FORCED WAVES IN A LOTKA-VOLTERRA COMPETITION-DIFFUSION MODEL WITH A SHIFTING HABITAT

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ABSTRACT. We establish the existence of traveling waves for a Lotka-Volterra competition-diffusion model with a shifting habitat. It is assumed that the growth rate of each species is nondecreasing along the x-axis, positive near ∞ and negative near $-\infty$, and shifting rightward at a speed c. We show that under appropriate conditions, for the case that one species is competitively stronger near ∞ and the case that both species coexist near ∞, there exists a critical number $\bar{c}(\infty)$ such that for any $c > \bar{c}(\infty)$ there exists a forced traveling wave with speed c connecting the origin and a semi-trivial steady state and for $c < \bar{c}(\infty)$ such a traveling wave does not exist. We also show that when a coexistence steady state exists, for any $c > 0$, there is a forced traveling wave with speed c connecting the origin and the coexistence steady state.

Key words: Reaction-diffusion equation, shifting habitat, competition, linear determinacy, forced traveling waves.

AMS Subject Classification (2000): 92D25, 92D40.

1. INTRODUCTION

In this paper we are concerned with the existence of forced traveling waves for the following competition-diffusion model

$$\begin{cases} u_t(t,x) = d_1 u_{xx}(t,x) + u(t,x)(r_1(x-ct) - u(t,x) - a_1 v(t,x)), \\ v_t(t,x) = d_2 v_{xx}(t,x) + v(t,x)(r_2(x-ct) - v(t,x) - a_2 u(t,x)). \end{cases} \tag{1.1}$$

This is a Lotka-Volterra type competition model. $u(t,x), v(t,x)$ denote the densities of two competing species, respectively, at time t and space x; $d_i > 0$ are diffusion coefficients; $a_i > 0$ represent interspecific competition coefficients; each $r_i(x-ct)$ describes a population growth rate as a function of $x-ct$, which is bounded and nondecreasing in $x-ct$, positive at ∞, and negative at $-\infty$; $c > 0$ is a speed at which the habitat shifts. Here the habitat in which two species grow and compete is shrinking in time. Model (1.1) has been used to investigate the impacts of climate change on dynamics of competing species [1, 18].

The problem of spreading speeds for (1.1) has been studied for two cases: (i) one species is competitively stronger and (ii) both species coexist in the habitat suitable for growth of both species. Case (ii) was studied by Zhang el al. [23] and Yuan et al. [21] where a lower bound $\bar{c}(\infty)$ of the speed at which one species spreads into its rival was obtained. Dong et al. [7] showed that $\bar{c}(\infty)$ serves as an upper bound of the speed for both Case (i) and Case (ii) under certain conditions. In this paper we demonstrate that $\bar{c}(\infty)$ also plays an important role in determining traveling waves for (1.1). We particularly show that under appropriate conditions, for case (i)

Date: April 30, 2021.

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and case (ii), if $c > \bar{c}(\infty)$ there exists a forced traveling wave with speed c connecting the origin and a semi-trivial steady state and if $c < \bar{c}(\infty)$ such a traveling wave does not exist. We also prove that when a coexistence steady state exists, for any $c > 0$, there always is a forced traveling wave with speed c connecting the origin and the coexistence steady state. Berestycki et al. [2] established the existence of traveling waves for (1.1) when $r_1(x - ct)$ is nonincreasing in $x - ct$, negative at ∞, and positive at $-\infty$, and $r_2(x - ct)$ is nondecreasing in $x - ct$, positive at ∞, and negative at $-\infty$. The existence and stability of traveling waves for scalar reaction-diffusion equations with shifting habitats related to (1.1) have been extensively studied; see, for example, Berestycki and Fang [4], Fang et al. [8], Bouhours and Giletti [6], Berestycki et al. [3], Berestycki and Rossi [5], Hamel [9], Hamel and Roques [10]. The reader is refereed to Hu et al. [11], Hu and Zou [12], Li et al. [15], Li and Wu [16], Wang et al. [17, 19], Wu et al. [20], and Zhang and Zhao [22] for studies in spreading speeds and traveling waves in temporal-spatial models with shifting habitats in other forms including integro-difference equations and integral-differential equations.

The paper is organized as follows. In the next section, we present the main results. Section 3 is devoted to constructing appropriate super-sub solutions. The proofs of main results are given in Section 4.

2. Main results

This section is devoted to stating the hypotheses and our main results. Now putting $u(t, x) = \phi(\xi), v(t, x) = \varphi(\xi)$ with $\xi = x - ct$ and plugging them into (1.1), we have

$$\begin{cases} d_1\phi''(\xi) + c\phi'(\xi) + \phi(\xi)(r_1(\xi) - \phi(\xi) - a_1\varphi(\xi)) = 0, \\ d_2\varphi''(\xi) + c\varphi'(\xi) + \varphi(\xi)(r_2(\xi) - \varphi(\xi) - a_2\phi(\xi)) = 0. \end{cases} \tag{2.1}$$

Define $\overline{\mu}(\infty) = \sqrt{(r_1(\infty) - a_1r_2(\infty))/d_1}$. We make the following hypotheses

(H): (i) For $i = 1, 2$, $r_i(\xi)$ is nondecreasing, bounded, and piecewise continuously differentiable function in ξ for $-\infty < \xi < \infty$, and $r_i(-\infty)$ and $r_i(\infty)$ satisfy $-\infty < r_i(-\infty) < 0 < r_i(\infty) < \infty$.

(ii) For $i = 1, 2$, there exist $\alpha_i > 0$ and A_i such that

$$\lim_{\xi\to\infty} \frac{r_i(\infty) - r_i(\xi)}{e^{-\alpha_i\xi}} = A_i,$$

where $\alpha_1 \geq \overline{\mu}(\infty)$, $\alpha_2 \geq 2\overline{\mu}(\infty)$.

We are interested in the following two cases:

Case (i): $r_1(\infty)/r_2(\infty) > \max\{a_1, 1/a_2\}$,

Case (ii): $a_1 < r_1(\infty)/r_2(\infty) < 1/a_2$.

In Case (i), u is competitively stronger than v near ∞, and in Case (ii) both species coexist near ∞. In the latter case for the system (1.1) with $r_i(x - ct)$ replaced by $r_i(\infty)$, $i = 1,\ 2$, the coexistence equilibrium (u^*, v^*) is given by

$$u^* = \frac{r_1(\infty) - a_1r_2(\infty)}{1 - a_1a_2},\ v^* = \frac{r_2(\infty) - a_2r_1(\infty)}{1 - a_1a_2}.$$

We introduce the following condition:

(LD): $\min\left\{1, 2 - \frac{d_2}{d_1}\right\} \geq \frac{(\max\{a_1a_2,1\}-1)r_2(\infty)}{r_1(\infty)-a_1r_2(\infty)}$.

This condition is similar to and stronger than the linear determinacy condition given by Lewis et al. [13] in studying the spreading speeds of (1.1) with $r_i(x-ct)$ replaced by $r_i(\infty)$. Linear determinacy condition has been used to study traveling waves for competition models with constant coefficients; see for example Li [14].

Let

$$\overline{c}(\infty) = 2\sqrt{d_1(r_1(\infty) - a_1 r_2(\infty))}.$$

It was shown in [7] that under appropriate conditions, $\overline{c}(\infty)$ is the speed at which one species spreads into its rival for model (1.1).

We are in a position to state our main results. The first theorem is for Case (i).

Theorem 2.1. *Consider Case (i). Assume (H) and (LD) hold.*
(i) *If* $c > \overline{c}(\infty)$, (1.1) *admits a forced traveling wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \varphi(x-ct))$ *with*

$$(\phi(-\infty), \varphi(-\infty)) = (0,0), \ (\phi(\infty), \varphi(\infty)) = (0, r_2(\infty)) \tag{2.2}$$

and $\phi(z_0) > 0$ *for some* $z_0 \in \mathbb{R}$.
(ii) *If* $c < \overline{c}(\infty)$, *there is no forced traveling wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \varphi(x-ct))$ *with* (2.2) *and* $\phi(z_0) > 0$ *for some* $z_0 \in \mathbb{R}$.

We present results for Case (ii).

Theorem 2.2. *Consider Case (ii). Assume (H) and (LD) hold.*
(i) *If* $c > \overline{c}(\infty)$, (1.1) *admits a forced traveling wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \varphi(x-ct))$ *with* (2.2) *and* $\phi(z_0) > 0$ *for some* $z_0 \in \mathbb{R}$.
(ii) *If* $c < \overline{c}(\infty)$, *there is no forced traveling wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \varphi(x-ct))$ *with* (2.2) *and* $\phi(z_0) > 0$ *for some* $z_0 \in \mathbb{R}$.

Let $\overline{c}^*(\infty) = 2\sqrt{d_2(r_2(\infty) - a_2 r_1(\infty))}, \overline{\mu}^*(\infty) = \sqrt{(r_2(\infty) - a_2 r_1(\infty))/d_2}$. If $\alpha_1 \geq 2\overline{\mu}^*(\infty), \alpha_2 \geq \overline{\mu}^*(\infty)$ in the (H) (ii), then by virtue of Theorem 2.2, we have following corollary.

Corollary 2.3. *Consider Case (ii). Assume*

$$\min\left\{1, 2 - \frac{d_1}{d_2}\right\} \geq \frac{(\max\{a_1 a_2, 1\} - 1) r_1(\infty)}{r_2(\infty) - a_2 r_1(\infty)}$$

and (H) hold. Then
(i) *if* $c > \overline{c}^*(\infty)$, (1.1) *admits a forced traveling wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \varphi(x-ct))$ *with*

$$(\phi(-\infty), \varphi(-\infty)) = (0,0), \ (\phi(\infty), \varphi(\infty)) = (r_1(\infty), 0) \tag{2.3}$$

and $\varphi(z_0) > 0$ *for some* $z_0 \in \mathbb{R}$.
(ii) *if* $c < \overline{c}^*(\infty)$, *there is no forced traveling wave solution* $(u(t,x), v(t,x)) = (\phi(x-ct), \varphi(x-ct))$ *with* (2.3) *and* $\varphi(z_0) > 0$ *for some* $z_0 \in \mathbb{R}$.

Remark 2.4. Corollary 2.3 can be obtained from Theorem 2.2. In fact, we consider following system

$$\begin{cases} v_t(t,x) = d_2 v_{xx}(t,x) + v(t,x)(r_2(x-ct) - v(t,x) - a_2 u(t,x)), \\ u_t(t,x) = d_1 u_{xx}(t,x) + u(t,x)(r_1(x-ct) - u(t,x) - a_1 v(t,x)). \end{cases} \tag{2.4}$$

Based on the assumption of Corollary 2.3, then it follows from Theorem 2.2 that

(i)′ if $c > \overline{c}^*(\infty)$, (2.4) admits a forced traveling wave solution $(v(t,x),u(t,x)) = (\varphi_1(x-ct),\phi_1(x-ct))$ with

$$(\varphi_1(-\infty),\phi_1(-\infty)) = (0,0),\ (\varphi_1(\infty),\phi_1(\infty)) = (0,r_1(\infty)) \tag{2.5}$$

and $\varphi_1(z_0) > 0$ for some $z_0 \in \mathbb{R}$.

(ii)′ if $c < \overline{c}^*(\infty)$, there is no forced traveling wave solution $(v(t,x),u(t,x)) = (\varphi_1(x-ct),\phi_1(x-ct))$ with (2.5) and $\varphi_1(z_0) > 0$ for some $z_0 \in \mathbb{R}$.

Obviously the statement (ii) of Corollary 2.3 comes from (ii)′. Let $\phi(\xi) = \phi_1(\xi), \varphi(\xi) = \varphi_1(\xi)$, then $(u(t,x),v(t,x)) = (\phi(x-ct),\varphi(x-ct))$ is a forced traveling wave solution of (1.1) and satisfies the statement (i) of Corollary 2.3.

Theorem 2.5. *Consider Case (ii). Assume (H) holds. Then for any $c > 0$,* (1.1) *admits a forced traveling wave solution $(u(t,x),v(t,x)) = (\phi(x-ct),\varphi(x-ct))$ with following boundary conditions*

$$(\phi(-\infty),\varphi(-\infty)) = (0,0),\ (\phi(\infty),\varphi(\infty)) = (u^*,v^*). \tag{2.6}$$

3. Super-sub solutions

In this section, we focus on constructing super-sub solutions for (2.1) and the corresponding cooperative system (3.1). In the first subsection, the super-sub solutions for Cases (i) and (ii) are discussed, and the super-sub solutions given in the second subsection are only applicable for Case (ii).

3.1. **Super-sub solutions for Case (i) and (ii).** We translate equation (2.1) into a cooperative system by letting $\psi(\xi) = r_2(\infty) - \varphi(\xi)$,

$$\begin{cases} d_1\phi''(\xi) + c\phi'(\xi) + f_1(\phi,\psi)(\xi) = 0, \\ d_2\psi''(\xi) + c\psi'(\xi) + f_2(\phi,\psi)(\xi) = 0, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} f_1(\phi,\psi)(\xi) =& \phi(\xi)[r_1(\xi) - a_1r_2(\infty) - \phi(\xi) + a_1\psi(\xi)], \\ f_2(\phi,\psi)(\xi) =& [r_2(\infty) - \psi(\xi)][r_2(\infty) - r_2(\xi) + a_2\phi(\xi) - \psi(\xi)]. \end{aligned}$$

For any $\mu > 0$, define

$$B(\mu) := \begin{pmatrix} d_1\mu^2 + r_1(\infty) - a_1r_2(\infty), & 0 \\ a_2r_2(\infty), & d_2\mu^2 - r_2(\infty) \end{pmatrix}.$$

Obviously,

$$\overline{c}(\infty) = \inf_{\mu>0} \frac{d_1\mu^2 + r_1(\infty) - a_1r_2(\infty)}{\mu},$$

the infimum occurs at $\overline{\mu}(\infty)$. Fix $c > \overline{c}(\infty)$, let $0 < \mu_c < \mu_1 < \min\{\overline{\mu}(\infty), 2\mu_c\}$ be constants such that $c = \frac{\lambda(\mu_c)}{\mu_c}$ and $\frac{\lambda(\mu_c)}{\mu_c} > \frac{\lambda(\mu_1)}{\mu_1} > \overline{c}(\infty)$, where $\lambda(\mu)$ is the principal eigenvalue of $B(\mu)$. Let ϕ_1, ϕ_2 be the positive eigenfunctions of $B(\mu_c)$ associated to $\lambda(\mu_c)$ and ψ_1, ψ_2 be the positive eigenfunctions of $B(\mu_1)$ associated to $\lambda(\mu_1)$. Under the condition (LD), some calculations yield that

$$\begin{aligned}\lambda(\mu) &= d_1\mu^2 + r_1(\infty) - a_1 r_2(\infty),\\ \phi_1 &= (d_1-d_2)\mu_c^2 + r_1(\infty) - a_1 r_2(\infty) + r_2(\infty),\ \phi_2 = a_2 r_2(\infty),\\ \psi_1 &= (d_1-d_2)\mu_1^2 + r_1(\infty) - a_1 r_2(\infty) + r_2(\infty),\ \psi_2 = a_2 r_2(\infty).\end{aligned}$$

Set

$$\begin{aligned}\beta &= \min\left\{\frac{\phi_1}{\phi_2}, \frac{\psi_1}{\psi_2}\right\},\ \vartheta = -(\lambda(\mu_1) - c\mu_1) > 0,\\ \xi_1 &= \frac{1}{\mu_c}\ln\frac{2[\epsilon_1^2\phi_1^2 + \beta^2\psi_1^2 + a_1\epsilon_1\beta(\phi_1\psi_2 + \phi_2\psi_1)]}{\vartheta\psi_1\beta},\\ \xi_2 &= \frac{1}{\mu_c}\ln\frac{2\epsilon_1\phi_2\psi_2\beta + a_2(\epsilon_1^2\phi_1\phi_2 + \beta^2\psi_1\psi_2)}{\vartheta\psi_2\beta},\end{aligned}$$

where $0 < \epsilon_1 < 1$ will be determined later. From (H) (ii), there exists $M_1 > 0$ such that for $\xi \geq M_1$,

$$r_1(\infty) - r_1(\xi) \leq (A_1+1)e^{-\alpha_1\xi}. \tag{3.2}$$

Let

$$\xi_0 = \max\{\xi_1, \xi_2, M_1\},\ \eta_0 = \beta e^{(\mu_1-\mu_c)\xi_0}.$$

We choose the above ϵ_1 such that

$$\min\left\{\frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_1}{\epsilon_1\phi_1}, \frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_2}{\epsilon_1\phi_2}\right\} \geq \max\{0,\xi_0\},\ \frac{\vartheta}{2}\eta_0\psi_1 \geq (A_1+1)\phi_1\epsilon_1. \tag{3.3}$$

Define

$$\begin{aligned}\underline{\phi}(\xi) &= \alpha\max\{0, \epsilon_1\phi_1 e^{-\mu_c\xi} - \eta_0\psi_1 e^{-\mu_1\xi}\},\ \forall\xi\in\mathbb{R},\\ \underline{\psi}(\xi) &= \begin{cases}\epsilon_0,\ \xi \leq \overline{\xi},\\ \alpha\big(\epsilon_1\phi_2 e^{-\mu_c\xi} - \eta_0\psi_2 e^{-\mu_1\xi}\big),\ \xi \geq \overline{\xi},\end{cases}\end{aligned}$$

where $\overline{\xi} = \frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_2\mu_1}{\epsilon_1\phi_2\mu_c}$, ϵ_0 is the maximum of $\alpha\big(\epsilon_1\phi_2 e^{-\mu_c\xi} - \eta_0\psi_2 e^{-\mu_1\xi}\big)$ in $\mathbb{R}$, and $0 < \alpha \leq 1$ is sufficiently small and satisfies

$$r_2(\infty) - r_2\big(\overline{\xi}\big) \geq \underline{\psi}(\bar{\xi}) = \epsilon_0. \tag{3.4}$$

Lemma 3.1. *Consider Case (i) and Case (ii). Assume (H) holds, then* $\big(\underline{\phi}(\xi), \underline{\psi}(\xi)\big)$ *is a sub-solution of* (3.1).

Proof. We divide our proof into Case I and Case II.
Case I: $\overline{\xi} \geq \frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_1}{\epsilon_1\phi_1}$. When $\xi > \overline{\xi}$, then $\underline{\phi}(\xi) = \alpha(\epsilon_1\phi_1 e^{-\mu_c\xi} - \eta_0\psi_1 e^{-\mu_1\xi})$, $\underline{\psi}(\xi) = \alpha(\epsilon_1\phi_2 \times e^{-\mu_c\xi} - \eta_0\psi_2 e^{-\mu_1\xi})$, by (3.2) and (3.3), we get

$$\begin{aligned}&d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + f_1\big(\underline{\phi},\underline{\psi}\big)(\xi)\\ =&\alpha\epsilon_1\phi_1 d_1\mu_c^2 e^{-\mu_c\xi} - \alpha\eta_0\psi_1 d_1\mu_1^2 e^{-\mu_1\xi} - \alpha c\mu_c\phi_1\epsilon_1 e^{-\mu_c\xi} + \alpha c\eta_0\psi_1\mu_1 e^{-\mu_1\xi}\\ &+ \alpha e^{-\mu_c\xi}\big[\epsilon_1\phi_1 - \eta_0\psi_1 e^{-(\mu_1-\mu_c)\xi}\big]\big[r_1(\xi) - a_1 r_2(\infty) - \alpha e^{-\mu_c\xi}(\epsilon_1\phi_1 - \eta_0\psi_1 e^{-(\mu_1-\mu_c)\xi})\\ &+ \alpha a_1 e^{-\mu_c\xi}(\epsilon_1\phi_2 - \eta_0\psi_2 e^{-(\mu_1-\mu_c)\xi})\big]\end{aligned}$$

$$
\begin{aligned}
\geq & e^{-\mu_c\xi}\Big\{\alpha\epsilon_1\phi_1\big[d_1\mu_c^2-c\mu_c+r_1(\xi)-a_1r_2(\infty)\big]-\alpha\eta_0\psi_1\big[d_1\mu_1^2-c\mu_1+r_1(\infty)-a_1r_2(\infty)\big]\\
&\times e^{-(\mu_1-\mu_c)\xi}-\alpha^2e^{-\mu_c\xi}\big[(\epsilon_1^2\phi_1^2+\eta_0^2e^{-2(\mu_1-\mu_c)\xi}\psi_1^2)+a_1\epsilon_1\eta_0e^{-(\mu_1-\mu_c)\xi}(\phi_1\psi_2+\phi_2\psi_1)\big]\Big\}\\
= & \alpha e^{-\mu_1\xi}\Big\{\vartheta\eta_0\psi_1-[r_1(\infty)-r_1(\xi)]\phi_1\epsilon_1e^{(\mu_1-\mu_c)\xi}-\alpha e^{-(2\mu_c-\mu_1)\xi}\big[(\epsilon_1^2\phi_1^2+\eta_0^2e^{-2(\mu_1-\mu_c)\xi}\psi_1^2)\\
&+a_1\epsilon_1\eta_0e^{-(\mu_1-\mu_c)\xi}(\phi_1\psi_2+\phi_2\psi_1)\big]\Big\}\\
\geq & \alpha e^{-\mu_1\xi}\Big\{\vartheta\eta_0\psi_1-(A_1+1)\phi_1\epsilon_1e^{-(\alpha_1-\mu_1+\mu_c)\xi}-\alpha e^{-(2\mu_c-\mu_1)\xi}\big[(\epsilon_1^2\phi_1^2+\beta^2\psi_1^2)\\
&+a_1\epsilon_1\beta(\phi_1\psi_2+\phi_2\psi_1)\big]\Big\}\\
\geq & \alpha e^{-\mu_1\xi}\Big\{\vartheta\eta_0\psi_1-(A_1+1)\phi_1\epsilon_1-\alpha e^{-(2\mu_c-\mu_1)\xi}\big[(\epsilon_1^2\phi_1^2+\beta^2\psi_1^2)+a_1\epsilon_1\beta(\phi_1\psi_2+\phi_2\psi_1)\big]\Big\}\\
\geq & \alpha e^{-\mu_1\xi}\Big\{\frac{\vartheta}{2}\eta_0\psi_1-e^{-(2\mu_c-\mu_1)\xi_0}\big[(\epsilon_1^2\phi_1^2+\beta^2\psi_1^2)+a_1\epsilon_1\beta(\phi_1\psi_2+\phi_2\psi_1)\big]\Big\}\geq 0. \qquad (3.5)
\end{aligned}
$$

When $\frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_1}{\epsilon_1\phi_1}<\xi\leq\overline{\xi}$, then $\underline{\phi}(\xi)=\alpha\big(\epsilon_1\phi_1e^{-\mu_c\xi}-\eta_0\psi_1e^{-\mu_1\xi}\big),\underline{\psi}(\xi)=\epsilon_0\geq\alpha\big(\epsilon_1\phi_2e^{-\mu_c\xi}-\eta_0\psi_2e^{-\mu_1\xi}\big)$. Since $f_1(\phi,\psi)$ is increasing in ψ, an argument similar to that used to show (3.5), we have

$$
\begin{aligned}
& d_1\underline{\phi}''(\xi)+c\underline{\phi}'(\xi)+f_1\big(\underline{\phi},\underline{\psi}\big)(\xi)\\
= & d_1\underline{\phi}''(\xi)+c\underline{\phi}'(\xi)+f_1\big(\alpha\big(\epsilon_1\phi_1e^{-\mu_c\xi}-\eta_0\psi_1e^{-\mu_1\xi}\big),\epsilon_0\big)\\
\geq & d_1\underline{\phi}''(\xi)+c\underline{\phi}'(\xi)+f_1\big(\alpha\big(\epsilon_1\phi_1e^{-\mu_c\xi}-\eta_0\psi_1e^{-\mu_1\xi}\big),\alpha\big(\epsilon_1\phi_2e^{-\mu_c\xi}-\eta_0\psi_2e^{-\mu_1\xi}\big)\big)\geq 0. \qquad (3.6)
\end{aligned}
$$

When $\xi<\frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_1}{\epsilon_1\phi_1}$, then $\underline{\phi}(\xi)=0,\underline{\psi}(\xi)=\epsilon_0$, so

$$d_1\underline{\phi}''(\xi)+c\underline{\phi}'(\xi)+f_1\big(\underline{\phi},\underline{\psi}\big)(\xi)=f_1(0,\epsilon_0)=0.$$

This, (3.5) and (3.6) imply for $\xi\neq\frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_1}{\epsilon_1\phi_1}$, we have

$$d_1\underline{\phi}''(\xi)+c\underline{\phi}'(\xi)+f_1\big(\underline{\phi},\underline{\psi}\big)(\xi)\geq 0. \qquad (3.7)$$

Next we want to show for $\xi\neq\bar{\xi}$,

$$d_2\underline{\psi}''(\xi)+c\underline{\psi}'(\xi)+f_2\big(\underline{\phi},\underline{\psi}\big)(\xi)\geq 0. \qquad (3.8)$$

When $\xi>\overline{\xi}$, then $\underline{\psi}(\xi)=\alpha(\epsilon_1\phi_2e^{-\mu_c\xi}-\eta_0\psi_2e^{-\mu_1\xi}),\underline{\phi}(\xi)=\alpha(\epsilon_1\phi_1e^{-\mu_c\xi}-\eta_0\psi_1e^{-\mu_1\xi})$, by (3.3), there holds

$$
\begin{aligned}
& d_2\underline{\psi}''(\xi)+c\underline{\psi}'(\xi)+f_2\big(\underline{\phi},\underline{\psi}\big)(\xi)\\
\geq & d_2\underline{\psi}''(\xi)+c\underline{\psi}'(\xi)+(r_2(\infty)-\underline{\psi}(\xi))(a_2\underline{\phi}(\xi)-\underline{\psi}(\xi))\\
= & \alpha\epsilon_1\phi_2e^{-\mu_c\xi}[d_2\mu_c^2-c\mu_c]-\alpha\eta_0\psi_2e^{-\mu_1\xi}[d_2\mu_1^2-c\mu_1]+\alpha a_2r_2(\infty)e^{-\mu_c\xi}\big(\epsilon_1\phi_1-\eta_0\psi_1\\
&\times e^{-(\mu_1-\mu_c)\xi}\big)-\alpha r_2(\infty)e^{-\mu_c\xi}\big(\epsilon_1\phi_2-\eta_0\psi_2e^{-(\mu_1-\mu_c)\xi}\big)+\underline{\psi}(\xi)(\underline{\psi}(\xi)-a_2\underline{\phi}(\xi))\\
= & \alpha\epsilon_1e^{-\mu_c\xi}\big[(d_2\mu_c^2-r_2(\infty))\phi_2+a_2r_2(\infty)\phi_1-c\mu_c\phi_2\big]-\alpha\eta_0e^{-\mu_1\xi}\big[(d_2\mu_1^2-r_2(\infty))\psi_2\\
&+a_2r_2(\infty)\psi_1-c\mu_1\psi_2\big]+\underline{\psi}(\xi)(\underline{\psi}(\xi)-a_2\underline{\phi}(\xi))\\
= & \alpha e^{-\mu_1\xi}\Big\{\eta_0\vartheta\psi_2+\alpha e^{-(2\mu_c-\mu_1)\xi}\big[(\epsilon_1\phi_2-\eta_0\psi_2e^{-(\mu_1-\mu_c)\xi})^2-a_2(\epsilon_1\phi_2-\eta_0\psi_2e^{-(\mu_1-\mu_c)\xi})\\
&\times(\epsilon_1\phi_1-\eta_0\psi_1e^{-(\mu_1-\mu_c)\xi})\big]\Big\}\\
\geq & \alpha e^{-\mu_1\xi}\Big\{\eta_0\vartheta\psi_2-\alpha e^{-(2\mu_c-\mu_1)\xi}\big[2\epsilon_1\phi_2\eta_0\psi_2e^{-(\mu_1-\mu_c)\xi}+a_2(\epsilon_1^2\phi_1\phi_2+\eta_0^2e^{-2(\mu_1-\mu_c)\xi}\psi_1\psi_2)\big]\Big\}
\end{aligned}
$$

$$\geq \alpha e^{-\mu_1\xi}\Big\{\eta_0\vartheta\psi_2 - e^{-(2\mu_c-\mu_1)\xi_0}\big[2\epsilon_1\phi_2\psi_2\beta + a_2(\epsilon_1^2\phi_1\phi_2 + \beta^2\psi_1\psi_2)\big]\Big\} \geq 0. \tag{3.9}$$

When $\xi < \overline{\xi}$, then $\underline{\psi}(\xi) = \epsilon_0, \underline{\phi}(\xi) \geq 0$. Since $f_2(\phi,\psi)$ is increasing in ϕ, by (3.4), we have

$$\begin{aligned} &d_2\underline{\psi}''(\xi) + c\underline{\psi}'(\xi) + f_2\big(\underline{\phi},\underline{\psi}\big)(\xi) \\ \geq& f_2(0,\epsilon_0) \\ =&(r_2(\infty)-\epsilon_0)(r_2(\infty)-r_2(\xi)-\epsilon_0) \\ \geq&(r_2(\infty)-\epsilon_0)\Big[r_2(\infty)-r_2\big(\overline{\xi}\big)-\epsilon_0\Big] \geq 0. \end{aligned}$$

This and (3.9) lead to (3.8).

Case II: $\frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_1}{\epsilon_1\phi_1} > \overline{\xi}$. When $\xi > \frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_1}{\epsilon_1\phi_1}$, then $\underline{\phi}(\xi) = \alpha\big(\epsilon_1\phi_1 e^{-\mu_c\xi} - \eta_0\psi_1 e^{-\mu_1\xi}\big), \underline{\psi}(\xi) = \alpha\big(\epsilon_1\phi_2 e^{-\mu_c\xi} - \eta_0\psi_2 e^{-\mu_1\xi}\big)$. An argument similar to that used to show (3.5) and (3.9), we have

$$\begin{cases} d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + f_1\big(\underline{\phi},\underline{\psi}\big)(\xi) \geq 0, \\ d_2\underline{\psi}''(\xi) + c\underline{\psi}'(\xi) + f_2\big(\underline{\phi},\underline{\psi}\big)(\xi) \geq 0. \end{cases} \tag{3.10}$$

When $\xi < \frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_1}{\epsilon_1\phi_1}$, then $\underline{\phi}(\xi) = 0, \underline{\psi}(\xi) \leq \epsilon_0$, so

$$d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + f_1\big(\underline{\phi},\underline{\psi}\big)(\xi) = f_1(0,\underline{\psi}(\xi)) = 0. \tag{3.11}$$

When $\bar{\xi} < \xi \leq \frac{1}{\mu_1-\mu_c}\ln\frac{\eta_0\psi_1}{\epsilon_1\phi_1}$, then $\underline{\phi}(\xi) = 0, \underline{\psi}(\xi) = \alpha\big(\epsilon_1\phi_2 e^{-\mu_c\xi} - \eta_0\psi_2 e^{-\mu_1\xi}\big)$. Since $f_2(\phi,\psi)$ is increasing in ϕ and $0 \geq \alpha\big(\epsilon_1\phi_1 e^{-\mu_c\xi} - \eta_0\psi_1 e^{-\mu_1\xi}\big)$, an argument similar to that used to show (3.9), we get

$$\begin{aligned} &d_2\underline{\psi}''(\xi) + c\underline{\psi}'(\xi) + f_2\big(\underline{\phi},\underline{\psi}\big)(\xi) \\ =&d_2\underline{\psi}''(\xi) + c\underline{\psi}'(\xi) + f_2\Big(0,\alpha\big(\epsilon_1\phi_2 e^{-\mu_c\xi} - \eta_0\psi_2 e^{-\mu_1\xi}\big)\Big) \\ \geq&d_2\underline{\psi}''(\xi) + c\underline{\psi}'(\xi) + f_2\Big(\alpha\big(\epsilon_1\phi_1 e^{-\mu_c\xi} - \eta_0\psi_1 e^{-\mu_1\xi}\big),\alpha\big(\epsilon_1\phi_2 e^{-\mu_c\xi} - \eta_0\psi_2 e^{-\mu_1\xi}\big)\Big) \geq 0. \end{aligned} \tag{3.12}$$

When $\xi < \overline{\xi}$, then $\underline{\phi}(\xi) = 0, \underline{\psi}(\xi) = \epsilon_0$, by (3.4), there holds

$$\begin{aligned} &d_2\underline{\psi}''(\xi) + c\underline{\psi}'(\xi) + f_2\big(\underline{\phi},\underline{\psi}\big)(\xi) \\ =&(r_2(\infty)-\epsilon_0)(r_2(\infty)-r_2(\xi)-\epsilon_0) \\ \geq&(r_2(\infty)-\epsilon_0)\Big[r_2(\infty)-r_2\big(\overline{\xi}\big)-\epsilon_0\Big] \geq 0. \end{aligned} \tag{3.13}$$

(3.10)-(3.13) imply that (3.7) and (3.8) hold. This completes the proof. □

For $\xi \in \mathbb{R}$, define

$$\overline{\phi}(\xi) = \min\big\{B\phi_1 e^{-\mu_c\xi}, r_1(\infty)\big\},\ \overline{\psi}(\xi) = \min\big\{B\phi_2 e^{-\mu_c\xi}, r_2(\infty)\big\},$$

where $B > 0$ is a constant and will be determined later.

Lemma 3.2. *Consider Case (i) and Case (ii). Assume (LD) and (H) hold, then there exists $B > 0$ such that $\big(\overline{\phi}(\xi),\overline{\psi}(\xi)\big)$ is a super-solution of* (3.1)*.*

Proof. Under the assumption (LD), we first prove

$$\phi_1 > \max\{a_1, 1/a_2\}\phi_2. \tag{3.14}$$

For $\mu > 0$, let

$$h(\mu) = (d_1-d_2)\mu^2 + r_1(\infty) - a_1 r_2(\infty) + r_2(\infty).$$

If $d_1 \le d_2$, then (LD) becomes

$$2 - \frac{d_2}{d_1} \ge \frac{(\max\{a_1a_2, 1\} - 1)r_2(\infty)}{r_1(\infty) - a_1r_2(\infty)}.$$

Hence

$$\begin{aligned} h(\overline{\mu}(\infty)) =&(d_1 - d_2)\overline{\mu}^2(\infty) + r_1(\infty) - a_1r_2(\infty) + r_2(\infty) \\ =&\Big(2 - \frac{d_2}{d_1}\Big)(r_1(\infty) - a_1r_2(\infty)) + r_2(\infty) \ge \max\{a_1a_2, 1\}r_2(\infty). \end{aligned}$$

Since $0 < \mu_c < \overline{\mu}(\infty)$ and the fact $\phi_2 = a_2r_2(\infty)$,

$$\phi_1 = h(\mu_c) > h(\overline{\mu}(\infty)) \ge \max\{a_1a_2, 1\}r_2(\infty) = \max\big\{a_1, 1/a_2\big\}\phi_2.$$

If $d_1 > d_2$, then (LD) becomes

$$1 \ge \frac{(\max\{a_1a_2, 1\} - 1)r_2(\infty)}{r_1(\infty) - a_1r_2(\infty)}.$$

Similarly from the above inequality,

$$\begin{aligned} \phi_1 = h(\mu_c) =&(d_1 - d_2)\mu_c^2 + r_1(\infty) - a_1r_2(\infty) + r_2(\infty) \\ >&(r_1(\infty) - a_1r_2(\infty)) + r_2(\infty) \\ \ge&\max\{a_1a_2, 1\}r_2(\infty) = \max\big\{a_1, 1/a_2\big\}\phi_2. \end{aligned}$$

Therefore (3.14) holds.

From (H) (ii), there exists $M_2 > 0$ such that for $\xi \ge M_2$,

$$r_2(\infty) - r_2(\xi) \le (A_2 + 1)e^{-\alpha_2\xi}. \tag{3.15}$$

Choose $B \ge \epsilon_1$, where ϵ_1 is given by (3.3), such that

$$\min\Big\{\frac{1}{\mu_c}\ln\frac{B\phi_1}{r_1(\infty)}, \frac{1}{\mu_c}\ln\frac{B\phi_2}{r_2(\infty)}\Big\} \ge \max\Big\{0, M_2, \frac{1}{\alpha_2 - 2\mu_c}\ln\frac{r_2(\infty)(A_2+1)}{B^2\phi_2(a_2\phi_1 - \phi_2)}\Big\}. \tag{3.16}$$

When $\xi > \frac{1}{\mu_c}\ln\frac{B\phi_1}{r_1(\infty)}$, then $\overline{\phi}(\xi) = B\phi_1e^{-\mu_c\xi}, \overline{\psi}(\xi) \le B\phi_2e^{-\mu_c\xi}$. By (3.14), we get

$$\begin{aligned} &d_1\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + f_1\big(\overline{\phi}, \overline{\psi}\big)(\xi) \\ =&B\phi_1e^{-\mu_c\xi}\big[d_1\mu_c^2 - c\mu_c\big] + B\phi_1e^{-\mu_c\xi}\big[r_1(\xi) - a_1r_2(\infty) - B\phi_1e^{-\mu_c\xi} + a_1\overline{\psi}(\xi)\big] \\ \le&B\phi_1e^{-\mu_c\xi}\big[d_1\mu_c^2 - c\mu_c + r_1(\infty) - a_1r_2(\infty)\big] + B\phi_1e^{-\mu_c\xi}\big[a_1B\phi_2e^{-\mu_c\xi} - B\phi_1e^{-\mu_c\xi}\big] \\ =&B^2\phi_1e^{-2\mu_c\xi}[a_1\phi_2 - \phi_1] \le 0. \end{aligned} \tag{3.17}$$

When $\xi < \frac{1}{\mu_c}\ln\frac{B\phi_1}{r_1(\infty)}$, then $\overline{\phi}(\xi) = r_1(\infty), \overline{\psi}(\xi) \le r_2(\infty)$, so

$$d_1\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + f_1\big(\overline{\phi}, \overline{\psi}\big)(\xi) \le f_1(r_1(\infty), r_2(\infty)) \le 0.$$

It follows from this and (3.17) for $\xi \ne \frac{1}{\mu_c}\ln\frac{B\phi_1}{r_1(\infty)}$, we have

$$d_1\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + f_1\big(\overline{\phi}, \overline{\psi}\big)(\xi) \le 0. \tag{3.18}$$

When $\xi > \frac{1}{\mu_c}\ln\frac{B\phi_2}{r_2(\infty)} \ge \frac{1}{\mu_c}\ln\frac{B\phi_1}{r_1(\infty)}$, then $\overline{\psi}(\xi) = B\phi_2e^{-\mu_c\xi} \le r_2(\infty), \overline{\phi}(\xi) = B\phi_1e^{-\mu_c\xi}$. By (3.15), (3.14) and (3.16), we have

$$\begin{aligned} &d_2\overline{\psi}''(\xi) + c\overline{\psi}'(\xi) + f_2\big(\overline{\phi}, \overline{\psi}\big)(\xi) \\ =&d_2\overline{\psi}''(\xi) + c\overline{\psi}'(\xi) + \big(r_2(\infty) - \overline{\psi}(\xi)\big)\big[r_2(\infty) - r_2(\xi) + a_2\overline{\phi}(\xi) - \overline{\psi}(\xi)\big] \end{aligned}$$

$$
\begin{aligned}
=&Be^{-\mu_c\xi}\big[d_2\mu_c^2\phi_2-c\mu_c\phi_2+a_2r_2(\infty)\phi_1-r_2(\infty)\phi_2\big]+\big[r_2(\infty)-B\phi_2e^{-\mu_c\xi}\big]\\
&\times\Big[r_2(\infty)-r_2(\xi)\Big]+B^2\phi_2e^{-2\mu_c\xi}[\phi_2-a_2\phi_1]\\
=&\big[r_2(\infty)-B\phi_2e^{-\mu_c\xi}\big]\big[r_2(\infty)-r_2(\xi)\big]+B^2\phi_2e^{-2\mu_c\xi}\big[\phi_2-a_2\phi_1\big]\\
\le&e^{-2\mu_c\xi}\big[r_2(\infty)(r_2(\infty)-r_2(\xi))e^{2\mu_c\xi}+B^2\phi_2(\phi_2-a_2\phi_1)\big]\\
\le&e^{-2\mu_c\xi}\big[r_2(\infty)(A_2+1)e^{-(\alpha_2-2\mu_c)\xi}-B^2\phi_2(a_2\phi_1-\phi_2)\big]\le0.
\end{aligned}
\tag{3.19}
$$

When $\xi>\frac{1}{\mu_c}\ln\frac{B\phi_1}{r_1(\infty)}\ge\frac{1}{\mu_c}\ln\frac{B\phi_2}{r_2(\infty)}$, then $\overline{\psi}(\xi)=B\phi_2e^{-\mu_c\xi},\overline{\phi}(\xi)=B\phi_1e^{-\mu_c\xi}$, an argument similar to that used to show (3.19), we get

$$
d_2\overline{\psi}''(\xi)+c\overline{\psi}'(\xi)+f_2\big(\overline{\phi},\overline{\psi}\big)(\xi)\le0. \tag{3.20}
$$

For $\xi\in\Big(\frac{1}{\mu_c}\ln\frac{B\phi_2}{r_2(\infty)},\frac{1}{\mu_c}\ln\frac{B\phi_1}{r_1(\infty)}\Big]$, then $\overline{\psi}(\xi)=B\phi_2e^{-\mu_c\xi},\overline{\phi}(\xi)=r_1(\infty)\le B\phi_1e^{-\mu_c\xi}$. Since $f_2(\phi,\psi)$ is increasing in ϕ, an argument similar to that used to show (3.19), we get

$$
\begin{aligned}
&d_2\overline{\psi}''(\xi)+c\overline{\psi}'(\xi)+f_2\big(\overline{\phi},\overline{\psi}\big)(\xi)\\
\le&d_2\overline{\psi}''(\xi)+c\overline{\psi}'(\xi)+f_2\big(\overline{\phi},B\phi_1e^{-\mu_c\xi}\big)(\xi)\le0.
\end{aligned}
$$

This, (3.19) and (3.20) imply for $\xi>\frac{1}{\mu_c}\ln\frac{B\phi_2}{r_2(\infty)}$, we have

$$
d_2\overline{\psi}''(\xi)+c\overline{\psi}'(\xi)+f_2\big(\overline{\phi},\overline{\psi}\big)(\xi)\le0. \tag{3.21}
$$

Further when $\xi<\frac{1}{\mu_c}\ln\frac{B\phi_2}{r_2(\infty)}$, then $\overline{\psi}(\xi)=r_2(\infty)$, therefore

$$
d_2\overline{\psi}''(\xi)+c\overline{\psi}'(\xi)+f_2\big(\overline{\phi},\overline{\psi}\big)(\xi)=f_2\big(r_2(\infty),\overline{\psi}\big)(\xi)=0.
$$

This and (3.21) indicate for $\xi\ne\frac{1}{\mu_c}\ln\frac{B\phi_2}{r_2(\infty)}$,

$$
d_2\overline{\psi}''(\xi)+c\overline{\psi}'(\xi)+f_2\big(\overline{\phi},\overline{\psi}\big)(\xi)\le0.
$$

This and (3.18) lead to the desired result. The proof is complete. □

3.2. **Super-sub solutions for Case (ii).**

In this subsection, we construct a pair of super-sub solution for non-cooperative system (2.1). This super-sub solutions are only applicable for Case (ii).

For convenience, denote

$$
\begin{aligned}
g_1(\phi,\varphi)(\xi)&=\phi(\xi)(r_1(\xi)-\phi(\xi)-a_1\varphi(\xi)),\ \forall\xi\in\mathbb{R},\\
g_2(\phi,\varphi)(\xi)&=\varphi(\xi)(r_2(\xi)-\varphi(\xi)-a_2\phi(\xi)),\ \forall\xi\in\mathbb{R}.
\end{aligned}
$$

For $\xi\in\mathbb{R}$, define

$$
\begin{aligned}
&\overline{\phi}(\xi)=\min\big\{r_1(\infty),u^*+\epsilon_1u^*e^{-\gamma\xi}\big\},\ \overline{\varphi}(\xi)=\min\big\{r_2(\infty),v^*+\epsilon_2v^*e^{-\gamma\xi}\big\},\\
&\underline{\phi}(\xi)=\max\big\{0,u^*-\epsilon_3u^*e^{-\gamma\xi}\big\},\ \underline{\varphi}(\xi)=\max\big\{0,v^*-\epsilon_4v^*e^{-\gamma\xi}\big\},
\end{aligned}
$$

where $\gamma>0$ will be determined later and $\epsilon_i>0,i=1,2,3,4$ satisfy $\epsilon_3>1,\epsilon_4>1$ and

$$
\epsilon_1u^*>a_1\epsilon_4v^*,\ \epsilon_2v^*>a_2\epsilon_3u^*,\ \epsilon_3u^*\ge a_1\epsilon_2v^*\ \epsilon_4v^*\ge a_2\epsilon_1u^*. \tag{3.22}
$$

Note that $a_1a_2\in(0,1)$, then $\epsilon_i,i=1,2,3,4$, are well defined.

Lemma 3.3. *Consider Case (ii). Assume (H) holds. For $c>0$, there exists $\gamma>0$ such that $\big(\overline{\phi}(\xi),\overline{\varphi}(\xi)\big),\big(\underline{\phi}(\xi),\underline{\varphi}(\xi)\big)$ is a pair of super-sub solution of* (2.1).

Proof. From (H) (ii), there exists $M > 0$ such that for $\xi \geq M$,

$$r_1(\infty) - r_1(\xi) \leq (A_1+1)e^{-\alpha_1\xi},\ r_2(\infty) - r_2(\xi) \leq (A_2+1)e^{-\alpha_2\xi}. \tag{3.23}$$

Further, we choose $\gamma > 0$ sufficiently small such that

$$\begin{aligned}\gamma < \min\Bigg\{&\frac{c\epsilon_1 + \sqrt{c^2\epsilon_1^2 + 4d_1\epsilon_1(\epsilon_1u^* - a_1\epsilon_4v^*)}}{2d_1\epsilon_1}, \frac{c\epsilon_2 + \sqrt{c^2\epsilon_2^2 + 4d_2\epsilon_2(\epsilon_2v^* - a_2\epsilon_3u^*)}}{2d_2\epsilon_2},\\ &\frac{\ln\epsilon_3}{M}, \frac{\ln\epsilon_4}{M}, \alpha_1, \alpha_2, \frac{c}{2d_1}, \frac{c}{2d_2}\Bigg\},\end{aligned} \tag{3.24}$$

where $\alpha_i, i = 1,2$ are given by (H) (ii).

When $\xi > \frac{1}{\gamma}\ln\frac{\epsilon_1u^*}{r_1(\infty)-u^*}$, then $\overline{\phi}(\xi) = u^* + \epsilon_1u^*e^{-\gamma\xi}$, $\underline{\varphi}(\xi) \geq v^* - \epsilon_4v^*e^{-\gamma\xi}$, by (3.22) and (3.24), we have

$$\begin{aligned}&d_1\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + g_1\big(\overline{\phi}, \underline{\varphi}\big)(\xi)\\ =&d_1\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + \overline{\phi}(\xi)\big[r_1(\xi) - \overline{\phi}(\xi) - a_1\underline{\varphi}(\xi)\big]\\ \leq&d_1\epsilon_1u^*\gamma^2e^{-\gamma\xi} - c\epsilon_1u^*\gamma e^{-\gamma\xi} + \big[u^* + \epsilon_1u^*e^{-\gamma\xi}\big]\big[r_1(\infty) - u^* - \epsilon_1u^*e^{-\gamma\xi}\\ &- a_1v^* + a_1\epsilon_4v^*e^{-\gamma\xi}\big]\\ =&\epsilon_1u^*e^{-\gamma\xi}\big[d_1\gamma^2 - c\gamma\big] + u^*e^{-\gamma\xi}\big(1 + \epsilon_1e^{-\gamma\xi}\big)(a_1\epsilon_4v^* - \epsilon_1u^*)\\ \leq&\epsilon_1u^*e^{-\gamma\xi}\big[d_1\gamma^2 - c\gamma\big] + u^*e^{-\gamma\xi}(a_1\epsilon_4v^* - \epsilon_1u^*)\\ =&u^*e^{-\gamma\xi}\big[d_1\epsilon_1\gamma^2 - c\epsilon_1\gamma + a_1\epsilon_4v^* - \epsilon_1u^*\big] \leq 0.\end{aligned} \tag{3.25}$$

When $\xi < \frac{1}{\gamma}\ln\frac{\epsilon_1u^*}{r_1(\infty)-u^*}$, then $\overline{\phi}(\xi) = r_1(\infty), \underline{\varphi}(\xi) \geq 0$, we have

$$\begin{aligned}&d_1\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + g_1\big(\overline{\phi}, \underline{\varphi}\big)(\xi)\\ =&d_1\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + \overline{\phi}(\xi)\big[r_1(\xi) - \overline{\phi}(\xi) - a_1\underline{\psi}(\xi)\big]\\ \leq&r_1(\infty)[r_1(\xi) - r_1(\infty)] \leq 0.\end{aligned}$$

This and (3.25) indicate for $\xi \neq \frac{1}{\gamma}\ln\frac{\epsilon_1u^*}{r_1(\infty)-u^*}$,

$$d_1\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + g_1\big(\overline{\phi}, \underline{\varphi}\big)(\xi) \leq 0. \tag{3.26}$$

Next, we want to show for $\xi \neq \frac{1}{\gamma}\ln\frac{\epsilon_2v^*}{r_2(\infty)-v^*}$,

$$d_2\overline{\varphi}''(\xi) + c\overline{\varphi}'(\xi) + g_2\big(\underline{\phi}, \overline{\varphi}\big)(\xi) \leq 0. \tag{3.27}$$

When $\xi > \frac{1}{\gamma}\ln\frac{\epsilon_2v^*}{r_2(\infty)-v^*}$, then $\overline{\varphi}(\xi) = v^* + \epsilon_2v^*e^{-\gamma\xi}, \underline{\phi}(\xi) \geq u^* - \epsilon_3u^*e^{-\gamma\xi}$, by (3.22) and (3.24), we have

$$\begin{aligned}&d_2\overline{\varphi}''(\xi) + c\overline{\varphi}'(\xi) + g_2\big(\underline{\phi}, \overline{\varphi}\big)(\xi)\\ =&d_2\overline{\varphi}''(\xi) + c\overline{\varphi}'(\xi) + \overline{\varphi}(\xi)\big[r_2(\xi) - \overline{\varphi}(\xi) - a_2\underline{\phi}(\xi)\big]\\ \leq&d_2\epsilon_2v^*\gamma^2e^{-\gamma\xi} - c\epsilon_2v^*\gamma e^{-\gamma\xi} + \big[v^* + \epsilon_2v^*e^{-\gamma\xi}\big]\big[r_2(\infty) - v^* - \epsilon_2v^*e^{-\gamma\xi}\\ &- a_2u^* + a_2\epsilon_3u^*e^{-\gamma\xi}\big]\\ =&\epsilon_2v^*e^{-\gamma\xi}\big[d_2\gamma^2 - c\gamma\big] + v^*e^{-\gamma\xi}\big(1 + \epsilon_2e^{-\gamma\xi}\big)(a_2\epsilon_3u^* - \epsilon_2v^*)\\ \leq&\epsilon_2v^*e^{-\gamma\xi}\big[d_2\gamma^2 - c\gamma\big] + v^*e^{-\gamma\xi}(a_2\epsilon_3u^* - \epsilon_2v^*)\\ =&v^*e^{-\gamma\xi}\big[d_2\epsilon_2\gamma^2 - c\epsilon_2\gamma + a_2\epsilon_3u^* - \epsilon_2v^*\big] \leq 0.\end{aligned} \tag{3.28}$$

When $\xi < \frac{1}{\gamma}\ln\frac{\epsilon_1 u^*}{r_1(\infty)-u^*}$, then $\overline{\varphi}(\xi) = r_2(\infty), \underline{\phi}(\xi) \geq 0$, so

$$\begin{aligned}&d_2\overline{\varphi}''(\xi) + c\overline{\varphi}'(\xi) + g_2\big(\underline{\phi}, \overline{\varphi}\big)(\xi)\\ =&d_2\overline{\varphi}''(\xi) + c\overline{\varphi}'(\xi) + \overline{\varphi}(\xi)\big[r_2(\xi) - \overline{\varphi}(\xi) - a_2\underline{\phi}(\xi)\big]\\ \leq& r_2(\infty)[r_2(\xi) - r_2(\infty)] \leq 0.\end{aligned}$$

This and (3.28) lead to (3.27).

It follows from $\epsilon_3 > 1$ that

$$\lim_{\gamma\to 0+}\gamma^2\epsilon_3^{\frac{\alpha_1}{\gamma}} = \lim_{\gamma\to 0+}\frac{\alpha_1^2\ln^2\epsilon_3}{2}\epsilon_3^{\frac{\alpha_1}{\gamma}} = +\infty.$$

Hence for sufficiently small $\gamma > 0$, we have

$$\frac{A_1+1}{d_1} \leq \gamma^2\epsilon_3^{\frac{\alpha_1}{\gamma}},$$

which equivalent to

$$(A_1+1)\epsilon_3^{1-\frac{\alpha_1}{\gamma}} \leq d_1\epsilon_3\gamma^2. \tag{3.29}$$

Note that $r_2(\infty) - v^* = a_2u^*$ and $\epsilon_2v^* \geq a_2\epsilon_3u^*$ from (3.22), then $\frac{1}{\gamma}\ln\frac{\epsilon_2v^*}{r_2(\infty)-v^*} \geq \frac{1}{\gamma}\ln\epsilon_3$. Hence for $\xi > \frac{1}{\gamma}\ln\frac{\epsilon_2v^*}{r_2(\infty)-v^*}$, then $\underline{\phi}(\xi) = u^* - \epsilon_3u^*e^{-\gamma\xi}$, $\overline{\varphi}(\xi) = v^* + \epsilon_2v^*e^{-\gamma\xi}$, by (3.22), (3.23), (3.29) and (3.24), we have

$$\begin{aligned}&d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + g_1\big(\underline{\phi}, \overline{\varphi}\big)(\xi)\\ =&d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + \underline{\phi}(\xi)\big[r_1(\xi) - \underline{\phi}(\xi) - a_1\overline{\varphi}(\xi)\big]\\ =&-d_1\epsilon_3u^*\gamma^2e^{-\gamma\xi} + c\epsilon_3u^*\gamma e^{-\gamma\xi} + u^*\big(1-\epsilon_3e^{-\gamma\xi}\big)\big[r_1(\xi) - u^* + \epsilon_3u^*e^{-\gamma\xi}\\ &-a_1v^* - a_1\epsilon_2v^*e^{-\gamma\xi}\big]\\ =&\epsilon_3u^*e^{-\gamma\xi}\big[-d_1\gamma^2 + c\gamma\big] + u^*\big(1-\epsilon_3e^{-\gamma\xi}\big)[r_1(\xi) - r_1(\infty)]\\ &+u^*e^{-\gamma\xi}\big(1-\epsilon_3e^{-\gamma\xi}\big)(\epsilon_3u^* - a_1\epsilon_2v^*)\\ \geq&\epsilon_3u^*e^{-\gamma\xi}[-d_1\gamma^2 + c\gamma] + u^*(r_1(\xi) - r_1(\infty))\\ \geq&\epsilon_3u^*e^{-\gamma\xi}\big[-d_1\gamma^2 + c\gamma\big] - u^*(A_1+1)e^{-\alpha_1\xi}\\ =&u^*e^{-\gamma\xi}\big[-d_1\epsilon_3\gamma^2 + c\epsilon_3\gamma - (A_1+1)e^{(\gamma-\alpha_1)\xi}\big]\\ \geq&u^*e^{-\gamma\xi}\Big[-d_1\epsilon_3\gamma^2 + c\epsilon_3\gamma - (A_1+1)e^{\frac{\gamma-\alpha_1}{\gamma}\ln\epsilon_3}\Big]\\ =&u^*e^{-\gamma\xi}\Big[-d_1\epsilon_3\gamma^2 + c\epsilon_3\gamma - (A_1+1)\epsilon_3^{1-\frac{\alpha_1}{\gamma}}\Big]\\ \geq&u^*e^{-\gamma\xi}\big[-2d_1\epsilon_3\gamma^2 + c\epsilon_3\gamma\big]\\ =&u^*\epsilon_3\gamma e^{-\gamma\xi}\big[-2d_1\gamma + c\big] \geq 0.\end{aligned} \tag{3.30}$$

When $\frac{1}{\gamma}\ln\epsilon_3 < \xi \leq \frac{1}{\gamma}\ln\frac{\epsilon_2v^*}{r_2(\infty)-v^*}$, then $\underline{\phi}(\xi) = u^* - \epsilon_3u^*e^{-\gamma\xi}, \overline{\varphi}(\xi) = r_2(\infty) \leq v^* + \epsilon_2v^*e^{-\gamma\xi}$. Since $g_1(\phi,\varphi)$ is decreasing in φ, an argument similar to that used to show (3.30), we get

$$\begin{aligned}&d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + g_1\big(\underline{\phi}, \overline{\varphi}\big)(\xi)\\ =&d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + g_1\big(\underline{\phi}, r_2(\infty)\big)(\xi)\\ \geq&d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + g_1\big(\underline{\phi}, v^* + \epsilon_2v^*e^{-\gamma\xi}\big)(\xi) \geq 0.\end{aligned} \tag{3.31}$$

When $\xi < \frac{1}{\gamma}\ln\epsilon_3$, then $\underline{\phi}(\xi)=0$, so

$$\begin{aligned}&d_1\underline{\phi}''(\xi)+c\underline{\phi}'(\xi)+g_1\big(\underline{\phi},\overline{\varphi}\big)(\xi)\\=&d_1\underline{\phi}''(\xi)+c\underline{\phi}'(\xi)+\underline{\phi}(\xi)\big[r_1(\xi)-\underline{\phi}(\xi)-a_1\overline{\varphi}(\xi)\big]=0.\end{aligned}$$

This, (3.30) and (3.31) imply for $\xi \neq \frac{1}{\gamma}\ln\epsilon_3$,

$$d_1\underline{\phi}''(\xi)+c\underline{\phi}'(\xi)+g_1\big(\underline{\phi},\overline{\varphi}\big)(\xi)\geq 0. \tag{3.32}$$

Next, we want to show for $\xi \neq \frac{1}{\gamma}\ln\epsilon_4$,

$$d_2\underline{\varphi}''(\xi)+c\underline{\varphi}'(\xi)+g_2\big(\overline{\phi},\underline{\varphi}\big)(\xi)\geq 0. \tag{3.33}$$

An argument similar to that used to show (3.29), for sufficiently small $\gamma>0$, we have

$$(A_2+1)\epsilon_4^{1-\frac{\alpha_2}{\gamma}}\leq d_2\epsilon_4\gamma^2. \tag{3.34}$$

Since $r_1(\infty)-u^*=a_1v^*$ and $\epsilon_1u^*\geq a_1\epsilon_4v^*$ from (3.22), then $\frac{1}{\gamma}\ln\frac{\epsilon_1u^*}{r_1(\infty)-u^*}\geq\frac{1}{\gamma}\ln\epsilon_4$. Hence for $\xi>\frac{1}{\gamma}\ln\frac{\epsilon_1u^*}{r_1(\infty)-u^*}$, then $\underline{\varphi}(\xi)=v^*-\epsilon_4v^*e^{-\gamma\xi}, \overline{\phi}(\xi)=u^*+\epsilon_1u^*e^{-\gamma\xi}$, by (3.22), (3.23), (3.34) and (3.24), we have

$$\begin{aligned}&d_2\underline{\varphi}''(\xi)+c\underline{\varphi}'(\xi)+g_2\big(\overline{\phi},\underline{\varphi}\big)(\xi)\\
=&d_2\underline{\varphi}''(\xi)+c\underline{\varphi}'(\xi)+\underline{\varphi}(\xi)\big[r_2(\xi)-\underline{\varphi}(\xi)-a_2\overline{\phi}(\xi)\big]\\
=&-d_2\epsilon_4v^*\gamma^2e^{-\gamma\xi}+c\epsilon_4v^*\gamma e^{-\gamma\xi}+v^*\big(1-\epsilon_4e^{-\gamma\xi}\big)\big[r_2(\xi)-v^*+\epsilon_4v^*e^{-\gamma\xi}\\
&-a_2u^*-a_2\epsilon_1u^*e^{-\gamma\xi}\big]\\
=&\epsilon_4v^*e^{-\gamma\xi}\big[-d_2\gamma^2+c\gamma\big]+v^*\big(1-\epsilon_4e^{-\gamma\xi}\big)[r_2(\xi)-r_2(\infty)]\\
&+v^*e^{-\gamma\xi}(1-\epsilon_4e^{-\gamma\xi})(\epsilon_4v^*-a_2\epsilon_1u^*)\\
\geq&\epsilon_4v^*e^{-\gamma\xi}\big[-d_2\gamma^2+c\gamma\big]+v^*(r_2(\xi)-r_2(\infty))\\
\geq&\epsilon_4v^*e^{-\gamma\xi}\big[-d_2\gamma^2+c\gamma\big]-v^*(A_2+1)e^{-\alpha_2\xi}\\
=&v^*e^{-\gamma\xi}\big[-d_2\epsilon_4\gamma^2+c\epsilon_4\gamma-(A_2+1)e^{(\gamma-\alpha_2)\xi}\big]\\
\geq&v^*e^{-\gamma\xi}\Big[-d_2\epsilon_4\gamma^2+c\epsilon_4\gamma-(A_2+1)e^{\frac{\gamma-\alpha_2}{\gamma}\ln\epsilon_4}\Big]\\
=&v^*e^{-\gamma\xi}\Big[-d_2\epsilon_4\gamma^2+c\epsilon_4\gamma-(A_2+1)\epsilon_4^{1-\frac{\alpha_2}{\gamma}}\Big]\\
\geq&v^*e^{-\gamma\xi}\big[-2d_2\epsilon_4\gamma^2+c\epsilon_4\gamma\big]\\
=&v^*\epsilon_4\gamma e^{-\gamma\xi}[-2d_2\gamma+c]\geq 0.\end{aligned} \tag{3.35}$$

When $\frac{1}{\gamma}\ln\epsilon_4<\xi\leq\frac{1}{\gamma}\ln\frac{\epsilon_1u^*}{r_1(\infty)-u^*}$, then $\underline{\varphi}(\xi)=v^*-\epsilon_4v^*e^{-\gamma\xi}$, $\overline{\phi}(\xi)=r_1(\infty)\leq u^*+\epsilon_1u^*e^{-\gamma\xi}$. Since $g_2(\phi,\varphi)$ is decreasing in ϕ, an argument similar to that used to show (3.35), we get

$$\begin{aligned}&d_2\underline{\varphi}''(\xi)+c\underline{\varphi}'(\xi)+g_2\big(\overline{\phi},\underline{\varphi}\big)(\xi)\\
=&d_2\underline{\varphi}''(\xi)+c\underline{\varphi}'(\xi)+g_2\big(r_1(\infty),\underline{\varphi}\big)(\xi)\\
\geq&d_2\underline{\varphi}''(\xi)+c\underline{\varphi}'(\xi)+g_2\big(u^*+\epsilon_1u^*e^{-\gamma\xi},\underline{\varphi}\big)(\xi)\geq 0.\end{aligned} \tag{3.36}$$

When $\xi<\frac{1}{\gamma}\ln\epsilon_4$, then $\underline{\varphi}(\xi)=0$, so

$$\begin{aligned}&d_2\underline{\varphi}''(\xi)+c\underline{\varphi}'(\xi)+g_2\big(\overline{\phi},\underline{\varphi}\big)(\xi)\\
=&d_2\underline{\varphi}''(\xi)+c\underline{\varphi}'(\xi)+\underline{\varphi}(\xi)\big[r_2(\xi)-\underline{\varphi}(\xi)-a_2\overline{\phi}(\xi)\big]=0.\end{aligned}$$

This, (3.35) and (3.36) lead to (3.33). Hence the desired results come from (3.26), (3.27), (3.32) and (3.33). The proof is complete. □

4. Proofs of Theorems

This section is devoted to proving the main results by applying comparison principle and Schauder's fixed point theorem. First, we introduce the following function spaces

$$\begin{aligned} X &= \big\{\Phi = (\phi, \psi) |\ \Phi \text{ is a continuous function from } \mathbb{R} \text{ to } \mathbb{R}^2\big\}, \\ X_r &= \big\{(\phi, \psi) \in X |\ 0 \le |\phi(\xi)| \le r_1(\infty), 0 \le |\varphi(\xi)| \le r_2(\infty) \text{ for all } \xi \in \mathbb{R}\big\}, \\ X_r^+ &= \big\{(\phi, \varphi) \in X_r |\ \phi(\xi) \ge 0, \varphi(\xi) \ge 0 \text{ for all } \xi \in \mathbb{R}\big\}. \end{aligned}$$

ρ is a constant with $\rho > \max\{a_1 r_2(\infty) + 2r_1(\infty) - r_1(-\infty), a_2 r_1(\infty) + 2r_2(\infty) - r_2(-\infty)\}$. For $i = 1, 2$, let

$$\lambda_{ij} = \frac{-c \pm \sqrt{c^2 + 4d_i\rho}}{2d_i},\ j = 1, 2. \tag{4.1}$$

Obviously $\lambda_{ij}, j = 1, 2$ are the solutions of

$$d_i\lambda^2 + c\lambda - \rho = 0$$

and we assume $\lambda_{i2} > \lambda_{i1}$ for $i = 1, 2$.

For $(\phi, \psi) \in X_r^+$, we consider the operator $P = (P_1, P_2) : X_r^+ \to X$ defined as follows

$$P_i[\phi, \psi](\xi) = \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})}\left[\int_{-\infty}^{\xi} e^{\lambda_{i1}(\xi - s)} + \int_{\xi}^{\infty} e^{\lambda_{i2}(\xi - s)}\right] F_i(\phi, \psi)(s)ds,\ \forall \xi \in \mathbb{R},$$

where

$$\begin{aligned} F_1(\phi, \psi)(\xi) :=& \phi(\xi)[\rho + r_1(\xi) - a_1 r_2(\infty) - \phi(\xi) + a_1\psi(\xi)], \\ F_2(\phi, \psi)(\xi) :=& \rho\psi(\xi) + (r_2(\infty) - \psi(\xi))[r_2(\infty) - r_2(\xi) + a_2\phi(\xi) - \psi(\xi)]. \end{aligned}$$

It is easy to check that the operator P satisfies

$$d_i P_i''[\phi, \psi](\xi) + cP_i'[\phi, \psi](\xi) - \rho P_i[\phi, \psi](\xi) + F_i(\phi, \psi)(\xi) = 0,\ \forall \xi \in \mathbb{R}.$$

Lemma 4.1. *The functions* $\big(\underline{\phi}(\xi), \underline{\psi}(\xi)\big)$ *and* $\big(\overline{\phi}(\xi), \overline{\psi}(\xi)\big)$ *defined in Lemmas 3.1, 3.2 satisfy*

$$P[\underline{\phi}, \underline{\psi}](\xi) \ge \big(\underline{\phi}(\xi), \underline{\psi}(\xi)\big),\ P[\overline{\phi}, \overline{\psi}](\xi) \le \big(\overline{\phi}(\xi), \overline{\psi}(\xi)\big),\ \forall \xi \in \mathbb{R}.$$

Proof. We only prove

$$P_1[\underline{\phi}, \underline{\psi}](\xi) \ge \underline{\phi}(\xi),\ \forall \xi \in \mathbb{R},$$

the other cases can be treated similarly.

Denote $\xi^* = \frac{1}{\mu_1 - \mu_c} \ln \frac{\eta_0\psi_1}{\epsilon_1\phi_1}$. For $\xi < \xi^*$, Lemma 3.1 implies

$$\begin{aligned} & P_1[\underline{\phi}, \underline{\psi}](\xi) \\ =& \frac{1}{d_1(\lambda_{12} - \lambda_{11})}\left[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi - s)} + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi - s)}\right] F_1\big(\underline{\phi}, \underline{\psi}\big)(s)ds \\ \ge& \frac{-1}{d_1(\lambda_{12} - \lambda_{11})}\left[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi - s)} + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi - s)}\right]\big[d_1\underline{\phi}''(s) + c\underline{\phi}'(s) - \rho\underline{\phi}(s)\big]ds \\ =& \frac{-1}{d_1(\lambda_{12} - \lambda_{11})}\Big[d_1\underline{\phi}'(\xi) + d_1\lambda_{11}\underline{\phi}(\xi) + d_1\lambda_{11}^2 \int_{-\infty}^{\xi} \underline{\phi}(s)e^{\lambda_{11}(\xi - s)}ds + c\underline{\phi}(\xi) \end{aligned}$$

$$
\begin{aligned}
&+ c\lambda_{11}\int_{-\infty}^{\xi}\underline{\phi}(s)e^{\lambda_{11}(\xi-s)}ds - \rho\int_{-\infty}^{\xi}\underline{\phi}(s)e^{\lambda_{11}(\xi-s)}ds + d_1e^{\lambda_{12}(\xi-\xi^*)}\\
&\times\left(\underline{\phi}'(\xi^*-)-\underline{\phi}'(\xi^*+)\right) - d_1\underline{\phi}'(\xi) - d_1\lambda_{12}\underline{\phi}(\xi) + d_1\lambda_{12}^2\int_{\xi}^{\infty}\underline{\phi}(s)e^{\lambda_{12}(\xi-s)}ds\\
&- c\underline{\phi}(\xi) + c\lambda_{12}\int_{\xi}^{\infty}\underline{\phi}(s)e^{\lambda_{12}(\xi-s)}ds - \rho\int_{\xi}^{\infty}\underline{\phi}(s)e^{\lambda_{12}(\xi-s)}ds\Big]\\
=&\frac{-1}{d_1(\lambda_{12}-\lambda_{11})}\Big[d_1\underline{\phi}(\xi)(\lambda_{11}-\lambda_{12}) + (d_1\lambda_{11}^2 + c\lambda_{11} - \rho)\int_{-\infty}^{\xi}\underline{\phi}(s)e^{\lambda_{11}(\xi-s)}ds\\
&+ (d_1\lambda_{12}^2 + c\lambda_{12} - \rho)\int_{\xi}^{\infty}\underline{\phi}(s)e^{\lambda_{12}(\xi-s)}ds + d_1e^{\lambda_{12}(\xi-\xi^*)}\left[\underline{\phi}'(\xi^*-)-\underline{\phi}'(\xi^*+)\right]\Big]\\
=&\underline{\phi}(\xi) + \frac{1}{\lambda_{12}-\lambda_{11}}e^{\lambda_{12}(\xi-\xi^*)}\left[\underline{\phi}'(\xi^*+)-\underline{\phi}'(\xi^*-)\right].
\end{aligned}
\tag{4.2}
$$

Similarly,

$$
\begin{cases}
P_1[\underline{\phi},\underline{\psi}](\xi) \geq \underline{\phi}(\xi) + \frac{1}{\lambda_{12}-\lambda_{11}}e^{\lambda_{11}(\xi-\xi^*)}\left[\underline{\phi}'(\xi^*+)-\underline{\phi}'(\xi^*-)\right], \text{ for } \xi > \xi^*.\\
P_1[\underline{\phi},\underline{\psi}](\xi) \geq \underline{\phi}(\xi) + \frac{1}{\lambda_{12}-\lambda_{11}}\left[\underline{\phi}'(\xi^*+)-\underline{\phi}'(\xi^*-)\right], \text{ for } \xi = \xi^*.
\end{cases}
\tag{4.3}
$$

Moreover, some calculations lead to $\underline{\phi}'(\xi^*-) = 0$, and by $\mu_1 > \mu_c > 0$, we have

$$
\begin{aligned}
\underline{\phi}'(\xi^*+) &= \lim_{\Delta\xi\to 0+}\frac{\underline{\phi}(\xi^*+\Delta\xi)-\underline{\phi}(\xi^*)}{\Delta\xi}\\
&=\lim_{\Delta\xi\to 0+}\frac{\epsilon_1\phi_1e^{-\mu_c(\xi^*+\Delta\xi)} - \eta_0\psi_1e^{-\mu_1(\xi^*+\Delta\xi)}}{\Delta\xi}\\
&=\eta_0\psi_1\mu_1e^{-\mu_1\xi^*} - \epsilon_1\phi_1\mu_ce^{-\mu_c\xi^*}\\
&\geq\mu_c\left(\eta_0\psi_1e^{-\mu_1\xi^*} - \epsilon_1\phi_1e^{-\mu_c\xi^*}\right)\\
&=\mu_c\underline{\phi}(\xi^*) = 0 = \underline{\phi}'(\xi^*-).
\end{aligned}
$$

Therefore, (4.2) and (4.3) indicate

$$
P_1[\underline{\phi},\underline{\psi}](\xi) \geq \underline{\phi}(\xi), \ \forall\xi\in\mathbb{R}.
$$

The proof is complete. □

Proof of Theorem 2.1. From Lemma 4.1, we have

$$
P[\underline{\phi},\underline{\psi}](\xi) \geq \left(\underline{\phi}(\xi),\underline{\psi}(\xi)\right),\ P[\overline{\phi},\overline{\psi}](\xi) \leq \left(\overline{\phi}(\xi),\overline{\psi}(\xi)\right),\ \forall\xi\in\mathbb{R}.
$$

Consider the sequence $\left\{(\phi_n(\xi),\psi_n(\xi))\right\}$ generated by P with $(\phi_0(\xi),\psi_0(\xi)) = \left(\overline{\phi}(\xi),\overline{\psi}(\xi)\right)$. By virtue of monotonicity of $F_i(\phi,\psi)$ in ϕ and ψ, we have

$$
\left(\underline{\phi}(\xi),\underline{\psi}(\xi)\right) \leq (\phi_{n+1}(\xi),\psi_{n+1}(\xi)) \leq (\phi_n(\xi),\psi_n(\xi)) \leq \left(\overline{\phi}(\xi),\overline{\psi}(\xi)\right),\ \forall\xi\in\mathbb{R}.
$$

It follows that as $n\to\infty$, $(\phi_n(\xi),\psi_n(\xi))$ approaches a function $(\phi(\xi),\psi(\xi))$, which is a fixed point for P and

$$
\left(\underline{\phi}(\xi),\underline{\psi}(\xi)\right) \leq (\phi(\xi),\psi(\xi)) \leq \left(\overline{\phi}(\xi),\overline{\psi}(\xi)\right),\ \forall\xi\in\mathbb{R}.
$$

Hence

$$
(\phi(\infty),\psi(\infty)) = (0,0),
$$

and $\phi(z_0) > 0$ for some $z_0\in\mathbb{R}$ by the definition of $\underline{\phi}(\xi)$.

Next, we consider the value of $(\phi(\xi),\psi(\xi))$ at $\xi\to-\infty$. Denote

$$H(\phi)(\xi)=\phi(\xi)(\rho+r_1(\xi)-\phi(\xi)),\ \forall\xi\in\mathbb{R}.$$

It follows from $0\le\psi(\xi)\le r_2(\infty)$ that

$$\begin{aligned}\phi(\xi)=&P_1[\phi,\psi](\xi)\\ =&\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{\xi}e^{\lambda_{11}(\xi-s)}+\int_{\xi}^{\infty}e^{\lambda_{12}(\xi-s)}\right]F_1(\phi,\psi)(s)ds\\ \le&\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{\xi}e^{\lambda_{11}(\xi-s)}+\int_{\xi}^{\infty}e^{\lambda_{12}(\xi-s)}\right]H(\phi)(s)ds\triangleq Q[\phi](\xi).\end{aligned}$$

Some calculations yield that

$$d_1Q''[\phi](\xi)+cQ'[\phi](\xi)-\rho Q[\phi](\xi)+H(\phi)(\xi)=0,\ \forall\xi\in\mathbb{R}.$$

Consider the sequence $\{U_n(\xi)\}$ generated by Q with $U_0(\xi)=r_1(\infty)$. Since $0\le\phi(\xi)\le r_1(\infty)$ and $\phi(\xi)\le Q[\phi](\xi)$. Then the monotonicity of the operator Q and induction show that

$$\phi(\xi)\le U_{n+1}(\xi)\le U_n(\xi)\le r_1(\infty),\ \forall\xi\in\mathbb{R}.$$

It follows that as $n\to\infty$, $U_n(\xi)$ approaches a function $U(\xi)$, which is a fixed point for Q and

$$\phi(\xi)\le U(\xi)\le r_1(\infty),\ \forall\xi\in\mathbb{R}.$$

By taking limit $\xi\to-\infty$ in $U(\xi)=Q[U](\xi)$, we get

$$U(-\infty)(r_1(-\infty)-U(-\infty))=0.$$

Then $U(-\infty)=0$ because of $0\le\phi(\xi)\le U(\xi)$. Therefore

$$\phi(-\infty)=0. \tag{4.4}$$

Further, since $\psi(\xi)=P_2[\phi,\psi](\xi)$, by taking limit $\xi\to-\infty$ in $\psi(\xi)=P_2[\phi,\psi](\xi)$, we find

$$(r_2(\infty)-\psi(-\infty))(r_2(\infty)-r_2(-\infty)+a_2\phi(-\infty)-\psi(-\infty))=0.$$

By virtue of $0\le\psi(\xi)\le r_2(\infty)$ for all $\xi\in\mathbb{R}$ and (4.4), it holds

$$\psi(-\infty)=r_2(\infty).$$

This proves the statement (i).

Assume that $c<\bar{c}(\infty)$ and $(\phi(\xi),\psi(\xi))$ is a fixed point for P with $(\phi(-\infty),\psi(-\infty))=(0,r_2(\infty)),(\phi(\infty),\psi(\infty))=(0,0)$ and $\phi(z_0)>0$ for some $z_0\in\mathbb{R}$. Let $u(t,x)=\phi(x-ct),w(t,x)=\psi(x-ct)$ for all t and x. Choose $u_1(x),w_1(x)\in C(\mathbb{R})$ with compact support such that $0\le u_1(x)\le u(0,x),0\le w_1(x)\le w(0,x)$ and $u_1(x)\not\equiv0,w_1(x)\not\equiv0$. Further, the solution of

$$\begin{cases}u_t(t,x)=d_1u_{xx}(t,x)+u(t,x)(r_1(x-ct)-a_1r_2(\infty)-u(t,x)+a_1w(t,x)),\\ w_t(t,x)=d_2w_{xx}(t,x)+(r_2(\infty)-w(t,x))(r_2(\infty)-r_2(x-ct)-w(t,x)+a_2u(t,x))\end{cases}$$

with the initial value $(u_1(x),w_1(x))$ is denoted by $(u_1(t,x),w_1(t,x))$. Comparison principle implies

$$\begin{cases}u_1(t,x)\le u(t,x),\ \forall t>0,x\in\mathbb{R},\\ w_1(t,x)\le w(t,x),\ \forall t>0,x\in\mathbb{R}.\end{cases} \tag{4.5}$$

Further, it follows from [7, Lemma 4.3] (or an argument similar to that of [21, Theorem 2.7]) that for any $\epsilon \in (0, (\bar{c}(\infty) - c)/2)$,

$$\lim_{t\to\infty}\left[\sup_{(c+\epsilon)t\le x\le(\bar{c}(\infty)-\epsilon)t}|r_1(\infty) - u_1(t,x)| + |r_2(\infty) - w_1(t,x)|\right] = 0.$$

Hence

$$\lim_{t\to\infty} u_1(t, (\bar{c}(\infty) - \epsilon)t) = r_1(\infty), \ \lim_{t\to\infty} w_1(t, (\bar{c}(\infty) - \epsilon)t) = r_2(\infty). \tag{4.6}$$

However,

$$\begin{aligned}\lim_{t\to\infty} u(t, (\bar{c}(\infty) - \epsilon)t) &= \lim_{t\to\infty} \phi((\bar{c}(\infty) - c - \epsilon)t) = \phi(\infty) = 0,\\ \lim_{t\to\infty} w(t, (\bar{c}(\infty) - \epsilon)t) &= \lim_{t\to\infty} \psi((\bar{c}(\infty) - c - \epsilon)t) = \psi(\infty) = 0,\end{aligned}$$

this and (4.6) contradict (4.5). Thus the statement (ii) is proved. □

Proof of Theorem 2.2. The statement (i) can be obtained by an argument similar to that used to show Theorem 2.1 (i), we only need to prove assertion (ii). For Case (ii), (4.5) still holds, that is,

$$\begin{cases} u_1(t,x) \le u(t,x), \ \forall t > 0, x \in \mathbb{R},\\ w_1(t,x) \le w(t,x), \ \forall t > 0, x \in \mathbb{R}.\end{cases} \tag{4.7}$$

Moreover, it follows from [7, Lemma 4.4] (or an argument similar to that of [21, Theorem 2.7]) that for any $\epsilon \in (0, (\bar{c}(\infty) - c)/2)$,

$$\lim_{t\to\infty}\inf_{(c+\epsilon)t\le x\le(\bar{c}(\infty)-\epsilon)t} u_1(t,x) \ge u^*, \ \lim_{t\to\infty}\inf_{(c+\epsilon)t\le x\le(\bar{c}(\infty)-\epsilon)t} w_1(t,x) \ge w^* = r_2(\infty) - v^*.$$

Hence

$$\liminf_{t\to\infty}\ u_1(t, (\bar{c}(\infty) - \epsilon)t) \ge u^*, \ \liminf_{t\to\infty}\ w_1(t, (\bar{c}(\infty) - \epsilon)t) \ge w^*. \tag{4.8}$$

However,

$$\begin{aligned}\lim_{t\to\infty} u(t, (\bar{c}(\infty) - \epsilon)t) &= \lim_{t\to\infty} \phi((\bar{c}(\infty) - c - \epsilon)t) = \phi(\infty) = 0,\\ \lim_{t\to\infty} w(t, (\bar{c}(\infty) - \epsilon)t) &= \lim_{t\to\infty} \psi((\bar{c}(\infty) - c - \epsilon)t) = \psi(\infty) = 0,\end{aligned}$$

this and (4.8) contradict (4.7). This proves statement (ii). □

Before proving Theorem 2.5, we first give a lemma, which plays an important role in the following proof.

Lemma 4.2. *Consider Case (ii). Assume (H) holds, then for $c > 0$, (2.1) admits a forced traveling wave $(\phi(\xi), \varphi(\xi))$ with*

$$(\phi(\infty), \varphi(\infty)) = (u^*, v^*).$$

Proof. Denote

$$\begin{aligned}B_\alpha(\mathbb{R}, \mathbb{R}^2) =& \Big\{\Phi \in X_r | \sup_{\xi\in\mathbb{R}} ||\Phi(\xi)||e^{-\alpha|\xi|} < \infty\Big\}, \ |\Phi|_\alpha = \sup_{\xi\in\mathbb{R}} ||\Phi(\xi)||e^{-\alpha|\xi|},\\ \Gamma =& \big\{(\phi, \varphi) \in X_r^+ | \ \underline{\phi}(\xi) \le \phi(\xi) \le \overline{\phi}(\xi), \underline{\varphi}(\xi) \le \varphi(\xi) \le \overline{\varphi}(\xi), \ \forall \xi \in \mathbb{R}\big\},\end{aligned}$$

where X_r, X_r^+ are given in the beginning of the section 4, $0 < \alpha < \min\{\lambda_{12}, \lambda_{22}\}, ||\cdot||$ is the supremum norm in $\mathbb{R}^2$ and the functions $(\underline{\phi}(\xi), \underline{\varphi}(\xi)), (\overline{\phi}(\xi), \overline{\varphi}(\xi))$ are given by Lemma 3.3.

Calculations lead to the $(B_\alpha(\mathbb{R},\mathbb{R}^2), |\cdot|_\alpha)$ is a Banach space, and further Γ is a nonempty convex bounded closed set with respect to the weighted norm.

For any $\phi, \varphi \in X_r^+$, consider the operator $T=(T_1,T_2): X_r^+ \to X$ defined as follows

$$T_i[\phi,\varphi](\xi) = \frac{1}{d_i(\lambda_{i2}-\lambda_{i1})}\Big[\int_{-\infty}^{\xi} e^{\lambda_{i1}(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_{i2}(\xi-s)}\Big] G_i(\phi,\varphi)(s)ds,\ \forall \xi \in \mathbb{R},$$

where

$$G_1(\phi,\varphi)(\xi) = \phi(\xi)[\rho + r_1(\xi) - \phi(\xi) - a_1\varphi(\xi)],$$
$$G_2(\phi,\varphi)(\xi) = \varphi(\xi)[\rho + r_2(\xi) - \varphi(\xi) - a_2\phi(\xi)],$$

and $\lambda_{ij}, i,j=1,2$ are given by (4.1). Some calculations yield that

$$d_i T_i''[\phi,\psi](\xi) + cT_i'[\phi,\psi](\xi) - \rho T_i[\phi,\psi](\xi) + G_i(\phi,\psi)(\xi) = 0,\ \forall \xi \in \mathbb{R}.$$

We first verify that the operator T satisfies $T\Gamma \subset \Gamma$. For any $(\phi,\varphi) \in \Gamma$, then

$$\underline{\phi}(\xi) \le \phi(\xi) \le \overline{\phi}(\xi),\ \underline{\varphi}(\xi) \le \varphi(\xi) \le \overline{\varphi}(\xi),\ \forall \xi \in \mathbb{R}. \tag{4.9}$$

From (4.9), the monotonicity of $G_1(\phi,\varphi)$ in ϕ and φ and Lemma 3.3, we have

$$\begin{aligned} T_1[\phi,\varphi](\xi) =& \frac{1}{d_1(\lambda_{12}-\lambda_{11})}\Big[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)}\Big] G_1(\phi,\varphi)(s)ds \\ \ge& \frac{1}{d_1(\lambda_{12}-\lambda_{11})}\Big[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)}\Big] G_1(\underline{\phi},\overline{\varphi})(s)ds \\ \ge& \frac{-1}{d_1(\lambda_{12}-\lambda_{11})}\Big[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)}\Big] \big[d_1\underline{\phi}''(s) + c\underline{\phi}'(s) - \rho\underline{\phi}(s)\big]ds. \end{aligned}$$

An argument similar to that used to show (4.2) and (4.3), we get

$$T_1[\phi,\varphi](\xi) \ge \underline{\phi}(\xi),\ \forall \xi \in \mathbb{R}.$$

Similarly, we can get

$$T_1[\phi,\varphi](\xi) \le \overline{\phi}(\xi),\ \underline{\varphi}(\xi) \le T_2[\phi,\varphi](\xi) \le \overline{\varphi}(\xi),\ \forall \xi \in \mathbb{R}.$$

Hence $T\Gamma \subset \Gamma$.

In the following, we will show the operator $T: \Gamma \to \Gamma$ is completely continuous in the sense of the weighted norm $|\cdot|_\alpha$. For any $\Phi_1 = (\phi_1,\varphi_1), \Phi_2 = (\phi_2,\varphi_2) \in \Gamma$ and any $\xi \in \mathbb{R}$,

$$\begin{aligned} &|G_1(\phi_1,\varphi_1)(\xi) - G_1(\phi_2,\varphi_2)(\xi)| \\ =&|(\rho + r_1(\xi))(\phi_1(\xi) - \phi_2(\xi)) - (\phi_1(\xi)+\phi_2(\xi))(\phi_1(\xi)-\phi_2(\xi)) \\ &- a_1\varphi_1(\xi)(\phi_1(\xi)-\phi_2(\xi)) - a_1\phi_2(\xi)(\varphi_1(\xi)-\varphi_2(\xi))| \\ \le&[\rho + 3r_1(\infty) + a_1r_2(\infty)]|\phi_1(\xi)-\phi_2(\xi)| + a_1r_1(\infty)|\varphi_1(\xi)-\varphi_2(\xi)|. \end{aligned} \tag{4.10}$$

$$\begin{aligned} &|G_2(\phi_1,\varphi_1)(\xi) - G_2(\phi_2,\varphi_2)(\xi)| \\ =&|(\rho + r_2(\xi))(\varphi_1(\xi) - \varphi_2(\xi)) - (\varphi_1(\xi)+\varphi_2(\xi))(\varphi_1(\xi)-\varphi_2(\xi)) \\ &- a_2\varphi_1(\xi)(\phi_1(\xi)-\phi_2(\xi)) - a_2\phi_2(\xi)(\varphi_1(\xi)-\varphi_2(\xi))| \\ \le&[\rho + 3r_2(\infty) + a_2r_1(\infty)]|\varphi_1(\xi)-\varphi_2(\xi)| + a_2r_2(\infty)|\phi_1(\xi)-\phi_2(\xi)|. \end{aligned} \tag{4.11}$$

Moreover for $\xi \le 0$, by segment integration and the choose of α, we get

$$\begin{aligned}
&\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} ds + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} ds \\
=&\frac{-1}{\alpha+\lambda_{11}} + \frac{1}{\alpha+\lambda_{12}} + \frac{2\alpha}{\lambda_{12}^2-\alpha^2} e^{(\alpha+\lambda_{12})\xi} \\
\le&\frac{-1}{\alpha+\lambda_{11}} + \frac{1}{\alpha+\lambda_{12}} + \frac{2\alpha}{\lambda_{12}^2-\alpha^2} \qquad (4.12)
\end{aligned}$$

and for $\xi > 0$,

$$\begin{aligned}
&\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} ds + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} ds \\
=&\frac{1}{\alpha-\lambda_{11}} + \frac{2\alpha}{\lambda_{11}^2-\alpha^2} e^{(\lambda_{11}-\alpha)\xi} + \frac{1}{\lambda_{12}-\alpha} \\
\le&\frac{1}{\alpha-\lambda_{11}} + \frac{2\alpha}{\lambda_{11}^2-\alpha^2} + \frac{1}{\lambda_{12}-\alpha}. \qquad (4.13)
\end{aligned}$$

Similarly for $\xi \le 0$,

$$\begin{aligned}
&\int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} ds + \int_{\xi}^{\infty} e^{\lambda_{22}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} ds \\
\le&\frac{-1}{\alpha+\lambda_{21}} + \frac{1}{\alpha+\lambda_{22}} + \frac{2\alpha}{\lambda_{22}^2-\alpha^2} \qquad (4.14)
\end{aligned}$$

and for $\xi > 0$,

$$\begin{aligned}
&\int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} ds + \int_{\xi}^{\infty} e^{\lambda_{22}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} ds \\
\le&\frac{1}{\alpha-\lambda_{21}} + \frac{2\alpha}{\lambda_{21}^2-\alpha^2} + \frac{1}{\lambda_{22}-\alpha}. \qquad (4.15)
\end{aligned}$$

Let

$$C_0 = \max_{i=1,2}\left\{\frac{-1}{\alpha+\lambda_{i1}} + \frac{1}{\alpha+\lambda_{i2}} + \frac{2\alpha}{\lambda_{i2}^2-\alpha^2}, \frac{1}{\alpha-\lambda_{i1}} + \frac{2\alpha}{\lambda_{i1}^2-\alpha^2} + \frac{1}{\lambda_{i2}-\alpha}\right\}. \qquad (4.16)$$

Therefore for $\xi \in \mathbb{R}$, by (4.10), (4.12) and (4.13), we have

$$\begin{aligned}
&\left|T_1[\phi_1,\varphi_1](\xi)e^{-\alpha|\xi|} - T_1[\phi_2,\varphi_2](\xi)e^{-\alpha|\xi|}\right| \\
=&\bigg|\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|}\right] \\
&\times \left[G_1(\phi_1,\varphi_1)(s) - G_1(\phi_2,\varphi_2)(s)\right] e^{-\alpha|s|}\bigg| ds \\
\le&\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\rho + 3r_1(\infty) + a_1 r_2(\infty) + a_1 r_1(\infty)\right]\left|\Phi_1-\Phi_2\right|_\alpha \\
&\times \left[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|} + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)} e^{-\alpha|\xi|} e^{\alpha|s|}\right] ds \\
\le&C_1|\Phi_1-\Phi_2|_\alpha, \qquad (4.17)
\end{aligned}$$

where

$$C_1 = \frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\rho + 3r_1(\infty) + a_1 r_2(\infty) + a_1 r_1(\infty)\right] C_0.$$

Similarly for $\xi \in \mathbb{R}$, by (4.11), (4.14) and (4.15), we have

$$\begin{aligned}
&\big|T_2[\phi_1,\varphi_1](\xi)e^{-\alpha|\xi|} - T_2[\phi_2,\varphi_2](\xi)e^{-\alpha|\xi|}\big| \\
=&\Bigg|\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\Big[\int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-s)}e^{-\alpha|\xi|}e^{\alpha|s|} + \int_{\xi}^{\infty} e^{\lambda_{22}(\xi-s)}e^{-\alpha|\xi|}e^{\alpha|s|}\Big] \\
&\times \big[G_2(\phi_1,\varphi_1)(s) - G_2(\phi_2,\varphi_2)(s)\big]e^{-\alpha|s|}\Bigg|ds \\
\le&\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\big[\rho + 3r_2(\infty) + a_2r_1(\infty) + a_2r_2(\infty)\big]\big|\Phi_1-\Phi_2\big|_\alpha \\
&\times \Big[\int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-s)}e^{-\alpha|\xi|}e^{\alpha|s|} + \int_{\xi}^{\infty} e^{\lambda_{22}(\xi-s)}e^{-\alpha|\xi|}e^{\alpha|s|}\Big]ds \\
\le& C_2|\Phi_1-\Phi_2|_\alpha,
\end{aligned} \tag{4.18}$$

where

$$C_2 = \frac{1}{d_2(\lambda_{22}-\lambda_{21})}\big[\rho + 3r_2(\infty) + a_2r_1(\infty) + a_2r_2(\infty)\big]C_0$$

and C_0 is given by (4.16). Let $C = \max\{C_1, C_2\}$, then (4.17) and (4.18) indicate

$$|T\Phi_1 - T\Phi_2|_\alpha \le C|\Phi_1-\Phi_2|_\alpha.$$

Hence T is a continuous operator on Γ.

For any $\Phi = (\phi,\varphi) \in \Gamma$, note

$$0 \le T_1[\phi,\varphi](\xi) \le r_1(\infty),\ 0 \le T_2[\phi,\varphi](\xi) \le r_2(\infty),\ \forall \xi \in \mathbb{R}.$$

Therefore we have $|T\Phi|_\alpha \le r_1(\infty) + r_2(\infty)$. Further for any $x > y$,

$$\begin{aligned}
&\big|T_1[\phi,\varphi](x) - T_1[\phi,\varphi](y)\big| \\
=&\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\Bigg\{\Bigg|\int_{-\infty}^{x} e^{\lambda_{11}(x-s)}G_1(\phi,\varphi)(s)ds - \int_{-\infty}^{y} e^{\lambda_{11}(y-s)}G_1(\phi,\varphi)(s)ds \\
&+ \int_{x}^{\infty} e^{\lambda_{12}(x-s)}G_1(\phi,\varphi)(s)ds - \int_{y}^{\infty} e^{\lambda_{12}(y-s)}G_1(\phi,\varphi)(s)ds\Bigg|\Bigg\} \\
\le&\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\Bigg\{\Bigg|\int_{-\infty}^{x} e^{\lambda_{11}(x-s)}G_1(\phi,\varphi)(s)ds - \int_{-\infty}^{y} e^{\lambda_{11}(y-s)}G_1(\phi,\varphi)(s)ds\Bigg| \\
&+ \Bigg|\int_{y}^{\infty} e^{\lambda_{12}(y-s)}G_1(\phi,\varphi)(s)ds - \int_{x}^{\infty} e^{\lambda_{12}(x-s)}G_1(\phi,\varphi)(s)ds\Bigg|\Bigg\} \\
=&\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\Bigg\{\Bigg|\int_{-\infty}^{y} e^{-\lambda_{11}s}G_1(\phi,\varphi)(s)\left[e^{\lambda_{11}x} - e^{\lambda_{11}y}\right]ds + \int_{y}^{x} e^{\lambda_{11}(x-s)}G_1(\phi,\varphi)(s)ds\Bigg| \\
&+ \Bigg|\int_{x}^{\infty} e^{-\lambda_{12}s}G_1(\phi,\varphi)(s)\left[e^{\lambda_{12}y} - e^{\lambda_{12}x}\right]ds + \int_{y}^{x} e^{\lambda_{12}(y-s)}G_1(\phi,\varphi)(s)ds\Bigg|\Bigg\} \\
\le&\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\Bigg\{\Bigg|\int_{y}^{x} e^{\lambda_{11}(x-s)}G_1(\phi,\varphi)(s)ds\Bigg| + \Bigg|\int_{y}^{x} e^{\lambda_{12}(y-s)}G_1(\phi,\varphi)(s)ds\Bigg|\Bigg\} \\
\le&\frac{2\rho r_1(\infty)}{d_1(\lambda_{12}-\lambda_{11})}(x-y).
\end{aligned} \tag{4.19}$$

For $x \le y$, an argument similar to that used to show (4.19), we have

$$\big|T_1[\phi,\varphi](x) - T_1[\phi,\varphi](y)\big| \le \frac{2\rho r_1(\infty)}{d_1(\lambda_{12}-\lambda_{11})}(y-x).$$

Therefore for $x, y \in \mathbb{R}$,

$$\big|T_1[\phi,\varphi](x) - T_1[\phi,\varphi](y)\big| \leq \frac{2\rho r_1(\infty)}{d_1(\lambda_{12}-\lambda_{11})}|x-y|.$$

Similarly for any $x, y \in \mathbb{R}$,

$$\big|T_2[\phi,\varphi](x) - T_2[\phi,\varphi](y)\big| \leq \frac{2\rho r_2(\infty)}{d_2(\lambda_{22}-\lambda_{21})}|x-y|.$$

Hence for any $x, y \in \mathbb{R}$,

$$\big|T[\Phi](x) - T[\Phi](y)\big| \leq C_3|x-y|.$$

where

$$C_3 = \max\left\{\frac{2\rho r_1(\infty)}{d_1(\lambda_{12}-\lambda_{11})}, \frac{2\rho r_2(\infty)}{d_2(\lambda_{22}-\lambda_{21})}\right\}.$$

Thus T is a completely continuous operator on Γ. Therefore, by virtue of Schauder's fixed point theorem, there exists $\Phi = (\phi, \varphi) \in \Gamma$ such that $T\Phi = \Phi$. Obviously,

$$\big(\underline{\phi}(\xi), \underline{\varphi}(\xi)\big) \leq (\phi(\xi), \varphi(\xi)) \leq \big(\overline{\phi}(\xi), \overline{\varphi}(\xi)\big),\ \forall \xi \in \mathbb{R},$$

and $\big(\underline{\phi}(\infty), \underline{\varphi}(\infty)\big) = (u^*, v^*)$, $\big(\overline{\phi}(\infty), \overline{\varphi}(\infty)\big) = (u^*, v^*)$. Hence

$$(\phi(\infty), \varphi(\infty)) = (u^*, v^*).$$

The proof is complete. □

Proof of Theorem 2.5. By virtue of Lemma 4.2, we only need to prove

$$(\phi(-\infty), \varphi(-\infty)) = (0, 0).$$

It follows from $0 \leq \varphi(\xi) \leq r_2(\infty)$ for all $\xi \in \mathbb{R}$ that

$$\begin{aligned}
\phi(\xi) =& T_1[\phi,\varphi](\xi) \\
=& \frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)}\right] G_1(\phi,\varphi)(s)ds \\
\leq& \frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)}\right] G(\phi)(s)ds \triangleq Q[\phi](\xi),
\end{aligned}$$

where

$$G(\phi)(\xi) = \phi(\xi)[\rho + r_1(\xi) - \phi(\xi)],\ \forall \xi \in \mathbb{R}.$$

An argument similar to that used to show (4.4), we can get $\phi(-\infty) = 0$. Further, similarly we can find $\varphi(-\infty) = 0$. The proof is complete. □

Acknowledgments

The authors would like to thank the anonymous referees for careful reading and valuable suggestions. The first author was supported by China Scholarship Council through grant (201806180083). The second author was partially supported by the National Science Foundation under Grant DMS-1515875 and Grant DMS-1951482. The third author was partially supported by NSF of China (11731005, 11671180).

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