Secondary Prospective Teachers' Strategies to Determine Equivalence of Conditional Statements

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Future mathematics teachers must be able to interpret a wide range of mathematical statements, in particular conditional statements. Literature shows that even when students are familiar with conditional statements and equivalence to the contrapositive, identifying other equivalent and non-equivalent forms can be challenging. As a part of a larger grant to enhance and study prospective secondary teachers' (PSTs') mathematical knowledge for teaching proof, we analyzed data from 26 PSTs working on a task that required rewriting a conditional statement in several different forms and then determining those that were equivalent to the original statement. We identified three key strategies used to make sense of the various forms of conditional statements and to identify equivalent and non-equivalent forms: meaning making, comparing truth-values and comparing to known syntactic forms. The PSTs relied both on semantic meaning of the statements and on their formal logical knowledge to make their judgments.

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Mathematics educators generally agree that teaching mathematics in ways that promote reasoning and proof can deepen students' understanding and support retention of knowledge (e.g., Hanna & deVillers, 2012; Harel, 2013). Furthermore, researchers and policy makers support integration of proof and reasoning across all grade levels and mathematical topics (NCTM, 2009; NGA & CCSSO, 2010). In order to bring this vision of mathematics teaching into reality, teachers need to have robust understanding of deductive reasoning, valid modes of inference, proof techniques, and other aspects, which comprise *knowledge of the logical aspects of proof* (Buchbinder & McCrone, 2018). Moreover, teachers must flexibly use this knowledge in the context of school mathematics.

As part of a larger grant to study and enhance prospective secondary teachers' (PSTs') mathematical knowledge for teaching proof, we designed instructional activities aiming to strengthen their knowledge of the logical aspects of proof. The activities were enacted in the capstone course, *Mathematical Reasoning and Proof for Secondary Teachers*, which is a blended content and pedagogy course developed as a part of the grant (Buchbinder & McCrone, 2018).

One set of such activities focused on conditional statements and logical equivalence, or the lack of thereof, between the implication $P \Rightarrow Q$ and various logical forms, such as contrapositive, converse, inverse, but also other forms such as *P if Q* or *P is necessary for Q*. The purpose of the activities was to push the PSTs thinking beyond typical "if-then" statements, in hopes that by identifying ways of moving their own understanding forward, the PSTs would strengthen their knowledge of the logical aspects of proof, which in turn, could inform their future teaching.

As researchers, we sought to understand the process by which the PSTs made sense of various logical forms of conditional statements and to identify strategies by which the PSTs determined logical equivalence. Our work was guided by the research question: How do PSTs make sense of various logical forms of conditional statements and establish equivalence between them?

Background and Theoretical Perspectives

A conditional statement (or a logical implication), denoted as $P \Rightarrow Q$, can be expressed in words as *If P then Q* or *P implies Q*. But many other equivalent forms are possible, for example, *Q if P*, *P only if Q*, *P is sufficient for Q* and *Q is necessary for P*. Formally, two statements are logically equivalent if they have the same truth tables. This definition can be used to determine that a contrapositive $\sim Q \Rightarrow \sim P$ is equivalent to $P \Rightarrow Q$. However, it cannot be applied directly to the logical forms mentioned above. The equivalence of these forms to the original implication is not obvious and many require a few logical steps to show. As an example, consider the form *P only if Q.* Taken by itself, this statement says that P is true only under the condition that Q is true. In other words, if Q is false, it cannot be the case that P is true (i.e., P is also false). Using such reasoning, one might recognize that *P only if Q* is equivalent to *not Q implies not P* which is the contrapositive of the statement *P implies Q*, and thus is equivalent to *P implies Q* (Chartrand, Polimeni & Zhang, 2018). This type of reasoning can be challenging, as it requires both syntactic understanding of the formal symbolic notation and the semantic understanding of the meaning of each logical form and the related wording (Weber & Alcock, 2004).

Studies have shown that mathematics majors and PSTs often misinterpret mathematical language and experience particular difficulties with conditional statements. These include confusion between an implication and its converse ($Q \Rightarrow P$) (Durand-Guerrier, 2003), interpreting an implication as a biconditional ($P \Leftrightarrow Q$) (Epp, 2003), difficulty understanding the equivalence between an implication and a contrapositive (Dawkins & Hub, 2017; Stylianides, Stylianides & Philippou, 2004) or between an implication and a disjunction ($\sim P \vee Q$) (Hawthorne & Rasmussen, 2015). However, we are not aware of studies that examined how undergraduates, in particular PSTs, make sense of logical equivalence of a broad range of logical forms of conditional statements. We see this as a crucial aspect of teacher preparation, as teachers need flexible knowledge to make sense of and rephrase their students' contributions into more precise mathematical statements in order to determine their validity.

Methods

The Conditional Statements activities took place in the capstone course, *Mathematical Reasoning and Proof for Secondary Teachers*, for which the first author served as the instructor. All PSTs enrolled in the course agreed to participate in the study: 15 PSTs in Fall 2017, and 11 PSTs in Fall 2018. All PSTs were native English speakers. At the time of data collection all PSTs were seniors, thus, they had completed most of their mathematics coursework, including several proof-based classes such as Mathematical Proof, Geometry and Abstract Algebra. Hence, the goal of the Conditional Statements activities was not to introduce PSTs to new material, but to help them refresh and strengthen their content knowledge of conditional statements. The content focus included the meaning and logical notation of the implication $P \Rightarrow Q$, recognition of hypothesis and conclusion in context specific statements worded in different forms, determining truth-value of conditional statements, and recognizing equivalent and non-equivalent logical forms such as a contrapositive $\sim Q \Rightarrow \sim P$ and a converse $Q \Rightarrow P$.

The activity that is the focus of this paper included the following tasks:

a. Working in small groups, determine which of the 11 given logical forms are equivalent to the original implication $P \Rightarrow Q$ and which are not (Fig. 1);
b. Create a poster display of the equivalent and non-equivalent statements and share those posters with other groups;

c. Discuss the answers as a whole class, clarify difficult items, and summarize main points about logical equivalence and its relationship to truth-value.

Each group received a different mathematical statement, but the logical forms were the same across all groups. The mathematical statements of each group were the following:

Group 1: A graph of an odd function passes through the origin (assume *f* is defined at 0).

Group 2: A number that is divisible by 6 is divisible by 3.

Group 3: Diagonals of a rectangle are congruent to each other.

Group 3

Given a true statement: ***Diagonals of a rectangle are congruent to each other.***

1. Rewrite the statement in an *If P then Q* form. Identify *P* and *Q*.

2. Using P and Q you defined above, write statements in each the forms presented below.

a) P if Q	
b) P only if Q	
c) P is necessary for Q	
d) To infer Q, it is sufficient to know P	
e) Q if P	
f) Q is sufficient for P	
g) If not P then not Q	
h) Not-Q implies not-P	
i) P is sufficient to infer Q	
j) Q is necessary for P	
k) P if and only if Q	

3. Write each statement on a separate index card using different color for true and false statements. Sort the cards in two groups: (i) cards with statements equivalent to the given statement, and (ii) cards with statements not equivalent to the given one. Be prepared to explain why placed the cards the way you did.

Figure 1. A worksheet of Group 3 (Geometry)

The inclusion of a mathematical context and symbolic form in one task aimed to support the PSTs flexible understanding and use of formal logical notation while grounding it in familiar content (Dawkins, 2017; Dubinsky & Yiparaki, 2000).

The tasks were enacted across two class periods and took about two hours to complete. The data were collected in the form of video recordings of each groups' work with a tabletop 360° video camera, a stationary camera to capture the whole class discussion, and PSTs' written work in the form of worksheets and posters. The video recording and their transcripts were the primary data source for this paper. Written artifacts served as a secondary data source.

To analyze the data, we divided the video transcripts into meaningful episodes, and used open coding (Wiersma & Jurs, 2005) and the constant comparative method (Strauss & Corbin, 1994) to identify categories of strategies the PSTs utilized to determine whether certain forms of conditional statements are logically equivalent to the implication $P \Rightarrow Q$.

Results

Our analysis revealed three main strategies in the PSTs approaches to determining logical equivalence: (1) meaning making, (2) comparing truth-values, and (3) comparing to known syntactic forms. These strategies are interrelated and some have further sub-categories, which we elaborate below. The two types of logical forms that appeared to present most challenges to the PSTs were *P only if Q*, and the statements that contained the language of *necessary* and *sufficient* condition. Thus, the data excerpts chosen to illustrate the strategies are taken from these types of statements as they provided the clearest evidence of the PSTs strategies to determine equivalence, although the strategies were observed throughout all types of logical forms. Note, the text in square brackets was added for clarification.

Strategy 1: Meaning Making

This strategy entails rephrasing of the statement (often multiple times) in search of a clearer meaning. This process included three types of sub-strategies: (a) introducing additional or substituting alternative words, (b) attempting to put a statement into an "if-then" form, and (c) relying on counterexamples and "feelings".

Rephrasing by adding or substituting words. When trying to interpret conditional statements that were worded as necessary or sufficient conditions the PSTs initially attempted to substitute open sentences for *P* and *Q*, but quickly discovered that this may result in a nonsensical statement, as shown below:

Excerpt #1. Sam: …how to write that… P is necessary for Q.

Bill: Yeah, I'm trying to think about a word…

Nate: That's a lot of is's. Cause "a quadrilateral is a rectangle is necessary for its diagonals are congruent …it's just awkwardly worded.

Laura: A quadrilateral must be a rectangle for the diagonals to be congruent.

Nate: mmm…nice change.

The PSTs also found it difficult to distinguish between necessary and sufficient conditions. The following excerpt illustrates the PSTs' confusion as they tried to word the statement: If a number is divisible by 6 then it is divisible by 3 in the form *Q is sufficient for P.*

Excerpt #2. Penny: It is sufficient to know that a number is divisible by 3…

Zoe: If it's divisible by 3 then it's satisfies that it's going to be divisible by, …no.

Linda: No, technically not.

Zoe: But that's not what it's saying, sufficient is like if it satisfies… that guarantees… so, if a number is divisible by 3, then it guarantees that it is divisible by 6.

Here, Penny tried to retain the wording of "*is sufficient*" while Zoe struggled a bit but then offered the alternate wording *if it satisfies Q, then it guarantees P*. Not all attempts to rephrase the statement preserved the meaning, but this strategy was highly prevalent and for the most part was efficient in PSTs' attempts to determine equivalence.

Rephrasing in "if-then" form. Much of the PSTs' rephrasing efforts were directed towards putting the statements into an "if-then" form (e.g., Zoe's attempts in Excerpt #2 above), even when this was not entirely appropriate. Consider the excerpt below:

Excerpt #3. Audrey: Umm…and then another one that was really confusing for us was *Q is necessary for P*. So we said, if a number is divisible by 3, it is necessary for it to be divisible by 6, but that's like opposite of what the [original] statement is saying. Cause if a number is divisible 3, it doesn't have to be divisible by 6.

Here, Audrey's group rephrased *Q is necessary for P* incorrectly as *if Q it is necessary for it to be P*, which seems to be further simplified as *if Q then P*. This changed the meaning of the statement indeed transforming it into one not equivalent to the original.

The strategy of rephrasing a statement in an "if-then" form worked for the *P if Q* statement. The PSTs put "if" at the beginning of the sentence: *if Q, P*, and added the words "follow" or "then" to correctly conclude that *P if Q* is equivalent to *if Q then P*, and thus not equivalent to the original implication. However, this strategy failed when interpreting *P only if Q* statement. The

PSTs either came to impasse, not knowing how to interpret *only if Q, P* or they simply dropped the word "only," wrongly transforming the statement into *if Q then P* (see Excerpt #5 below)

Counterexamples and "Feelings." The PSTs constructed counterexamples to help them make sense of the statements. For example, Emily used the counterexample of $y = x^2$, a non-odd function passing through the origin to help her make sense of the form *P if Q* and to conclude that it is not equivalent to *P implies Q*. She said: "No, it's not equivalent. [A function] x^2 can pass through the origin. It [*P if Q*] is saying if a graph passes through the origin, then the function is odd. It's flipping it."

In both cohorts, the PSTs came up with the same counterexamples: $y = x^2$ for the statement about functions, an isosceles trapezoid to show that a quadrilateral which is not a rectangle can have congruent diagonals, and numbers 9 and 15, which are divisible by 3 but not divisible by 6. Once someone in the group introduced a counterexample into a public space, the members of the group repeatedly used it to make sense of various forms of conditional statements. However, the PSTs occasionally misused these counterexamples in statements. For instance, Derick attempted to use the function $y = x^2$ to disprove a statement "A graph of the function passes through the origin if the function is odd", which has a form *Q if P*. Derick noticed that the function $y = x^2$ satisfies *Q,* i.e. passes through the origin, but is not an odd function, and wrongly attempted to use it to disprove *Q if P*, without considering that it is equivalent to *P implies Q,* which makes his counterexample non-applicable.

As much as the use of counterexamples was prevalent in the PSTs' strategies, most frequently the PSTs simply relied on their perception to make a claim about the equivalence of statements, without providing any justification, as the next excerpt shows:

Excerpt # 4. Rebecca: Alright, next, *to infer Q is sufficient to know P*.

Logan: Logically, I think it is [equivalent].

Dylan: Really? This is definitely not logically equivalent. Knowing the function is odd is sufficient to infer the graph passes through the origin... Maybe it is.

Grace: Wait, I actually think that's equivalent. It's like exactly the same.

Dylan: Yeah, yeah actually it is equivalent. Oh yeah.

It can be argued that such words as "logically, I think" or "I feel like" (see excerpt #5 below) were used merely as a figure of speech. However, in the absence of any other justifications, we tend to interpret them as discursive mechanisms for convincing oneself or others. We also acknowledge that the PSTs' feelings were probably based on their prior knowledge of both the mathematical content, the formal logical notation, and the counterexamples discussed in their groups. All these could be used to create mental representations of statements, which could then be compared to one another to determine equivalence. However, not surprisingly, the "feelings" were not always reliable, and often led to incorrect conclusions.

Strategy 2: Comparing Truth-Values

A second key strategy the PSTs used to decide if the statements are equivalent or not, was to compare their truth-values. In group discussions, the PSTs quickly agreed that equivalent statements must have the same truth-values and that statements that have different truth-values cannot be equivalent. These conclusions were not obvious to everyone, but the PSTs eventually resolved any initial doubts within their groups, without the instructor's intervention.

A more difficult point to agree upon was whether having the same truth-value made statements equivalent. This point required more discussion and negotiation, especially, since all statements in the worksheets which had the same truth-value to the original implication were also equivalent to it. Thus, the PSTs had to come up with their own examples of statements that have the same truth-value but are not equivalent. This was not always easy, as can be seen in Dylan's comment: "I think, we agreed that if it's equivalent, the truth value has to be the same, but you can have a case that the truth value is the same, but it's not equivalent to the statement. So we were saying we found an if and only if case that was true, it's not going to be equivalent to your original statement, but you can have the same truth value."

If the truth-values could not be used to establish equivalence, the PSTs referred back to relying on meaning making and "feelings", as the next excerpt shows:

Excerpt # 5: Nate: I feel like the *only if* would be the same as saying *if Q then P*. So I think, yeah, the *P only if Q* is like saying *if Q then P*. Which is not necessarily equivalent.

Audrey: I feel like that shouldn't be equivalent.

Nate: It just can be true. Cause I feel like, I feel like…

Audrey: That's true. But doesn't mean that it's equivalent.

While attempting to rephrase *P only if Q* in an "if-then" form, Nate mistakenly transforms it into *Q implies P*, but then correctly concludes that the rephrased statement is not equivalent to *P implies Q*. Both Nate and Audrey appeal to their feelings to seal the conclusion.

Strategy 3: Compare to Known Syntactic Forms

Since all PSTs had previously successfully completed proof-oriented courses, they often drew on their familiarity with logical notation, conditional statements, and other relevant concepts. For instance, the PSTs were aware that an implication is equivalent to a contrapositive, and used it in their work, as Dana's comment shows: "I think I came to the decision that *P only if Q* is not-equivalent [to *P implies Q*]. So *P only if Q* is the same as saying *if Q is true, then P is true*. So then the contrapositive of that would be *not P implies not Q*, and that is not equivalent to *P implies Q* in the first place."

Dana's first step is an incorrect interpretation of *P only if Q* as *If Q then P*. But then she correctly states that the contrapositive of *If Q then P* is *not P implies not Q*, and correctly concludes that it cannot be equivalent to *P implies Q*.

Our PSTs also seem to have a good grasp of non-equivalence of an implication to its converse, as illustrated below.

Excerpt # 6: Angela: 'Cause doesn't the converse like by definition is the opposite truth-value of the original. That's what the converse is? Or no?

Sam: No, you can't determine truth-value by the converse. But the converse could be true in some cases. I think, yeah. It's not necessarily true, but it could be true… Technically the converse is a whole different statement in itself.

Bill: 'Cause we're reversing p and q.

Sam: Yeah, like they're never gonna be logically equivalent.

Here Sam correctly explained that although a converse can have the same truth-value as the implication, it cannot be equivalent to it. Bill justified that by saying that the converse "reverses *P* and *Q*". The other members of the group accepted this explanation. Similar conversations occurred in all other groups, and were resolved correctly without the instructor's intervention. The PSTs also recalled that a biconditional entails the truth of both an implication and its converse, and used it to justify non-equivalence of a biconditional and an implication. Note that in both excerpts above, the PSTs operated solely within a syntactic domain, without invoking semantic meaning of the statements and without appealing to examples.

The Missing Strategy: Negation

Visibly absent from PSTs' strategies was the use of negation. In our data from both cohorts, we recorded only one instance of the use of negation to make sense of *P only if Q*. Except for that instance, the PSTs avoided using negation despite having the relevant prior knowledge. The forms for which the use of negation was critical were *P only if Q*, and the *necessary* condition. Unless prompted by the instructor, the PSTs seemed unable or unwilling to introduce negation to interpret "only if" or "necessary" as "otherwise" or "if not".

Discussion

In this paper, we examined data from two cohorts of prospective secondary teachers' interactions with Conditional Statements activities in the context of a capstone course aimed to enhance their content and pedagogical knowledge of proof. The key feature of the activities was the inclusion of both the formal logical notation and mathematical context from the secondary curriculum: number and operation, geometry and functions. Our study focused on understanding how PSTs make sense of various logical forms of conditional statements and establish equivalence between them. We identified three main strategies the PSTs used to establish equivalence, or the lack thereof, between a broad range of logical forms: (1) meaning making, (2) comparing truth-values, and (3) comparing to known syntactic forms.

Semantic strategies, what we term as *meaning making*, were prevalent in the PSTs' approaches, concurring with the literature (e.g., Dubinsky & Yiparaki, 2000). The PSTs rephrased the statements multiple times to make them more comprehensible either by adding or substituting words or by putting the statements into an "if-then' form. The PSTs used counterexamples to make sense of the statements akin to experts' use of counterexamples (cf. Lockwood, Ellis, & Lynch, 2016). Our data show that in many cases these strategies proved to be helpful in determining equivalence of the various logical forms, but not always. At times, the PSTs would fall back on feelings, relying on their perceptions of the content of statements. This phenomenon reflects the nature of our data, which was captured during the natural discourse among the PSTs. We suspect that the PSTs' perceptions were grounded in prior knowledge, and the PSTs would be able to mathematically justify their thinking, if requested.

The two logical forms that were most challenging for the PSTs to interpret were: *P only if Q* and *P is necessary for Q*. Interpreting these forms requires the use of negation, which did not come naturally to our PSTs, as consistent with the literature (Dawkins, 2017). However contrary to prior studies (e.g., Dawkins & Hub, 2017; Durand-Guerrier, 2003) our PSTs seemed knowledgeable about an implication being equivalent to a contrapositive and non-equivalent to a converse. The PSTs often relied on this formal logical knowledge to determine whether certain logical forms are equivalent or not, without assessing the semantic meaning of these statements.

The studies on knowledge retention in the area of formal logic are scarce, thus, our study adds to this literature by demonstrating knowledge retention of logical aspects of proof, beyond the typical introduction to proof course. Specifically, our study identified particular aspects of this knowledge that were retained and flexibly used by PSTs to make sense of a broad range of logical forms of conditional statements, and determine their logical equivalence. Our data show that the PSTs can apply their knowledge of formal logic to the secondary school context, and use it to interpret a wide range of logical forms of conditional statements, which may come up in classroom discourse. Another contribution to the literature is the design of the Conditional Statements activity that uncovered these strategies and elicited a range of rich discussion among PSTs around the various logical forms of conditional statements.

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