and therefore

$$\tau((b \otimes 1)P(u \otimes 1)w) = \tau(b)\tau(P(1 \otimes w)) = \tau(b)\int_{\mathbb{T}} P(z) dm$$

for any polynomial P. Similarly, $\tau(bP(u\otimes w)^*) = \tau(b)\int_T P(\bar{z})dm$ for any polynomial P. Since polynomials of z and z^{-1} are dense in $C(\mathbb{T})$, one has

$$\tau((b\otimes 1)f((u\otimes 1)w)) = \tau(b)\tau(f(1\otimes w)) = \tau(b)\int_{\mathbb{T}} fdm, \quad f \in C(\mathbb{T}),$$

as desired.

For the second part of this lemma, assume that A has continuous scale. Then, for any $\delta > 0$ and any integer $N_1 \geq 1$, we can choose $e_1, e \in A_+$ such that $1 \geq e \geq e_1$, $e_1 e = ee_1 = e_1$ $\tau(e^k) \geq \tau(e_1^{N_1}) > 1-\delta$ for all $\tau \in T(A)$ and $k \in \mathbb{N}$. Fix N and $\varepsilon > 0$. A simple calculation shows the second part of the lemma follows by choosing sufficiently small δ and large N_1. $\square$

Proposition 14.6. *Let C be a non-unital amenable simple C^*-algebra and let U be an infinite dimensional UHF-algebra. For any $\delta > 0$, $\delta_c > 0$, $1 > \sigma_1, \sigma_2 > 0$, any finite subset $\mathcal{G} \subset \tilde{C}\otimes C(\mathbb{T})$, any finite $\mathcal{G}_c \subset \tilde{C}$, any finite subset $\mathcal{H}_1 \subset C(\mathbb{T})_+ \setminus \{0\}$ and any finite subset $\mathcal{H}_2 \subset (C\otimes C(\mathbb{T}))_{s.a.}$ and any integer $N \geq 1$, there exist $\delta_1 > 0$ and a finite subset $\mathcal{G}_1 \subset C$ satisfying the following: For any unital $\mathcal{G}_1$-δ_1-multiplicative contractive completely positive linear map $L : C \to A$ and a unitary $u \in \tilde{A}$ with $\|[L(g), u]\| < \delta_1$ for all $g \in \mathcal{G}_1$, where A is another non-unital C^*-algebra with $T(A) \neq \emptyset$ and with continuous scale, there exists a positive element $e \in A$ with $\|e\| = 1$ and $h \in U$ satisfying the following: there are two unital $\mathcal{G}$-δ-multiplicative completely positive contractive linear maps $L_1, L_2 : \tilde{C}\otimes C(\mathbb{T}) \to \tilde{B}$ such that*

$$|\tau(L_1(f)) - \tau(L_2(f))| < \sigma_1 \text{ for all } f \in \mathcal{H}_2,\ \tau \in T(B), \text{ and} \tag{e14.96}$$

$$\tau(g(u\exp(\sqrt{-1}e\otimes h))) \geq \sigma_2(\int gdm) \text{ for all } g \in \mathcal{H}_1,\ \tau \in T(B), \tag{e14.97}$$

where $B = A\otimes U$ and m is the normalized Lebesgue measure on $\mathbb{T}$, and

$$\|L_i(g\otimes 1_{C(\mathbb{T})}) - L^{\sim}(g)\otimes 1_U\| < \delta_c \text{ for all } g \in \mathcal{G}_c,\ i = 1, 2, \tag{e14.98}$$

$$\|L_1(g\otimes z^j) - L^{\sim}(g)(u\exp(\sqrt{-1}e\otimes h))^j\| < \delta_c \text{ for all } g \in \mathcal{G}_c \text{ and} \tag{e14.99}$$

$$\|L_2(g\otimes z^j) - L(g)^{\sim}\exp(\sqrt{-1}e\otimes h)^j\| < \delta_c \text{ for all } g \in \mathcal{G}_c \tag{e14.100}$$

and for all $0 < |j| \leq N$, where $L^{\sim} : \tilde{C} \to \tilde{A}$ is the unital extension of L. Moreover, $\tau(e\otimes h) = 0$ for all $\tau \in A\otimes U$.

Proof. Without loss of generality, we may assume that there are finite subsets $\mathcal{G}_c, \mathcal{H}_{c,1} \subset \tilde{C}$ such that $\mathcal{G} = \{c\otimes 1_{C(\mathbb{T})} : c \in \mathcal{G}_c\} \cup \{1, 1_{\tilde{C}}\otimes z, 1_{\tilde{C}}\otimes z^*\}$ and $\mathcal{H}_2 = \{c\otimes 1_{C(\mathbb{T})} : c \in \mathcal{H}_{c,1}\} \cup \{1\otimes b : b \in \mathcal{H}_T\}$, where $\mathcal{H}_T \subset C(\mathbb{T})_{s.a.}$. We may assume that $1_{\tilde{C}} \in \mathcal{G}_c$, $1_{\tilde{C}} \in \mathcal{H}_{c,1}$ and $1_{C(\mathbb{T})} \in \mathcal{H}_T$. We may also assume that $\|a\| \leq 1$ for all $a \in \mathcal{G}_c \cup \mathcal{H}_2$. Put

$$\mathcal{G}_0 = \{cd\otimes gf : c, d \in \mathcal{G}_c \cup \mathcal{H}_{c,1},\ g, f \in \{z, z^*\} \cup \mathcal{H}_T\}.$$

Fix $\delta, \delta_c > 0$, $\sigma_1, \sigma_2 > 0$. Put $\varepsilon = \min\{\delta/4, \delta_c/4, \sigma_1/4, \sigma_2/4\}$.

Let $\delta_1' > 0$ and $\mathcal{G}_{0m} \subset \tilde{C}$ be a finite subset such that there is a $\mathcal{G}_0$-ε-multiplicative completely positive contractive linear map $L' : \tilde{C}\otimes C(\mathbb{T}) \to D$, for any C^*-algebra D and any $\mathcal{G}_{0m}$-δ_1'-multiplicative completely positive contractive linear map $L'' : \tilde{C} \to D$, such that

$$\|L'(g\otimes 1_{C(\mathbb{T})}) - L''(g)\| < \varepsilon \text{ for all } g \in \mathcal{G}_0. \tag{e14.101}$$

Let $\mathcal{G}_1 = \mathcal{G}_0 \cup \mathcal{G}_{0m}$ and $\delta_1 = \min\{\delta_1'/4, \varepsilon/4\}$.

Now suppose that $L : \tilde{C} \to \tilde{A}$ is a $\mathcal{G}_1$-δ_1-multiplicative completely positive contractive linear map and $u \in \tilde{A}$ is a unitary. Without loss of generality, we may assume that there are positive elements $e_1, e \in A$ with $\|e_1\| = 1 = \|e\|$ such that

$$e_1 L(g) = L(g)e = L(g) \text{ for all } g \in C, ee_1 = e_1 e = e_1 \text{ and } \tau_U(e_1) > 1-\varepsilon. \tag{e14.102}$$

Furthermore, without loss of generality, we may assume that $L(c) = e_1 L(c)e_1$ for all $c \in C$. Let $h \in U$ be as in 14.5. Let $v = \exp(\sqrt{-1}e\otimes h)$. Note that $\tau_U(e^j) > 1-\varepsilon$ for all $j \in \mathbb{N}$. We can choose e so that both (e14.94) and (e14.95) hold. This lemma then follows from an easy application of 14.5 and Lemma 2.8 of [35] (with $L_1 = \Phi_{v_1,L}$ and $L_2 = \Phi_{v_2,L}$, where $v_1 = u(\exp(ie\otimes h))$ and $v_2 = \exp(ie\otimes h)$). $\square$

Corollary 14.7. *Let C be a non-unital separable C^*-algebra. Suppose that there is an embedding $\varphi : C \to \mathcal{W}$. Then $C(\mathbb{T}, \tilde{C})^o$ satisfies the condition in 9.3. Moreover, there exists an embedding $\Phi : C(\mathbb{T}, \tilde{C})^o \to \mathcal{W}$ which maps strictly positive elements to strictly positive elements.*

Proof. Let $\{e_n\}$ be an approximate identity for $\mathcal{W}$ such that $e_{n+1}e_n = e_n$ fort all n. Let $W_1 = \overline{e_n\mathcal{W}e_n}$ Then there exists an isomorphism $\psi_w : \mathcal{W} \to W_1$. Put $\varphi_0 = \psi_w \circ \varphi$. Therefore there is $e \in \mathcal{W}_+$ with $\|e\| = 1$ such that $e\varphi_1(c) = \varphi_1(c)e = \varphi_1(c)$ for all $c \in C$. Let $\mathcal{U}$ be a UHF-algebra of infinite type. Choose $h \in \mathcal{U}_{s.a.}$ with $\mathrm{sp}(h) = [-\pi, \pi]$ and $t_{\mathcal{U}}(h) = 0$, where $t_{\mathcal{U}}$ is the unique tracial state of $\mathcal{U}$. Define $x = \sum_{n=1}^{\infty}\frac{(\sqrt{-1}e\otimes h)^n}{n!} \in \mathcal{W}\otimes\mathcal{U}$ and $u = 1_{\tilde{\mathcal{W}}} + x \in \tilde{\mathcal{W}}\otimes\mathcal{U}$. Note that $u\varphi_1(c) = \varphi_1(c)u$

for all $c \in C$. Define $\Psi : C(\mathbb{T}, \tilde{C}) \to \tilde{\mathcal{W}} \otimes \mathcal{U}$ by

$$\Psi(f \otimes 1_{\tilde{C}}) = f(u) \text{ for all } f \in C(\mathbb{T}) \text{ and } \Psi(1_{C(\mathbb{T})} \otimes c) = \varphi_1(c) \text{ for all } c. \tag{e14.103}$$

This gives a homomorphism $\Phi : C(\mathbb{T}, \tilde{C})^o \to \mathcal{W} \otimes \mathcal{U}$. By the proof of 14.5, we have, for all $c \in C$ and $f \in C(\mathbb{T})$,

$$(t_{\mathcal{W}} \otimes t_{\mathcal{U}})(\Psi(f \otimes c)) = t_{\mathcal{W}}(\varphi_1(c)) \int_{\mathbb{T}} f dm, \tag{e14.104}$$

where m is the normalized Lebesgue measure on $\mathbb{T}$. It follows that, for any $f \in (C(\mathbb{T}) \otimes \tilde{C})_+$,

$$(t_{\mathcal{W}} \otimes t_{\mathcal{U}})(\Psi(f)) = \int_{\mathbb{T}} t_w(\varphi_1(f(t)))dm. \tag{e14.105}$$

This implies Φ is injective. Note that $\mathcal{W} \otimes \mathcal{U} \cong \mathcal{W}$. By replacing $\mathcal{W}$ by $\overline{\Phi(C(\mathbb{T}, \tilde{C})^o)\mathcal{W}\Phi(C(\mathbb{T}, \tilde{C})^o)}$, we may assume that Φ maps strictly positive elements to strictly positive elements. It follows from 5.6 of [15] that $C(\mathbb{T}, \tilde{C})^o$ satisfies the condition in 9.3. $\square$

In what follows, if A is a unital C^*-algebra, u is a unitary and p is a projection in A such that $\|[p, u]\| < \delta$ for some sufficiently small δ, then $(1-p)+pu$ is close to a unitary with the form $(1-p)+v$, where v is a unitary in pAp. As before this unitary may be chosen to be $\lceil (1-p)+pu \rceil$ (see 2.1). Moreover, when $\lceil (1-p)+pu \rceil$ is written we also assume that $\|[p, u]\|$ is sufficiently small so the notation makes sense. Therefore, if $L : A \to B$ is a map which is η-$\mathcal{F}$-multiplicative, $p \in M_n(\tilde{A})$ is a projection and $u \in \tilde{B}$ is a unitary such that $\|[L(x), u]\| < \eta$ for all $x \in \mathcal{F}$ for some sufficiently small η and some large $\mathcal{F} \subset A$, then $L(p)$ is close to a projection and $\|[L(p), u_{\underline{n}}]\| < \delta$, where $u_{\underline{n}} = u \otimes 1_{M_n}$. So (e14.108) makes sense. Similar items will appear again later.

Lemma 14.8. *Let $A \in \mathcal{A}$ be a separable simple C^*-algebra with continuous scale. For any $1 > \varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, $\sigma > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\{p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_k\}$ of projections of $M_N(\tilde{A})$ (for some integer $N \geq 1$) such that $\{[p_1]-[q_1], [p_2]-[q_2], \dots, [p_k]-[q_k]\}$ generates a free subgroup G_u of $K_0(A)$, and a finite subset $\mathcal{P} \subset \underline{K}(A)$, satisfying the following:*

Suppose that $\varphi : A \to B \otimes V$ is a homomorphism which maps strictly positive elements to strictly positive elements, where $B \in \mathcal{D}$ has continuous scale and V is a UHF-algebra of infinite type. If $u \in U(\widetilde{B \otimes V})$ is a unitary such that

$$\|[\varphi(x), u]\| < \delta \text{ for all } x \in \mathcal{G}, \tag{e14.106}$$

$$\mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = 0, \tag{e14.107}$$

$$\mathrm{dist}(\overline{\lceil((1-\varphi^{\sim}(p_i)) + \varphi^{\sim}(p_i)u_{\underline{N}})((1-\varphi^{\sim}(q_i)) + \varphi^{\sim}(q_i)u_{\underline{N}}^*)\rceil}, \bar{1}) < \sigma \text{ and} \tag{e14.108}$$

$$\mathrm{dist}(\bar{u}, \bar{1}) < \sigma, \tag{e14.109}$$

(where $u_{\underline{N}} = u \otimes 1_{M_N}$), then there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\} \subset U_0(\widetilde{B \otimes V})$ such that

$$u(0) = u, \quad u(1) = 1 \tag{e14.110}$$

$$\|[\varphi(a), u(t)]\| < \varepsilon \text{ for all } a \in \mathcal{F} \text{ and for all } t \in [0, 1]. \tag{e14.111}$$

Proof. Without loss of generality, one only has to prove the statement with assumption that $u \in CU(B \otimes V)$ as (e14.109) is assumed. Since $B \otimes V \otimes V \cong B \otimes V$, to simplify notation, without loss of generality, we may assume that $B = B \otimes V$. In particular, $K_0(\tilde{B})$ is weakly unperforated (see 5.5).

In what follows we will use the fact that every C*-algebra in $\mathcal{D}$ has stable rank one (11.5 of [15]). We will also use z for the generator unitary function on the unit circle. Let $A_2 = C(\mathbb{T}) \otimes \tilde{A}$ and m is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Define

$$\Delta(\hat{h}) = (1/4) \inf\{\int_{\mathbb{T}} \tau(h(t))dm : \tau \in T(A)\} \tag{e14.112}$$

for $h \in (A_2)_+^{\mathbf{1}} \setminus \{0\}$ (note, by the assumption, $T(A)$ is compact). Let $\mathcal{F}_1 = \{x \otimes f : x \in \mathcal{F}, f = 1, z, z^*\}$. To simplify notation, without loss of generality, we may assume that $\mathcal{F} \subset A^{\mathbf{1}}$. Let $1 > \delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset A_2$ be a finite subset (in place of $\mathcal{G}$), $1/4 > \gamma_1 > 0$, $1/4 > \gamma_2 > 0$, $\mathcal{P}' \subset \underline{K}(A_2)$ (in place of $\mathcal{P}$) be a finite subset, $\mathcal{H}_1 \subset (A_2)_+^{\mathbf{1}} \setminus \{0\}$ be a finite subset, $\mathcal{H}_2 \subset (A_2)_{s.a.}$ be a finite subset and $\mathcal{U} \subset J_c(K_1(A_2))$ (for some integer $N \geq 1$) be a finite subset as required by 14.4 for $\varepsilon/16$ (in place of ε), $\mathcal{F}_1$ (in place of $\mathcal{F}$), Δ and A_2 (in place of A). Here we assume that $[L]|_{\mathcal{P}'}$ is well defined whenever L is a $\mathcal{G}_1$-δ_1-multiplicative completely positive contractive linear map from A_2. Moreover,

$$[L_1]|_{\mathcal{P}'} = [L_2]|_{\mathcal{P}'}, \tag{e14.113}$$

if both L_1 and L_2 are $\mathcal{G}_1$-δ_1-multiplicative completely positive contractive linear maps from A_2 to a unital C^*-algebra and $\|L_1(g) - L_2(g)\| < \delta_1$ for all $g \in \mathcal{G}_1$.

Without loss of generality, we may assume that $\mathcal{G}_1 = \{z \otimes 1_{\tilde{A}}, 1_{C(\mathbb{T})} \otimes a : a \in \mathcal{G}_{1A}\}$, $\mathcal{H}_1 = \{h' \otimes 1_{\tilde{A}}, 1_{C(\mathbb{T})} \otimes h'' : h' \in \mathcal{H}_{1T}$ and $h'' \in \mathcal{H}_{1A}\}$, $\mathcal{H}_2 = \{h_1 \otimes 1_{\tilde{A}}, 1_{C(\mathbb{T})} \otimes h_2 : h_1 \in \mathcal{H}_{2T}$ and $h_2 \in \mathcal{H}_{2A}\}$, where $\mathcal{H}_{1T} \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$, $\mathcal{H}_{2T} \subset C(\mathbb{T})_{s.a.}$, $\mathcal{G}_{1,A} \subset \tilde{A}$, $\mathcal{H}_{1A} \subset A_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{H}_{2A}$ are finite subsets. Furthermore, we may also assume that elements in $\mathcal{H}_{1T}$ and $\mathcal{H}_{2T}$ are polynomials of z and z^* of degree no more than N_1 and all coefficients with absolute values no more than M. In addition, we assume that $\mathcal{H}_{1A} \subset \mathcal{H}_{2A}$. We may assume that $\mathcal{P}' = \mathcal{P}_1 \cup \boldsymbol{\beta}(\mathcal{P}_2) \cup \boldsymbol{\beta}([1_{\tilde{A}}])$, where $\mathcal{P}_1, \mathcal{P}_2 \subset \underline{K}(A)$ are finite subsets. We further assume that

$$\mathrm{Bott}(\varphi, v(0))|_{\mathcal{P}_2} = \mathrm{Bott}(\varphi, v(t))|_{\mathcal{P}_2}, \tag{e14.114}$$

if $\|[\varphi(a), v(t)]\| < \delta_1$ for all $a \in \mathcal{G}_{1A}$ and for any continuous path of unitaries $\{v(t) : t \in [0, 1]\}$.

We may further assume that,

$$\mathcal{U} = \mathcal{U}_1 \cup \{\overline{1 \otimes z}\} \cup \mathcal{U}_2, \tag{e14.115}$$

where $\mathcal{U}_1 = \{\overline{1_{C(\mathbb{T})} \otimes a} : a \in \mathcal{U}_1' \subset U(\tilde{A})\}$ and $\mathcal{U}_1'$ is a finite subset, $\mathcal{U}_2 \subset U(M_N(A_2))/CU(M_N(A_2))$ is a finite subset whose elements represent a finite subset of $\boldsymbol{\beta}(K_0(A))$. So we may assume that $\mathcal{U}_2 \in J_c(\boldsymbol{\beta}(K_0(A)))$.

We may even assume that $\mathcal{U}_2 = \mathcal{U}_{2f} \sqcup \mathcal{U}_{2t}$, where $\mathcal{U}_{2f} = \{J_c(g_{1,f}), J_c(g_{2,f}), \ldots, J_c(g_{m(f),f})\}$ and $\mathcal{U}_{2t} = \{J_c(g_{1,t}), J_c(g_{2,t}), \ldots, J_c(g_{m(t),t})\}$, where $\mathcal{P}' \cap \boldsymbol{\beta}(K_0(A)) = \{g_{i,f}, g_{j,t} : 1 \le i \le m(f),\ 1 \le j \le m(t)\}$. Moreover, $\{g_{1,f}, g_{2,f}, \ldots, g_{m(f),f}\}$ is a set of free generators of a finitely generated free subgroup of $\boldsymbol{\beta}(K_0(A))$ and $\{g_{1,t}, g_{2,t}, \ldots, g_{m(t),t}\}$ are generators for a finite subgroup of $\boldsymbol{\beta}(K_0(A))$. Since J_c is a homomorphism, we may assume that there is an integer $k_m \ge 1$ such that $k_m J_c(g_{j,t}) = 0$ in $U(M_N(A_2))/CU(M_N(A_2))$. Since $g_{i,f} \in \boldsymbol{\beta}(K_0(A))$, we may write that

$$g_{i,f} = [(1 \otimes (1 - p_i) + z \otimes p_i)(1 \otimes (1 - q_i) + z^* \otimes q_i)], \quad i = 1, 2, \ldots, m(f). \tag{e14.116}$$

Write $p_s = (a_{i,j}^{p_s})_{N \times N}$ and $q_s = (a_{i,j}^{q_s})_{N \times N}$ as matrices over $\tilde{A}$. Let $w_l = (b_{i,j}^l)_{N \times N}$ be unitaries in $M_N(\tilde{A})$ such that $\overline{w_l} = J_c(g_{j,t})$, $l = 1, 2, \ldots, m(t)$.

We assume that $(2\delta_1, \mathcal{P}, \mathcal{G}_1)$ is a KL-triple for A_2, $(2\delta_1, \mathcal{P}_1, \mathcal{G}_{1A})$ is a KL-triple for A (see 2.12 of [18], for example). We may also choose σ_1 and σ_2 such that

$$0 < \sigma_1 < (1/4)\min\{\gamma_1/16, \inf\{\Delta(\hat{f}) : f \in \mathcal{H}_1\}\}/4M(N+1) \text{ and} \tag{e14.117}$$

$$\sigma_2 = 1 - \gamma_2/16(N+1)M. \tag{e14.118}$$

Choose $\delta_2 > 0$ and a finite subset $\mathcal{G}_{2A} \subset \tilde{A}$ (and denote $\mathcal{G}_2 := \{g \otimes f : g \in \mathcal{G}_{2A}, f = \{1, z, z^*\}\}$) such that, for any two unital $\mathcal{G}_2$-δ_2-multiplicative contractive completely positive linear maps $\Psi_1, \Psi_2 : C(\mathbb{T}) \otimes \tilde{A} \to \tilde{C}$ (any unital C^*-algebra C), any $\mathcal{G}_{2A}$-δ_2-multiplicative contractive completely positive linear map $\Psi_0 : \tilde{A} \to \tilde{C}$ and unitary $W \in \tilde{C}$ $(1 \le i \le k)$, if

$$\|\Psi_0(g) - \Psi_1(g \otimes 1)\| < \delta_2 \text{ for all } g \in \mathcal{G}_{2A} \tag{e14.119}$$

$$\|\Psi_1(z \otimes 1_{\tilde{A}}) - W\| < \delta_2 \text{ and } \|\Psi_1(g) - \Psi_2(g)\| < \delta_2 \text{ for all } g \in \mathcal{G}_2, \tag{e14.120}$$

then ($\underline{W} = W \otimes 1_{M_N}$)

$$\lceil(1 - \Psi_0(p_i) + \Psi_0(p_i)\underline{W})(1 - \Psi_0(q_i) + \Psi_0(q_i)\underline{W}^*)\rceil \tag{e14.121}$$

$$\approx_{\frac{\gamma_2}{2^{10}}} \lceil\Psi_1(((1 - p_i) + z \otimes p_i)((1 - q_i) + z^* \otimes q_i))\rceil, \tag{e14.122}$$

$$\|\lceil\Psi_1(x)\rceil - \lceil\Psi_2(x)\rceil\| < \gamma_2/2^{10} \text{ for all } x \in \mathcal{U}_2', \tag{e14.123}$$

$$\Psi_1(((1 - p_i) + z \otimes p_i)((1 - q_i) + z^* \otimes q_i)) \tag{e14.124}$$

$$\approx_{\frac{\gamma_2}{2^{10}}} \Psi_1(((1 - p_i) + z \otimes p_i))\Psi_1(((1 - q_i) + z^* \otimes q_i)), \tag{e14.125}$$

furthermore for $d_i^{(1)} = p_i$, $d_i^{(2)} = q_i$, there are projections $\bar{d}_i^{(j)} \in M_N(\tilde{C})$ and unitaries $\bar{z}_i^{(j)} \in \bar{d}_i^{(j)} M_N(\tilde{C})\bar{d}_i^{(j)}$ such that

$$\Psi_1(((1 - d_i^{(j)}) + z \otimes d_i^{(j)})) \approx_{\frac{\gamma_2}{2^{12}}} (1 - \bar{d}_i^{(j)}) + \bar{z}_i^{(j)} \text{ and} \tag{e14.126}$$

$$\bar{d}_i^{(j)} \approx_{\frac{\gamma_2}{2^{12}}} \Psi_1(d_i^{(j)}),\ \bar{z}_i^{(1)} \approx_{\frac{\gamma_2}{2^{12}}} \Psi_1(z \otimes p_i), \text{ and } \bar{z}_i^{(2)} \approx_{\frac{\gamma_2}{2^{12}}} \Psi_1(q_i \otimes z^*), \tag{e14.127}$$

where $1 \le i \le k$, $j = 1, 2$.

Let $\delta_3 > 0$ and let $\mathcal{G}_3 \subset C(\mathbb{T}, \tilde{A})^o$ be a finite subset required by 11.6 for $C = C(\mathbb{T}, \tilde{A})^o$, $\gamma_2/2$ (in place of ε) and for all unitaries in $\mathcal{U}_{2t}$. Without loss of generality, we may write $\mathcal{G}_3 = \mathcal{G}_{3A} \cup \{1, z, z^*\}$, where $\mathcal{G}_{3A}$ is a finite subset of A.

Choose $\delta_A = \min\{\varepsilon/16, \delta_1/16, \delta_2/16, \sigma_1/4, \sigma_2/4\}/8M(N+1)^3$ and

$$\mathcal{G}_A = \mathcal{F} \cup \mathcal{G}_{1A} \cup \mathcal{G}_{2A} \cup \mathcal{H}_{1A} \cup \mathcal{H}_{2A} \cup \mathcal{U}_1' \cup \{a_{i,j}^{p_s}, a_{i,j}^{q_s}, b_{i,j}^l : 1 \le s \le, 1 \le l \le m(t),\ 1 \le i, j \le N\}.$$

Let $\mathcal{G}_A' \subset A$ be a finite subset such that every element $a \in \mathcal{G}_A$ has the form $a = \lambda + b$ for some $\lambda \in \mathbb{C}$ and $b \in \mathcal{G}_A'$. Let $\mathcal{G}_4 = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{U}_1$.

Let $\delta_4 > 0$ (in place of δ_1) and a finite subset $\mathcal{G}_5$ (in place of $\mathcal{G}_1$) be as required by 14.6 for A (in place of C), $\delta_1/4$ (in place of δ), δ_A (in place of δ_c), σ_1, σ_2, $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{G}_4$ (in place of $\mathcal{G}$), $\mathcal{G}_A$ (in place of $\mathcal{G}_c$) and N_1.

By choosing even smaller δ_4, without loss of generality, we may assume that $\mathcal{G}_5 = \{a \otimes f : g \in \mathcal{G}_{5A}$ and $f = 1, z, z^*\}$ with a large finite subset $\mathcal{G}_{5A} \supset \mathcal{G}_A$. Let $\mathcal{G}'_{5A} \subset A$ be a finite subset such that every element $g \in \mathcal{G}_{5A}$ has the form $g = \lambda + x$ for some $\lambda \in \mathbb{C}$ and $x \in \mathcal{G}'_{5A}$.

Choose $\sigma > 0$ so it is smaller than $\min\{\sigma_1/16, \varepsilon/16, \sigma_2/16, \delta_2/16, \delta_3/16, \delta_4/16, \delta_A/4\}$.

Let $\delta = \sigma$ and $\mathcal{G} = \mathcal{G}'_{5A} \cup \mathcal{G}_A$.

Now suppose that $\varphi : A \to B$ is a homomorphism and $u \in CU(\tilde{B})$ which satisfy the assumption (e14.106)–(e14.108) for the above mentioned δ, σ, $\mathcal{G}$, $\mathcal{P}$, p_i, and q_i. There is an isomorphism $s : V \otimes V \to V$. Moreover, $s \circ \imath$ is approximately unitarily equivalent to the identity map on V, where $\imath : V \to V \otimes V$ is defined by $\imath(a) = a \otimes 1$ (for all $a \in V$). To simplify notation, without loss of generality, we may assume that $\varphi(A) \subset B \otimes 1 \subset B \otimes V$. Suppose that $u \in U(B) \otimes 1_V$ is a unitary which satisfies the assumption. As mentioned at the beginning, we may assume that $u \in CU(B) \otimes 1_V$.

Applying 14.6, we obtain $e \in (B)_+$ with $\|e\| = 1$ and $h \in V_{s.a}$ satisfying the conclusions of 14.6. Note that we may assume, without loss of generality, that

$$e\varphi^{\sim}(g) = \varphi^{\sim}(g)e \text{ for all } g \in \mathcal{G}_{3A} \cup \mathcal{G}_{5A} \text{ and} \tag{e14.128}$$
$$e\varphi(g) = \varphi(g)e = \varphi(g) \text{ for all } g \in \mathcal{G}'_{3A} \cup \mathcal{G}'_{5A}. \tag{e14.129}$$

In particular, for $E = \mathrm{diag}(\overbrace{e, e, \ldots, e}^{N})$ and $y = p_i,\ q_i,\ i = 1, 2, \ldots, m(f)$,

$$(\varphi^{\sim} \otimes \mathrm{id}_{M_N})(y)E = E(\varphi^{\sim} \otimes \mathrm{id}_{M_N})(y). \tag{e14.130}$$

Put $v_1 = u\exp(ie \otimes h)$ and $v_2 = \exp(\widetilde{ie \otimes h})$. Note that $\mathrm{sp}(h) = [-\pi, \pi]$ and $t_V(h) = 0$ and where t_V is the unique tracial state of V. Let $L_1, L_2 : C(\mathbb{T}) \otimes \tilde{A} \to \widetilde{B \otimes V}$ be given by 14.6 such that

$$|\tau(L_1(f)) - \tau(L_2(f))| < \sigma_1 \text{ for all } f \in \mathcal{H}_2,\ \ \tau \in T(B), \tag{e14.131}$$
$$\tau(g(v_1)) \ge \sigma_2(\int gdm) \text{ for all } g \in \mathcal{H}_1,\ \ \tau \in T(B), \text{ and} \tag{e14.132}$$

$$\|L_i(c \otimes 1_{C(\mathbb{T})}) - \varphi^{\sim}(c) \otimes 1_V\| < \delta_A \text{ for all } c \in \mathcal{G}_c,\ \ i = 1, 2, \tag{e14.133}$$
$$\|L_1(c \otimes z^j) - \varphi^{\sim}(c)(u\exp(\sqrt{-1}e \otimes h))^j\| < \delta_A \text{ for all } c \in \mathcal{G}_c \text{ and} \tag{e14.134}$$
$$\|L_2(c \otimes z^j) - \varphi(c)^{\sim}\exp(ie \otimes h)^j\| < \delta_A \text{ for all } c \in \mathcal{G}_c \tag{e14.135}$$

and for all $0 < |j| \le N_1$, where $\varphi^{\sim} : \tilde{A} \to \widetilde{B \otimes V}$. Note by (e14.133)–(e14.135), we may write $L_1 = \Phi_{v_1,\varphi}$ and $L_2 = \Phi_{v_2,\varphi}$. Let $u(t) = \exp(\sqrt{-1}3t(e \otimes h))$ for $t \in [0, 1/3]$. Then

$$\|[\varphi(a),\ u(t)]\| < \delta_c \text{ for all } a \in \mathcal{G}_c. \tag{e14.136}$$

In particular,

$$\mathrm{Bott}(\varphi, v_1)|_{\mathcal{P}_2} = 0. \tag{e14.137}$$

Exactly the same reason, we have that

$$\mathrm{Bott}(\varphi, v_2)|_{\mathcal{P}_2} = 0. \tag{e14.138}$$

This implies

$$[L_1]|_{\beta(\mathcal{P}_2)} = [L_2]|_{\beta(\mathcal{P}_2)}. \tag{e14.139}$$

We also have

$$[L_1]|_{\mathcal{P}_1} = [\varphi]|_{\mathcal{P}_1} = [L_2]|_{\mathcal{P}_1} \text{ and } [v_1] = [v_2] = 0. \tag{e14.140}$$

Therefore

$$[L_1]|_{\mathcal{P}'} = [L_2]|_{\mathcal{P}'}. \tag{e14.141}$$

Then, by (e14.132) and the choice of δ_A, we compute (as in (e14.94)) that

$$\tau(L_i(h)) \ge \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1,\ \ i = 1, 2. \tag{e14.142}$$

We also have

$$\mathrm{dist}(L_1^{\dagger}(x), L_2^{\dagger}(x)) < 2\delta_A \text{ for all } x \in \mathcal{U}_1 \cup \{\overline{z \otimes 1_{\tilde{A}}}\}. \tag{e14.143}$$

Write $V_2 = \mathrm{diag}(\overbrace{v_2, v_2, \ldots, v_2}^{N})$ and $H = \mathrm{diag}(\overbrace{h, h, \ldots, h}^{N})$. As always, we will write $\varphi^{\sim}(y)$ for $\varphi^{\sim} \otimes \mathrm{id}_{M_N}(y)$ for $y = p_i,\ q_i$, $i = 1, 2, \ldots, m(f)$. By (e14.130),

$$\psi^{\sim}(p_i)V_2 = \exp(i\psi^{\sim}(p_i)E \otimes H) \text{ and } \psi^{\sim}(q_i)V_2 = \exp(i\psi^{\sim}(q_i)E \otimes H), \tag{e14.144}$$

$i = 1, 2, \ldots, m(f)$. However,

$$\tau(\psi(q_i)E \otimes H) = \tau(\psi(q_i)E)\tau_V(H) = 0 \text{ for all } \tau \in T(B \otimes V). \tag{e14.145}$$

In the next few lines, $\mathbf{1} = 1_{M_N}$. Therefore

$$\psi^{\sim}(p_i)V_2 + (\mathbf{1} - \psi^{\sim}(p_i)),\ \psi^{\sim}(q_i)V_2 + (\mathbf{1} - \psi^{\sim}(q_i)) \in CU(M_N(\widetilde{B \otimes V})),$$

$i = 1, 2, \ldots, m(f)$. This implies that

$$L_2^{\dagger}(x) = \bar{1} \text{ for all } x \in \mathcal{U}_{2f}. \tag{e14.146}$$

with $x = ((\mathbf{1} - p_i) + p_i \otimes z)((\mathbf{1} - q_i) + q_i \otimes z^*)$, one then computes from (e14.125) and from the assumption (e14.108) that

$$\overline{\langle L_1(x)\rangle} \approx_{\gamma_2/2^{10}} \overline{(\bar{z}_i^{(1)} \otimes v_2 + (\mathbf{1} - \bar{p}_i))(\bar{z}_i^{(2)} \otimes v_2 + (\mathbf{1} - \bar{q}_i))} \tag{e14.147}$$

$$= \overline{(\bar{z}_i^{(1)} + (\mathbf{1} - \bar{p}_i))(\bar{p}_i V_2 + (\mathbf{1} - \bar{p}_i) \otimes 1_V)(\bar{z}_i^{(2)} + (\mathbf{1} - \bar{q}_i))(\bar{q}_i V_2 + (1 - \bar{q}_i))} \tag{e14.148}$$

$$= \overline{(\bar{z}_i^{(1)} + (\mathbf{1} - \bar{p}_i))(\bar{z}_i^{(2)} + (\mathbf{1} - \bar{q}_i))} \approx_{\sigma} \bar{1}. \tag{e14.149}$$

where $\bar{p}_i, \bar{q}_i, \bar{z}_i^{(1)}, \bar{z}_i^{(2)}$ are as above (see the lines below (e14.125)), replacing Ψ_1 by L_1. It follows that

$$\mathrm{dist}(L_1^{\dagger}(x), \bar{1}) < \gamma_2/4 \text{ for all } x \in \{\overline{1 \otimes z}\} \cup \mathcal{U}_{2f}. \tag{e14.150}$$

By the choice of δ_3 and $\mathcal{G}_4$, and by applying 11.6, we also have

$$\mathrm{dist}(\overline{\lceil L_1(w_l)\rceil}, \overline{\lceil L_2(w_l^*)\rceil}) < \gamma_2/2, \quad l = 1, 2, \ldots, m(t). \tag{e14.151}$$

Combining (e14.146), (e14.150) and (e14.151), we obtain that

$$\mathrm{dist}(L_1^{\dagger}(w), L_2^{\dagger}(w)) < \gamma_2 \text{ for all } w \in \mathcal{U}. \tag{e14.152}$$

By (e14.141), (e14.131), (e14.142) and (e14.152), and by applying 14.4, we obtain a unitary $W_1 \in \widetilde{B \otimes V}$ such that

$$\|W_1{}^* L_2(f) W_1 - L_1(f)\| < \varepsilon/16 \text{ for all } f \in \mathcal{F}_1. \tag{e14.153}$$

Therefore

$$\|[L_1(a),\ W_1{}^* v_2 W_1]\| < \varepsilon/8 \text{ and } \|L_1(a) - W_1{}^* L_1(a) W_1\| < \varepsilon/8 \text{ for all } a \in \mathcal{F} \tag{e14.154}$$

$$\text{and } \|v_1 - W_1{}^* v_2 W_1\| < \varepsilon/8. \tag{e14.155}$$

Let $v_1^* W_1{}^* v_2 W_1 = \exp(\sqrt{-1}h_1)$ for some $h_1 \in \tilde{B}_{s.a.}$ such that $\|h_1\| \leq 2\arcsin(\varepsilon/16)$. Now define $u(t) = u\exp(i3t(e \otimes h))$ for $t \in [0, 1/3]$, $u(t) = u(1/3)\exp(i3(t - 1/3)h_1)$ for $t \in (1/3, 2/3]$ and $u(t) = u(2/3)W_1{}^* \exp(\sqrt{-1}(3(t - 2/3))(e \otimes h))W_1$ for $t \in (2/3, 1]$. So $\{u(t) : t \in [0, 1]\}$ is a continuous path of unitaries in $\widetilde{B \otimes V}$ such that $u(0) = u$ and $u(1) = 1_{\tilde{B}}$. Moreover, we estimate, by (e14.106), (e14.154) and (e14.154) that

$$\|[\varphi(a),\ u(t)]\| < \varepsilon \text{ for all } a \in \mathcal{F}. \quad \Box \tag{e14.156}$$

Corollary 14.9. *Let $A \in \mathcal{M}_0$ with continuous scale. For any $1 > \varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset A$ satisfying the following:*

Let $B = B_1 \otimes V$, where $B_1 \in \mathcal{M}_0$ with continuous scale which satisfies the UCT and V is UHF-algebras of infinite type. Suppose that $\varphi : A \to B$ is a homomorphism.

If $u \in U(\tilde{B})$ is a unitary such that

$$\|[\varphi(x),\ u]\| < \delta \text{ for all } x \in \mathcal{G}, \tag{e14.157}$$

there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\} \subset U(\tilde{B})$ such that

$$u(0) = u, \quad u(1) = 1_{\tilde{B}}, \tag{e14.158}$$

$$\|[\varphi(a),\ u(t)]\| < \varepsilon \text{ for all } a \in \mathcal{F} \text{ and for all } t \in [0, 1]. \tag{e14.159}$$

Proof. Fix a finite subset $\mathcal{G}' \subset A^{\mathbf{1}}$ and $\varepsilon' > 0$, there exist positive elements $e', e'', e''' \in B \backslash \{0\}$ with $\|e'\| = 1 = \|e''\| = \|e'''\|$, $e'e'' = e''e' = e'$ such that $e'''e' = e'e''' = 0$, and

$$\|\varphi(g)e' - \varphi(g)\| < \varepsilon'/2 \text{ and } \|e'\varphi(g) - \varphi(g)\| < \varepsilon'/2 \text{ for all } g \in \mathcal{G}'. \tag{e14.160}$$

Let $\pi^{B\sim} : \tilde{B} \to \mathbb{C}$ be the quotient map. Without loss of generality, we may assume that $\pi^{B\sim}(u) = 1_{\mathbb{C}}$. Since $U(\tilde{B}) = U_0(\tilde{B})$, we may write $u = \prod_{i=1}^{m} \exp(ih_{j0})$ for some $h_{j0} \in \tilde{B}_{s.a.}$. Write $h_{j,0} = r_j + h'_{j0}$, where $r_j \in \mathbb{R}$ and $h'_{j0} \in B_{s.a.}$. Note that $\sum_{j=1}^{m} r_j = 2\pi k_u$ for $k_u \in \mathbb{Z}$. Therefore $u = \prod_{j=1}^{m} \exp(ih'_{j0})$. We may also assume, without loss of generality, that

$u = 1 + x$, where $x \in \overline{e''Be''}$. It is easy to find an element $h_0 \in \overline{e'''Be''}$ such that $\tau(h_0) = \sum_{j=1}^{m} \tau(h'_{j0})$ for $\tau \in T(B)$. Let $u_0(t) = \exp(-\sqrt{-1}th_0)$ for all $t \in [0, 1]$. Note that

$$uu_0(0) = u \text{ and } uu_0(1) \in CU(\tilde{B}). \tag{e14.161}$$

Moreover, by (e14.160),

$$\|\varphi(g)u_0(t) - u_0(t)\varphi(g)\| < 2\varepsilon' \text{ for all } g \in \mathcal{G}' \text{ and } \text{ for all } t \in [0, 1]. \tag{e14.162}$$

In other words, we have just reduced the general case to the case that $u \in CU(\tilde{B})$. In other words, we may assume that, without loss of generality, that $\bar{u} = \bar{1}$.

Now we will apply 14.8. Note, from the above, we may assume (e14.109) holds. Since $K_0(A) = \{0\}$, (e14.108) automatically holds. Since both A and B are KK-contractible, (e14.107) also holds. □

Lemma 14.10. *Let $A \in \mathcal{B}_T$ have continuous scale. For any finite subset $\mathcal{P} \subset \underline{K}(A)$, there exist $\delta_0 > 0$ and a finite subset $\mathcal{G}_0 \subset A$ satisfy the following: For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any homomorphism $\varphi : A \to B = B_1 \otimes Q$ which maps strictly positive elements to strictly positive elements, where $B_1 \cong B_1 \otimes \mathcal{Z}_0 \in \mathcal{D}_0$ has continuous scale, suppose that $u \in U(\tilde{B})$ satisfies*

$$\|[\varphi(g), u]\| < \delta_0 \text{ for all } g \in \mathcal{G}_0. \tag{e14.163}$$

Then there exists another unitary $v \in U(\tilde{B})$ such that

$$\|[\varphi(g), v]\| < \min\{\varepsilon, \delta_0\} \text{ for all } g \in \mathcal{G}_0 \cup \mathcal{F} \text{ and} \tag{e14.164}$$

$$\mathrm{Bott}(\varphi, uv)|_{\mathcal{P}} = 0 \text{ and } [uv] = 0 \text{ in } K_1(B). \tag{e14.165}$$

Proof. Define $\Delta_1(\hat{h}) = \inf\{\tau(h) : \tau \in T(A)\}$ for $h \in A_+^{\mathbf{1}} \setminus \{0\}$. Let $\Delta = \Delta_1/2$. Let $T : A_+^{\mathbf{1}} \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$ be the map given by Δ as in 14.1. Let $\mathcal{P}$ be given.

Write $A = \cup_{n=1}^{\infty} A_n$, where $A_n = A(W, \alpha_n) \oplus W_n$ as in Section 7. Without loss of generality, we may assume $\mathcal{F} \subset A_{N'}$ for some integer N' and $\mathcal{P} \subset [\imath'](\mathcal{P}_{N'})$ for some finite subset $\mathcal{P}_{N'} \subset \underline{K}(A_{N'})$, where $\imath' : A_{N'} \to A$ is the embedding.

Let $\delta_0 > 0$ and let $\mathcal{G}_0 \subset A_{N'}$ be a finite subset satisfying the following: $\mathrm{Bott}(L, w)|_{\mathcal{P}}$ is well defined for any $\mathcal{G}_0$-δ_0-multiplicative completely positive contractive linear map $L : A \to C$ and any unitary $w \in \tilde{C}$ with $\|[L(g), w]\| < 2\delta_0$ for all $g \in \mathcal{G}_0$. Moreover, if w' is another unitary, we also require that

$$\mathrm{Bott}(L, ww')|_{\mathcal{P}} = \mathrm{Bott}(L, w)|_{\mathcal{P}} + \mathrm{Bott}(\varphi, w')|_{\mathcal{P}}, \tag{e14.166}$$

when $\|[L(g), w']\| < 2\delta_0$ for all $g \in \mathcal{G}_0$.

Let φ and u be given satisfying the assumption for the above $\mathcal{G}_0$ and δ_0.

Now fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$.

Let $\varepsilon_1 = \min\{\delta_0/4, \varepsilon/16\}$ and $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{G}_0$. Let $\delta_1 > 0$ (in place of δ), $\gamma > 0$, $\eta > 0$, $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_1 \subset \underline{K}(A)$ (in place $\mathcal{P}$) be a finite subset, $\mathcal{U} \subset U(\tilde{A})$ be a finite subset, $\mathcal{H}_1 \subset A_+ \setminus \{0\}$ be a finite subset, and $\mathcal{H}_2 \subset A_{s.a.}$ be a finite subset required by 5.3 for the above T (and for $\mathbf{T}(n, k) = n$ as $K_0(\tilde{B}_1)$ is weakly unperforated). Let us assume that $\mathcal{P}_1$ contains the set $\{[u] \in K_1(A) \subset \underline{K}(A) : u \in \mathcal{U}\}$, by enlarger $\mathcal{P}_1$ if necessary.

Without loss of generality, we may assume that $\mathcal{P}_1 \subset [\imath](\mathcal{P}_N)$ for some finite subset $\mathcal{P}_N \subset \underline{K}(A_N)$, where $N \geq N'$ and $\imath : A_N \to A$ is the embedding. We assume that $\delta_1 < \delta_0$. Without loss of generality, by choosing large N, we may assume that $\mathcal{G}_1 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \subset (A_N)_+^{\mathbf{1}}$. We may also assume that $\mathcal{U} \subset U(\tilde{A}_N)$. Write $w = \lambda_w + \alpha(w)$, where $\lambda_w \in \mathbb{T} \subset \mathbb{C}$ and $\alpha(w) \in A_N$. As in the remark of 5.3, we may assume that $[w] \neq 0$ and $[w] \in \mathcal{P}_N$ for all $w \in \mathcal{U}$. Let G_u be the subgroup generated by $\{\overline{w} : w \in \mathcal{U}\}$. We may view $G_u \subset J_c(K_1(A))$ (see the statement of 14.2). Moreover, for any $\mathcal{G}_1$-δ_1-multiplicative completely positive contractive linear map L' from A_N to a non-unital C^*-algebra$C$ induces a homomorphism $\lambda' : G_u \to U(\tilde{C})/CU(\tilde{C})$ (see 14.5 of [38]). Furthermore, since $K_i(A_N)$ is finitely generated, $i = 0, 1$, we may assume, with even smaller δ_1 and larger $\mathcal{G}_1$, that $[\Phi_{u', L'}]$defines an element in $KL(C(\mathbb{T}, \tilde{A}_N), \tilde{C})$, if $\|[L'(g), u']\| < \delta_1$ for all $g \in \mathcal{G}_1$.

Set $\mathcal{G} = \mathcal{F}_1 \cup \mathcal{G}_1 \cup \{\alpha(w) : w \in \mathcal{U}\}$ and set a rational number

$$0 < \sigma_0 < \min\{\inf\{\Delta(\hat{h}) : h \in \mathcal{H}_1\}, \gamma/4\}.$$

Choose $\delta = \min\{\varepsilon_1/16, \delta_1/16, \gamma/16, \eta/16\}$. We may write $u = 1_{\tilde{B}} + \alpha(u)$, where $\alpha(u) \in B$. Since $B \otimes Q \cong B$, $K_i(B)$ is divisible ($i = 0, 1$). Therefore $KL(A, B) = \mathrm{Hom}(K_*(A), K_*(B))$ and there is $\kappa \in KL(C(\mathbb{T}, \tilde{A}_N), \tilde{B})$ such that

$$[\Phi_{u, \varphi \circ \imath}]|_{\mathcal{P}_{N'} \cup \beta(\mathcal{P}_{N'})} = \kappa|_{\mathcal{P}_{N'} \cup \beta(\mathcal{P}_{N'})} \text{ and } [u] = \kappa([z \otimes 1_{\tilde{A}_N}]). \tag{e14.167}$$

Note that $B \cong B \otimes \mathcal{Z}_0$. Define $\psi_{b,w} : B \otimes \mathcal{Z}_0 \to B \otimes \mathcal{W}$ by letting $\psi_{b,w}(b \otimes a) = b \otimes \psi_{z,w}(a)$ for all $b \in B$ and $a \in \mathcal{Z}_0$, where $\varphi_{z,w} : \mathcal{Z}_0 \to \mathcal{W}$ is a homomorphism defined in 8.12. Note also $\mathcal{W} \otimes Q \cong \mathcal{W}$. There is a homomorphism $\psi_{\sigma,\mathcal{W}} : \mathcal{W} \to \mathcal{W}$ such that $d_{t_{\mathcal{W}}}(\psi_{\sigma,\mathcal{W}}(e_W)) = 1 - \sigma_0$ and

$$t_{\mathcal{W}}(\psi_{\sigma,\mathcal{W}}(a)) = (1 - \sigma_0)t_{\mathcal{W}}(a) \text{ for all } a \in \mathcal{W}. \tag{e14.168}$$

Let $\varphi_{w,z}$ be as in 8.12. Note that $t_{\mathcal{W}} = t_Z \circ \varphi_{w,z}$ and $t_Z = t_{\mathcal{W}} \circ \varphi_{z,w}$, where $t_{\mathcal{W}}$ and t_Z are tracial states of $\mathcal{W}$ and $\mathcal{Z}_0$, respectively. Let $\psi_{b,\sigma} : B \to B$ be defined by $\psi_{b,\sigma}(b \otimes a) = b \otimes \varphi_{w,z} \circ \psi_{\sigma,\mathcal{W}} \circ \varphi_{z,w}(a)$ for all $b \in B$ and $a \in \mathcal{Z}_0$. Note that $\psi_{b,\sigma}(B)^{\perp} \neq \{0\}$.

Let $\varphi_\sigma = \psi_{b,\sigma} \circ \varphi$ and $u_\sigma \in \tilde{B}$ satisfy $\alpha(u_\sigma) = \psi_{b,\sigma}(\alpha(u))$. Then, by (e14.168)

$$|\tau \circ \varphi(a) - \tau \circ \varphi_\sigma(a)| \le \sigma_0 |\tau(a)| \text{ for all } a \in A. \tag{e14.169}$$

In particular,

$$\tau \circ \psi_\sigma(h) \ge (1-\sigma_0)\tau(\varphi(h)) \ge (1-\sigma_0)\Delta_1(\hat{h}) \text{ for all } h \in (A_+)^{\mathbf{1}} \setminus \{0\}. \tag{e14.170}$$

Choose two mutually orthogonal non-zero positive elements $e_1, e_2 \in \psi_{b,\sigma}(B)^\perp$. Note that

$$\sum_{i=1}^{2} \tau(e_i) < \sigma_0 \text{ for all } \tau \in T(B). \tag{e14.171}$$

Consider C^*-algebra $C_0 = C(\mathbb{T}, \tilde{A}_N)^o$. By 14.7, $C(\mathbb{T}, \tilde{A})^o$ satisfies the condition in 9.3. It follows from 10.8 that there exists an asymptotic completely positive contractive linear maps $L_n : C_0 \to B \otimes M_{k(n)}$ such that

$$[L_n^{\sim}]|_{\mathcal{P}\cup\boldsymbol{\beta}(\mathcal{P}\cup\{[1_{\tilde{C}}]\})} = \kappa^{\circledast}|_{\mathcal{P}\cup\boldsymbol{\beta}(\mathcal{P}\cup\{[1_{\tilde{C}}]\})}, \tag{e14.172}$$

where $k(n) \to \infty$ and where

$$\kappa^{\circledast}|_{\underline{K}(A_N)} = \kappa|_{\underline{K}(A_N)} \text{ and } \kappa^{\circledast}|_{\boldsymbol{\beta}(\underline{K}(\tilde{A}_N))} = -\kappa|_{\boldsymbol{\beta}}(\underline{K}(\tilde{A_N})). \tag{e14.173}$$

In particular,

$$\kappa^{\circledast}(\boldsymbol{\beta}([1_{\tilde{A}_N}])) = -\kappa(\boldsymbol{\beta}([1_{\tilde{A}_N}])) = -[u]. \tag{e14.174}$$

For each n, there are two sequences of completely positive contractive linear maps $\psi_{0,m} : B \otimes M_{k(n)} \to B_{0,m} \subset B \otimes M_{k(n)}$ and $\psi_{1,m} : B \otimes M_{k(n)} \to D_m \subset B \otimes M_{k(n)}$ such that

$$\lim_{m\to\infty} \|x - (\psi_{0,m}(x) \oplus \psi_{1,m}(x))\| = 0 \text{ for all } x \in B \otimes M_{k(n)}, \tag{e14.175}$$

$$\lim_{m\to\infty} \|\psi_{i,m}(ab) - \psi_{i,m}(a)\psi_{i,m}(b)\| = 0 \text{ for all } a, b \in B \otimes M_{k(n)}, \quad i = 0, 1, \tag{e14.176}$$

$$\lim_{m\to\infty} \sup\{d_\tau(e_{b,m}) : \tau \in T(B)\} = 0, \tag{e14.177}$$

where $D_m \in C_0^0$, $B_{0,m} \perp D_m$, and $e_{b,m} \in B_{0,m}$ is a strictly positive element of $B_{0,m}$. Since $K_i(D_m) = \{0\}$, $i = 0, 1$, by choosing sufficiently large n and m, put $L_n' = \psi_{0,m} \circ L_n$, we may assume that $L_n'^{\sim}$ is $\mathcal{G}$-$\delta/2$-multiplicative (with embedding $\imath : C_0 \to C(\mathbb{T}, \tilde{A})^o$) and

$$[L_n'^{\sim} \circ \imath]|_{\mathcal{P}\cup\boldsymbol{\beta}(\mathcal{P}\cup\{[1_{\tilde{A}_N}]\})} = \kappa^{\circledast}|_{\mathcal{P}\cup\boldsymbol{\beta}(\mathcal{P}\cup\{[1_{\tilde{A}_N}]\})}. \tag{e14.178}$$

Moreover, by (e14.177), we may assume that $e_{b,m} \lesssim e_{0,1}$, where $e_{0,1} \in B$, $e_{0,1}e_1 = e_1 e_{0,1} = e_{0,1}$. Since B has almost stable rank one, there is a unitary $w_1 \in \tilde{B}$ such that $\mathrm{Ad}\, w_1 \circ L_n'(a) \in B_{0,e} = \overline{e_1 B e_1}$ for all $a \in A$. Put $L_n'' = \mathrm{Ad}\, w_1 \circ L_n'^{\sim}$. Let $u_0 \in \widetilde{B_{0,e}}$ such that $u_0 = 1_{\tilde{B}_{0,e}} + \alpha(u_0)$ for some $\alpha(u_0) \in (B_{0,e})_{s.a.}$ and

$$\|L_n''(z \otimes 1_{\tilde{A}_N}) - u_0\| < \delta/16. \tag{e14.179}$$

It follows from (e14.178) and (e14.174) that

$$[u_0] = \kappa^{\circledast}(\boldsymbol{\beta}([1_{\tilde{A}_N}])) = -[u] \in K_1(B) \tag{e14.180}$$

Define $L : A \to B$ by (for some sufficiently large n as specified above)

$$L(a) = L_n''(a) \oplus \psi_{b,\sigma} \circ \varphi(a) \text{ for all } a \in A. \tag{e14.181}$$

It is ready to check that L is $\mathcal{G}_1$-δ-multiplicative. Let $\lambda' : G_u \to U(\tilde{B})/CU(\tilde{B})$ be a homomorphism induced by L. Let $\lambda = \varphi^{\dagger}|_{\overline{\mathcal{U}}} - \lambda'$. Since $\psi_{b,\sigma} \circ \varphi$ factors through $B \otimes \mathcal{W}$, $[\psi_{b,\sigma} \circ \varphi] = 0$. By (e14.178) and (e14.167) and the fact that $\mathcal{P}_1$ contains the set $\{[u] \in K_1(A) \subset \underline{K}(A) : u \in \mathcal{U}\}$, we know that $[L]|_{\{[u], u \in G_u\}} = [\varphi]|_{\{[u], u\in G_u\}}$. Consequently, the map λ maps G_u into $U_0(\tilde{B})/CU(\tilde{B})$. Since $U_0(\tilde{B})/CU(\tilde{B})$ is divisible, we may extend λ to a map from $J_c(K_1(A))$ into $\mathrm{Aff}(T(\tilde{B}))/\mathbb{Z}$. Choose a non-zero element $e_0 \in B$ with $e_0 e_2 = e_2 e_0 = e_0$ such that $d_\tau(e_0)$ is continuous on $T(B)$. Let $\lambda_T : T(\overline{e_0 B e_0}) \to T(A)$ be an affine continuous map defined by $\lambda_T(t) = \tau_A$ for all $t \in T(\overline{e_0 B e_0})$, where τ_A is a fixed trace in $T(A)$. Define $\lambda_{cu} : U(\tilde{A})/CU(\tilde{A}) \to U_0(\widetilde{e_0 B e_0})/CU(\widetilde{e_0 B e_0})$ by $\lambda_{cu}|_{J_c(K_1(\tilde{A}))} = \lambda$ and $\lambda_{cu}|_{U_0(\tilde{A})/CU(\tilde{A})} = \lambda_T^{\sharp}$, i.e., $\lambda_{cu}(f)(t) = f(\lambda_T(t))$ for all $t \in T(\overline{e_0 B e_0})$. Define $\lambda_K : \underline{K}(A) \to \underline{K}(\overline{e_0 B e_0})$ by $\lambda_K = 0$. Then $(\lambda_J, \lambda_{cu}, \lambda_T)$ is compatible. It follows from 12.8 that there exists a homomorphism $\varphi_{cu} : A \to \overline{e_0 B e_0}$ such that $([\varphi_{cu}], \varphi_{cu}^{\dagger}, (\varphi_{cu})_T) = (\lambda_K, \lambda_{cu}, \lambda_T)$. (Note that $\varphi_{c,u} \perp L$, since $e_1, e_2 \in \psi_{b,\sigma}(B)^\perp$ and $e_1 e_2 = 0$.)

Now define $\Phi : A \to B$ by $\Phi(a) = \varphi_{cu}(a) \oplus L(a)$ for $a \in A$. Then Φ is $\mathcal{G}_1$-δ-multiplicative,

$$\tau \circ \Phi(h) \ge \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1, \qquad \text{(by (e14.170))} \tag{e14.182}$$

$$\|\tau\circ\Phi(h)-\tau\circ\varphi(h)\|<\gamma \ \text{ for all } \ h\in\mathcal{H}_2, \tag{e14.183}$$
$$[\Phi]|_{\mathcal{P}}=[\varphi]|_{\mathcal{P}} \ \text{ and} \tag{e14.184}$$
$$\Phi^{\dagger}(\bar{w})=\lambda(\bar{w})+\lambda'(\bar{w})=\varphi^{\dagger}(\bar{w}) \ \text{ for all } \ w\in\mathcal{U}. \tag{e14.185}$$

Let $v'=1_{\tilde{B}}+\alpha(u_0)+\psi_{b,\sigma}(\alpha(u))$. By (e14.179), (e14.181), and $[\psi_{b,\sigma}\circ\varphi]=0$, We have

$$\mathrm{Bott}(\Phi,v')|_{\mathcal{P}}=\mathrm{Bott}(L_n''|_A,u_0)|_{\mathcal{P}}=[L_n'']\circ\beta|_{\mathcal{P}} \tag{e14.186}$$

Combining with (e14.178) and (e14.173), one obtains

$$\mathrm{Bott}(\Phi,v')|_{\mathcal{P}}=\kappa^{\circledast}\circ\beta|\mathcal{P}=-\kappa\circ\beta|_{\mathcal{P}}. \tag{e14.187}$$

By (e14.182), Φ is also T-$\mathcal{H}_1$-full. By applying 5.3, we obtain a unitary $W\in\tilde{B}$ such that

$$\|W^*\Phi(f)W-\varphi(f)\|<\varepsilon_1 \ \text{ for all } \ f\in\mathcal{F}\cup\mathcal{G}_0. \tag{e14.188}$$

Let $v=W^*(1_{\tilde{B}}+\alpha(u_0)+\psi_{b,\sigma}(\alpha(u))W$. Then v is a unitary. It follows from (e14.180) that

$$[v]=-[u] \tag{e14.189}$$

We have

$$\|[\varphi(f),v]\|<\varepsilon_1+\delta \ \text{ for all } \ f\in\mathcal{F}\cup\mathcal{G}_0. \tag{e14.190}$$

Note that $\mathrm{Bott}(\varphi,v)=\mathrm{Bott}(\Phi,v')$. Recall that from (e14.167), $\mathrm{Bott}(\varphi,u)|_{\mathcal{P}}=\kappa\circ\beta|_{\mathcal{P}}$. By (e14.187) and (e14.189), we then compute that

$$\mathrm{Bott}(\varphi,uv)|_{\mathcal{P}}=\mathrm{Bott}(\varphi,u)|_{\mathcal{P}}+\mathrm{Bott}(\varphi,v)|_{\mathcal{P}}=0 \ \text{ and } \ [uv]=0. \quad\Box \tag{e14.191}$$

Remark 14.11. Lemma 14.10 still holds by replacing Q by any UHF-algebra of infinite type if $K_i(A)$ is finitely generated. It should be noted that δ_0 and $\mathcal{G}_0$ are independent of ε and $\mathcal{F}$.

Lemma 14.12. *Let $A\in\mathcal{B}_T$ have continuous scale. For any $\varepsilon>0$ and any finite subset $\mathcal{F}\subset A$, there exist $\delta>0$ and a finite subset $\mathcal{G}\subset A$ satisfying the following: Suppose that $\varphi: A\to B\cong B\otimes\mathcal{W}$, where $B\in\mathcal{D}_0$ with continuous scale, is a homomorphism which maps strictly positive elements to strictly positive elements and suppose that there is a unitary $u\in\tilde{B}$ such that*

$$\|[\varphi(g),\ u]\|<\delta \ \text{ for all } \ g\in\mathcal{G}. \tag{e14.192}$$

Then there exists a continuous path of unitaries $\{u(t): t\in[0,1]\}\subset\tilde{B}$ such that $u(0)=u$, $u(1)=1_{\tilde{B}}$ and

$$\|[\varphi(f),\ u(t)]\|<\varepsilon \ \text{ for all } \ f\in\mathcal{F}. \tag{e14.193}$$

Proof. Note that, by 13.4, $A\cong A\otimes\mathcal{Z}_0$. We identify A with $A\otimes\mathcal{Z}_0$. Let $\varphi_{w,z}:\mathcal{W}\to\mathcal{Z}_0$ be defined in 8.12. Let $\psi_{w,a}: A\otimes\mathcal{W}\to A\otimes\mathcal{Z}_0$ defined by $\psi_{w,a}(a\otimes w)=a\otimes\varphi_{w,z}$ for all $a\in A$ and $w\in\mathcal{W}$. Put $A_1=A\otimes\mathcal{W}$. Fix $\varepsilon>0$ and a finite subset $\mathcal{F}\subset A$.

Note $T(A)=T(A\otimes\mathcal{W})$ and $\rho_{\tilde{A}}(K_0(\widetilde{A\otimes\mathcal{W}}))=\mathbb{Z}$. It follows from 12.8 that there exists a homomorphism $h_{a,w}: A\to A\otimes\mathcal{W}$ such that $(h_{a,w})_T=\mathrm{id}_{T(A)}$ and $h^{\dagger}_{a,w}|_{J_c(K_1(A))}=\bar{1}$ and $h^{\dagger}_{a,w}|_{\mathrm{Aff}(T(\tilde{A}))/\mathbb{Z}}=\mathrm{id}_{\mathrm{Aff}(T(\tilde{A}))/Z}$.

Let $\mathcal{F}_1=h_{a,w}(\mathcal{F})$. Choose $\mathcal{G}_w\in A\otimes\mathcal{W}$ and $\delta_W>0$ which are required by 14.9 for $A\otimes\mathcal{W}\in\mathcal{M}_0$, $\mathcal{F}_1$ and $\varepsilon/16$.

Suppose that $\psi: A\to B$ is a homomorphism which maps strictly positive elements to strictly positive elements and suppose that there is a unitary $v\in\tilde{B}$ such that

$$\|[\psi(g),\ v]\|<\delta_W/2 \ \text{ for all } \ g\in\psi_{w,a}(\mathcal{G}_w) \tag{e14.194}$$

and suppose that $\psi^{\dagger}$ maps $J_c(K_1(A))$ to $\bar{1}$.

Consider homomorphism $\psi': A\to B$ defined by $\psi'=\psi\circ\psi_{w,a}\circ h_{a,w}$. Note that $[\psi']=[\psi]$ in $KL(A,B)$ (since $B\cong B\otimes\mathcal{W}$) and $\tau\circ\psi'=\tau\circ\psi$ for all $\tau\in T(B)$ and $\psi^{\dagger}=(\psi')^{\dagger}$ (Note that $(\psi')^{\dagger}$ maps $J_c(K_1(A))$ to $\bar{1}$). Therefore, by 5.3 (and 5.6), there is a unitary $V\in\tilde{B}$ such that

$$\|V^*\psi'(g)V-\psi(g)\|<\min\{\delta_W/2,\varepsilon/16\} \ \text{ for all } \ g\in\psi_{w,a}(\mathcal{G}_w)\cup\mathcal{F}. \tag{e14.195}$$

Define $\psi_W: A\otimes\mathcal{W}\to B$ by $\psi_W=\mathrm{Ad}\,V\circ\psi\circ\psi_{w,a}$. Then

$$\|[\psi_W(g),\ v]\|<\delta \ \text{ for all } \ g\in\mathcal{G}_W. \tag{e14.196}$$

Note $A\otimes\mathcal{W}\in\mathcal{M}_0$. It follows from 14.9 that there exists a continuous path of unitaries $\{v(t): t\in[0,1]\}\subset U(\tilde{B})$ with $v(0)=u$ and $u(1)=1_{\tilde{B}}$ such that

$$\|[\psi_W(g),\ v(t)]\|<\varepsilon/16 \ \text{ for all } \ g\in\mathcal{F}_1. \tag{e14.197}$$

Therefore,

$$\|[\mathrm{Ad}\, V\circ \psi'(f),\, v(t)]\| < \varepsilon/16 \ \text{ for all } \ f\in \mathcal{F}. \tag{e14.198}$$

It follows from this and (e14.195) that

$$\|[\psi(f),\ v(t)]\| < \varepsilon/8 \ \text{ for all } \ f\in \mathcal{F}. \tag{e14.199}$$

Now we consider the general case that $\psi^{\dagger}(J_c(K_1(A)))\neq \bar{1}$. Let

$$\Delta_A(\hat{a}) = \inf\{\tau(a) : \tau\in T(A)\} \ \text{ for all } \ a\in A_+^{\mathbf{1}}\setminus\{0\} \ \text{ and} \tag{e14.200}$$

$$\Delta_0(\hat{c}) = \inf\{(\int f dm)\Delta_A(a) : c\geq f\otimes a,\ \ f\in C(\mathbb{T}),\ \ a\in A\} \tag{e14.201}$$

for all $c\in C(\mathbb{T},\tilde{A})^o$, where m is the normalized Haar measure on $\mathbb{T}$. Put $\Delta=\Delta_0/2$.

Put $A_c = C(\mathbb{T},\tilde{A})^o$. Let $\mathcal{H}_1\subset (\tilde{A_c})_+^{\mathbf{1}}\setminus\{0\}$ be a finite subset, $\gamma_1>0$, $\gamma_2>0$, $\delta_c>0$, $\mathcal{G}_1\subset \tilde{A_c}$ (in place of $\mathcal{G}$) and $\mathcal{P}\subset \underline{K}(\tilde{A_c})$, $\mathcal{H}_2\subset A_c$ and $\mathcal{U}\subset J_c(K_1(\tilde{A_c}))$ be finite subsets with $[\mathcal{U}]\subset \mathcal{P}$ be required by 14.4 for $\min\{\delta_W/4, \varepsilon/16\}$ (in place of ε) and $\psi_{a,w}(\mathcal{G}_W)$ (in place of $\mathcal{F}$) and Δ. With smaller $\delta_c>0$, γ_i, without loss of generality, we may assume that $\mathcal{H}_1=\{g\otimes 1_{\tilde{A}} : g\in \mathcal{H}_{1,T}\}\cup\{1_{C(\mathbb{T})}\otimes a : a\in \mathcal{H}_{1,A}\}$, and $\mathcal{G}_1=\{g\otimes 1_{\tilde{A}} : g\in \mathcal{G}_{1,T}\}\cup\{1_{C(\mathbb{T})}\otimes a : a\in \mathcal{G}_{1,A}\}$, $\mathcal{H}_2=\{g\otimes 1_{\tilde{A}} : g\in \mathcal{H}_{2,T}\}\cup\{1_{C(\mathbb{T})}\otimes a : a\in \mathcal{H}_{2,A}\}$, where $\mathcal{H}_{1,T}, \mathcal{H}_{2,T}, \mathcal{G}_{1,T}\subset C(\mathbb{T})$, and $\mathcal{H}_{1,A}, \mathcal{H}_{2,A}, \mathcal{G}_{1,A}\subset \tilde{A}$ are finite subsets.

Let $\mathcal{G}'=\mathcal{G}_{1,A}\cup\mathcal{F}$ and $\delta'=\min\{\delta_c/2, \delta_W/2, \varepsilon/16\}$. Let $0<\delta<\delta'$ and $\mathcal{G}\supset\mathcal{G}'$ be finite subset such that any $\mathcal{G}$-δ-multiplicative completely positive contractive linear map L' from A to a C^*-algebra C and any unitary $u'\in\tilde{C}$ with property $\|[L'(g),\ u']\|<2\delta$ for all $g\in\mathcal{G}$ gives a $\mathcal{G}_1$-δ-multiplicative completely positive contractive linear map from $C(\mathbb{T},\tilde{A})$ to $\tilde{C}$.

Suppose that $\varphi : A\to B$ is a homomorphism which maps strictly positive elements to strictly positive elements and $u\in\tilde{B}$ such that

$$\|[\varphi(g),\ u]\| < \delta \ \text{ for all } \ g\in\mathcal{G}. \tag{e14.202}$$

Note that $B\otimes Q\otimes Q\cong B$. We may assume that $\varphi(A)\subset B\otimes 1_Q\otimes 1_Q$ and $u\in\tilde{B}\otimes 1_Q\otimes 1_Q$. Let $\{e_n\}$ be an approximate identity for A. Consider $v_n=u(\exp(ie_n\otimes h))$, where $h\in Q\otimes 1_Q$ with $\mathrm{sp}(h)=[-\pi,\pi]$ and $t_Q(h)=0$ and where t_Q is the tracial state of Q. Let $p_k, q_{1,k}, q_{2,k}\in 1_Q\otimes Q$ be mutually orthogonal projections with $t_Q(p_k)=1-1/k$, $t_Q(q_{i,k})=1/2k, i=1,2,$ and $p_k\oplus q_{1,k}\oplus q_{2,k}=1_Q\otimes 1_Q$, $k=1,2,\ldots$. Put $B_k=B\otimes p_k$, $B_{i,k}=B\otimes q_{i,k}$, $i=1,2$, $k=1,2,\ldots$. By 12.8, there are homomorphisms $\Psi_{i,k} : A\to B_{i,k}$ such that $\tau(\Psi_{i,k}(a))=(1/2k)\tau(\varphi(a))$ for all $a\in A$ and

$$\Psi_{1,k}^{\dagger}|_{J_c(K_1(A))} = -(1-\frac{1}{k})\varphi^{\dagger}|_{J_c(K_1(A))} \ \text{ and } \ \Psi_{2,k}^{\dagger}|_{J_c(K_1(A))} = (1-\frac{1}{k})\varphi^{\dagger}|_{J_c(K_1(A))}, \tag{e14.203}$$

$k=1,2,\ldots$. Define $\psi'_{n,k} : A\to C_k := B_k\oplus B_{1,k}$ by $\psi'_{n,k}(a)=\varphi(a)\otimes p_k\oplus\Psi_{1,k}(a)$ for all $a\in A$, and define $\psi_{n,k} : A\to B\otimes 1_Q\otimes 1_Q$ by $\psi_{n,k}(a)=\psi'_{n,k}(a)\oplus\Psi_{2,k}(a)$ for all $a\in A$, $k=1,2,\ldots$. Write $v_n=\lambda+\alpha(v_n)$ for some $\lambda\in\mathbb{T}$ and $\alpha(v_n)\in B\otimes 1_Q\otimes 1_Q$. Let $v_{n,k}=\lambda\cdot 1_{\tilde{C}_k}+\alpha(v_n)(p_k\oplus q_{1,k})$ and $w_{n,k}=\lambda\cdot 1_{\tilde{B}}+\alpha(v_n)(p_k\oplus q_{1,k})$. Choose a completely positive contractive linear map $L_{n,k}=\Phi_{w_{n,k},\psi_{n,k}} : C(\mathbb{T},\tilde{A})^o\to B\otimes Q\otimes Q$ induced by the unitary $w_{n,k}$ and $\psi_{n,k}$. Let $\Phi_{v_n,\varphi} : C(\mathbb{T},\tilde{A})^o\to B\otimes Q\otimes Q$ be the one induced by v_n and φ.

Note that $U(\tilde{B})/CU(\tilde{B})=\mathrm{Aff}(T(\tilde{B}))/\mathbb{Z}$. By applying 14.5, for all sufficiently large n and k (we then fix a pair n and k)

$$\tau(L_{n,k}(h))\geq\Delta_0(\hat{h})/2=\Delta(\hat{h}) \ \text{ for all } \ \tau\in T(B) \ \text{ and } \ \text{ for all } \ h\in\mathcal{H}_1, \tag{e14.204}$$

$$|\tau(L_{n,k}(h))-\tau(\Phi_{v_n,\varphi})(h)| < \gamma_1 \ \text{ for all } \ h\in\mathcal{H}_2 \ \text{ and} \tag{e14.205}$$

$$\mathrm{dist}(L_{n,k}^{\dagger}(\bar{w}), \Phi_{v_n,\varphi}^{\dagger}(\bar{w})) < \gamma_2 \ \text{ for all } \ w\in\mathcal{U}. \tag{e14.206}$$

It follows from 14.4 that there exists a unitary $U\in B\otimes\widetilde{Q\otimes Q}$ such that

$$\|U^*\psi_{n,k}(g)U-\varphi(g)\| < \min\{\delta_W/4, \varepsilon/16\} \ \text{ for all } \ g\in\psi_{w,a}(\mathcal{G}_W) \ \text{ and} \tag{e14.207}$$

$$\|U^*w_{n,k}U-v_n\| < \min\{\delta_W/4, \varepsilon/16\}. \tag{e14.208}$$

Now consider $\mathrm{Ad}\,U\circ\psi'_{n,k} : A\to D_k := U^*C_kU$ and the unitary $U^*v_{n,k}U\in\tilde{D}_k$. Note, by (e14.203), $(\mathrm{Ad}\,U\circ\psi'_{n,k})^{\dagger}|_{J_c(K_1(A))}=\bar{1}$. So we reduce this case to the case that has been proved. Thus there is a continuous path of unitaries $\{V(t) : t\in[2/3,1]\}\subset\tilde{D}_k$ such that $V(2/3)=U^*v_{n,k}U$ and $V(1)=1_{\tilde{D}_k}$ and

$$\|[\mathrm{Ad}\,U\circ\psi'_{n,k}(f),\ V(t)]\| < \varepsilon/8 \ \text{ for all } \ f\in\mathcal{F}. \tag{e14.209}$$

Note that $U^*w_{n,k}U=\lambda\circ 1_{\tilde{B}}+U^*\alpha(v_{n,k})U$. Write $V(t)=\lambda(t)\cdot 1_{\tilde{D}_k}+\alpha(V(t))$ for some $\lambda(t)\in\mathbb{T}$ and $\alpha(V(t))\in D_k$. Put $Z(t)=\lambda(t)\cdot 1_{\tilde{B}}+\alpha(V(t))$. Then $Z(2/3)=U^*w_{n,k}U$ and $Z(1)=1_{\tilde{B}}$. Since $B_{2,k}\perp C_k$, we have that

$$\|[\mathrm{Ad}\,U\circ\psi_{n,k}(g),\ Z(t)]\| < \varepsilon/8 \ \text{ for all } \ f\in\mathcal{F}. \tag{e14.210}$$

By (e14.208), we may write $v_n^*U^*w_{n,k}U=\exp(ib)$ for some $b\in\tilde{B}_{s.a.}$ with $\|b\|\leq 2\arcsin(\varepsilon/32)$. Define $Z(t)=v_n\exp(\sqrt{-1}(3(t-1/3)b))$ for $t\in[1/3,2/3)$. Then $Z(1/3)=v_n$. We also have

$$\|[\mathrm{Ad}\,U\circ\psi_{n,k}(g),\ Z(t)]\| < \varepsilon/8 \ \text{ for all } \ t\in[1/3,1]. \tag{e14.211}$$

It follows that

$$\|[\varphi(g),\ Z(t)]\| < \varepsilon/8 + \varepsilon/16 \ \text{ for all } \ t \in [1/3, 1]. \tag{e14.212}$$

Define $Z(t) = u(\exp(3\sqrt{-1}te_n \otimes h))$ for $t \in [0, 1/3)$. Then $Z(0) = u$ and $\{Z(t) : t \in [0, 1]\}$ is a continuous path of unitaries in $\tilde{B}$. Moreover,

$$\|[\varphi(g),\ Z(t)]\| < \varepsilon \ \text{ for all } \ g \in \mathcal{F} \ \text{ and } \ t \in [0, 1]. \quad \square \tag{e14.213}$$

Theorem 14.13. *Let $A \in \mathcal{B}_T$ have continuous scale. Let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset, let $\{p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k\}$ be projections of $M_s(\tilde{A})$ (for some integer $s \geq 1$) such that $\{[p_1]-[q_1], [p_2]-[q_2], \ldots, [p_k]-[q_k]\} \subset \mathcal{P}$ generates a free subgroup G_{u0} of $K_0(A)$, let $\sigma > 0$, $\varepsilon_0 > 0$ and $\mathcal{F}_0 \subset A$ be a finite subset. There exist $\delta_0 > 0$ and $\mathcal{G}_0 \subset A$ such that the following hold: For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any homomorphism $\varphi : A \to B = B_1 \otimes Q$ which maps strictly positive elements to strictly positive elements, where $B_1 \cong B_1 \otimes \mathcal{Z}_0 \in \mathcal{D}_0$ has continuous scale, and any unitary $u \in U(\tilde{B})$ such that*

$$\|[\varphi(g),\ u]\| < \delta_0 \ \text{ for all } \ g \in \mathcal{G}_0, \tag{e14.214}$$

there exists a continuous path of unitaries $\{v(t) : t \in [0, 1]\} \subset U(\tilde{B})$ such that

$$\|[\varphi(g),\ v(0)]\| < \varepsilon \ \text{ for all } \ g \in \mathcal{G}_0 \cup \mathcal{F}, \tag{e14.215}$$

$$\|[\varphi(f),\ v(t)]\| < \varepsilon_0 \ \text{ for all } \ f \in \mathcal{F}_0, \tag{e14.216}$$

$$\mathrm{Bott}(\varphi,\ uv(1))|_{\mathcal{P}} = 0, \ \ [uv(1)] = 0 \ \text{ and} \tag{e14.217}$$

$$\mathrm{dist}(\overline{\lceil((1_s - \varphi(p_i)) + (uv(1))_{\underline{s}}\varphi(p_i))((1_s - \varphi(q_i)) + (uv(1))_{\underline{s}}^*\varphi(q_i))\rceil},\ \bar{1}) < \sigma, \tag{e14.218}$$

where $1_s = 1_{M_s}$ and $(uv(1))_{\underline{s}} = uv(1) \otimes 1_{M_s}$.

Proof. Define $\Delta_1(\hat{h}) = \inf\{\tau(h) : \tau \in T(A)\}$ for $h \in A_+^{\mathbf{1}} \setminus \{0\}$. Let $\Delta = \Delta_1/2$. Let $T : A_+^{\mathbf{1}} \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$ be the map given by Δ as in 14.1. Let ε_0, σ, $\mathcal{F}_0$, $\mathcal{P}$ and $\{p_1, \ldots, p_k, q_1, q_2, \ldots, q_k\} \subset M_s(\tilde{A})$ be given. In what follows, if v' is a unitary, $v'_{\underline{s}} = v' \otimes 1_{M_s}$.

Write $p_l = (a_{i,j}^{p_l})_{s\times s}$ and $q_l = (a_{i,j}^{q_l})_{s\times s}$, where $a_{i,j}^{p_l}, a_{i,j}^{q_l} \in \tilde{A}$, $1 \leq i,j \leq s$, $1 \leq l \leq k$. Let $\mathcal{F}_p$ be a finite subset in A such that $a_{i,j}^{p_l}, a_{i,j}^{q_l} \in \mathbb{C}\cdot 1 + \mathcal{F}_p$.

In what follows, if $L' : A \to C'$ is a map, we will continue to use L' for $L'^{\sim} : \tilde{A} \to \tilde{C}'$ and $L' \otimes \mathrm{id}_{M_s}$ as well as $L'^{\sim} \otimes \mathrm{id}_{M_s}$ when it is convenient. Moreover, $1_s := 1_{M_s}$.

Let $\delta_0' > 0$ and let $\mathcal{G}_0' \subset A$ be a finite subset satisfying the following: $\mathrm{Bott}(L,\ w)|_{\mathcal{P}}$ is well defined for any $\mathcal{G}_0'$-δ_0'-multiplicative completely positive contractive linear map $L : A \to C$ and any unitary $w \in \tilde{C}$ with $\|[L(g),\ w]\| < 2\delta_0'$ for all $g \in \mathcal{G}_0$. Also, if w' is another unitary, we also require that

$$\mathrm{Bott}(L,\ ww')|_{\mathcal{P}} = \mathrm{Bott}(L,\ w)|_{\mathcal{P}} + \mathrm{Bott}(\varphi,\ w')|_{\mathcal{P}}, \tag{e14.219}$$

when $\|[L(g),\ w']\| < \delta_0'$ for all $g \in \mathcal{G}_0'$. Moreover, for any $\mathcal{G}_0'$-δ_0'-multiplicative completely positive contractive linear map L' from A to a non-unital C^*-algebra C' induces a homomorphism $\lambda' : G_u \to U(\tilde{C})/CU(\tilde{C})$ (see 14.5 of [38]). Furthermore, using 14.5 of [38] again, we assume that, for any unitary $w' \in M_s(\tilde{C})$ with the property that $\|[L'(g),\ w']\| < 2\delta_0'$ for all $g \in \mathcal{G}_0'$, $\Phi_{w',L'}$ induces a homomorphism $\lambda_{L',w'}$ from G_{u0} to $U(\tilde{C})/CU(\tilde{C})$ and, for $1 \leq i \leq k$,

$$\mathrm{dist}(\overline{\lceil(1_s - L'(p_i) + w_{\underline{s}}'L'(p_i))(1_s - L'(q_i) + (w_{\underline{s}}')^*L'(q_i))\rceil},\ \lambda_{L',w'}([p_i] - [q_i])) < \sigma/16, \tag{e14.220}$$

where $w_{\underline{s}}' = w' \otimes 1_s$. We may assume that δ_0' is smaller than δ_0 in 14.10 and $\mathcal{G}_0'$ is larger than $\mathcal{G}_0$ in 14.10 for the above $\mathcal{P}$.

Let $\delta_W > 0$ and let $\mathcal{G}_W \subset A$ be a finite subset required by 14.12 for $\min\{\varepsilon_0/4, \delta_0'/2\}$ (in place of ε) and $\mathcal{F}_0 \cup \mathcal{G}_0'$. Put $\delta_0'' = \min\{\delta_0'/4, d_W/4\}$ and $\mathcal{G}_0'' = \mathcal{G}_0' \cup \mathcal{G}_W \cup \mathcal{F}_0 \cup \mathcal{F}_p$.

Let $\varepsilon_1 = \min\{\delta_0''/4, \varepsilon_0/16, \sigma/16\}/2^{10}(s+1)^2$. Let $\delta_1 > 0$ (in place of δ), $\gamma > 0$, $\eta > 0$, $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_1 \subset \underline{K}(A)$ (in place $\mathcal{P}$) be a finite subset, $\mathcal{U} \subset U(\tilde{A})$ be a finite subset, $\mathcal{H}_1 \subset A_+ \setminus \{0\}$ be a finite subset, and $\mathcal{H}_2 \subset A_{s.a.}$ be a finite subset required by 5.3 for ε_1 (in place of ε) and $\mathcal{G}_0''$ (in place of $\mathcal{F}$) the above T (and $\mathbf{T}(n, k) = n$).

We assume that $\delta_1 < \delta_0''$ and that $\mathcal{G}_1 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \subset (A)_+^{\mathbf{1}}$. Write $w = \lambda_w + \alpha(w)$, where $\lambda_w \in \mathbb{T} \subset \mathbb{C}$ and $\alpha(w) \in A$. As in the remark of 5.3, we may assume that $[w] \neq 0$ and $[w] \in \mathcal{P}$ for all $w \in \mathcal{U}$. Let G_u be the subgroup generated by $\{\overline{w} : w \in \mathcal{U}\}$. We may view $G_u \subset J_c(K_1(A))$ (see the statement of 14.2).

Note that $B \cong B \otimes \mathcal{Z}_0$. Define $\psi_{b,W} : B \otimes \mathcal{Z}_0 \to B \otimes \mathcal{W}$ by letting $\psi_{b,W}(b \otimes a) = b \otimes \varphi_{z,w}(a)$ for all $b \in B$ and $a \in \mathcal{Z}_0$, where $\varphi_{z,w} : \mathcal{Z}_0 \to \mathcal{W}$ is a homomorphism defined in 8.12. Note that, by 6.8 of [16], $B \otimes \mathcal{W}$ is in $\mathcal{M}_0$ with continuous scale.

Set $\mathcal{G}_2 = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \{\alpha(w) : w \in \mathcal{U}\}$ and set a rational number

$$0 < \sigma_0 < \min\{\inf\{\Delta(\hat{h}) : h \in \mathcal{H}_1\}, \gamma/4\}.$$

Let $\{e_n\}$ be an approximate identity for A such that $e_{n+1}e_n = e_n$ and $(e_nAe_n)^{\perp} \neq \{0\}$ for all n. Define $\psi_n(a) = \varphi(e_nae_n)$ for all $a \in A$. Then $\lim_{n\to\infty} \|\psi_n(a) - \varphi(a)\| = 0$ for all $a \in A$. Choose a sufficiently large n such that $\psi_n|_{\mathcal{P}} = \varphi|_{\mathcal{P}}$. Therefore, without loss of generality, we may assume that there are $e_A, e_A' \in A_+$ with $\|e_A'\| = \|e_A\| = 1$ such that, with $\psi_A(a) = \varphi(e_A a e_A)$

$$e_A g = g e_A = g \text{ for all } g \in \mathcal{G}_2, \ \ e_A' e_A = e_A, (\overline{e_A' A e_A'})^{\perp} \neq \{0\} \text{ and } \varphi|_{\mathcal{P}} = \psi_A|_{\mathcal{P}}. \tag{e14.221}$$

Choose a pair of mutually orthogonal non-zero positive elements $e_0, e_0' \in (\overline{e_A' A e_A'})^{\perp}$ such that

$$d_\tau(e_0 + e_0') < \sigma_0 \text{ for all } \tau \in T(A). \tag{e14.222}$$

Choose an integer $K \geq 1$ such that

$$1/K < \min\{\sigma_0/4, \inf\{d_\tau(e_0) : \tau \in T(A)\}\} \tag{e14.223}$$

and choose $\delta_0 = \min\{\varepsilon_1/16, \delta_1/16, \gamma/16, \eta/16\}/64(s+1)^3(K+1)^2$. Put $\mathcal{G}_0 = \mathcal{G}_2 \cup \{e_A, e_A', e_0, e_0'\}$.

Now let φ and u be given satisfying the assumption for the above $\mathcal{G}_0$ and δ_0. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. We may write $u = 1_{\tilde{B}} + \alpha(u)$, where $\alpha(u) \in B$. Put $\mathcal{Q} = \mathcal{P} \cup \boldsymbol{\beta}(\mathcal{P})$.

Note also $\mathcal{W} \otimes Q \cong \mathcal{W}$. Let $e_q \in Q$ be a projection with $t_U(e_q) = 1/K$, where t_Q is the tracial state of Q. Define $\psi_{1/K,W} : \mathcal{W} \to \mathcal{W} \otimes Q$ by $\psi_{1/K,W}(a) = a \otimes e_q$ for all $a \in \mathcal{W}$. Then

$$t_{\mathcal{W}}(\psi_{1/K,W}(a)) = (1/K) t_{\mathcal{W}}(a) \text{ for all } a \in \mathcal{W}. \tag{e14.224}$$

Let $\varphi_{w,z}$ be as in 8.12. Note that $t_{\mathcal{W}} = t_Z \circ \varphi_{w,z}$ and $t_Z = t_{\mathcal{W}} \circ \varphi_{z,w}$, where $t_{\mathcal{W}}$ and t_Z are tracial states of $\mathcal{W}$ and $\mathcal{Z}_0$, respectively. Let $\psi_{b,1/K} : B \to B$ be defined by $\psi_{b,1/K}(b \otimes a) = b \otimes \varphi_{w,z} \circ \psi_{1/K,W} \circ \varphi_{z,w}(a)$ for all $b \in B$ and $a \in \mathcal{Z}_0$. Let $\psi_{b,w,1/K} : B \to B \otimes \mathcal{W} \otimes e_q$ be defined by $\psi_{b,w,1/K}(b \otimes a) = b \otimes \psi_{1/K,W} \circ \varphi_{z,w}(a)$ for all $b \in B$ and $a \in \mathcal{Z}_0$.

By applying 14.10, there is a unitary $v_1 \in \tilde{B}$ such that

$$\|[\varphi(g),\ v_1]\| < \min\{\delta_0, \varepsilon\} \text{ for all } g \in \mathcal{F} \cup \mathcal{G}_0 \text{ and} \tag{e14.225}$$

$$\mathrm{Bott}(\varphi,\ uv_1)|_{\mathcal{P}} = 0 \text{ and } [uv_1] = 0. \tag{e14.226}$$

Note that

$$\|[\varphi(g),\ uv_1]\| < \delta_0 + \min\{\delta_0, \varepsilon\} \text{ for all } g \in \mathcal{G}_0. \tag{e14.227}$$

We may write $uv_1 = 1_{\tilde{B}} + \alpha(uv_1)$ for some $\alpha(uv_1) \in B$. Define $\psi' : A \to B$ by $\psi'(a) = \psi_{b,1/K} \circ \varphi(a)$ for all $a \in A$. Using (e14.223), by replacing ψ' by $\mathrm{Ad}\, w_1 \circ \psi'$ for some unitary w_1, we may assume that $\psi'(A) \subset B_0 := \overline{e_{0,b} B e_{0,b}}$, where $e_{0,b} = \varphi(e_0)$. Let $v_2' = 1_{\tilde{B}} + \psi_{b,1/K}(\alpha(uv_1))$, $v_2 = ((v_2')^*)^K$ and $v_2'' = 1_{\tilde{B}_0} + \psi_{b,1/K}(\alpha(uv_1))$. Note that $[\psi']|_{\mathcal{P}} = 0$, since it factors through $B \otimes \mathcal{W}$. Moreover

$$\mathrm{Bott}(\psi',\ v_2'')|_{\mathcal{P}} = 0 \text{ and } \mathrm{Bott}(\psi', (v_2'')^K)|_{\mathcal{P}} = 0. \tag{e14.228}$$

Let $\lambda_{\varphi,uv_1} : G_{u0} \to U(M_s(\tilde{B}))/CU(M_s(\tilde{B}))$ be the homomorphism induced by φ and uv_1, via a map $\Phi_{uv_1,\varphi}$. Then (e14.226) implies that λ_{φ,uv_1} maps G_{u0} to $\mathrm{Aff}(T(\tilde{B}))/\mathbb{Z}$ (see also [21]). Let $\lambda_{\psi',v_2'} : G_{u0} \to \mathrm{Aff}(T(\tilde{B}))/\mathbb{Z}$ be the homomorphism induced by $\Phi_{v_2',\psi'}$. Since $\tau \circ \psi_{b,1/K}(b) = (1/K)\tau(b)$ for all $b \in B$ and for all $\tau \in T(B)$, it is straightforward that we may write

$$\lambda_{\psi',v_2'}([p_i] - [q_i]) = (1/K)\lambda_{\varphi,uv_1}([p_i] - [q_i]), \tag{e14.229}$$

$i = 1, 2, \ldots, k$. It follows that, by the choice of δ_1 and δ_2, since $v_2 = ((v_2')^*)^K$,

$$\mathrm{dist}(\zeta_i',\ -(\lambda_{\varphi,uv_1}([p_i] - [q_i]))) < \sigma/16, \tag{e14.230}$$

where $\zeta_i' = \overline{\lceil((1_s - \psi'(p_i)) + (v_2)_{\underline{s}}\psi'(p_i))((1 - \psi'(q_i) + (v_2^*)_{\underline{s}}\psi'(q_i)))\rceil}$, $i = 1, 2, \ldots, k$. As in the proof of 14.10, by applying 12.8, we obtain a homomorphism, $\psi_{cu} : A \to \overline{e_{b,0}' B e_{b,0}'}$, where $e_{b,0}' = \varphi(e_0')$, such that

$$[\psi_{cu}] = 0 \text{ in } KL(A, B) \text{ and } \psi_{cu}^{\dagger} = -(\psi')^{\dagger}. \tag{e14.231}$$

Define $\psi : A \to B$ by $\psi(a) = \psi_{cu}(a) \oplus \psi'(a) \oplus \varphi(e_A a e_A)$ for all $a \in A$. Then ψ is $\mathcal{G}_2$-$2\delta_2$-multiplicative (see the last part of (e14.221)),

$$\tau \circ \psi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1, \tag{e14.232}$$

$$|\tau \circ \psi(h) - \tau \circ \varphi(h)| < \gamma \text{ for all } h \in \mathcal{H}_2, \tag{e14.233}$$

$$[\psi]|_{\mathcal{P}} = [\varphi]|_{\mathcal{P}} \text{ and} \tag{e14.234}$$

$$\psi^{\dagger}(\bar{w}) = -(\psi')^{\dagger}(\bar{w}) + ((\psi')^{\dagger}(\bar{w}) + \varphi^{\dagger}(\bar{w})) = \varphi^{\dagger}(\bar{w}) \text{ for all } w \in \mathcal{U}. \tag{e14.235}$$

By (e14.232), ψ is T-$\mathcal{H}_1$-full. By applying 5.3 (as $K_0(\tilde{B})$ is weakly unperforated), we obtain a unitary $U \in \tilde{B}$ such that

$$\|U^*\psi(f)U - \varphi(f)\| < \varepsilon_1 \text{ for all } f \in \mathcal{G}_0'. \tag{e14.236}$$

Let $v = v_1 U^*(v_2)U$. Then v is a unitary. We have

$$\|[\varphi(f), v]\| < 2\varepsilon_1 + (K+1)\delta_0 \text{ for all } f \in \mathcal{G}_0'. \tag{e14.237}$$

We then compute that, by (e14.226), (e14.236) and (e14.228), and by the fact that $\varphi(e_A)v_2 = v_2\varphi(e_A) = \varphi(e_A)$,

$$\begin{aligned}\mathrm{Bott}(\varphi, uv)|_{\mathcal{P}} &= \mathrm{Bott}(\varphi, uv_1)|_{\mathcal{P}} + \mathrm{Bott}(\varphi, U^*v_2U)|_{\mathcal{P}} &\text{(e14.238)}\\ &= 0 + \mathrm{Bott}(\psi, v_2)|_{\mathcal{P}} &\text{(e14.239)}\\ &= 0 + \mathrm{Bott}(\varphi(e_A\cdot e_A), 1) + \mathrm{Bott}(\psi', v_2)|_{\mathcal{P}} = 0. &\text{(e14.240)}\end{aligned}$$

Put $\Psi = \mathrm{Ad}\, U\circ\psi$, $\psi'' = \mathrm{Ad}\, U\circ\psi'$ and $u_2 = U^*v_2U$. Put $\varepsilon_s = s^2\varepsilon_1$ and $\delta_2 = (K+1)\delta_0$. We have (recall $w'_{\underline{s}} = w'\otimes 1_s$)

$$\begin{aligned}&(1_s - \varphi(p_i)) + (uv)_{\underline{s}}\varphi(p_i) &\text{(e14.241)}\\ &= (1_s - \varphi(p_i)) + (uv_1u_2)_{\underline{s}}\varphi(p_i) &\text{(e14.242)}\\ &\approx_{\varepsilon_s} (1_s - \varphi(p_i)) + (uv_1)_{\underline{s}}(u_2)_{\underline{s}}\Psi(p_i) \qquad \text{(using (e14.236))} &\text{(e14.243)}\\ &\approx_{2s^2\delta_2} (1_s - \varphi(p_i)) + (uv_1)_{\underline{s}}\Psi(p_i)(u_2)_{\underline{s}}\Psi(p_i) &\text{(e14.244)}\\ &\approx_{2\varepsilon_s} (1_s - \varphi(p_i))(1_s - \Psi(p_i)) + (uv_1)_{\underline{s}}\varphi(p_i)\Psi(p_i)(u_2)_{\underline{s}}\Psi(p_i) &\text{(e14.245)}\\ &\approx_{2\varepsilon_s} ((1_s - \varphi(p_i)) + (uv_1)_{\underline{s}}\varphi(p_i))((1_s - \Psi(p_i)) + (u_2)_{\underline{s}}\Psi(p_i)). &\text{(e14.246)}\end{aligned}$$

Similarly,

$$(1_s - \varphi(q_i)) + (uv)_{\underline{s}}\varphi(q_i) \approx_{6\varepsilon_s} ((1_s - \varphi(q_i)) + (uv_1)_{\underline{s}}\varphi(q_i))((1_s - \Psi(q_i)) + (u_2)_{\underline{s}}\Psi(q_i)). \tag{e14.247}$$

Put

$$Z_i = \lceil ((1_s - \Psi(p_i)) + (u_2)_{\underline{s}}\Psi(p_i))((1_s - \Psi(q_i)) + (u_2)^*_{\underline{s}}\Psi(q_i))\rceil.$$

Then, since we have assumed that $\psi'(A) \subset \overline{e_{0,b}Be_{0,b}}$, one computes, by (e14.221), that

$$\overline{Z_i} = \zeta'_i, \quad i = 1, 2, \dots, k. \tag{e14.248}$$

Then, in $U(M_s(\tilde{B}))/CU(M_s(\tilde{B}))$, for $i = 1, 2, \dots, k$, by (e14.246) and (e14.247),

$$\begin{aligned}&\overline{\lceil((1_s - \varphi(p_i)) + (uv)_{\underline{s}}\varphi(p_i))((1_s - \varphi(q_i)) + (uv)^*_{\underline{s}}\varphi(q_i))\rceil} &\text{(e14.249)}\\ &\approx_{12\varepsilon_s} \overline{\lceil((1_s - \varphi(p_i)) + (uv_1)_{\underline{s}}\varphi(p_i))Z_i((1_s - \varphi(q_i)) + (uv_1)^*_{\underline{s}}\varphi(q_i))\rceil} &\text{(e14.250)}\\ &= \overline{\lceil((1_s - \varphi(p_i)) + (uv_1)_{\underline{s}}\varphi(p_i))((1_s - \varphi(q_i)) + (uv_1)^*_{\underline{s}}\varphi(q_i))\rceil}\ \overline{Z_i} &\text{(e14.251)}\\ &\approx \lambda_{\varphi, uv_1}([p_i] - [q_i])\overline{Z_i} \approx_{\sigma/16} \bar{1}. \qquad \text{(see (e14.230))} &\text{(e14.252)}\end{aligned}$$

Now back to ψ'. Let $\varphi_{00} : A \to B_W := B\otimes\mathcal{W}\otimes e_q$ be defined by $\varphi_{00} = \psi_{b,w,1/K}\circ\varphi$. Then

$$\|[\varphi_{00}(g),\ ((v''_2)^*)^K]\| < 2K\delta_0 < \delta_1/2 \ \text{ for all }\ g \in \mathcal{G}_0. \tag{e14.253}$$

By the choice of δ_W and $\mathcal{G}_W$ and by applying 14.12, there exists a continuous path of unitaries $\{V(t) : t \in [0,1]\}$ in $\widetilde{B\otimes\mathcal{W}\otimes e_q}$ such that $V(0) = 1_{\tilde{B}_W}$, $V(1) = (v''^*_2)^K$ and

$$\|[\varphi_{00}(g),\ V(t)]\| < \min\{\varepsilon_0/4, \delta'_0/2\} \ \text{ for all }\ g \in \mathcal{F}\cup\mathcal{G}'_0. \tag{e14.254}$$

Write $V(t) = \lambda(t)\cdot 1_{\tilde{B}_W} + \alpha(V(t))$ for some $\lambda(t)\in\mathbb{T}$ and $\alpha(V(T))\in B_W$. Put

$$v(t) = v_1U^*(\lambda(t)\cdot 1_{\tilde{B}} + \alpha(V(t)))U \ \text{ for all }\ t\in[0,1]. \tag{e14.255}$$

Then we have

$$\|[\varphi(f),\ v(t)]\| < \min\{\varepsilon_0, \delta''_0\} \ \text{ for all }\ f\in\mathcal{F}_0. \tag{e14.256}$$

Note that $v(0) = v_1$ and $v(1) = v$. So, (e14.216) holds. Also, by (e14.225), (e14.215) holds and, by (e14.240), (e14.217) holds. Moreover, by the choice of ε_1 and by (e14.252), (e14.218) also holds. □

Corollary 14.14. *Let $A \in \mathcal{B}_T$ have continuous scale. For any $1 > \varepsilon_0 > 0$ and any finite subset $\mathcal{F}_0 \subset A$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following:*

For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ and any homomorphism $\varphi : A \to B \otimes Q$ which maps strictly positive elements to strictly positive elements, where $B \cong B \otimes \mathcal{Z}_0 \in \mathcal{D}_0$ has continuous scale. If $u \in U(\widetilde{B \otimes Q})$ is a unitary such that

$$\|[\varphi(x),\ u]\| < \delta \ \text{for all}\ x \in \mathcal{G}, \tag{e14.257}$$

there exists a unitary $v \in \widetilde{B \otimes Q}$ such that

$$\|[\varphi(f),\ v]\| < \varepsilon \ \text{for all}\ f \in \mathcal{F}, \tag{e14.258}$$

and there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\} \subset U_0(\widetilde{B \otimes Q})$ such that

$$u(0) = uv, \quad u(1) = 1 \tag{e14.259}$$

$$\|[\varphi(a),\ u(t)]\| < \varepsilon_0 \ \text{for all}\ a \in \mathcal{F}_0 \ \text{and for all}\ t \in [0, 1]. \tag{e14.260}$$

Proof. This is a combination of 14.13 and 14.8. Let $\varepsilon_0 > 0$ and $\mathcal{F}_0$ be given. Let $\delta_1 > 0$, $\sigma > 0$, $\mathcal{G}_1 \subset A$ be a finite subset, let $\{p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k\}$ be projections of $M_N(\tilde{A})$ (for some integer $N \geq 1$) such that $\{[p_1] - [q_1], [p_2] - [q_2], \ldots, [p_k] - [q_k]\}$ generates a free subgroup G_u of $K_0(A)$, and $\mathcal{P} \subset \underline{K}(A)$ be finite subset required by 14.8.

Let $\delta_0 > 0$ and $\mathcal{G}_0$ be required by 14.13 for $\min\{\delta_1, \varepsilon_0\}$ (in place of ε_0), σ and $\mathcal{G}_1 \cup \mathcal{F}_0$ (in place of $\mathcal{F}_0$) and $\mathcal{P}$ and G_u.

Now suppose that φ and u satisfy the assumption for this pair of δ_0 and $\mathcal{G}_0$. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be given. Then, by applying 14.13, there is a unitary $v \in \tilde{B}_1 = B \otimes Q$ and a continuous path of unitaries $\{v(t) : t \in [0, 1/2]\} \subset \tilde{B}_1$ such that $v(0) = v$,

$$\|[\varphi(f),\ v]\| < \varepsilon \ \text{for all}\ f \in \mathcal{F}, \tag{e14.261}$$

$$\|[\varphi(g),\ v(t)]\| < \varepsilon_0 \ \text{for all}\ g \in \mathcal{F}_0 \tag{e14.262}$$

$$\mathrm{Bott}(\varphi,\ uv(1/2))|_{\mathcal{P}} = \{0\},\ [uv(1/2)] = 0 \ \text{and} \tag{e14.263}$$

$$\mathrm{dist}(\overline{\lceil((1_s - \varphi(p_i)) + (uv(1/2))_{\underline{s}}\varphi(p_i))((1_s - \varphi(q_i)) + (uv(1/2))_{\underline{s}}^*\varphi(q_i))\rceil},\ \bar{1}) < \sigma, \tag{e14.264}$$

where $1_s = 1_{M_s}$ and $(uv(1/2))_{\underline{s}} = uv(1/2) \otimes 1_{M_s}$. Note, since B is non-unital, it is easy to see that we may assume, without loss of generality, that everything mentioned above lie in $M_N(\tilde{B}_0)$, where B_0 is a hereditary C^*-subalgebra of B so that $B_0^{\perp} \neq \{0\}$. By the proof of 14.9, therefore one may assume $uv(1/2) \in CU(\tilde{B})$. It follows from 14.8 that there is a continuous path of unitaries $\{u(t) : t \in [1/2, 1]\} \subset \tilde{B}_1$ such that $u(1/2) = uv(1/2)$, $u(1) = 1_{\tilde{B}_1}$ and

$$\|[\varphi(f),\ u(t)]\| < \varepsilon_0 \ \text{for all}\ f \in \mathcal{F}_0 \ \text{for all}\ t \in [1/2, 1]. \tag{e14.265}$$

Finally, define $u(t) = uv(t)$ for $t \in [0, 1/2]$. □

15. Finite nuclear dimension

The following proposition follows from the definition immediately.

Proposition 15.1. *Let A be a non-unital separable amenable simple C^*-algebra. Then A has tracially approximate divisible property in the sense of 10.1 of [15] if and only if the following holds:*

For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any integer $n \geq 1$ and any non-zero elements $a_0 \in A_+ \setminus \{0\}$, there are mutually orthogonal positive elements e_i, $i = 0, 1, 2, \ldots, n$, elements w_i, $i = 1, 2, \ldots, n$, such that $w_i^* w_i = e_1^2$, $w_i w_i^* = e_i^2$, $i = 1, 2, \ldots, n$, $e_0 \lesssim a_0$ and

$$\|x - \sum_{i=0}^{n} e_i x e_i\| < \varepsilon \ \text{and}\ \|w_i x - x w_i\| < \varepsilon, \quad 1 \leq i \leq n, \ \text{for all}\ x \in \mathcal{F}. \tag{e15.1}$$

Theorem 15.2. *Let A be a non-unital separable simple C^*-algebra with continuous scale and with finite nuclear dimension which satisfies the UCT. Suppose that $T(A) \neq \emptyset$ and every tracial state of A is a $\mathcal{W}$-trace. Then $A \in \mathcal{D}_0$.*

Proof. Since every tracial state of A is a $\mathcal{W}$-trace, by 12.3 of [15] (see 18.3 of [18]), $K_0(A) = \ker \rho_A$. Suppose that A is tracially approximately divisible. Then, since we assume that every tracial state of A is a $\mathcal{W}$ trace, by 3.12 of this paper and 6.5 of [16] and the proof of 18.6 of [18], $A \in \mathcal{D}_0$. Therefore it suffices to show that A is tracially approximately divisible.

It follows from [57] that $A \cong A \otimes \mathcal{Z}$. Put $B = A \otimes \mathcal{Z}_0$, $B_q = B \otimes Q$ and $A_q = A \otimes Q$. Pick a pair of relatively prime supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$ such that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} = Q$. Let

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} = \{f \in C([0, 1], Q) : f(0) \in M_{\mathfrak{p}} \ \text{and}\ f(1) \in M_{\mathfrak{q}}\} \ \text{and} \tag{e15.2}$$

$$D \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} = \{f \in C([0, 1], D \otimes Q) : f(0) \in D \otimes M_{\mathfrak{p}} \ \text{and}\ f(1) \in D \otimes M_{\mathfrak{q}}\} \tag{e15.3}$$

for any C^*-algebra D. Note, by [53], $\mathcal{Z}$ is a stationary inductive limit of $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ with trace collapsing connecting map.

Let $\varepsilon > 0$, let $\mathcal{F} \subset A \otimes \mathcal{Z}$, let $a_0 \in (A \otimes \mathcal{Z})_+ \setminus \{0\}$, and let $n \geq 1$ be an integer. Put $\eta = \inf\{d_\tau(a_0) : \tau \in T(A \otimes \mathcal{Z})\}$. Since A is assumed to have continuous scale, one may find a positive element $f_e \in A \otimes \mathcal{Z}$ with $\|f_e\| = 1$ such that

$$\tau(f_e) > 1 - \eta/16(n+1)^3 \text{ for all } \tau \in T(A \otimes \mathcal{Z}). \tag{e15.4}$$

We assume that $f_e \in \mathcal{F}$. Without loss of generality, we may also assume that $\mathcal{F} \subset A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. We may further assume, without loss of generality, that there is $0 < 1/2 < d_0 < 1$ such that

$$f(t) = f(1) \text{ for all } t > d_0 \tag{e15.5}$$

and for all $f \in \mathcal{F}$. Note $\mathrm{Aff}(T(B)) = \mathrm{Aff}(T(A))$ and $U(\tilde{B})/CU(\tilde{B}) = U(\tilde{A})/CU(\tilde{A})$. There is a KK-equivalence $\kappa \in KL(B, A)$ which is compatible to the identifications $\kappa_T : \mathrm{Aff}(T(B)) \to \mathrm{Aff}(T(A))$ and $\kappa_{cu} : U(\tilde{B})/CU(\tilde{B}) \to U(\tilde{A})/CU(\tilde{A})$ above. We will consider the triple $(\kappa, \kappa_T, \kappa_{cu})$. Let $\varphi_{\mathfrak{p}} : B \otimes M_{\mathfrak{p}} \to A \otimes M_{\mathfrak{p}}$ and $\varphi_{\mathfrak{q}} : B \otimes M_{\mathfrak{q}} \to A \otimes M_{\mathfrak{q}}$ be isomorphisms given by 13.1 and induced by $(\kappa \otimes [\mathrm{id}_{M_{\mathfrak{p}}}], \kappa_T, \kappa_{cu} \otimes (\mathrm{id}_{M_{\mathfrak{p}}})_{cu})$, and by $(\kappa \otimes [\mathrm{id}_{M_{\mathfrak{q}}}], \kappa_T, \kappa_{cu} \otimes (\mathrm{id}_{M_{\mathfrak{q}}})_{cu})$. Let $\psi_{\mathfrak{p}} : B \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} = B \otimes Q \to A \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} = A \otimes Q$ given by $\psi_{\mathfrak{p}} = \varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$ and let $\psi_{\mathfrak{q}} = \varphi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}} : B \otimes Q \to A \otimes Q$. Then

$$([\psi_{\mathfrak{q}}], (\psi_{\mathfrak{q}})_T, \psi_{\mathfrak{q}}^\dagger) = ([\psi_{\mathfrak{p}}], (\psi_{\mathfrak{p}})_T, \psi_{\mathfrak{p}}^\dagger). \tag{e15.6}$$

Let $\mathcal{F}_1 = \{f(1) : f \in \mathcal{F}\}$ in $A \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$. Let $\mathcal{G}_{1,b} = \{\psi_{\mathfrak{q}}^{-1}(f) : f \in \mathcal{F}_1\} \subset B \otimes Q$. Fix an $\varepsilon > 0$. Put $C_{00} = C_0((0,1]) \oplus M_n(C_0((0,1]))$ and $C_g = \{(f, 0), (0, f \otimes e_{i,i}), (0, f \otimes e_{1,i}) : 1 \leq i \leq n\}$ form a set of generators, where $f \in C_0((0,1])$ is the identity function on $[0,1]$ and $\{e_{i,j}\}_{1 \leq i,j \leq n}$ is a system of matrix units for M_n. It is well known that C_{00} is semi-projective. Let $\delta_c > 0$ satisfy the following: if $L : C_{00} \to C'$ is a C_g-δ_c-multiplicative completely positive contractive linear map for a C^*-algebra C', there exists a homomorphism $h_c : C_{00} \to C'$ such that

$$\|h_c(g) - L(g)\| < \min\{\varepsilon, \eta\}/64(n+1)^3 \text{ for all } g \in C_g. \tag{e15.7}$$

Let $\varepsilon_0 = \min\{\varepsilon/(n+1)^3 16, \delta_c/4, \eta/(n+1)^3 16\}$.

Let $\delta > 0$ and $\mathcal{G} \subset A \otimes Q$ be a finite subset required by 14.14 for ε_0 and $\mathcal{F}_1$. Without loss of generality, we may assume that $\mathcal{G} \subset (A \otimes Q)^{\mathbf{1}}$ and $\mathcal{F}_1 \subset \mathcal{G}$. Let $\varepsilon_1 = \min\{\varepsilon_0/2, \delta/4\}$ and $\mathcal{G}_1 = \psi_q^{-1}(\mathcal{G}) \cup \mathcal{G}_{1,b} \subset B \otimes Q$.

It follows from 5.3 (see 5.6) that there exists a unitary $u \in \widetilde{A \otimes Q}$ such that

$$\|u^* \psi_{\mathfrak{p}}(g) u - \psi_{\mathfrak{q}}(g)\| < \varepsilon_1/4 \text{ for all } g \in \mathcal{G}_1. \tag{e15.8}$$

Write $u = \lambda + \alpha(u)$ for some $\alpha(u) \in A \otimes Q$. Choose $e_{00}, e_{01} \in (A \otimes Q)_+$ with $\|e_{00}\| = \|e_{01}\| = 1$ such that $e_{00}e_{01} = e_{00}$ and $\|e_{00}x - x\| < \varepsilon_1/16$ and $\|x - xe_{00}\| < \varepsilon_1/16$ for all $x \in \mathcal{G}_1$ and $x = \alpha(u)$. We also assume that there is a non-zero $e_{00}' \in A \otimes Q$ such that $e_{00}'e_{01} = 0$. There is a unitary $u_1 \in \mathbb{C} \cdot 1_{\widetilde{A \otimes Q}} + \overline{e_{00}(A \otimes Q)e_{00}}$ such that $\|u_1 - u\| < \varepsilon/8$. Since $A \otimes Q \in \mathcal{D}_0$, by 11.5 of [15], it has stable rank one. Thus there is a unitary $u_2 \in \mathbb{C} \cdot 1_{\widetilde{A \otimes Q}} + \overline{e_{00}'(A \otimes Q)e_{00}'}$ such that $[u_2] = -[u]$ in $K_1(A)$. Put $u_3 = uu_2$. Then, since $e_{00}'e_{01} = 0$, by (e15.8),

$$\|u_3^* \psi_{\mathfrak{p}}(g) u_3 - \psi_{\mathfrak{q}}(g)\| < \varepsilon_1/2 \text{ for all } g \in \mathcal{G}_1. \tag{e15.9}$$

But now $u_3 \in U_0(\widetilde{A \otimes Q})$. There is a continuous path of unitaries $\{u(t) : t \in [0, d]\} \subset U(\widetilde{A \otimes Q})$ such that $u(0) = 1$ and $u(t) = u_3$ for all $t \in [d, 1]$ and for some $0 < d_0 < d < 1$. Define

$$\gamma(f)(t) = \begin{cases} \psi_{\mathfrak{p}}^{-1}(u(t)f(t)u(t)^*) & t \in [0, d]; \\ \frac{(1-t)}{1-d}\psi_{\mathfrak{p}}^{-1}(u(d)f(d)u(d)^*) + \frac{(t-d)}{1-d}\psi_{\mathfrak{q}}^{-1}(f(1)) & t \in (d, 1]. \end{cases} \tag{e15.10}$$

Note that $\gamma(f) \in B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. For $f \in \mathcal{F}$, let $g = \psi_{\mathfrak{q}}^{-1}(f(1)) = \psi_{\mathfrak{q}}^{-1}(f(d))$, by (e15.9),

$$\|g - \psi_{\mathfrak{p}}^{-1}(u(d)\psi_{\mathfrak{q}}(g)u(d)^*)\| < \varepsilon_1/2. \tag{e15.11}$$

In other words, if $f \in \mathcal{F}$,

$$\|\psi_{\mathfrak{p}}^{-1}(u(d)f(d)u(d)^*) - \psi_{\mathfrak{q}}^{-1}(f(1))\| < \varepsilon_1/2 \tag{e15.12}$$

Let $\mathcal{F}_2 = \{\gamma(f) : f \in \mathcal{F}\} \subset B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. Note that B is a simple C^*-algebra and $B \cong B \otimes \mathcal{Z}_0$, B has tracially approximate divisible property (see 8.9). Since $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ is unital and B has tracially approximate divisible property, there exist mutually orthogonal positive elements e_i, $i = 0, 1, 2, \ldots, n$, elements w_i, $i = 1, 2, \ldots, n$, in $B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that $w_i^* w_i = e_1^2$, $w_i w_i^* = e_i^2$, $e_0 e_i = 0$, $i = 1, 2, \ldots, n$, and

$$\|x - \sum_{i=0}^{n} e_i x e_i\| < \varepsilon_1/4, \ \|xw_i - w_i x\| < \varepsilon_1/4, \ 1 \leq i \leq n \text{ for all } x \in \mathcal{F}_2 \text{ and} \tag{e15.13}$$

$$d_\tau(e_0) \leq \eta/4 \text{ for all } \tau \in T(B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}). \tag{e15.14}$$

Since $f_e \in \mathcal{F}$, (e15.13) also implies that

$$\sum_{i=1}^{n} \tau(e_i) \geq 1 - \varepsilon_0/4 - \eta/16n^2 \text{ for all } \tau \in T(B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}). \tag{e15.15}$$

Without loss of generality, we may assume that $e_i(t) = e_i(1)$ and $w_i(t) = w_i(1)$ for all $t \in [d_1, 1]$ for some $d_1 > d > d_0$.

Let $\mathcal{G}_2 = \mathcal{G}_1 \cup \{e_i(1), w_i(1) : 1 \le i \le n\}$. By applying 5.3 again, we obtain another unitary $u_4 \in \widetilde{A \otimes Q}$ such that

$$\|u_4^*(u_3^*\psi_p(g)u_3)u_4 - \psi_q(g)\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_2. \tag{e15.16}$$

Therefore (see also (e15.9)), for any $g \in \mathcal{G}_1$,

$$\|[\mathrm{Ad}\, u_3 \circ \psi_p(g),\ u_4]\| < \varepsilon_1. \tag{e15.17}$$

It follows from 14.14 that there exist a unitary u_5 and a continuous path of unitaries $\{v(t) : t \in [d_1, r]\}$ in $\widetilde{A \otimes Q}$ (for some $1 > r > d_1$) with $v(r) = u_4u_5$ and $v(d_1) = 1_{\widetilde{A\otimes Q}}$ such that

$$\|[\mathrm{Ad}\, u_3 \circ \psi_p(g),\ u_5]\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_2 \text{ and} \tag{e15.18}$$

$$\|[\mathrm{Ad}\, u_3 \circ \psi_p(f),\ v(t)]\| < \varepsilon_0 \text{ for all } f \in \mathcal{F}_1 \text{ and } t \in [d_1, r]. \tag{e15.19}$$

It follows from (e15.16) and (e15.18) that

$$\|v(r)^*(u_3^*\psi_p(g)u_3)v(r) - \psi_q(g)\| < \varepsilon_1/8 \text{ for all } g \in \mathcal{G}_2. \tag{e15.20}$$

Now define

$$b_i' = \begin{cases} u^*(t)\psi_p(e_i(t))u(t) & t \in [0, d_1], \\ v^*(t)u_3^*\psi_p(e_i(t))u_3v(t) & t \in [d_1, r],\ i = 0, 1, 2, \ldots, n \text{ and} \\ (\frac{(1-t)}{1-r} v(r)^*u_3^*\psi_p(e_i(1))u_3v(r)) + \frac{(t-r)}{1-r}\psi_q(e_i(1)) & t \in (r, 1], \end{cases} \tag{e15.21}$$

$$z_i' = \begin{cases} u^*(t)\psi_p(w_i(t))u(t) & t \in [0, d_1], \\ v^*(t)u_3^*\psi_p(w_i(t))u_3v(t) & t \in [d_1, r], \quad i = 1, 2, \ldots, n. \\ (\frac{(1-t)}{1-r} v(r)^*u_3^*\psi_p(w_i(1))u_3v(r)) + \frac{(t-r)}{1-r}\psi_q(w_i(1)) & t \in (r, 1], \end{cases} \tag{e15.22}$$

From the definition of e_i and w_i, and (e15.20) we have

$$\|(z_i')^*z_i' - (b_1')^2\| < \varepsilon_1,\ \|z_i'(z_i')^* - (b_i')^2\| < \varepsilon_1, \text{ for all } i \ge 1 \text{ and } \|b_i'b_l'\| < \varepsilon_1 \text{ for all } i \ne l. \tag{e15.23}$$

For the next few estimates, recall that $f(t) = f(1)$ for all $t \in [d_0, 1]$, $e_i(t) = e_i(1)$ for all $t \in [d_1, 1]$, and $u(t) = u(d)$ for all $t \in [d, d_1]$.

For $t \in [0, d_0]$, since $\gamma(f) \in \mathcal{F}_2$, by (e15.13),

$$\|f(t) - \sum_{i=0}^n b_i'(t)f(t)b_i'(t)\| < \varepsilon_1 \text{ for all } f \in \mathcal{F}. \tag{e15.24}$$

For $t \in [0, d_1]$, by the definition of $\gamma(f)$, by (e15.12), the definition of b_i', and (e15.13), we have

$$f(t) \approx_{\varepsilon_1/2} u(t)^*\psi_p(\gamma(f(t)))u(t) \approx_{\varepsilon_1/4} \sum_{i=0}^n b_i'(t)u(t)^*\psi_p(\gamma(f(t)))u(t)b_i'(t) \tag{e15.25}$$

$$\approx_{\varepsilon_1/2} \sum_{i=0}^n b_i'(t)f(t)b_i'(t) \quad \text{for all } f \in \mathcal{F}. \tag{e15.26}$$

For $t \in [d_1, r]$, by (e15.5), (e15.9), (e15.19), and (e15.13), with $g = \psi_q^{-1}(f(1))$,

$$f(t) = f(1) \approx_{\varepsilon_1} u_3^*\psi_p(g)u_3 \approx_{\varepsilon_0} v(t)^*u_3^*\psi_p(g)u_3v(t) \tag{e15.27}$$

$$\approx_{\varepsilon_1/2} \mathrm{Ad}\, u_3v(t) \circ \psi_p(\gamma(f(t))) \approx_{\varepsilon_1/4} \mathrm{Ad}\, u_3v(t) \circ \psi_p(\sum_{i=0}^n e_i(t)\gamma(f(t))e_i(t)) \tag{e15.28}$$

$$\approx_{\varepsilon_1} \sum_{i=0}^n b_i'(t)f(t)b_i'(t). \tag{e15.29}$$

On $[r, 1]$, by the above, and by (e15.20), as $e_i(1) \in \mathcal{G}_2$,

$$f(t) = f(1) \approx_{3\varepsilon_1} \sum_{i=0}^n b_i'(r)f(r)b_i'(r) \approx_{4(n+1)\varepsilon_1/8} \sum_{i=0}^n b_i'(t)f(t)b_i'(t). \tag{e15.30}$$

Coming all the four estimates above, we have that

$$\|f - \sum_{i=0}^n b_i'fb_i'\| < (n+1)\varepsilon_1 + \varepsilon_0 < \varepsilon/16(n+1)^2 \text{ for all } f \in \mathcal{F}. \tag{e15.31}$$

We also compute that

$$\|z_i'f - fz_i'\| < 2\varepsilon_1 + \varepsilon_0, \quad 1 \le i \le n, \quad \text{for all } f \in \mathcal{F}. \tag{e15.32}$$

By the semi-projectivity of C_{00} and (e15.23), and choice of δ_c ($\varepsilon_1 < \delta_c$), we obtain $b_i, z_j \in A \otimes \mathcal{Z}_{p,q}$, $i = 0, 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, such that

$$\|b_i - b_i'\| < \min\{\varepsilon, \eta\}/(64n^2) \text{ and } \|z_j - z_j'\| < \min\{\varepsilon, \eta\}/(64n^2), \tag{e15.33}$$

$$b_ib_l = 0 \text{ if } i \neq l, \quad z_i^*z_i = b_1^2, \quad z_iz_i^* = b_i^2, \tag{e15.34}$$

$i, l = 0, 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. By (e15.31) and (e15.32),

$$\|f - \sum_{i=0}^n b_ifb_i\| < \varepsilon \text{ and } \|z_if - fz_i\| < \varepsilon, \quad 1 \le i \le n, \quad \text{for all } x \in \mathcal{F}. \tag{e15.35}$$

We also estimate, by (e15.15), that

$$\tau(\sum_{i=1}^n b_i) > 1 - \eta/2 \text{ for all } \tau \in T(A \otimes \mathcal{Z}). \tag{e15.36}$$

It follows that $d_\tau(b_0) < \eta$ for all $\tau \in T(A \otimes \mathcal{Z})$. This implies that $b_0 \lesssim a_0$. Therefore $A \otimes \mathcal{Z}$ has the tracial approximate divisible property (see 15.1). $\square$

Lemma 15.3. *Let A be a separable simple $\mathcal{Z}$-stable C^*-algebra with continuous scale, and with $T(A) \neq \emptyset$, and $QT(A) = T(A)$. Let $x \in \ker \rho_A$. Then there exists a homomorphism $\psi : A \to M_4(A)$ which maps strictly positive elements to strictly positive elements, $\psi_{*0}(x) = 0$ and $(\tau \otimes Tr)(\psi(a)) = 4\tau(a)$ for all $a \in A$ and $\tau \in T(A)$. where Tr is the standard trace on M_4.*

Proof. We first assume that A is stably projectionless. By A8 of the appendix of [16], there exists a projection $p \in M_r(\tilde{A})$ such that $[p] = [1_A] - x$ in $K_0(\tilde{A})$ for some integer $r > 0$. By A6 of the appendix of [16], we may assume that $p \in M_2(\tilde{A})$. Denote by $1_2 \in M_2(\tilde{A})$ the identity of $M_2(\tilde{A})$. Put $q = 1_2 - p$. Note that $p + q = 1_2$. Write $\{e_{ij}\}_{2\times 2}$ as the matrix unit for M_2. By replacing p by Z^*pZ, where Z is a unitary matrix with scalar entires, we may assume that $\pi(p) = e_{11}$, where π is the map induced by the quotient map $\tilde{A} \to \mathbb{C}$. Later we will also use π for the quotient map $M_2(M(A)) \to M_2(M(A)/A)$. Note that we also have $\pi(q) = e_{22}$.

We have $\tau(p) = \tau(q)$ for all $\tau \in T(A)$. Let $A_1 = pM_2(A)p$ and $A_2 = qM_2(A)q$. Let $a_p \in A_1$ and $a_q \in A_2$ be strict positive element of A_1 and A_2, respectively. We also assume that $0 \le a_p \le 1$ and $0 \le a_q \le 1$. Then

$$d_\tau(a_p) = \tau(p) = \tau(q) = d_\tau(a_q) \text{ for all } \tau \in T(A). \tag{e15.37}$$

Note that we assume that A is stably projectionless and $\mathcal{Z}$-stable. Then, by Theorem 1.2 of [50], $a_p \sim a_q$ in $Cu(A)$. Also, by [50], A almost has stable rank one. By 2.6 there is a partial isometry $w \in M_2(A)^{**}$ such that $w^*a, aw \in M_2(A)$ and $ww^*a = aww^* = a$ for all $a \in A_1$ and $wb, bw^* \in M_2(A)$ for all $b \in A_2$ such that $w^*a_pw := b_q$ is a strictly positive element of A_2.

Moreover,

$$w^*(a_p)^{1/n}w = b_q^{1/n} \text{ for all } n. \tag{e15.38}$$

Consider $W = pwq + qw^*p$. Then, for any $a \in M_2(A)$, $W^*a, aW \in M_2(A)$. In fact, we may write

$$a = pap + paq + qap + qaq$$

for any $a \in M_2(A)$. Then, for any $a \in M_2(A)$,

$$W^*a = qw^*pap + qw^*paq + pwqap + pwqaq \in M_2(A) \text{ and } aW \in M_2(A). \tag{e15.39}$$

(Note that $M_2(A)$ is an ideal in $M_2(\tilde{A})$ and $p, q \in M_2(\tilde{A})$.) Therefore $W \in M(M_2(A)) = M_2(M(A))$. Since $p + q = 1_2$, $a_p + b_q$ is a strictly positive element of $M_2(A)$. Hence $a_q^{1/n} + b_q^{1/n} \to 1_2$ in the strict topology. We also have, by (e15.38),

$$W^*(a_p^{1/n} + b_q^{1/n})W = w^*a_p^{1/n}w + wb_q^{1/n}w^* = b_q^{1/n} + w(w^*a_q^{1/n}w)w^* \tag{e15.40}$$

$$= b_q^{1/n} + (ww^*)a_p^{1/n}(ww^*) = b_q^{1/n} + a_q^{1/n}. \tag{e15.41}$$

It follows that $W^*W = 1_2$. As $W^* = W$, W is a self adjoint unitary in $M_2(M(A))$ and $\varphi(a) = W^*aW$ for all $a \in A$ defines an automorphism of $M_2(A)$. Note that $a_p^{1/n}$ converges strictly to the identity of $M(A_1)$. Note also

$$a_p^{1/n}(a_p + b_q)^{1/2} = a_p^{1/n}a_p^{1/2} \to 1_{\tilde{A}_1}(a_p + b_q)^{1/2} \text{ and } (a_p + b_q)^{1/2}a_p^{1/n} \to (a_p + b_q)^{1/2}1_{\tilde{A}_1}$$

(in norm). It follows that $a_p^{1/n}$ converges to a projection $p' \in M_2(M(A))$ strictly. Exactly the same argument shows that $b_p^{1/n}$ converges strictly to a projection $q' \in M_2(M(A))$. Since $a_p + b_q$ is a strictly positive element of $M_2(A)$, this implies that

$p' + q' = 1_2$. Since $p \in M_2(M(A))$ and $pa_p^{1/n} = a_p^{1/n}$, $p \geq p'$. Also $pq' = q'p = 0$. Similarly $q \geq q'$ and $qp' = p'q = 0$. Since $p + q = 1_2$ and $p' + q' = 1_2$, this implies that $p = p'$ and $q = q'$. Since $W \in M_2(M(A))$, it follows that

$$W^*pW = q. \tag{e15.42}$$

We now show that $W^*M_2(\tilde{A})W = M_2(\tilde{A})$. Write

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}. \tag{e15.43}$$

Note that $\pi(p) = e_{11}$ and $\pi(q) = e_{22}$. Since W is a self adjoint unitary, by (e15.42), $\pi(W)^*e_{11}\pi(W) = e_{22}$, $\pi(W)^*e_{22}\pi(W) = e_{11}$, and $e_{11}\pi(W) = \pi(W)e_{22}$. Hence

$$\pi(w_{11}) = 0 = \pi(w_{22}). \tag{e15.44}$$

In other words,

$$\pi(W) = \begin{pmatrix} 0 & \pi(w_{12}) \\ \pi(w_{21}) & 0 \end{pmatrix}. \tag{e15.45}$$

Then, since W is a unitary,

$$\pi(W^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\pi(W) = \begin{pmatrix} \pi(w_{21}^*) & 0 \\ 0 & \pi(w_{12}^*) \end{pmatrix}\pi(W) = \begin{pmatrix} 0 & \pi(w_{21}^*w_{21}) \\ \pi(w_{12}^*w_{12}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $e_{1,1}$, $e_{2,2}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ generates M_2, this implies that $\pi(W^*)s\pi(W) \in M_2$ for any scalar matrix s. Therefore $W^*M_2(\tilde{A})W = M_2(\tilde{A})$. This extends φ from $M_2(\tilde{A})$ to $M_2(\tilde{A})$ as an isomorphism. It follows that $\varphi_{*0}(2[1_{\tilde{A}}]) = 2[1_{\tilde{A}}]$. Put $y = \varphi_{*0}([1_{\tilde{A}}]) - [1_{\tilde{A}}]$. Then $2y = 0$. Also

$$\varphi_{*0}(x) = \varphi_{*0}([1_{\tilde{A}}] - [p]) = \varphi_{*0}([1_{\tilde{A}}]) - [q] \tag{e15.46}$$
$$= (\varphi_{*0}([1_{\tilde{A}}]) - [1_{\tilde{A}}]) + [1_{\tilde{A}}] - (2[1_{\tilde{A}}] - [p]) \tag{e15.47}$$
$$= y + ([p] - [1_{\tilde{A}}]) = y - x. \tag{e15.48}$$

Define $\psi : A \to M_4(A)$ by

$$\psi(a) = \mathrm{diag}(a, a, \varphi(a), \varphi(a)) \text{ for all } a \in A. \tag{e15.49}$$

Note that

$$\psi_{*0}(x) = 2x + 2(y - x) = 2x - 2x = 0. \tag{e15.50}$$

In case that A is not stably projectionless, let $e \in M_m(A)$ be a nonzero projection. Put $B = eM_m(A)e$. Then B is a unital simple C^*-algebra with nonzero quasidiagonal traces. Then the conclusion follows from 5.5 of [9]. □

Corollary 15.4. *Let A be a separable simple C^*-algebra which is $\mathcal{Z}$-stable and $QT(A) = T(A) \neq \emptyset$. For any finitely generated subgroup $G_0 \subset \ker\rho_A$, there exist an integer $m \geq 1$ and a homomorphism $\psi : A \to M_m(A)$ which maps strictly positive elements to strictly positive elements, $\psi_{*0}(x) = 0$ for all $x \in G_0$ and $(\tau \otimes Tr_m)(\psi(a)) = m\tau(a)$ for all $a \in A$ and $\tau \in T(A)$, where Tr_m is the standard trace on M_m.*

Proof. Let $x_1, x_2, \ldots, x_k \in G_0$ be a set of generators of G_0. We prove the corollary by induction. By 15.3, it holds for $k = 1$. Suppose that it holds for all integers $1 \leq k' < k$. Let $G_{0,1} \subset G_0$ which is generated by $x_1, x_2, \ldots, x_{k-1}$. By the inductive assumption, there exists a homomorphism $\psi_1 : A \to M_{m'}(A)$ which maps strictly positive elements to strictly positive elements, $(\psi_1)_{*0}(x) = 0$ for all $x \in G_{0,1}$ and $(\tau \otimes Tr_{m'})(\psi_1(a)) = m'\tau(a)$ for all $a \in A$.

Let $y = (\psi_1)_{*0}(x_k)$ and $B = M_{m'}(A)$. Lemma 15.3 shows that there exists a homomorphism $\varphi : B \to M_4(B)$ such that φ maps strictly positive elements to strictly positive elements, $(\varphi)_{*0}(y) = 0$ and $(\tau \otimes Tr_4)(\psi(b)) = 4\tau(a)$ for all $b \in A$ and $\tau \in T(B)$. Let $m = 4m'$. Define $\psi : A \to M_m(A)$ by

$$\psi(a) = \varphi \circ \psi_1(a) \text{ for all } a \in A. \tag{e15.51}$$

Then $(\psi)_{*0}(x) = 0$ for all $x \in G_0$. Lemma follows. □

Theorem 15.5. *Let A be a finite separable simple C^*-algebras with finite nuclear dimension which satisfies the UCT. Suppose that $T(A) \neq \emptyset$ and $K_0(A) = \ker\rho_A$. Then $gTR(A) \leq 1$.*

Proof. Note that, by [59], that A is $\mathcal{Z}$-stable. It follows that there exists $e \in A_+ \setminus \{0\}$ such that eAe has continuous scale. Without loss of generality, we may assume that $A = eAe$. It follows from [58] that every tracial state of A is quasidiagonal.

We will prove that every tracial state of A is a $\mathcal{W}$-trace. Then 15.2 applies. We will follow exactly the same proof of 7.4 of [16].

As in the proof of 7.4 of [16], it suffices to show that every tracial state of $A \otimes Q$ is a $\mathcal{W}$-trace. Therefore, from now on, in this proof, we assume that $A = A \otimes Q$. Let $K_0(A) = \cup_{n=1}^{\infty} G_n$, where $G_n \subset G_{n+1}$ is a sequence of finitely generated subgroups. Let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset and $G_{\mathcal{P}}$ be the subgroup generated by $\mathcal{P}$. We may assume that $\mathcal{P} \cap K_0(A) \subset G_n$ for some integer $n \geq 1$. It follows from 15.4 that there exists a homomorphism $\varphi_n : A \to A \otimes M_m(\mathbb{C}) \to A \otimes Q$ which maps strictly positive elements to strictly positive elements, $\varphi_{*0}(x) = 0$ for all $x \in G_n$ and $\tau(\varphi(a)) = \tau(a)$ for all $a \in A$ and $\tau \in T(A \otimes Q)$. By [58], every tracial state of $A \otimes Q$ is quasidiagonal, there exists a sequence of completely positive contractive linear maps $\psi_k : A \otimes Q \to Q$ such that

$$\lim_{k\to\infty} \|\psi_k(ab) - \psi_k(a)\psi_k(b)\| = 0 \text{ and } \lim_{k\to\infty} \tau(\psi_k(a)) = \tau(a) \text{ for all } a, b \in A. \tag{e15.52}$$

For each n choose $k(n)$ such that $L_n : A \otimes Q \to Q$ defined by $L_n(a) = \psi_{k(n)} \circ \varphi_n(a)$ for all $a \in A$ has the property that $[L_n]|_{G_n} = 0$ and

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \text{ and } \lim_{n\to\infty} \mathrm{tr}_Q(L_n(a)) = \tau(a) \text{ for all } a \in A, \tag{e15.53}$$

where tr_Q is the unique trace on Q. Since both $A \otimes Q$ and Q are divisible, and $K_1(Q) = \{0\}$, for any finite subset $\mathcal{P} \subset \underline{K}(A)$, $[L_n]|_{\mathcal{P}} = \{0\}$ for all sufficiently large n.

By Lemma 7.2 and the proof of 7.4 of [16] there exists a sequence of completely positive contractive linear maps $\Phi_n : A \to \mathcal{W}$ such that

$$\lim_{n\to\infty} \|\Phi_n(ab) - \Phi_n(a)\Phi_n(b)\| = 0 \text{ and } \tau(a) = \lim_{n\to\infty} t_{\mathcal{W}} \circ \Phi_n(a) \text{ for all } a \in A. \tag{e15.54}$$

To see this, let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset. Then $[L_n]|_{\mathcal{P}} = 0$ for all sufficiently large n. In the proof of 7.4 of [16] let us replace ψ there by L_n above. Since $[L_n]|_{\mathcal{P}} = 0$, we obtain $[\Psi_0]|_{\mathcal{P}} = [\Psi_1]|_{\mathcal{P}}$ with L_n in place of ψ—namely, Ψ_0 is defined to be m copies of L_n (in place of ψ there) and Ψ_1 is defined to be $m+1$ copies of L_n (in place of ψ there). The fact $[\Psi_0]|_{\mathcal{P}} = [\Psi_1]|_{\mathcal{P}}$ is used to connect Ψ_0 and Ψ_1. Thus, by the same proof of 7.4 of [16]), we can construct $\{\Phi_n\}$ as required. This proves that every tracial state of eAe is a $\mathcal{W}$-trace. It follows from 15.2 that $eAe \in \mathcal{D}_0$. □

Theorem 15.6. *Let A_1 and A_2 be two separable simple C^*-algebras with finite nuclear dimension which satisfy the UCT. Suppose that $K_0(A_i) = \ker \rho_{A_i}$ ($i = 0, 1$). Then $A_1 \cong A_2$ if and only if*

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \cong (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B). \tag{e15.55}$$

Moreover, in case that $\tilde{T}(A) \neq \{0\}$, both A and B are stably isomorphic to one of B_T constructed in Section 7.

Proof. Since A and B have finite nuclear dimension, A and B are both stably finite or purely infinite (which are the case that $\tilde{T}(A) = \tilde{T}(B) = \{0\}$). By [26] and [46], we may assume that A and B are stably finite and by [3], $\tilde{T}(A), \tilde{T}(B) \neq \{0\}$.

It follows from [57] that both A and B are $\mathcal{Z}$-stable. Let $e_A \in A_+$ with $\|e_A\| = 1$ and $e_B \in B_+$ with $\|e_B\| = 1$ such that both $A_0 := \overline{e_A A e_A}$ and $B_0 := \overline{e_B B e_B}$ have continuous scales (see 5.2 of [15]). It follows from 15.5 that both A_0 and B_0 are in $\mathcal{D}_0$ which implies $gTR(A) \leq 1$ and $gTR(B) \leq 1$. Then Theorem 13.2 applies. □

Corollary 15.7. *Let A be a stably finite separable simple C^*-algebras with finite nuclear dimension which satisfies the UCT. Then the following are equivalent:*

(1) A is isomorphic to $\mathcal{Z}_0$;
(2) A has a unique tracial state, $K_0(A) = \ker \rho_A = \mathbb{Z}$ and $K_1(A) = \{0\}$ and
(3) A is stably projectionless and has a unique tracial state, $K_0(A) = \mathbb{Z}$ and $K_1(A) = \{0\}$.

Proof. The equivalence of (1) and (2) follows from 15.6. It is obvious that (2) implies (3). If A is stably projectionless, then, by A8 of [16], $K_0(A) = \mathbb{Z} = \ker \rho_A$. Therefore (3) implies (2). □

Finally we offer the following result (as Theorem 1.2).

Theorem 15.8. *Let A and B be two separable simple C^*-algebras with finite nuclear dimension which satisfy the UCT. Then $A \otimes \mathcal{Z}_0 \cong B \otimes \mathcal{Z}_0$ if and only if*

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \cong (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B). \tag{e15.56}$$

(We emphasise that there is no order on K_0-groups. Also in case that $\tilde{T}(A) = \{0\}$, we view $\Sigma_A = 0$.)

Proof. First, if A is infinite, it follows $\tilde{T}(A) = \{0\}$. Moreover since A has finite nuclear dimension, it is purely infinite. Since $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$, $\tilde{T}(B) = \{0\}$. So B is also not stably finite. As B has finite nuclear dimension, B is also purely infinite. Thus, the infinite case is covered by the classification of non-unital purely infinite simple C^*-algebras (see [26] and [46]).

We now assume both A and B are finite. We only need to show the "if" part.

Put $A_1 = A \otimes \mathcal{Z}_0$ and $B_1 = B \otimes \mathcal{Z}_0$. Then we have, ignoring the order structure on $K_0(A)$,

$$(K_0(A_1), K_1(A_1), \tilde{T}(A_1), \Sigma_{A_1}) = (K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \tag{e15.57}$$

$$(K_0(B_1), K_1(B_1), \tilde{T}(B_1), \Sigma_{B_1}) = (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B). \tag{e15.58}$$

Let $e_A \in (\mathrm{Ped}(A_1))_+$ with $\|e_A\| = 1$ and $e_B \in (\mathrm{Ped}(B_1))_+$ with $\|e_B\| = 1$ such that $A_0 = \mathrm{Ped}(A_0)$ and $B_0 = \mathrm{Ped}(B_0)$, where $A_0 := e_A(A_1)e_A$ and $B_0 := e_B(B_1)e_B$. It follows from Proposition 12.5 of [15] that all tracial states of $A_0 \otimes \mathcal{Z}_0$ and $B_0 \otimes \mathcal{Z}_0$ are $\mathcal{W}$ traces. It follows from 6.6 of [16] (see 17.6 and the proof of 18.6 of [18]) that $A_0 \otimes \mathcal{Z}_0, B_0 \otimes \mathcal{Z}_0 \in \mathcal{D}_0$. Note A_0 and B_0 are hereditary C^*-subalgebras of $A_1 \otimes \mathcal{Z}_0$ and $B_1 \otimes \mathcal{Z}_0$, respectively. Note also $A_1 \otimes \mathcal{Z}_0 \cong A_1$ and $B_1 \otimes \mathcal{Z}_0 \cong B_1$, by 13.4. Therefore $gTR(A_1) \le 1$ and $gTR(B_1) \le 1$. Since $A_0, B_0 \in \mathcal{D}_0$, by 8.5, $K_0(A_1) = \ker \rho_{A_1}$ and $K_0(B_1) = \ker \rho_{B_1}$. Thus the theorem follows from (e15.57), (e15.58) and Theorem 13.2. □

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Appendix

In this appendix, we show that separable amenable C^*-algebra in $\mathcal{D}$ are $\mathcal{Z}$-stable. The proof is a non-unital version of Matui and Sato's proof in [42] which is identical to the unital case with only a few modification. We will follow steps of their proof as well as the notation in [42].

Lemma A.1 (*cf. 2.4 of [42]*)**.** *Let A be a separable simple C^*-algebra with continuous scale and with $T(A) \neq \emptyset$ and let $a \in A_+ \setminus \{0\}$. Then there exists $\alpha > 0$ such that*

$$\alpha \liminf_{n\to\infty} \inf_{\tau \in T(A)} \tau(f_n) \le \liminf_{n\to\infty} \inf_{\tau\in T(A)} \tau(f_n^{1/2} a f_n^{1/2}) \tag{eA.1}$$

for any central sequence $(f_n)_n$ of positive contractions of A.

Proof. By 5.6 of [15], A is strongly uniformly full in A. Therefore there are $M(a), N(a) > 0$ such that, for $b \in A_+$ with $\|b\| \le 1$ and for any $\varepsilon > 0$, there are $x_i \in A$ with $\|x_i\| \le M(a)$, $i = 1, 2, \ldots, N(a)$ such that

$$\|\sum_{i=1}^{N(a)} x_i^* a x_i - b\| < \varepsilon. \tag{eA.2}$$

Put $\alpha_0 = M(a)^2 N(a)$ and $\alpha = \frac{4}{3\alpha_0}$. Let $\{f_n\}_n$ be given. We may assume that

$$\liminf_{n\to\infty} \inf_{\tau\in T(A)} \tau(f_n) = \beta > 0,$$

otherwise there is nothing to prove. Since A has continuous scale, there exists $e \in A_+$ with $\|e\| = 1$ such that

$$\tau((1 - e^{1/2})c(1 - e^{1/2})) < \beta/8 \ \text{ for all } \ \tau \in T(A) \tag{eA.3}$$

for any $c \in A_+$ with $\|c\| = 1$. Then there are $y_i \in A$ such that $\|y_i\| \le M(a)$, $i = 1, 2, \ldots, N(a)$ such that

$$\|\sum_{i=1}^{N(a)} y_i^* a y_i - e\| < \beta/8, \quad i = 1, 2, \ldots. \tag{eA.4}$$

One also has that

$$\tau((1 - e)f_n) < \beta/8, \quad n \in \mathbb{N}. \tag{eA.5}$$

Then, keeping in mind that $(f_n)_n$ is a central sequence,

$$\begin{aligned}\beta &= \liminf_{n\to\infty}\inf_{\tau\in T(A)}\tau(f_n) \le \liminf_{n\to\infty}\inf_{\tau\in T(A)}\tau(ef_n)+\beta/8 \le \liminf_{n\to\infty}\inf_{\tau\in T(A)}\sum_{i=1}^{N(a)}\tau(y_i^*ay_if_n)+\beta/4\\ &= \liminf_{n\to\infty}\inf_{\tau\in T(A)}\sum_{i=1}^{N(a)}\tau(y_i^*a^{1/2}f_na^{1/2}y_i)+\beta/4 = \liminf_{n\to\infty}\inf_{\tau\in T(A)}\sum_{i=1}^{N(a)}\tau(f_n^{1/2}a^{1/2}y_iy_i^*a^{1/2}f_n^{1/2})+\beta/4\\ &\le \alpha_0\liminf_{n\to\infty}\inf_{\tau\in T(A)}\tau(f_n^{1/2}af_n^{1/2})+\beta/4.\end{aligned}$$

Thus

$$3\beta/4 \le \alpha_0\liminf_{n\to\infty}\inf_{\tau\in T(A)}\tau(f_n^{1/2}af_n^{1/2}). \quad \square \tag{eA.6}$$

Definition A.2 (*2.1 of [42]*)**.** Let A be a separable C^*-algebra with $T(A)\neq\emptyset$ and let $\varphi : A\to A$ be a completely positive linear map. Suppose that $T(A)$ is compact. Recall that φ is said to be excised in small central sequence if for any central sequence $(e_n)_n$ and $(f_n)_n$ of positive contractions in A satisfying

$$\lim_{n\to\infty}\sup_{\tau\in T(A)}\tau(e_n)=0 \ \text{ and } \ \lim_{m\to\infty}\liminf_{n\to\infty}\inf_{\tau\in T(A)}\tau(f_n^m)>0, \tag{eA.7}$$

there exists $s_n\in A$ with $\|s_n\|\le\|\varphi\|^{1/2}$ and $n\in\mathbb{N}$ such that

$$\lim_{n\to\infty}\|s_n^*as_n-\varphi(a)e_n\|=0 \ \text{ for all } \ a\in A \ \text{ and } \ \lim_{n\to\infty}\|f_ns_n-s_n\|=0. \tag{eA.8}$$

Lemma A.3 (*2.5 of [42]*)**.** *Let A be a separable simple C^*-algebra with $T(A)\neq\emptyset$ with continuous scale. Suppose also that A has the strict comparison for positive elements. Let $(e_n)_n$ and $(f_n)_n$ be as (eA.7). Then for any $a\in A_+$ with $\|a\|=1$, there exists a sequence $(r_n)_n$ in A such that*

$$\lim_{n\to\infty}\|r_n^*f_n^{1/2}af_n^{1/2}r_n-e_n\|=0 \ \text{ and } \ \limsup_{n\to\infty}\|r_n\|=\limsup_{n\to\infty}\|e_n\|^{1/2}. \tag{eA.9}$$

Proof. The proof of this is exactly the same as that of Lemma 2.5 of [42] using A.1 instead of 2.4 in [42]. $\square$

Proposition A.4 (*2.2 of [42]*)**.** *Let A be a separable amenable simple C^*-algebra with $T(A)\neq\emptyset$ and with continuous scale. Suppose that A has strict comparison for positive elements. Let ω be a non-zero pure state of A, $c_i, d_i\in A$, $i=1,2,\dots,N$. Then a completely positive linear map $\varphi : A\to A$ defined by $\varphi(a)=\sum_{i,j=1}^N\omega(d_i^*ad_j)c_i^*c_j$ can be excised by small central sequences.*

Proof. Let $\varepsilon>0$ and let $\mathcal{F}\subset A$ be a finite subset. It suffices to show that there exist $s_n\in A$, $n\in\mathbb{N}$, such that $\|s_n\|\le\|\varphi\|^{1/2}+\varepsilon$ and

$$\lim_{n\to\infty}\|s_n^*as_n-\varphi(a)e_n\|<\varepsilon \ \text{ and } \ \lim_{n\to\infty}\|f_ns_n-s_n\|=0. \tag{eA.10}$$

Let $\mathcal{G}=\{d_i^*ad_j : a\in\mathcal{F}, 1\le i,j\le N\}$ and let $\delta=\varepsilon/N^2$.

By Proposition 2.2 of [1], there is $a\in A_+$ with $\|a\|=1$ such that $\|a(\omega(x)-x)a\|<\delta$ for all $x\in\mathcal{G}$. Let $\{e_n\}_n$ and $\{f_n\}_n$ be as in (eA.7). By 2.3 of [42], there is a central sequence $\{\tilde f_n\}_n$ of positive contractions of A such that $\{\tilde f_nf_n\}_n=\{f_n\}_n$ in A_∞ and

$$\lim_{m\to\infty}\liminf_{n\to\infty}\inf_{\tau\in T(A)}\tau(\tilde f_n^m)=\lim_{m\to\infty}\liminf_{n\to\infty}\inf_{\tau\in T(A)}\tau(f_n^m). \tag{eA.11}$$

Applying A.3 to $\{e_n\}_n$, $\{f_n\}_n$, and a^2, we obtain $r_n\in A$, $n\in\mathbb{N}$, satisfying

$$\lim_{n\to\infty}\|r_n^*\tilde f_n^{1/2}a^2\tilde f_n^{1/2}r_n-e_n\|=0 \ \text{ and } \ \limsup_{n\to\infty}\|r_n\|\le 1. \tag{eA.12}$$

Define

$$s_n=\sum_{i=1}^N d_ia\tilde f_n^{1/2}r_nc_i,\quad n=1,2,\dots. \tag{eA.13}$$

The rest of the proof is exactly the same as that of proof of Proposition 2.2 in [42] with one exception. We need to address the norm of s_n. Note that, by (eA.10),

$$\|s_n^*bs_n\|\le\|\varphi\|+\varepsilon \ \text{ for all } \ b\in A_+^{\mathbf{1}}. \tag{eA.14}$$

Therefore by replacing s_n by E_ns_n for some $E_n\in A_+^{\mathbf{1}}$ as subsequence of an approximate identity of A, we may assume $\|s_n\|\le\|\varphi\|^{1/2}$. $\square$

Lemma A.5 (*3.1 of [42]*)**.** *Let A be a separable amenable simple non-elementary C^*-algebra, and let ω be a non-zero pure state of A. Then any completely positive contractive linear map $\varphi : A \to A$ can be approximated point-wisely in norm by completely positive contractive linear maps ψ of the from*

$$\psi(a) = \sum_{l=1}^{N}\sum_{i,j=1}^{N} \omega(d_i^* a d_j) c_{l,i}^* c_{l,j} \text{ for all } a \in A, \tag{eA.15}$$

where $c_{l,i},\ d_i \in A$, $l, i = 1, 2, \ldots, N$.

Proof. The proof is identical to that of 3.1 of [42]. Unital condition can be easily removed. In the first place that unital condition is mentioned, by using an approximate identity $\{e_n\}$ of A, and consider $\rho(e_n)^{-1/2}\rho(\cdot)\rho(e_n)^{-1/2}$ and $\sigma(\rho(e_n)^{1/2} \cdot \rho(e_n)^{1/2})$ for some large n, we can assume that $\rho(e_n)$ is the unit of M_N, by considering a hereditary C^*-subalgebra of a full matrix algebras exactly the way as described in that proof. Then, since we assume that A is simple and non-elementary, $\pi(A)$ does not contain any non-zero compact operators on $\mathcal{H}$ in the second paragraph of that proof. So Voiculescu theorem applies. The rest of proof are unchanged. □

Lemma A.6. *Let $A \in \mathcal{D}$ be separable C^*-algebra with continuous scale. Then, for any integer $k \geq 1$, there exists an order zero c.p.c. map $\psi : M_k \to A_\infty \cap A'$ such that*

$$\lim_{n\to\infty} \inf\{|\tau(c_n^m) - 1/k| : \tau \in T(A)\} = 0 \text{ for all } m \in \mathbb{N}, \tag{eA.16}$$

where $c_n = \psi(e)$ and $e \in M_k$ is a minimal rank one projection of M_k.

Proof. This proof can be extracted from the proof of 10.4 of [15]. First keep in mind, by 9.4 of [15], A has strict comparison for positive elements. In the case that $A \in \mathcal{D}_0$, this directly follows from 10.7 of [15]. In this case, by 10.7 of [15], there are two sequences of C^*-subalgebras $A_{0,n}$, $M_k(D_n)$ of A, two sequences of completely positive contractive linear maps $\varphi_n^{(0)} : A \to A_{0,n}$ and $\varphi_n^{(1)} : A \to D_n \in \mathcal{C}_0^{0'}$ with $M_k(D_n) \perp A_{0,m}$ satisfy the following:

$$\lim_{n\to\infty} \|\varphi_n^{(i)}(ab) - \varphi_n^{(i)}(a)\varphi_n^{(i)}(b)\| = 0 \text{ for all } a, b \in A, \ i = 0, 1, \tag{eA.17}$$

$$\lim_{n\to\infty} \|a - (\varphi_n^{(0)}(a)) \oplus \mathrm{diag}(\overbrace{\varphi_n^{(1)}(a), \varphi_n^{(1)}(a), \ldots, \varphi_n^{(1)}(a)}^{k})\| = 0 \text{ for all } a \in A, \tag{eA.18}$$

$$\lim_{n\to\infty} \sup_{\tau\in T(A)} d_\tau(c_n) = 0, \tag{eA.19}$$

$$\tau(f_{1/4}(\psi_n^{(1)}(a_0))) \geq d \text{ for all } \tau \in T(D_n) \tag{eA.20}$$

and $\varphi_n^{(1)}(a_0)$ is a strictly positive element in D_n, where c_n is a strictly positive element of $A_{0,n}$ and $1 > d > 0$. It is easy to see (see the proof of 9.1 of [15]) that

$$\lim_{n\to\infty} \sup\{|\tau(a) - \tau \circ \mathrm{diag}(\overbrace{\varphi_n^{(1)}(a), \varphi_n^{(1)}(a), \ldots, \varphi_n^{(1)}(a)}^{k})| : \tau \in T(A)\} = 0 \text{ for all } a \in A. \tag{eA.21}$$

Let $e_{0,n}$ and $e_{1,n}$ be approximate identities for $A_{0,n}$ and D_n, respectively. Define $e_{j,l,n} = f_{1/2l}(e_{j,n})$, $j = 0, 1$, $l \in \mathbb{N}$. Then $\{e_{0,l,n}\}_l$ and $\{e_{1,l,n}\}_l$ are approximate identities for $A_{0,n}$ and D_n, respectively. Define $\bar{e}_{1,l,n} = \mathrm{diag}(\overbrace{e_{1,m,n}, e_{1,m,n}, \ldots, e_{1,m,n}}^{k})$. Put $E_{l,n} = e_{0,l,n} + \bar{e}_{1,l,n}$. Then since $T(A)$ is compact, as we assume A has continuous scale, $\lim_{l\to\infty} \sup_{\tau\in T(A)} \tau(E_{m,n}) = 1$.

Therefore, by (eA.19), it is easy to choose a subsequence j_n such that

$$\lim_{n\to\infty} \sup_{\tau\in T(A)} |\tau(e_{1,j_n,n}^m) - 1/k| = 0 \text{ for all } m \in \mathbb{N}, \tag{eA.22}$$

and by (eA.18), $\{e_{1,j_n,n}\}$ is a central sequence. Note that we identify $e_{1,j_n,n}$ with $\mathrm{diag}(e_{1,j_n,n}, \overbrace{0, \ldots, 0}^{k-1}) \subset M_k(D_n)$. Put $e_{1,j_n,n,i} = \mathrm{diag}(\overbrace{0, \ldots, 0,}^{i-1}, e_{1,j_n,n,i}, 0, \ldots, 0)$, $i = 1, 2, \ldots, k$. There are $w_{i,n} \in M_k(D_n)$ such that $w_{i,n}^* w_{i,n} = e_{1,j_n,n,1}$ and $w_{i,n}w_{i,n}^* = e_{1,j_n,n,i}$, $i = 2, 3, \ldots, k$. Since A is stably projectionless, the C^*-subalgebra generated by $e_{1,j_n,n,i}$ and $w_{i,n}$ is isomorphic to $C_0(C(0, 1], M_k)$. Note $\{w_{i,n}\}$ can be chosen to be central (by (eA.17) and (eA.18). Put $c_n = e_{1,j_n,n}$. We obtain a completely positive contractive linear map $\psi : M_k \to A_\infty \cap A'$.

In the case that $A \in \mathcal{D}$, $M_k(D_n)$ is replaced by D_n and (eA.18) is replaced by

$$\lim_{n\to\infty} \|a - \mathrm{diag}(\varphi_n^{(0)}(a), \mathrm{diag}(\varphi_n^{(1)}(a)))\| = 0 \text{ for all } a \in A. \tag{eA.23}$$

But, as in the proof of 10.4 of [15], the algebra D in that proof is $\mathcal{Z}$-stable. Therefore, in the proof of 10.2 of [15], one has that (as (e.10.6) there)

$$\|[\varphi_{n,m}(x), y]\| < \varepsilon/16K^2 \text{ for all } x \in \mathcal{F} \tag{eA.24}$$

and $y \in \{d''^{1/2}, d'', v'', e''_j, w''_j, j = 1, 2, \ldots, K\}$. Note that one can choose $K = nk$ and using n copies of e''_j and w''_j, the same argument above also produces the completely positive contractive linear map map φ from M_k. □

Lemma A.7. *Let A be a separable amenable simple C^*-algebra in $\mathcal{D}$ with continuous scale. Then every completely positive linear map $\varphi : A \to A$ can be excised by small central sequences.*

Proof. Let $\varphi : A \to A$ be a completely positive contractive linear map (so we assume $\|\varphi\| = 1$ without loss of generality). Let $\{e_n\}_n$ and $\{f_n\}_n$ be as in A.2. By A.1, we may assume that there exist a pure state ω of A and $c_{l,i}d_i \in A$, $l, i = 1, 2, \ldots, N$, such that

$$\varphi(a) = \sum_{l=1}^{N}\sum_{i,j=1}^{N} \omega(d_i^* a d_j) c_{l,i}^* c_{l,j} \text{ for all } a \in A. \tag{eA.25}$$

Set $\varphi_l(a) = \sum_{i,j=1}^{N} \omega(d_i^* a d_j) c_{l,i}^* c_{l,j}$ for all $a \in A$, $l = 1, 2, \ldots, N$. Thus $\varphi = \sum_{l=1}^{N} \varphi_l$. Note that Lemma 3.4 of [42] holds for non-unital case, in particular, holds for the case $A \in \mathcal{D}$ which can also be directly proved by repeatedly using the construction in A.6 in $\overline{f_n A f_n}$. Therefore we also have a central sequence $\{f_{l,n}\}_n$, $l = 1, 2, \ldots, N$, of positive contractions in A such that $\{f_n f_{l,n}\}_n = \{f_{l,n}\}$, $\{f_{l,n} f_{l',n}\}_n = 0$, $l \neq l'$, $l = 1, 2, \ldots, N$, in $A_\infty \cap A'$, and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \inf_{\tau\in T(A)} \tau(f_{l,n}^m) > 0. \tag{eA.26}$$

Applying A.4 to φ_l, $\{e_n\}_n$ and $\{f_{l,n}\}_n$, we obtain a sequence $\{s_{l,n}\}_n$ in $A^{\mathbf{1}}$ such that

$$\lim_{n\to\infty} \|s_{l,n}^* a s_{l,n} - \varphi_l(a) e_n\| = 0 \text{ and } \lim_{n\to\infty} \|f_n s_{l,n} - s_{l,n}\| = 0. \tag{eA.27}$$

Put $s_n = \sum_{l=1}^{N} s_{l,n}$. One estimates that (recall that $\|s_{l,n}\| \le 1$)

$$\begin{aligned}
\|f_n s_n - s_n\| &\le \sum_{l=1}^{N} \|f_n s_{l,n} - s_{l,n}\| \\
&\le \sum_{l=1}^{N} (\|f_n s_{l,n} - f_n f_{l,n} s_{l,n}\| + \|f_n f_{l,n} s_{l,n} - f_{l,n} s_{l,n}\| + \|f_{l,n} s_{l,n} - s_{l,n}\|) \\
&\le \sum_{l=1}^{N} (\|f_n\| \|s_{l,n} - f_{l,n} s_{l,n}\| + \|f_n f_{l,n} - f_{l,n}\| \|s_{l,n}\| + \|f_{l,n} s_{l,n} - s_{l,n}\|) \to 0,
\end{aligned}$$

as $n \to \infty$. If $l \neq l'$, then, since $\{f_{l,n}\}_n$ is central and $\{f_{l,n} f_{l',n}\}_n = 0$ in A_∞,

$$\lim_{n\to\infty} \|s_{l,n}^* a s_{l',n}\| = \lim_{n\to\infty} \|s_{l,n}^* f_{l,n} a f_{l',n} s_{l,n}\| = 0. \tag{eA.28}$$

Therefore, for all $a \in A$,

$$\lim_{n\to\infty} \|s_n^* a s_n - \varphi(a) e_n\| = \lim_{n\to\infty} \|\sum_{l=1}^{N} s_{l,n}^* a s_{l,n} - \varphi_l(a) e_n\| = 0. \quad \Box \tag{eA.29}$$

Definition A.8 (*cf. 4.1 of [42]*)**.** Let A be a separable C^*-algebra with $T(A) \neq \emptyset$ and with $T(A)$ compact. We say A has property (SI) if for any central sequence $\{e_n\}_n$ and $\{f_n\}_n$ which satisfy (eA.7), there exists a central sequence $\{s_n\}_n$ in A such that

$$\lim_{n\to\infty} \|f_n s_n - s_n\| = 0 \text{ and } \{s_n^* s_n\}_n - \{e_n\}_n \in A^\perp, \tag{eA.30}$$

where $A^\perp = \{\{b_n\}_n \in A_\infty : \{b_n\}_n A = A\{b_n\}_n = 0\}$.

Lemma A.9. *Let A be a separable amenable C^*-algebra in $\mathcal{D}$ with continuous scale. Then A has (SI).*

Proof. Let $\{e_n\}_n$ and $\{f_n\}_n$ be as in (eA.7). Then, by A.7, id_A can be excised in small central sequences. Thus there is a sequence $s'_n \in A^{\mathbf{1}}$ such that $\lim_{n\to\infty} \|(s'_n)^* a (s'_n) - a e_n\| = 0$ for all $a \in A$ and $\lim_{n\to\infty} \|f_n s_n - s_n\| = 0$. Fix an approximate identity $\{d_n\}$ of A. By passing to s'_{n_k}, e'_{n_k} and f_{n_k}, if necessary, we may assume further that

$$\lim_{n\to\infty} \|(s'_n)^* d_n (s'_n) - d_n e_n\| = 0 \text{ and } \lim_{n\to\infty} \|f_n d_n^{1/2} - d_n^{1/2} f_n\| = 0. \tag{eA.31}$$

Define $s_n = d_n^{1/2}s_n'$, $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} \|s_n^*s_n - d_ne_n\| = 0 \ \text{ and } \ \lim_{n\to\infty} \|f_ns_n - s_n\| = \lim_{n\to\infty} \|d_n^{1/2}(f_ns_n' - s_n')\| = 0. \tag{eA.32}$$

Moreover, for any $a \in A$, since $\{d_n\}$ is an approximate identity for A,

$$\lim_{n\to\infty} \|a(s_n^*s_n) - ae_n\| \le \lim_{n\to\infty} \|a(s_n')^*d_n(s_n') - ad_ne_n\| + \lim_{n\to\infty} \|ad_ne_n - ae_n\| = 0. \tag{eA.33}$$

It follows that $\{s_n^*s_n\}_n - \{e_n\}_n \in A^\perp$. Moreover, for $a \in A$, by (eA.33),

$$\lim_{n\to\infty} \|[s_n,\, a]\|^2 = \lim_{n\to\infty} \|as_n^*s_na - a^*s_n^*as_n - s_n^*a^*s_na + s_n^*a^*as_n\| \tag{eA.34}$$

$$= \lim_{n\to\infty} \|as_n^*s_na - a^*e_na\| = \lim_{n\to\infty} \|a(s_n^*s_n - e_n)a\| = 0. \tag{eA.35}$$

Therefore $\{s_n\}_n$ is a central sequence. $\square$

Theorem A.10. *Every separable amenable C^*-algebra in $\mathcal{D}$ is $\mathcal{Z}$-stable.*

Proof. Let $A \in \mathcal{D}$. It suffices to show that a non-zero hereditary C^*-subalgebra of A is $\mathcal{Z}$-stable. Therefore, by 11.7 of [15], we may assume that A has continuous scale.

Fix any integer $k > 1$. By Lemma A.6, we obtain a central sequence $\{c_{i,n}\}_n$ in A, $i = 1, 2, \ldots, k$, such that $\{c_{i,n}c_{j,n}^*\}_n = \delta_{i,j}\{c_{1,n}^2\}_n$ in A_∞ and

$$\lim_{n\to\infty} \sup_{\tau\in T(A)} |\tau(c_{1,n}^m) - 1/k| = 0 \ \text{ for all } \ m \in \mathbb{N}. \tag{eA.36}$$

Thus we obtain an order zero completely positive contractive linear map $\varphi : M_k \to A_\infty \cap A'$ such that $\varphi(e) = \{c_{1,n}\}_n$ for a minimal projection $e \in M_k$. Let $\{d_n\}$ be an approximate identity for A. Then $\{d_n\}_n$ is a central sequence. Then $\overline{\{d_n\}_n}$ is the identity of $A_\infty \cap A'/A^\perp$, where $\overline{\{d_n\}_n}$ is the image of $\{d_n\}_n$ in $A_\infty \cap A'/A^\perp$. We may choose such $\{d_n\}$ so that $\{d_n - \sum_{i=1}^N c_{i,n}^*c_{i,n}\}_n \in (A_\infty)_+$. Note that, since A has continuous scale, $\lim_{n\to\infty}\sup_{\tau\in T(A)} \tau(d_n) = 1$. Let $\{e_n\}$ be a central sequence of positive contraction such that $\{e_n\}_n = \{d_n - \sum_{i=1}^k c_{i,n}^*c_{i,n}\}_n$. As in A.6 $\{c_{i,n}\}_n$ can be chosen so that

$$\limsup_{n\to\infty} \sup_{\tau\in T(A)} \tau(e_n) = 0 \tag{eA.37}$$

which can also be computed directly from (eA.36). Then, we also have

$$\lim_{m\to\infty} \liminf_{n\to\infty} \inf_{\tau\in T(A)} \tau(c_{1,n}^m) = 1/k. \tag{eA.38}$$

By the property (SI), we obtain a central sequence $\{s_n\}$ in $A^{\mathbf{1}}$ such that

$$\{s_n^*s_n\}_n - \{e_n\}_n \in A^\perp \ \text{ and } \ \lim_{n\to\infty} \{c_{1,n}s_n\}_n = \{s_n\}_n \ \text{ in } \ A_\infty. \tag{eA.39}$$

Thus we obtain an order zero completely positive contractive linear map $\Phi : M_k \to A_\infty \cap A'/A^\perp$ induced by φ and $s = \overline{\{s_n\}_n} \in A_\infty \cap A'/A^\perp$ such that,

$$s^*s + \Phi(1_{M_k}) = 1 \ \text{ and } \ \Phi(e)s = s \ \text{ in } \ A_\infty \cap A'/A^\perp \tag{eA.40}$$

This implies that $A \otimes \mathcal{Z} \cong A$ as in the proof of (iv) $\Longrightarrow$ (i) in section 4 of [42], see also, for example, Proposition 5.3 and 5.6 of [57]. $\square$

Remark A.11. More general result related to this appendix will appear elsewhere.

References

[1] C.A. Akemann, J. Anderson, G.K. Pedersen, Excising states of C^*-algebras, Canad. J. Math. 38 (1986) 1239–1260.
[2] B. Blackadar, K-Theory for Operator Algebras, second ed., in: Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, ISBN: 0-521-63532-2, 1998, p. xx+300.
[3] B. Blackadar, M. Rørdam, Extending states on preordered semigroups and the existence of quasitraces on C^*-algebras, J. Algebra 152 (1992) 240–247.
[4] L.G. Brown, Stable isomorphism of hereditary subalgebras of C^*-algebras, Pacific J. Math. 71 (1977) 335–348.
[5] N.P. Brown, A.S. Toms, Three applications of the Cuntz semigroup, Int. Math. Res. Not. IMRN (19) (2007) rnm068, 14 pp..
[6] M.-D. Choi, E.G. Effros, The completely positive lifting problem for C^*-algebras, Ann. of Math. 104 (1976) 585–609.
[7] A. Ciuperca, G.A. Elliott, A remark on invariants for C^*-algebras of stable rank one, Int. Math. Res. Not. IMRN (5) (2008) rnm158, 33 pp.
[8] K. Coward, G.A. Elliott, C. Ivanescu, The cuntz semigroup as an invariant for C^*-algebras, J. Reine Angew. Math. 623 (2008) 161–193.
[9] M. Dădărlat, Morphisms of simple tracially AF algebars, Internat. J. Math. 9 (2004) 919–957.
[10] M. Dădărlat, T. Loring, A universal multicoefficient theorem for the Kasparov groups, Duke Math. J. 84 (1996) 355–377.
[11] G.A. Elliott, The classification problem for amenable C^*-algebras, in: Proceedings of the International Conference of Mathematics, Vol.1, 2, Birkhusr, Zurich, Basel, 1994, pp. 922–932, 1995.

[12] G.A. Elliott, An invariant for simple C*-algebras, in: Canadian Mathematical Society. 1945–1995, Vol. 3, Canadian Math. Soc, Ottawa, ON, 1996, pp. 61–90.
[13] G.A. Elliott, G. Gong, On the classification of C^*-algebras of real rank zero. II, Ann. of Math. (2) 144 (1996) 497–610.
[14] G.A. Elliott, G. Gong, H. Lin, Z. Niu, On the classification of simple amenable C^*-algebras with finite decomposition rank, II, arXiv:1507.03437, 0000.
[15] G.A. Elliott, G. Gong, H. Lin, Z. Niu, Simple stably projectionless C^*-algebras of generalized tracial rank one, J. Non-Commutative Geom. 14 (2020) 251–347.
[16] G.A. Elliott, G. Gong, H. Lin, Z. Niu, The classification of simple separable KK-contractible C^*-algebras with finite nuclear dimension, J. Geometry and Physics 158 (2020).
[17] G.A. Elliott, K. Thomsen, The state space of the K_0-group of a simple separable C*-algebra, Geom. Funct. Anal. 4 (5) (1994) 522–538.
[18] G. Gong, H. Lin, On classification of non-unital simple amenable C^*-algebras, I, preprint, arXiv:1611.04440, 0000.
[19] G. Gong, H. Lin, Almost multiplicative morphisms and K-theory, Internat. J. Math. 11 (2000) 983–1000.
[20] G. Gong, H. Lin, Z. Niu, Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras, preprint, arXiv:1501.00135, 0000.
[21] G. Gong, H. Lin, Y. Xue, Determinant rank of C*-algebras, Pacific J. Math. 274 (2015) 405–436.
[22] N. Higson, A characterization of KK-theory, Pacific J. Math. 126 (1987) 253–276.
[23] S. Hu, H. Lin, Y. Xue, Limits of homomorphisms with finite-dimensional range, Internat. J. Math. 16 (2005) 807–821.
[24] J. Hua, H. Lin, Rotation algebras and Exel trace formula, Canad. J. Math. 67 (2015) 404–423.
[25] B. Jacelon, A simple, monotracial, stably projectionless C^*-algebra, J. Lond. Math. Soc. 87 (2013) 365–383.
[26] E. Kirchberg, N.C. Phillips, Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$, J. Reine Angew. Math. 525 (2000) 17–53.
[27] H. Lin, Bounded module maps and pure completely positive maps, J. Oper. Theory 26 (1991) 121–138.
[28] H. Lin, Simple C^*-algebras with continuous scales and simple corona algebras, Proc. Amer. Math. Soc. 112 (1991) 871–880.
[29] H. Lin, Exponential rank of C^*-algebras with real rank zero and the Brown-Pedersen conjectures, J. Funct. Anal. 114 (1) (1993) 1–11.
[30] H. Lin, An Introduction to the Classification of Amenable C^*-algebras, World Scientific Publishing Co., Inc., River Edge, NJ, ISBN: 981-02-4680-3, 2001, p. xii+320.
[31] H. Lin, A separable Brown-Douglass-Fillmore Therem and weak stability, Trans. Amer. Math. Soc. 356 (2004) 2889–2925.
[32] H. Lin, Simple corona C^*-algebras, Proc. Amer. Math. Soc. 132 (2004) 3215–3224.
[33] H. Lin, Simple nuclear C^*-algebras of tracial topological rank one, J. Funct. Anal. 251 (2007) 601–679.
[34] H. Lin, Asymptotically unitary equivalence and asymptotically inner automorphisms, Amer. J. Math. 131 (2009) 1589–1677.
[35] H. Lin, Approximate homotopy of homomorphisms from C(X) into a simple C^*-algebra, Mem. Amer. Math. Soc. 205 (963) (2010) vi+131,
[36] H. Lin, Cuntz semigroups of C*-algebras of stable rank one and projective Hilbert modules, 2010, http://arxiv.org/abs/1001.4558.
[37] H. Lin, Homotopy of unitaries in simple C^*-algebras with tracial rank one, J. Funct. Anal. 258 (2010) 1822–1882.
[38] H. Lin, Locally AH algebras, Mem. Amer. Math. Soc. 235 (1107) (2015) vi+109, ISBN: 978-1-4704-1466-5; 978-1-4704-2225-7.
[39] H. Lin, From the Basic Homotopy Lemma To the Classification of C^*-algebras, in: a CBMS Lectures Notes, vol. 121, Amer. Math. Soc., Providence, RI, ISBN: 978-1-4704-3490-8, 2017, p. vi+240.
[40] H. Lin, Homomorphisms from AH-algebras, J. Topol. Anal. 9 (2017) 67–125, arXiv:1102.4631v1(2011).
[41] T.A. Loring, Lifting Solutions To Perturbing Problems in C^*-lgebras, in: Fields Institute Monographs, vol. 8, American Mathematical Society, Providence, RI, ISBN: 0-8218-0602-5, 1997, p. x+165.
[42] H. Matui, Y. Sato, Strict comparison and $\mathcal{Z}$-absorption of nuclear C^*-algebras, Acta Math. 209 (1) (2012) 179–196.
[43] G.K. Pedersen, Measure theory for C^*-algebras, Math. Scand. 19 (1966) 131–145.
[44] G.K. Pedersen, Measure theory for C^*-algebras. III, Math. Scand. 25 (1969) 71–93.
[45] G.K. Pedersen, Saw*-algebras and corona C^*-algebras, contributions to noncommutative topology, J. Oper. Theory 15 (1986) 15–32.
[46] N.C. Phillips, A classification theorem for nuclear purely infinite simple C^*-algebras, Doc. Math. 5 (2000) 49–114.
[47] S. Razak, On the classification of simple stably projectionless C^*-algebras, Canad. J. Math. 54 (2002) 138–224.
[48] M. Rieffel, The homotopy groups of the unitary groups of noncommutative tori, J. Oper. Theory 17 (1987) 237–254.
[49] L. Robert, Classification of inductive limits of 1-dimensional NCCW complexes, Adv. Math. 231 (2012) 2802–2836.
[50] L. Robert, Remarks on $\mathcal{Z}$-stable projectionless C^*-algebras, Glasg. Math. J. 58 (2016) 273–277.
[51] M. Rørdam, On the structure of simple C^*-algebras tensored with a UHF-algebra. II, J. Funct. Anal. 107 (1992) 255–269.
[52] M. Rørdam, The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras, Internat. J. Math. 15 (2004) 1065–1084.
[53] M. Rørdam, W. Winter, The Jiang-Su algebra revisited, J. Reine Angew. Math. 642 (2010) 129–155.
[54] J. Rosenberg, C. Schochet, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor, Duke Math. J. 55 (1987) 431–474.
[55] L. Santiago, A classification of inductive limits of splitting interval algebras, 2010, preprint, arXiv:1011.6559.
[56] K. Thomsen, Traces, unitary characters and crossed products by $\mathbb{Z}$, Publ. Res. Inst. Math. Sci. 31 (1995) 1011–1029.
[57] A. Tikuisis, Nuclear dimension, $\mathcal{Z}$-stability and algebraic simplicity for stably projectionless C^*-algebras, Math. Ann. 358 (2014) 729–778.
[58] A. Tikuisis, S. White, W. Winter, Quasidiagonality of nuclear C*-algebras, Ann. of Math. 185 (2017) 229–284.
[59] W. Winter, Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras, Invent. Math. 187 (2012) 259–342.
[60] W. Winter, J. Zacharias, The nuclear dimension of C^*-algebras, Adv. Math. 224 (2010) 461–498.