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Sub-Hermitian geometry and the quantitative Newlander-Nirenberg theorem

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A R T I C L E   I N F O

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A B S T R A C T

Given a finite collection of C^1 complex vector fields on a C^2 manifold M such that they and their complex conjugates span the complexified tangent space at every point, the classical Newlander-Nirenberg theorem gives conditions on the vector fields so that there is a complex structure on M with respect to which the vector fields are $T^{0,1}$. In this paper, we give intrinsic, diffeomorphic invariant, necessary and sufficient conditions on the vector fields so that they have a desired level of regularity with respect to this complex structure (i.e., smooth, real analytic, or have Zygmund regularity of some finite order). By addressing this in a quantitative way we obtain a holomorphic analog of the quantitative theory of sub-Riemannian geometry initiated by Nagel, Stein, and Wainger. We call this sub-Hermitian geometry. Moreover, we proceed more generally and obtain similar results for manifolds which have an associated formally integrable elliptic structure. This allows us to introduce a setting which generalizes both the real and complex theories.

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1. Introduction

Let M be a C^2 manifold and let $L_1, \ldots, L_m$ be C^1 complex vector fields on M. Suppose, $\forall \zeta \in M$,

- $L_1(\zeta), \ldots, L_m(\zeta), \overline{L_1}(\zeta), \ldots, \overline{L_m}(\zeta)$ span $\mathbb{C}T_\zeta M$.
- $[L_j, L_k](\zeta) \in \operatorname{span}_{\mathbb{C}} \{L_1(\zeta), \ldots, L_m(\zeta)\}$, $\forall 1 \leq j, k \leq m$.
- $\operatorname{span}_{\mathbb{C}} \{L_1(\zeta), \ldots, L_m(\zeta)\} \bigcap \operatorname{span}_{\mathbb{C}} \left\{\overline{L_1}(\zeta), \ldots, \overline{L_m}(\zeta)\right\} = \{0\}$.

Under these conditions, the classical Newlander-Nirenberg Theorem (see [13]) states that M can be given the structure of a complex manifold such that $L_1(\zeta), \ldots, L_m(\zeta)$ form a spanning set of $T^{0,1}_\zeta(M)$, $\forall \zeta \in M$; and this is the unique such complex structure on M. For $s > 0$ we let $\mathscr{C}^s$ denote the Zygmund[2] space of order s (see Section 2.1), $\mathscr{C}^\infty$ denote the space of smooth functions, and $\mathscr{C}^\omega$ the space of real analytic functions. For $s \in (0, \infty] \cup \{\omega\}$ if M is known to be a $\mathscr{C}^{s+2}$ manifold[3] and $L_1, \ldots, L_m$ are known to be $\mathscr{C}^{s+1}$ vector fields on M, then it is a result of Malgrange [15] that the complex structure on M is compatible with the original $\mathscr{C}^{s+2}$ manifold structure, and therefore $L_1, \ldots, L_m$ are also $\mathscr{C}^{s+1}$ with respect to the complex structure on M–and this is the best one can say in general regarding the regularity of the vector fields $L_1, \ldots, L_m$ with respect to the complex structure.[4]

In this paper, we proceed in a different direction and only assume M is a C^2 manifold and $L_1, \ldots, L_m$ are C^1 vector fields on M as above, and investigate the following two closely related questions for $s \in (1, \infty] \cup \{\omega\}$:

(i) When are the vector fields, $L_1, \ldots, L_m$, $\mathscr{C}^{s+1}$ with respect to the above complex structure on M? We present necessary and sufficient conditions for this to hold, which are intrinsic to the C^2 structure on M (and can be checked locally in any C^2 coordinate system on M).

(ii) Under the conditions we give for (i), how can we pick a holomorphic coordinate system near each point so that the vector fields $L_1, \ldots, L_m$ are normalized in this coordinate system in a way which is useful for applying techniques from analysis? See Section 1.2.2 for an example of what we mean by "normalized".

The real analogs of the above two questions were answered in a work of Stovall and the author [27,31,32]. The coordinate charts in those papers were seen as scaling maps in sub-Riemannian geometry. The quantitative study of scaling maps in sub-Riemannian

[2] For non-integer exponents, the Zygmund space agrees with the Hölder space. More precisely, for $m \in \mathbb{N}$ and $a \in (0, 1)$, the Zygmund space $\mathscr{C}^{m+a}$ is locally the same as the Hölder space $C^{m,a}$ (see [36, Theorem 1.118 (i)]). For $a \in \{0, 1\}$, these spaces differ: $C^{m+1,0} \subsetneq C^{m,1} \subsetneq \mathscr{C}^{m+1}$.

[3] We use the convention $\infty + 1 = \infty + 2 = \infty$ and $\omega + 1 = \omega + 2 = \omega$.

[4] [15] used Hölder spaces with non-integer exponents instead of Zygmund spaces, though the proof extends to Zygmund spaces. See [33] for a further discussion in the setting of Zygmund spaces.

geometry began with the foundational work of Nagel, Stein, and Wainger [19] and the closely related work of C. Fefferman and Sánchez-Calle [10], and was furthered by Tao and Wright [34, Section 4] and the author [28], and most recently in the above mentioned series of papers [27,31,32]. Since Nagel, Stein, and Wainger's original work, these ideas have had many applications. They have been particularly useful in the study of partial differential equations defined by vector fields; see the notes at the end of Chapter 2 of [30] for some comments on this history.

When applying these ideas to questions in several complex variables (when working on, for example, a complex manifold) a problem immediately arises. The scaling maps studied by Nagel, Stein, and Wainger (and in the subsequent works described above) are not holomorphic. Thus, if one tries to rescale questions using these maps, one destroys any holomorphic aspects of the questions under consideration. Nevertheless, scaling techniques are one of the main tools needed to prove the quantitative estimates required to apply the theory of singular integrals to partial differential operators. Thus, when working in the complex category, one needs a different approach than the one given by Nagel, Stein, and Wainger to be able to scale with holomorphic maps. Some authors use ad hoc methods to create these scaling maps for the particular problem they wish to study (e.g., by using non-isotropic dilations determined by the Taylor series of some ingredients in the problem)–see, e.g., [18, Section 3], [4, Section 3.3.2], and [2, Section 2.1].

A main goal of this paper is to adapt the results of Nagel, Stein, and Wainger [19] (and more generally, the results of [27,31,32]) to the complex category. Thus, in an appropriate setting, one obtains *holomorphic* scaling maps adapted to a collection of complex vector fields. Much as the theory of Nagel, Stein, and Wainger allows one to quantitatively study sub-Riemannian geometry on a real manifold, the theory in this paper allows one to quantitatively study certain sub-Riemannian geometries on a complex manifold which are well adapted to the complex structure, using only holomorphic maps. We call such geometries sub-Hermitian.

While the complex setting is easier to understand, we proceed more generally than above. Instead of working with the category of complex manifolds, we work more generally in the category of real manifolds endowed with an elliptic structure; we call these manifolds E-manifolds (see Section 6). This allows us to state a general theorem which implies both the results in the complex setting, as well as generalizes the results from the real setting in [27,31,32]. The more general results apply, in some cases, to CR manifolds (see Section 6.1 for the relationship between E-manifolds and CR manifolds and Section 8.4 for a discussion of our results in a setting on CR manifolds).

Our main result in the complex setting can be seen as a diffeomorphic invariant,[5] quantitative version of the classical Newlander-Nirenberg theorem [21], while the more general main result in the elliptic setting can be seen as a diffeomorphic invariant, quantitative version of Nirenberg's theorem on the integrability of elliptic structures [23].

[5] Here, by diffeomorphic invariant, we mean that all of the quantitative estimates are invariant under arbitrary C^2 diffeomorphisms. See Section 4.3.

1.1. Comparison with previous results

The results in this paper can be compared to previous work in two ways:

- We provide a quantitatively diffeomorphic invariant approach to the classical Newlander-Nirenberg theorem, and more generally Nirenberg's theorem on the integrability of elliptic structures.
- We provide a holomorphic analog of the quantitative theory of sub-Riemannian geometry due to Nagel, Stein, and Wainger [19]; and more generally results on "E-manifolds." See Section 6 for the definition of E-manifolds.

We have already described the second point, so we focus on the first.

In previous results on the Newlander-Nirenberg theorem, one is given complex vector fields $L_1, \ldots, L_m$, as described at the start of the introduction, with some fixed regularity (e.g., in $\mathscr{C}^{s+1}$ for some $s > 0$). Given a fixed point $\zeta_0 \in M$, the goal is to find a $\mathscr{C}^{s+2}$ coordinate chart $\Phi : B_{\mathbb{C}^n}(1) \to W$ (where W is a neighborhood of ζ_0), such that $\Phi^* L_1, \ldots, \Phi^* L_m$ are $T^{0,1}$ (i.e., are spanned by $\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}$); in this case $\Phi^* L_1, \ldots, \Phi^* L_m$ are $\mathscr{C}^{s+1}$. $\mathscr{C}^{s+2}$ is the optimal possible regularity for Φ (in general), and was established by Malgrange [15].

Our results take a different perspective. In this paper, the vector fields are only assumed to be C^1, and we ask the question as to when it is possible to choose a C^2 coordinate chart Φ so that the vector fields are $\mathscr{C}^{s+1}$ and $T^{0,1}$. Our results imply the above classical results on the Newlander-Nirenberg theorem[6] but are more general: our results are invariant under arbitrary C^2 diffeomorphisms (whereas previous results are only invariant under $\mathscr{C}^{s+2}$ diffeomorphisms).

Remark 1.1. The main results of this paper are in Section 4. There are many aspects of the main results which are important for applications. Some of these are:

- They are invariant under arbitrary C^2 diffeomorphisms (see Section 4.3). For example, this allows us to understand the regularity of a given collection of C^1 complex vector fields, satisfying the conditions of the Newlander-Nirenberg theorem, with respect to the induced complex structure. See, e.g., Section 3.1 and more generally Section 7.1.
- They are quantitative. This allows us to view the induced coordinate charts as scaling maps in "sub-Hermitian geometry" (see Section 3.2.2) and more generally "sub-E geometry" (see Section 7.2). The quantitative nature of our results also has some applications to singular foliations; see Section 4.4.
- Instead of dealing with complex structures, we state our results in the context of elliptic structures (see Section 6). This allows us to state a general theorem which

[6] At least for $s > 1$.

includes both the complex setting and the real setting of [27,31,32] as special cases. This more general setting applies, in some instances, to CR-manifolds.

Because we include all these considerations into our main results, the statements of these results are quite technical. In Section 1.2 we state some simple corollaries of the main results of this paper which are less technical, to help give the reader an idea of the types of results we are interested in. Furthermore, we describe several more significant consequences of the main results in Section 3. We hope that if the reader reads these results before the main results, it will make the main results easier to digest.

1.2. Some simple corollaries

Before we introduce all the relevant function spaces and notation, in this section we present some easy to understand corollaries of our main result to help give the reader an idea of the direction of this paper. Here, we only consider the smooth setting; precise statements of more general results appear later in the paper. We also only consider the complex setting in this section; the more general setting of E-manifolds is described in Sections 6 and 7. There are two, related, ways in which the main result of this paper (Theorem 4.5) can be understood. Below we give examples of these two perspectives. The main result addresses both of these perspectives simultaneously, and we will see that it also applies to several other situations.

1.2.1. Smoothness in the Newlander-Nirenberg theorem

Let $L_1, \ldots, L_m$ be C^1 complex vector fields defined on an open set $W \subseteq \mathbb{C}^n$. Fix a point $\zeta_0 \in U$. We wish to understand when the following goal is possible:

Goal 1.2. Find a C^2 diffeomorphism $\Phi : U \to W'$, where $U \subseteq \mathbb{C}^n$ is open and $W' \subseteq W$ is an open set containing ζ_0 such that:

- The vector fields $\Phi^* L_1, \ldots, \Phi^* L_m$ are C^∞ vector fields on U.
- $\forall \zeta \in U$,

$$\mathrm{span}_{\mathbb{C}}\{\Phi^* L_1(\zeta), \ldots, \Phi^* L_m(\zeta)\} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \overline{z}_1}, \ldots \frac{\partial}{\partial \overline{z}_n} \right\}.$$

There are some obvious necessary conditions for Goal 1.2 to be possible. Namely, that there be an open neighborhood $W'' \subseteq W$ containing ζ_0, such that the following holds:

(i) $\mathrm{span}_{\mathbb{C}}\{L_1(\zeta), \ldots, L_m(\zeta)\} \bigcap \mathrm{span}_{\mathbb{C}}\{\overline{L_1}(\zeta), \ldots, \overline{L_m}(\zeta)\} = \{0\}$, $\forall \zeta \in W''$.

(ii) $\mathrm{span}_{\mathbb{C}}\{L_1(\zeta), \ldots, L_m(\zeta), \overline{L_1}(\zeta), \ldots, \overline{L_m}(\zeta)\} = \mathbb{C}T_\zeta W''$, $\forall \zeta \in W''$.

(iii) $[L_j, L_k] = \sum_{l=1}^m c_{j,k}^{1,l} L_l$ and $[L_j, \overline{L_k}] = \sum_{l=1}^m c_{j,k}^{2,l} L_l + \sum_{l=1}^m c_{j,k}^{3,l} \overline{L_l}$, where $c_{j,k}^{1,l}, c_{j,k}^{2,l}, c_{j,k}^{3,l} : W'' \to \mathbb{C}$ and satisfy the following: for any sequence $V_1, \ldots, V_K \in \{L_1, \ldots, L_m, \overline{L_1}, \ldots, \overline{L_m}\}$, of any length $K \in \mathbb{N}$, we have

$$V_1 V_2 \cdots V_K c_{j,k}^{p,l}$$

defines a continuous function $W'' \to \mathbb{C}$, $1 \leq p \leq 3$, $1 \leq j, k, l \leq m$.

That these conditions are necessary to achieve Goal 1.2 is clear: if Goal 1.2 holds, the above conditions all clearly hold for the vector fields $\Phi^* L_1, \ldots, \Phi^* L_m$. Indeed, for the vector fields $\Phi^* L_1, \ldots, \Phi^* L_m$ one may take $c_{j,k}^{p,l}$ to be C^∞ functions on U. The above conditions are all invariant under C^2 diffeomorphisms, and therefore if they hold for $\Phi^* L_1, \ldots, \Phi^* L_m$, they must also hold for the original vector fields $L_1, \ldots, L_m$. Our first corollary of Theorem 4.5 says the following:

Corollary 1.3. *The above necessary conditions are also sufficient to obtain Goal 1.2.*

See Section 3.1.2 for a more general version of Corollary 1.3.

1.2.2. Normalizing vector fields

Suppose one is given complex vector fields $L_1, \ldots, L_m$ on an open set $W \subseteq \mathbb{C}^n$ of the form:

$$L_j = \sum_{k=1}^n b_j^k \frac{\partial}{\partial \overline{z}_k}, \quad b_j^k \in C^\infty(W), \tag{1.1}$$

and such that $\forall \zeta \in W$,

$$\operatorname{span}_{\mathbb{C}}\{L_1(\zeta), \ldots, L_m(\zeta)\} = \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \overline{z}_1}, \ldots \frac{\partial}{\partial \overline{z}_n}\right\}. \tag{1.2}$$

Assign to each L_j a formal degree $d_j \in [1, \infty)$. For $\delta \in (0, 1]$ (we think of δ as small), one tends to think of the vector fields $\delta^{d_1} L_1, \ldots, \delta^{d_m} L_m$ as being small. Fix a point $\zeta_0 \in W$. Our next goal is to find a holomorphic coordinate system, near ζ_0, in which the vector fields are not small. More precisely, we wish to understand when the following goal is possible:

Goal 1.4. For each $\delta \in (0, 1]$ find a biholomorphism $\Phi_\delta : B_{\mathbb{C}^n}(1) \to W_\delta$ with $\Phi_\delta(0) = \zeta_0$, where $B_{\mathbb{C}^n}(1)$ is the unit ball in $\mathbb{C}^n$ and $W_\delta \subseteq W$ is an open neighborhood of ζ_0, such that:

- $\Phi_\delta^* \delta^{d_1} L_1, \ldots, \Phi_\delta^* \delta^{d_m} L_m$ are C^∞ vector fields, uniformly in $\delta \in (0, 1]$, in the sense that

$$\max_{1 \leq j \leq m} \sup_{\delta \in (0,1]} \|\Phi_\delta^* \delta^{d_j} L_j\|_{C^k(B_{\mathbb{C}^n}(1); \mathbb{C}^n)} < \infty, \quad \forall k \in \mathbb{N}.$$

- Because Φ_δ is a biholomorphism, we have

$$\operatorname{span}_{\mathbb{C}}\{\Phi_\delta^*\delta^{d_1}L_1(z),\ldots,\Phi_\delta^*\delta^{d_m}L_m(z)\}=\operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial\overline{z}_1},\cdots\frac{\partial}{\partial\overline{z}_n}\right\},\quad \forall z\in B_{\mathbb{C}^n}(1).$$

We ask that this be true uniformly in δ in the sense

$$\inf_{\delta\in(0,1]}\max_{j_1,\ldots,j_n\in\{1,\ldots,m\}}\inf_{z\in B_{\mathbb{C}^n}(1)}\left|\det\left(\Phi_\delta^*\delta^{d_{j_1}}L_{j_1}(z)|\cdots|\Phi_\delta^*\delta^{d_{j_n}}L_{j_n}(z)\right)\right|>0;$$

where the matrix $\left(\Phi_\delta^*\delta^{d_{j_1}}L_{j_1}(z)|\cdots|\Phi_\delta^*\delta^{d_{j_n}}L_{j_n}(z)\right)$ is the $n\times n$ matrix whose columns are given by the coefficients of the vector fields $\Phi_\delta^*\delta^{d_{j_k}}L_{j_k}(z)$, written as linear combinations of $\frac{\partial}{\partial\overline{z}_1},\ldots,\frac{\partial}{\partial\overline{z}_n}$.

Goal 1.4 can be thought of as rescaling the vector fields so that they are "normalized". Indeed, the vector fields $\Phi_\delta^*\delta^{d_1}L_m,\ldots,\Phi_\delta^*\delta^{d_m}L_m$ are C^∞ uniformly in δ and span $T^{0,1}B_{\mathbb{C}^n}(1)$ uniformly in δ. In short, we have changed coordinates near ζ_0 to turn the case of δ small back into a situation similar to $\delta=1$. Notice that Goal 1.2 is trivial in the situation we are considering; nevertheless we will see that the necessary and sufficient condition for when to Goal 1.4 is possible looks very similar to the necessary and sufficient conditions for when Goal 1.2 is possible.

There is an obvious necessary condition for Goal 1.4 to be possible. Namely, that for every $\delta\in(0,1]$, there is an open neighborhood $W_\delta'\subseteq W$ of ζ_0 such that the follow holds. For every $\delta\in(0,1]$, $[\delta^{d_j}L_j,\delta^{d_k}L_k]=\sum_{l=1}^m c_{j,k}^{1,l,\delta}\delta^{d_l}L_l$ and $[\delta^{d_j}L_j,\delta^{d_k}\overline{L_k}]=\sum_{l=1}^m c_{j,k}^{2,l,\delta}\delta^{d_l}L_l+\sum_{l=1}^m c_{j,k}^{3,l,\delta}\delta^{d_l}\overline{L_l}$, where $c_{j,k}^{1,l,\delta},c_{j,k}^{2,l,\delta},c_{j,k}^{3,l,\delta}:W_\delta'\to\mathbb{C}$ and satisfy the following: for any sequence $V_1^\delta,\ldots,V_K^\delta\in\{\delta^{d_1}L_1,\ldots,\delta^{d_m}L_m,\delta^{d_1}\overline{L_1},\ldots,\delta^{d_m}\overline{L_m}\}$, of any length $K\in\mathbb{N}$, we have

$$\sup_{\delta\in(0,1]}\|V_1^\delta\cdots V_K^\delta c_{j,k}^{p,l,\delta}\|_{C(W_\delta')}<\infty,$$

$\forall 1\leq p\leq 3$, $1\leq j,k,l\leq m$. That this condition is necessary is clear: if Φ_δ exists as in Goal 1.4, then one may write

$$[\Phi_\delta^*\delta^{d_j}L_j,\Phi_\delta^*\delta^{d_k}L_k]=\sum_{l=1}^m\hat{c}_{j,k}^{1,l,\delta}\Phi_\delta^*\delta^{d_l}L_l,$$

$$[\Phi_\delta^*\delta^{d_j}L_j,\Phi_\delta^*\delta^{d_k}\overline{L_k}]=\sum_{l=1}^m\hat{c}_{j,k}^{2,l,\delta}\Phi_\delta^*\delta^{d_l}L_l+\sum_{l=1}^m\hat{c}_{j,k}^{3,l,\delta}\Phi_\delta^*\delta^{d_l}\overline{L_l},$$

where $\hat{c}_{j,k}^{p,l,\delta}\in C^\infty(B_{\mathbb{C}^n}(1))$, *uniformly in* $\delta\in(0,1]$. Setting $c_{j,k}^{p,l,\delta}:=\hat{c}_{j,k}^{p,l,\delta}\circ\Phi_\delta^{-1}$ and $W_\delta':=\Phi_\delta(B^n(1))$, we see that the above condition is necessary. It is also necessary that the set W_δ' must not be too small: it must essentially contain a sub-Riemannian ball adapted to the vector fields $\delta^{d_1}L_1,\ldots,\delta^{d_m}L_m$. This is somewhat technical to make precise (see Remark 1.7 for a precise statement), and the reader may wish to skip this on a first reading. Our next corollary is that the above necessary condition is also sufficient.

Corollary 1.5. *Once the requirement on the size of W'_δ described above is made precise (see Remark 1.7), the above necessary condition is also sufficient for Goal 1.4 to be possible.*

Proof. This follows from Theorem 4.5, using Lemma 4.13. □

At first glance, it may be hard to see Corollary 1.5 as a consequence of Theorem 4.5. Indeed, the thrust of Corollary 1.5 is that we have a result which is "uniform in δ". In Theorem 4.5, there is no parameter similar to δ for the results to be uniform in: there is just one finite list of vector fields, which does not depend on any variable like δ. The key is that we keep careful track of what all the estimates in Theorem 4.5 depend on. Because of this, we may apply Theorem 4.5 to each of the lists $\delta^{d_1}L_1, \ldots, \delta^{d_m}L_m$, for $\delta \in (0,1]$, and obtain results which are uniform in δ–this is because we can see from the dependances of the estimates in Theorem 4.5 that they do not depend on $\delta \in (0,1]$, when applied to $\delta^{d_1}L_1, \ldots, \delta^{d_m}L_m$.

Thus, to proceed in this way, it is essential to keep careful track of what each constant depends on in Theorem 4.5. This is notationally cumbersome, but is justified because it applies not only to results like Corollary 1.5, but also to much more complicated situations. For example, one might consider vector fields that depend on δ in a more complicated way than above, or consider the multi-parameter case $\delta \in (0,1]^\nu$, or look for results which are uniform in the base point ζ_0. All of these are possible, and follow from Theorem 4.5 in the same way Corollary 1.5 does. See, for example, Section 8.

For a setting which generalizes Corollary 1.5 and which appears in several complex variables, see Section 8. For some more significant results similar to, but slightly different than Corollary 1.5, see Section 3.2.2–there we will see similar ideas as providing holomorphic scaling maps adapted sub-Riemannian geometries on a complex manifold.

Remark 1.6. In light of the above discussion, one way to think about one aspect of Theorem 4.5 is the following. Suppose you are given smooth vector fields $L_1, \ldots, L_m$ of the form described in (1.1) satisfying (1.2). But suppose the vector fields have very large coefficients, or very small coefficients (for example, in the above setting the coefficients were very small when δ was small). Theorem 4.5 provides necessary and sufficient conditions on when one can apply a holomorphic change of variables to normalize the coefficients in the way described above.

Remark 1.7. The size of W'_δ can be described as follows. There exists $\xi > 0$ (independent of $\delta \in (0,1]$) such that

$$B_{\delta^{d_1}L_1,\ldots,\delta^{d_m}L_m}(\zeta_0, \xi) \subseteq W'_\delta. \tag{1.3}$$

See (2.2) and (2.4) for the definition of this ball. In the above description of necessity of our condition for Goal 1.4, we chose $W'_\delta = \Phi_\delta(B^n(1))$. Thus, to prove the necessity of (1.3), under the conclusions of Goal 1.4, we wish to show

$$B_{\delta^{d_1}L_1,\ldots,\delta^{d_m}L_m}(\zeta_0,\xi) \subseteq \Phi_\delta(B^n(1)), \tag{1.4}$$

for some $\xi > 0$, independent of $\delta \in (0,1]$. Once we prove (1.4), it will show (1.3) is necessary for Goal 1.4 to hold. To see (1.4), note that the Picard-Lindelöf Theorem shows that there exists $\xi > 0$, independent of $\delta \in (0,1]$ such that

$$B_{\Phi_\delta^*\delta^{d_1}L_1,\ldots,\Phi_\delta^*\delta^{d_m}L_m}(0,\xi) \subseteq B_{\mathbb{C}^n}(1/2). \tag{1.5}$$

Applying Φ_δ to both sides of (1.5) implies (1.4), which completes the proof of necessity.

2. Function spaces

In this section, we introduce the function spaces which are used in this paper. We make a distinction between function spaces on open subsets of $\mathbb{R}^n$ and function spaces on a C^2 manifold M. $\mathbb{R}^n$ is endowed with its usual real analytic structure, and it makes sense to consider all the usual function spaces on an open subset of $\mathbb{R}^n$. Since M is merely a C^2 manifold, it does not make sense to consider, for example, C^∞ functions on M. However, if we are given a finite collection of C^1 vector fields on M, it makes sense to consider functions which are C^∞ with respect to these vector fields, and that is how we will proceed. The following function spaces were defined in [27], and we refer the reader there for a more detailed discussion. Throughout the paper, given a Banach space $\mathscr{X}$, we denote by $B_{\mathscr{X}}(r)$ the ball of radius $r > 0$ centered at $0 \in \mathscr{X}$.

2.1. Function spaces on Euclidean space

Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set (we will almost always be considering the case when Ω is a ball in $\mathbb{R}^n$). We have the following classical spaces of functions on Ω:

$$C(\Omega) = C^0(\Omega) := \{f : \Omega \to \mathbb{C} \bigm| f \text{ is continuous and bounded}\},$$
$$\|f\|_{C(\Omega)} = \|f\|_{C^0(\Omega)} := \sup_{x\in\Omega} |f(x)|.$$

For $m \in \mathbb{N}$, (we use the convention $0 \in \mathbb{N}$)

$$C^m(\Omega) := \{f \in C(\Omega) \bigm| \partial_x^\alpha f \in C(\Omega), \forall |\alpha| \leq m\}, \quad \|f\|_{C^m(\Omega)} := \sum_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{C(\Omega)}.$$

Next we define the classical Hölder spaces. For $s \in [0,1]$,

$$\begin{aligned} \|f\|_{C^{0,s}(\Omega)} &:= \|f\|_{C(\Omega)} + \sup_{\substack{x,y\in\Omega\\ x\neq y}} |x-y|^{-s}|f(x)-f(y)|, \\ C^{0,s}(\Omega) &:= \{f \in C(\Omega) : \|f\|_{C^{0,s}(\Omega)} < \infty\}. \end{aligned} \tag{2.1}$$

For $m \in \mathbb{N}$, $s \in [0,1]$,

$$\|f\|_{C^{m,s}(\Omega)} := \sum_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{C^{0,s}(\Omega)}, \quad C^{m,s}(\Omega) := \{f \in C^m(\Omega) : \|f\|_{C^{m,s}(\Omega)} < \infty\}.$$

Next, we turn to the classical Zygmund spaces. Given $h \in \mathbb{R}^n$ define $\Omega_h := \{x \in \mathbb{R}^n : x, x+h, x+2h \in \Omega\}$. For $s \in (0,1]$ set

$$\|f\|_{\mathscr{C}^s(\Omega)} := \|f\|_{C^{0,s/2}(\Omega)} + \sup_{\substack{0\neq h\in\mathbb{R}^n\\ x\in\Omega_h}} |h|^{-s}|f(x+2h)-2f(x+h)+f(x)|,$$
$$\mathscr{C}^s(\Omega) := \{f \in C(\Omega) : \|f\|_{\mathscr{C}^s(\Omega)} < \infty\}.$$

For $m \in \mathbb{N}$, $s \in (0,1]$, set

$$\|f\|_{\mathscr{C}^{m+s}(\Omega)} := \sum_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{\mathscr{C}^s(\Omega)}, \quad \mathscr{C}^{m+s}(\Omega) := \{f \in C^m(\Omega) : \|f\|_{\mathscr{C}^{m+s}(\Omega)} < \infty\}.$$

We set

$$\mathscr{C}^\infty(\Omega) := \bigcap_{s>0} \mathscr{C}^s(\Omega), \quad C^\infty(\Omega) := \bigcap_{m\in\mathbb{N}} C^m(\Omega).$$

If Ω is a ball, $\mathscr{C}^\infty(\Omega) = C^\infty(\Omega)$.

Finally, we turn to spaces of real analytic functions. Given $r > 0$, we define

$$\|f\|_{C^{\omega,r}(\Omega)} := \sum_{\alpha\in\mathbb{N}^n} \frac{\|\partial_x^\alpha f\|_{C(\Omega)}}{\alpha!} r^{|\alpha|}, \quad C^{\omega,r}(\Omega) := \{f \in C^\infty(\Omega) : \|f\|_{C^{\omega,r}(\Omega)} < \infty\}.$$

We set

$$C^\omega(\Omega) := \bigcup_{r>0} C^{\omega,r}(\Omega), \quad \mathscr{C}^\omega(\Omega) := C^\omega(\Omega).$$

We also define another space of real analytic functions. We define $\mathscr{A}^{n,r}$ to be the space of those $f \in C(B_{\mathbb{R}^n}(r))$ such that $f(t) = \sum_{\alpha\in\mathbb{N}^n} \frac{c_\alpha}{\alpha!} t^\alpha$, where

$$\|f\|_{\mathscr{A}^{n,r}} := \sum_{\alpha\in\mathbb{N}^n} \frac{|c_\alpha|}{\alpha!} r^{|\alpha|} < \infty.$$

See Lemma 9.1 (vi) and (vii) for the relationship between $\mathscr{A}^{n,r}$ and C^ω.

For $s \in (0,\infty] \cup \{\omega\}$, we say $f \in \mathscr{C}^s_{\mathrm{loc}}(\Omega)$ if $\forall x \in \Omega$, there exists an open ball $B \subseteq \Omega$, centered at x, such that $f\big|_B \in \mathscr{C}^s(B)$. It is immediate to verify that $\mathscr{C}^\infty_{\mathrm{loc}}(\Omega)$ is the usual space of smooth functions on Ω and $\mathscr{C}^\omega_{\mathrm{loc}}(\Omega)$ is the usual space of real analytic functions on Ω.

If $\mathscr{X}$ is a Banach Space, we define the same spaces taking values in $\mathscr{X}$ in the obvious way, and denote these spaces by $C(\Omega;\mathscr{X})$, $C^m(\Omega;\mathscr{X})$, $C^{m,s}(\Omega;\mathscr{X})$, $\mathscr{C}^s(\Omega;\mathscr{X})$, $C^{\omega,r}(\Omega;\mathscr{X})$, $C^\omega(\Omega;\mathscr{X})$, and $\mathscr{A}^{n,r}(\mathscr{X})$. Given a complex vector field X on Ω, we identify $X = \sum_{j=1}^n a_j(x)\frac{\partial}{\partial x_j}$ with the function $(a_1,\ldots,a_n) : \Omega \to \mathbb{C}^n$. It therefore makes sense to consider quantities like $\|X\|_{\mathscr{C}^s(\Omega;\mathbb{C}^n)}$. When $\mathscr{X}$ is clear from context, we sometimes suppress it and write, e.g., $\|f\|_{\mathscr{C}^s(\Omega)}$ instead of $\|f\|_{\mathscr{C}^s(\Omega;\mathscr{X})}$ for readability considerations.

2.2. Function spaces on manifolds

Let $W_1,\ldots,W_N$ be C^1 real vector fields on a connected C^2 manifold M. Define the Carnot-Carathéodory ball associated to $W_1,\ldots,W_N$, centered at $x \in M$, of radius $\delta > 0$ by

$$B_W(x,\delta) := \Bigg\{ y \in M \,\Bigg|\, \exists \gamma : [0,1] \to M, \gamma(0) = x, \gamma(1) = y, \gamma'(t) = \sum_{j=1}^N a_j(t)\delta W_j(\gamma(t)), \\ a_j \in L^\infty([0,1]), \Bigg\| \sum_{j=1}^N |a_j|^2 \Bigg\|_{L^\infty} < 1 \Bigg\}, \tag{2.2}$$

and for $y \in M$, set

$$\rho(x,y) := \inf\{\delta > 0 : y \in B_W(x,\delta)\}. \tag{2.3}$$

ρ is an extended metric: it is possible that $\rho(x,y) = \infty$ for some $x, y \in M$. When $\rho(x,y) = \infty$, we define $\rho(x,y)^{-s} = 0$ for $s > 0$ and $\rho(x,y)^0 = 1$. See Remark 2.6 for the precise definition of $\gamma'(t)$ used in (2.2).

We use ordered multi-index notation W^α. Here, α denotes a list of elements of $\{1,\ldots,N\}$ and $|\alpha|$ denotes the length of the list. For example, $W^{(2,1,3,1)} = W_2W_1W_3W_1$ and $|(2,1,3,1)| = 4$.

Associated to the vector fields $W_1,\ldots,W_N$, we have the following function spaces on M.

$$C(M) = C^0_W(M) := \{ f : M \to \mathbb{C} \mid f \text{ is bounded and continuous}\},$$
$$\|f\|_{C(M)} = \|f\|_{C^0_W(M)} := \sup_{x\in M} |f(x)|.$$

For $m \in \mathbb{N}$, we define

$$C_W^m(M) := \{f \in C(M) : W^\alpha f \text{ exists and } W^\alpha f \in C(M), \forall |\alpha| \leq m\},$$
$$\|f\|_{C_W^m(M)} := \sum_{|\alpha|\leq m} \|W^\alpha f\|_{C(M)}.$$

For $s \in [0,1]$ we define the Hölder spaces associated to $W_1, \ldots, W_N$ by

$$\|f\|_{C_W^{0,s}(M)} := \|f\|_{C(M)} + \sup_{\substack{x,y\in M\\ x\neq y}} \rho(x,y)^{-s}|f(x) - f(y)|,$$
$$C_W^{0,s}(M) := \{f \in C(M) : \|f\|_{C_W^{0,s}(M)} < \infty\}.$$

For $m \in \mathbb{N}$ and $s \in [0,1]$, set

$$\|f\|_{C_W^{m,s}(M)} := \sum_{|\alpha|\leq m} \|W^\alpha f\|_{C_W^{0,s}(M)}, \quad C_W^{m,s}(M) := \{f \in C_W^m(M) : \|f\|_{C_W^{m,s}(M)} < \infty\}.$$

Next, we turn to the Zygmund spaces associated to $W_1, \ldots, W_N$. For this, we use the Hölder spaces $C^{0,s}([a,b])$ for a closed interval $[a,b] \subset \mathbb{R}$; $\|\cdot\|_{C^{0,s}([a,b])}$ is defined via the formula (2.1). Given $h > 0$, $s \in (0,1)$, define

$$\mathcal{P}_{W,s}^h := \left\{\gamma : [0,2h] \to M \,\middle|\, \gamma'(t) = \sum_{j=1}^N d_j(t) W_j(\gamma(t)), d_j \in C^{0,s}([0,2h]), \right.$$
$$\left. \sum_{j=1}^q \|d_j\|_{C^{0,s}([0,2h])}^2 < 1 \right\}.$$

For $s \in (0,1]$ set

$$\|f\|_{\mathscr{C}_W^s(M)} := \|f\|_{C_W^{0,s/2}(M)} + \sup_{\substack{h>0\\ \gamma\in\mathcal{P}_{W,s/2}^h}} h^{-s}|f(\gamma(2h)) - 2f(\gamma(h)) + f(\gamma(0))|,$$

and for $m \in \mathbb{N}$,

$$\|f\|_{\mathscr{C}_W^{m+s}(M)} := \sum_{|\alpha|\leq m} \|W^\alpha f\|_{\mathscr{C}_W^s(M)},$$

and we set

$$\mathscr{C}_W^{m+s}(M) := \{f \in C_W^m(M) : \|f\|_{\mathscr{C}_W^{m+s}(M)} < \infty\}.$$

Set

$$\mathscr{C}_W^\infty(M) := \bigcap_{s>0} \mathscr{C}_W^s(M) \text{ and } C_W^\infty(M) := \bigcap_{m\in\mathbb{N}} C_W^m(M).$$

We have $\mathscr{C}_W^\infty(M) = C_W^\infty(M)$; indeed, $\mathscr{C}_W^\infty(M) \subseteq C_W^\infty(M)$ is obvious, while the reverse containment follows from Lemma 9.1.

Finally, we turn to functions which are real analytic with respect to $W_1, \ldots, W_N$. Given $r > 0$, we set

$$\|f\|_{C_W^{\omega,r}(M)} := \sum_{m=0}^{\infty} \frac{r^m}{m!} \sum_{|\alpha|=m} \|W^\alpha f\|_{C(M)},$$

$$C_W^{\omega,r}(M) := \{f \in C_W^\infty(M) : \|f\|_{C_W^{\omega,r}(M)} < \infty\};$$

this definition was introduced in greater generality by Nelson [20]. We set $C_W^\omega(M) := \bigcup_{r>0} C_W^{\omega,r}(M)$, and $\mathscr{C}_W^\omega(M) := C_W^\omega(M)$.

Given $x_0 \in M$ and $r > 0$ we define $\mathscr{A}_W^{x_0,r}$ to be the space of those $f \in C(M)$ such that $h(t_1, \ldots, t_N) := f(e^{t_1 W_1 + \cdots + t_N W_N} x_0) \in \mathscr{A}^{N,r}$ (here, we are assuming $e^{t_1 W_1 + \cdots + t_N W_N} x_0$ exists for $(t_1, \ldots, t_N) \in B_{\mathbb{R}^N}(r)$–see Definition 4.1). We set $\|f\|_{\mathscr{A}_W^{x_0,r}} := \|h\|_{\mathscr{A}^{N,r}}$. Note that $\|f\|_{\mathscr{A}_W^{x_0,r}}$ depends only on the values of $f(y)$ where $y = e^{t_1 W_1 + \cdots + t_N W_N} x_0$ and $(t_1, \ldots, t_N) \in B^N(r)$; thus this is merely a semi-norm.

An important property of the above spaces and norms is that they are invariant under diffeomorphisms.

Proposition 2.1. *Let L be another C^2 manifold, let $\Phi : M \to L$ be a C^2 diffeomorphism, and let $\Phi_* W$ denote the list of vector fields $\Phi_* W_1, \ldots, \Phi_* W_N$. Then, the map $f \mapsto f \circ \Phi$ is an isometric isomorphism between the following spaces: $C_{\Phi_* W}^m(L) \to C_W^m(M)$, $C_{\Phi_* W}^{m,s}(L) \to C_W^{m,s}(M)$, $\mathscr{C}_{\Phi_* W}^s(L) \to \mathscr{C}_W^s(M)$, $C_{\Phi_* W}^{\omega,r}(L) \to C_W^{\omega,r}(M)$, and $\mathscr{A}_{\Phi_* W}^{\Phi(x_0),r} \to \mathscr{A}_W^{x_0,r}$.*

Proof. This is immediate from the definitions. $\square$

Remark 2.2. Informally, Proposition 2.1 says that the spaces described in this section are "coordinate-free". One can locally compute the norms in any C^2 coordinate system, and one gets the same result no matter what coordinate system is used.

Remark 2.3. When we write Vf for a C^1 vector field V and $f : M \to \mathbb{R}$, we define this as $Vf(x) := \frac{d}{dt}\big|_{t=0} f(e^{tV} x)$. When we say Vf exists, it means that this derivative exists in the classical sense, $\forall x$. If we have several C^1 vector fields $V_1, \ldots, V_K$, we define $V_1 V_2 \cdots V_K f := V_1(V_2(\cdots V_K(f)))$ and to say that this exists means that at each stage the derivative exists.

Remark 2.4. All of the above function spaces can be defined, with the same formulas, with M replaced by $B_W(x, \delta)$, whether or not $B_W(x, \delta)$ is a manifold. Indeed, for a function $f : B_W(x, \delta) \to \mathbb{C}$, one may define $W_j f(x) := \frac{d}{dt}\big|_{t=0} f(e^{tW_j} x)$. Using this one may define all the above norms, with the same formulas, for M replaced by $B_W(x, \delta)$. See [27, Section 2.2.1] for a further discussion of this.

Remark 2.5. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set. Let ∇ denote the list of vector fields $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$. We have $\mathscr{A}_{\nabla}^{0,r} = \mathscr{A}^{n,r}$ and $C_{\nabla}^{\omega,r}(\Omega) = C^{\omega,r}(\Omega)$, with equality of norms.

Remark 2.6. In (2.2) (and in the rest of the paper), $\gamma'(t)$ is defined as follows. In the case that M is an open subset $\Omega \subseteq \mathbb{R}^n$ and $\gamma : [a,b] \to \Omega$, $\gamma'(t) = \sum_{j=1}^q a_j(t) X_j(\gamma(t))$ is defined to mean $\gamma(t) = \gamma(a) + \int_a^t \sum_j a_j(s) X_j(\gamma(s))\, ds$; note that this definition is local in t (equivalently, we are requiring that γ be absolutely continuous and have the desired derivative almost everywhere). For an abstract C^2 manifold, this is interpreted locally. I.e., if $\gamma : [a,b] \to M$, we say $\gamma'(t) = \sum_{j=1}^q a_j(t) X_j(\gamma(t))$ if $\forall t_0 \in [a,b]$, there is an open neighborhood N of $\gamma(t_0)$ and a C^2 diffeomorphism $\Psi : N \to \Omega$, where $\Omega \subseteq \mathbb{R}^n$ is open, such that $(\Psi \circ \gamma)'(t) = \sum_{j=1}^q a_j(t)(\Psi_* X_j)(\Psi \circ \gamma(t))$ for t near t_0 ($t \in [a,b]$).

2.2.1. Complex vector fields

Let M be a C^2 manifold, let $L_1, \ldots, L_m$ be complex C^1 vector fields on M (i.e., $L_1, \ldots, L_m$ take values in the complexified tangent space), and let $X_1, \ldots, X_q$ be real C^1 vector fields on M. We denote by X, L the list $X_1, \ldots, X_q, L_1, \ldots, L_m$. Associated to X, L we define the list of real vector fields $W_1, \ldots, W_{q+2m} = X_1, \ldots, X_q, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_m), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_m)$. Set

$$B_{X,L}(x,\delta) := B_W(x,\delta). \tag{2.4}$$

We define $C_{X,L}^m(M) := C_W^m(M)$, with equality of norms. We similarly define $C_{X,L}^{m,s}(M)$, $\mathscr{C}_{X,L}^s(M)$, $C_{X,L}^{\omega,r}(M)$, $\mathscr{A}_{X,L}^{x_0,r}$, $C_{X,L}^\infty(M)$, and $C_{X,L}^\omega(M)$. We will often consider the case when $q = 0$, and in that case we just write $C_L^m(M)$ instead of $C_{X,L}^m(M)$, and similarly for $C_L^{m,s}(M)$, $\mathscr{C}_L^s(M)$, $C_L^{\omega,r}(M)$, $\mathscr{A}_L^{x_0,r}$, $C_L^\infty(M)$, and $C_L^\omega(M)$.

Remark 2.7. The factor 2 in $2\mathrm{Re}(L_j)$ and $2\mathrm{Im}(L_j)$ in the definition of W is not an essential point. It is chosen so that if $M = \mathbb{R}^q \times \mathbb{C}^m$, with coordinates $(t_1, \ldots, t_q, z_1, \ldots, z_m)$, and if $X_k = \frac{\partial}{\partial t_k}$ and $L_j = \frac{\partial}{\partial \bar{z}_j}$, then $W = \nabla$, where ∇ denotes the gradient on $\mathbb{R}^{q+2m} \cong \mathbb{R}^q \times \mathbb{C}^m$.

3. Corollaries of the main result

Our main result (Theorem 4.5) concerns the existence of a certain coordinate chart which satisfies good quantitative properties. This coordinate chart is useful in two, related, ways:

- It is a coordinate system in which given vector fields have the optimal level of regularity.
- It normalizes vector fields in a way which is useful for applying techniques from analysis. When viewed in this light, it can be seen as a scaling map for sub-Riemannian, or sub-Hermitian, geometries.

In this section, we present two corollaries of our main result, which separate the above two uses. In each of these corollaries, we present the real setting (which is known) and the complex setting (which is new). In Section 7, we will revisit these corollaries and present a setting which unifies both the real and complex settings.

3.1. Optimal smoothness

3.1.1. The real case

Let $W_1, \ldots, W_N$ be C^1 real vector fields on a C^2 manifold M of dimension n, which span the tangent space at every point. In this section, we describe when there is a smoother structure on M with respect to which $W_1, \ldots, W_N$ have a desired level of regularity. These results were proved in [27,31,32] (though in Section 10.1, we will see them as corollaries of the main result of this paper), and they set the stage for the results in the complex setting in Section 3.1.2.

Theorem 3.1 *(The local theorem). For $x_0 \in M$, $s \in (1, \infty] \cup \{\omega\}$, the following three conditions are equivalent:*

(i) There is an open neighborhood $V \subseteq M$ of x_0 and a C^2 diffeomorphism $\Phi : U \to V$ where $U \subseteq \mathbb{R}^n$ is open, such that $\Phi^* W_1, \ldots, \Phi^* W_N \in \mathscr{C}^{s+1}(U; \mathbb{R}^n)$.

(ii) Re-order the vector fields so that $W_1(x_0), \ldots, W_n(x_0)$ are linearly independent. There is an open neighborhood $V \subseteq M$ of x_0 such that:
- *$[W_i, W_j] = \sum_{k=1}^n \hat{c}_{i,j}^k W_k$, $1 \leq i, j \leq n$, where $\hat{c}_{i,j}^k \in \mathscr{C}_W^s(V)$.*
- *For $n+1 \leq j \leq N$, $W_j = \sum_{k=1}^n b_j^k W_k$, where $b_j^k \in \mathscr{C}_W^{s+1}(V)$.*

(iii) There exists an open neighborhood $V \subseteq M$ of x_0 such that $[W_i, W_j] = \sum_{k=1}^N c_{i,j}^k W_k$, $1 \leq i, j \leq N$, where $c_{i,j}^k \in \mathscr{C}_W^s(V)$.

Remark 3.2. Note that Theorem 3.1 (ii) and (iii) can be checked in any C^2 coordinate system (see Proposition 2.1 and Remark 2.2), while Theorem 3.1 (i) gives the existence of a "nice" coordinate system.

Theorem 3.3 *(The global theorem). For $s \in (1, \infty] \cup \{\omega\}$, the following two conditions are equivalent:*

(i) There exists a $\mathscr{C}^{s+2}$ atlas on M, compatible with its C^2 structure, such that $W_1, \ldots, W_N$ are $\mathscr{C}^{s+1}$ vector fields with respect to this atlas.

(ii) For each $x_0 \in M$, any of the three equivalent conditions from Theorem 3.1 hold for this choice of x_0.

Furthermore, under these conditions, the $\mathscr{C}^{s+2}$ manifold structure induced by the atlas in (i) is unique, in the sense that if there is another $\mathscr{C}^{s+2}$ atlas on M, compatible with its C^2 structure, and such that $W_1, \ldots, W_N$ are locally $\mathscr{C}^{s+1}$ with respect to this second *atlas, then the identity map $M \to M$ is a $\mathscr{C}^{s+2}$ diffeomorphism between these two $\mathscr{C}^{s+2}$ manifold structures on M. Finally, when $s \in (1,\infty]$, there is a third equivalent condition*

(iii) $[W_i, W_j] = \sum_{k=1}^N c_{i,j}^k W_k$, $1 \le i,j \le N$, *where* $\forall x_0 \in M$, $\exists V \subseteq M$ *open with* $x_0 \in V$ *such that* $c_{i,j}^k\big|_V \in \mathscr{C}^s_W(V)$, $1 \le i,j,k \le N$.

Remark 3.4. Theorems 3.1 and 3.3 are stated for $s > 1$. It would be desirable to have the same results for $s > 0$, but our proof runs into technical difficulties for $s \in (0,1]$. See [31] for details. Similar remarks hold for many of the main results in this paper; in particular, the same remark holds for the main result of the paper: Theorem 4.5.

3.1.2. The complex case

Let M be a C^2 manifold and let $L_1, \ldots, L_m$ be complex C^1 vector fields on M. We assume:

- $\forall \zeta \in M$, $\operatorname{span}_{\mathbb{C}} \left\{L_1(\zeta), \ldots, L_m(\zeta), \overline{L_1}(\zeta), \ldots, \overline{L_m}(\zeta)\right\} = \mathbb{C}T_\zeta M$.
- $\forall \zeta \in M$, $\operatorname{span}_{\mathbb{C}} \{L_1(\zeta), \ldots, L_m(\zeta)\} \cap \operatorname{span}_{\mathbb{C}} \left\{\overline{L_1}(\zeta), \ldots, \overline{L_m}(\zeta)\right\} = \{0\}$.

By Lemma B.1 and the above assumptions we have, $\forall \zeta \in M$,

$$\begin{aligned}\dim M &= \dim \operatorname{span}_{\mathbb{C}} \left\{L_1(\zeta), \ldots, L_m(\zeta), \overline{L_1}(\zeta), \ldots, \overline{L_m}(\zeta)\right\} \\ &= 2 \dim \operatorname{span}_{\mathbb{C}} \{L_1(\zeta), \ldots, L_m(\zeta)\}.\end{aligned}$$

In particular, let $n := \dim \operatorname{span}_{\mathbb{C}} \{L_1(\zeta), \ldots, L_m(\zeta)\}$, then n does not depend on ζ and $\dim M = 2n$.

Theorem 3.5 *(The local theorem). Fix $\zeta_0 \in M$ and $s \in (1,\infty] \cup \{\omega\}$. The following three conditions are equivalent:*

(i) *There exists an open neighborhood $V \subseteq M$ of ζ_0 and a C^2 diffeomorphism $\Phi : U \to V$, where $U \subseteq \mathbb{C}^n$ is open, such that $\forall z \in U$, $1 \le j \le m$,*

$$\Phi^* L_j(z) \in \operatorname{span}_{\mathbb{C}} \left\{\frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\},$$

and $\Phi^* L_j \in \mathscr{C}^{s+1}(U; \mathbb{C}^n)$.

(ii) *Reorder $L_1, \ldots, L_m$ so that $L_1(\zeta_0), \ldots, L_n(\zeta_0)$ are linearly independent. There exists a neighborhood $V \subseteq M$ of ζ_0 such that:*
- $[L_j, L_k] = \sum_{l=1}^n \hat{c}_{j,k}^{1,l} L_l$ *and* $[L_j, \overline{L_k}] = \sum_{l=1}^n \hat{c}_{j,k}^{2,l} L_l + \sum_{l=1}^n \hat{c}_{j,k}^{3,l} \overline{L_l}$, *where* $\hat{c}_{j,k}^{a,l} \in \mathscr{C}^s_L(V)$, $1 \le j,k,l \le n$, $1 \le a \le 3$.
- $L_j = \sum_{l=1}^n b_j^l L_l$, *where* $b_j^l \in \mathscr{C}^{s+1}_L(V)$, $n+1 \le j \le m$, $1 \le l \le n$.

(iii) There exists a neighborhood $V \subseteq M$ *of* ζ_0 *such that* $[L_j, L_k] = \sum_{l=1}^m c_{j,k}^{1,l} L_l$ *and* $[L_j, \overline{L_k}] = \sum_{l=1}^m c_{j,k}^{2,l} L_l + \sum_{l=1}^m c_{j,k}^{3,l} \overline{L_l}$*, where* $c_{j,k}^{a,l} \in \mathscr{C}_L^s(V)$, $1 \leq a \leq 3$, $1 \leq j, k, l \leq m$.

Theorem 3.6 *(The global theorem). For* $s \in (1, \infty] \cup \{\omega\}$ *the following two conditions are equivalent:*

(i) There exists a complex manifold structure on M*, compatible with its* C^2 *structure, such that* $L_1, \dots, L_m$ *are* $\mathscr{C}^{s+1}$ *vector fields on* M *(with respect to this complex structure), and* $\forall \zeta \in M$,

$$\operatorname{span}_{\mathbb{C}} \{L_1(\zeta), \dots, L_m(\zeta)\} = T_\zeta^{0,1} M.$$

(ii) For each $\zeta_0 \in M$*, any of the three equivalent conditions from Theorem 3.5 hold for this choice of* ζ_0.

Furthermore, under these conditions, the complex manifold structure in (i) is unique, in the sense that if M *has another complex manifold structure satisfying the conditions of (i), then the identity map* $M \to M$ *is a biholomorphism between these two complex structures. Finally, when* $s \in (1, \infty]$*, there is a third equivalent condition:*

(iii) $[L_j, L_k] = \sum_{l=1}^m c_{j,k}^{1,l} L_l$ *and* $[L_j, \overline{L_k}] = \sum_{l=1}^m c_{j,k}^{2,l} L_l + \sum_{l=1}^m c_{j,k}^{3,l} \overline{L_l}$*, where* $\forall \zeta \in M$*, there exists an open neighborhood* $V \subseteq M$ *of* ζ *such that* $c_{j,k}^{a,l}\big|_V \in \mathscr{C}_L^s(V)$, $1 \leq a \leq 3$, $1 \leq j, k, l \leq m$.

Remark 3.7. Theorem 3.6 can be seen as a version of the Newlander-Nirenberg theorem (with sharp regularity in terms of Zygmund spaces), which is invariant under arbitrary C^2 diffeomorphisms.

Remark 3.8. Because the Zygmund space $\mathscr{C}^{m+\alpha}$ is (locally) the same as the Hölder space $C^{m,\alpha}$ for $m \in \mathbb{N}$, $\alpha \in (0,1)$, one can obtain analogs of Theorems 3.5 and 3.6 using the easier to understand Hölder spaces, as long as one avoids integer exponents. This is carried out in Section 14. For integer exponents, the use of Zygmund spaces is essential, as Theorem 3.5 does not hold if we replace the Zygmund spaces $\mathscr{C}^{s+1}$ (for $s \in \mathbb{N}$) with C^{s+1} or $C^{s,1}$; this is described in Lemma 14.4. As a consequence, Zygmund spaces are also essential in the main theorem of this paper (Theorem 4.5). The reason our proof requires Zygmund spaces when considering integer exponents is because it relies on nonlinear elliptic PDEs (via the results from [31,33]). As is well-known, the regularity theory of elliptic PDEs works best when using Zygmund spaces instead of C^m spaces or Lipschitz spaces.

3.2. Geometries defined by vector fields

We present the basic results concerning sub-Riemannian and sub-Hermitian geometry in this section. The results on sub-Riemannian geometry are just a reprise (in a slightly different language) of the main results of Nagel, Stein, and Wainger's work [19].[7] The results on sub-Hermitian geometry can be seen as holomorphic analogs of these results. In this section, we present these ideas in these two simple settings. In Section 7.2 we generalize these results to a single unified result on "E-manifolds".

3.2.1. Sub-Riemannian geometry: the results of Nagel, Stein, and Wainger

In this section, we describe the main results of the foundational paper of Nagel, Stein, and Wainger [19]. This describes how the existence of certain coordinate charts (like the ones developed in our main theorem) can be viewed as scaling maps in sub-Riemannian geometry. The results in this section set the stage for the results in the complex setting in Section 3.2.2.

Let $W_1, \ldots, W_N$ be C^∞ real vector fields on a connected, C^∞ manifold M of dimension n which span the tangent space at every point. To each W_j we assign a formal degree $d_j \in [1, \infty)$. We assume

$$[W_j, W_k] = \sum_{d_l \leq d_j + d_k} c_{j,k}^l W_l, \quad c_{j,k}^l \in C^\infty(M).$$

We write (W, d) for the list $(W_1, d_1), \ldots, (W_N, d_N)$ and for $\delta > 0$ write $\delta^d W$ for the list $\delta^{d_1} W_1, \ldots, \delta^{d_N} W_N$. The sub-Riemannian ball associated to (W, d) centered at $x_0 \in M$ of radius $\delta > 0$ is defined by

$$B_S(x_0, \delta) := B_{\delta^d W}(x_0, 1),$$

where the later ball is defined by (2.2). $B_S(x_0, \delta)$ is an open subset of M. We define $\rho_S(x, y) := \inf\{\delta > 0 : y \in B_S(x, \delta)\}$; ρ is a metric on M and is called a *sub-Riemannian metric*. For the relationship between this definition of a sub-Riemannian metric and some of the other common definitions, see [19].

We define another metric on M, which will turn out to be equal to ρ_S, as follows. We say $\rho_F(x, y) < \delta$ if and only if there exists $K \in \mathbb{N}$, smooth functions $f_1, \ldots, f_K : B_{\mathbb{R}}(1/2) \to M$, and $\delta_1, \ldots, \delta_K > 0$ with $\sum \delta_l \leq \delta$ such that:

[7] We present results on sub-Riemannian geometry which are essentially those of Nagel, Stein, and Wainger, however the main results of this paper (even in this real setting) imply many results which are beyond those that are implied by Nagel, Stein, and Wainger's methods. In the real setting, this is described in the series [27,31,32]. We present the corollaries in this section in the simplest possible setting (as opposed to a very general setting) to help the reader understand the thrust of our main theorem, Theorem 4.5, which is stated in some generality. For example, even if one only considers real vector fields, the main results of this paper imply (and are stronger than) the results in the multi-parameter setting of [28], which could not be achieved by the methods of [19]. We also present a more complicated example in the complex setting in Section 8.

- $f_j'(t) = \sum_{l=1}^N s_j^l(t)\delta_j^{d_l} W_l(f_j(t))$, with $\left\|\sum_l |s_j^l|^2\right\|_{L^\infty(B_{\mathbb{R}}(1/2))} < 1$.
- $f_j(B_{\mathbb{R}}(1/2)) \bigcap f_{j+1}(B_{\mathbb{R}}(1/2)) \neq \emptyset$, $1 \leq j \leq K-1$.
- $x \in f_1(B_{\mathbb{R}}(1/2))$, $y \in f_K(B_{\mathbb{R}}(1/2))$.

ρ_F is clearly an extended metric. Once we prove ρ_F and ρ_S are equal, it will then follow that ρ_F is a metric.

Fix a strictly positive, C^∞ density ν on M.[8] For $x \in M$, $\delta > 0$, set

$$\Lambda(x,\delta) := \max_{j_1,\ldots,j_n \in \{1,\ldots,N\}} \nu(x)(\delta^{d_{j_1}} X_{j_1}(x), \ldots, \delta^{d_{j_N}} X_{j_N}(x)).$$

The next result follows from the methods of [19] (though we prove it directly by seeing is as a special case of the result in Section 7.2).

Theorem 3.9 *([19]).*

(a) $\forall x, y \in M$, $\rho_S(x,y) = \rho_F(x,y)$.

Fix a compact set $\mathcal{K} \subseteq M$. *There exists* $\delta_0 = \delta_0(\mathcal{K}) \in (0,1]$ *such that the following holds. We write* $A \lesssim B$ *for* $A \leq CB$, *where* C *can be chosen independent of* $x, y \in \mathcal{K}$ *and* $\delta > 0$. *We write* $A \approx B$ *for* $A \lesssim B$ *and* $B \lesssim A$.

(b) $\nu(B_S(x,\delta)) \approx \Lambda(x,\delta)$, $\forall x \in \mathcal{K}, \delta \in (0,\delta_0]$.
(c) $\nu(B_S(x,2\delta)) \lesssim \nu(B_S(x,\delta))$, $\forall x \in \mathcal{K}, \delta \in (0,\delta_0/2]$.

For each $x \in \mathcal{K}$, $\delta \in (0,1]$, *there exists* $\Phi_{x,\delta} : B_{\mathbb{R}^n}(1) \to B_S(x,\delta)$ *such that:*

(d) $\Phi_{x,\delta}(B_{\mathbb{R}^n}(1)) \subseteq M$ *is open and* $\Phi_{x,\delta} : B_{\mathbb{R}^n}(1) \to \Phi_{x,\delta}(B_{\mathbb{R}^n}(1))$ *is a* C^∞ *diffeomorphism.*
(e) $\Phi_{x,\delta}^*\nu = h_{x,\delta}\sigma_{\mathrm{Leb}}$, *where* $h_{x,\delta} \in C^\infty(B_{\mathbb{R}^n}(1))$, $h_{x,\delta}(t) \approx \Lambda(x,\delta)$ $\forall t$, *and* $\|h_{x,\delta}\|_{C^m(B_{\mathbb{R}^n}(1))} \lesssim \Lambda(x,\delta)$, $\forall m$ *(where the implicit constant depends on* m, *but not on* $x \in \mathcal{K}$ *or* $\delta \in (0,1]$*). Here, and in the rest of the paper,* σ_{Leb} *denotes the usual Lebesgue density on* $\mathbb{R}^n$.

Let $Y_j^{x,\delta} := \Phi_{x,\delta}^*\delta^{d_j}W_j$, *so that* $Y_j^{x,\delta}$ *is a* C^∞ *vector field on* $B_{\mathbb{R}^n}(1)$.

(f) $\|Y_j^{x,\delta}\|_{C^m(B_{\mathbb{R}^n}(1);\mathbb{R}^n)} \lesssim 1$, $\forall x \in \mathcal{K}, \delta \in (0,1], m \in \mathbb{N}$, *where the implicit constant depends on* m, *but not on* x *or* δ.

[8] The results that follow are local and do not depend on the choice of ν, so long as it is strictly positive and smooth.

(g) $Y_1^{x,\delta}(u), \ldots, Y_N^{x,\delta}(u)$ *span the tangent space uniformly in* u, x, δ *in the sense that*

$$\max_{j_1,\ldots,j_n\in\{1,\ldots,N\}} \inf_{u\in B_{\mathbb{R}^N}(1)} \left|\det\left(Y_{j_1}^{x,\delta}(u)|\cdots|Y_{j_n}^{x,\delta}(u)\right)\right| \approx 1, \quad x\in\mathcal{K}, \delta\in(0,1].$$

(h) $\exists\epsilon\approx 1$ *such that* $B_S(x,\epsilon\delta)\subseteq\Phi_{x,\delta}(B_{\mathbb{R}^n}(1))\subseteq B_S(x,\delta)$, $\forall x\in\mathcal{K}, \delta\in(0,1]$.

Remark 3.10. The most important aspects of Theorem 3.9 are (f) and (g); and these allow us to see the maps $\Phi_{x,\delta}$ as "scaling maps". Indeed, for δ small, one tends to think of $\delta^{d_j}W_j$ as a "small" vector field. However, $\Phi_{x,\delta}$ gives a coordinate system in which $\delta^{d_j}W_j$ is of "unit size": not only are $\Phi_{x,\delta}^*\delta^{d_1}W_1, \ldots, \Phi_{x,\delta}^*\delta^{d_N}W_N$ smooth uniformly in x and δ (i.e., (f)), but they also span the tangent space uniformly in x and δ (i.e., (g)). See [27, Section 7.1.1] for some more comments in this direction.

Remark 3.11. (c) is the main estimate needed to show that the balls $B_S(x,\delta)$ when paired with the density ν locally give a space of homogeneous type. Because of this, one has access to the Calderón-Zygmund theory of singular integrals with respect to these balls. This has had many uses: see the remarks at the end of Chapter 2 of [30] for a history of these ideas.

3.2.2. Sub-Hermitian geometry

Let M be a connected complex manifold of complex dimension n. Let $L_1, \ldots, L_m$ be C^∞, $T^{0,1}$ vector fields on M such that $\forall\zeta\in M$, $\operatorname{span}_{\mathbb{C}}\{L_1(\zeta),\ldots,L_m(\zeta)\} = T^{0,1}_\zeta M$. Our goal in this section is to describe a complex analog of the results in Section 3.2.1 with respect to the vector fields $L_1, \ldots, L_m$. The main point is to achieve as much as possible using only *holomorphic* maps, so that these results can be applied to questions in several complex variables.

To each L_j we assign a formal degree $\beta_j\in[1,\infty)$. We assume

$$[L_j, L_k] = \sum_{\beta_l\leq\beta_j+\beta_k} c_{j,k}^{1,l}L_l, \quad [L_j,\overline{L_k}] = \sum_{\beta_l\leq\beta_j+\beta_k} c_{j,k}^{2,l}L_l + \sum_{\beta_l\leq\beta_j+\beta_k} c_{j,k}^{3,l}\overline{L_l}, \quad c_{j,k}^{a,l}\in C^\infty(M).$$

Let $(W_1,d_1),\ldots,(W_{2m},d_{2m}) = (2\mathrm{Re}(L_1),\beta_1),\ldots,(2\mathrm{Re}(L_m),\beta_m),(2\mathrm{Im}(L_1),\beta_1),\ldots,(2\mathrm{Im}(L_m),\beta_m)$. Fix a strictly positive, smooth density ν on M. It is immediate to verify that the list $(W_1,d_1),\ldots,(W_{2m},d_{2m})$ satisfies all the hypotheses of Section 3.2.1. Thus we obtain balls $B_S(\zeta,\delta)$ and an associated metric $\rho_S=\rho_F$, and Theorem 3.9 applies. The main problem is that the definitions of ρ_S and ρ_F use the underlying smooth structure on M and not the complex structure, and the scaling maps $\Phi_{x,\delta}$ from Theorem 3.9 are only guaranteed to be smooth, not holomorphic. In particular, when rescaling $\delta^{\beta_j}L_j$ by computing $\Phi_{x,\delta}^*\delta^{\beta_j}L_j$ we do not know that $\Phi_{x,\delta}^*\delta^{\beta_j}L_j$ continues to be a $T^{0,1}$ vector field; i.e., we do not know $\Phi_{x,\delta}^*\delta^{\beta_j}L_j$ is spanned by $\frac{\partial}{\partial\bar z_1},\ldots,\frac{\partial}{\partial\bar z_n}$. The results in this section fix these problems.

First, we define a metric using the complex structure on M, which we will see is locally equivalent to $\rho_S = \rho_F$. This metric is obtained by taking the definition for ρ_F, and rewriting it with holomorphic maps in place of smooth maps. We say $\rho_H(\zeta_1, \zeta_2) < \delta$ if and only if there exists $K \in \mathbb{N}$, holomorphic functions $f_1, \ldots, f_K : B_{\mathbb{C}}(1/2) \to M$, and $\delta_1, \ldots, \delta_K > 0$ with $\sum_{l=1}^{K} \delta_l \leq \delta$ such that:

- $df_j(z)\frac{\partial}{\partial \overline{z}} = \sum_{l=1}^{m} s_j^l(z, \overline{z})\delta_j^{\beta_l} L_l(f_j(z))$, with $\|\sum_l |s_j^l|^2\|_{L^\infty(B_{\mathbb{C}}(1/2))} < 1$.
- $f_j(B_{\mathbb{C}}(1/2)) \bigcap f_{j+1}(B_{\mathbb{C}}(1/2)) \neq \emptyset$, $1 \leq j \leq K-1$.
- $\zeta_1 \in f_1(B_{\mathbb{C}}(1/2))$, $\zeta_2 \in f_K(B_{\mathbb{C}}(1/2))$.

ρ_H is clearly an extended metric; once we show it is locally equivalent to ρ_S, it will follow that ρ_H is a metric.

Theorem 3.12.

(a) $\forall \zeta_1, \zeta_2 \in M$, $\rho_S(\zeta_1, \zeta_2) = \rho_F(\zeta_1, \zeta_2) \leq \rho_H(\zeta_1, \zeta_2)$.

Fix a compact set $\mathcal{K} \subseteq M$. *We write* $A \lesssim B$ *for* $A \leq CB$ *where* C *can be chosen independent of* $\zeta, \zeta_1, \zeta_2 \in \mathcal{K}$ *and* $\delta \in (0, 1]$. *We write* $A \approx B$ *for* $A \lesssim B$ *and* $B \lesssim A$.

(b) $\rho_H(\zeta_1, \zeta_2) \lesssim \rho_S(\zeta_1, \zeta_2)$, $\forall \zeta_1, \zeta_2 \in \mathcal{K}$, *and therefore* ρ_H *and* ρ_S *are equivalent on compact sets.*

(c) All of the conclusions of Theorem 3.9 hold (when applied to $(W_1, d_1), \ldots, (W_{2m}, d_{2m})$*) and (by identifying* $\mathbb{R}^{2n} \cong \mathbb{C}^n$*) the maps* $\Phi_{\zeta,\delta} : B_{\mathbb{C}^n}(1) \to B_S(\zeta, \delta) \subseteq M$ *can be taken to be holomorphic.*

Because $\Phi_{\zeta,\delta}$ *is holomorphic,* $\Phi_{\zeta,\delta}^* \delta^{\beta_j} L_j$ *is a* $T^{0,1}$ *vector field; in other words,* $\Phi_{\zeta,\delta}^* \delta^{\beta_j} L_j(z) \in \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\}$, $\forall z \in B_{\mathbb{C}^n}(1)$. *We can thus think of* $\Phi_{\zeta,\delta}^* \delta^{\beta_j} L_j$ *as a map* $B_{\mathbb{C}^n}(1) \to \mathbb{C}^n$.

(d) $\|\Phi_{\zeta,\delta}^* \delta^{\beta_j} L_j\|_{C^k(B_{\mathbb{C}^n}(1);\mathbb{C}^n)} \lesssim 1$, $\forall \zeta \in \mathcal{K}, \delta \in (0, 1], k \in \mathbb{N}$, *where the implicit constant depends on* k, *but not on* $\zeta \in \mathcal{K}$ *or* $\delta \in (0, 1]$.

(e) $\Phi_{x,\delta}^* \delta^{\beta_1} L_1(z), \ldots, \Phi_{x,\delta}^* \delta^{\beta_m} L_m(z)$ *span* $T_z^{0,1}\mathbb{C}^n$ *uniformly in* z, ζ, δ *in the sense that*

$$\max_{j_1,\ldots,j_n \in \{1,\ldots m\}} \inf_{z \in B_{\mathbb{C}^n}(1)} \left| \det \left(\Phi_{\zeta,\delta}^* \delta^{\beta_{j_1}} L_{j_1}(z) | \ldots | \Phi_{\zeta,\delta}^* \delta^{\beta_{j_n}} L_{j_n}(z) \right) \right| \approx 1,$$

$$\forall \zeta \in \mathcal{K}, \delta \in (0, 1].$$

Remark 3.13. In Theorem 3.12 we described a C^∞ version of sub-Hermitian geometry. With a very similar proof one can obtain a similar real analytic version; see Remark 7.7. One can also obtain results for vector fields with only a finite level of smoothness; see Remark 7.9.

Remark 3.14. In the above discussion, we studied the vector fields $\delta^{\beta_1} L_1, \ldots, \delta^{\beta_m} L_m$. In many applications, the vector fields depend on δ in a more complicated way (such an example is given in Section 8). Furthermore, in some applications, δ ranges over $(0,1]^\mu$ instead of $(0,1]$ (as studied in the real setting in [28]). Our proof methods allow us to study such settings in the same way; see Remark 7.8. We stated results in this setting for simplicity of presentation, so that the reader can easily see the main ideas.

4. The main results

Let $X_1, \ldots, X_q$ be real C^1 vector fields on a C^2 manifold $\mathfrak{M}$ and let $L_1, \ldots, L_m$ be complex C^1 vector fields on $\mathfrak{M}$. For each $x \in \mathfrak{M}$, set $\mathscr{L}_x := \operatorname{span}_{\mathbb{C}}\{L_1(x), \ldots, L_m(x), X_1(x), \ldots, X_q(x)\}$, $\mathscr{X}_x := \operatorname{span}_{\mathbb{C}}\{X_1(x), \ldots, X_q(x)\}$.

Fix $x_0 \in \mathfrak{M}$, $\xi > 0$. Set $r := \dim \mathscr{X}_{x_0}$ and $n + r := \dim \mathscr{L}_{x_0}$. Our goal in this section is to choose a "coordinate system" $\Phi : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to B_{X,L}(x_0, \xi)$ so that $\Phi^* X_1, \ldots, \Phi^* X_q, \Phi^* L_1, \ldots, \Phi^* L_m$ have a desired level of regularity and $\forall (t,z) \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$,

$$\begin{aligned} &\operatorname{span}_{\mathbb{C}}\{\Phi^* X_1(t,z), \ldots, \Phi^* X_q(t,z), \Phi^* L_1(t,z), \ldots, \Phi^* L_m(t,z)\} \\ &= \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\}, \end{aligned}$$

where we have given $\mathbb{R}^r \times \mathbb{C}^n$ coordinates $(t_1, \ldots, t_r, z_1, \ldots, z_n)$. Finally, we wish to pick this coordinate system so that $\Phi^* X_1, \ldots, \Phi^* X_q, \Phi^* L_1, \ldots, \Phi^* L_m$ are normalized in a way which is useful for applying techniques from analysis.

Let $Z_1, \ldots, Z_{q+m} := X_1, \ldots, X_q, L_1, \ldots, L_m$. Our three main *algebraic* assumptions are as follows:

(i) $\forall x \in B_{X,L}(x_0, \xi)$, $\mathscr{L}_x \bigcap \overline{\mathscr{L}_x} = \mathscr{X}_x$.
(ii) $[Z_j, Z_k] = \sum_{l=1}^{m+q} c_{j,k}^{1,l} Z_l$ and $[Z_j, \overline{Z_k}] = \sum_{l=1}^{m+q} c_{j,k}^{2,l} Z_l + \sum_{l=1}^{m+q} c_{j,k}^{3,l} \overline{Z_l}$, where $c_{j,k}^{a,l} \in C(B_{X,L}(x_0, \xi))$, $1 \leq a \leq 3$, $1 \leq j,k,l \leq q+m$ (here we are giving $B_{X,L}(x_0, \xi)$ the topology induced by the associated metric (2.3)).
(iii) $x \mapsto \dim \mathscr{L}_x$, $B_{X,L}(x_0, \xi) \to \mathbb{N}$, is constant in x (it follows from the other assumptions that this is equivalent to the map $x \mapsto \dim \mathscr{X}_x$ being constant in x; see Section 4.2).

Under the above hypotheses, $B_{X,L}(x_0, \xi)$ is a C^2, injectively immersed submanifold of $\mathfrak{M}$ (see Proposition A.1), and $\mathbb{C}T_x B_{X,L}(x_0, \xi) = \mathscr{L}_x + \overline{\mathscr{L}_x}$, $\forall x \in B_{X,L}(x_0, \xi)$. In particular, using Lemma B.1,

$$\dim B_{X,L}(x_0, \xi) = \dim T_{x_0} B_{X,L}(x_0, \xi) = \dim(\mathscr{L}_{x_0} + \overline{\mathscr{L}_{x_0}}) = 2 \dim \mathscr{L}_{x_0} - \dim \mathscr{X}_{x_0} = 2n + r.$$

Henceforth we view $X_1, \ldots, X_q, L_1, \ldots, L_m$ as C^1 vector fields on $B_{X,L}(x_0, \xi)$.

For $a, b \in \mathbb{N}$, we set

$$\mathcal{I}(a,b) := \{(i_1, i_2, \ldots, i_a) : i_1, \ldots, i_a \in \{1, \ldots, b\}\} = \{1, \ldots, b\}^a. \tag{4.1}$$

For $K = (k_1, \ldots, k_{r_1}) \in \mathcal{I}(r_1, q)$, we write X_K for the list $X_{k_1}, \ldots, X_{k_{r_1}}$ and for $J = (j_1, \ldots, j_{n_1}) \in \mathcal{I}(n_1, m)$ we write L_J for the list $L_{j_1}, \ldots, L_{j_{n_1}}$. We write $\bigwedge X_K := X_{k_1} \wedge X_{k_2} \wedge \cdots \wedge X_{k_{r_1}}$ and $\bigwedge L_J := L_{j_1} \wedge L_{j_2} \wedge \cdots \wedge L_{j_{n_1}}$.

Fix $\zeta \in (0,1]$, $K_0 \in \mathcal{I}(r,q)$, $J_0 \in \mathcal{I}(n,m)$ such that

$$\max_{\substack{K \in \mathcal{I}(r_1,q), J \in \mathcal{I}(n_1,m) \\ r_1+n_1=r+n}} \left| \frac{(\bigwedge X_K(x_0)) \bigwedge (\bigwedge L_J(x_0))}{(\bigwedge X_{K_0}(x_0)) \bigwedge (\bigwedge L_{J_0}(x_0))} \right| \leq \zeta^{-1}. \tag{4.2}$$

See Appendix B.2 for the definition of this quotient. Such a choice of J_0, K_0, and ζ always exist; see Remark B.6. One cannot necessarily choose K_0, J_0 so that (4.2) holds with $\zeta = 1$, however if $n = 0$ or $r = 0$ (the two most important special cases) one always can–see Remark B.6. Without loss of generality, reorder $X_1, \ldots, X_q$ and $L_1, \ldots, L_m$ so that $K_0 = (1, 2, \ldots, r)$, $J_0 = (1, 2, \ldots, n)$.

Let $W_1, \ldots, W_{2m+q}$ denote the list of vector fields $X_1, \ldots, X_q, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_m)$, $2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_m)$; and order $W_1, \ldots, W_{2m+q}$ so that

$$W_1, \ldots, W_{2n+r} = X_1, \ldots, X_r, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_n). \tag{4.3}$$

Define $\mathscr{W}_x := \mathrm{span}_{\mathbb{R}}\{W_1(x), \ldots, W_{2m+q}(x)\} = (\mathscr{L}_x + \overline{\mathscr{L}_x}) \cap T_x B_{X,L}(x_0, \xi)$. Set $P_0 := (1, \ldots, 2n+r) \in \mathcal{I}(2n+r, 2m+q)$ and for any $P = (p_1, \ldots, p_{2n+r}) \in \mathcal{I}(2n+r, 2m+q)$ we write W_P for the list $W_{p_1}, \ldots, W_{p_{2n+r}}$ and set $\bigwedge W_P = W_{p_1} \wedge W_{p_2} \wedge \cdots \wedge W_{p_{2n+r}}$. In particular,

$$\begin{aligned} \bigwedge W_{P_0} &= X_1 \wedge X_2 \wedge \cdots \wedge X_r \wedge 2\mathrm{Re}(L_1) \wedge 2\mathrm{Re}(L_2) \wedge \cdots \wedge 2\mathrm{Re}(L_n) \\ &\quad \wedge 2\mathrm{Im}(L_1) \wedge 2\mathrm{Im}(L_2) \wedge \cdots \wedge 2\mathrm{Im}(L_n) \\ &= \left(\bigwedge X_{K_0}\right) \bigwedge \left(\bigwedge 2\mathrm{Re}(L)_{J_0}\right) \bigwedge \left(\bigwedge 2\mathrm{Im}(L)_{J_0}\right), \end{aligned}$$

where $\bigwedge 2\mathrm{Re}(L)_{J_0}$ and $\bigwedge 2\mathrm{Im}(L)_{J_0}$ are defined in the obvious way; see (B.2). Note that $B_{W_{P_0}}(x_0, \xi)$ and $B_{X_{K_0}, L_{J_0}}(x_0, \xi)$ are (by definition) equal; see (2.4).

Definition 4.1. For $x \in \mathfrak{M}$, $U \subseteq \mathfrak{M}$, and $\eta > 0$, we say W_{P_0} satisfies $\mathcal{C}(x, \eta, U)$ if for every $a \in B^{2n+r}(\eta)$ the expression

$$e^{a_1 W_1 + a_2 W_2 + \cdots + a_{2n+r} W_{2n+r}} x$$

exists in U. More precisely, consider the differential equation

$$\frac{\partial}{\partial r} E(r) = a_1 W_1(E(r)) + \cdots + a_{2n+r} W_{2n+r}(E(r)), \quad E(0) = x.$$

We assume that a solution $E : [0,1] \to U$ exists for this differential equation. We have $E(r) = e^{ra_1 W_1 + \cdots + ra_{2n+r} W_{2n+r}} x$.

We fix the following two quantities:

- Fix $\eta > 0$ so that W_{P_0} satisfies $\mathcal{C}(x_0, \eta, \mathfrak{M})$.
- Fix $\delta_0 > 0$ such that $\forall \delta \in (0, \delta_0]$, the following holds. If $z \in B_{X_{K_0}, L_{J_0}}(x_0, \xi)$ is such that W_{P_0} satisfies $\mathcal{C}(z, \delta, B_{X_{K_0}, L_{J_0}}(x_0, \xi))$ and if $t \in B_{\mathbb{R}^{2n+r}}(\delta)$ is such that $e^{t_1 W_1 + \cdots + t_{2n+r} W_{2n+r}} z = z$ and if $W_1(z), \ldots, W_{2n+r}(z)$ are linearly independent, then $t = 0$.

Such a choice of η, δ_0 always exist (see Lemma 4.13). These constants are invariant under C^2 diffeomorphisms, and our quantitative results will be in terms of these constants; see [27, Section 4.1] for a detailed discussion of η and δ_0.

In our main result, we keep track of what parameters each estimate depends on.[9] To ease notation, we introduce various notions of "admissible constants". These will be constants which only depend on certain parameters.[10]

Definition 4.2. We say C is a 0-admissible constant if C can be chosen to depend only on upper bounds for m, q, ζ^{-1}, ξ^{-1}, and $\|c_{j,k}^{a,l}\|_{C(B_{X_{K_0}, L_{J_0}}(x_0,\xi))}$, $1 \leq j,k,l \leq m+q$, $1 \leq a \leq 3$.

Fix $s_0 \in (1,\infty) \cup \{\omega\}$; when $s_0 \in (1,\infty)$ the following result concerns the setting of $\mathscr{C}^s$ for $s \in [s_0, \infty]$ (and the results are stronger the closer s_0 is to 1, but the constants depend on the choice of s_0). When $s_0 = \omega$ the following result concerns the real analytic setting. Thus, there are two cases in what follows: when $s_0 \in (1,\infty)$ and when $s_0 = \omega$.

Definition 4.3. If $s_0 \in (1,\infty)$, for $s \in [s_0, \infty)$, if we say C is a $\{s\}$-admissible constant, it means that we assume $c_{j,k}^{a,l} \in \mathscr{C}^s_{X_{K_0}, L_{J_0}}(B_{X_{K_0}, L_{J_0}}(x_0, \xi))$, for $1 \leq j,k,l \leq m+q$, $1 \leq a \leq 3$. C can then be chosen to depend only on s, s_0, and upper bounds for m, q, ζ^{-1}, ξ^{-1}, η^{-1}, δ_0^{-1}, and $\|c_{j,k}^{a,l}\|_{\mathscr{C}^s_{X_{K_0}, L_{J_0}}(B_{X_{K_0}, L_{J_0}}(x_0,\xi))}$, $1 \leq j,k,l \leq m+q$, $1 \leq a \leq 3$. For $s \in (0, s_0)$, we define $\{s\}$-admissible constants to be $\{s_0\}$-admissible constants.

Definition 4.4. If $s_0 = \omega$, and if we say C is an $\{\omega\}$-admissible constant, it means that we assume $c_{j,k}^{a,l} \in \mathscr{A}^{x_0,\eta}_{X_{K_0}, L_{J_0}}$, $1 \leq j,k,l \leq m+q$, $1 \leq a \leq 3$. C can be chosen to depend only on upper bounds for m, q, ζ^{-1}, ξ^{-1}, η^{-1}, δ_0^{-1}, and $\|c_{j,k}^{a,l}\|_{\mathscr{A}^{x_0,\eta}_{X_{K_0}, L_{J_0}}}$, $1 \leq j,k,l \leq m+q$, $1 \leq a \leq 3$.

[9] Keeping track of constants in our main theorem is essential for applications. For example, to prove the results in Sections 3.2.1, 3.2.2, and 7.2 we will apply Theorem 4.5 infinitely many times, and the constants must be uniform over all these applications.

[10] The various notions of admissible constants may vary from section to section, but we are explicit about how they are defined whenever they are used. See Remark 11.2 for how this varying notation is exploited in the proofs.

Whenever we define a notion of $*$-admissible constant (where $*$ can be any symbol), we write $A \lesssim_* B$ for $A \leq CB$, where C is a positive $*$-admissible constant. We write $A \approx_* B$ for $A \lesssim_* B$ and $B \lesssim_* A$.

In what follows, we give $\mathbb{R}^r \times \mathbb{C}^n$ coordinates (t, z), where $t = (t_1, \ldots, t_r) \in \mathbb{R}^r$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We write $\frac{\partial}{\partial t}$ for the column vector $[\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}]^\top$ and $\frac{\partial}{\partial \bar{z}}$ for the column vector $[\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}]^\top$.

Theorem 4.5. *There exists a* 0*-admissible constant* $\chi \in (0, \xi]$ *such that:*

(i) $\forall y \in B_{X_{K_0}, L_{J_0}}(x_0, \chi)$,

$$\Big(\bigwedge X_{K_0}(y)\Big) \bigwedge \Big(\bigwedge L_{J_0}(y)\Big) \neq 0, \quad \bigwedge W_{P_0}(y) \neq 0.$$

In particular, $X_{K_0}(y), L_{J_0}(y)$ *is a basis for* $\mathscr{L}_y$ *and* $W_{P_0}(y)$ *is a basis for* $\mathscr{W}_y$*. Recall,*

$$\mathscr{W}_y = \operatorname{span}_{\mathbb{R}}\{W_1(y), \ldots, W_{2m+q}(y)\}.$$

(ii) $\forall y \in B_{X_{K_0}, L_{J_0}}(x_0, \chi)$,

$$\max_{\substack{J \in \mathcal{I}(n_1, m), K \in \mathcal{I}(r_1, q) \\ n_1 + r_1 = n + r}} \left| \frac{(\bigwedge X_K(y)) \bigwedge (\bigwedge L_J(y))}{(\bigwedge X_{K_0}(y)) \bigwedge (\bigwedge L_{J_0}(y))} \right| \approx_0 1,$$

$$\max_{P \in \mathcal{I}(2n+r, 2m+q)} \left| \frac{\bigwedge W_P(y)}{\bigwedge W_{P_0}(y)} \right| \approx_0 1.$$

(iii) $\forall \chi' \in (0, \chi]$, $B_{X_{K_0}, L_{J_0}}(x_0, \chi')$ *is an open subset of* $B_{X,L}(x_0, \xi)$*, and is therefore a submanifold.*

For the rest of the theorem, we assume:

- *If* $s_0 \in (1, \infty)$*, we assume* $c_{j,k}^{a,l} \in \mathscr{C}^{s_0}_{X_{K_0}, L_{J_0}}(B_{X_{K_0}, L_{J_0}}(x_0, \xi))$, $\forall 1 \leq j, k, l \leq m + q$, $1 \leq a \leq 3$.
- *If* $s_0 = \omega$*, we assume* $c_{j,k}^{a,l} \in \mathscr{A}^{x_0, \eta}_{X_{K_0}, L_{J_0}}$, $\forall 1 \leq j, k, l \leq m + q$, $1 \leq a \leq 3$.

There exists a C^2 *map* $\Phi : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to B_{X_{K_0}, L_{J_0}}(x_0, \chi)$ *and* $\{s_0\}$*-admissible constants* $\xi_1, \xi_2 > 0$ *such that:*

(iv) $\Phi(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))$ *is an open subset of* $B_{X_{K_0}, L_{J_0}}(x_0, \chi)$ *and is therefore a submanifold of* $B_{X,L}(x_0, \xi)$*.*

(v) $\Phi : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to \Phi(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))$ *is a* C^2*-diffeomorphism.*

(vi) $B_{X,L}(x_0, \xi_2) \subseteq B_{X_{K_0}, L_{J_0}}(x_0, \xi_1) \subseteq \Phi(B_{\mathbb{R}^r \times \mathbb{C}^n}(1)) \subseteq B_{X_{K_0}, L_{J_0}}(x_0, \chi) \subseteq B_{X,L}(x_0, \xi)$*.*

(vii) $\Phi(0) = x_0$*.*

There exists an $\{s_0\}$-admissible constant $K \geq 1$ and a matrix $\mathcal{A} : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to \mathbb{M}^{(n+r)\times(n+r)}(\mathbb{C})$ such that:

(viii) $\mathcal{A}(0) = 0$.

(ix)

$$\begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial \bar{z}} \end{bmatrix} = K^{-1}(I + \mathcal{A}) \begin{bmatrix} \Phi^* X_{K_0} \\ \Phi^* L_{J_0} \end{bmatrix},$$

where we have written $\Phi^* X_{K_0}$ for the column vector of vector fields $[\Phi^* X_1, \ldots, \Phi^* X_r]^\top$ and similarly for $\Phi^* L_{J_0}$.

(x)
- *If $s_0 \in (1, \infty)$, $\|\mathcal{A}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{M}^{(n+r)\times(n+r)})} \lesssim_{\{s\}} 1$, $\forall s \in (0, \infty)$, and $\|\mathcal{A}\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{M}^{(n+r)\times(n+r)})} \leq \frac{1}{4}$.*
- *If $s_0 = \omega$, $\|\mathcal{A}\|_{\mathscr{A}^{r+2n,1}(\mathbb{M}^{(n+r)\times(n+r)})} \leq \frac{1}{4}$, where we have identified $\mathbb{R}^r \times \mathbb{C}^n \cong \mathbb{R}^{r+2n}$.*

Note that in either case, this implies the matrix $(I + \mathcal{A}(\zeta))$ is invertible, $\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$.

(xi) $\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$, $1 \leq k \leq q$, $1 \leq j \leq m$,

$$\Phi^* X_k(\zeta) \in \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r} \right\},$$

$$\Phi^* L_j(\zeta) \in \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n} \right\}.$$

(xii)
- *If $s_0 \in (1, \infty)$, we have $\forall s \in (0, \infty)$, $1 \leq k \leq q$, $1 \leq j \leq m$,*

$$\|\Phi^* X_k\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{R}^r)} \lesssim_{\{s\}} 1, \quad \|\Phi^* L_j\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{C}^{r+n})} \lesssim_{\{s\}} 1.$$

- *If $s_0 = \omega$, we have for $1 \leq k \leq q$, $1 \leq j \leq m$,*

$$\|\Phi^* X_k\|_{\mathscr{A}^{2n+r,1}(\mathbb{R}^r)} \lesssim_{\{\omega\}} 1, \quad \|\Phi^* L_j\|_{\mathscr{A}^{2n+r,1}(\mathbb{C}^{r+n})} \lesssim_{\{\omega\}} 1.$$

Remark 4.6. In the language of Section 6, the map $\Phi : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to B_{X,L}(x_0, \xi)$ from Theorem 4.5 is an E-map; where $B_{X,L}(x_0, \xi)$ is given the E-manifold structure with the associated elliptic structure $\mathscr{L}$. In particular, when $r = 0$, $\mathscr{L}$ is a complex structure and the E-manifold structure on $B_{X,L}(x_0, \xi)$ is the complex manifold structure associated to $\mathscr{L}$ (via the Newlander-Nirenberg theorem). In this case, $\Phi : B_{\mathbb{C}^n}(1) \to B_{X,L}(x_0, \xi)$ is a holomorphic map (see Remark 6.12). This is particularly important for applications to several complex variables. For example this is used in Sections 3.1.2, 3.2.2, and 8.2 to guarantee the desired coordinate charts are holomorphic.

4.1. Densities

In many applications, one wishes to change variables in an integral using the coordinate chart given in Theorem 4.5 (see, e.g., the settings in Sections 3.2.1, 3.2.2, and 7.2[11]). Thus, it is important to understand pullbacks of certain densities via the map Φ. We present such results in this section. We refer the reader to [11] for a quick introduction to densities (see also [22] where densities are called 1-densities). In this section, we take all the assumptions as in Theorem 4.5 and let Φ be as in that theorem.

Let $\chi \in (0, \xi]$ be as in Theorem 4.5 and let ν be a real C^1 density on $B_{X_{K_0}, L_{J_0}}(x_0, \chi)$. Suppose, for $1 \leq k \leq r$, $1 \leq j \leq n$,

$$\mathcal{L}_{X_k} \nu = f_k^1 \nu, \quad \mathcal{L}_{L_j} \nu = f_j^2 \nu, \quad f_k^1, f_j^2 \in C(B_{X_{K_0}, L_{J_0}}(x_0, \chi)),$$

where $\mathcal{L}_V$ denotes the Lie derivative with respect to V, and $\mathcal{L}_{L_j}$ is defined as $\mathcal{L}_{\mathrm{Re} L_j} + i\mathcal{L}_{\mathrm{Im} L_j}$.

Definition 4.7. If we say C is a $[s_0; \nu]$-admissible constant, it means C is a $\{s_0\}$-admissible constant which is also allowed to depend on upper bounds for $\|f_k^1\|_{C(B_{X_{K_0}, L_{J_0}}(x_0, \chi))}$, $1 \leq k \leq r$, and $\|f_j^2\|_{C(B_{X_{K_0}, L_{J_0}}(x_0, \chi))}$, $1 \leq j \leq n$. This definition applies in either case: $s_0 \in (1, \infty)$ or $s_0 = \omega$.

Definition 4.8. If $s_0 \in (1, \infty)$, for $s > 0$, if we say C is a $\{s; \nu\}$-admissible constant it means that we assume $f_k^1, f_j^2 \in \mathscr{C}^s_{X_{K_0}, L_{J_0}}(B_{X_{K_0}, L_{J_0}}(x_0, \chi))$, $1 \leq k \leq r$, $1 \leq j \leq n$. C is allowed to depend on anything an $\{s\}$-admissible constant is allowed to depend on, and is also allowed to depend on upper bounds for $\|f_k^1\|_{\mathscr{C}^s_{X_{K_0}, L_{J_0}}(B_{X_{K_0}, L_{J_0}}(x_0, \chi))}$, $1 \leq k \leq r$, and $\|f_j^2\|_{\mathscr{C}^s_{X_{K_0}, L_{J_0}}(B_{X_{K_0}, L_{J_0}}(x_0, \chi))}$, $1 \leq j \leq n$. For $s \leq 0$, we define $\{s; \nu\}$-admissible constants to be $[s_0; \nu]$-admissible constants.

If $s_0 = \omega$ we fix some $r_0 > 0$; the results which follow depend on the choice of r_0.

Definition 4.9. If $s_0 = \omega$, if we say C is an $\{\omega; \nu\}$-admissible constant, it means that we assume $f_k^1, f_j^2 \in \mathscr{A}^{x_0, r_0}_{X_{K_0}, L_{J_0}}$. C is allowed to depend on anything an $\{\omega\}$-admissible constant may depend on, and is allowed to depend on upper bounds for r_0^{-1}, $\|f_k^1\|_{\mathscr{A}^{x_0, r_0}_{X_{K_0}, L_{J_0}}}$, $1 \leq k \leq r$, and $\|f_j^2\|_{\mathscr{A}^{x_0, r_0}_{X_{K_0}, L_{J_0}}}$, $1 \leq j \leq n$.

Theorem 4.10. *Define $h \in C^1(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))$ by $\Phi^* \nu = h \sigma_{\mathrm{Leb}}$, where σ_{Leb} denotes the usual Lebesgue density on $\mathbb{R}^r \times \mathbb{C}^n$.*

[11] For example, such changes of variables were important in the study of multi-parameter singular integrals and singular radon transforms in [30,24,29,26].

(i) $\forall \zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$,

$$h(\zeta) \approx_{[s_0;\nu]} \nu(X_1, \ldots, X_r, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_n))(x_0).$$

In particular, $h(\zeta)$ always has the same sign, and is either never zero or always zero.

(ii) • *If $s_0 \in (1, \infty)$, for $s > 0$,*

$$\|h\|_{\mathscr{C}^s(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_{\{s-1;\nu\}} |\nu(X_1, \ldots, X_r, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_n))(x_0)|.$$

• *If $s_0 = \omega$,*

$$\|h\|_{\mathscr{A}^{2n+r, \min\{1, r_0\}}} \lesssim_{\{\omega;\nu\}} |\nu(X_1, \ldots, X_r, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_n))(x_0)|.$$

Corollary 4.11. *Let $\xi_2 > 0$ be as in Theorem 4.5. Then,*

$$\begin{aligned} &\nu(B_{X_{K_0}, L_{J_0}}(x_0, \xi_2)) \approx_{[s_0;\nu]} \nu(B_{X,L}(x_0, \xi_2)) \\ &\approx_{[s_0;\nu]} \nu(X_1, \ldots, X_r, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_n))(x_0), \end{aligned} \tag{4.4}$$

and therefore

$$\begin{aligned} &\left|\nu(B_{X_{K_0}, L_{J_0}}(x_0, \xi_2))\right| \approx_{[s_0;\nu]} |\nu(B_{X,L}(x_0, \xi_2)| \\ &\approx_{[s_0;\nu]} |\nu(X_1, \ldots, X_r, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_n))(x_0)| \\ &\approx_0 \max_{K \in \mathcal{I}(r,q), J \in \mathcal{I}(n,m)} |\nu(X_K, 2\mathrm{Re}(L)_J, 2\mathrm{Im}(L)_J)(x_0)| \\ &\approx_0 \max_{P \in \mathcal{I}(2n+r, 2m+q)} |\nu(W_P)(x_0)|. \end{aligned} \tag{4.5}$$

4.2. Some comments on the assumptions

Because $W_1, \ldots, W_{2m+q}$ span the tangent space of $B_{X,L}(x_0, \xi)$ at every point (see Proposition A.1) and $B_{X,L}(x_0, \xi)$ is a $2n + r$ dimensional manifold, it follows that $x \mapsto \dim \mathscr{W}_x$, taking $B_{X,L}(x_0, \xi) \to \mathbb{N}$, is constant. However, the hypothesis that $x \mapsto \dim \mathscr{L}_x$ is constant does not follow from the other assumptions. The next example elucidates this:

Example 4.12. On $\mathbb{C}$, consider the vector fields $L_1 = \frac{\partial}{\partial z}$, $L_2 = \overline{z}\frac{\partial}{\partial \overline{z}}$, $X_1 = z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}$, and $X_2 = \frac{1}{i}\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right)$. We then have

$$[L_1, L_2] = [X_1, X_2] = [L_2, X_1] = [L_2, X_2] = 0, \quad [L_1, X_1] = L_1, \quad [L_1, X_2] = \frac{1}{i} L_1,$$

and the vector fields $L_1, L_2, X_1, X_2, \overline{L_1}, \overline{L_2}$ span the complexified tangent space at every point (in fact L_1 and $\overline{L_1}$ do). However,

$$\dim \operatorname{span}_{\mathbb{C}}\{L_1(z), L_2(z), X_1(z), X_2(z)\} = \begin{cases} 2, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

The assumption that $x \mapsto \dim \mathscr{L}_x$ is constant is equivalent to the assumption that $x \mapsto \dim \mathscr{X}_x$ is constant. Indeed, by Lemma B.1, and the fact that $\dim \mathscr{W}_x = 2n + r = \dim B_{X,L}(x_0, \xi)$, we have

$$2n+r = \dim \mathscr{W}_x = \dim(\mathscr{L}_x + \overline{\mathscr{L}_x}) = 2\dim(\mathscr{L}_x) - \dim(\mathscr{L}_x \bigcap \overline{\mathscr{L}_x}) = 2\dim(\mathscr{L}_x) - \dim(\mathscr{X}_x).$$

In particular, in the two most important special cases $\mathscr{L}_x = \mathscr{X}_x\ \forall x$, or $\mathscr{X}_x = \{0\}\ \forall x$, the hypothesis that $x \mapsto \dim \mathscr{L}_x$ is constant does follow from the other assumptions.

A choice of $\eta, \delta_0 > 0$, as in the hypotheses of Theorem 4.5, always exist. In fact, they can be chosen uniformly on compact sets, as the next lemma shows.

Lemma 4.13. *Let $W = W_1, \ldots, W_N$ be a list of C^1 vector fields on a C^2 manifold M and let $\mathcal{K} \Subset M$ be a compact set.*

(i) $\exists \eta > 0$ *such that* $\forall x_0 \in \mathcal{K}$, *$W$ satisfies $\mathcal{C}(x_0, \eta, M)$.*
(ii) $\exists \delta_0 > 0$ *such that* $\forall \theta \in S^{N-1}$ *if $x \in \mathcal{K}$ is such that $\theta_1 W_1(x) + \cdots + \theta_N W_N(x) \neq 0$, then* $\forall r \in (0, \delta_0]$,

$$e^{r\theta_1 W_1 + \cdots + r\theta_N W_N} x \neq x.$$

Proof. (i) is a simple consequence of the Phragmén–Lindelöf Principle. (ii) is proved in [27, Proposition 4.14]. □

Despite the fact that a choice of $\eta, \delta_0 > 0$ always exist (as described in Lemma 4.13), η and δ_0 are diffeomorphic invariant quantities,[12] and the proof of existence of these constants in Lemma 4.13 depends on the C^1 norms of the vector fields $W_1, \ldots, W_N$ in some fixed coordinate system (which is not a diffeomorphic invariant quantity). Thus, we state all of our results in terms of η and δ_0 to preserve the quantitative diffeomorphism invariance. See Section 4.3.

4.3. Diffeomorphism invariance

The main results of this paper are invariant under arbitrary C^2 diffeomorphisms. This is true quantitatively. For example, consider Theorem 4.5. Let $X_1, \ldots, X_q, L_1, \ldots, L_m$

[12] I.e., η and δ_0 remain unchanged when the entire setting is pushed forward under a C^2 diffeomorphism.

be the vector fields on $\mathfrak{M}$ from Theorem 4.5 and let $\Psi : \mathfrak{M} \to \mathfrak{N}$ be a C^2 diffeomorphism. Then $X_1, \ldots, X_q, L_1, \ldots, L_m$ satisfy the conditions of Theorem 4.5 at the point $x_0 \in \mathfrak{M}$ if and only if $\Psi_* X_1, \ldots, \Psi_* X_q, \Psi_* L_1, \ldots, \Psi_* L_m$ satisfy the conditions at $\Psi(x_0)$. Moreover, any constant which is $*$-admissible (where $*$ is any symbol) with respect to $X_1, \ldots, X_q, L_1, \ldots, L_m$ is $*$-admissible with respect to $\Psi_* X_1, \ldots, \Psi_* X_q, \Psi_* L_1, \ldots, \Psi_* L_m$. Finally, if Φ is the map guaranteed by Theorem 4.5 when applied to $X_1, \ldots, X_q, L_1, \ldots, L_m$, then $\Psi \circ \Phi$ is the map given by Theorem 4.5 when applied to $\Psi_* X_1, \ldots, \Psi_* X_q, \Psi_* L_1, \ldots, \Psi_* L_m$ (as can be seen by tracing through the proof). Thus the main results (and, indeed, the entire proofs) are invariant under arbitrary C^2 diffeomorphisms.

4.4. The Frobenius theorem and singular foliations

We now describe a consequence of Theorem 4.5 which is not used in the rest of the paper: it provides coordinate charts on leaves of singular foliations, which behave well in a quantitative way near singular points.

Let $\mathfrak{M}$ be a smooth manifold and let $X_1, \ldots, X_q$ be real C^∞ vector fields on $\mathfrak{M}$. Suppose

$$[X_j, X_k] = \sum_{l=1}^{q} c_{j,k}^l X_l, \quad c_{j,k}^l \in C^\infty(\mathfrak{M}).$$

For $x \in \mathfrak{M}$, let $\mathscr{X}_x := \mathrm{span}_{\mathbb{R}}\{X_1(x), \ldots, X_q(x)\}$. Under these hypotheses, the real Frobenius theorem applies to the distribution $\mathscr{X}$ to foliate $\mathfrak{M}$ into leaves. The tangent bundle to each leaf is given by $\mathscr{X}$ restricted to the leaf. Note that this may be a singular foliation: different leaves may have different dimensions, since $x \mapsto \dim \mathscr{X}_x$ might not be constant in x. For $x \in \mathfrak{M}$, let Leaf_x denote the leaf passing through x; thus Leaf_x is an injectively immersed C^∞ submanifold of $\mathfrak{M}$.

Definition 4.14. We say $x \in \mathfrak{M}$ is a singular point of the foliation if $x \mapsto \dim \mathscr{L}_x$ is not constant on any neighborhood of x (equivalently, if $x \mapsto \dim \mathrm{Leaf}_x$ is not constant on any neighborhood of x).

Leaf_x is a manifold, and is therefore defined by an atlas. For applications in analysis, it is sometimes important to have quantitative control of the charts which define the atlas. An interesting aspect of Theorem 4.5 is that it yields coordinate charts which behave well whether or not one is near a singular point.

Indeed, let $\mathcal{K} \Subset \mathfrak{M}$ be a compact set. Lemma 4.13 and some straightforward estimates show that Theorem 4.5 (in the case $m = 0$) applies to the vector fields $X_1, \ldots, X_q$ (with, e.g., $s_0 = 3/2$), *uniformly* for $x_0 \in \mathcal{K}$. Thus, any constant which is $\{s\}$-admissible (for any $s \in (0, \infty)$) in the sense of Theorem 4.5 can be taken independent of $x_0 \in \mathcal{K}$. The map Φ provided by Theorem 4.5 can be seen as a coordinate chart on Leaf_{x_0}, centered at x_0, which has good estimates which are uniform in x_0. In particular, as $x_0 \in \mathcal{K}$ approaches a singular point in $\mathcal{K}$, the estimates remain uniform.

The above holds in the complex setting as well. Again let $\mathfrak{M}$ be a smooth manifold, and let $L_1, \ldots, L_m$ be C^∞ complex vector fields on $\mathfrak{M}$. Suppose

$$[L_j, L_k] = \sum_{l=1}^m c_{j,k}^{1,l} L_l, \quad [L_j, \overline{L_k}] = \sum_{l=1}^m c_{j,k}^{2,l} L_l + \sum_{l=1}^m c_{j,k}^{3,l} \overline{L_l}, \quad c_{j,k}^{a,l} \in C^\infty(\mathfrak{M}).$$

For $x \in \mathfrak{M}$, set $\mathscr{L}_x := \operatorname{span}_{\mathbb{C}}\{L_1(x), \ldots, L_m(x)\}$; we assume $\mathscr{L}_x \bigcap \overline{\mathscr{L}_x} = \{0\}$, $\forall x \in \mathfrak{M}$. Under the above assumptions, the real Frobenius theorem applies to the distribution $\mathscr{L} + \overline{\mathscr{L}}$ to foliate $\mathfrak{M}$ into leaves; as before this may be a singular foliation. Let Leaf_x denote the leaf passing through x. For each $x \in \mathfrak{M}$, $\mathscr{L}$ (restricted to Leaf_x) defines a complex structure on Leaf_x, and the classical Newlander-Nirenberg theorem therefore gives Leaf_x the structure of a complex manifold. As in the real case, Theorem 4.5 applies uniformly as the base point x_0 ranges over compact sets (in this case, we take $q = 0$), in the sense that $\{s\}$-admissible constants (for any $s \in (0, \infty)$) may be taken independent of x_0 as x_0 ranges over a compact set. The map Φ provided by Theorem 4.5 can be seen as a holomorphic coordinate chart near x_0, which has estimates which are uniform on compact sets; whether or not that compact set contains a singular point.

Remark 4.15. A previous (and weaker) version of the above ideas (originally described in [28]) was an essential point in the work of the author and Stein on singular Radon transforms [24,29,26,25]. For example, a corollary of one of the main results of [25] is the following. Suppose $\gamma_t(x)$ is real analytic function defined on a neighborhood of the origin of $(t, x) \in \mathbb{R}^N \times \mathbb{R}^n$, mapping to $\mathbb{R}^n$, and satisfying $\gamma_0(x) \equiv x$. Define an operator acting on functions $f(x)$ defined near the origin in $x \in \mathbb{R}^n$ by

$$Tf(x) = \psi(x) \int f(\gamma_t(x)) K(t)\, dt,$$

where $K(t)$ is a Calderón-Zygmund kernel supported near $t = 0$, and $\psi \in C_0^\infty(\mathbb{R}^n)$ is supported near $x = 0$. Then, $T : L^p \to L^p$, for $1 < p < \infty$; see [25] for a more precise statement and further details. This result does not follow from the foundational work of Christ, Nagel, Stein, and Wainger on singular Radon transforms [8]; however, the only additional ingredient necessary to conclude this result (beyond the theory in [8]) is the above described uniformity of coordinate charts near singular points (though the theory in [25] proceeds by proving a more general result, and concluding the above result as a corollary).

Remark 4.16. One way to view the above discussion is that Theorem 4.5 is quantitatively invariant under C^2 diffeomorphisms (see Section 4.3), and being "nearly" a singular point is not a diffeomorphically invariant concept. Indeed, consider the real case described above. Fix $x_0 \in \mathfrak{M}$ and let $k := \dim \mathscr{X}_{x_0} = \dim \mathrm{Leaf}_{x_0}$ and $N := \dim \mathfrak{M}$. Pick a coordinate system on $\mathfrak{M}$ near x_0. In this coordinate system, we may think of $X_1(x_0), \ldots, X_q(x_0)$ as vectors in $\mathbb{R}^N$ which span a k dimensional subspace of $\mathbb{R}^N$. Let $\sigma := \min |\det B|$, where B ranges over all $k \times k$ submatrices of the $N \times q$ matrix $(X_1(x_0)| \cdots |X_q(x_0))$. Then, $\sigma > 0$. One might say x_0 is "nearly" a singular point if σ is small. However, σ is not invariant under diffeomorphisms: the above procedure depended on the choice of coordinate system. This is one way of intuitively understanding why the estimates in Theorem 4.5 do not depend on a lower bound for $\sigma > 0$.

Remark 4.17. While we described the above for smooth vector fields, similar remarks hold for vector fields with a finite level of smoothness using the same ideas.

4.5. Proof outline

Theorem 4.5 is the central result of this paper. If all we wanted was a coordinate system like Φ in which the vector fields were normalized and had the desired regularity, but did not have the key property given in Theorem 4.5 (xi),[13] then Theorem 4.5 would be an easy consequence of the main results in [27,31] applied to $W_1, \ldots, W_{2m+q}$ (see Section 12 for a detailed statement of this). In particular, in the case when $q = 0$ (and M is given the complex structure induced by $\mathscr{L}$ via the Newlander-Nirenberg theorem–see Remark 4.6), then if we did not require that Φ be holomorphic, Theorem 4.5 would be a simple consequence of the results in [27,31].

The proof proceeds as follows. We apply the results from [27,31] (see Section 12) to yield a candidate chart Φ_0 satisfying all the conclusions of Theorem 4.5 without the key property discussed above. Then, we apply the main technical result of [33] to obtain another map Φ_1 such that if we set $\Phi = \Phi_0 \circ \Phi_1$, Φ satisfies all the conclusions of Theorem 4.5.

As described above, in this paper we construct the map Φ as a composition of two maps $\Phi = \Phi_0 \circ \Phi_1$. When $s_0 \in (1, \infty)$, Φ_0 is constructed in [31] as a composition of three maps (one of which was a simple dilation map). When $s_0 \in (1, \infty)$, Φ_1 was constructed in [33] as a composition of four maps (two of which were simple dilation maps). Thus, if $s_0 \in (1, \infty)$, when all the proofs are unraveled, Φ is a composition of seven maps, three of which are simple dilation maps. When $s_0 = \omega$, Φ is considerably simpler.

5. Notation

If $f : M \to N$ is a C^1 map between C^1 manifolds, we write $df(x) : T_x M \to T_x N$ for the usual differential. We extend this to be a complex linear map $df(x) : \mathbb{C}T_x M \to \mathbb{C}T_x N$, where $\mathbb{C}T_x M = T_x M \otimes_{\mathbb{R}} \mathbb{C}$ denotes the complexified tangent space. Even if the manifold M has additional structure (e.g., in the case of a complex manifold), $df(x)$ is defined in terms of the underlying real manifold structure.

[13] And replacing $\frac{\partial}{\partial \bar{z}}$ with $\frac{\partial}{\partial x}$ in Theorem 4.5 (ix), where $x \in \mathbb{R}^{2n}$.

When working on $\mathbb{R}^r \times \mathbb{C}^n$ we will often use coordinates (t, z) where $t = (t_1, \dots, t_r) \in \mathbb{R}^r$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. We write

$$\frac{\partial}{\partial t} = \begin{bmatrix} \frac{\partial}{\partial t_1} \\ \frac{\partial}{\partial t_2} \\ \vdots \\ \frac{\partial}{\partial t_r} \end{bmatrix}, \quad \frac{\partial}{\partial \overline{z}} = \begin{bmatrix} \frac{\partial}{\partial \overline{z}_1} \\ \frac{\partial}{\partial \overline{z}_2} \\ \vdots \\ \frac{\partial}{\partial \overline{z}_n} \end{bmatrix}.$$

At times we will instead use coordinates (u, w) where $u \in \mathbb{R}^r$ and $w \in \mathbb{C}^n$ and define $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial \overline{w}}$ similarly.

We identify $\mathbb{R}^r \times \mathbb{R}^{2n} \cong \mathbb{R}^r \times \mathbb{C}^n$ via the map $(t_1, \dots, t_r, x_1, \dots, x_{2n}) \mapsto (t_1, \dots, t_r, x_1 + ix_{n+1}, \dots, x_n + ix_{2n})$. Thus, given a function $G(t, z) : \mathbb{R}^r \times \mathbb{C}^n \to \mathbb{R}^s \times \mathbb{C}^m$, we may also think of G as function $G(t, x) = (G_1(t, x), \dots, G_{s+2m}(t, x)) : \mathbb{R}^r \times \mathbb{R}^{2n} \to \mathbb{R}^s \times \mathbb{R}^{2m}$. For such a function, we write

$$d_{(t,x)}G = \begin{bmatrix} \frac{\partial G_1}{\partial t_1} & \cdots & \frac{\partial G_1}{\partial t_r} & \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_{2n}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_{s+2m}}{\partial t_1} & \cdots & \frac{\partial G_{s+2m}}{\partial t_r} & \frac{\partial G_{s+2m}}{\partial x_1} & \cdots & \frac{\partial G_{s+2m}}{\partial x_{2n}} \end{bmatrix}.$$

We write $I_{N\times N} \in \mathbb{M}^{N\times N}$ to denote the $N \times N$ identity matrix, and $0_{a\times b} \in \mathbb{M}^{a\times b}$ to denote the $a \times b$ zero matrix.

6. E-manifolds

The results in this paper simultaneously deal with the setting of real vector fields (on a real manifold) and the setting of complex vector fields (on a complex manifold). It is more convenient to work in a category of manifolds which contains both real manifolds and complex manifolds as full subcategories. We define these manifolds here, and call them E-manifolds.[14] This category of manifolds was also used in [33], and we refer the reader to that reference for a more detailed description.

Remark 6.1. "E" in the name E-manifolds stands for "elliptic". Indeed, using the terminology of [35, Definition I.2.3], a complex manifold is a manifold endowed with a complex structure, a CR-manifold is a manifold endowed with a CR structure, and an E-manifold is a manifold endowed with an elliptic structure; see Theorem 6.22 and [33] for a more detailed discussion. Unfortunately, the name "elliptic manifold" is already taken by an unrelated concept.

[14] The manifold structure we discuss here is well-known to experts, but we could not find a name for the category of such manifolds, and decided to call them E-manifolds for lack of a better name.

Definition 6.2. Let $U_1 \subseteq \mathbb{R}^{r_1} \times \mathbb{C}^{n_1}$ and $U_2 \subseteq \mathbb{R}^{r_2} \times \mathbb{C}^{n_2}$ be open sets. We give $\mathbb{R}^{r_1} \times \mathbb{C}^{n_1}$ coordinates (t, z) and $\mathbb{R}^{r_2} \times \mathbb{C}^{n_2}$ coordinates (u, w). We say a C^1 map $f : U_1 \to U_2$ is an E-map if

$$df(t,z)\frac{\partial}{\partial t_k}, df(t,z)\frac{\partial}{\partial \overline{z}_j} \in \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{r_2}}, \frac{\partial}{\partial \overline{w}_1}, \ldots, \frac{\partial}{\partial \overline{w}_{n_2}} \right\},$$
$$\forall (t,z) \in U_1, 1 \leq k \leq r_1, 1 \leq j \leq n_1.$$

For $s \in (1, \infty] \cup \{\omega\}$, we say f is a $\mathscr{C}^s_{\mathrm{loc}}$ E-map if f is an E-map and $f \in \mathscr{C}^s_{\mathrm{loc}}(U_1; \mathbb{R}^{r_2} \times \mathbb{C}^{n_2})$.

Remark 6.3. Suppose $U_1, U_2 \subseteq \mathbb{R}^r \times \mathbb{C}^n$ and $f : U_1 \to U_2$ is an E-map which is also a C^1-diffeomorphism. Then, $f^{-1} : U_2 \to U_1$ is an E-map.

Remark 6.4. Note that when $r_1 = r_2 = 0$, if $U_1 \subseteq \mathbb{R}^0 \times \mathbb{C}^{n_1} \cong \mathbb{C}^{n_1}$, $U_2 \subseteq \mathbb{R}^0 \times \mathbb{C}^{n_2} \cong \mathbb{C}^{n_2}$, then $f : U_1 \to U_2$ is an E-map if and only if it is holomorphic.

Definition 6.5. Let M be a Hausdorff, paracompact topological space and fix $n, r \in \mathbb{N}$, $s \in (1, \infty] \cup \{\omega\}$. We say $\{(\phi_\alpha, V_\alpha) : \alpha \in \mathcal{I}\}$ (where $\mathcal{I}$ is some index set) is a $\mathscr{C}^s$ E-atlas of dimension (r, n) if $\{V_\alpha : \alpha \in \mathcal{I}\}$ is an open cover for M, $\phi_\alpha : V_\alpha \to U_\alpha$ is a homeomorphism where $U_\alpha \subseteq \mathbb{R}^r \times \mathbb{C}^n$ is open, and $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(V_\beta \cap V_\alpha) \to U_\beta$ is a $\mathscr{C}^s_{\mathrm{loc}}$ E-map, $\forall \alpha, \beta$.

Definition 6.6. A $\mathscr{C}^s$ E-manifold M of dimension (r, n) is a Hausdorff, paracompact topological space M endowed with a $\mathscr{C}^s$ E-atlas of dimension (r, n).

Remark 6.7. On may analogously define C^m E-manifolds in the obvious way. C^∞ E-manifolds and $\mathscr{C}^\infty$ E-manifolds are the same (because $\mathscr{C}^\infty_{\mathrm{loc}}$ is the usual space of smooth functions).

Definition 6.8. For $s \in (0, \infty] \cup \{\omega\}$, let M and N be $\mathscr{C}^{s+1}$ E-manifolds with $\mathscr{C}^{s+1}$ E-atlases $\{(\phi_\alpha, V_\alpha)\}$ and $\{(\psi_\beta, W_\beta)\}$, respectively. We say $f : M \to N$ is a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ E-map if $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ E-map, $\forall \alpha, \beta$.

Lemma 6.9. *For $s \in (0, \infty] \cup \{\omega\}$, let M_1, M_2, and M_3 be $\mathscr{C}^{s+1}$ E-manifolds and $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ be $\mathscr{C}^{s+1}_{\mathrm{loc}}$ E-maps. Then, $f_2 \circ f_1 : M_1 \to M_3$ is a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ E-map.*

Proof. See [33, Lemma 4.10] for a proof of this standard result. □

Lemma 6.10. *For $s \in (0, \infty] \cup \{\omega\}$, let M_1 and M_2 be $\mathscr{C}^{s+1}$ E-manifolds and let $f : M_1 \to M_2$ be a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ E-map which is also a C^1 diffeomorphism. Then, $f^{-1} : M_2 \to M_1$ is a $\mathscr{C}^{s+1}_{\mathrm{loc}}$ E-map.*

Proof. See [33, Lemma 4.11] for a proof of this standard result. □

Definition 6.11. Suppose $s \in (0,\infty] \cup \{\omega\}$, and M_1 and M_2 are $\mathscr{C}^{s+1}$ E-manifolds. We say $f : M_1 \to M_2$ is a $\mathscr{C}^{s+1}$ E-diffeomorphism if $f : M_1 \to M_2$ is invertible and $f : M_1 \to M_2$ and $f^{-1} : M_2 \to M_1$ are $\mathscr{C}^{s+1}_{\mathrm{loc}}$ E-maps.

Remark 6.12. For $s \in (1,\infty] \cup \{\omega\}$ the category of $\mathscr{C}^s$ E-manifolds, whose objects are $\mathscr{C}^s$ E-manifolds and morphisms are $\mathscr{C}^s_{\mathrm{loc}}$ E-maps, contains both $\mathscr{C}^s$ real manifolds and complex manifolds as full subcategories. The real manifolds of dimension r are those with E-dimension $(r,0)$, while the complex manifolds of complex dimension n are those with E-dimension $(0,n)$. That complex manifolds (with morphisms given by holomorphic maps) embed as a *full* subcategory follows from Remark 6.4. The isomorphisms in the category of $\mathscr{C}^s$ E-manifolds are the $\mathscr{C}^s$ E-diffeomorphisms.

Remark 6.13. Note that open subsets of $\mathbb{R}^r \times \mathbb{C}^n$ are $\mathscr{C}^\omega$ E-manifolds of dimension (r,n), by using the atlas consisting of one coordinate chart (the identity map). Henceforth, we give such sets this E-manifold structure.

As mentioned above, $\mathscr{C}^s$ E-manifolds of dimension $(r,0)$ are exactly the $\mathscr{C}^s$ manifolds of dimension r, in the usual sense (in particular, one may take Definition 6.6 in the case $n = 0$ as the definition of a $\mathscr{C}^s$ manifolds of dimension r). There is a natural forgetful functor taking $\mathscr{C}^s$ E-manifolds of dimension (r,n) to $\mathscr{C}^s$ manifolds of dimension $2n+r$. Thus, one may define any of the usual objects from manifolds on E-manifolds. For example, we have the following standard definitions on $\mathscr{C}^s$ manifolds, and therefore on $\mathscr{C}^s$ E-manifolds.[15]

Definition 6.14. For $s \in (0,\infty] \cup \{\omega\}$ let M be a $\mathscr{C}^{s+1}$ manifold of dimension r, with $\mathscr{C}^{s+1}$ atlas $\{(\phi_\alpha, V_\alpha)\}$; here $\phi_\alpha : V_\alpha \to U_\alpha$ is a $\mathscr{C}^{s+1}$ diffeomorphism and $U_\alpha \subseteq \mathbb{R}^r$ is open. We say a complex vector field X on M is a $\mathscr{C}^s$ vector field if $(\phi_\alpha)_* X \in \mathscr{C}^s_{\mathrm{loc}}(U_\alpha; \mathbb{C}^r)$, $\forall \alpha$.

Definition 6.15. For $s \in (0,\infty] \cup \{\omega\}$, a $\mathscr{C}^s$ sub-bundle $\mathscr{L}$ of $\mathbb{C}TM$ of rank $m \in \mathbb{N}$ is a disjoint union

$$\mathscr{L} = \bigcup_{\zeta \in M} \mathscr{L}_\zeta \subseteq \mathbb{C}TM$$

such that:

- $\forall \zeta \in M$, $\mathscr{L}_\zeta$ is an m-dimensional vector subspace of $\mathbb{C}T_\zeta M$.

[15] The following standard definitions can all be found in [35] in the case $s = \infty$, and in [33] for finite levels of smoothness.

- $\forall \zeta_0 \in M$, there exists an open neighborhood $U \subseteq M$ of ζ_0 and a finite collection of complex $\mathscr{C}^s$ vector fields $L_1, \ldots, L_K$ on U, such that $\forall \zeta \in U$,

$$\operatorname{span}_{\mathbb{C}}\{L_1(\zeta), \ldots, L_K(\zeta)\} = \mathscr{L}_\zeta.$$

Definition 6.16. For a $\mathscr{C}^s$ sub-bundle $\mathscr{L}$ of $\mathbb{C}TM$, we define $\overline{\mathscr{L}}$ by $\overline{\mathscr{L}_\zeta} = \{\overline{z} : z \in \mathscr{L}_\zeta\}$. It is easy to see that $\overline{\mathscr{L}}$ is a $\mathscr{C}^s$ sub-bundle of $\mathbb{C}TM$.

Definition 6.17. Let $W \subseteq M$ be open, L a complex vector field on W, and $\mathscr{L}$ a $\mathscr{C}^s$ sub-bundle of $\mathbb{C}TM$. We say L is a section of $\mathscr{L}$ over W if $\forall \zeta \in W$, $L(\zeta) \in \mathscr{L}_\zeta$. We say L is a $\mathscr{C}^s$ section of $\mathscr{L}$ over W if L is a section of $\mathscr{L}$ over W and L is a $\mathscr{C}^s$ complex vector field on W.

Definition 6.18. Let $\mathscr{L}$ be a $\mathscr{C}^{s+1}$ sub-bundle of $\mathbb{C}TM$. We say $\mathscr{L}$ is a $\mathscr{C}^{s+1}$ formally integrable structure if the following holds. For all $W \subseteq M$ open, and all $\mathscr{C}^{s+1}$ sections L_1 and L_2 of $\mathscr{L}$ over W, we have $[L_1, L_2]$ is a section of $\mathscr{L}$ over W.

Definition 6.19. Let $\mathscr{L}$ be a $\mathscr{C}^{s+1}$ formally integrable structure on M. We say $\mathscr{L}$ is a $\mathscr{C}^{s+1}$ elliptic structure if $\mathscr{L}_\zeta + \overline{\mathscr{L}_\zeta} = \mathbb{C}T_\zeta M$, $\forall \zeta \in M$.

For $s \in (0, \infty] \cup \{\omega\}$, on a $\mathscr{C}^{s+2}$ E-manifold of dimension (r, n), there is a naturally associated $\mathscr{C}^{s+1}$ elliptic structure on M defined as follows. Let (ϕ_α, V_α) be an E-atlas for M. For $\zeta \in M$ let $\zeta \in V_\alpha$ for some α. We set:

$$\begin{aligned}\mathscr{L}_\zeta := \operatorname{span}_{\mathbb{C}}\Big\{ & d\Phi_\alpha^{-1}(\Phi_\alpha(\zeta))\frac{\partial}{\partial t_1}, \ldots, d\Phi_\alpha^{-1}(\Phi_\alpha(\zeta))\frac{\partial}{\partial t_r}, d\Phi_\alpha^{-1}(\Phi_\alpha(\zeta))\frac{\partial}{\partial \overline{z}_1}, \ldots, \\ & d\Phi_\alpha^{-1}(\Phi_\alpha(\zeta))\frac{\partial}{\partial \overline{z}_n}\Big\}.\end{aligned}$$

It is straightforward to check that $\mathscr{L}_\zeta \subseteq \mathbb{C}T_{x_0}M$ is well-defined[16] and $\mathscr{L} = \bigcup_{\zeta \in M} \mathscr{L}_\zeta$ is a $\mathscr{C}^{s+1}$ elliptic structure on M. As remarked above, an E-manifold of dimension $(0, n)$ is a complex manifold; in this case $\mathscr{L}$ equals $T^{0,1}M$.

Definition 6.20. We call $\mathscr{L}$ the elliptic structure associated to the E-manifold M.

Lemma 6.21. *Suppose M and $\widehat{M}$ are $\mathscr{C}^s$ E-manifolds with associated elliptic structures $\mathscr{L}$ and $\widehat{\mathscr{L}}$. Then a $\mathscr{C}^s_{\mathrm{loc}}$ map $f : M \to \widehat{M}$ is a $\mathscr{C}^s_{\mathrm{loc}}$ E-map if and only if $df(x)\mathscr{L}_x \subseteq \widehat{\mathscr{L}}_{f(x)}$, $\forall x \in M$.*

Proof. This follows immediately from the definitions. □

[16] I.e., $\mathscr{L}_\zeta$ does not depend on which α we pick with $\zeta \in V_\alpha$.

It turns out that the elliptic structure $\mathscr{L}$ associated to the E-manifold M uniquely determines the E-manifold structure as the following theorem shows.

Theorem 6.22. *Let $s \in (0,\infty] \cup \{\omega\}$ and let M be a $\mathscr{C}^{s+2}$ manifold. For each $\zeta \in M$, let $\mathscr{L}_\zeta$ be a vector subspace of $\mathbb{C}T_\zeta M$, and let $\mathscr{L} = \bigcup_{\zeta\in M} \mathscr{L}_\zeta$. The following are equivalent:*

(i) There is a $\mathscr{C}^{s+2}$ E-manifold structure M, compatible with its $\mathscr{C}^{s+2}$ structure, such that $\mathscr{L}$ is the $\mathscr{C}^{s+1}$ elliptic structure associated to M.
(ii) $\mathscr{L}$ is a $\mathscr{C}^{s+1}$ elliptic structure.

Moreover, under these conditions, the E-manifold structure given in (i) is unique in the sense that of M is given another $\mathscr{C}^{s+2}$ E-manifold structure, compatible with its $\mathscr{C}^{s+2}$ structure, with respect to which $\mathscr{L}$ is the associated elliptic sub-bundle, then the identity map $M \to M$ is a $\mathscr{C}^{s+2}$ E-diffeomorphism between these two $\mathscr{C}^{s+2}$ E-manifold structures on M.

Comments on the proof. The case $s = \omega$ of this result is classical. The case $s = \infty$ is due to Nirenberg [23]. In the special case of complex manifolds (i.e., E-manifolds of dimension $(0, n)$) this is the Newlander-Nirenberg Theorem [21] with sharp regularity, as proved by Malgrange [15]. The full result can be found in [33, Theorem 4.18]. □

6.1. CR manifolds

There is another, related, category of manifolds of substantial interest, where the original scaling maps of Nagel, Stein, and Wainger [19] have been widely used: CR manifolds. Theorem 6.22 characterizes E-manifolds as manifolds endowed with an elliptic structure. CR manifolds are defined in a similar way.

Definition 6.23. Let $\mathscr{W}$ be a $\mathscr{C}^{s+1}$ formally integrable structure on M. We say $\mathscr{W}$ is a $\mathscr{C}^{s+1}$ CR structure if $\mathscr{W}_\zeta \cap \overline{\mathscr{W}_\zeta} = \{0\}$, $\forall \zeta \in M$.

Definition 6.24. A $\mathscr{C}^{s+2}$ CR manifold M is a $\mathscr{C}^{s+2}$ manifold M endowed with at $\mathscr{C}^{s+1}$ CR structure on M.

Notice that a CR structure, $\mathscr{W}$, is an elliptic structure if and only if $\mathscr{W}_\zeta \oplus \overline{\mathscr{W}_\zeta} = \mathbb{C}T_\zeta M$, $\forall \zeta \in M$. This is precisely the definition of a complex structure. There does not seem to be a natural way, given an arbitrary E-manifold, to see it as a CR manifold. Nor is there a natural way, given an arbitrary CR manifold, to see it as an E-manifold. Nevertheless, many of the classical examples of CR manifolds can be naturally given the structure of an E-manifold. Indeed, given a CR structure $\mathscr{W}$, it is often the case that there is another sub-bundle, $\mathscr{T}$, of $\mathbb{C}TM$ such that $\mathscr{W} \oplus \mathscr{T}$ is an elliptic structure.

The simplest example of this is the three dimensional Heisenberg group $\mathbb{H}^1$. As a manifold, $\mathbb{H}^1$ is diffeomorphic to $\mathbb{C}\times\mathbb{R}$ and we give it coordinates $(z, t) \in \mathbb{C}\times\mathbb{R}$. We give $\mathbb{H}^1$

a CR structure by setting $\mathscr{W}_{(z,t)} := \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t} \right\}$. By setting $\mathscr{T}_{(z,t)} := \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t} \right\}$, we have $\mathscr{W} \oplus \mathscr{T}$ is an elliptic structure on $\mathbb{H}^1$. In many examples of CR manifolds one has a similar setting: there are local coordinates $(z_1, \ldots, z_n, t_1, \ldots, t_r) \in \mathbb{C}^n \times \mathbb{R}^r$ such that the CR structure, $\mathscr{W}$, is contained in the span of $\frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}$ in such at way that if one takes $\mathscr{T}_{(z,t)} := \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r} \right\}$, then $\mathscr{W} \oplus \mathscr{T}$ is an elliptic structure. See Section 8.4 for a discussion of one way the results of this paper can be applied to CR manifolds.

Remark 6.25. A major distinction between CR structures and elliptic structures is that elliptic structures of dimension (r, n) have a single canonical example. Indeed, $\mathbb{R}^r \times \mathbb{C}^n$ is naturally an E-manifold with associated elliptic structure $\widehat{\mathscr{L}}$ given by

$$\widehat{\mathscr{L}_\zeta} = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r} \right\}, \quad \forall \zeta \in \mathbb{R}^r \times \mathbb{C}^n.$$

Theorem 6.22 shows that given any elliptic structure $\mathscr{L}$, there is a local coordinate system in which $\mathscr{L}$ is given by $\widehat{\mathscr{L}}$, where $n + r = \dim \mathscr{L}_\zeta$ and $r = \dim \left(\mathscr{L}_\zeta \cap \overline{\mathscr{L}_\zeta} \right)$ (here, n and r are constant in ζ–see [33, Section 3]). Theorem 4.5 can be thought of as a quantitative, diffeomorphic invariant version of a coordinate system which sees an elliptic structure as this canonical example. Since there is no similar canonical example of a CR structure, it is not immediately clear what an analog of Theorem 4.5 would be for general CR structures.

7. Corollaries revisited

In this section, we generalize the results from Section 3 using the language of E-manifolds. This unifies the complex and real settings.

7.1. Optimal smoothness

Let $X_1, \ldots, X_q$ be real C^1 vector fields on a connected C^2 manifold M and let $L_1, \ldots, L_m$ be complex C^1 vector fields on M. For $x \in M$ set

$$\mathscr{L}_x := \operatorname{span}_{\mathbb{C}} \{ X_1(x), \ldots, X_q(x), L_1(x), \ldots, L_m(x) \}, \quad \mathscr{X}_x := \operatorname{span}_{\mathbb{C}} \{ X_1(x), \ldots, X_q(x) \}. \tag{7.1}$$

We assume:

- $\mathscr{L}_x + \overline{\mathscr{L}_x} = \mathbb{C}T_x M$, $\forall x \in M$.
- $\mathscr{X}_x = \mathscr{L}_x \cap \overline{\mathscr{L}_x}$, $\forall x \in M$.

Theorem 7.1 *(The local theorem). Fix $x_0 \in M$, $s \in (1, \infty] \cup \{\omega\}$, and set $r := \dim \mathscr{X}_{x_0}$ and $n + r := \dim \mathscr{L}_{x_0}$. The following three conditions are equivalent:*

(i) *There exists an open neighborhood $V \subseteq M$ of x_0 and a C^2 diffeomorphism $\Phi : U \to V$, where $U \subseteq \mathbb{R}^r \times \mathbb{C}^n$ is open, such that $\forall (t,z) \in U$, $1 \le k \le q$, $1 \le j \le m$,*

$$\Phi^* X_k(t,z) \in \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r} \right\},$$

$$\Phi^* L_j(t,z) \in \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\},$$

and $\Phi^* X_k \in \mathscr{C}^{s+1}(U;\mathbb{R}^r)$, $\Phi^* L_j \in \mathscr{C}^{s+1}(U;\mathbb{C}^{r+n})$.

(ii) *Reorder $X_1, \ldots, X_q$ so that $X_1(x_0), \ldots, X_r(x_0)$ are linearly independent, and reorder $L_1, \ldots, L_m$ so that $L_1(x_0), \ldots, L_n(x_0), X_1(x_0), \ldots, X_r(x_0)$ are linearly independent. Let $\widehat{Z}_1, \ldots, \widehat{Z}_{n+r}$ denote the list $X_1, \ldots, X_r, L_1, \ldots, L_n$, and let $Y_1, \ldots, Y_{m+q-(r+n)}$ denote the list $X_{r+1}, \ldots, X_q, L_{n+1}, \ldots, L_m$. There exists an open neighborhood $V \subseteq M$ of x_0 such that:*

- *$[\widehat{Z}_j, \widehat{Z}_k] = \sum_{l=1}^{n+r} \hat{c}_{j,k}^{1,l} \widehat{Z}_l$, and $[\widehat{Z}_j, \overline{\widehat{Z}_k}] = \sum_{l=1}^{n+r} \hat{c}_{j,k}^{2,l} \widehat{Z}_l + \sum_{l=1}^{n+r} \hat{c}_{j,k}^{3,l} \overline{\widehat{Z}_l}$, where $\hat{c}_{j,k}^{a,l} \in \mathscr{C}^s_{X,L}(V)$, $1 \le j,k,l \le n+r$, $1 \le a \le 3$.*
- *$Y_j = \sum_{l=1}^{n+r} b_j^l \widehat{Z}_l$, where $b_j^l \in \mathscr{C}^{s+1}_{X,L}(V)$, $1 \le j \le m+q-(r+n)$, $1 \le l \le n+r$.*

Furthermore, the map $x \mapsto \dim \mathscr{L}_x$, $V \to \mathbb{N}$ is constant in x.

(iii) *Let $Z_1, \ldots, Z_{m+q}$ denote the list $X_1, \ldots, X_q, L_1, \ldots, L_m$. There exists a neighborhood $V \subseteq M$ of x_0 such that $[Z_j, Z_k] = \sum_{l=1}^{m+q} c_{j,k}^{1,l} Z_l$ and $[Z_j, \overline{Z_k}] = \sum_{l=1}^{m+q} c_{j,k}^{2,l} Z_l + \sum_{l=1}^{m+q} c_{j,k}^{3,l} \overline{Z_l}$, where $c_{j,k}^{a,l} \in \mathscr{C}^s_{X,L}(V)$, $1 \le a \le 3$, $1 \le j,k,l \le m+q$. Furthermore, the map $x \mapsto \dim \mathscr{L}_x$, $V \to \mathbb{N}$ is constant in x.*

Theorem 7.2 *(The global theorem). For $s \in (1,\infty] \cup \{\omega\}$ the following two conditions are equivalent:*

(i) *There exists a $\mathscr{C}^{s+2}$ E-manifold structure on M, compatible with its C^2 structure, such that $X_1, \ldots, X_q, L_1, \ldots, L_m$ are $\mathscr{C}^{s+1}$ vector fields on M and $\mathscr{L}$ (as defined in (7.1)) is the associated elliptic structure (see Definition 6.20).*

(ii) *For each $x_0 \in M$, any of the three equivalent conditions from Theorem 7.1 hold for this choice of x_0.*

Furthermore, under these conditions, the $\mathscr{C}^{s+2}$ E-manifold structure in (i) is unique, in the sense that if M has another $\mathscr{C}^{s+2}$ E-manifold structure satisfying the conclusions of (i), then the identity map $M \to M$ is a $\mathscr{C}^{s+2}$ E-diffeomorphism between these two E-manifold structures. Finally, when $s \in (1,\infty]$, there is a third equivalent condition:

(iii) *Let $Z_1, \ldots, Z_{m+q}$ denote the list $X_1, \ldots, X_q, L_1, \ldots, L_m$. Then, $[Z_j, Z_k] = \sum_{l=1}^{m+q} c_{j,k}^{1,l} Z_l$ and $[Z_j, \overline{Z_k}] = \sum_{l=1}^{m+q} c_{j,k}^{2,l} Z_l + \sum_{l=1}^{m+q} c_{j,k}^{3,l} \overline{Z_l}$, where $\forall x \in M$, there exists an open neighborhood $V \subseteq M$ of x such that $c_{j,k}^{a,l}\big|_V \in \mathscr{C}^s_{X,L}(V)$, $1 \le a \le 3$, $1 \le j,k,l \le m+q$. Furthermore, the map $x \mapsto \dim \mathscr{L}_x$, $M \to \mathbb{N}$ is constant.*

Remark 7.3. For a discussion of results like Theorems 7.1 and 7.2 using the easier to understand Hölder spaces, see Section 14.

7.2. Sub-E geometry

Let M be a connected C^∞ E-manifold of dimension (r,n) and let $\mathscr{L}$ be the associated elliptic structure. For $x\in M$, set $\mathscr{X}_x := \mathscr{L}_x \cap \overline{\mathscr{L}_x}$, so that $r=\dim \mathscr{X}_x$ and $n+r=\dim\mathscr{L}_x$, $\forall x\in M$. Fix a strictly positive C^∞ density ν on M.[17] Suppose $X_1,\ldots,X_q$ are C^∞ real vector fields on M and $L_1,\ldots,L_m$ are C^∞ complex vector fields on M such that $\mathscr{X}_x=\operatorname{span}_{\mathbb{C}}\{X_1(x),\ldots,X_q(x)\}$ and $\mathscr{L}_x=\operatorname{span}_{\mathbb{C}}\{X_1(x),\ldots,X_q(x),L_1(x),\ldots,L_m(x)\}$, $\forall x\in M$.

To each X_k, we assign a formal degree $\beta_k\in[1,\infty)$, and to each L_j we assign a formal degree $\beta_{j+q}\in[1,\infty)$. We let $Z_1,\ldots,Z_{m+q}$ denote the list $X_1,\ldots,X_q,L_1,\ldots,L_m$, so that Z_j has assigned formal degree β_j.

We assume:

$$[Z_j,Z_k]=\sum_{\beta_l\leq\beta_j+\beta_k} c_{j,k}^{1,l} Z_l,\quad [Z_j,\overline{Z_k}]=\sum_{\beta_l\leq\beta_j+\beta_k} c_{j,k}^{2,l}Z_l+\sum_{\beta_l\leq\beta_j+\beta_k} c_{j,k}^{3,l}\overline{Z_l},\quad c_{j,k}^{a,l}\in C^\infty(M). \tag{7.2}$$

For $\delta\in(0,1]$ write $\delta^\beta X$ for the list $\delta^{\beta_1}X_1,\ldots,\delta^{\beta_q}X_q$ and write $\delta^\beta L=\delta^{\beta_{q+1}}L_1,\ldots,$ $\delta^{\beta_{q+m}}L_m$. Using the notation from Section 4 it makes sense to write, for $K\in\mathcal{I}(r_1,q)$, $J\in\mathcal{I}(n_1,m)$, $\left(\bigwedge(\delta^\beta X)_K\right)\bigwedge\left(\bigwedge(\delta^\beta L)_J\right)$. We assume: $\forall\mathcal{K}\Subset M$ compact, $\exists\zeta\in(0,1]$ such that $\forall x\in\mathcal{K}$, $\delta\in(0,1]$, $\exists K_0(x,\delta)\in\mathcal{I}(r,q)$, $J_0(x,\delta)\in\mathcal{I}(n,m)$ such that

$$\sup_{\substack{x\in\mathcal{K}\\ \delta\in(0,1]}}\ \max_{\substack{K\in\mathcal{I}(r_1,q),J\in\mathcal{I}(n_1,m)\\ r_1+n_1=r+n}}\left|\frac{\left(\bigwedge(\delta^\beta X(x))_K\right)\bigwedge\left((\delta^\beta L(x))_J\right)}{\left(\bigwedge(\delta^\beta X(x))_{K_0(x,\delta)}\right)\bigwedge\left((\delta^\beta L(x))_{J_0(x,\delta)}\right)}\right|\leq\zeta^{-1}. \tag{7.3}$$

Remark 7.4. The existence of $K_0(x,\delta)$, $J_0(x,\delta)$, and ζ as in (7.3) does not follow from the other hypotheses. However, it is immediate to see that if $r=0$ or $n=0$, one may always find $J_0(x,\delta)$ and $K_0(x,\delta)$ so that (7.3) holds with $\zeta=1$. This accounts for the two most important special cases: the ones in Sections 3.2.1 and 3.2.2.

Under these hypotheses, we will study two metrics on M (and show these two metrics are equivalent on compact sets). The first metric is a standard sub-Riemannian metric and we will define it in two different ways, denoted by ρ_S and ρ_F. We will show that $\rho_S=\rho_F$. Both of the definitions ρ_S and ρ_F are defined extrinsically: they are defined by using the underlying manifold structure on M using maps which are not necessarily E-maps. The second metric, ρ_H, has a definition which is similar to that of ρ_F, but it is defined intrinsically on M: it is defined entirely within the category of E-manifolds.

[17] The results that follow are local and do not depend on the choice of density.

For $x \in M$, $\delta > 0$ set $B_S(x,\delta) := B_{\delta^\beta X, \delta^\beta L}(x,1)$ (where the later ball is defined in (2.4)) and set $\rho_S(x,y) := \inf\{\delta > 0 : y \in B_S(x,\delta)\}$.

Let $(W_1, d_1), \ldots, (W_{2m+q}, d_{2m+q})$ denote the list of vector fields with formal degrees

$$(X_1,\beta_1),\ldots,(X_q,\beta_q),(2\mathrm{Re}(L_1),\beta_{q+1}),\ldots,(2\mathrm{Re}(L_m),\beta_{q+m}),(2\mathrm{Im}(L_1),\beta_{q+1}),\ldots,$$
$$(2\mathrm{Im}(L_m),\beta_{q+m}).$$

We say $\rho_F(x,y) < \delta$ if and only if $\exists K \in \mathbb{N}$, C^∞ functions $f_1,\ldots,f_K : B_{\mathbb{R}}(1/2) \to M$, and $\delta_1,\ldots,\delta_K > 0$ with $\sum_{j=1}^K \delta_j \leq \delta$, such that:

- $f_j'(t) = \sum_{l=1}^{2m+q} s_j^l(t)\delta_j^{d_l} W_l(f_j(t))$, with $\|\sum_l |s_j^l|^2\|_{L^\infty(B_{\mathbb{R}}(1/2))} < 1$.
- $f_j(B_{\mathbb{R}}(1/2)) \cap f_{j+1}(B_{\mathbb{R}}(1/2)) \neq \emptyset$, $1 \leq j \leq K-1$.
- $x \in f_1(B_{\mathbb{R}}(1/2))$, $y \in f_K(B_{\mathbb{R}}(1/2))$.

Set $B_F(x,\delta) := \{y \in M : \rho_F(x,y) < \delta\}$.

Finally, we define ρ_H. We say $\rho_H(x,y) < \delta$ if and only if $\exists K \in \mathbb{N}$, C^∞ E-maps $f_1,\ldots,f_K : B_{\mathbb{R}\times\mathbb{C}}(1/2) \to M$, and $\delta_1,\ldots,\delta_K$ with $\sum_{j=1}^K \delta_j \leq \delta$, such that:

(1) Because f_j is an E-map, we may write

$$df_j(t,z)\frac{\partial}{\partial t} = \sum_{k=1}^q s_{j,1}^k(t,z)\delta_j^{\beta_k} X_k(f_j(t,z)) + \sum_{l=1}^m s_{j,1}^{l+q}(t,z)\delta_j^{\beta_{l+q}} \frac{2}{\sqrt{2}} L_l(f_j(t,z)),$$
$$df_j(t,z)\frac{2}{\sqrt{2}}\frac{\partial}{\partial \bar{z}} = \sum_{k=1}^q s_{j,2}^k(t,z)\delta_j^{\beta_k} X_k(f_j(t,z)) + \sum_{l=1}^m s_{j,2}^{l+q}(t,z)\delta_j^{\beta_{l+q}} \frac{2}{\sqrt{2}} L_l(f_j(t,z)).$$

The choice of s_j's is not necessarily unique. Let $S_j(t,z)$ denote the $(q+2m) \times 3$ matrix such that the (l,a) component of $S_j(t,z)$ is given by

$$\begin{cases} s_{j,a}^l(t,z), & 0 \leq l \leq m+q, a = 1,2 \\ 0, & m+q+1 \leq l \leq 2m+q, a = 1,2 \\ \overline{s_{j,a}^l(t,z)}, & 0 \leq l \leq q \text{ or } m+q+1 \leq l \leq 2m+q, a = 3 \\ 0, & q+1 \leq l \leq m+q, a = 3. \end{cases}$$

In particular, $S_j(t,z)$ is a matrix representation of $df_j(t,z)$ thought of as taking the basis $\frac{\partial}{\partial t}, \frac{2}{\sqrt{2}}\frac{\partial}{\partial \bar{z}}, \frac{2}{\sqrt{2}}\frac{\partial}{\partial z}$ to the spanning set

$$\delta_j^{\beta_1} X_1, \ldots, \delta_j^{\beta_q} X_q, \delta_j^{\beta_{q+1}} \frac{2}{\sqrt{2}} L_1, \ldots, \delta_j^{\beta_{q+m}} \frac{2}{\sqrt{2}} L_m, \delta_j^{\beta_{q+1}} \frac{2}{\sqrt{2}} \overline{L_1}, \ldots, \delta_j^{\beta_{q+m}} \frac{2}{\sqrt{2}} \overline{L_m}.$$

We assume

$$\|S_j\|_{L^\infty(B_{\mathbb{R}\times\mathbb{C}}(1/2);\mathbb{M}^{(q+2m)\times 3})} < 1.$$

The choice of S_j may not be unique,[18] and we only ask for the existence of such an S_j.

(2) $f_j(B_{\mathbb{R}\times\mathbb{C}}(1/2)) \cap f_{j+1}(B_{\mathbb{R}\times\mathbb{C}}(1/2)) \neq \emptyset$, $1 \leq j \leq K-1$.

(3) $x \in f_1(B_{\mathbb{R}\times\mathbb{C}}(1/2))$, $y \in f_K(B_{\mathbb{R}\times\mathbb{C}}(1/2))$.

Set $B_H(x,\delta) := \{y \in M : \rho_H(x,y) < \delta\}$.

Remark 7.5. A consequence of (1) is the following. We identify $\mathbb{R}\times\mathbb{C}$ with $\mathbb{R}^3$ in the usual way. Let $\widehat{S}_j(t,x_1,x_2)$ be a $(2m+q)\times 3$ matrix representation of $df_j(t,x_1,x_2)$ thought as taking the basis $\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ to the spanning set $\delta_j^{d_1} W_1(f_j(t,x_1,x_2)), \ldots, \delta_j^{d_{2m+q}} W_{2m+q}(f_j(t,x_1,x_2))$. Then if (1) holds we may choose $\widehat{S}_j$ so that

$$\|\widehat{S}_j\|_{L^\infty(B_{\mathbb{R}^3}(1/2);\mathbb{M}^{(q+2m)\times 3})} < 1. \tag{7.4}$$

Define, for $x \in M$, $\delta > 0$,

$$\Lambda(x,\delta) := \max_{j_1,\ldots,j_{2n+r}\in\{1,\ldots,2m+q\}} \nu(x)(\delta^{d_{j_1}} W_{j_1}(x), \ldots, \delta^{d_{j_{2n+r}}} W_{j_{2n+r}}(x)).$$

Theorem 7.6.

(a) $\forall x,y \in M$, $\rho_S(x,y) = \rho_F(x,y) \leq \rho_H(x,y)$.

Fix a compact set $\mathcal{K} \Subset M$. *We write* $A \lesssim B$ *for* $A \leq CB$, *where* C *is a positive constant which can be chosen independent of* $x,y \in \mathcal{K}$, $\delta > 0$. *We write* $A \approx B$ *for* $A \lesssim B$ *and* $B \lesssim A$. *There exists* $\delta_1 \approx 1$ *such that:*

(b) $\rho_H(x,y) \lesssim \rho_S(x,y)$, *and therefore* ρ_S *and* ρ_H *are equivalent on compact sets.*

(c) $\nu(B_S(x,\delta)) \approx \nu(B_H(x,\delta)) \approx \Lambda(x,\delta)$, $x \in \mathcal{K}$, $\delta \in (0,\delta_1]$.

(d) $\nu(B_S(x,2\delta)) \lesssim \nu(B_S(x,\delta))$, $\forall x \in \mathcal{K}$, $\delta \in (0,\delta_1/2]$; *the same holds with* B_S *replaced by* B_H.[19]

For each $x \in \mathcal{K}$, $\delta \in (0,1]$, *there exists a* C^∞ *E-map* $\Phi_{x,\delta} : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to B_S(x,\delta)$ *such that*

(e) $\Phi_{x,\delta}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1)) \subseteq M$ *is open and* $\Phi_{x,\delta} : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to \Phi(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))$ *is a* C^∞ *diffeomorphism.*

(f) $\Phi_{x,\delta}^*\nu = h_{x,\delta}\sigma_{\mathrm{Leb}}$, *where* σ_{Leb} *denotes the usual Lebesgue density on* $\mathbb{R}^r\times\mathbb{C}^n$, $h_{x,\delta} \in C^\infty(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))$, *and* $\|h_{x,\delta}\|_{C^m(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim \Lambda(x,\delta)$, $\forall m$ *(where the implicit*

[18] The choice of S_j is not unique if $m+q > n+r$.

[19] This is the key estimate that shows that the balls $B_S(x,\delta)$, when paired with the density ν, locally give a space of homogeneous type.

constant may depend on m). Also, $h_{x,\delta}(t,z) \approx \Lambda(x,\delta)$, $\forall (t,z) \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$, *where the implicit constant does not depend on* $x \in \mathcal{K}$, $\delta \in (0,1]$, *or* $(t,z) \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$.

Let $\widehat{Z}_j^{x,\delta} := \Phi_{x,\delta}^* \delta^{\beta_j} Z_j$, *so that* $\widehat{Z}_j^{x,\delta}$ *is a* C^∞ *vector field on* $B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$.

(g) $\widehat{Z}_j^{x,\delta}(t,z) \in \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n} \right\}$, $\forall (t,z) \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$.

In light of (g), we may think of $\widehat{Z}_j^{x,\delta}$ *as a map* $B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to \mathbb{C}^{r+n}$, *and we henceforth do this.*

(h) $\widehat{Z}_1^{x,\delta}(t,z), \ldots, \widehat{Z}_{m+q}^{x,\delta}(t,z)$ *span* $\mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n} \right\}$ *uniformly in* t, z, x, δ *in the sense that*

$$\max_{j_1,\ldots,j_{n+r}\in\{1,\ldots,m+q\}} \inf_{(t,z)\in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)} \left| \det \left(\widehat{Z}_{j_1}^{x,\delta}(t,z) | \cdots | \widehat{Z}_{j_{n+r}}^{x,\delta}(t,z) \right) \right| \approx 1,$$
$$\forall x \in \mathcal{K}, \delta \in (0,1].$$

In fact, for $x \in \mathcal{K}$, $\delta \in (0,1]$,

$$\max_{\substack{k_1,\ldots,k_r\in\{1,\ldots,q\}\\ j_1,\ldots,j_n\in\{1,\ldots,m\}}} \inf_{(t,z)\in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)} \left| \det\left(\Phi_{x,\delta}^*\delta^{\beta_{k_1}} X_{k_1}(t,z) | \cdots | \Phi_{x,\delta}^*\delta^{\beta_{k_r}} X_{k_r}(t,z) | \Phi_{x,\delta}^*\delta^{\beta_{j_1+q}} L_{j_1}(t,z) | \cdots | \Phi_{x,\delta}^*\delta^{\beta_{j_n+q}} L_{j_n}(t,z) \right) \right| \approx 1.$$

(i) $\|\widehat{Z}_j^{x,\delta}\|_{C^k(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{C}^{r+n})} \lesssim 1$, $\forall x \in \mathcal{K}$, $\delta \in (0,1]$ *(where the implicit constant may depend on* $k \in \mathbb{N}$*).*

(j) $\exists R \approx 1$ *such that* $\Phi_{x,\delta}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1)) \subseteq B_H(x, R\delta)$, $x \in \mathcal{K}$, $\delta \in (0,1]$.

(k) $\exists \epsilon \approx 1$ *such that* $B_S(x,\epsilon\delta) \subseteq \Phi_{x,\delta}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1)) \subseteq B_S(x,\delta)$, $x \in \mathcal{K}$, $\delta \in (0,1]$.

Remark 7.7. In Theorem 7.6 we stated a result for C^∞ vector fields. A similar result, with a similar proof, can be stated for real analytic vector fields, where one can ensure the map $\Phi_{x,\delta}$ is real analytic and the vector fields $\widehat{Z}_j^{x,\delta}$ are real analytic in a quantitative way. This proceeds by using the case $s_0 = \omega$ in Theorem 4.5 (instead of $s_0 \in (1,\infty)$). In the setting of real vector fields, this was done in [32]. We leave the details to the interested reader.

Remark 7.8. In this section, we described geometries where the vector fields at scale δ where given by $\delta^{\beta_1} X_1, \ldots, \delta^{\beta_q} X_q, \delta^{\beta_{q+1}} L_1, \ldots, \delta^{\beta_{q+m}} L_m$, for some fixed vector fields $X_1, \ldots, X_q, L_1, \ldots, L_m$. It is straightforward to generalize Theorem 7.6 to work in a setting where the vector fields have a more complicated dependance on δ. In this setting, one would take, for each $\delta \in (0,1]$, a collection of vector fields $X_1^\delta, \ldots, X_q^\delta, L_1^\delta, \ldots, L_m^\delta$

and place appropriate axioms on these vector fields so that the proof of Theorem 7.6 works uniformly for $\delta \in (0,1]$. This approach was described in the real setting in [27,32]. An example in the complex setting is described in Section 8. Using the same ideas, the results in this paper generalize the result in the multi-parameter stetting of [28]. Here, we fix some $\mu \in \mathbb{N}$, $\mu \geq 1$ and for each $\delta \in (0,1]^\mu$ we are given vector fields $X_1^\delta, \ldots, X_q^\delta, L_1^\delta, \ldots, L_m^\delta$ and proceed in the same way. We leave further details to the interested reader.

Remark 7.9. The assumption that the vector fields are C^∞ is not essential. In fact, because Theorem 4.5 is stated for C^1 vector fields, one need only assume the given vector fields are C^1. Then, as in Remark 7.8, one assumes that the hypotheses of Theorem 4.5 hold uniformly in the relevant parameters. See [27, Section 7.3] for a description of this in the real setting.

8. An example from several complex variables

In Sections 1.2.2, 3.2, and 7.2 we described how to use the coordinate system Φ from Theorem 4.5 as a generalized scaling map. In these settings, we applied Theorem 4.5 to a family of vector fields which depended on $\delta \in (0,1]$. For example, in Section 3.2.2, the vector fields were $\delta^{\beta_1} L_1, \ldots, \delta^{\beta_m} L_m$, where $L_1, \ldots, L_m$ were sections of $T^{0,1}M$ satisfying certain properties (and M was a complex manifold). In these settings, the vector fields depend on δ in a very simple way; and we presented results in these settings for simplicity. However, Theorem 4.5 allows one to consider vector fields which depend on δ (and on the base point) in much more complicated ways. This can be important in applications, and to describe these ideas we present an important setting which arises in several complex variables: extremal bases.

Extremal bases were first used by McNeal [16] to study Bergman kernels and invariant metrics associated to convex domains of finite type; see also [12]. More generally, extremal bases can be used to study lineally convex domains [9] (see also [4, Section 7.1]). They can also be used to study Bergman and Szegö kernels and invariant metrics on pseudoconvex domains of finite type with comparable eigenvalues [14,5,6,7]. Finally, they have been used to study pseudoconvex domains of finite type with locally diagonalizable Levi forms [3,2,4]. All of these settings have been generalized to one abstract setting by Charpentier and Dupain [4]. The presentation below is closely related to the ideas of [4], though expressed in a different way.

As can be seen from the above mentioned works, extremal bases are closely related to a notion of distance in many complex domains; and scaling techniques are central in using extremal bases to study objects like Bergman and Szegö kernels (many of the above papers use some kind of scaling). See [17] for a particularly straightforward explanation of the form scaling takes in some of these examples. In this section, we show how to use Theorem 4.5 to understand this scaling in a more abstract way. The idea is to rephrase the notion of an extremal basis in a way which is *quantitatively* invariant under arbitrary biholomorphisms. We hope that this will give the reader some idea of how to apply the results of this paper to questions in several complex variables, perhaps even beyond the setting of extremal bases.

Following the philosophy of this paper, we describe the scaling associated to extremal bases in three steps:

- Extremal bases at the unit scale: because we wish to scale a small scale into the unit scale, first we must introduce what we mean by the unit scale. This is the setting where classical techniques from several complex variables can be used to prove estimates.
- Extremal bases in a biholomorphic invariant setting: using Theorem 4.5, we rephrase the unit scale from the previous point in a way which is quantitatively invariant under arbitrary biholomorphisms. Because of this, we completely remove the notion of “scale,” because that notion depends on a choice of coordinate system.
- Extremal bases at small scales: here we introduce the notion of extremal bases at small scales, by seeing it as a special case of the biholomorphically invariant version of the previous point. Because of this, it will immediately follow that the setting of small scales is biholomorphically equivalent to the unit scale.

After introducing these three steps, we describe the similar setting of CR manifolds.

Though it will not play a role in our discussion, the setting to keep in mind is the following. $\mathfrak{M}$ is a complex manifold, and $\Omega = \{\zeta \in \mathfrak{M} : \rho(\zeta) < 0\}$ is relatively compact domain, where $\rho \in C^\infty(\mathfrak{M}; \mathbb{R})$ is a defining function of Ω such that $d\rho(\zeta) \neq 0$, $\forall \zeta \in \partial\Omega = \{\rho = 0\}$. All of the above mentioned papers concern pseudoconvex domains near points of finite type.

While the scaling maps of Nagel, Stein, and Wainger [19] have long been used in such problems (see, e.g., [14]), we will see that the results of this paper allow us to have similar scaling maps which are *holomorphic*, as opposed to the smooth maps given in [19]; thus they do not destroy the complex nature of the problem.

Remark 8.1. Other than a new way of viewing extremal bases, the perspective here may not bring much new to this well-studied concept. However, we hope the general outline may be useful for other problems in several complex variables. Indeed, as we will explain, the idea is to take a known result at the unit scale, rewrite it in a way which is quantitatively invariant under biholomorphisms (using Theorem 4.5). This then automatically gives a quantitative result at small scales, since the notion of scale is not invariant under biholomorphisms.

8.1. Extremal bases at the unit scale

Fix $n \in \mathbb{N}$ and let $\rho \in C^\infty(B_{\mathbb{C}^n}(1); \mathbb{R})$ satisfy $\rho(0) = 0$ and $d\rho(\zeta) \neq 0$, $\forall \zeta \in B_{\mathbb{C}^n}(1)$. Let L_n be a smooth section of $T^{0,1}B_{\mathbb{C}^n}(1)$ (i.e., L_n is a complex vector field spanned by

$\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}$) be such that $L_n\rho(\zeta) \neq 0$, $\forall \zeta \in B_{\mathbb{C}^n}(1)$. For example, one often takes

$$L_n = \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial \overline{z}_j},$$

so that $L_n\rho = \sum \left| \frac{\partial \rho}{\partial z_j} \right|^2$.

Let $L_1, \ldots, L_{n-1}$ be smooth sections of $T^{0,1}B_{\mathbb{C}^n}(1)$ such that $L_j\rho = 0$ on $B_{\mathbb{C}^n}(1)$, and such that $L_1(\zeta), \ldots, L_n(\zeta)$ span $T^{0,1}_\zeta B_{\mathbb{C}^n}$, $\forall \zeta \in B_{\mathbb{C}^n}(1)$. Given $\theta = (\theta_1, \ldots, \theta_{n-1}) \in \mathbb{C}^{n-1}$ with $|\theta| = 1$, set $L_\theta = \sum_{j=1}^{n-1} \theta_j L_j$. Set $\mathcal{Z}_1^\theta = \{[L_\theta, \overline{L_\theta}]\}$, and recursively set $\mathcal{Z}_j^\theta = \{[L_\theta, Z], [\overline{L_\theta}, Z] : Z \in \mathcal{Z}_{j-1}^\theta\}$ for $j \geq 2$.

Definition 8.2. We say $L_1, \ldots, L_n, \rho$ is an extremal system if there exists $K \in \mathbb{N}$ such that

$$\{L_1, \ldots, L_{n-1}, \overline{L_1}, \ldots, \overline{L_{n-1}}, L_n\} \bigcup \left(\bigcup_{j=1}^{K} \mathcal{Z}_j^\theta \right)$$

spans $\mathbb{C}T_\zeta B_{\mathbb{C}^n}(1)$, $\forall \zeta \in B_{\mathbb{C}^n}(1)$, $|\theta| = 1$.

Along with an extremal system, strictly plurisubharmonic functions are often used. Thus, we assume we are given a function $H \in C^3(B_{\mathbb{C}^n}(1); \mathbb{R})$ such that $\partial\overline{\partial}H$ is strictly positive definite on $B_{\mathbb{C}^n}(1)$.

Remark 8.3. Given an extremal system and plurisubharmonic function, as above, there are many estimates one can prove using now standard techniques (usually, this occurs under the additional qualitative assumption that the domain is weakly pseudoconvex, see the above mentioned works for details). These estimates often depend on the following quantities (or something similar):

(i) Upper bounds for n and K.
(ii) Upper bounds for $\|\rho\|_{C^N(B_{\mathbb{C}^n}(1))}$ and $\max_{1 \leq j \leq n} \|L_j\|_{C^N(B_{\mathbb{C}^n}(1); \mathbb{C}^n)}$, where N can be chosen to depend only on upper bounds for K, n, and the particular estimate being shown.
(iii) A lower bound, > 0, for $\inf_{\zeta \in B_{\mathbb{C}^n}(1)} |L_n\rho(\zeta)|$.
(iv) A lower bound, > 0, for $\inf_{\zeta \in B_{\mathbb{C}^n}(1)} |\det(L_1(\zeta)| \cdots |L_n(\zeta))|$, where this matrix has columns $L_1, \ldots, L_n$, written in terms of $\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}$.
(v) A lower bound, > 0, for

$$\inf_{\substack{\zeta \in B_{\mathbb{C}^n}(1) \\ |\theta|=1}} \max_{Z \in \bigcup_{j=1}^{K} \mathcal{Z}_j^\theta} \left| \det \left(L_1(\zeta) | \cdots | L_n(\zeta) | \overline{L_1}(\zeta) | \cdots | \overline{L_{n-1}}(\zeta) | Z(\zeta) \right) \right|,$$

where in the above matrix, the vector fields are written in terms of $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}$.

(vi) An upper bound for $\|H\|_{C^3(B_{\mathbb{C}^n}(1))}$.

(vii) A lower bound, > 0, for the quadratic form $\partial\overline{\partial}H$ on $B_{\mathbb{C}^n}(1)$. Equivalently, using (ii) and (iv), for $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n$ with $|\omega| = 1$, set $L_\omega = \sum \omega_j L_j$. The estimates may depend on a lower bound, > 0, for:

$$\inf_{\substack{\zeta \in B_{\mathbb{C}^n}(1) \\ |\omega|=1}} \left\langle \partial\overline{\partial}H(\zeta); L_\omega(\zeta), \overline{L_\omega}(\zeta) \right\rangle. \tag{8.1}$$

Thus, if one has an infinite collection of extremal systems and plurisubharmonic functions, such that the above quantities can be chosen uniformly over this infinite collection, then one can prove the above mentioned estimates, uniformly over the infinite collection. See [17] for some easy to understand examples of such estimates. Note that all of the above quantities except for (i) depend on the choice of coordinate system: if one applies a biholomorphism to this setting, it destroys all of the above constants. The next section fixes this problem.

Remark 8.4. In (vii), we assumed that the quadratic form $\partial\overline{\partial}H$ was bounded away from 0. In the famous work of Catlin [1], subelliptic estimates are shown using plurisubharmonic functions where this bound can be chosen very large. While the form being positive definite does not depend on the choice of holomorphic coordinate system, the lower bound for the form does depend on the choice of coordinate system. In Section 8.3, we will assume the existence of a plurisubharmonic function adapted to each scale; those adapted to a small scale will have a large lower bound when viewed in a fixed coordinate system independent of the scale (see Remark 8.8).

8.2. Extremal bases invariant under biholomorphisms

In this section, we present extremal bases again. Qualitatively, this is exactly the same as what is written in Section 8.1; the difference here is that our quantitative assumptions will be written in a way which is invariant under biholomorphisms (as opposed to the quantitative assumptions in Remark 8.3 which depended on the choice of coordinate system).

Let $\mathfrak{M}$ be a complex manifold of complex dimension n. Fix a point $\zeta_0 \in \mathfrak{M}$ and $\rho \in C^\infty(\mathfrak{M}; \mathbb{R})$ with $\rho(\zeta_0) = 0$. Let $L_1, \ldots, L_n$ be smooth sections of $T^{0,1}\mathfrak{M}$ and fix $\xi > 0$. We take the following assumptions and definitions:

(i) $\forall \zeta \in B_L(\zeta_0, \xi)$, $\mathrm{span}_{\mathbb{C}}\{L_1(\zeta), \ldots, L_n(\zeta)\} = T^{0,1}\mathfrak{M}$.

(ii) $c_1 := \inf_{\zeta \in B_L(\zeta_0,\xi)} |L_n\rho(\zeta)| > 0$.

(iii) For $1 \leq j \leq n-1$, $L_j\rho(\zeta) = 0$ for $\zeta \in B_L(\zeta_0, \xi)$.

(iv) Due to (i), we may write $[L_j, L_k] = \sum_{l=1}^n c_{j,k}^{1,l} L_l$ and $[L_j, \overline{L_k}] = \sum_{l=1}^n c_{j,k}^{2,l} L_l + \sum_{j=1}^n c_{j,k}^{3,l} \overline{L_l}$. For each $N \in \mathbb{N}$, take C_N (which we assume to be finite[20]) so that

$$\|c_{j,k}^{a,l}\|_{C_L^N(B_L(\zeta_0,\xi))}, \sum_{m=1}^N \|L_n^m \rho\|_{C(B_L(\zeta_0,\xi))} \leq C_N, \quad 1 \leq j,k,l \leq n, 1 \leq a \leq 3.$$

(v) For each $\theta \in \mathbb{C}^{n-1}$ with $|\theta| = 1$, define $\mathcal{Z}_j^\theta$ in terms of $L_1, \ldots, L_{n-1}$ as in Section 8.1. We assume that there exists $K \in \mathbb{N}$ such that $\forall \zeta \in B_L(\zeta_0, \xi)$,

$$\overline{L_n}(\zeta) = \sum_{j=1}^n a_j^{1,\theta}(\zeta) L_j(\zeta) + \sum_{j=1}^{n-1} a_j^{2,\theta}(\zeta)\overline{L_j}(\zeta) + \sum_{Z \in \bigcup_{j=1}^K \mathcal{Z}_j^\theta} b_Z^\theta(\zeta) Z(\zeta),$$

with

$$D := \sup_{\substack{\zeta \in B_L(\zeta_0,\xi) \\ |\theta|=0, Z \in \bigcup_{j=1}^K \mathcal{Z}_j^\theta}} |a_j^{1,\theta}(\zeta)| + |a_j^{2,\theta}(\zeta)| + |b_Z^\theta(\zeta)| < \infty.$$

(vi) Let $\eta, \delta_0 > 0$ be as in Theorem 4.5. In applications, usually $L_1, \ldots, L_n$ are often given in a coordinate system in which their C^1 norms are very small, and then η and δ_0 can be bounded below in terms of the C^1 norms in this coordinate system (see, e.g., Lemma 4.13 and the discussion following it).

(vii) We suppose we are given a function $H \in C_L^3(B_L(\zeta_0, \xi); \mathbb{R})$. For $\omega \in \mathbb{C}^n$ with $|\omega| = 1$, define $L_\omega = \sum_{j=1}^n \omega_j L_j$. We assume,

$$c_2 := \inf_{\substack{\zeta \in B_L(\zeta_0,\xi) \\ |\omega|=1}} \left\langle \partial\overline{\partial} H(\zeta); L_\omega(\zeta), \overline{L_\omega}(\zeta) \right\rangle > 0.$$

Proposition 8.5. *In the above setting, there is a biholomorphism $\Phi : B_{\mathbb{C}^n}(1) \to \Phi(B_{\mathbb{C}^n}(1)) \subseteq B_L(\zeta_0, \xi)$, with $\Phi(0) = \zeta_0$, such that $\Phi^* L_1, \ldots, \Phi^* L_n, \Phi^* \rho$ is an extremal system with plurisubharmonic function $\Phi^* H$. Moreover, all of the estimates described in Remark 8.3 can be bounded in terms of upper bounds for n, K, c_1^{-1}, C_N (where N can be chosen to depend only on n, K, and the particular estimate being shown), D, η^{-1}, δ_0^{-1}, ξ^{-1}, c_2^{-1}, and $\|H\|_{C_L^3(B_L(\zeta_0,\xi))}$.*

Proof. This follows immediately from Theorem 4.5 (and that theorem includes more properties of Φ), by applying the theorem to $L_1, \ldots, L_n$ (we are talking $m = n$ and $q = 0$)–that Φ is holomorphic is the essence of Remark 4.6.

[20] C_N can always be chosen to be finite, so long as $B_L(\zeta_0, \xi)$ is relatively compact in $\mathfrak{M}$, which can be guaranteed by taking ξ small enough; though it is the particular value of C_N which is important, not just that it is finite.

There are two parts which do not follow directly from the statement of Theorem 4.5. To estimate $\|\Phi^*\rho\|_{C^N(B_{\mathbb{C}^N}(1))}$ we would like to have estimates on $\|\rho\|_{C^N_L(B_L(\zeta_0,\xi))}$. To obtain this, note that it follows easily from (iii) and (iv) that $\|\rho\|_{C^N_L(B_L(\zeta_0,\xi))} \approx \sum_{m=1}^N \|L_n^m \rho\|_{C(B_L(\zeta_0,\xi))} \leq C_N$.

The other part that does not follow directly from the statement of Theorem 4.5 is a lower bound for (8.1). The key here is that this quantity is invariant under biholomorphisms. Indeed, since Φ is a biholomorphism,

$$\left\langle \partial\overline{\partial}(\Phi^* H)(z); (\Phi^* L_\omega)(z), (\Phi^* \overline{L_\omega})(z) \right\rangle = \left\langle \partial\overline{\partial} H(\Phi(z)); L_\omega(\Phi(z)), \overline{L_\omega}(\Phi(z)) \right\rangle .$$

Thus, (8.1) is bounded below by c_2. □

Definition 8.6. Let $\mathcal{I}$ be an index set, and suppose for each $\iota \in \mathcal{I}$, we are given $\mathfrak{M}^\iota$, ζ_0^ι, $L_1^\iota, \ldots, L_n^\iota$, ρ^ι, and H^ι as above, satisfying the above estimates uniformly (i.e., there are upper bounds for the quantities discussed in Proposition 8.5 which can be chosen independent of ι). We say that the collection $L_1^\iota, \ldots, L_n^\iota$, ζ_0^ι, ρ^ι, H^ι is an extremal system with adapted plurisubharmonic function, uniformly in ι.

Remark 8.7. Combining Definition 8.6 and Proposition 8.5, we see that if $L_1^\iota, \ldots, L_n^\iota$, ζ_0^ι, ρ^ι, H^ι is an extremal system with adapted plurisubharmonic function, uniformly in ι, then for each $\iota \in \mathcal{I}$, there exists a biholomorphism $\Phi_\iota : B_{\mathbb{C}^n}(1) \to \Phi_\iota(B_{\mathbb{C}^n}(1)) \subseteq \mathfrak{M}^\iota$, with $\Phi_\iota(0) = \zeta_0^\iota$ such that $\Phi_\iota^* L_1^\iota, \ldots, \Phi_\iota^* L_n^\iota$, $\Phi_\iota^* \rho^\iota$, $\Phi_\iota^* H^\iota$ is an extremal system with plurisubharmonic function on $B_{\mathbb{C}^n}(1)$, satisfying all of the estimates outlined in Remark 8.3, uniformly in ι.

8.3. Extremal bases at small scales

Let $\mathfrak{M}$ be a complex manifold of complex dimension n and fix $\zeta_0 \in \mathfrak{M}$, and let $\rho \in C^\infty(\mathfrak{M}; \mathbb{R})$. Let $\mathcal{K} \subseteq \{\zeta \in \mathfrak{M} : \rho(\zeta) = 0\}$, and suppose L_n is a smooth section of $T^{0,1}\mathfrak{M}$ such that $\inf_{\zeta \in \mathcal{K}} |L_n \rho(\zeta)| > 0$. To work at scale $\delta \in (0, 1]$, we wish to replace ρ with $\delta^{-2}\rho$ and L_n with $\delta^2 L_n$.

To do this, we assume that for each $\delta \in (0, 1]$, $\zeta_0 \in \mathcal{K}$, we are given smooth sections of $T^{0,1}\mathfrak{M}$ defined near ζ_0, $L_1^{\delta,\zeta}, \ldots, L_{n-1}^{\delta,\zeta}$, and a real valued smooth function $H^{\zeta_0,\delta}$ defined near ζ_0, such that

$$L_1^{\delta,\zeta_0}, \ldots, L_{n-1}^{\delta,\zeta_0}, \delta^2 L_n, \delta^{-2}\rho, H^{\zeta_0,\delta}, \zeta_0$$

is an extremal system with adapted plurisubharmonic function, uniformly in $\delta \in (0, 1]$, $\zeta_0 \in \mathcal{K}$. Thus, using Remark 8.7, for each $\delta \in (0, 1]$, $\zeta_0 \in \mathcal{K}$, there is a biholomorphism $\Phi_{\zeta_0,\delta} : B_{\mathbb{C}^n}(1) \to \Phi_{\zeta_0,\delta}(B_{\mathbb{C}^n}(1))$, with $\Phi_{\zeta_0,\delta}(0) = \zeta_0$ and such that $\Phi_{\zeta_0,\delta}^* L_1^{\delta,\zeta_0}, \ldots, \Phi_{\zeta_0,\delta}^* L_{n-1}^{\delta,\zeta_0}, \Phi_{\zeta_0,\delta}^* \delta^2 L_n$, $\Phi_{\zeta_0,\delta}^* \delta^{-2}\rho$, $\Phi_{\zeta_0,\delta}^* H^{\zeta_0,\delta}$ is an extremal system at the unit scale, uniformly in $\zeta_0 \in \mathcal{K}$ and $\delta \in (0, 1]$ (in the sense that the constants described in Remark 8.3 can be chosen uniformly in ζ_0 and δ).

If one imagines $\mathfrak{M}$ has having some fixed coordinate system, independent of δ, then $\Phi_{\zeta,\delta}$ takes points which look to be of distance $\approx \delta^2$ from $\{\rho = 0\}$ and "rescales" them to have distance ≈ 1 from $\{\Phi^*_{\zeta,\delta}\rho = 0\}$.

Remark 8.8. In the coordinate system $\Phi_{\zeta,\delta}$, the quadratic form $\partial\overline{\partial}\Phi^*_{\zeta,\delta}H^{\zeta,\delta}$ is bounded above and below, uniformly in $\zeta \in \mathcal{K}$, $\delta \in (0,1]$. However, if one starts with a fixed coordinate system on $\mathfrak{M}$ (independent of $\delta \in (0,1]$), then in terms of this coordinate system, $\partial\overline{\partial}H^{\zeta,\delta}$ has a large lower bound (as $\delta \to 0$), since the vector fields $L_1^{\delta,\zeta}, \ldots, L_{n-1}^{\delta,\zeta}, \delta^2 L_n$ are small in this fixed coordinate system.

In applications, a major difficulty is showing such an extremal basis and adapted plurisubharmonic function exists at each scale. As mentioned before, this has been done in many settings, and was abstracted in [4]. Indeed, if $L_1, \ldots, L_n$ is a (M, K, ζ, δ) extremal basis as in [4, Definition 3.1], then one can obtain an extremal basis at scale δ (in the sense discussed here) by considering $F(L_1, \zeta, \delta)^{-1/2}L_1, \ldots, F(L_{n-1}, \zeta, \delta)^{-1/2}L_{n-1}, \delta^2 L_n$ and $\delta^{-2}\rho$, where these terms are all defined in [4] (adapted plurisubharmonic functions are described in [4, Section 5]). While the results in this paper do not help find such an extremal basis, perhaps they will help make clear what to look for in other similar situations. Indeed, once one has a result on the unit scale, to translate the result to small scales it often suffices to write the unit scale result in a way which is invariant under biholomorphisms using Theorem 4.5, as we did in Section 8.2. Then one can immediately translate the setting to small scales as we did in Section 8.3.

8.4. CR manifolds

Instead of studying extremal bases on a neighborhood of a point on the boundary of a complex domain, one could try to work directly on the boundary by working with abstract CR manifolds; this is the approach taken, for example, in [14] (which addressed the "comparable eigenvalue" setting).

Let $\mathfrak{M}$ be a smooth manifold of dimension $2n-1$ endowed with a smooth CR structure $\mathscr{W}$ of rank $n-1$ (so that $\mathscr{W} \oplus \overline{\mathscr{W}}$ has codimension 1 in $\mathbb{C}T\mathfrak{M}$). Fix a point $\zeta_0 \in \mathfrak{M}$ and pick a smooth real vector field T, defined near ζ_0, such that

$$\mathscr{W}_{\zeta_0} + \overline{\mathscr{W}_{\zeta_0}} + \mathrm{span}_{\mathbb{C}}\{T(\zeta_0)\} = \mathbb{C}T\mathfrak{M}.$$

If we pick smooth sections $L_1, \ldots, L_{n-1}$ of $\mathscr{W}$ near ζ_0, these can play the role that the vector fields of the same name did in Section 8.2. We then take the same assumptions as (iv), (i), and (vi) from Section 8.2, where we replace $\overline{L_n}$ with T, L_n with 0, and remove ρ from (iv); thus we obtain constants ξ, C_N, D, K, η, and δ_0 as in those assumptions.

In this setting, Theorem 4.5 applies with $X_1, \ldots, X_q = \mathrm{Re}(L_1), \mathrm{Im}(L_1), \ldots, \mathrm{Re}(L_{n-1}), \mathrm{Im}(L_{n-1}), T$, to obtain a C^∞ diffeomorphism[21] $\Phi : B_{\mathbb{R}\times\mathbb{C}^{n-1}}(1) \to \Phi(B_{\mathbb{R}\times\mathbb{C}^{n-1}}(1))$ with

[21] Throughout we identify $\mathbb{R}^{2n-1}$ with $\mathbb{R} \times \mathbb{C}^{n-1}$.

$\Phi(0) = \zeta_0$ and such that $\Phi^* L_1, \ldots, \Phi^* L_{n-1}, \Phi^* T$ satisfy good estimates at the unit scale. Namely, we obtain Remark 8.3 (ii) and (v), where in (ii) we replace L_n with T and ρ with 0, and in (v) we replace L_n with 0 and the vector fields are written in terms of the standard basis for vector fields on $\mathbb{R} \times \mathbb{C}^{n-1} \cong \mathbb{R}^{2n-1}$–and these quantities can be estimated in terms of upper bounds for ξ^{-1}, n, K, C_N (where N can be chosen to depend only on n, K, and the estimate at hand), D, η^{-1}, and δ_0^{-1}.

Thus, Theorem 4.5 allows us to rewrite a setting at the "unit scale" in a way which is invariant under arbitrary diffeomorphisms; which, as in Section 8.3, allows us to view these maps as scaling maps. In Section 8.2 we asked that the map be bioholomorphic. It does not make a priori sense to insist that the map Φ given here is a CR map because $\mathbb{R} \times \mathbb{C}^{n-1}$ does not have a canonical CR structure. We could give $B_{\mathbb{R}\times\mathbb{C}^{n-1}}(1)$ the CR structure $\Phi^*\mathscr{W}$, and then Φ is automatically a CR map, but this does not add much useful information.

However, in the special case that one can choose T so that the bundle $\mathscr{W}_\zeta + \mathbb{C}T(\zeta)$ is formally integrable (and therefore an elliptic structure), then we can do more. In this case, we can choose Φ so that $\Phi^* L_1, \ldots, \Phi^* L_{n-1}, \Phi^* T$ are all spanned by $\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}, \frac{\partial}{\partial t}$ (where $\mathbb{R} \times \mathbb{C}^{n-1}$ is given coordinates $(t, z_1, \ldots, z_n)$). One can see this by applying Theorem 4.5 to the vector fields $L_1, \ldots, L_{n-1}$ and $X_1 = T$ (with $m = n-1$ and $q = 1$). In many of the standard examples of CR manifolds, one can find such a T–see Section 6.1.

9. Function spaces revisited

In this section we present the basic properties of the function spaces introduced in Section 2; most of these properties were proved in [27,31,32], and we refer the reader to those references for proofs and a further discussion of the results not proved here. We take $W_1, \ldots, W_N$ to be real C^1 vector fields on a C^2 manifold M as in Section 2.

Lemma 9.1.

(i) *For* $0 \leq s_1 \leq s_2 \leq 1$, $m \in \mathbb{N}$, $\|f\|_{C^{m,s_1}_W(M)} \leq 3\|f\|_{C^{m,s_2}_W(M)}$.
(ii) $\|f\|_{C^{m,1}_W(M)} \leq \|f\|_{C^{m+1}_W(M)}$.
(iii) *For* $s \in (0,1]$, $m \in \mathbb{N}$, $\|f\|_{\mathscr{C}^{s+m}_W(M)} \leq 5\|f\|_{C^{m,s}_W(M)}$.
(iv) *For* $0 < s_1 \leq s_2 < \infty$, $\|f\|_{\mathscr{C}^{s_1}_W(M)} \leq 15\|f\|_{\mathscr{C}^{s_2}_W(M)}$.
(v) *If* $U \subseteq M$ *is an open set, then* $\|f\|_{C^{m,s}_W(U)} \leq \|f\|_{C^{m,s}_W(M)}$ *and* $\|f\|_{\mathscr{C}^{s}_W(U)} \leq \|f\|_{\mathscr{C}^{s}_W(M)}$.
(vi) $C^{\omega,r}(B_{\mathbb{R}^n}(r)) \subseteq \mathscr{A}^{n,r}$ *and* $\|f\|_{\mathscr{A}^{n,r}} \leq \|f\|_{C^{\omega,r}(B_{\mathbb{R}^n}(r))}$.
(vii) $\mathscr{A}^{n,r} \subseteq C^{\omega,r/2}(B_{\mathbb{R}^n}(r/2))$ *and* $\|f\|_{C^{\omega,r/2}(B^n(r/2))} \leq \|f\|_{\mathscr{A}^{n,r}}$.
(viii) *Suppose* $W = W_1, \ldots, W_N$ *satisfies* $\mathcal{C}(x_0, r, M)$. *Then,* $C^{\omega,r}_W(M) \subseteq \mathscr{A}^{x_0,r}_W$ *and* $\|f\|_{\mathscr{A}^{x_0,r}_W} \leq \|f\|_{C^{\omega,r}_W(M)}$.
(ix) *For any* $s \in (0,r)$, $W_j : C^{\omega,r}_W(M) \to C^{\omega,s}_W(M)$. *In particular,* $W_j : C^{\omega}_W(M) \to C^{\omega}_W(M)$.
(x) *For any* $s \in (1,\infty] \cup \{\omega\}$, $W_j : \mathscr{C}^{s}_W(M) \to \mathscr{C}^{s-1}_W(M)$.

Proof. (i), (ii), (iii), (iv), and (v) are contained in [27, Lemma 8.1]. (vi), (vii), (viii), (ix) are contained in [32]. For $s \in (1,\infty]$, (x) follows immediately from the definitions. For $s=\omega$, (x) follows from (ix); where we are using the convention $\omega = \omega - 1$. □

Remark 9.2. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. In analogy with Lemma 9.1 (iii), for $m \in \mathbb{N}$, $s \in [0,1]$ with $m+s>0$, we have $C^{m,s}(\Omega) \subseteq \mathscr{C}^{m+s}(\Omega)$. If Ω is a bounded Lipschitz domain and $s \in (0,1)$, then we have the reverse containment as well $\mathscr{C}^{m+s}(\Omega) \subseteq C^{m,s}(\Omega)$ (see [36, Theorem 1.118 (i)]). Because of this, one might hope for the reverse inequality to the one in Lemma 9.1 (iii) for $s \in (0,1)$. One can obtain such an estimate, but it requires additional hypotheses on the vector fields. This is discussed in [31].

Proposition 9.3. *The spaces $C^{m,s}_W(M)$, $\mathscr{C}^s_W(M)$, $C^{m,s}(\Omega)$, $\mathscr{C}^s(\Omega)$, $C^{\omega,r}_W(M)$, $\mathscr{A}^{x_0,r}_W$, $C^{\omega,r}(M)$, and $\mathscr{A}^{n,r}$ are algebras. In fact, if $\mathscr{Y}$ denotes any one of these spaces, then*

$$\|fg\|_{\mathscr{Y}} \leq C_{\mathscr{Y}} \|f\|_{\mathscr{Y}} \|g\|_{\mathscr{Y}}.$$

When $\mathscr{Y} \in \{C^{\omega,r}_W(M), \mathscr{A}^{x_0,r}_W, C^{\omega,r}(M), \mathscr{A}^{n,r}\}$, we may take $C_{\mathscr{Y}} = 1$; i.e., these spaces are Banach algebras.[22] *When $\mathscr{Y} \in \{C^{m,s}_W(M), \mathscr{C}^s_W(M), C^{m,s}(\Omega), \mathscr{C}^s(\Omega)\}$, these spaces have multiplicative inverses for functions which are bounded away from zero: if $f \in \mathscr{Y}$ with $\inf_x |f(x)| \geq c_0 > 0$, then $f(x)^{-1} = \frac{1}{f(x)} \in \mathscr{Y}$. Furthermore, $\|f(x)^{-1}\|_{\mathscr{Y}} \leq C$ where C can be chosen to depend only on $\mathscr{Y}$, c_0, and an upper bound for $\|f\|_{\mathscr{Y}}$.*

Proof. The proofs for $C^{m,s}_W(M)$ and $C^{m,s}(\Omega)$ are straightforward and standard, and we leave the proofs to the reader. The results for $\mathscr{C}^s_W(M)$ and $\mathscr{C}^s(\Omega)$ are in [27, Proposition 8.3]. The results for $C^{\omega,r}_W(M)$, $\mathscr{A}^{x_0,r}_W$, $C^{\omega,r}(M)$, and $\mathscr{A}^{n,r}$ are in [32]. □

Remark 9.4. For $s \in (0,\infty] \cup \{\omega\}$, suppose $A \in \mathscr{C}^s(\Omega; \mathbb{M}^{k\times k})$ is such that $\inf_{t\in\Omega} |\det A(t)| > 0$. Then it follows that $A(\cdot)^{-1} \in \mathscr{C}^s(\Omega; \mathbb{M}^{k\times k})$; where we write $A(\cdot)^{-1}$ for the function $t \mapsto A(t)^{-1}$. Indeed, for $s \in (0,\infty]$, this follows from Proposition 9.3 using the cofactor representation of $A(\cdot)^{-1}$. For $s=\omega$, this is standard. When $s \in (0,\infty)$, $\|A(\cdot)^{-1}\|_{\mathscr{C}^s(\Omega)}$ can be bounded in terms of s, k, n, a lower bound for $\inf_{t\in\Omega} |\det A(t)| > 0$, and an upper bound for $\|A\|_{\mathscr{C}^s(\Omega)}$.

Lemma 9.5. *Let $D_1, D_2 > 0$, $s_1 > 0$, $s_2 \geq s_1$, $s_2 > 1$, $f \in \mathscr{C}^{s_1}(B_{\mathbb{R}^n}(D_1))$, $g \in \mathscr{C}^{s_2}(B_{\mathbb{R}^m}(D_2); \mathbb{R}^n)$ with $g(B_{\mathbb{R}^m}(D_2)) \subseteq B_{\mathbb{R}^n}(D_1)$. Then, $f\circ g \in \mathscr{C}^{s_1}(B_{\mathbb{R}^m}(D_2))$ and $\|f\circ g\|_{\mathscr{C}^{s_1}(B_{\mathbb{R}^m}(D_2))} \leq C \|f\|_{\mathscr{C}^{s_1}(B_{\mathbb{R}^n}(D_1))}$, where C can be chosen to depend only on s_1, s_2, D_1, D_2, m, n, and an upper bound for $\|g\|_{\mathscr{C}^{s_2}(B_{\mathbb{R}^m}(D_2))}$.*

Proof. This is proved in [31]. □

[22] This remains true for the analogous spaces taking values in a Banach algebra.

Lemma 9.6. *Let $\eta_1, \eta_2 > 0$, $n_1, n_2 \in \mathbb{N}$, and let $\mathscr{X}$ be a Banach space. Suppose $f \in \mathscr{A}^{n_1,\eta_1}(\mathscr{X})$, $g \in \mathscr{A}^{n_2,\eta_2}(\mathbb{R}^{n_1})$ with $\|g\|_{\mathscr{A}^{n_2,\eta_2}(\mathbb{R}^{n_1})} \leq \eta_1$. Then, $f \circ g \in \mathscr{A}^{n_2,\eta_2}(\mathscr{X})$ with $\|f \circ g\|_{\mathscr{A}^{n_2,\eta_2}} \leq \|f\|_{\mathscr{A}^{n_1,\eta_1}}$.*

Proof. This is immediate from the definitions. □

Lemma 9.7. *Fix $0 < \eta_2 < \eta_1$, and suppose $f \in \mathscr{A}^{n,\eta_1}(\mathscr{X})$, where $\mathscr{X}$ is a Banach space. Then, for each $j = 1, \ldots, n$, $\frac{\partial}{\partial t_j} f(t) \in \mathscr{A}^{n,\eta_2}(\mathscr{X})$ and $\|\frac{\partial}{\partial t_j} f\|_{\mathscr{A}^{n,\eta_2}} \leq C\|f\|_{\mathscr{A}^{n,\eta_1}}$, where C can be chosen to depend only on η_1 and η_2.*

Proof. Without loss of generality, we prove the result for $j = 1$. We let e_1 denote the first standard basis element: $e_1 = (1, 0, \ldots, 0) \in \mathbb{N}^n$. Suppose $f(t) = \sum c_\alpha \frac{t^\alpha}{\alpha!}$. Then, $\frac{\partial}{\partial t_1} f(t) = \sum_{\alpha_1 > 0} c_\alpha \frac{t^{\alpha - e_1}}{(\alpha - e_1)!}$. Hence,

$$\begin{aligned}\left\|\frac{\partial}{\partial t_1} f\right\|_{\mathscr{A}^{n,\eta_2}} &= \sum_{\alpha_1>0} \frac{|c_\alpha|}{(\alpha - e_1)!} \eta_2^{|\alpha - e_1|} = \sum_{\alpha} \frac{|c_\alpha|}{\alpha!} \eta_1^{|\alpha|} \left(\frac{\eta_2}{\eta_1}\right)^{|\alpha|} \frac{\alpha_1}{\eta_1} \\ &\leq \left(\sup_\alpha \left(\frac{\eta_2}{\eta_1}\right)^{|\alpha|} \frac{\alpha_1}{\eta_1}\right) \|f\|_{\mathscr{A}^{n,\eta_1}},\end{aligned}$$

completing the proof. □

Proposition 9.8. *Let $Y_1, \ldots, Y_N$ be C^1 vector fields on an open ball $B \subseteq \mathbb{R}^n$. Suppose $Y_1, \ldots, Y_N$ span the tangent space at every point in the sense that for $1 \leq j \leq n$,*

$$\frac{\partial}{\partial t_j} = \sum_{k=1}^{N} b_j^k Y_k, \quad b_j^k \in C(B).$$

Fix $s \in (0, \infty] \cup \{\omega\}$ and suppose $Y_k \in \mathscr{C}^{s-1}(B; \mathbb{R}^n)$, $b_j^k \in \mathscr{C}^{s-1}(B)$, $\forall j, k$. Then, $\mathscr{C}^s(B) = \mathscr{C}^s_Y(B)$. Here we use the convention that for $s \in (-1, 0]$, $\mathscr{C}^s(B) := C^{0,(s+1)/2}(B)$.

Proof. The case $s \in (0, \infty]$ is contained in [27, Proposition 8.12], while the case $s = \omega$ is discussed in [32]. The case $s = \omega$ is part of a more general result due to Nelson [20, Theorem 2]. [27,32] also contain quantitative versions of this result. □

10. Proofs of corollaries

10.1. Optimal smoothness

In this section, we prove Theorems 7.1 and 7.2, and describe how Theorems 3.1, 3.3, 3.5, and 3.6 are consequences of Theorems 7.1 and 7.2.

Proof of Theorem 7.1. (i)⇒(ii): Suppose the conditions of (i) hold and without loss of generality we may assume $0 \in U$ and $\Phi(0) = x_0$; reorder the vector fields as in (ii). Because $\dim \mathscr{X}_{x_0} = r$ and $\dim \mathscr{L}_{x_0} = n + r$, we have

$$\operatorname{span}_{\mathbb{R}}\{\Phi^* X_1(0,0), \ldots, \Phi^* X_r(0,0)\} = \operatorname{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}\right\},$$

$$\operatorname{span}_{\mathbb{C}}\{\Phi^* X_1(0,0), \ldots, \Phi^* X_r(0,0), \Phi^* L_1(0,0), \ldots, \Phi^* L_n(0,0)\}$$
$$= \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\}.$$

Writing X_{K_0} for the column vector of vector fields $[X_1, \ldots, X_r]^\top$ and L_{J_0} for the column vector $[L_1, \ldots, L_n]^\top$ and using the hypotheses of (i), we may write

$$\begin{bmatrix} \Phi^* X_{K_0} \\ \Phi^* L_{J_0} \end{bmatrix} = B \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial \overline{z}} \end{bmatrix},$$

where $B \in \mathscr{C}^{s+1}(U; \mathbb{M}^{(n+r)\times(n+r)})$ is such that $B(0,0)$ is invertible. Letting $U_0 \subseteq U$ be a sufficiently small open ball centered at $(0,0)$, we have that $|\det B(t,z)|$ is bounded away from 0 on U_0. Thus, on U_0, B is invertible and $B(\cdot)^{-1} \in \mathscr{C}^{s+1}(U_0; \mathbb{M}^{(n+r)\times(n+r)})$ (see Remark 9.4); and we have

$$B^{-1} \begin{bmatrix} \Phi^* X_{K_0} \\ \Phi^* L_{J_0} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial \overline{z}} \end{bmatrix}. \tag{10.1}$$

Thus, for $x \in \Phi(U_0)$,

$$\dim \mathscr{L}_x \geq \dim \operatorname{span}_{\mathbb{C}}\{X_1(x), \ldots, X_r(x), L_1(x), \ldots, L_n(x)\}$$
$$= \dim \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\} = n + r.$$

The hypothesis (i) implies for $x \in \Phi(U) = V$,

$$\dim \mathscr{L}_x = \dim \operatorname{span}_{\mathbb{C}}\{X_1(x), \ldots, X_q(x), L_1(x), \ldots, L_m(x)\}$$
$$\leq \dim \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\} = n + r.$$

This shows that the map $x \mapsto \dim \mathscr{L}_x$, $\Phi(U_0) \to \mathbb{N}$ is the constant function $n + r$.

Since $\Phi^* \widehat{Z}_j \in \operatorname{span}_{\mathbb{C}}\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\}$ we can think of $\Phi^* \widehat{Z}_j$ as a function taking values in $\mathbb{C}^{n+r}$. We have $\Phi^* \widehat{Z}_j \in \mathscr{C}^{s+1}(U; \mathbb{C}^{n+r})$, and therefore $[\Phi^* \widehat{Z}_j, \Phi^* \widehat{Z}_k] \in \mathscr{C}^{s}(U; \mathbb{C}^{n+r})$ and it follows from (10.1) and Proposition 9.3 that

$$[\Phi^*\widehat{Z}_j, \Phi^*\widehat{Z}_k] = \sum_{l=1}^{n+r} \tilde{c}_{j,k}^{1,l} \Phi^*\widehat{Z}_l, \quad \tilde{c}_{j,k}^{1,l} \in \mathscr{C}^s(U_0). \tag{10.2}$$

Similarly, since $[\Phi^*\widehat{Z}_j, \Phi^*\overline{\widehat{Z}_k}] \in \mathscr{C}^s(U;\mathbb{C}^{2n+r})$, we have

$$[\Phi^*\widehat{Z}_j, \Phi^*\overline{\widehat{Z}_k}] = \sum_{l=1}^{n+r} \tilde{c}_{j,k}^{2,l} \Phi^*\widehat{Z}_l + \sum_{l=1}^{n+r} \tilde{c}_{j,k}^{3,l} \Phi^*\overline{\widehat{Z}_l}, \quad \tilde{c}_{j,k}^{2,l}, \tilde{c}_{j,k}^{3,l} \in \mathscr{C}^s(U_0). \tag{10.3}$$

Furthermore, since $\Phi^*Y_j(t,z) \in \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right\}$ and $\Phi^*Y_j \in \mathscr{C}^{s+1}(U; \mathbb{C}^{r+n})$, (10.1) and Proposition 9.3 imply

$$\Phi^*Y_j = \sum_{l=1}^{n+r} \tilde{b}_j^l \Phi^*\widehat{Z}_l, \quad \tilde{b}_j^l \in \mathscr{C}^{s+1}(U_0). \tag{10.4}$$

Proposition 9.8, combined with (10.1), shows $\tilde{c}_{j,k}^{a,l} \in \mathscr{C}^s(U_0) = \mathscr{C}^s_{\Phi^*X,\Phi^*L}(U_0)$ and $\tilde{b}_j^l \in \mathscr{C}^{s+1}(U_0) = \mathscr{C}^{s+1}_{\Phi^*X,\Phi^*L}(U_0)$, $\forall j,k,l,a$. Let $\hat{c}_{j,k}^{a,l} := \tilde{c}_{j,k}^{a,l} \circ \Phi^{-1}$, $b_j^l := \tilde{b}_j^l \circ \Phi^{-1}$, and $V_0 := \Phi(U_0)$. Proposition 2.1 shows $\hat{c}_{j,k}^{a,l} \in \mathscr{C}^s_{X,L}(V_0)$ and $b_j^l \in \mathscr{C}^{s+1}_{X,L}(V_0)$. Pushing forward (10.2), (10.3), and (10.4) via Φ gives

$$[\widehat{Z}_j, \widehat{Z}_k] = \sum_{l=1}^{n+r} \hat{c}_{j,k}^{1,l} \widehat{Z}_l, \quad [Z_j, \overline{\widehat{Z}_k}] = \sum_{l=1}^{n+r} \hat{c}_{j,k}^{2,l} \widehat{Z}_l + \sum_{l=1}^{n+r} \hat{c}_{j,k}^{3,l} \overline{\widehat{Z}_l}, \quad Y_j = \sum_{l=1}^{n+r} b_j^l \widehat{Z}_l.$$

Along with the above remarks on $\hat{c}_{j,k}^{a,l}$ and b_j^l, this completes the proof of (ii) with V replaced by V_0.

(ii)⇒(iii): Suppose (ii) holds. First, we wish to show that

$$[Z_j, Z_k] = \sum_{l=1}^{m+q} c_{j,k}^{1,l} Z_l, \quad c_{j,k}^{1,l} \in \mathscr{C}^s_{X,L}(V). \tag{10.5}$$

Z_j and Z_k are each either of the form $\widehat{Z}_l$ or Y_l for some l (where $\widehat{Z}_l$ and Y_l are as in (ii)). When Z_j and Z_k are both of the form $\widehat{Z}_l$ for some l, (10.5) is contained in (ii). We address the case when $Z_j = Y_{l_1}$, $Z_k = Y_{l_2}$ for some l_1, l_2. The remaining case (when $Z_j = \widehat{Z}_{l_1}$ and $Z_k = Y_{l_2}$) is similar, and we leave it to the reader. We have,

$$\begin{aligned}
[Z_j, Z_k] = [Y_{l_1}, Y_{l_2}] &= \left[\sum_{l_3} b_{l_1}^{l_3} \widehat{Z}_{l_3}, \sum_{l_4} b_{l_2}^{l_4} \widehat{Z}_{l_4}\right] \\
&= \sum_{l_3,l_4} b_{l_1}^{l_3} b_{l_2}^{l_4} [\widehat{Z}_{l_3}, \widehat{Z}_{l_4}] + \sum_{l_3,l_4} b_{l_1}^{l_3} (\widehat{Z}_{l_3} b_{l_2}^{l_4}) \widehat{Z}_{l_4} - \sum_{l_3,l_4} b_{l_2}^{l_4} (\widehat{Z}_{l_4} b_{l_1}^{l_3}) \widehat{Z}_{l_3}.
\end{aligned}$$

Using Lemma 9.1 (x) and Proposition 9.3, we have $b_{l_1}^{l_3}(\widehat{Z}_{l_3} b_{l_2}^{l_4}), b_{l_2}^{l_4}(\widehat{Z}_{l_4} b_{l_1}^{l_3}) \in \mathscr{C}^s_{X,L}(V)$. Also, we have

$$\sum_{l_3,l_4} b_{l_1}^{l_3} b_{l_2}^{l_4} [\widehat{Z}_{l_3}, \widehat{Z}_{l_4}] = \sum_{l_3,l_4} \sum_{l_5} b_{l_1}^{l_3} b_{l_2}^{l_4} \hat{c}_{l_3,l_4}^{1,l_5} \widehat{Z}_{l_5},$$

and by Proposition 9.3, $b_{l_1}^{l_3} b_{l_2}^{l_4} \hat{c}_{l_3,l_4}^{1,l_5} \in \mathscr{C}^s_{X,L}(V)$. Combining the above remarks, we have

$$[Z_j, Z_k] = \sum_{l=1}^{n+r} c_{j,k}^{1,l} \widehat{Z}_l, \quad c_{j,k}^{1,l} \in \mathscr{C}^s_{X,L}(V).$$

Since each $\widehat{Z}_l$ is of the form $Z_{l'}$ for some l', (10.5) follows. A similar proof shows

$$[Z_j, \overline{Z_k}] = \sum_l c_{j,k}^{2,l} Z_l + \sum_l c_{j,k}^{3,l} \overline{Z_l}, \quad c_{j,k}^{2,l}, c_{j,k}^{3,l} \in \mathscr{C}^s_{X,L}(V),$$

and we leave the details to the reader. This completes the proof of (iii).

(iii)⇒(i): This is a consequence of Theorem 4.5; and we include a few remarks on this. First, a choice of $\eta, \delta_0 > 0$ as in the hypotheses of Theorem 4.5 always exist; see Lemma 4.13. A choice of J_0, K_0, and $\zeta > 0$ as in the hypotheses also always exist; see Remark B.6. We take $\xi > 0$ so small $B_{X,L}(x_0, \xi) \subseteq V$.

First we address the case $s \in (1, \infty]$. In this case, pick $s_0 \in (1, s] \setminus \{\infty\}$ (the choice of s_0 does not matter). We have, directly from the definitions

$$\begin{aligned} c_{j,l}^{a,l} &\in \mathscr{C}^s_{X,L}(V) \subseteq \mathscr{C}^s_{X,L}(B_{X,L}(x_0,\xi)) \subseteq \mathscr{C}^s_{X_{K_0},L_{J_0}}(B_{X_{K_0},L_{J_0}}(x_0,\xi)) \\ &\subseteq \mathscr{C}^{s_0}_{X_{K_0},L_{J_0}}(B_{X_{K_0},L_{J_0}}(x_0,\xi)). \end{aligned}$$

Thus, all of the hypotheses of Theorem 4.5 hold for this choice of s_0. The map guaranteed by Theorem 4.5 satisfies the conclusions of (i) and this completes the proof in the case $s \in (1, \infty]$.

When $s = \omega$, we wish to apply Theorem 4.5 in the case $s_0 = \omega$. There is a slight discrepancy between the hypotheses of Theorem 4.5 and (iii). Namely, we are currently assuming $c_{j,k}^{a,l} \in C^{\omega,r_0}_{X,L}(V)$ for some $r_0 > 0$, while Theorem 4.5 assumes $c_{j,k}^{a,l} \in \mathscr{A}^{x_0,\eta}_{X_{K_0},L_{J_0}}$ and $c_{j,k}^{a,l}$ is continuous near x_0. However, $c_{j,k}^{a,l} \in C^{\omega,r_0}_{X,L}(V)$ clearly implies $c_{j,k}^{a,l}$ is continuous near x_0, and using Lemma 9.1 (viii) we have $C^{\omega,\eta}_{X,L} \subseteq C^{\omega,\eta}_{X_{K_0},L_{J_0}} \subseteq \mathscr{A}^{x_0,\eta}_{X_{K_0},L_{J_0}}$, so by shrinking η so that $\eta \leq r_0$, these hypotheses follow. With these remarks, Theorem 4.5 applies to yield the coordinate chart Φ as in that theorem, which satisfies all the conclusions of (i). This completes the proof. □

Before we prove Theorem 7.2, we require two lemmas.

Lemma 10.1. *Fix $s \in (0, \infty] \cup \{\omega\}$ and suppose M_1 and M_2 are $\mathscr{C}^{s+2}$ manifolds. Let $Z_1, \dots, Z_N$ be complex $\mathscr{C}^{s+1}$ vector fields on M_1 such that $Z_1, \dots Z_N, \overline{Z_1}, \dots, \overline{Z_N}$ span the complexified tangent space to M_1 at every point. Let $\Psi : M_1 \to M_2$ be a C^2 diffeomorphism such that $\Psi_* Z_j$ is a $\mathscr{C}^{s+1}$ vector field, $\forall 1 \leq j \leq N$. Then, Ψ is a $\mathscr{C}^{s+2}$ diffeomorphism.*

Proof. By taking real an imaginary parts, it suffices to prove the result in the case $Z_1, \ldots, Z_N$ are real and span the tangent space at every point. In the case $s \in (0, \infty]$, this is proved in [31]. In the case $s = \omega$, this is proved in [32]. $\square$

Lemma 10.2. $\left(T_{(t_0,z_0)}(\mathbb{R}^r \times \mathbb{C}^n)\right) \cap \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\} = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r} \right\}$.

Proof. This is immediate. $\square$

Proof of Theorem 7.2. (i)⇒(ii): The inverses of the coordinate charts from the atlas given in (i) satisfy the conditions in Theorem 7.1 (i) (this uses Lemma 10.2); and so (ii) follows.

(ii)⇒(i): Assume that (ii) holds. Using the characterization in Theorem 7.1 (iii), we have that $x \mapsto \dim \mathscr{L}_x$, $M \to \mathbb{N}$ is locally constant, and since M is connected, $x \mapsto \dim \mathscr{L}_x$, $M \to \mathbb{N}$ is constant. By the discussion in Section 4.2 we also have $x \mapsto \dim \mathscr{X}_x$, $M \to \mathbb{N}$ is constant. Set $r := \dim \mathscr{X}_x$ and $n+r := \dim \mathscr{L}_x$ (so that n and r do not depend on x, by the above discussion). Now, we use the characterization given in Theorem 7.1 (i). Thus, for each $x \in M$, there is a neighborhood $V_x \subseteq M$ of x and a C^2 diffeomorphism $\Phi_x : U_x \to V_x$, where $U_x \subseteq \mathbb{R}^r \times \mathbb{C}^n$ is open, such that $\forall (t, z) \in U_x$, $1 \le k \le q$, $1 \le j \le m$,

$$\Phi_x^* X_k(t, z) \in \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r} \right\},$$
$$\Phi_x^* L_j(t, z) \in \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\},$$

and $\Phi_x^* X_k \in \mathscr{C}^{s+1}(U_x; \mathbb{R}^r)$, $\Phi_x^* L_j \in \mathscr{C}^{s+1}(U_x; \mathbb{C}^{r+n})$. Our desired atlas is $\{(\Phi_x^{-1}, V_x) : x \in M\}$–once we show this is a $\mathscr{C}^{s+2}$ E-atlas, (i) will follow. For $x, y \in M$, set $\Psi_{x,y} := \Phi_y^{-1} \circ \Phi_x : \Phi_x^{-1}(V_y \cap V_x) \to U_y$; we wish to show that $\Psi_{x,y}$ is a $\mathscr{C}^{s+2}_{\mathrm{loc}}$ E-map. Note that

$$\begin{aligned} d\Psi_{x,y}(t,z)(\Phi_x^* X_k)(t,z) &= (\Phi_y^* X_k)(\Psi_{x,y}(t,z)), \\ d\Psi_{x,y}(t,z)(\Phi_x^* L_j)(t,z) &= (\Phi_y^* L_j)(\Psi_{x,y}(t,z)), \quad \forall j, k. \end{aligned} \tag{10.6}$$

In other words,

$$(\Psi_{x,y})_* \Phi_x^* X_k = \Phi_y^* X_k, \quad (\Psi_{x,y})_* \Phi_x^* L_j = \Phi_y^* L_j, \quad \forall j, k. \tag{10.7}$$

Since $\dim \mathscr{L}_y = n + r$, $\forall y \in M$, we have $\forall (t, z) \in U_x$,

$$\begin{aligned} &\operatorname{span}_{\mathbb{C}} \{\Phi_x^* X_1(t,z), \ldots, \Phi_x^* X_q(t,z), \Phi_x^* L_1(t,z), \ldots, \Phi_x^* L_m(t,z)\} \\ &= \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\}. \end{aligned} \tag{10.8}$$

Combining (10.8) and (10.6) shows that $\Psi_{x,y}$ is an E-map. (10.8) implies

$$\Phi_x^* X_1(t,z),\ldots,\Phi_x^* X_q(t,z),\Phi_x^* L_1(t,z),\ldots,\Phi_x^* L_m(t,z),\overline{\Phi_x^* L_1}(t,z),\ldots,\overline{\Phi_x^* L_m}(t,z)$$

span the complexified tangent space at every point of U_x. Since these vector fields are also $\mathscr{C}^{s+1}$ by hypothesis, (10.7) and Lemma 10.1 show that $\Psi_{x,y}$ is $\mathscr{C}^{s+2}_{\mathrm{loc}}$. This completes the proof of (i).

(iii)⇒(ii): This is obvious, and holds for $s \in (0,\infty] \cup \{\omega\}$.

(i)⇒(iii), for $s \in (0,\infty]$: Assuming that (i) holds (where M is an E-manifold of dimension (r,n)), a simple partition of unity argument shows that we may write $[Z_j, Z_k] = \sum_{l=1}^{m+q} c_{j,k}^{1,l} Z_l$ and $[Z_j, \overline{Z_k}] = \sum_{l=1}^{m+q} c_{j,k}^{2,l} Z_l + \sum_{l=1}^{m+q} c_{j,k}^{3,l} \overline{Z_l}$, where $c_{j,k}^{a,l} : M \to \mathbb{C}$ and $c_{j,k}^{a,l}$ are locally in $\mathscr{C}^s$. We wish to show $\forall x_0 \in M$, $\exists V \subseteq M$ open with $x_0 \in V$ and $c_{j,k}^{a,l}\big|_V \in \mathscr{C}^s_{X,L}(V)$. Fix $x_0 \in M$ and let $W \subseteq M$ be a neighborhood of x_0 such that there is a $\mathscr{C}^{s+2}$ diffeomorphism $\Phi : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to W$ with $\Phi(0) = x_0$. Let $Y_1,\ldots,Y_{q+2m}$ denote the list $\Phi^* X_1,\ldots,\Phi^* X_r, 2\Phi^*\mathrm{Re}(L_1),\ldots,2\Phi^*\mathrm{Re}(L_m), 2\Phi^*\mathrm{Im}(L_1),\ldots,2\Phi^*\mathrm{Im}(L_m)$. $Y_1,\ldots,Y_{q+2m}$ are $\mathscr{C}^{s+1}$ vector fields on $B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$ and span the tangent space at every point. We conclude $Y_1,\ldots,Y_{q+2m}$ satisfy all the hypotheses of Proposition 9.8 with $B := B_{\mathbb{R}^r\times\mathbb{C}^n}(1/2)$. Thus, by Proposition 9.8, $c_{j,k}^{a,l} \circ \Phi \in \mathscr{C}^s(B) = \mathscr{C}^s_Y(B)$. Proposition 2.1 shows $c_{j,k}^{a,l} \in \mathscr{C}^s_{X,L}(\Phi(B))$, completing the proof with $V = \Phi(B)$.

Finally, we turn to the uniqueness claimed in the theorem; that under the equivalent hypotheses (i) and (ii), the E-manifold structure given in (i) is unique. Indeed, suppose there are two such structures on M. Under these conditions, the identity map $M \to M$ is $\mathscr{C}^{s+2}_{\mathrm{loc}}$ by Lemma 10.1 (here we have applied Lemma 10.1 with the vector fields $X_1,\ldots,X_q,L_1,\ldots,L_m$). That the identity map is a $\mathscr{C}^{s+2}_{\mathrm{loc}}$ E-map follows from Lemma 6.21. It follows that the identity map is a $\mathscr{C}^{s+2}$ E-diffeomorphism, as claimed. □

Proof of Theorems 3.1 and 3.3. In the setting of Theorems 3.1 and 3.3, because $W_1,\ldots,W_N$ span the tangent space at every point, we have $\dim \mathrm{span}_{\mathbb{R}}\{W_1(x),\ldots, W_N(x)\} = \dim M = n$, $\forall x$; in particular, the map $x \mapsto \dim \mathrm{span}_{\mathbb{R}}\{W_1(x),\ldots,W_N(x)\}$ is constant. With this in mind, Theorems 3.1 and 3.3 are immediate consequences of the case $m = 0$ of Theorems 7.1 and 7.2. □

Proof of Theorems 3.5 and 3.6. In the setting of Theorems 3.5 and 3.6, we have $\dim \mathscr{L}_\zeta = n$, $\forall \zeta \in M$. Thus, the map $\zeta \mapsto \dim \mathscr{L}_\zeta$ is constant. Also, in the context of Theorem 3.6, E-maps are holomorphic (and E-diffeomorphisms are biholomorphisms); this is because complex manifolds embed into E-manifolds as a *full* sub-category (see Remark 6.12). With these remarks in hand, Theorems 3.5 and 3.6 are immediate consequences of the case $q = 0$ of Theorems 7.1 and 7.2. □

10.2. Sub-E geometry

In this section, we prove Theorem 7.6. In light of Remark 7.4, Theorem 3.9 is a special case of Theorem 7.6. Theorem 3.12 is also a special case of Theorem 7.6:

Proof of Theorem 3.12. In light of Remark 7.4, the hypotheses of Theorem 3.12 imply the hypotheses of Theorem 7.6. The main issue in seeing Theorem 3.12 as a special case of Theorem 7.6 is that the definitions of ρ_H in the two theorems are not obviously the same. However, if M is a complex manifold and $f(t,z) : B_{\mathbb{R}\times\mathbb{C}}(1/2) \to M$ is an E-map, then f must be constant in t and is therefore a holomorphic map $B_{\mathbb{C}}(1/2) \to M$. Indeed, $df(t,z)\frac{\partial}{\partial t}$ is both a $T^{0,1}_{f(t,z)}$ tangent vector and a real tangent vector, and we conclude $df(t,z)\frac{\partial}{\partial t} \equiv 0$. Using this, it is easy to see that the definition of ρ_H in Theorem 7.6 is the same as the definition of ρ_H in Theorem 3.12 when M is a complex manifold. □

The rest of this section is devoted to the proof of Theorem 7.6.

Lemma 10.3. $\lim_{y\to x} \rho_F(x,y) = 0$, *where the limit is taken in the usual topology on M–recall, M is a manifold and therefore comes equipped with a topology which we are referring to as the "usual topology."*

Proof. Fix $\epsilon > 0$; we wish to find a neighborhood $N \subseteq M$ of x such that $\forall y \in N$, $\rho_F(x,y) < \epsilon$. Reorder $W_1, \ldots, W_{2m+q}$ so that $W_1(x), \ldots, W_{2n+r}(x)$ form a basis for T_xM and set

$$\Psi(t_1, \ldots, t_{2n+r}) := e^{t_1W_1+\cdots+t_{2n+r}W_{2n+r}}x.$$

Since $\frac{\partial}{\partial t_j}\big|_{t=0}\Psi(t) = W_j(x)$ it follows from the inverse function theorem that there exists an open neighborhood U of $0 \in \mathbb{R}^{2n+r}$ such that $\Psi(U)$ is open and $\Psi : U \to \Psi(U)$ is a C^∞ diffeomorphism. Set $0 < c \leq (32(2n+r))^{-1/2}$ and let $B := \{t = (t_1, \ldots, t_{2n+r}) : |t_j| < c\epsilon^{d_j}\}$; take c so small that $B \subseteq U$ and set $N = \Psi(B)$. N is clearly open since Ψ is a diffeomorphism. Thus, it remains to show $N \subseteq B_F(x,\epsilon)$. Take $y \in N$, so that there exists $t \in B$ with $y = \Psi(t)$. Define $f : B_{\mathbb{R}}(1/2) \to M$ by

$$f(s) := e^{4s(t_1W_1+\cdots+t_{2n+r}W_{2n+r})}x,$$

so that $f \in C^\infty$, $f(0) = x$, $f(1/4) = y$, and

$$f'(s) = \sum_{j=1}^{2n+r} 4t_jW_j(f(s)) = \sum_{j=1}^{2n+r} 4\frac{t_j}{\epsilon^{d_j}}\epsilon^{d_j}W_j(f(s)).$$

Since

$$\sum_{j=1}^{2n+r}\left(4\frac{t_j}{\epsilon^{d_j}}\right)^2 \leq \sum_{j=1}^{2n+r}\frac{1}{2(2n+r)} \leq \frac{1}{2} < 1,$$

it follows that $\rho_F(x,y) < \epsilon$, completing the proof. □

Lemma 10.4. *The metric topology induced by ρ_F is the same as the usual topology on M.*

Proof. Lemma 10.3 shows that the usual topology on M is finer than the metric topology induced ρ_F. That the metric topology induced by ρ_F is finer than the usual topology is a straightforward application of the Phragmén-Lindelöf Theorem; and we leave the details to the reader.[23] $\square$

Proof of Theorem 7.6 (a). We begin by showing $\rho_F \leq \rho_S$. Suppose $\rho_S(x,y) < \delta$. Then, there exists $\gamma : [0,1] \to M$, $\gamma(0) = x$, $\gamma(1) = y$, $\gamma'(t) = \sum a_j(t)\delta^{d_j} W_j(\gamma(t))$, $\|\sum |a_j|^2\|_{L^\infty([0,1])} < 1$. For $\sigma > 0$, let $\gamma_\sigma : [0,1] \to M$ be functions such that $\gamma_\sigma\big|_{(0,1)} \in C^\infty$, $\gamma_\sigma \xrightarrow{\sigma\to 0} \gamma$ in $C([0,1])$, and $\gamma_\sigma'(t) = \sum b_j^\sigma(t)(\delta+\sigma)^{d_j} W_j(\gamma_\sigma(t))$ with $\left\|\sum |b_j^\sigma|^2\right\|_{L^\infty} < 1$–this can be achieved by simple argument using mollifiers and the fact that $W_1, \ldots, W_{2m+q}$ are smooth and span the tangent space at every point. Set $x_\sigma := \gamma_\sigma(\sigma)$, $y_\sigma := \gamma_\sigma(1-\sigma)$, so that $\lim_{\sigma\downarrow 0} x_\sigma = x$ and $\lim_{\sigma\downarrow 0} y_\sigma = y$. Using the function $f_\sigma : B_{\mathbb{R}}(1/2) \to M$ given by $f_\sigma(t) := \gamma_\sigma(t+1/2)$, it follows from the definition of ρ_F that $\rho_F(x_\sigma, y_\sigma) < \delta + \sigma$. Thus, we have

$$\rho_F(x,y) \leq \rho_F(x,x_\sigma) + \rho_F(x_\sigma,y_\sigma) + \rho_F(y_\sigma,y) < \delta + \sigma + \rho_F(x,x_\sigma) + \rho_F(y_\sigma,y) \xrightarrow{\sigma\to 0} \delta,$$

where in the last step we have used Lemma 10.3. We conclude $\rho_F(x,y) \leq \rho_S(x,y)$.

Next, we show $\rho_S \leq \rho_F$. Suppose $\rho_F(x,y) < \delta$ and let $f_1, \ldots, f_K, \delta_1, \ldots, \delta_K$ be as in the definition of ρ_F. For $w_1, w_2 \in f_j(B_{\mathbb{R}}(1/2))$, we will show $\rho_S(w_1,w_2) < \delta_j$. Notice, this will complete the proof since we may find $\xi_1, \ldots, \xi_{L+1}$ with $\xi_j, \xi_{j+1} \in f_j(B_{\mathbb{R}}(1/2))$, $x = \xi_1$, $y = \xi_{L+1}$, and so using the triangle inequality for ρ_S, we have

$$\rho_S(x,y) \leq \sum_{j=1}^{L} \rho_S(\xi_j,\xi_{j+1}) \leq \sum_{j=1}^{L} \rho_F(\xi_j,\xi_{j+1}) < \sum_{j=1}^{L} \delta_j \leq \delta,$$

which will prove $\rho_S(x,y) \leq \rho_F(x,y)$.

Given $w_1, w_2 \in f_j(B_{\mathbb{R}}(1/2))$, we have $w_1 = f_j(t_1)$, $w_2 = f_j(t_2)$ for some $t_1, t_2 \in B_{\mathbb{R}}(1/2)$. Set $\gamma(r) := f_j((1-r)t_1 + rt_2)$. Then,

$$\gamma'(r) = f_j'((1-r)t_1 + rt_2)(t_1 - t_2) = \sum_{l=1}^{m+q} (t_1 - t_2) s_j^l(t) \delta_j^{d_l} W_l(f_j(t)),$$

with $\|\sum_l |s_j^l|^2\|_{L^\infty} < 1$. Since $|t_1 - t_2| < 1$, it follows from the definition of ρ_S that $\rho_S(w_1,w_2) < \delta_j$, completing the proof of $\rho_S \leq \rho_F$.

Finally, we show $\rho_F \leq \rho_H$. Suppose $\rho_H(x,y) < \delta$. Take $\delta_1, \ldots, \delta_K$ and $f_1, \ldots, f_K$ as in the definition of ρ_H. We will show that if $w_1, w_2 \in f_j(B_{\mathbb{C}\times\mathbb{R}}(1/2))$, then $\rho_F(w_1,w_2) < \delta_j$. The result will then follow from the triangle inequality, just as in the proof of $\rho_S \leq \rho_F$.

[23] Another way to prove Lemma 10.4 is as follows. We see below in the proof of Theorem 7.6 (a) that $\rho_S \leq \rho_F$–and the proof of this inequality does not use Lemma 10.4. Thus the metric topology induced by ρ_F is finer than the metric topology induced by ρ_S. That the metric topology induced by ρ_S is finer than the usual topology follows from [27, Lemma A.1]. Alternatively, one can easily adapt the proof of [27, Lemma A.1] to directly prove that the metric topology induced by ρ_F is finer than the usual topology.

Let $w_1 = f_j(\xi_1)$ and $w_2 = f_j(\xi_2)$ with $\xi_1, \xi_2 \in B_{\mathbb{R}\times\mathbb{C}}(1/2)$. Fix $\epsilon > 0$ small (depending on ξ_1, ξ_2) and set $\eta(r) := (\frac{1}{2} - (1+\epsilon)r)\xi_1 + (\frac{1}{2} + (1+\epsilon)r)\xi_2$. Note (if $\epsilon > 0$ is small enough), $\eta : B_{\mathbb{R}}(1/2) \to B_{\mathbb{R}\times\mathbb{C}}(1/2)$. Set $g(r) := f_j(\eta(r))$. Let $\xi_3 = (1+\epsilon)(\xi_2 - \xi_1)$, and we henceforth think of ξ_3 as an element of $B_{\mathbb{R}^3}(1)$, by identifying $\mathbb{R}\times\mathbb{C}$ with $\mathbb{R}^3$. We have

$$g'(r) = df_j(\eta(r))\eta'(r) = df_j(\eta(r))\xi_3.$$

Let $\widehat{S}_j(t, x_1, x_2)$ be the matrix from Remark 7.5. We have

$$g'(r) = \sum_{l=1}^{2m+q} (\widehat{S}_j(t, x_1, x_2)\xi_3)_l \delta_j^{d_l} W_l(g(r)),$$

where $(\widehat{S}_j(t, z)\xi_3)_l$ denotes the l-th component of the vector $\widehat{S}_j(t, z)\xi_3$. Since $|\xi_3| < 1$ and using (7.4), we have

$$\left\| \sum_l \left| (\widehat{S}_j(\cdot)\xi_3)_l \right|^2 \right\|_{L^\infty} < 1.$$

Since $g(-1/(2(1+\epsilon))) = w_1$ and $g(1/(2(1+\epsilon)) = w_2$, it follows that $\rho_F(w_1, w_2) < \delta_j$, as desired. □

Completion of the proof of Theorem 7.6. We will prove the theorem by applying Theorems 4.5 and 4.10 and Corollary 4.11 to $\delta^\beta X, \delta^\beta L$, as the base point x_0 ranges over $\mathcal{K}$ and as δ ranges over $(0, 1]$ (where $\delta^\beta X$ and $\delta^\beta L$ are defined in Section 7.2). Thus, our first goal is to show that the hypotheses of these results are satisfied uniformly for $x_0 \in \mathcal{K}$ and $\delta \in (0, 1]$; so that any type of admissible constant in those results can be chosen independently of $x_0 \in \mathcal{K}$ and $\delta \in (0, 1]$. For notational simplicity, we turn to calling the base point x instead of x_0.

For $\delta \in (0, 1]$, we multiply both sides of (7.2) by $\delta^{\beta_j+\beta_k}$ to see

$$[\delta^{\beta_j} Z_j, \delta^{\beta_k} Z_k] = \sum_{\beta_l \le \beta_j+\beta_k} (\delta^{\beta_j+\beta_k-\beta_l} c_{j,k}^{1,l})\delta^{\beta_l} Z_l,$$

$$[\delta^{\beta_j} Z_j, \delta^{\beta_k} \overline{Z_k}] = \sum_{\beta_l \le \beta_j+\beta_k} (\delta^{\beta_j+\beta_k-\beta_l} c_{j,k}^{2,l})\delta^{\beta_l} Z_l + \sum_{\beta_l \le \beta_j+\beta_k} (\delta^{\beta_j+\beta_k-\beta_l} c_{j,k}^{3,l})\delta^{\beta_l} \overline{Z_l}.$$

Setting $Z_j^\delta := \delta^{\beta_j} Z_j$ and

$$c_{j,k}^{a,l,\delta} := \begin{cases} \delta^{\beta_j+\beta_k-\beta_l} c_{j,k}^{a,l} & \text{if } \beta_l \le \beta_j + \beta_k \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$[Z_j^\delta, Z_k^\delta] = \sum_l c_{j,k}^{1,\delta} Z_l^\delta, \quad [Z_j^\delta, \overline{Z_k^\delta}] = \sum_l c_{j,k}^{2,\delta} Z_l^\delta + \sum_l c_{j,k}^{3,\delta} \overline{Z_l^\delta}.$$

With this notation, $\delta^\beta X, \delta^\beta L$ is the same as the list $Z_1^\delta, \ldots, Z_{m+q}^\delta$.

For $\delta \in (0,1]$, $c_{j,k}^{a,l,\delta} \in C^\infty$ and $Z_l^\delta \in C^\infty$, *uniformly in* δ. Thus if $\Omega \Subset M$ is a relatively compact open set with $\mathcal{K} \subseteq \Omega$, we have, directly from the definitions,

$$\|c_{j,k}^{a,l,\delta}\|_{C^p_{\delta^\beta X, \delta^\beta L}(\Omega)} \lesssim 1, \quad \forall j,k,l,a, \quad \forall p \in \mathbb{N},$$

where the implicit constant may depend on p, but does not depend on $\delta \in (0,1]$. It follows from Lemma 9.1 (ii) and (iii) that

$$\|c_{j,k}^{a,l,\delta}\|_{\mathscr{C}^s_{\delta^\beta X, \delta^\beta L}(\Omega)} \lesssim 1, \quad \forall j,k,l,a, \quad s > 0, \delta \in (0,1], \tag{10.9}$$

where the implicit constant may depend on s, but does not depend on $\delta \in (0,1]$. We take $\xi \in (0,1]$ so small $B_{X,L}(x,\xi) \subseteq \Omega$, $\forall x \in \mathcal{K}$; as a consequence, $B_{\delta^\beta X, \delta^\beta L}(x,\xi) \subseteq B_{X,L}(x,\xi) \subseteq \Omega$, $\forall x \in \mathcal{K}$, $\delta \in (0,1]$. By Lemma 9.1 (v) and (10.9) we have

$$\left\| c_{j,k}^{a,l,\delta} \right\|_{\mathscr{C}^s_{\delta^\beta X, \delta^\beta L}(B_{\delta^\beta X, \delta^\beta L}(x,\xi))} \lesssim 1, \quad \forall j,k,l,a, \quad s > 0, \delta \in (0,1], x \in \mathcal{K},$$

where the implicit constant does not depend on $\delta \in (0,1]$ or $x \in \mathcal{K}$. We also have $\mathcal{L}_{Z_j^\delta} \nu = f_j^\delta \nu$ where $f_j^\delta \in C^\infty$ uniformly for $\delta \in (0,1]$ (this follows directly from the definitions and the fact that ν is a strictly positive, C^∞ density). Similar to the above discussion, we have

$$\|f_j^\delta\|_{\mathscr{C}^s_{\delta^\beta X, \delta^\beta L}(B_{\delta^\beta X, \delta^\beta L}(x,\xi))} \lesssim 1, \quad \forall j, \quad s > 0, \delta \in (0,1], x \in \mathcal{K}.$$

The existence of $\eta > 0$ and $\delta_0 > 0$ (independent of $x \in \mathcal{K}$ and $\delta \in (0,1]$) as in the hypotheses of Theorem 4.5 (when applied to $\delta^\beta X, \delta^\beta L$ at the base point x) follows from Lemma 4.13; indeed Lemma 4.13 directly gives the existence of these constants for $x \in \mathcal{K}$ when $\delta = 1$ and it is immediate from the definitions of η and δ_0 that the same constants may be used $\forall \delta \in (0,1]$. The existence of $J_0 = J_0(x,\delta) \in \mathcal{I}(r,q)$, $K_0 = K_0(x,\delta) \in \mathcal{I}(n,m)$, and $\zeta \in (0,1]$ (independent of $x \in \mathcal{K}$, $\delta \in (0,1]$) as in Theorem 4.5 (when applied to $\delta^\beta X, \delta^\beta L$ at the base point x) follows from the hypothesis (7.3).

Thus, Theorems 4.5 and 4.10 and Corollary 4.11 apply (with, e.g., $s_0 = 3/2$–the choice of $s_0 \in (1,\infty)$ is irrelevant for what follows), uniformly for $x \in \mathcal{K}$, $\delta \in (0,1]$. In particular, any positive $\{s\}$-admissible constant from those results (for any $s > 0$) can be chosen independent of $x \in \mathcal{K}$, $\delta \in (0,1]$ (and is therefore ≈ 1 in the sense of this theorem); and similarly for any other kind of admissible constant. We let $\xi_2 \approx 1$ ($0 < \xi_2 \leq \xi \leq 1$) and $K \approx 1$ be the constants of the same name from Theorem 4.5, and let $\Phi_{x,\delta} : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to B_{\delta^\beta X, \delta^\beta L}(x,\xi)$ be the map guaranteed by Theorem 4.5 when applied to $\delta^\beta X, \delta^\beta L$ at the base point $x \in \mathcal{K}$.

We turn to proving (k). By Theorem 4.5 (vi) we have

$$B_{\delta^\beta X,\delta^\beta L}(x,\xi_2)\subseteq \Phi_{x,\delta}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))\subseteq B_{\delta^\beta X,\delta^\beta L}(x,\xi)\subseteq B_{\delta^\beta X,\delta^\beta L}(x,1)=B_S(x,\delta).$$

We set $\epsilon=\xi_2$, and the proof of (k) will be complete once we show

$$B_S(x,\xi_2\delta)\subseteq B_{\delta^\beta X,\delta^\beta L}(x,\xi_2). \tag{10.10}$$

Take $y\in B_S(x,\xi_2\delta)$. Thus, $\exists\gamma:[0,1]\to M$, $\gamma(0)=x$, $\gamma(1)=y$, $\gamma'(t)=\sum a_j(t)\xi_2^{d_j}\delta^{d_j}W_j(\gamma(t))$, with $\|\sum|a_j|^2\|_{L^\infty([0,1])}<1$. Hence,

$$\gamma'(t)=\sum_j (a_j(t)\xi_2^{d_j-1})\xi_2\delta^{d_j}W_j(\gamma(t)),\quad \left\|\sum\left|a_j\xi_2^{d_j-1}\right|^2\right\|_{L^\infty}<1.$$

It follows that $y=\gamma(1)\in B_{\delta^\beta X,\delta^\beta L}(x,\xi_2)$, completing the proof of (k).

(g) follows from Theorem 4.5 (xi). (h) follows from Theorem 4.5 (ix) using the fact that if $\mathcal{A}$ is as in that result, $\|\mathcal{A}(t,z)\|_{\mathbb{M}^{(n+r)\times(n+r)}}\leq\frac{1}{4}$, $\forall t,z$ by Theorem 4.5 (x) and therefore $I+\mathcal{A}(t,z)$ is invertible with $\|(I+\mathcal{A}(t,z))^{-1}\|_{\mathbb{M}^{(n+r)\times(n+r)}}\leq\frac{4}{3}$, $\forall t,z$.

Since $\|\cdot\|_{C^k}\leq\|\cdot\|_{\mathscr{C}^{k+1}}$, $\forall k\in\mathbb{N}$, by definition, (i) follows from Theorem 4.5 (xii). Similarly, (f) follows from Theorem 4.10 (i) and (ii).

(e) follows from Theorem 4.5 (iv) and (v); except that (v) only guarantees $\Phi_{x,\delta}$ is a C^2 diffeomorphism. That $\Phi_{x,\delta}$ is C^∞ follows by combining (i) and Lemma 10.1.

Next, we prove (j). Let $y\in\Phi_{x,\delta}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))$. We will show $y\in B_H(x,R\delta)$ for some $R\approx 1$ to be chosen later. By (h) and (i) we may write

$$\frac{\partial}{\partial t_k}=\sum_{l=1}^{m+q}a^l_{k,x,\delta}\widehat{Z}^{x,\delta}_l,\quad \frac{\partial}{\partial\overline{z}_j}=\sum_{l=1}^{m+q}b^l_{j,x,\delta}\widehat{Z}^{x,\delta}_l,$$

where,

$$\|a^l_{k,x,\delta}\|_{C^p(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))},\|b^l_{j,x,\delta}\|_{C^p(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}\lesssim 1,\quad\forall p\in\mathbb{N}.$$

We have

$$\begin{aligned}d\Phi_{x,\delta}(t,z)\frac{\partial}{\partial t_k}&=\sum_{l=1}^{m+q}a^l_{k,x,\delta}(t,z)Z^\delta_l(\Phi_{x,\delta}(t,z)),\\ d\Phi_{x,\delta}(t,z)\frac{\partial}{\partial\overline{z}_j}&=\sum_{l=1}^{m+q}b^l_{j,x,\delta}(t,z)Z^\delta_l(\Phi_{x,\delta}(t,z)).\end{aligned}$$

Let $y=\Phi_{x,\delta}(t_0,z_0)$ for some $(t_0,z_0)\in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$. Define

$$f(s,w):=\Phi_{x,\delta}\left(2s\frac{t_0}{|t_0|},2w\frac{z_0}{|z_0|}\right)$$

so that $f : B_{\mathbb{R}\times\mathbb{C}}(1/2) \to M$, $f(0,0) = x$ and $y \in f(B_{\mathbb{R}\times\mathbb{C}}(1/2))$. We have

$$df(s,w)\frac{\partial}{\partial s} = \sum_{l=1}^{m+q} \tilde{a}_l(s,w) Z_l^\delta(f(s,w)), \quad df(s,w)\frac{\partial}{\partial \overline{w}} = \sum_{l=1}^{m+q} \tilde{b}_l(s,w) Z_l^\delta(f(s,w)),$$

where

$$\tilde{a}_l(s,w) := \sum_{k=1}^{r} a_{k,x,\delta}^l \left(2s\frac{t_0}{|t_0|}, 2w\frac{z_0}{|z_0|}\right)\left(2\frac{t_0}{|t_0|}\right)_k,$$

where $\left(2\frac{t_0}{|t_0|}\right)_k$ denotes the kth component of $2\frac{t_0}{|t_0|}$; and $\tilde{b}_l$ is defined similarly. In particular

$$\|\tilde{a}_l\|_{L^\infty(B_{\mathbb{R}\times\mathbb{C}}(1/2))}, \|\tilde{b}_l\|_{L^\infty(B_{\mathbb{R}\times\mathbb{C}}(1/2))} \lesssim 1.$$

For $R \geq 1$ set $\tilde{a}_l^R := \tilde{a}_l/(R^{\beta_l})$, so that we have

$$df(s,w)\frac{\partial}{\partial s} = \sum_{l=1}^{m+q} \tilde{a}_l^R(s,w) Z_l^{R\delta}(f(s,w)), \quad df(s,w)\frac{\partial}{\partial \overline{w}} = \sum_{l=1}^{m+q} \tilde{b}_l^R(s,w) Z_l^{R\delta}(f(s,w)).$$

By taking R to be a sufficiently large admissible constant, we see that f satisfies the hypotheses of the definition of ρ_H with $K = 1$ (i.e., we are using $f_1 = f$ and $\delta_1 = R\delta$). This proves $y \in f(B_{\mathbb{R}\times\mathbb{C}}(1/2)) \subseteq B_H(x, R\delta)$, completing the proof of (j).

We turn to (b). Because $\mathcal{K}$ is compact with respect to the usual topology on M, ρ_F induces the usual topology on M (Lemma 10.4), and $\rho_F = \rho_S$, it follows from Lemma 10.4 that $\mathcal{K}$ is compact with respect to the metric topology induced by ρ_S. A simple compactness argument shows that to prove (b), it suffices to show that there exists $\epsilon' > 0$ such that if $\rho_S(x,y) < \epsilon'$, $x, y \in \mathcal{K}$, then $\rho_H(x,y) \lesssim \rho_S(x,y)$. We take $\epsilon' = \epsilon$, where $\epsilon > 0$ is from (k). If $\rho_S(x,y) < \epsilon\delta$ (for some $\delta \in (0,1]$), we have (by (k) and (j)) $y \in B_S(x, \epsilon\delta) \subseteq \Phi_{x,\delta}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1)) \subseteq B_H(x, R\delta)$. Hence $\rho_H(x,y) \leq R\delta$. We conclude that if $\rho_S(x,y) < \epsilon$ with $x, y \in \mathcal{K}$, then $\rho_H(x,y) \leq \frac{R}{\epsilon}\rho_S(x,y)$. This completes the proof of (b).

Next we prove (c). Corollary 4.11 shows

$$\nu(B_{\delta^\beta X, \delta^\beta L}(x, \xi_2)) \approx \Lambda(x,\delta) \approx \Lambda(x, \epsilon\delta), \tag{10.11}$$

where in the second $\approx$, we have used the formula for Λ and the fact that $\epsilon \approx 1$. Using this, (10.10), and the fact that we chose $\epsilon = \xi_2$, we have

$$\nu(B_S(x, \epsilon\delta)) \leq \nu(B_{\delta^\beta X, \delta^\beta L}(x, \xi_2)) \lesssim \Lambda(x, \epsilon\delta). \tag{10.12}$$

Conversely, again using (10.11), we have

$$\Lambda(x,\delta) \lesssim \nu(B_{\delta^\beta X,\delta^\beta L}(x,\xi_2)) \leq \nu(B_{\delta^\beta X,\delta^\beta L}(x,1)) = \nu(B_S(x,\delta)). \tag{10.13}$$

Since (10.12) and (10.13) hold $\forall \delta \in (0,1]$, it follows that $\nu(B_S(x,\delta)) \approx \Lambda(x,\delta)$, $\forall \delta \in (0,\epsilon]$. By (a) we have (for $\delta \in (0,\epsilon]$),

$$\nu(B_H(x,\delta)) \leq \nu(B_S(x,\delta)) \approx \Lambda(x,\delta). \tag{10.14}$$

By (j) and (k), we have (for $\delta \in (0,1]$)

$$\Lambda(x,R\delta) \approx \Lambda(x,\epsilon\delta) \approx \nu(B_S(x,\epsilon\delta)) \leq \nu(B_H(x,R\delta)), \tag{10.15}$$

where in the first $\approx$, we have used $R,\epsilon \approx 1$ and the formula for Λ. Combining (10.14) and (10.15), we have for $\delta \in (0, \min\{\epsilon, 1/R\}]$,

$$\nu(B_H(x,\delta)) \approx \Lambda(x,\delta).$$

This completes the proof of (c). (d) is a consequence of (c) and the formula for Λ. □

11. Nirenberg's theorem for elliptic structures

In this section, we present the main technical result from [33]. This can be seen as a sharp (in terms of regularity) version of Nirenberg's theorem that formally integrable elliptic structures are integrable [23]. Here, unlike the setting of Theorem 4.5, we assume the vector fields already have the desired regularity, and that we have good estimates on the coefficients in a given coordinate system. The goal is to pick a new coordinate system in which the vector fields are spanned by $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}$, while maintaining the regularity of the vector fields.

Fix $s_0 \in (0,\infty) \cup \{\omega\}$ and let $X_1, \ldots, X_r, L_1, \ldots, L_n$ be complex vector fields on $B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$ with:

- If $s_0 \in (0,\infty)$, $X_k, L_j \in \mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1); \mathbb{C}^{r+2n})$.
- If $s_0 = \omega$, $X_k, L_j \in \mathscr{A}^{r+2n,1}(\mathbb{C}^{r+2n})$.

We suppose:

- $X_k(0) = \frac{\partial}{\partial t_k}$, $L_j(0) = \frac{\partial}{\partial \bar{z}_j}$.
- $\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$, $[X_{k_1}, X_{k_2}](\zeta), [X_k, L_j](\zeta), [L_{j_1}, L_{j_2}](\zeta) \in \operatorname{span}_{\mathbb{C}}\{X_1(\zeta), \ldots, X_r(\zeta), L_1(\zeta), \ldots, L_n(\zeta)\}$.

Under these hypotheses, Nirenberg's theorem[24] implies that there exists a map $\Phi_1 : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$, with $\Phi_1(0) = 0$, Φ_1 is a diffeomorphism onto its image (which

[24] Originally, Nirenberg considered only the case of C^∞ vector fields and worked in the case when $X_1, \ldots, X_r$ were real.

is an open neighborhood of $0 \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$), and such that $\Phi_1^* X_k(u,w), \Phi_1^* L_j(u,w) \in \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_r},\frac{\partial}{\partial \overline{w}_1},\ldots,\frac{\partial}{\partial \overline{w}_r}\right\}$, $\forall (u,w)$ (here we are giving the domain space $\mathbb{R}^r \times \mathbb{C}^n$ coordinates (u,w)). In [33] this is improved to a quantitative version which gives Φ_1 the optimal regularity (namely, when $s_0 \in (0,\infty)$, Φ_1 is in $\mathscr{C}^{s_0+2}$, and when $s_0 = \omega$, Φ_1 is real analytic). Unlike the results in the rest of this paper, the results in this section are not quantitatively diffeomorphically invariant: the estimates depend on the particular coordinate system we are using (the standard coordinate system on $\mathbb{R}^r \times \mathbb{C}^n$).

Definition 11.1. If $s_0 \in (0,\infty)$, for $s \geq s_0$ if we say C is an $\{s\}$-admissible constant, it means that we assume $X_k, L_j \in \mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{C}^{r+2n})$, $\forall j,k$. C can then be chosen to depend only on n, r, s, s_0, and upper bounds for $\|X_k\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}$ and $\|L_j\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}$, $1 \leq k \leq r$, $1 \leq j \leq n$. For $s \leq s_0$, we define $\{s\}$-admissible constants to be $\{s_0\}$-admissible constants.

Remark 11.2. In Definition 11.1 we have defined admissible constants differently than they were defined in Definitions 4.2, 4.3, and 4.4. This reuse of notation is justified when we turn to the proof of the main theorem (Theorem 4.5). Indeed, when we apply Theorem 11.4 in the proof Theorem 4.5, we apply it to a choice of vector fields in such a way that constants which are admissible in the sense of Theorem 11.4 are admissible in the sense of Theorem 4.5. Thus, *in the particular application* of Theorem 11.4 used to prove Theorem 4.5, the definitions of admissible constants do coincide.

Definition 11.3. If $s_0 = \omega$, we say C is an $\{\omega\}$-admissible constant if C can be chosen to depend only on n, r, and upper bounds for $\|X_k\|_{\mathscr{A}^{2n+r,1}}$, $\|L_j\|_{\mathscr{A}^{2n+r,1}}$, $1 \leq k \leq r$, $1 \leq j \leq n$.

Theorem 11.4. *There exists an $\{s_0\}$-admissible constant $K_1 \geq 1$ and a map $\Phi_1 : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$ such that*

(i) • *If $s_0 \in (0,\infty)$, $\Phi_1 \in \mathscr{C}^{s_0+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{R}^r\times\mathbb{C}^n)$ and $\|\Phi_1\|_{\mathscr{C}^{s+2}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_{\{s\}} 1$, $\forall s > 0$.*
• *If $s_0 = \omega$, $\Phi_1 \in \mathscr{A}^{2n+r,2}(\mathbb{R}^r\times\mathbb{C}^n)$ and $\|\Phi_1\|_{\mathscr{A}^{2n+r,2}} \leq 1$.*

(ii) *$\Phi_1(0) = 0$ and $d_{(t,x)}\Phi_1(0) = K_1^{-1} I_{(r+2n)\times(r+2n)}$. See Section 5 for the notation $d_{(t,x)}$.*

(iii) *$\forall \zeta \in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$, $\det d_{(t,x)}\Phi_1(\zeta) \approx_{\{s_0\}} 1$.*

(iv) *$\Phi_1(B_{\mathbb{R}^r\times\mathbb{C}^n}(1)) \subseteq B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$ is an open set and $\Phi_1 : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to \Phi_1(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))$ is a diffeomorphism.*[25]

[25] By diffeomorphism we mean that $\Phi_1 : B_{\mathbb{R}^r\times\mathbb{C}^n}(1) \to \Phi_1(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))$ is a bijection and $d\Phi_1$ is everywhere nonsingular.

(v)

$$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = K_1^{-1}(I+\mathcal{A}) \begin{bmatrix} \Phi_1^* X \\ \Phi_1^* L \end{bmatrix},$$

where $\mathcal{A} : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to \mathbb{M}^{(n+r)\times(n+r)}(\mathbb{C})$, $\mathcal{A}(0) = 0$ *and*

- *If* $s_0 \in (0,\infty)$, $\|\mathcal{A}\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{M}^{(n+r)\times(n+r)})} \lesssim_{\{s\}} 1$, $\forall s > 0$ *and*

$$\|\mathcal{A}\|_{\mathscr{C}^{s_0+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{M}^{(n+r)\times(n+r)})} \leq \frac{1}{4}.$$

- *If* $s_0 = \omega$, $\|\mathcal{A}\|_{\mathscr{A}^{2n+r,1}(\mathbb{M}^{(n+r)\times(n+r)})} \leq \frac{1}{4}$.

In either case, note that this implies $(I+\mathcal{A})$ *is an invertible matrix on* $B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$.

(vi) Suppose Z *is another complex vector field on* $B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$. *Then,*

- *If* $s_0 \in (0,\infty)$, $\|\Phi_1^* Z\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))} \lesssim_{\{s\}} \|Z\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}$, $\forall s > 0$.
- *If* $s_0 = \omega$, $\|\Phi_1^* Z\|_{\mathscr{A}^{2n+r,1}} \lesssim_{\{\omega\}} \|Z\|_{\mathscr{A}^{2n+r,1}}$.

Proof. This is [33, Theorem 7.3]. □

12. The real case

The case when $m = 0$ of Theorem 4.5 (i.e., when there are no complex vector fields), was the subject of the series [27,31,32]. In this section, we present a simplified version of this for use in proving Theorem 4.5.

Let $W_1, \ldots, W_Q$ be C^1 real vector fields on a C^2 manifold $\mathfrak{M}$. Fix $x_0 \in \mathfrak{M}$ and let $N := \dim \operatorname{span}_{\mathbb{R}}\{W_1(x_0), \ldots, W_Q(x_0)\}$. Fix $\xi, \zeta \in (0,1]$. We assume that on $B_W(x_0,\xi)$, the W_j satisfy

$$[W_j, W_k] = \sum_{l=1}^{Q} c_{j,k}^l W_l, \quad c_{j,k}^l \in C(B_W(x_0,\xi)),$$

where $B_W(x_0,\xi)$ is given the metric topology induced by the corresponding sub-Riemannian metric (2.3). Under the above hypotheses, $B_W(x_0,\xi)$ is a C^2, injectively immersed submanifold of $\mathfrak{M}$ of dimension N and $T_x B_W(x_0,\xi) = \operatorname{span}_{\mathbb{R}}\{W_1(x), \ldots, W_Q(x)\}$, $\forall x \in B_W(x_0,\xi)$ (see Proposition A.1). Henceforth we view $W_1, \ldots, W_Q$ as C^1 vector fields on $B_W(x_0,\xi)$.

Let $P_0 \in \mathcal{I}(N,Q)$ be such that $\bigwedge W_{P_0}(x_0) \neq 0$ and moreover

$$\max_{P\in\mathcal{I}(N,Q)} \left| \frac{\bigwedge W_P(x_0)}{\bigwedge W_{P_0}(x_0)} \right| \leq \zeta^{-1}.$$

Without loss of generality, reorder the vector fields so that $P_0 = (1, \ldots, N)$.

We take $\eta > 0$ and $\delta_0 > 0$ as in Theorem 4.5; i.e.,

- Fix $\eta > 0$ so that W_{P_0} satisfies $\mathcal{C}(x_0, \eta, \mathfrak{M})$.
- Fix $\delta_0 > 0$ such that $\forall \delta \in (0, \delta_0]$, the following holds. If $z \in B_{W_{P_0}}(x_0, \xi)$ is such that W_{P_0} satisfies $\mathcal{C}(z, \delta, B_{W_{P_0}}(x_0, \xi))$ and if $t \in B_{\mathbb{R}^{2n+r}}(\delta)$ is such that $e^{t_1 W_1 + \cdots + t_{2n+r} W_N} z = z$ and if $W_1(z), \ldots, W_N(z)$ are linearly independent, then $t = 0$.

Definition 12.1. We say C is a 0-admissible constant if C can be chosen to depend only on upper bounds for Q, ζ^{-1}, ξ^{-1}, and $\|c_{j,k}^l\|_{C(B_{W_{P_0}}(x_0,\xi))}$, $1 \leq j, k, l \leq Q$.

Fix $s_0 \in (1, \infty) \cup \{\omega\}$.

Definition 12.2. Suppose $s_0 \in (1, \infty)$. For $s \in [s_0, \infty)$ if we say C is an $\{s\}$-admissible constant it means that we assume $c_{j,k}^l \in \mathscr{C}_{W_{P_0}}^s(B_{W_{P_0}}(x_0, \xi))$. C is allowed to depend only on s, s_0, and upper bounds for ζ^{-1}, ξ^{-1}, η^{-1}, δ_0^{-1}, Q, and $\|c_{j,k}^l\|_{\mathscr{C}_{W_{P_0}}^s(B_{W_{P_0}}(x_0,\xi))}$, $1 \leq j, k, l \leq Q$. For $s \in (0, s_0)$, we define $\{s\}$-admissible constants to be $\{s_0\}$-admissible constants.

Definition 12.3. Suppose $s_0 = \omega$. If we say C is an $\{s_0\}$-admissible constant it means that we assume $c_{j,k}^l \in \mathscr{A}_{W_{P_0}}^{x_0,\eta}$. C is allowed to depend only on anything a 0-admissible constant may depend on, as well as upper bounds for η^{-1}, δ_0^{-1}, and $\|c_{j,k}^l\|_{\mathscr{A}_{W_{P_0}}^{x_0,\eta}}$, $1 \leq j, k, l \leq Q$.

Proposition 12.4. *There exists a 0-admissible constant $\chi \in (0, \xi]$ such that*

(i) $\forall y \in B_{W_{P_0}}(x_0, \chi)$, $\bigwedge W_{P_0}(y) \neq 0$.
(ii) $\forall y \in B_{W_{P_0}}(x_0, \chi)$,

$$\max_{P \in \mathcal{I}(N,Q)} \left| \frac{\bigwedge W_P(y)}{\bigwedge W_{P_0}(y)} \right| \approx_0 1.$$

(iii) $\forall \chi' \in (0, \chi]$, $B_{W_{P_0}}(x_0, \chi')$ *is an open subset of* $B_W(x_0, \chi)$, *and is therefore a submanifold.*

For the remainder of the proposition, we assume:

- *If* $s_0 \in (1, \infty)$, *we assume* $c_{j,k}^l \in \mathscr{C}_{W_{P_0}}^{s_0}(B_{W_{P_0}}(x_0, \xi))$.
- *If* $s_0 = \omega$, *we assume* $c_{j,k}^l \in \mathscr{A}_{W_{P_0}}^{x_0,\eta}$.

There exists a C^2 *map* $\Phi_0 : B_{\mathbb{R}^N}(1) \to B_{W_{P_0}}(x_0, \chi)$ *such that:*

(iv) $\Phi_0(B_{\mathbb{R}^N}(1))$ *is an open subset of* $B_{W_{P_0}}(x_0, \chi)$ *and is therefore a submanifold.*
(v) $\Phi_0(0) = x_0$.
(vi) $\Phi_0 : B_{\mathbb{R}^N}(1) \to \Phi_0(B_{\mathbb{R}^N}(1))$ *is a* C^2 *diffeomorphism.*
(vii) • *If* $s_0 \in (1, \infty)$, $\|\Phi_0^* W_j\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^N}(1);\mathbb{R}^N)} \lesssim_{\{s\}} 1$, $\forall s > 0$, $1 \leq j \leq Q$.

- *If* $s_0 = \omega$, $\|\Phi_0^* W_j\|_{\mathscr{A}^{N,1}(\mathbb{R}^N)} \lesssim_{\{\omega\}} 1$, $1 \leq j \leq Q$.

(viii) There exists an $\{s_0\}$*-admissible constant* $K_0 \geq 1$ *such that*

$$\Phi_0^* W_{P_0} = K_0(I + \mathcal{A}_0)\frac{\partial}{\partial t},$$

where $\mathcal{A}_0 : B_{\mathbb{R}^N}(1) \to \mathbb{M}^{N\times N}(\mathbb{R})$, $\mathcal{A}_0(0) = 0$, $\sup_{t\in B_{\mathbb{R}^N}(1)} \|\mathcal{A}_0(t)\|_{\mathbb{M}^{N\times N}} \leq \frac{1}{2}$, *and:*

- *If* $s_0 \in (1, \infty)$, $\|\mathcal{A}_0\|_{\mathscr{C}^s(B_{\mathbb{R}^N}(1);\mathbb{M}^{N\times N})} \lesssim_{\{s\}} 1$, $\forall s > 0$.
- *If* $s_0 = \omega$, $\|\mathcal{A}_0\|_{\mathscr{A}^{N,1}(\mathbb{M}^{N\times N})} \leq \frac{1}{2}$.

12.1. Densities

We take the same setting as Proposition 12.4, and let $\chi \in (0, \xi]$ be as in that proposition. Let ν be a real C^1 density on $B_{W_{P_0}}(x_0, \chi)$ and suppose for $1 \leq j \leq N$ (recall, we are assuming $P_0 = (1, \ldots, N)$),

$$\mathcal{L}_{W_j}\nu = f_j\nu, \quad f_j \in C(B_{W_{P_0}}(x_0, \chi)).$$

Definition 12.5. If we say C is a $[s_0; \nu]$-admissible constant, it means that C is a $\{s_0\}$-admissible constant, which is also allowed to depend on upper bounds for $\|f_j\|_{C(B_{W_{P_0}}(x_0,\chi))}$, $1 \leq j \leq N$. This definition holds in both cases: $s_0 \in (1, \infty)$ and $s_0 = \omega$.

Definition 12.6. If $s_0 \in (1, \infty)$, for $s > 0$ if we say C is an $\{s; \nu\}$-admissible constant it means that $f_j \in \mathscr{C}^s_{W_{P_0}}(B_{W_{P_0}}(x_0, \chi))$. C is then allowed to depend on anything an $\{s\}$-admissible constant may depend on, and is allowed to depend on upper bounds for $\|f_j\|_{\mathscr{C}^s_{W_{P_0}}(B_{W_{P_0}}(x_0,\chi))}$, $1 \leq j \leq N$. For $s \leq 0$, we define $\{s; \nu\}$-admissible constants to be $[s_0; \nu]$-admissible constants.

If $s_0 = \omega$, we fix some number $r_0 > 0$.

Definition 12.7. If $s_0 = \omega$ and if we say C is a $\{\omega; \nu\}$-admissible constant, it means that we assume $f_j \in \mathscr{A}^{x_0,r_0}_{W_{P_0}}$, $1 \leq j \leq N$. C is then allowed to depend on anything a $\{\omega\}$-admissible constant may depend on, and is allowed to depend on upper bounds for r_0^{-1} and $\|f_j\|_{\mathscr{A}^{x_0,r_0}_{W_{P_0}}}$, $1 \leq j \leq N$.

Proposition 12.8. *Define* $h_0 \in C^1(B_{\mathbb{R}^N}(1))$ *by* $\Phi_0^*\nu = h_0\sigma_{\mathrm{Leb}}$. *Then,*

(a) $h_0(t) \approx_{[s_0;\nu]} \nu(W_1, \ldots, W_N)(x_0)$, $\forall t \in B_{\mathbb{R}^N}(1)$.[26] *In particular,* $h_0(t)$ *always has the same sign, and is either never zero or always zero.*

[26] Recall, we are assuming without loss of generality that $P_0 = (1, \ldots, N)$.

(b) • *If* $s_0 \in (1,\infty)$, *for* $s>0$,

$$\|h_0\|_{\mathscr{C}^s(B_{\mathbb{R}^N}(1))} \lesssim_{\{s-1;\nu\}} |\nu(W_1,\ldots,W_N)(x_0)|.$$

• *If* $s_0=\omega$,

$$\|h_0\|_{\mathscr{A}^{N,\min\{1,r_0\}}} \lesssim_{\{\omega;\nu\}} |\nu(W_1,\ldots,W_N)(x_0)|.$$

12.2. Proofs

In this section, we discuss the proofs of Proposition 12.4 and Proposition 12.8. When $s_0 \in (1,\infty)$, Proposition 12.4 and Proposition 12.8 follow directly from the main results in [31], and so we focus on the case $s_0=\omega$.

The main results of [32] are very similar to Proposition 12.4 and Proposition 12.8 when $s_0=\omega$. (i), (ii), and (iii) of Proposition 12.4 are directly contained in [32]. The main result of [32] shows that there exists an $\{\omega\}$-admissible constant $\hat{\eta}\in(0,1]$ and a map

$$\widehat{\Phi}: B_{\mathbb{R}^N}(\hat{\eta}) \to B_{W_{P_0}}(x_0,\xi)$$

such that

- $\widehat{\Phi}(B_{\mathbb{R}^N}(\hat{\eta}))$ is an open subset of $B_{W_{P_0}}(x_0,\chi)$ and is therefore a submanifold of $B_W(x_0,\xi)$.
- $\widehat{\Phi}: B_{\mathbb{R}^N}(\hat{\eta}) \to \widehat{\Phi}(B_{\mathbb{R}^N}(\hat{\eta}))$ is a C^2 diffeomorphism, and $\widehat{\Phi}(0)=x_0$.
- $\|\widehat{\Phi}^*W_j\|_{\mathscr{A}^{N,\hat{\eta}}(\mathbb{R}^N)} \lesssim_{\{\omega\}} 1$, $1\le j\le Q$.
- $$\widehat{\Phi}^*W_{P_0} = (I+\widehat{\mathcal{A}})\frac{\partial}{\partial t},$$
  where $\widehat{\mathcal{A}}: B_{\mathbb{R}^n}(\hat{\eta}) \to \mathbb{M}^{N\times N}(\mathbb{R})$, $\widehat{\mathcal{A}}(0)=0$, and $\|\widehat{\mathcal{A}}\|_{\mathscr{A}^{N,\hat{\eta}}} \le \frac{1}{2}$.

Define $\Psi: B_{\mathbb{R}^N}(1)\to B_{\mathbb{R}^N}(\hat{\eta})$ by $\Psi(t)=\hat{\eta}t$, and set $\Phi_0 := \widehat{\Phi}\circ\Psi$. The remainder of Proposition 12.4 follows from the above properties of $\widehat{\Phi}$, with $K_0 := \hat{\eta}^{-1}$ and $\mathcal{A}_0 := \widehat{\mathcal{A}}\circ\Psi$.

Now let ν be a real C^1 density on $B_{W_{P_0}}(x_0,\chi)$ as in Proposition 12.8. The main result on densities in [32] shows that if $\hat{h}\in C^1(B_{\mathbb{R}^n}(\hat{\eta}))$ is defined by $\widehat{\Phi}^*\nu = \hat{h}\sigma_{\mathrm{Leb}}$, then

- $\hat{h}(t) \approx_{[\omega;\nu]} \nu(W_1,\ldots,W_N)(x_0)$, $\forall t\in B_{\mathbb{R}^n}(\hat{\eta})$.
- $\hat{h}\in\mathscr{A}^{N,\min\{\hat{\eta},r_0\}}$ and $\|\hat{h}\|_{\mathscr{A}^{N,\min\{\hat{\eta},r_0\}}} \lesssim_{\{\omega;\nu\}} 1$.

Note that $h_0 = (\hat{h}\circ\Psi)\det d\Psi = \hat{\eta}^N\hat{h}\circ\Psi$. Since $\hat{\eta}\approx_{\{\omega\}} 1$, Proposition 12.8 follows from the above estimates on $\hat{h}$.

13. Proofs of the main results

In this section, we prove Theorem 4.5, Theorem 4.10, and Corollary 4.11.

Note that, by the definitions, $B_{W_{P_0}}(x_0,\xi)=B_{X_{K_0},L_{J_0}}(x_0,\xi)$, $\mathscr{C}^s_{W_{P_0}}(U)=\mathscr{C}^s_{X_{K_0},L_{J_0}}(U)$, and $\mathscr{A}^{r,\eta}_{W_{P_0}}=\mathscr{A}^{r,\eta}_{X_{K_0},L_{J_0}}$ (with equality of norms). It follows from the hypotheses that we may write (for $1\le j,k\le 2m+q$)

$$[W_j,W_k]=\sum_{l=1}^{2m+q}\tilde{c}^l_{j,k}W_l,\quad \tilde{c}^l_{j,k}\in C(B_{W_{P_0}}(x_0,\xi)),$$

$\|\tilde{c}^l_{j,k}\|_{C(B_{W_{P_0}}(x_0,\xi))}\lesssim_0 1$, and

- If $s_0\in(1,\infty)$, $\|\tilde{c}^l_{j,k}\|_{\mathscr{C}^s_{W_{P_0}}(B_{W_{P_0}}(x_0,\chi))}\lesssim_{\{s\}}1$, $\forall s>0$.
- If $s_0=\omega$, $\|\tilde{c}^l_{j,k}\|_{\mathscr{A}^{x_0,\eta}_{W_{P_0}}}\lesssim_{\{\omega\}}1$.

Recall,

$$W_{P_0}=W_1,\dots,W_{2n+r}=X_1,\dots,X_r,2\mathrm{Re}(L_1),\dots,2\mathrm{Re}(L_n),2\mathrm{Im}(L_1),\dots,2\mathrm{Im}(L_n). \tag{13.1}$$

Combining (4.2) with Proposition B.5 (i), we see

$$\max_{P\in\mathcal{I}(2n+r,2m+q)}\left|\frac{\bigwedge W_P(x_0)}{\bigwedge W_{P_0}(x_0)}\right|\le\left(2\zeta^{-1}\sqrt{2n+r}\right)^{2n+r}\lesssim_0 1. \tag{13.2}$$

In light of these remarks, and the definition of η and δ_0, Proposition 12.4 applies to the vector fields $W_1,\dots,W_{2m+q}$ (with $N=2n+r$) and any constant which is $*$-admissible in the sense of Proposition 12.4 is $*$-admissible in the sense of this section (where $*$ is any symbol).

We take the 0-admissible constant $\chi\in(0,\xi]$ from Proposition 12.4. By Proposition 12.4 (i) and (ii), $\forall y\in B_{W_{P_0}}(x_0,\chi)$, $\bigwedge W_{P_0}(y)\neq 0$ and

$$\max_{P\in\mathcal{I}(N,Q)}\left|\frac{\bigwedge W_P(y)}{\bigwedge W_{P_0}(y)}\right|\approx_0 1. \tag{13.3}$$

By hypothesis, $\dim\mathscr{L}_y=\dim\mathscr{L}_{x_0}=2n+r$ and $\dim\mathscr{X}_y=\dim\mathscr{X}_{x_0}=r$, $\forall y\in B_{W_{P_0}}(x_0,\chi)\subseteq B_{X,L}(x_0,\xi)$. Combining this with (13.3), Proposition B.5 (ii) implies $\forall y\in B_{W_{P_0}}(x_0,\chi)=B_{X_{K_0},L_{J_0}}(x_0,\chi)$,

$$\left(\bigwedge X_{K_0}(y)\right)\bigwedge\left(\bigwedge L_{J_0}(y)\right)\neq 0,$$

and moreover

$$\max_{\substack{J\in\mathcal{I}(n_1,m),K\in\mathcal{I}(r_1,q)\\ n_1+r_1=n+r}}\left|\frac{(\bigwedge X_K(y))\bigwedge(\bigwedge L_J(y))}{(\bigwedge X_{K_0}(y))\bigwedge(\bigwedge L_{J_0}(y))}\right|\lesssim_0 1. \tag{13.4}$$

Since the left hand side of (13.4) is ≥ 1, it follows that the left hand side of (13.4) is $\approx_0 1$. Theorem 4.5 (i) and (ii) follow. Theorem 4.5 (iii) follows from Proposition 12.4 (iii). Since $\dim \mathscr{L}_x = \dim \mathscr{L}_{x_0} = 2n+r$, $\forall x \in B_{X,L}(x_0,\xi)$, Theorem 4.5 (i) implies that $X_1(x),\ldots,X_r(x),L_1(x),\ldots,L_n(x)$ form a basis for $\mathscr{L}_x$, $\forall x\in B_{X_{K_0},L_{J_0}}(x_0,\chi)$. In particular, for $x\in B_{X_{K_0},L_{J_0}}(x_0,\chi)$, $1\leq k,k_1,k_2\leq q$, $1\leq j,j_1,j_2\leq m$,

$$\begin{aligned}&X_k(x),L_j(x),[X_{k_1},X_{k_2}](x),[L_{j_1},L_{j_2}](x),[X_k,L_j](x)\\&\in\mathscr{L}_x=\operatorname{span}_{\mathbb{C}}\{X_1(x),\ldots,X_r(x),L_1(x),\ldots,L_n(x)\}.\end{aligned} \tag{13.5}$$

Let $\Phi_0 : B_{\mathbb{R}^{r+2n}}(1)\rightarrow B_{X_{K_0},L_{J_0}}(x_0,\chi)$ be the map from Proposition 12.4.

- If $s_0\in(1,\infty)$, Proposition 12.4 (vii) gives $\|\Phi_0^*W_j\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^{2n+r}}(1))}\lesssim_{\{s\}}1$, $1\leq j\leq 2m+q$, and therefore

$$\|\Phi_0^*X_k\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^N}(1))},\|\Phi_0^*L_j\|_{\mathscr{C}^{s+1}(B_{\mathbb{R}^{2n+r}}(1))}\lesssim_{\{s\}}1,\quad 1\leq j\leq m,1\leq k\leq q. \tag{13.6}$$

- If $s_0=\omega$, Proposition 12.4 (vii) gives $\|\Phi_0^*W_j\|_{\mathscr{A}^{2n+r,1}}\lesssim_{\{\omega\}}1$, $1\leq j\leq 2m+q$, and therefore

$$\|\Phi_0^*X_k\|_{\mathscr{A}^{2n+r,1}},\|\Phi_0^*L_j\|_{\mathscr{A}^{2n+r,1}}\lesssim_{\{\omega\}}1,\quad 1\leq j\leq m,1\leq k\leq q. \tag{13.7}$$

We identify $\mathbb{R}^{r+2n}\cong\mathbb{R}^r\times\mathbb{C}^n$, via the map $(t_1,\ldots,t_r,x_1,\ldots,x_{2n})\mapsto(t_1,\ldots,t_r,x_1+ix_{n+1},\ldots,x_n+ix_{2n})$. Let $K_2\geq 1$ be the $\{s_0\}$-admissible constant called K_0 in Proposition 12.4. By Proposition 12.4 (viii) (and since $P_0=(1,\ldots,2n+r)$), we have

$$\Phi_0^*K_2^{-1}W_j(0)=\begin{cases}\frac{\partial}{\partial t_j} & 1\leq j\leq r,\\ \frac{\partial}{\partial x_{j-r}} & r+1\leq j\leq 2n.\end{cases}$$

Using this and (13.1) shows, for $1\leq k\leq r$, $1\leq j\leq n$,

$$\Phi_0^*K_2^{-1}X_k(0)=\frac{\partial}{\partial t_k},\quad \Phi_0^*K_2^{-1}L_j(0)=\frac{\partial}{\partial \bar{z}_j}.$$

Pulling (13.5) back via Φ_0 (and multiplying by K_2^{-2}), we have for $1\leq k,k_1,k_2\leq r$, $1\leq j,j_1,j_2\leq n$, $\zeta\in B_{\mathbb{R}^r\times\mathbb{C}^n}(1)$,

$$\begin{aligned}&\left[\Phi_0^*K_2^{-1}X_{k_1},\Phi_0^*K_2^{-1}X_{k_2}\right](\zeta),\left[\Phi_0^*K_2^{-1}L_{j_1},\Phi_0^*K_2^{-1}L_{j_2}\right](\zeta),\left[\Phi_0^*K_2^{-1}X_k,\Phi_0^*K_2^{-1}L_j\right](\zeta)\\&\in\operatorname{span}_{\mathbb{C}}\left\{\Phi_0^*K_2^{-1}X_1(\zeta),\ldots\Phi_0^*K_2^{-1}X_r(\zeta),\Phi_0^*K_2^{-1}L_1(\zeta),\ldots,\Phi_0^*K_2^{-1}L_n(\zeta)\right\}.\end{aligned}$$

The above remarks show that Theorem 11.4 applies to the vector fields

$$\Phi_0^* K_2^{-1} X_1, \ldots, \Phi_0^* K_2^{-1} X_r, \Phi_0^* K_2^{-1} L_1, \ldots, \Phi_0^* K_2^{-1} L_n,$$

and any constant which is $\{s\}$-admissible in the sense of Theorem 11.4 is $\{s\}$-admissible in the sense of this section. We let $K_1 \geq 1$ be the $\{s_0\}$-admissible constant from Theorem 11.4, and $\Phi_1 : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$ and $\mathcal{A} : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to \mathbb{M}^{(r+n)\times(r+n)}(\mathbb{C})$ be as in Theorem 11.4. Set $K = K_2 K_1$ and $\Phi = \Phi_0 \circ \Phi_1$. Note that $\Phi^* = \Phi_1^* \Phi_0^*$. Theorem 4.5 (iv) follows from Theorem 11.4 (iv) and Proposition 12.4 (iv) and (vi). Theorem 4.5 (v) follows from Theorem 11.4 (i) and (iv) and Proposition 12.4 (vi). Theorem 4.5 (vii) follows from Theorem 11.4 (ii) and Proposition 12.4 (v). Theorem 4.5 (viii) and (x) follow from Theorem 11.4 (v).

Using Theorem 11.4 (v) we have

$$\begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = K_1^{-1}(I + \mathcal{A}) \begin{bmatrix} \Phi_1^* \Phi_0^* K_2^{-1} X_{K_0} \\ \Phi_1^* \Phi_0^* K_2^{-1} L_{J_0} \end{bmatrix} = K^{-1}(I + \mathcal{A}) \begin{bmatrix} \Phi^* X_{K_0} \\ \Phi^* L_{J_0} \end{bmatrix}.$$

Theorem 4.5 (ix) follows.

Because $X_1(x), \ldots, X_r(x), L_1(x), \ldots, L_n(x)$ forms a basis for $\mathscr{L}_x$, $\forall x \in B_{X_{K_0}, L_{J_0}}(x_0, \chi)$,

$$\Phi^* X_1(\zeta), \ldots, \Phi^* X_r(\zeta), \Phi^* L_1(\zeta), \ldots, \Phi^* L_n(\zeta)$$

forms a basis for $(\Phi^* \mathscr{L})_\zeta$, $\forall \zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$. Theorem 4.5 (x) (which we have already shown) implies that

$$\sup_{\zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)} \|\mathcal{A}(\zeta)\|_{\mathbb{M}^{(r+n)\times(r+n)}} \leq \frac{1}{4}.$$

In particular, the matrix $I + \mathcal{A}(\zeta)$ is invertible, $\forall \zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$. Hence, Theorem 4.5 (ix) (which we have already proved) implies, $\forall \zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$,

$$\begin{aligned} (\Phi^* \mathscr{L})_\zeta &= \operatorname{span}_{\mathbb{C}} \{\Phi^* X_1(\zeta), \ldots, \Phi^* X_r(\zeta), \Phi^* L_1(\zeta), \ldots, \Phi^* L_n(\zeta)\} \\ &= \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\}. \end{aligned}$$

Since $X_k(x), L_j(x) \in \mathscr{L}_x$, $\forall x$, $1 \leq k \leq q$, $1 \leq j \leq m$, it follows that for $1 \leq k \leq q$, $1 \leq j \leq m$ and $\forall \zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1)$ we have

$$\Phi^* X_k(\zeta), \Phi^* L_j(\zeta) \in \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\}.$$

Because $\Phi^* X_k$ is a real vector field, we conclude for $1 \leq k \leq q$,

$$\Phi^* X_k(\zeta) \in \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r} \right\}, \quad \forall \zeta \in B_{\mathbb{R}^r \times \mathbb{C}^n}(1).$$

Theorem 4.5 (xi) follows. Since $\Phi^* = \Phi_1^* \Phi_0^*$, Theorem 4.5 (xii) follows by combining Theorem 11.4 (vi), (13.6), and (13.7).

All that remains of Theorem 4.5 is (vi). We already have, by the range of Φ_0, that $\Phi(B_{\mathbb{R}^r\times\mathbb{C}^n}(1)) \subseteq B_{X_{K_0},L_{J_0}}(x_0,\chi) \subseteq B_{X,L}(x_0,\xi)$ and the final two containments in (vi) follow. Let $\xi_1 \in (0,\xi]$ be a constant to be chosen later, and suppose $y \in B_{X_{K_0},L_{J_0}}(x_0,\xi_1) = B_{W_{P_0}}(x_0,\xi_1)$. Thus, there exists $\gamma : [0,1] \to B_{W_{P_0}}(x_0,\xi_1)$ with $\gamma(0) = x_0$, $\gamma(1) = y$, $\gamma'(t) = \sum_{j=1}^{2n+r} b_j(t)\xi_1 W_j(\gamma(t))$, $\|\sum |b_j(t)|^2\|_{L^\infty} < 1$. Define

$$t_0 := \sup \left\{ t \in [0,1] : \gamma(t') \in \Phi(B_{\mathbb{R}^r\times\mathbb{C}^n}(1/2)), \forall 0 \le t' \le t \right\}.$$

We want to show that by taking $\xi_1 > 0$ to be a sufficiently small $\{s_0\}$-admissible constant, we have $t_0 = 1$ and $\gamma(1) \in \Phi(B_{\mathbb{R}^r\times\mathbb{C}^n}(1/2))$. Note that $t_0 \ge 0$, since $\gamma(0) = x_0 = \Phi(0)$.

Suppose $t_0 < 1$. Then $|\Phi^{-1}(\gamma(t_0))| = 1/2$. Using that $\|\Phi^* W_j\|_{C(B_{\mathbb{R}^r\times\mathbb{C}^n}(1);\mathbb{R}^{r+2n})} \lesssim_{\{s_0\}} 1$ (by Theorem 4.5 (xii) and the definition of the W_j), and $\Phi(0) = x_0$ (by Theorem 4.5 (vii)) and therefore $\Phi^{-1}(\gamma(0)) = \Phi^{-1}(x_0) = 0$, we have

$$1/2 = |\Phi^{-1}(\gamma(t_0))| = \left| \int_0^{t_0} \frac{d}{dt} \Phi^{-1} \circ \gamma(t)\, dt \right| = \left| \int_0^{t_0} \sum_{j=1}^{2n+r} b_j(t)\xi_1 (\Phi^* W_j)(\Phi^{-1}\circ\gamma(t))\, dt \right|$$
$$\lesssim_{\{s_0\}} \xi_1.$$

This a contradiction if ξ_1 is a sufficiently small $\{s_0\}$-admissible constant, which proves the second containment in Theorem 4.5 (vi). The existence of $\xi_2 > 0$ as in Theorem 4.5 (vi) follows from [27, Lemma 9.35]. This completes the proof of Theorem 4.5.

Now let ν be a density as in Section 4.1. Proposition 12.8 applies to ν, and any constant which is $[s_0;\nu]$ or $\{s;\nu\}$-admissible in the sense of that proposition is $[s_0;\nu]$ or $\{s;\nu\}$-admissible, respectively, in the sense of this section. Let h_0 be as in Proposition 12.8 so that $\Phi_0^*\nu = h_0\sigma_{\text{Leb}}$. Thus,

$$h\sigma_{\text{Leb}} = \Phi^*\nu = \Phi_1^* h_0 \sigma_{\text{Leb}} = (h_0 \circ \Phi_1) \det d\Phi_1 \sigma_{\text{Leb}}.$$

We conclude $h = (h_0 \circ \Phi_1)\det d\Phi_1$. Proposition 12.8 (a) combined with Theorem 11.4 (iii) yields Theorem 4.10 (i).

Combining Proposition 12.8 (b) with Theorem 11.4 (i) (and using Lemmas 9.5 and 9.6) shows:

- If $s_0 \in (1,\infty)$, for $s > 0$,

$$\|h_0 \circ \Phi_1\|_{\mathscr{C}^s(B_{\mathbb{R}^r\times\mathbb{C}^n}(1))}$$
$$\lesssim_{\{s-1;\nu\}} |\nu(X_1,\ldots,X_r, 2\mathrm{Re}(L_1),\ldots,2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1),\ldots,2\mathrm{Im}(L_n))(x_0)|.$$

- If $s_0 = \omega$,

$$\begin{aligned}&\|h_0 \circ \Phi_1\|_{\mathscr{A}^{2n+r,\min\{1,r_0\}}} \\ &\quad \lesssim_{\{\omega;\nu\}} |\nu(X_1, \ldots, X_r, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_n))(x_0)|.\end{aligned}$$

Also by Theorem 11.4 (i) (and using Proposition 9.3 and Lemma 9.7) we have:

- If $s_0 \in (1, \infty)$, for $s > 0$, $\|\det d\Phi_1\|_{\mathscr{C}^s(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \lesssim_{\{s-1\}} 1$.
- If $s_0 = \omega$, $\|\det d\Phi_1\|_{\mathscr{A}^{2n+r,1}} \lesssim_{\{\omega\}} 1$.

Combining the above estimates and using Proposition 9.3 yields Theorem 4.10 (ii).

Finally, we turn to Corollary 4.11. To prove this, we introduce a corollary of Theorem 4.5.

Corollary 13.1. *Let Φ, ξ_1, and ξ_2 be as in Theorem 4.5. Then, there exist $\{s_0\}$-admissible constants $0 < \xi_4 \leq \xi_3 \leq \xi_2$ and a map $\widehat{\Phi} : B_{\mathbb{R}^r \times \mathbb{C}^n}(1) \to B_{X_{K_0}, L_{J_0}}(x_0, \xi_2)$, which satisfies all the same estimates as Φ, so that*

$$\begin{aligned}&B_{X,L}(x_0, \xi_4) \subseteq B_{X_{K_0}, L_{J_0}}(x_0, \xi_3) \subseteq \widehat{\Phi}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1)) \subseteq B_{X_{K_0}, L_{J_0}}(x_0, \xi_2) \subseteq B_{X,L}(x_0, \xi_2) \\ &\subseteq B_{X_{K_0}, L_{J_0}}(x_0, \xi_1) \subseteq \Phi(B_{\mathbb{R}^r \times \mathbb{C}^n}(1)) \subseteq B_{X_{K_0}, L_{J_0}}(x_0, \chi) \subseteq B_{X_{K_0}, L_{J_0}}(x_0, \xi).\end{aligned}$$

Proof. After applying Theorem 4.5 to obtain Φ, ξ_1, and ξ_2, we apply Theorem 4.5 again with ξ replaced by ξ_2 to obtain ξ_3, ξ_4, and $\widehat{\Phi}$ as above. □

Proof of Corollary 4.11. Using Theorem 4.10 (i), we have

$$\begin{aligned}\nu(\Phi(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))) &= \int_{\Phi(B_{\mathbb{R}^r \times \mathbb{C}^n}(1))} \nu = \int_{B_{\mathbb{R}^r \times \mathbb{C}^n}(1)} \Phi^*\nu = \int_{B_{\mathbb{R}^r \times \mathbb{C}^n}(1)} h(t,x)\, dt\, dx \\ &\approx_{[s_0;\nu]} \nu(X_1, \ldots, X_r, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_n))(x_0),\end{aligned}$$

with the same result with Φ replaced by $\widehat{\Phi}$, where $\widehat{\Phi}$ is as in Corollary 13.1. Since

$$\widehat{\Phi}(B_{\mathbb{R}^r \times \mathbb{C}^n}(1)) \subseteq B_{X_{K_0}, L_{J_0}}(x_0, \xi_2) \subseteq B_{X,L}(x_0, \xi_2) \subseteq \Phi(B_{\mathbb{R}^r \times \mathbb{C}^n}(1)),$$

and since $h(t,x)$ always has the same sign (Theorem 4.10 (i)), (4.4) follows.

We turn to (4.5). It follows from the definitions that

$$\begin{aligned}&|\nu(X_1, \ldots, X_r, 2\mathrm{Re}(L_1), \ldots, 2\mathrm{Re}(L_n), 2\mathrm{Im}(L_1), \ldots, 2\mathrm{Im}(L_n))(x_0)| \\ &\leq \max_{K \in \mathcal{I}(r,q), J \in \mathcal{I}(n,m)} |\nu(X_K, 2\mathrm{Re}(L)_J, 2\mathrm{Im}(L)_J)(x_0)| \\ &\leq \max_{P \in \mathcal{I}(2n+r, 2m+q)} |\nu(W_P)(x_0)|.\end{aligned}$$

Thus, with (4.4) in hand, to prove (4.5) it suffices to show

$$\max_{P\in\mathcal{I}(2n+r,2m+q)} |\nu(W_P)(x_0)| \lesssim_0 |\nu(X_1,\ldots,X_r,2\mathrm{Re}(L_1),\ldots,2\mathrm{Re}(L_n),2\mathrm{Im}(L_1),\ldots,$$
$$2\mathrm{Im}(L_n))(x_0)|;$$

i.e., we wish to show

$$\max_{P\in\mathcal{I}(2n+r,2m+q)} |\nu(W_P)(x_0)| \lesssim_0 |\nu(W_{P_0})(x_0)|\,. \tag{13.8}$$

Since $W_{P_0}(x_0)$ forms a basis for the tangent space $T_{x_0}B_{X,L}(x_0,\xi)$, if the right hand side is 0, the left hand side must be zero as well. If the right hand side is nonzero, it follows from Lemma B.4 that

$$\max_{P\in\mathcal{I}(2n+r,2m+q)} \frac{|\nu(W_P)(x_0)|}{|\nu(W_{P_0})(x_0)|} = \max_{P\in\mathcal{I}(2n+r,2m+q)} \left|\frac{\bigwedge W_P(x_0)}{\bigwedge W_{P_0}(x_0)}\right| \lesssim_0 1,$$

where the final inequality follows from (13.2). (13.8) follows, which completes the proof. □

Remark 13.2. The most important special case of Theorem 4.5 is the case when $r=0$. In that case, we can always pick J_0 so that (4.2) holds with $\zeta=1$. However, even in this case, because of (13.1), we require Proposition 12.4 in the general case $\zeta\in(0,1]$. Thus, even for the reader only interested in Theorem 4.5 in the case $\zeta=1$, it is important that we at least have Proposition 12.4 for general $\zeta\in(0,1]$. In any case, having Theorem 4.5 for general $\zeta\in(0,1]$ gives additional, convenient flexibility in applications, even when $r=0$.

14. Hölder spaces

Let $\Omega\subset\mathbb{R}^n$ be a bounded, Lipschitz domain. It follows immediately from the definitions that for $g\in\mathbb{N}$, $s\in[0,1]$ with $g+s>0$, we have the containment $C^{g,s}(\Omega)\subseteq\mathscr{C}^{g+s}(\Omega)$. For $g\in\mathbb{N}$, $s\in(0,1)$ we also have the reverse containment $\mathscr{C}^{g+s}(\Omega)\subseteq C^{g,s}(\Omega)$; this follows easily from [36, Theorem 1.118 (i)].

When we move to the corresponding spaces with respect to C^1 real vector fields $W_1,\ldots,W_N$ on a C^2 manifold M, we have similar results. For any $g\in\mathbb{N}$, $s\in[0,1]$, $g+s>0$, we have $C^{g,s}_W(M)\subseteq\mathscr{C}^{g+s}_W(M)$; this follows from Lemma 9.1. The reverse containment for $g\in\mathbb{N}$, $s\in(0,1)$ requires more hypotheses on the vector fields. This is described in [31].

In a similar vein, we can create Hölder versions of Theorem 7.1 and Theorem 7.2. We present these here.

Let $X_1,\ldots,X_q$ be real C^1 vector fields on a connected C^2 manifold M and let $L_1,\ldots,L_m$ be complex C^1 vector fields on M. For $x\in M$ set

$$\mathscr{L}_x := \operatorname{span}_{\mathbb{C}}\{X_1(x), \ldots, X_q(x), L_1(x), \ldots, L_m(x)\}, \quad \mathscr{X}_x := \operatorname{span}_{\mathbb{C}}\{X_1(x), \ldots, X_q(x)\}. \tag{14.1}$$

We assume:

- $\mathscr{L}_x + \overline{\mathscr{L}_x} = \mathbb{C}T_xM$, $\forall x \in M$.
- $\mathscr{X}_x = \mathscr{L}_x \cap \overline{\mathscr{L}_x}$, $\forall x \in M$.

Corollary 14.1 *(The local result). Fix $x_0 \in M$, $g \in \mathbb{N}$, $g \geq 1$, $s \in (0,1)$, and set $r := \dim \mathscr{X}_{x_0}$ and $n + r := \dim \mathscr{L}_{x_0}$. The following three conditions are equivalent:*

(i) There exists an open neighborhood $V \subseteq M$ of x_0 and a C^2 diffeomorphism $\Phi : U \to V$, where $U \subseteq \mathbb{R}^r \times \mathbb{C}^n$ is open, such that $\forall (t,z) \in U$, $1 \leq k \leq q$, $1 \leq j \leq m$,

$$\Phi^* X_k(t,z) \in \operatorname{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}\right\},$$
$$\Phi^* L_j(t,z) \in \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\},$$

and $\Phi^* X_k \in C^{g+1,s}(U;\mathbb{R}^r)$, $\Phi^* L_j \in C^{g+1,s}(U;\mathbb{C}^{r+n})$.

(ii) Reorder $X_1, \ldots, X_q$ so that $X_1(x_0), \ldots, X_r(x_0)$ are linearly independent, and reorder $L_1, \ldots, L_m$ so that $L_1(x_0), \ldots, L_n(x_0), X_1(x_0), \ldots, X_r(x_0)$ are linearly independent. Let $\widehat{Z}_1, \ldots, \widehat{Z}_{n+r}$ denote the list $X_1, \ldots, X_r, L_1, \ldots, L_n$, and let $Y_1, \ldots, Y_{m+q-(r+n)}$ denote the list $X_{r+1}, \ldots, X_q, L_{n+1}, \ldots, L_m$. There exists an open neighborhood $V \subseteq M$ of x_0 such that:
  - *$[\widehat{Z}_j, \widehat{Z}_k] = \sum_{l=1}^{n+r} \hat{c}_{j,k}^{1,l} \widehat{Z}_l$, and $[\widehat{Z}_j, \overline{\widehat{Z}_k}] = \sum_{l=1}^{n+r} \hat{c}_{j,k}^{2,l} \widehat{Z}_l + \sum_{l=1}^{n+r} \hat{c}_{j,k}^{3,l} \overline{\widehat{Z}_l}$, where $\hat{c}_{j,k}^{a,l} \in C_{X,L}^{g,s}(V)$, $1 \leq j,k,l \leq n+r$, $1 \leq a \leq 3$.*
  - *$Y_j = \sum_{l=1}^{n+r} b_j^l \widehat{Z}_l$, where $b_j^l \in C_{X,L}^{g+1,s}(V)$, $1 \leq j \leq m+q-(r+n)$, $1 \leq l \leq n+r$.*

  Furthermore, the map $x \mapsto \dim \mathscr{L}_x$, $V \to \mathbb{N}$ is constant in x.

(iii) Let $Z_1, \ldots, Z_{m+q}$ denote the list $X_1, \ldots, X_q, L_1, \ldots, L_m$. There exists a neighborhood $V \subseteq M$ of x_0 such that $[Z_j, Z_k] = \sum_{l=1}^{m+q} c_{j,k}^{1,l} Z_l$ and $[Z_j, \overline{Z_k}] = \sum_{l=1}^{m+q} c_{j,k}^{2,l} Z_l + \sum_{l=1}^{m+q} c_{j,k}^{3,l} \overline{Z_l}$, where $c_{j,k}^{a,l} \in C_{X,L}^{g,s}(V)$, $1 \leq a \leq 3$, $1 \leq j,k,l \leq m+q$. Furthermore, the map $x \mapsto \dim \mathscr{L}_x$, $V \to \mathbb{N}$ is constant in x.

Proof. (i)⇒(ii)⇒(iii) has a nearly identical proof to the corresponding parts of Theorem 7.1, and we leave the details to the reader. Assume (iii) holds. Then, since $C_{X,L}^{g,s}(V) \subseteq \mathscr{C}_{X,L}^{g+s}(V)$ (by Lemma 9.1 (iii)) we have that Theorem 7.1 (iii) holds (with s replaced by $g+s$). Therefore Theorem 7.1 (i) holds (again, with s replaced by $g+s$); we may shrink U in Theorem 7.1 (i) so that it is a Euclidean ball. This establishes all of (i), except that it shows $\Phi^* X_k \in \mathscr{C}^{g+s+1}(U;\mathbb{R}^r)$, $\Phi^* L_j \in \mathscr{C}^{g+s+1}(U;\mathbb{C}^{r+n})$ instead of $\Phi^* X_k \in C^{g+1,s}(U;\mathbb{R}^r)$, $\Phi^* L_j \in C^{g+1,s}(U;\mathbb{C}^{r+n})$. However, since U is a ball and $s \in (0,1)$ (this is the only place we use $s \neq 0, 1$), it follows from [36, Theorem 1.118

(i)] that $\mathscr{C}^{g+s+1}(U;\mathbb{R}^r) = C^{g+1,s}(U;\mathbb{R}^r)$. This establishes (iii)⇒(i) and completes the proof. □

Remark 14.2. The only place where $g \geq 1$, $s \neq 0,1$ was used in the proof of Corollary 14.1 was the implication (iii)⇒(i). The implications (i)⇒(ii)⇒(iii) hold for $g \in \mathbb{N}$ and $s \in [0,1]$ with the same proof.

Corollary 14.3 *(The global result). For $g \in \mathbb{N}$, $g \geq 1$, $s \in (0,1)$ the following three conditions are equivalent:*

(i) There exists a $C^{g+2,s}$ E-manifold structure on M, compatible with its C^2 structure, such that $X_1,\ldots,X_q,L_1,\ldots,L_m$ are $C^{g+1,s}$ vector fields on M and $\mathscr{L}$ (as defined in (14.1)) is the associated elliptic structure (see Definition 6.20).
(ii) For each $x_0 \in M$, any of the three equivalent conditions from Corollary 14.1 hold for this choice of x_0.
(iii) Let $Z_1,\ldots,Z_{m+q}$ denote the list $X_1,\ldots,X_q,L_1,\ldots,L_m$. Then, $[Z_j,Z_k] = \sum_{l=1}^{m+q} c_{j,k}^{1,l} Z_l$ and $[Z_j,\overline{Z_k}] = \sum_{l=1}^{m+q} c_{j,k}^{2,l} Z_l + \sum_{l=1}^{m+q} c_{j,k}^{3,l} \overline{Z_l}$, where $\forall x \in M$, there exists an open neighborhood $V \subseteq M$ of x such that $c_{j,k}^{a,l}\big|_V \in C_{X,L}^{g,s}(V)$, $1 \leq a \leq 3$, $1 \leq j,k,l \leq m+q$. Furthermore, the map $x \mapsto \dim \mathscr{L}_x$, $M \to \mathbb{N}$ is constant.

Furthermore, under these conditions, the $C^{g+2,s}$ E-manifold structure in (i) is unique, in the sense that if M has another $C^{g+2,s}$ E-manifold structure satisfying the conclusions of (i), then the identity map $M \to M$ is a $C^{g+2,s}$ E-diffeomorphism between these two E-manifold structures.

Proof. With Corollary 14.1 in hand, the proof is nearly identical to the proof of Theorem 7.2, and we leave the details to the interested reader. □

When $s \in \{0,1\}$, the use of Zygmund spaces (as in Theorem 7.1 and Theorem 7.2) is essential. Indeed, the above results do not hold with $s \in \{0,1\}$, at least in the special case when $n \geq 1$, $r = 0$. This follows from the next lemma.

Lemma 14.4. *Fix $n \geq 1$, $g \in \mathbb{N}$. There exists an open neighborhood $V' \subseteq \mathbb{C}^n$ of 0 and complex vector fields $L_1,\ldots,L_n \in C^{g+1}(V';\mathbb{C}^{2n})$ such that*

(i) For every $\zeta \in V$, $L_1(\zeta),\ldots,L_n(\zeta),\overline{L_1}(\zeta),\ldots,\overline{L_n}(\zeta)$ form a basis for $\mathbb{C}T_\zeta V'$.
(ii) $[L_j,L_k] = \sum_{l=1}^m c_{j,k}^{1,l} L_l$ and $[L_j,\overline{L_k}] = \sum_{l=1}^m c_{j,k}^{2,l} L_l + \sum_{l=1}^m c_{j,k}^{3,l} \overline{L_l}$, where $c_{j,k}^{a,l} \in C_L^g(V')$, $1 \leq a \leq 3$, $1 \leq j,k,l \leq n$.
(iii) There does not exist a C^2 diffeomorphism $\Phi : U \to V$, where $V \subseteq V'$ is an open neighborhood of 0 and $U \subseteq \mathbb{C}^n$ is open such that $\Phi^* L_1,\ldots,\Phi^* L_n \in C^{g,1}(U)$ and $\forall \zeta \in U$,

$$\Phi^* L_1(\zeta), \ldots, \Phi^* L_n(\zeta) \in \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\}.$$

Proof. The idea of the proof is that our results (e.g., Theorem 3.5) imply the sharp regularity of the classical Newlander-Nirenberg theorem, and it is a result of Liding Yao [37] that sharp results for the classical Newlander-Nirenberg theorem require Zygmund spaces. Indeed, he exhibits a set complex vector fields $L_1, \ldots, L_n \in C^{g+1}(V'; \mathbb{C}^{2n})$, defined on an open neighborhood V' of the origin in $\mathbb{C}^n$ such that

- For every $\zeta \in V$, $L_1(\zeta), \ldots, L_n(\zeta), \overline{L_1}(\zeta), \ldots, \overline{L_n}(\zeta)$ form a basis for $\mathbb{C}T_\zeta V'$.
- There does not exist a $C^{g+1,1}$ diffeomorphism $\Phi : U \to V$, where $V \subseteq V'$ is an open neighborhood of 0 and $U \subseteq \mathbb{C}^n$ is open such that $\forall \zeta \in U$,

$$\Phi^* L_1(\zeta), \ldots, \Phi^* L_n(\zeta) \in \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\}.$$

To see why this choice of $L_1, \ldots, L_n$ satisfies the conclusion of them lemma, note that

$$[L_j, L_k] = \sum_{l=1}^m c_{j,k}^{1,l} L_l, \quad [L_j, \overline{L_k}] = \sum_{l=1}^m c_{j,k}^{2,l} L_l + \sum_{l=1}^m c_{j,k}^{3,l} \overline{L_l},$$
$$c_{j,k}^{a,l} \in C^g(V'), \quad 1 \leq a \leq 3, 1 \leq j, k, l \leq n.$$

Since $L_1, \ldots, L_n \in C^{g+1}(V'; \mathbb{C}^{2n})$, it follows immediately from the definitions that $C^g(V') \subseteq C^g_L(V')$. Now suppose, for contradiction, $\Phi : U \to V$ is a C^2 diffeomorphism as in (iii). Then, the obvious of analog of Lemma 10.1 for Hölder spaces shows $\Phi \in C^{g+1,1}(U)$, contradicting the choice of $L_1, \ldots, L_n$. $\square$

Appendix A. Immersed submanifolds

Let $W_1, \ldots, W_N$ be real C^1 vector fields on a C^2 manifold $\mathfrak{M}$. For $x, y \in \mathfrak{M}$, define $\rho(x, y)$ as in (2.3). Fix $x_0 \in \mathfrak{M}$ and let $Z := \{y \in \mathfrak{M} : \rho(x_0, y) < \infty\}$. ρ is a metric on Z, and we give Z the topology induced by ρ (this is finer[27] than the topology as a subspace of $\mathfrak{M}$, and may be strictly finer). Let $M \subseteq Z$ be a connected open subset of Z containing x_0. We give M the topology of a subspace of Z.

Proposition A.1. *Suppose $[W_j, W_k] = \sum_{l=1}^N c_{j,k}^l W_l$, where $c_{j,k}^l : M \to \mathbb{R}$ are locally bounded. Then, there is a C^2 manifold structure on M (compatible with its topology) such that:*

- *The inclusion $M \hookrightarrow \mathfrak{M}$ is a C^2 injective immersion.*

[27] See [27, Lemma A.1] for a proof that this topology is finer than the subspace topology.

- $W_1, \ldots, W_N$ *are* C^1 *vector fields tangent to* M.
- $W_1, \ldots, W_N$ *span the tangent space at every point of* M.

Furthermore, this C^2 *structure is unique in the sense that if* M *is given another* C^2 *structure (compatible with its topology) such that the inclusion map* $M \hookrightarrow \mathfrak{M}$ *is a* C^2 *injective immersion, then the identity map* $M \to M$ *is a* C^2 *diffeomorphism between these two* C^2 *structures on* M.

Proposition A.1 is standard; see [27, Appendix A] for a proof.

Appendix B. Linear algebra

B.1. Real and complex vector spaces

Let $\mathscr{V}$ be a real vector space and let $\mathscr{V}^{\mathbb{C}} = \mathscr{V} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. We consider $\mathscr{V} \hookrightarrow \mathscr{V}^{\mathbb{C}}$ as a real subspace by identifying v with $v \otimes 1$. There are natural maps:

$$\mathrm{Re} : \mathscr{V}^{\mathbb{C}} \to \mathscr{V}, \quad \mathrm{Im} : \mathscr{V}^{\mathbb{C}} \to \mathscr{V}, \quad \text{complex conjugation} : \mathscr{V}^{\mathbb{C}} \to \mathscr{V}^{\mathbb{C}},$$

defined as follows. Every $v \in \mathscr{V}^{\mathbb{C}}$ can be written uniquely as $v = v_1 \otimes 1 + v_2 \otimes i$, with $v_1, v_2 \in V$. Then, $\mathrm{Re}(v) := v_1$, $\mathrm{Im}(v) := v_2$, and $\overline{v} := v_1 \otimes 1 - v_2 \otimes i$.

Lemma B.1. *Let* $\mathscr{L} \subseteq \mathscr{V}^{\mathbb{C}}$ *be a finite dimensional complex subspace. Then,* $\dim(\mathscr{L} + \overline{\mathscr{L}}) + \dim(\mathscr{L} \bigcap \overline{\mathscr{L}}) = 2\dim(\mathscr{L})$.

Proof. It is a standard fact that $\dim(\mathscr{L} + \overline{\mathscr{L}}) + \dim(\mathscr{L} \bigcap \overline{\mathscr{L}}) = \dim(\mathscr{L}) + \dim(\overline{\mathscr{L}})$. Using that $w \mapsto \overline{w}$, $\mathscr{L} \to \overline{\mathscr{L}}$ is an anti-linear isomorphism, the result follows. □

Lemma B.2. *Let* $\mathscr{L} \subseteq \mathscr{V}^{\mathbb{C}}$ *be a finite dimensional subspace. Let* $x_1, \ldots, x_r \in \mathscr{L} \bigcap \overline{\mathscr{L}} \bigcap V$ *be a basis for* $\mathscr{L} \bigcap \overline{\mathscr{L}}$ *and let* $l_1, \ldots, l_n \in \mathscr{L}$. *The following are equivalent:*

(i) $x_1, \ldots, x_r, \mathrm{Re}(l_1), \ldots, \mathrm{Re}(l_n), \mathrm{Im}(l_1), \ldots, \mathrm{Im}(l_n)$ *is a basis for* $\mathscr{L} + \overline{\mathscr{L}}$.
(ii) $x_1, \ldots, x_r, l_1, \ldots, l_n$ *is a basis for* $\mathscr{L}$.

Proof. Clearly $r = \dim(\mathscr{L} \cap \overline{\mathscr{L}})$.

(i)⇒(ii): Suppose (i) holds. Then $\dim(\mathscr{L}+\overline{\mathscr{L}}) = 2n+r$. Lemma B.1 implies $\dim(\mathscr{L}) = n + r$. Thus, once we show $x_1, \ldots, x_r, l_1, \ldots, l_n$ are linearly independent, they will form a basis. Suppose

$$\sum_{k=1}^{r} a_k x_k + \sum_{j=1}^{n} (b_j + ic_j) l_j = 0, \tag{B.1}$$

with $a_k \in \mathbb{C}$, $b_j, c_j \in \mathbb{R}$. We wish to show $a_k = b_j = c_j = 0$, $\forall j, k$. Applying Re to (B.1), we see

$$\sum_{k=1}^{n} \mathrm{Re}(a_k) x_k + \sum_{j=1}^{n} b_j \mathrm{Re}(l_j) - \sum_{j=1}^{n} c_j \mathrm{Im}(l_j) = 0.$$

Since $x_1, \ldots, x_r, \mathrm{Re}(l_1), \ldots, \mathrm{Re}(l_n), \mathrm{Im}(l_1), \ldots, \mathrm{Im}(l_n)$ are linearly independent by hypothesis, we see $\mathrm{Re}(a_k) = b_j = c_j = 0$, $\forall j, k$. Plugging this into (B.1) we have

$$\sum_{k=1}^{r} i\mathrm{Im}(a_k) x_k = 0.$$

Since $x_1, \ldots, x_r$ are linearly independent, we see $\mathrm{Im}(a_k) = 0$, $\forall k$. Thus, $a_k = b_j = c_j = 0$, $\forall j, k$ and (ii) follows.

(ii)$\Rightarrow$(i): Suppose $x_1, \ldots, x_r, l_1, \ldots, l_n$ form a basis for $\mathscr{L}$. Then, $\dim(\mathscr{L}) = n + r$ and Lemma B.1 shows $\dim(\mathscr{L} + \overline{\mathscr{L}}) = 2n + r$. Thus, once we show $x_1, \ldots, x_r, \mathrm{Re}(l_1), \ldots, \mathrm{Re}(l_n), \mathrm{Im}(l_1), \ldots, \mathrm{Im}(l_n)$ span $\mathscr{L} + \overline{\mathscr{L}}$ it will follow that they are a basis. But it is immediate to verify that $\mathrm{Re}(\mathscr{L})$ spans $\mathscr{L} + \overline{\mathscr{L}}$, thus since $\mathrm{Re}(x_j) = x_j$, $\mathrm{Re}(ix_j) = 0$, and $\mathrm{Re}(-il_j) = \mathrm{Im}(l_j)$, it follows that $x_1, \ldots, x_r, \mathrm{Re}(l_1), \ldots, \mathrm{Re}(l_n), \mathrm{Im}(l_1), \ldots, \mathrm{Im}(l_n)$ span $\mathscr{L} + \overline{\mathscr{L}}$, which completes the proof. $\square$

Lemma B.3. *Let $\mathscr{L} \subseteq \mathscr{V}^{\mathbb{C}}$ be a finite dimensional complex subspace. Suppose $x_1, \ldots, x_r \in \mathscr{L} \bigcap \overline{\mathscr{L}} \bigcap \mathscr{V}$ is a basis for $\mathscr{L} \bigcap \overline{\mathscr{L}}$ and extend this to a basis $x_1, \ldots, x_r, l_1, \ldots, l_n \in \mathscr{L}$. Suppose $z \in \mathscr{L}$ and*

$$\mathrm{Re}(z) = \sum_{k=1}^{r} a_k x_k + \sum_{j=1}^{n} b_j \mathrm{Re}(l_j) + \sum_{j=1}^{n} c_j \mathrm{Im}(l_j),$$
$$\mathrm{Im}(z) = \sum_{k=1}^{r} d_k x_k + \sum_{j=1}^{n} e_j \mathrm{Re}(l_j) + \sum_{j=1}^{n} f_j \mathrm{Im}(l_j),$$

with $a_k, b_j, c_j, d_k, e_j, f_j \in \mathbb{R}$. Then,

$$z = \sum_{k=1}^{r} (a_k + i d_k) x_k + \sum_{j=1}^{n} (b_j - i c_j) l_j.$$

Proof. Set $z_0 = \sum_{k=1}^{r} (a_k + i d_k) x_k + \sum_{j=1}^{n} (b_j - i c_j) l_j$; we wish to show $z = z_0$. Clearly $\mathrm{Re}(z - z_0) = 0$. We have

$$\mathrm{Im}(z - z_0) = \sum_{j=1}^{n} (e_j + c_j) \mathrm{Re}(l_j) + \sum_{j=1}^{n} (f_j - b_j) \mathrm{Im}(l_j)$$
$$\in \mathrm{span}_{\mathbb{C}} \{\mathrm{Re}(l_1), \ldots, \mathrm{Re}(l_n), \mathrm{Im}(l_1), \ldots, \mathrm{Im}(l_n)\}.$$

However, since $\mathrm{Re}(z - z_0) = 0$,

$$\mathrm{Im}(z - z_0) = \frac{1}{i}(z - z_0) = -\frac{1}{i}(\overline{z} - \overline{z_0}) \in \mathscr{L} \bigcap \overline{\mathscr{L}} = \mathrm{span}_{\mathbb{C}}\{x_1, \ldots, x_r\}.$$

Thus,

$$\mathrm{Im}(z - z_0) \in \mathrm{span}_{\mathbb{C}}\{\mathrm{Re}(l_1), \ldots, \mathrm{Re}(l_n), \mathrm{Im}(l_1), \ldots, \mathrm{Im}(l_n)\} \bigcap \mathrm{span}_{\mathbb{C}}\{x_1, \ldots, x_r\}.$$

Since $x_1, \ldots, x_r, \mathrm{Re}(l_1), \ldots, \mathrm{Re}(l_n), \mathrm{Im}(l_1), \ldots, \mathrm{Im}(l_n)$ are linearly independent (by Lemma B.2), it follows that $\mathrm{Im}(z - z_0) = 0$, which completes the proof. □

B.2. Wedge products

Let $\mathscr{Z}$ be a one dimensional vector space over a field $\mathbb{F}$ (we will always be using $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). For $z_1, z_2 \in \mathscr{Z}$, $z_1 \neq 0$ we set

$$\frac{z_2}{z_1} := \frac{\lambda(z_2)}{\lambda(z_1)} \in \mathbb{F},$$

where $\lambda : \mathscr{Z} \to \mathbb{F}$ is any non-zero linear functional. It is easy to see that $\frac{z_2}{z_1}$ is independent of the choice of λ.

Let $\mathscr{W}$ be an N-dimensional vector space over $\mathbb{F}$, so that $\bigwedge^N \mathscr{W}$ is a one-dimensional vector space over $\mathbb{F}$. Let $w_1, \ldots, w_N \in \mathscr{W}$ be a basis for $\mathscr{W}$ and let $w_1', \ldots, w_N' \in \mathscr{W}$. Using the above definition, it makes sense to consider

$$\frac{w_1' \wedge w_2' \wedge \cdots \wedge w_N'}{w_1 \wedge w_2 \wedge \cdots \wedge w_N}.$$

Lemma B.4. *In the above setting, the following three quantities are equal:*

(i) $\frac{w_1' \wedge w_2' \wedge \cdots \wedge w_N'}{w_1 \wedge w_2 \wedge \cdots \wedge w_N}$.
(ii) $\det(B)$, *where B is the linear transformation defined by $Bw_j = w_j'$.*
(iii) $\det(C)$, *where C is the $N \times N$ matrix with components c_j^k, where $w_j' = \sum c_j^k w_k$.*

Proof. Clearly (ii) and (iii) are equal. To see that (i) and (ii) are equal, let B be as in (ii). Then, we have

$$\begin{aligned}\frac{w_1' \wedge w_2' \wedge \cdots \wedge w_N'}{w_1 \wedge w_2 \wedge \cdots \wedge w_N} &= \frac{(Bw_1) \wedge (Bw_2) \wedge \cdots \wedge (Bw_N)}{w_1 \wedge w_2 \wedge \cdots \wedge w_N} = \frac{\det(B)(w_1 \wedge w_2 \wedge \cdots \wedge w_N)}{w_1 \wedge w_2 \wedge \cdots \wedge w_N} \\ &= \det(B),\end{aligned}$$

completing the proof. □

Let $\mathscr{V}$ be a real vector space and let $\mathscr{V}^{\mathbb{C}}$ be its complexification. Let $\mathscr{L} \subseteq \mathscr{V}^{\mathbb{C}}$ be a finite dimensional subspace and let $\mathscr{X} := \mathscr{L} \bigcap \overline{\mathscr{L}}$; note that $\mathscr{X} = \overline{\mathscr{X}}$. Set $r = \dim(\mathscr{X})$ and $n + r = \dim(\mathscr{L})$. Set $\mathscr{W} := (\mathscr{L} + \overline{\mathscr{L}}) \bigcap \mathscr{V} = \mathrm{span}_{\mathbb{R}}\{\mathrm{Re}(l) : l \in \mathscr{L}\} \subseteq \mathscr{V}$ (so that $\mathscr{W}$ is a real vector space). By Lemma B.1, $\dim(\mathscr{W}) = 2n + r$.

Fix $x_1, \ldots, x_q \in \mathscr{X} \bigcap \mathscr{V}$ and $l_1, \ldots, l_m \in \mathscr{L}$ such that $\mathscr{X} = \mathrm{span}_{\mathbb{C}}\{x_1, \ldots, x_q\}$ and $\mathscr{L} = \mathrm{span}_{\mathbb{C}}\{x_1, \ldots, x_q, l_1, \ldots, l_m\}$. For $K = (k_1, \ldots, k_{r_1}) \in \mathcal{I}(r_1, q)$ (where $\mathcal{I}(r_1, q) = \{1, \ldots, q\}^{r_1}$; see (4.1)), set $\bigwedge X_K := x_{k_1} \wedge x_{k_2} \wedge \cdots \wedge x_{k_{r_1}}$. For $J = (j_1, \ldots, j_{n_1}) \in \mathcal{I}(n_1, m)$ set

$$\begin{aligned}\bigwedge L_J := l_{j_1} \wedge l_{j_2} \wedge \cdots \wedge l_{j_{n_1}}, \quad \bigwedge 2\mathrm{Re}(L)_J := 2\mathrm{Re}(l_{j_1}) \wedge 2\mathrm{Re}(l_{j_2}) \wedge \cdots \wedge 2\mathrm{Re}(l_{j_{n_1}}),\\ \bigwedge 2\mathrm{Im}(L)_J := 2\mathrm{Im}(l_{j_1}) \wedge 2\mathrm{Im}(l_{j_2}) \wedge \cdots \wedge 2\mathrm{Im}(l_{j_{n_1}}).\end{aligned} \tag{B.2}$$

Let $w_1, \ldots, w_{2m+q}$ denote the list $x_1, \ldots, x_q, 2\mathrm{Re}(l_1), \ldots, 2\mathrm{Re}(l_m), 2\mathrm{Im}(l_1), \ldots, 2\mathrm{Im}(l_m)$, so that $\mathscr{W} = \mathrm{span}_{\mathbb{R}}\{w_1, \ldots, w_{2m+q}\}$. For $P = (p_1, \ldots, p_{2n+r}) \in \mathcal{I}(2n + r, 2m + q)$, we set $\bigwedge W_P := w_{p_1} \wedge w_{p_2} \wedge \cdots \wedge w_{p_{2n+r}}$.

Proposition B.5. *Fix $\zeta \in (0, 1]$, $J_0 \in \mathcal{I}(n, m)$, $K_0 \in \mathcal{I}(r, q)$.*

(i) Suppose $(\bigwedge X_{K_0}) \bigwedge (\bigwedge L_{J_0}) \neq 0$ and moreover,

$$\max_{\substack{J \in \mathcal{I}(n_1, m), K \in \mathcal{I}(r_1, q)\\ n_1 + r_1 = n + r}} \left| \frac{(\bigwedge X_K) \bigwedge (\bigwedge L_J)}{(\bigwedge X_{K_0}) \bigwedge (\bigwedge L_{J_0})} \right| \leq \zeta^{-1}.$$

Then, $(\bigwedge X_{K_0}) \bigwedge (\bigwedge 2\mathrm{Re}(L)_{J_0}) \bigwedge (\bigwedge 2\mathrm{Im}(L)_{J_0}) \neq 0$ and moreover,

$$\max_{P \in \mathcal{I}(2n+r, 2m+q)} \left| \frac{\bigwedge W_P}{(\bigwedge X_{K_0}) \bigwedge (\bigwedge 2\mathrm{Re}(L)_{J_0}) \bigwedge (\bigwedge 2\mathrm{Im}(L)_{J_0})} \right| \leq \left(2\zeta^{-1}\sqrt{2n + r}\right)^{2n+r}. \tag{B.3}$$

(ii) Conversely, suppose $(\bigwedge X_{K_0}) \bigwedge (\bigwedge 2\mathrm{Re}(L)_{J_0}) \bigwedge (\bigwedge 2\mathrm{Im}(L)_{J_0}) \neq 0$ and moreover,

$$\max_{P \in \mathcal{I}(2n+r, 2m+q)} \left| \frac{\bigwedge W_P}{(\bigwedge X_{K_0}) \bigwedge (\bigwedge 2\mathrm{Re}(L)_{J_0}) \bigwedge (\bigwedge 2\mathrm{Im}(L)_{J_0})} \right| \leq \zeta^{-1}.$$

Then, $(\bigwedge X_{K_0}) \bigwedge (\bigwedge L_{J_0}) \neq 0$ and moreover,

$$\max_{\substack{J \in \mathcal{I}(n_1, m), K \in \mathcal{I}(r_1, q)\\ n_1 + r_1 = n + r}} \left| \frac{(\bigwedge X_K) \bigwedge (\bigwedge L_J)}{(\bigwedge X_{K_0}) \bigwedge (\bigwedge L_{J_0})} \right| \leq \left(4\zeta^{-1}\sqrt{n + r}\right)^{n+r}. \tag{B.4}$$

Remark B.6. A choice of K_0, J_0, and ζ as in (i) or (ii) always exist: take $K_0 = (k_1, \ldots, k_r)$ and $J_0 = (j_1, \ldots, j_n)$ so that $x_{k_1}, \ldots, x_{k_r}, l_{j_1}, \ldots, l_{j_n}$ form a basis for $\mathscr{L}$. With this choice, the conditions for (i) and (ii) then hold for some $\zeta \in (0, 1]$. If $\mathscr{X} \bigcap \mathrm{span}_{\mathbb{C}}\{l_1, \ldots, l_m\} =$

$\{0\}$, one may pick J_0 and K_0 so that the conditions of (i) hold with $\zeta = 1$. This occurs in the two most important special cases: $r = 0$ or $m = 0$.

Remark B.7. The estimates (B.3) and (B.4) are not optimal; however, we do not know the optimal estimates, and so content ourselves with proving the simplest estimates which are sufficient for our purposes.

Proof. Suppose K_0, J_0, and ζ are as in (i); let $K_0 = (k_1, \ldots, k_r)$, $J_0 = (j_1, \ldots, j_n)$. Since $\dim \mathscr{L} = n + r$ and since $x_{k_1}, \ldots, x_{k_r}, l_{j_1}, \ldots, l_{j_n}$ are linearly independent, it follows that $x_{k_1}, \ldots, x_{k_r}, l_{j_1}, \ldots, l_{j_n}$ are a basis for $\mathscr{L}$. By Lemma B.2, $x_{k_1}, \ldots, x_{k_r}$, $2\mathrm{Re}(l_{j_1}), \ldots, 2\mathrm{Re}(l_{j_n}), 2\mathrm{Im}(l_{j_1}), \ldots, 2\mathrm{Im}(l_{j_n})$ are a basis for $\mathscr{W}$, and therefore $(\bigwedge X_{K_0}) \bigwedge (\bigwedge 2\mathrm{Re}(L)_{J_0}) \bigwedge (\bigwedge 2\mathrm{Im}(L)_{J_0}) \neq 0$.

Let $P = (p_1, \ldots, p_{2n+r}) \in \mathcal{I}(2n + r, 2m + q)$. We claim, for $t = 1, \ldots, 2n + r$,

$$w_{p_t} = \sum_{\alpha=1}^{r} a_t^\alpha x_{k_\alpha} + \sum_{\beta=1}^{n} b_t^\beta 2\mathrm{Re}(l_{j_\beta}) + \sum_{\beta=1}^{n} c_t^\beta 2\mathrm{Im}(l_{j_\beta}), \quad \frac{1}{2}|a_t^\alpha|, |b_t^\beta|, |c_t^\beta| \leq \zeta^{-1}, \forall t, \alpha, \beta. \tag{B.5}$$

By its definition $w_{p_t} = 2\mathrm{Re}(z)$, where $z \in \{\frac{1}{2}x_1, \ldots, \frac{1}{2}x_q, l_1, \ldots, l_m, -il_1, \ldots, -il_m\}$. Using Cramer's rule, we have

$$\begin{aligned} z &= \sum_{\alpha=1}^{r} \frac{x_{k_1} \wedge \cdots \wedge x_{k_{\alpha-1}} \wedge z \wedge x_{k_{\alpha+1}} \wedge \cdots \wedge x_{k_r} \wedge l_{j_1} \wedge \cdots \wedge l_{j_n}}{x_{k_1} \wedge \cdots \wedge x_{k_r} \wedge l_{j_1} \wedge \cdots \wedge l_{j_n}} x_{k_\alpha} \\ &\quad + \sum_{\beta=1}^{n} \frac{x_{k_1} \wedge \cdots \wedge x_{k_r} \wedge l_{j_1} \wedge \cdots \wedge l_{j_{\beta-1}} \wedge z \wedge l_{j_{\beta+1}} \wedge \cdots \wedge l_{j_n}}{x_{k_1} \wedge \cdots \wedge x_{k_r} \wedge l_{j_1} \wedge \cdots \wedge l_{j_n}} l_{j_\beta} \\ &=: \sum_{\alpha=1}^{r} d_\alpha x_{k_\alpha} + \sum_{\beta=1}^{n} e_\beta l_{j_\beta}, \end{aligned}$$

where $|d_\alpha|, |e_\beta| \leq \zeta^{-1}$ by hypothesis. Thus,

$$\begin{aligned} w_{p_t} = 2\mathrm{Re}(z) = z + \overline{z} &= \sum_{\alpha=1}^{r} (d_\alpha + \overline{d_\alpha}) x_{k_\alpha} + \sum_{\beta=1}^{n} (e_\beta l_{j_\beta}) + (\overline{e_\beta} \overline{l_{j_\beta}}) \\ &= \sum_{\alpha=1}^{r} 2\mathrm{Re}(d_\alpha) x_{k_\alpha} + \sum_{\beta=1}^{n} \mathrm{Re}(e_\beta) 2\mathrm{Re}(l_{j_\beta}) + \sum_{\beta=1}^{n} -\mathrm{Im}(e_\beta) 2\mathrm{Im}(l_{j_\beta}). \end{aligned}$$

(B.5) follows.

Using (B.5), Lemma B.4 shows

$$\frac{\bigwedge W_P}{(\bigwedge X_{K_0}) \bigwedge (\bigwedge 2\mathrm{Re}(L)_{J_0}) \bigwedge (\bigwedge 2\mathrm{Im}(L)_{J_0})}$$

is equal to the determinant of a $(2n+r)\times(2n+r)$ matrix, all of whose components are bounded by $2\zeta^{-1}$. (B.3) now follows from Hadamard's inequality.

Suppose K_0, J_0, and ζ are as in (ii); let $K_0=(k_1,\ldots,k_r)$, $J_0=(j_1,\ldots,j_n)$. Since

$$x_{k_1},\ldots,x_{k_r},2\mathrm{Re}(l_{j_1}),\ldots,2\mathrm{Re}(l_{j_n}),2\mathrm{Im}(l_{j_1}),\ldots,2\mathrm{Im}(l_{j_n})$$

are linearly independent, and since $\dim\mathscr{W}=2n+r$, it follows that they are a basis for $\mathscr{W}$. By Lemma B.2, $x_{k_1},\ldots,x_r,l_{j_1},\ldots,l_{j_n}$ is a basis for $\mathscr{L}$, and therefore $(\bigwedge X_{K_0})\bigwedge(\bigwedge L_{J_0})\neq 0$.

Let $J\in\mathcal{I}(n_1,m), K\in\mathcal{I}(r_1,q)$ with $n_1+r_1=n+r$. $(\bigwedge X_K)\bigwedge(\bigwedge L_J)=z_1\wedge z_2\wedge\cdots\wedge z_{n+r}$, where each z_t is of the form x_k or l_j for some j or k. We claim

$$z_t=\sum_{\alpha=1}^{r} g_t^\alpha x_{k_\alpha}+\sum_{\beta=1}^{n} h_t^\beta l_{j_\beta},\quad |g_t^\alpha|,|h_t^\beta|\leq 4\zeta^{-1},\forall t,\alpha,\beta. \tag{B.6}$$

Indeed, suppose $z_t=l_j$ for some j. Then,

$$2\mathrm{Re}(z_t)=\sum_{\alpha=1}^{r} a_t^\alpha x_{k_\alpha}+\sum_{\beta=1}^{n} b_t^\beta 2\mathrm{Re}(l_{j_\beta})+\sum_{\beta=1}^{n} c_t^\beta 2\mathrm{Im}(l_{j_\beta}),$$

where, by Cramer's rule,

$$a_t^\alpha=\frac{\bigwedge W_{P_{t,\alpha}}}{(\bigwedge X_{K_0})\bigwedge(\bigwedge 2\mathrm{Re}(L)_{J_0})\bigwedge(\bigwedge 2\mathrm{Im}(L)_{J_0})},$$

and $\bigwedge W_{P_{t,\alpha}}$ is defined by replacing x_{k_α} with $2\mathrm{Re}(z_t)$ in $(\bigwedge X_{K_0})\bigwedge(\bigwedge 2\mathrm{Re}(L)_{J_0})\bigwedge(\bigwedge 2\mathrm{Im}(L)_{J_0})$, and therefore $|a_t^\alpha|\leq\zeta^{-1}$, by hypothesis. Similarly, $|b_t^\beta|,|c_t^\beta|\leq\zeta^{-1}$. Similarly,

$$2\mathrm{Im} z_t=\sum_{\alpha=1}^{r} d_t^\alpha x_{k_\alpha}+\sum_{\beta=1}^{n} e_t^\beta 2\mathrm{Re}(l_{j_\beta})+\sum_{\beta=1}^{n} f_t^\beta 2\mathrm{Re}(l_{j_\beta}),\quad |d_t^\alpha|,|e_t^\beta|,|f_t^\beta|\leq\zeta^{-1},\forall t,\alpha,\beta.$$

(B.6) now follows from Lemma B.3 (with, in fact, $4\zeta^{-1}$ replaced by $2\zeta^{-1}$). A similar proof works when $z_t=x_k$ for some k, yielding (B.6).

Using (B.6), Lemma B.4 shows

$$\frac{(\bigwedge X_K)\bigwedge(\bigwedge L_J)}{(\bigwedge X_{K_0})\bigwedge(\bigwedge L_{J_0})}$$

is equal to the determinant of an $(n+r)\times(n+r)$ matrix, all of whose components are bounded by $4\zeta^{-1}$. (B.4) now follows from Hadamard's inequality. □

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