# STABILITY FOR PRODUCT GROUPS AND PROPERTY $(\tau)$

ADRIAN IOANA

Abstract. We study the notion of permutation stability (or P-stability) for countable groups. Our main result provides a wide class of non-amenable product groups which are not P-stable. This class includes the product group $\Sigma\times\Lambda$, whenever $\Sigma$ admits a non-abelian free quotient and $\Lambda$ admits an infinite cyclic quotient. In particular, we obtain that the groups $\mathbb{F}_m\times\mathbb{Z}^d$ and $\mathbb{F}_m\times\mathbb{F}_n$ are not P-stable, for any integers $m,n\geq 2$ and $d\geq 1$. This implies that P-stability is not closed under the direct product construction, which answers a question of Becker, Lubotzky and Thom. The proof of our main result relies on a construction of asymptotic homomorphisms from $\Sigma\times\Lambda$ to finite symmetric groups starting from sequences of finite index subgroups in $\Sigma$ and $\Lambda$ with and without property $(\tau)$. Our method is sufficiently robust to show that the groups covered are not even flexibly P-stable, thus giving the first such non-amenable residually finite examples.

## 1. Introduction and statement of main results

The notion of permutation stability has been developed in a series of works [GR09, AP14, BLT18]. A countable group $\Gamma$ is *stable in permutations* (or P-*stable*) if any "almost homomorphism" from $\Gamma$ to a finite symmetric group is "close" to a homomorphism. To make this precise, we endow the symmetric group $\text{Sym}(X)$ of any finite set $X$ with the normalized Hamming metric:

$$\mathrm{d_H}(\sigma,\tau)=\frac{1}{|X|}\cdot|\{x\in X\mid \sigma(x)\neq\tau(x)\}|.$$

Hereafter, we will use the same formula to define the normalized Hamming distance between any maps $\sigma$ and $\tau$ with domain (but not necessarily co-domain) equal to $X$. P

**Definition 1.1.** A sequence of maps $\sigma_n:\Gamma\to\text{Sym}(X_n)$, for some finite sets $X_n$, is called an *asymptotic homomorphism* if $\lim\limits_{n\to\infty}\mathrm{d_H}(\sigma_n(gh),\sigma_n(g)\sigma_n(h))=0$, for every $g,h\in\Gamma$. The group $\Gamma$ is called P-*stable*[1] if for any asymptotic homomorphism $\sigma_n:\Gamma\to\text{Sym}(X_n)$, there exists a sequence of homomorphisms $\tau_n:\Gamma\to\text{Sym}(X_n)$ such that $\lim\limits_{n\to\infty}\mathrm{d_H}(\sigma_n(g),\tau_n(g))=0$, for every $g\in\Gamma$.

More generally, one can define stability with respect to any class $\mathcal{C}$ of metric groups endowed with bi-invariant metrics (see [AP14, AP17, CGLT17, Th17]). While this notion has only been formalized recently, in the case when $\Gamma=\mathbb{Z}^2$ and $\mathcal{C}$ consists of groups of matrices, the stability problem has been studied extensively in the literature. Indeed, this problem is equivalent to the well-known question (posed in [Ro69] for the normalized Hilbert-Schmidt norm and in [Ha76] for the operator norm) of whether "almost commuting" matrices are "close" to commuting matrices. The answer depends both on the groups of matrices considered and the norms chosen (see the introduction of [AP14]). For instance, if $\mathcal{C}$ is the class of unitary groups $\{\mathrm{U}(n)\mid n\in\mathbb{N}\}$, then the answer is positive if one uses the normalized Hilbert-Schmidt norm [HL08, Gl10] and negative if one uses the operator norm [Vo83]. Recently, the stability problem with respect to unitary groups has been investigated for general countable groups $\Gamma$ in [HS17, ESS18] and for other matrix norms in [CGLT17, LO18].

The author was supported in part by NSF Career Grant DMS #1253402 and NSF FRG Grant #1854074.

[1] Definition 1.1 agrees with the definitions of P-stability given in [AP14] when $\Gamma$ is finitely presented and in [BLT18] when $\Gamma$ is finitely generated, see Lemma 3.1.

At the same time, there has been a surge of interest in the study of permutation stability. This started with the works of Glebsky and Rivera [GR09] who observed that finite groups are P-stable[2], and of Arzhantseva and Păunescu [AP14] who proved that abelian groups are P-stable (see [BM18] for a quantitative approach to these results). In [BLT18], Becker, Lubotzky and Thom obtained a characterization of P-stability for amenable groups in terms of invariant random subgroups. This has since been used to provide many new classes of P-stable amenable groups, including polycyclic groups and the Baumslag-Solitar groups $\mathrm{BS}(1,n)$ [BLT18], the first Grigorchuk group [Zh19], the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ [LL19a] and an uncountable family of 2-generated groups [LL19b]. On the other hand, Becker and Lubotzky [BL18] proved that groups $\Gamma$ that have property $(\tau)$ are not P-stable by removing one point from a set on which $\Gamma$ acts and deforming the action to get an almost action. This motivated them to define two flexible variants of P-stability (see Definition 1.2). Subsequently, Lazarovich, Levit and Minsky proved that surface groups are flexibly P-stable [LLM19].

The study of P-stability is motivated in part by the longstanding problem of whether any countable group is sofic. By an observation in [GR09], in order to find a non-sofic group, it is enough to find a group that is both P-stable and non-residually finite[2]. We note that this point of view was used by De Chiffre, Glebsky, Lubotzky and Thom in their breakthrough work [CGLT17] to construct non-Frobenius-approximable groups. Very recently, Burton and Bowen proved that the existence of non-sofic groups would also follow from the flexible P-stability of $\mathrm{PSL}_d(\mathbb{Z})$ for $d \geq 5$ [BB19].

The above results have led to a much better understanding of permutation stability, by providing several classes of P-stable and non-P-stable groups, as well as potential applications of this notion. However, in spite of the progress made, the following basic question posed in [BLT18] is open: is P-stability closed under direct products? While P-stability is clearly closed under free products, but not under the amalgamated free product or semi-direct product constructions by results in [BL18], the situation remained unclear for direct products.

We settle this question in the negative here, by giving the first examples of P-stable groups whose direct product is not P-stable (see Corollary B and the paragraph following it). This will be deduced from our first main result (Theorem A) which provides a general criterion for non-P-stability of direct products of groups. Moreover, our method of proof is sufficiently robust to address the flexible versions of P-stability introduced in [BL18], allowing to prove the following:

A

**Theorem A.** *Let $\Sigma$ and $\Lambda$ be finitely generated groups. Assume that $\Sigma$ admits a free non-abelian quotient and $\Lambda$ does not have property $(\tau)$. Then $\Sigma \times \Lambda$ is not very flexibly P-stable.*

Before presenting several concrete examples of groups covered by Theorem A, let us discuss the notions used in its statement and an equivalent formulation of it.

A countable group $\Lambda$ has Lubotzky's *property* $(\tau)$ if the representation of $\Lambda$ on $\bigoplus_{[\Lambda:\Delta]<\infty} \ell^2_0(\Lambda/\Delta)$, where $\Delta$ runs through all finite index subgroups of $\Lambda$ and $\ell^2_0(\Lambda/\Delta) := \ell^2(\Lambda/\Delta) \ominus \mathbb{C}\mathbf{1}_{\Lambda/\Delta}$, does not have almost invariant vectors [Lu94]. Property $(\tau)$ is a weaker version of property (T) which is satisfied by any irreducible lattice in a product of second countable, locally compact non-compact groups, at least one of which has property (T) [LZ89]. In the opposite direction, any group admitting an infinite, residually finite amenable quotient group does not have property $(\tau)$ [LW93, LZ03]. FP

[2]note1 The results referenced here are stated in [GR09] using the notion of stability in permutations for presentations of groups, see Remark 1.5. In the form presented here, they follow from [AP14], where it was shown that stability is a group property, i.e., it is independent of the choice of the presentation.

**Definition 1.2.** A countable group $\Gamma$ is called *flexibly* P-*stable* if for any asymptotic homomorphism $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$, there exist a sequence of finite sets $Y_n$ and homomorphisms $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ such that $X_n \subset Y_n$, for every $n$, $\lim_{n\to\infty} \mathrm{d_H}(\sigma_n(g), \tau_n(g)_{|X_n}) = 0$, for every $g \in \Gamma$, and $\lim_{n\to\infty} \frac{|Y_n|}{|X_n|} = 1$.

The group $\Gamma$ is called *very flexibly* P-*stable* if for any asymptotic homomorphism $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$, there exist a sequence of finite sets $Y_n$ and homomorphisms $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ such that $X_n \subset Y_n$, for every $n$, and $\lim_{n\to\infty} \mathrm{d_H}(\sigma_n(g), \tau_n(g)_{|X_n}) = 0$, for every $g \in \Gamma$.

**Remark 1.3.** A group $\Gamma$ is very flexibly P-stable if any asymptotic homomorphism is essentially obtained by restricting homomorphisms $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$ to "almost invariant" sets $X_n \subset Y_n$, i.e., such that $|\tau_n(g)X_n \triangle X_n|/|X_n| \to 0$, for every $g \in \Gamma$. If the sets $X_n$ are obtained by removing $o(|Y_n|)$ points from $Y_n$ (in which case they are trivially almost invariant), then $\Gamma$ is flexibly P-stable.

It is unclear how much weaker these notions are than P-stability. On the one hand, P-stability coincides with flexible P-stability for amenable groups and flexible P-stability coincides with very flexible P-stability for groups with property ($\tau$) (see Lemma 3.2). On the other hand, it is open whether groups with property ($\tau$) can be flexibly P-stable and whether surface groups are P-stable (see [BL18, LLM19]). Moreover, while very flexible P-stability is inherited by subgroups of finite index (see Lemma 3.3), we do not know if this holds for P-stability or flexible P-stabillity.

Since very flexible P-stability passes to finite index subgroups, Theorem A implies the following seemingly stronger statement: the product between a large group and a group without property ($\tau$) (and any group containing such a product as a finite index subgroup) is not very flexibly P-stable. Recall that a group is called *large* if one of its finite index subgroups admits a non-abelian free quotient. By [BP78] any finitely presented group with at least two more generators than relators is large; for more recent examples of large groups, see [La07] and the references therein.

Theorem A thus provides a wide class of groups, including the product of any large group and any group having an infinite, residually finite amenable quotient, which are not very flexibly P-stable. As an immediate consequence, we derive the following concrete examples of non-P-stable groups:B

**Corollary B.** *The following groups are not very flexibly P-stable:*

(1) $\mathbb{F}_m \times \mathbb{Z}^d$, *for every integers* $m \geq 2$ *and* $d \geq 1$.
(2) $\mathbb{F}_m \times \mathbb{F}_n$, *for every integers* $m, n \geq 2$.
(3) *the Baumslag-Solitar group* $\mathrm{BS}(m, n)$, *for every integers* $m, n$ *with* $|m| = |n| \geq 2$.
(4) *the braid group* $\mathrm{B}_n$ *and pure braid group* $\mathrm{PB}_n$, *for every integer* $n \geq 3$.

Since free groups are obviously stable and abelian groups are stable by [AP14], (1) and (2) imply that P-stability is not closed under direct products, thus answering Becker, Lubotzky and Thom's question [BLT18] in the negative. Moreover, we deduce that a direct product of P-stable groups need not even be very flexibly P-stable. However, since the groups we treat are not amenable, this leaves open the question of whether the product of two P-stable amenable groups is P-stable [BLT18].

In [AP14, Example 7.3] it was shown that $\mathrm{BS}(m, n)$ is P-stable if $m = n = \pm 1$ but not P-stable if $|m| \neq |n|$ and $|m|, |n| \geq 2$, while [BLT18, Theorem 1.2 (ii)] established that $\mathrm{BS}(1, n)$ is stable for every $n \in \mathbb{Z}$. Part (3) of Corollary B completes the classification of P-stability of the Baumslag-Solitar groups $\mathrm{BS}(m, n)$ by addressing the remaining case when $|m| = |n| \geq 2$.

To put Corollary B into a better perspective, let us indicate several additional consequences of it. First, as remarked in [BL18, Section 4.4] (extending observations made in [GR09, AP14]), any group which is sofic and non-residually finite is not very flexibly P-stable. By [BLT18, Theorem 1.2 (iii)], there are amenable residually finite groups which are not P-stable and thus not flexibly P-stable.

Corollary B gives the first examples of non-amenable residually finite groups that are not flexibly P-stable, and the first examples of residually finite groups that are not very flexibly P-stable. BB

**Remark 1.4.** A countable group $\Gamma$ is called *Hilbert-Schmidt stable* (or HS-*stable*) if it is stable with respect to the class of unitary groups $\{(\mathrm{U}(n), \mathrm{d}_{\mathrm{HS}}) \mid n \in \mathbb{N}\}$ endowed with the normalized Hilbert-Schmidt distance given by $\mathrm{d}_{\mathrm{HS}}(T,S) = \|T-S\|_{\mathrm{HS}}$ for $T,S \in \mathrm{U}(n)$, where $\|V\|_{\mathrm{HS}} = \sqrt{\frac{1}{n}\cdot \mathrm{Tr}(V^*V)}$. Since the normalized Hamming distance can be expressed in terms of the normalized Hilbert-Schmidt distance, the study of P-stability and HS-stability are similar in flavor [AP14].

In spite of the similarity between these notions, Corollary B highlights a surprising difference between them, by providing, to our knowledge, the first examples of HS-stable groups which are not P-stable. By [HS17, Theorem 1], the product of two HS-stable groups is HS-stable provided that one of the groups is abelian (by [IS19, Corollary D] the same holds if one of the groups is amenable). Consequently, $\mathbb{F}_m \times \mathbb{Z}^d$ is HS-stable but not P-stable, for any integers $m \geq 2$ and $d \geq 1$.

Note that is an open question whether HS-stability is closed under direct products. It seems likely that this question has a negative answer, and moreover that $\mathbb{F}_m \times \mathbb{F}_n$ is not HS-stable, for $m,n \geq 2$. Supporting evidence is provided by [IS19, Theorem E] which shows that $\mathbb{F}_m \times \mathbb{F}_n$ is not stable with respect to the class $\{(\mathrm{U}(M), \|\cdot\|_2) \mid (M,\tau) \text{ tracial von Neumann algebra}\}$ of unitary groups of tracial von Neumann algebras endowed with their 2-norms, $\|T\|_2 = \sqrt{\tau(T^*T)}$. equations

**Remark 1.5.** Let $R \subset \mathbb{F}_k$ be a finite set, for $k \in \mathbb{N}$. The system of equations $(\star)$ $r(\tau_1, ..., \tau_k) = e$, for every $r \in R$, is called P-stable if for every $\varepsilon > 0$, there is $\delta > 0$ such that the following holds: for any finite set $X$ and $\sigma_1, ..., \sigma_k \in \mathrm{Sym}(X)$ satisfying $\mathrm{d}_{\mathrm{H}}(r(\sigma_1, ..., \sigma_k), \mathrm{Id}_X) < \delta$, for every $r \in R$, $(\star)$ has a solution $\tau_1, ..., \tau_k \in \mathrm{Sym}(X)$ such that $\mathrm{d}_{\mathrm{H}}(\sigma_i, \tau_i) \leq \varepsilon$, for every $1 \leq i \leq k$ (see [GR09, AP14]). A finitely presented group $\Gamma = \langle \mathbb{F}_k | R\rangle$ is P-stable if and only if $R$ is P-stable [AP14]. The P-stability of $\mathbb{Z}^2$ proved in [AP14] is thus equivalent to the P-stability of the system $[a,b] = aba^{-1}b^{-1} = e$.

On the other hand, since the groups $\mathbb{F}_2 \times \mathbb{Z}$ and $\mathbb{F}_2 \times \mathbb{F}_2$ are not P-stable by Corollary B, we conclude that the systems $[a_1, b] = [a_2, b] = e$ and $[a_1, b_1] = [a_1, b_2] = [a_2, b_1] = [a_2, b_2] = e$ are not P-stable.

Corollary B also implies the existence of universal sofic groups which fail a certain lifting property for commuting subgroups. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and $(X_n)$ finite sets with $\lim_{n\to\mathcal{U}} |X_n| = +\infty$. Define the metric ultraproduct group $\prod_{\mathcal{U}} \mathrm{Sym}(X_n) := \big(\prod_n \mathrm{Sym}(X_n)\big)/\mathcal{N}$, where $\mathcal{N}$ is the subgroup of $(\sigma_n) \in \prod_n \mathrm{Sym}(X_n)$ satisfying $\lim_{n\to\mathcal{U}} \mathrm{d}_{\mathrm{H}}(\sigma_n, \mathrm{Id}_{X_n}) = 0$. Since a countable group is sofic if and only if it embeds into $\prod_{\mathcal{U}} \mathrm{Sym}(X_n)$ [ES04], the latter is called a *universal sofic group*. comutant

**Corollary C.** *There exist countable commuting subgroups $\Sigma, \Lambda$ of a universal sofic group $\prod_{\mathcal{U}} \mathrm{Sym}(X_n)$ such that the following holds: there are no commuting subgroups $\Sigma_n, \Lambda_n$ of $\mathrm{Sym}(X_n)$, for all $n \in \mathbb{N}$, such that $\Sigma \subset \prod_{\mathcal{U}} \Sigma_n$ and $\Lambda \subset \prod_{\mathcal{U}} \Lambda_n$.*

We end the introduction by discussing a weakening of the notion of P-stability found by considering asymptotic homomorphisms that are sofic approximations [AP14]. Let $\Gamma$ be a countable group.

**Definition 1.6.** An asymptotic homomorphism $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ is called a *sofic approximation* of $\Gamma$ if $\lim_{n\to\infty} \mathrm{d}_{\mathrm{H}}(\sigma_n(g), \mathrm{Id}_{X_n}) = 1$, for every $g \in \Gamma \setminus \{e\}$. The group $\Gamma$ is called *weakly* P-*stable* (respectively, *weakly flexibly* P-*stable* or *weakly very flexibly* P-*stable*) if the condition from Definition 1.1 (respectively, the conditions from Definition 1.2) holds for any sofic approximation $(\sigma_n)$ of $\Gamma$.

The notion of weak P-stability is in general strictly weaker than that of P-stability. More precisely, [AP14, Theorem 7.2] shows that any finitely presented, residually finite amenable group is weakly P-stable, whereas [BLT18, Theorem 1.2 (iii)] proves that there is such a group which is not P-stable.

Our last main result provides a class of non-amenable groups which are not weakly P-stable:

**Theorem D.** *Any group which has a subgroup of finite index isomorphic to* $\mathbb{F}_m \times \mathbb{Z}^d$ *or to* $\mathbb{F}_m \times \mathbb{F}_n$, *for some integers* $m, n \geq 2$ *and* $d \geq 1$, *is not weakly very flexibly* P*-stable. In particular, any group from Corollary B, parts (1)-(3), is not weakly very flexibly* P*-stable.*

Theorem D implies that the Baumslag-Solitar BS$(m, n)$ group is not weakly P-stable, whenever $|m| = |n| \geq 2$. This settles a question posed by Arzhantseva and Păunescu in [AP14, Example 7.3]. As a special case of Theorem D, we deduce that $\mathbb{F}_2 \times \mathbb{F}_2$ and $\mathbb{F}_2 \times \mathbb{Z}$ are not weakly flexibly P-stable, which answers a question of Bowen (see [Bo17, Problem 4]). The question of whether $\mathbb{F}_2 \times \mathbb{Z}$ is weakly flexibly P-stable was also emphasized by Bowen and Burton in [BB19] who pointed out that this seems to be the most elementary group for which weak flexible P-stability was unknown (note that the notion of flexible stability used in [BB19] is what we call here weak flexible stability).

**Remark 1.7.** The above results hold when (weak) very flexible P-stability is replaced by an even weaker notion. Thus, we say that a countable group $\Gamma$ is *extremely flexibly* P*-stable* if for any asymptotic homomorphism $\sigma_n : \Gamma \to \text{Sym}(X_n)$, there exist a sequence of not necessarily finite sets $Y_n$ and homomorphisms $\tau_n : \Gamma \to \text{Sym}(Y_n)$ such that $X_n \subset Y_n$, for every $n$, and $\lim\limits_{n\to\infty} \text{d}_\text{H}(\sigma_n(g), \tau_n(g)_{|X_n}) = 0$, for every $g \in \Gamma$. The group $\Gamma$ is *weakly extremely flexibly* P*-stable* if this holds for any sofic approximation $(\sigma_n)$ of $\Gamma$. Then the proofs of Theorem A, Corollary B and Theorem D, which do not use that the involved sets $Y_n$ are finite, show that the groups considered therein are not extremely flexibly P-stable and not weakly extremely flexibly P-stable, respectively.

**Comments on the proof of Theorem A.** We end the introduction with an outline of the proof of Theorem A under the following additional assumption: there exist a group $\Gamma$, a sequence $\{\Gamma_n\}_{n=1}^\infty$ of finite index normal subgroups of $\Gamma$, and homomorphisms $q_n : \Lambda \to \Gamma/\Gamma_n$ such that $\Sigma = \Gamma * \mathbb{Z}$,

- $\Gamma$ has property ($\tau$) with respect to $\{\Gamma_n\}_{n=1}^\infty$, and
- $\Lambda$ does not have property ($\tau$) with respect to $\{\ker(q_n)\}_{n=1}^\infty$.

This assumption holds for $\Sigma = \mathbb{F}_3$ and $\Lambda = \mathbb{Z}$, by taking $\{\Gamma_n\}_{n=1}^\infty$ be a sequence of finite index normal subgroups of $\Gamma = \mathbb{F}_2$ with property ($\tau$) and $q_n : \Lambda \to \Gamma/\Gamma_n$ homomorphisms with $|q_n(\Lambda)| \to +\infty$. More generally, we use Kassabov's theorem [Ka05] (that the symmetric groups $\{\text{Sym}(n)\}_{n=1}^\infty$ admit Cayley graphs which form a bounded degree expander family) to conclude that there is $L \geq 2$ such that the assumption is satisfied when $\Sigma = \mathbb{F}_{L+1}$, $\Gamma = \mathbb{F}_L$ and $\Lambda$ is any group without property ($\tau$).

Next, let $X_n = \Gamma/\Gamma_n$, $p_n : \Gamma \twoheadrightarrow X_n$ be the quotient homomorphism and view $\Gamma \times \Lambda$ as a subgroup of $\Sigma \times \Lambda$. We define the left-right multiplication action $\sigma_n : \Gamma \times \Lambda \to \text{Sym}(X_n)$ by letting

$$\sigma_n(g, h)x = p_n(g)xq_n(h)^{-1}, \text{ for every } g \in \Gamma, h \in \Lambda, x \in X_n.$$

There are two main ingredients in the proof of Theorem A.

The first is a rigidity result for asymptotic homomorphisms $\widetilde{\sigma}_n : \Sigma \times \Lambda \to \text{Sym}(X_n)$ extending $\sigma_n$, i.e., $\widetilde{\sigma_n}_{|\Gamma\times\Lambda} = \sigma_n$. Assume there are homomorphisms $\tau_n : \Sigma \times \Lambda \to \text{Sym}(Y_n)$, with $Y_n \supset X_n$ finite, such that $\text{d}_\text{H}(\widetilde{\sigma}_n(g), \tau_n(g)_{|X_n}) \to 0$, for all $g \in \Sigma \times \Lambda$. Using the property ($\tau$) assumption, we prove that there must be homomorphisms $\overline{\sigma}_n : \Sigma \times \Lambda \to \text{Sym}(X_n)$ extending $\sigma_n$ such that $\text{d}_\text{H}(\widetilde{\sigma}_n(g), \overline{\sigma}_n(g)) \to 0$, for all $g \in \Sigma \times \Lambda$ (see Theorem 5.1). In other words, if $\widetilde{\sigma}_n$ is close to the restriction to $X_n$ of a homomorphism, then $\widetilde{\sigma}_n$ is close to a homomorphism which extends $\sigma_n$.

The second ingredient in the proof of Theorem A is the construction of a "non-trivial" asymptotic homomorphism $\widetilde{\sigma}_n : \Sigma \times \Lambda \to \text{Sym}(X_n)$ extending $\sigma_n$. Using that $\Lambda$ does not have property ($\tau$) with respect to $\{\ker(q_n)\}_{n=1}^\infty$, we construct in Lemma 6.1 a permutation $\rho_n \in \text{Sym}(X_n)$ such that

(1) $\mathrm{d_H}(\rho_n \circ \sigma_n(e,h), \sigma_n(e,h) \circ \rho_n) \to 0$, for every $h \in \Lambda$, and
(2) $\max\{\mathrm{d_H}(\rho_n \circ \sigma_n(e,h), \sigma_n(e,h) \circ \rho_n) \mid h \in \Lambda\} \geq \frac{1}{126}$, for infinitely many $n$.

Specifically, we first find $A_n \subset X_n$ which is almost invariant under the right multiplication action of $\Lambda$ and satisfies $\frac{|A_n|}{|X_n|} \in (\frac{1}{7}, \frac{1}{6})$ for $n$ large (see Lemma 2.8). After replacing $A_n$ with a subset, we may assume that $A_n \cap g_n A_n = \emptyset$, for $g_n \in X_n$. We then show that $\rho_n$ defined by $\rho_n(x) = g_n x$ if $x \in A_n$, $\rho_n(x) = g_n^{-1} x$ if $x \in g_n A_n$, and $\rho_n(x) = x$ if $x \notin A_n \cup g_n A_n$, satisfies conditions (1) and (2).

Finally, condition (1) allows us to define an asymptotic homomorphism $\widetilde{\sigma}_n : \Sigma \times \Lambda \to \mathrm{Sym}(X_n)$ which extends $\sigma_n$ by letting $\widetilde{\sigma}_n(t,e) = \rho_n$, where $t \in \mathbb{Z}$ is a generator. On the other hand, (2) guarantees that $\widetilde{\sigma}_n$ is not close to any homomorphism $\overline{\sigma}_n : \Sigma \times \Lambda \to \mathrm{Sym}(X_n)$ which extends $\sigma_n$. But then the first ingredient above implies that $\Sigma \times \Lambda$ is not very flexibly P-stable, as desired.

In the general case, when $\Sigma$ is only assumed to have a non-abelian free quotient, after replacing it with a finite index subgroup, we may assume that there is an onto homomorphism $\pi : \Sigma \to \mathbb{F}_{L+1}$. Let $\widetilde{\sigma}_n : \mathbb{F}_{L+1} \times \Lambda \to \mathrm{Sym}(X_n)$ be the asymptotic homomorphism constructed above which witnesses that $\mathbb{F}_{L+1} \times \Lambda$ is not very flexibly P-stable. Then we analyze the asymptotic homomorphism $\widetilde{\sigma}_n \circ (\pi \times \mathrm{Id}_\Lambda) : \Sigma \times \Lambda \to \mathrm{Sym}(X_n)$ to show that $\Sigma \times \Lambda$ is not very flexibly P-stable.

**Acknowledgements.** I would like to thank Goulnara Arzhantseva and Pieter Spaas for several helpful comments and corrections, and Lewis Bowen and Andreas Thom for stimulating discussions.

## 2. Preliminaries

In this section, we first recall some notation and then gather several results that will be needed later. Let $X$ be a finite set. We denote by $\mathrm{B}(\ell^2(X))$ the algebra of all linear maps $T : \ell^2(X) \to \ell^2(X)$ and by $\{\delta_x\}_{x \in X}$ the usual orthonormal basis of $\ell^2(X)$.

The *normalized Hilbert-Schmidt norm* of $T \in \mathrm{B}(\ell^2(X))$ is given by

$$\|T\|_{\mathrm{HS}} = \sqrt{\frac{1}{|X|} \cdot \mathrm{Tr}(T^*T)} = \sqrt{\frac{1}{|X|} \cdot \sum_{x,y \in X} |\langle T\delta_x, \delta_y \rangle|^2}.$$

Let $U : \mathrm{Sym}(X) \to \mathrm{U}(\ell^2(X))$ be the group homomorphism given by $U_\sigma(\delta_x) = \delta_{\sigma(x)}$, for all $x \in X$. Hereafter, we view $\mathrm{Sym}(X)$ as a subgroup of $\mathrm{U}(\ell^2(X))$, via the embedding $U$. Note that

$$\|U_\sigma - U_\tau\|_{\mathrm{HS}} = \sqrt{2 \cdot \mathrm{d_H}(\sigma, \tau)}, \text{ for every } \sigma, \tau \in \mathrm{Sym}(X).$$

2.1. **On the distance to invariant sets.** Next, we record the following well-known fact. component

**Lemma 2.1.** *Let $Y$ be a set, $X \subset Y$ be a finite subset and $H < \mathrm{Sym}(Y)$ be a finite subgroup. Then there exists an $H$-invariant subset $X_0 \subset Y$ such that $|X_0 \triangle X| \leq 2 \cdot \max_{h \in H} |X \triangle hX|$.*

*Proof.* Put $\varepsilon = \max_{h \in H} |X \triangle hX|$ and define the $H$-invariant function $f = \frac{1}{|H|} \sum_{h \in H} \mathbf{1}_{hX} \in \ell^1(Y)$. Since $\|\mathbf{1}_X - \mathbf{1}_{hX}\|_1 = |X \triangle hX| \leq \varepsilon$, for every $h \in H$, we get that $\|\mathbf{1}_X - f\|_1 \leq \varepsilon$. Then the set $X_0 = \{y \in Y \mid f(y) \geq \frac{1}{2}\}$ is $H$-invariant and since

$$\|\mathbf{1}_X - f\|_1 = \sum_{y \in Y \setminus X} |f(y)| + \sum_{y \in X} |f(y) - 1| \geq \frac{1}{2}|(Y \setminus X) \cap X_0| + \frac{1}{2}|X \setminus X_0| = \frac{1}{2}|X_0 \triangle X|,$$

the conclusion follows. ■

2.2. **A commutator calculation.** comutator

**Lemma 2.2.** *Let $G$ be a finite group, $g, h \in G$ and $A \subset G$ be a set such that $A \cap gA = \emptyset$. Define $\sigma, \tau \in \mathrm{Sym}(G)$ by letting $\tau(x) = xh^{-1}$, for all $x \in G$, and* $\sigma(x) = \begin{cases} gx, \text{ if } x \in A \\ g^{-1}x, \text{ if } x \in gA \\ x, \text{ otherwise} \end{cases}$.

*Then we have that* $\mathrm{d_H}(\sigma \circ \tau, \tau \circ \sigma) = \begin{cases} \frac{2|A \setminus Ah| + |(A \cup gA) \setminus (A \cup gA)h|}{|G|}, \text{ if } g^2 \neq e \\ \frac{2|(A \cup gA) \setminus (A \cup gA)h|}{|G|}, \text{ if } g^2 = e. \end{cases}$

*Proof.* Note that $(\sigma \circ \tau)(x) = \begin{cases} gxh^{-1}, \text{ if } x \in Ah \\ g^{-1}xh^{-1}, \text{ if } x \in gAh \\ xh^{-1}, \text{ otherwise} \end{cases}$ and $(\tau \circ \sigma)(x) = \begin{cases} gxh^{-1}, \text{ if } x \in A \\ g^{-1}xh^{-1}, \text{ if } x \in gA \\ xh^{-1}, \text{ otherwise} \end{cases}$.

From this we derive that

$$\{x \in G \mid (\sigma \circ \tau)(x) \neq (\tau \circ \sigma)(x)\} = \begin{cases} (Ah \setminus A) \cup (gAh \setminus gA) \cup \big((A \cup gA) \setminus (Ah \cup gAh)\big), \text{ if } g^2 \neq e \\ (Ah \cup gAh) \triangle (A \cup gA), \text{ if } g^2 = e \end{cases}$$

which clearly implies the conclusion. ■

2.3. **Kazhdan constants.** We continue by recalling the notion of a Kazhdan constant and two well-known facts which we prove for completeness.

**Definition 2.3.** Let $G$ be a finite group and $S$ be a set of generators. The *Kazhdan constant* $\kappa(G, S)$ is the largest constant $\kappa > 0$ such that $\kappa \cdot \|\xi\| \leq \max_{g \in S} \|\pi(g)\xi - \xi\|$, for every $\xi \in \mathcal{H}$ and unitary representation $\pi : G \to \mathrm{U}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ without non-zero invariant vectors. expansion

**Lemma 2.4.** *Let $G$ be a finite group and $S$ be a set of generators. Then for every subset $A \subset G$ we have that $\kappa(G, S)^2 \cdot |A| \cdot |G \setminus A| \leq \max_{g \in S} |gA \triangle A| \cdot |G|$.*

*Proof.* Let $\lambda : G \to \mathrm{U}(\ell^2(G))$ be the left regular representation. Put $\xi = \mathbf{1}_A - \frac{|A|}{|G|} \cdot \mathbf{1}_G \in \ell^2(G) \ominus \mathbb{C}\mathbf{1}_G$. Then the conclusion is equivalent to the inequality $\kappa(G, S) \cdot \|\xi\|_2 \leq \max_{g \in S} \|\lambda(g)\xi - \xi\|_2$, which holds since the restriction of $\lambda$ to $\ell^2(G) \ominus \mathbb{C}\mathbf{1}_G$ has no non-zero invariant vectors. ■ almostinv

**Lemma 2.5.** *Let $G$ be a finite group and $S$ be a set of generators. Then for every unitary representation $\pi : G \to \mathrm{U}(\mathcal{H})$ and $\xi \in \mathcal{H}$ we have that $\kappa(G, S) \cdot \max_{g \in G} \|\pi(g)\xi - \xi\| \leq 2 \cdot \max_{g \in S} \|\pi(g)\xi - \xi\|$.*

*Proof.* Let $\mathcal{H}^G$ be the subspace of $\mathcal{H}$ consisting of $\pi(G)$-invariant vectors. Let $\xi \in \mathcal{H}$ and write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \mathcal{H} \ominus \mathcal{H}^G$ and $\xi_2 \in \mathcal{H}^G$. Then $\|\pi(g)\xi - \xi\| = \|\pi(g)\xi_1 - \xi_1\| \leq 2 \cdot \|\xi_1\|$, for every $g \in G$. Since the restriction of $\pi$ to $\mathcal{H} \ominus \mathcal{H}^G$ has no non-zero invariant vectors, we get that $\kappa(G, S) \cdot \|\xi_1\| \leq \max_{g \in S} \|\pi(g)\xi_1 - \xi_1\|$ and the conclusion follows. ■

2.4. **Property** ($\tau$)**.** We now recall an equivalent formulation of property ($\tau$) for a finitely generated group $\Gamma$ with respect to a sequence of finite index normal subgroups $\{\Gamma_n\}_{n=1}^{\infty}$ [Lu94]. Let $S$ be a finite set of generators of $\Gamma$ and denote by $p_n : \Gamma \to \Gamma/\Gamma_n$ the quotient homomorphism. tau

**Definition 2.6.** We say that $\Gamma$ has *property ($\tau$)* with respect to $\{\Gamma_n\}_{n=1}^{\infty}$ if $\inf_n \kappa(\Gamma/\Gamma_n, p_n(S)) > 0$.

**Remark 2.7.** If $\lim_{n\to\infty}\kappa(\Gamma/\Gamma_n,p_n(S))=0$, then there are sets $C_n\subset\Gamma/\Gamma_n$ with $0<|C_n|<\frac{|\Gamma/\Gamma_n|}{2}$ which are almost invariant, in the sense that $\lim_{n\to\infty}|p_n(g)C_n\triangle C_n|/|C_n|=0$, for every $g\in\Gamma$ (see [LZ03, Proposition 2.5]). Moreover, if the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is a decreasing chain, Abért and Elek proved that one can choose $C_n$ such that the sequence $\{|C_n|/|\Gamma/\Gamma_n|\}_{n=1}^{\infty}$ converges to any prescribed limit in $[0,\frac{1}{2}]$ (see [AE10, Theorem 4]).

The next lemma, which is of independent interest and will be used in the proof of Lemma 6.1, generalizes this result to arbitrary, not necessarily decreasing, sequences of normal subgroups. AE

**Lemma 2.8.** *In the above setting, assume that* $\lim_{n\to\infty}\kappa(\Gamma/\Gamma_n,p_n(S))=0$. *Let* $0<\alpha<\beta\leq\frac{1}{2}$. *Then for large enough $n$ there is $C_n\subset\Gamma/\Gamma_n$ such that*

$$\alpha\leq\frac{|C_n|}{|\Gamma/\Gamma_n|}\leq\beta \text{ and } \lim_{n\to\infty}\frac{|p_n(g)C_n\triangle C_n|}{|\Gamma/\Gamma_n|}=0, \text{ for every } g\in\Gamma.$$

*Proof.* If $\{\Gamma_n\}_{n=1}^{\infty}$ is a descending chain, the lemma is a direct consequence of [AE10, Theorem 4]. In general, denote $G_n=\Gamma/\Gamma_n$ for $n\geq 1$. The proof is based on the following:

**Claim.** Let $D_n\subset G_n$ be a sequence of sets such that $\lim_{n\to\infty}\frac{|p_n(g)D_n\triangle D_n|}{|D_n|}=0$, for every $g\in\Gamma$, and $0<|D_n|<\frac{3|G_n|}{4}$, for every $n\geq 1$. Then for any large enough $n$ we can find $h_n\in G_n$ such that

$$\frac{|D_n|^2}{4|G_n|}\leq|D_nh_n\cap D_n|\leq\frac{3|D_n|}{4}.$$

*Proof of the claim.* Assume that the claim is false. After passing to a subsequence, we may assume that for every $n\geq 1$ and $h\in G_n$ we have $|D_nh\cap D_n|<\frac{|D_n|^2}{4|G_n|}$ or $|D_nh\cap D_n|>\frac{3|D_n|}{4}$. Let $H_n$ be the set of $h\in G_n$ such that $|D_nh\cap D_n|>\frac{3|D_n|}{4}$. If $h,h'\in H_n$, then $|D_nhh'\cap D_n|>\frac{|D_n|}{2}>\frac{|D_n|^2}{4|G_n|}$ and hence $hh'\in H_n$. This implies that $H_n$ is a subgroup of $G_n$. Next, since

$$|D_n|^2=\sum_{h\in G_n}|D_nh\cap D_n|=\sum_{h\in H_n}|D_nh\cap D_n|+\sum_{h\in G_n\setminus H_n}|D_nh\cap D_n|\leq|D_n|\cdot|H_n|+\frac{|D_n|^2}{4|G_n|}\cdot|G_n|,$$

we get that $|H_n|\geq\frac{3|D_n|}{4}$. On the other hand, since

$$\sum_{x\in D_n}|H_n\cap x^{-1}D_n|=\sum_{h\in H_n}|D_nh\cap D_n|\geq\frac{3|D_n|}{4}\cdot|H_n|,$$

we can find $x_n\in D_n$ such that $|x_nH_n\cap D_n|=|H_n\cap x_n^{-1}D_n|\geq\frac{3|H_n|}{4}$. In particular, $|D_n|\geq\frac{3|H_n|}{4}$. Since $|D_n|\leq\frac{4|H_n|}{3}$, we get $|x_nH_n\triangle D_n|=|D_n|+|H_n|-2|x_nH_n\cap D_n|\leq|D_n|-\frac{|H_n|}{2}\leq\frac{5|H_n|}{6}$. Thus, for every $g\in\Gamma$ we have

$$|p_n(g)x_nH_n\triangle x_nH_n|\leq 2|x_nH_n\triangle D_n|+|p_n(g)D_n\triangle D_n|\leq\frac{5|H_n|}{3}+|p_n(g)D_n\triangle D_n|.$$

Since $\lim_{n\to\infty}\frac{|p_n(g)D_n\triangle D_n|}{|D_n|}=0$ and $\frac{|D_n|}{|H_n|}\leq\frac{4}{3}$, it follows that $\limsup_{n\to\infty}\frac{|p_n(g)x_nH_n\triangle x_nH_n|}{|H_n|}\leq\frac{5}{3}<2$. Thus, for every $g\in\Gamma$ we have $p_n(g)\in x_nH_nx_n^{-1}$, for $n$ large enough. Since $\Gamma$ is finitely generated, we get that $H_n=G_n$, for $n$ large enough. This contradicts that $|H_n|\leq\frac{4|D_n|}{3}<|G_n|$, for any $n$. □

Now, let $L$ be the set of $\ell\in[0,\frac{1}{2}]$ for which there is a sequence of nonempty sets $D_n\subset G_n$ with

$$\limsup_{n\to\infty}\frac{|D_n|}{|G_n|}=\ell \quad\text{and}\quad \lim_{n\to\infty}\frac{|p_n(g)D_n\triangle D_n|}{|D_n|}=0, \text{ for every } g\in\Gamma.$$

Since $\lim_{n\to\infty} \kappa(G_n, p_n(S)) = 0$ we have that $L \neq \emptyset$ (see, e.g., [LZ03, Proposition 2.5]).

We claim that $\inf L = 0$. If $0 \in L$, there is nothing to prove. Otherwise, let $\ell \in L\setminus\{0\}$ and $D_n \subset G_n$ be sets witnessing that $\ell \in L$. By the above claim, for every $n$ large enough we can find $h_n \in G_n$ such that $\frac{|D_n|^2}{4|G_n|} \leq |D_n h_n \cap D_n| \leq \frac{3|D_n|}{4}$. For every $n \geq 1$, define $D'_n = \begin{cases} D_n h_n \cap D_n, \text{ if } \frac{|D_n|}{|G_n|} > \frac{\ell}{2} \\ D_n, \text{ if } \frac{|D_n|}{|G_n|} \leq \frac{\ell}{2}. \end{cases}$
If $\frac{|D_n|}{|G_n|} > \frac{\ell}{2}$, then $p_n(g)D'_n \triangle D'_n \subset (p_n(g)D_n \triangle D_n) \cup (p_n(g)D_n \triangle D_n)h_n$ and hence we get that

$$\frac{|p_n(g)D'_n \triangle D'_n|}{|D'_n|} \leq \frac{2|p_n(g)D_n \triangle D_n|}{\frac{|D_n|^2}{4|G_n|}} \leq \frac{16}{\ell}\,\frac{|p_n(g)D_n \triangle D_n|}{|D_n|}.$$

From this it follows that $\lim_{n\to\infty} \frac{|p_n(g)D'_n \triangle D'_n|}{|D'_n|} = 0$, for every $g \in \Gamma$. Thus, $\ell' = \limsup_{n\to\infty} \frac{|D'_n|}{|G_n|} \in L$. Since $\frac{|D'_n|}{|G_n|} \leq \max\{\frac{3|D_n|}{4|G_n|}, \frac{\ell}{2}\}$, for every $n$, we conclude that $\ell' \leq \frac{3\ell}{4}$. This implies that $\inf L = 0$.

Let now $0 < \alpha < \beta \leq \frac{1}{2}$. Since $\inf L = 0$, we can find a sequence of sets $D_n \subset G_n$ such that $\frac{|D_n|}{|G_n|} \leq \min\{\beta - \alpha, \alpha\}$, for $n$ large enough, and $\lim_{n\to\infty} \frac{|p_n(g)D_n \triangle D_n|}{|D_n|} = 0$, for every $g \in \Gamma$.

For $n \geq 1$, let $k_n = \left\lceil \frac{\log(1-\alpha)}{\log(1-\frac{|D_n|}{|G_n|})} \right\rceil$ be the smallest integer such that $1 - \big(1 - \frac{|D_n|}{|G_n|}\big)^{k_n} \geq \alpha$. Let $m_n \geq 1$ be the smallest integer for which there exists a set $F_n \subset G_n$ of cardinality $m_n$ such that $C_n := D_n F_n$ satisfies $\frac{|C_n|}{|G_n|} \geq \alpha$. By [AE10, Lemma 2.3] we have that $m_n \leq k_n$. Then $\frac{|C_n|}{|G_n|} < \beta$, for all $n$. Indeed, if $g \in F_n$, then the minimality of $m_n$ implies that $\frac{|D_n(F_n\setminus\{g\})|}{|G_n|} < \alpha$ and thus

$$\frac{|C_n|}{|G_n|} \leq \frac{|D_n(F_n \setminus \{g\})|}{|G_n|} + \frac{|D_n g|}{|G_n|} < \alpha + (\beta - \alpha) = \beta.$$

Finally, if $g \in \Gamma$, then $p_n(g)C_n \triangle C_n \subset \cup_{h\in F_n}(p_n(g)D_n h \triangle D_n h)$ and thus

$$\frac{|p_n(g)C_n \triangle C_n|}{|G_n|} \leq \frac{m_n\,|p_n(g)D_n \triangle D_n|}{|G_n|} \leq \frac{k_n|D_n|}{|G_n|}\,\frac{|p_n(g)D_n \triangle D_n|}{|D_n|}$$

Since the sequence $\{\frac{k_n|D_n|}{|G_n|}\}_{n=1}^{\infty}$ is bounded, this implies that $\lim_{n\to\infty} \frac{|p_n(g)C_n \triangle C_n|}{|C_n|} = 0$, for every $g \in \Gamma$, which finishes the proof of the lemma. ■

## 3. Basic results on P-stability

In this section, we record three results on the general theory of P-stability. Note that with one exception, Lemma 3.3, these results will not be needed in the rest of the paper.

3.1. **Equivalence of definitions of P-stability.** The notion of P-stability was introduced in [AP14, Definition 3.2] (see also [GR09]) for finitely presented groups, and generalized to finitely generated groups in [BLT18, Definition 3.11]. Our next result provides an equivalent formulation of P-stability, in the sense of Definition 1.1, for general groups. This implies that for finitely generated groups the notions of P-stability given by [BLT18, Definition 3.11] and Definition 1.1 coincide.

Let $\Gamma$ be a countable group and $S$ a set of generators. Denote by $\{\bar{s}\}_{s\in S}$ the free generators of $\mathbb{F}_S$ and by $\pi : \mathbb{F}_S \to \Gamma$ the onto homomorphism given by $\pi(\bar{s}) = s$, for every $s \in S$.

equivalence

**Lemma 3.1.** *The group* $\Gamma$ *is* P*-stable if and only if the following condition is satisfied:*
$(\star)$ *for every* $T \subset S$ *finite and* $\varepsilon > 0$*, there are* $E \subset \ker \pi$ *finite and* $\delta > 0$ *such that for any finite set* $X$ *and homomorphism* $\rho : \mathbb{F}_S \to \mathrm{Sym}(X)$ *satisfying* $\mathrm{d_H}(\rho(g), \mathrm{Id}_X) \leq \delta$*, for all* $g \in E$*, there is a homomorphism* $\tau : \Gamma \to \mathrm{Sym}(X)$ *satisfying* $\mathrm{d_H}(\rho(\bar{s}), \tau(s)) \leq \varepsilon$*, for all* $s \in T$*.*

*Moreover, if* $S$ *is finite, then* $\Gamma$ *is* P*-stable if and only if* $(\star)$ *is satisfied for* $T = S$*.*

*Proof.* In the above notation, let $E_n \subset \ker \pi$ be an increasing sequence of sets with $\cup_n E_n = \ker(\pi)$. Let $p : \Gamma \to \mathbb{F}_S$ be a map such that $p(s) = \bar{s}$, for any $s \in S$, and $\pi(p(g)) = g$, for any $g \in \Gamma$.

If $(\star)$ fails, then there exist $T \subset S$ finite, $\varepsilon > 0$ and homomorphisms $\rho_n : \mathbb{F}_S \to \mathrm{Sym}(X_n)$, with $X_n$ finite, such that $\max\{\mathrm{d_H}(\rho_n(g), \mathrm{Id}_{X_n}) \mid g \in E_n\} \leq \frac{1}{n}$ and $\max\{\mathrm{d_H}(\rho_n(\bar{s}), \tau_n(s)) \mid s \in T\} > \varepsilon$, for any $n \in \mathbb{N}$ and homomorphism $\tau_n : \Gamma \to \mathrm{Sym}(X_n)$. Define $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ by $\sigma_n(g) = \rho_n(p(g))$. If $g, h \in \Gamma$, then $p(gh)^{-1}p(g)p(h) \in \ker \pi$, hence $p(gh)^{-1}p(g)p(h) \in E_{n_0}$, for some $n_0 \in \mathbb{N}$. Therefore,

$$\mathrm{d_H}(\sigma_n(gh), \sigma_n(g)\sigma_n(h)) = \mathrm{d_H}(\rho_n(p(gh)^{-1}p(g)p(h)), \mathrm{Id}_{X_n}) \leq \frac{1}{n}, \text{ for every } n \geq n_0.$$

Then $(\sigma_n)_{n \in \mathbb{N}}$ is an asymptotic homomorphism of $\Gamma$. On the other hand, as $\sigma_n(s) = \rho_n(\bar{s})$, for every $s \in S$, we get that $\max\{\mathrm{d_H}(\sigma_n(s), \tau_n(s)) | s \in T\} > \varepsilon$, for any homomorphism $\tau_n : \Gamma \to \mathrm{Sym}(X_n)$ and $n \in \mathbb{N}$. This implies that $\Gamma$ is not P-stable.

Conversely, if $\Gamma$ is not P-stable, then there are an asymptotic homomorphism $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$, $T \subset S$ finite and $\varepsilon > 0$ such that $\max\{\mathrm{d_H}(\sigma_n(s), \tau_n(s)) \mid s \in T\} > \varepsilon$, for any $n \in \mathbb{N}$ and homomorphism $\tau_n : \Gamma \to \mathrm{Sym}(X_n)$. Let $\rho_n : \mathbb{F}_S \to \mathrm{Sym}(X_n)$ be the homomorphism given by $\rho_n(\bar{s}) = \sigma_n(s)$, for all $s \in S$. Let $g \in \ker \pi$ and write $g = \bar{s}_1^{\varepsilon_1} ... \bar{s}_k^{\varepsilon_k}$, for $s_1, ..., s_k \in S$ and $\varepsilon_1, ..., \varepsilon_k \in \{\pm 1\}$. Then $\rho_n(g) = \sigma_n(s_1)^{\varepsilon_1} ... \sigma_n(s_k)^{\varepsilon_k}$. Since $s_1^{\varepsilon_1} ... s_k^{\varepsilon_k} = e$ and $(\sigma_n)$ is an asymptotic homomorphism, we get that $\mathrm{d_H}(\rho_n(g), \mathrm{Id}_{X_n}) \to 0$. Since $\max_{s \in T} \mathrm{d_H}(\rho_n(\bar{s}), \tau_n(s)) > \varepsilon$, for any $n \in \mathbb{N}$ and homomorphism $\tau_n : \Gamma \to \mathrm{Sym}(X_n)$, we get that $(\star)$ is not satisfied. This finishes the proof of the lemma. ■

### 3.2. **Comparisons between various versions of P-stability.**

vs

**Lemma 3.2.** *Let* $\Gamma$ *be a countable group.*

1. *If* $\Gamma$ *is amenable, then it is* P*-stable if and only if it is flexibly* P*-stable.*
2. *If* $\Gamma$ *has property* $(\tau)$*, then it is flexibly* P*-stable if and only if it is very flexibly* P*-stable.*

*Proof.* (1) Assume that $\Gamma$ is a flexibly P-stable amenable group. In order to conclude that $\Gamma$ is P-stable, it is sufficient to prove the following claim:

**Claim.** Let $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ be an asymptotic homomorphism and $0 < \varepsilon < 1$. Then we can find a subsequence $(\sigma_{n_k})$ of $(\sigma_n)$ and homomorphisms $\tau_k : \Gamma \to \mathrm{Sym}(X_{n_k})$, for any $k \in \mathbb{N}$, such that $\limsup_{k \to \infty} \mathrm{d_H}(\sigma_{n_k}(g), \tau_k(g)) \leq \varepsilon$, for every $g \in \Gamma$.

To prove this claim we treat separately two cases. Firstly, assume that $N := \sup_n |X_n| < +\infty$. Since $\Gamma$ is flexibly P-stable, there are homomorphisms $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$, with $Y_n \supset X_n$ finite, such that $|Y_n|/|X_n| \to 1$ and $\mathrm{d_H}(\sigma_n(g), \tau_n(g)_{|X_n}) \to 0$, for every $g \in \Gamma$. Thus, $|Y_n|/|X_n| < 1 + \frac{1}{N}$ and therefore $Y_n = X_n$, for $n$ large. This clearly implies the claim.

Secondly, assume that $\sup_n |X_n| = +\infty$. After replacing $(\sigma_n)$ with a subsequence, we may suppose that $|X_n| \to +\infty$. Since $\Gamma$ is amenable, by using Ornstein and Weiss' theorem [OW80] (similarly to the proof of [BLT18, Proposition 6.5]), we can find a subsequence $(\sigma_{n_k})$ of $(\sigma_n)$ and $A_k \subset X_{n_k}$, for any $k \in \mathbb{N}$, such that $|\sigma_{n_k}(g)A_k \triangle A_k|/|X_{n_k}| \to 0$, for every $g \in \Gamma$, and $|A_k|/|X_{n_k}| \to \lambda := 1 - \varepsilon$.

For $k \in \mathbb{N}$, let $\rho_k : \Gamma \to \mathrm{Sym}(A_k)$ be a map such that $\rho_k(g)$ agrees with $\sigma_{n_k}(g)$ on $A_k \cap \sigma_{n_k}(g)^{-1}A_k$, for every $g \in \Gamma$. Then $(\rho_k)$ is an asymptotic homomorphism. Since $\Gamma$ is flexibly P-stable, there are $Y_k \supset A_k$ finite and homomorphisms $\zeta_k : \Gamma \to \mathrm{Sym}(Y_k)$ such that $\mathrm{d_H}(\rho_k(g), \zeta_k(g)_{|A_k}) \to 0$, for every $g \in \Gamma$, and $|Y_k|/|A_k| \to 1$. Since $|A_k|/|X_{n_k}| \to \lambda < 1$, we have $|Y_k| < |X_{n_k}|$ and so we may assume that $Y_k \subset X_{n_k}$, for $k$ large. If $\tau_k : \Gamma \to \mathrm{Sym}(X_{n_k})$ is the homomorphism given by $\tau_k(g)_{|Y_k} = \zeta_k(g)$ and $\tau_k(g)_{|X_{n_k}\setminus Y_k} = \mathrm{Id}_{X_{n_k}\setminus Y_k}$, then $\limsup_{k\to\infty} \mathrm{d_H}(\sigma_{n_k}(g), \tau_k(g)) \leq \lim_{k\to\infty} |X_{n_k} \setminus A_k|/|X_{n_k}| = \varepsilon$, for every $g \in \Gamma$. This finishes the proof of the claim and of part (1).

(2) Assume that $\Gamma$ is a very flexibly P-stable group with property ($\tau$). Let $\sigma_n : \Gamma \to \mathrm{Sym}(X_n)$ be an asymptotic homomorphism. Then we can find homomorphisms $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$, with $Y_n \supset X_n$, for every $n \in \mathbb{N}$, such that $\mathrm{d_H}(\sigma_n(g), \tau_n(g)_{|X_n}) \to 0$, for any $g \in \Gamma$. Since

$$\{x \in X_n \mid \tau_n(g)x \notin X_n\} \subset \{x \in X_n \mid \sigma_n(g)x \neq \tau_n(g)x\},$$

we get that $|\tau_n(g)X_n \triangle X_n|/|X_n| \to 0$, for any $g \in \Gamma$. Since $\Gamma$ has property ($\tau$), Lemma 2.5 implies that $\sup\{|\tau_n(g)X_n \triangle X_n|/|X_n| \mid g \in \Gamma\} \to 0$. By Lemma 2.1, there is a $\tau_n(\Gamma)$-invariant set $Z_n \subset Y_n$ such that $|Z_n \triangle X_n|/|X_n| \to 0$. Let $T_n = X_n \cup Z_n$ and $\rho_n : \Gamma \to \mathrm{Sym}(T_n)$ be the homomorphism given by $\rho_n(g)_{|Z_n} = \tau_n(g)_{|Z_n}$ and $\rho_n(g)_{|X_n\setminus Z_n} = \mathrm{Id}_{X_n\setminus Z_n}$. Then we have $X_n \subset T_n$, $|T_n|/|X_n| \to 1$, and $\mathrm{d_H}(\sigma_n(g), \rho_n(g)_{|X_n}) \to 0$, for every $g \in \Gamma$. This shows that $\Gamma$ is flexibly P-stable. ■

3.3. **Subgroups of finite index and very flexible P-stability.** We end this section by proving that very flexible P-stability passes to subgroups of finite index: finindex

**Lemma 3.3.** *Let $\Gamma_0 < \Gamma$ be a finite index inclusion of countable groups. If $\Gamma$ is very flexibly* P*-stable, then so is $\Gamma_0$. Moreover, if $\Gamma$ is weakly very flexibly* P*-stable, then so is $\Gamma_0$.*

The proof is based on a simple induction argument (compare with [ESS18, Proposition 4.12]). Let $s : \Gamma/\Gamma_0 \to \Gamma$ be a map such that $s(e\Gamma_0) = e$ and $s(g\Gamma_0) \in g\Gamma_0$, for all $g \in \Gamma$. Then $c : \Gamma \times \Gamma/\Gamma_0 \to \Gamma_0$ given by $c(g, h\Gamma_0) = s(gh\Gamma_0)^{-1}g\ s(h\Gamma_0)$ is a cocycle for the left multiplication action $\Gamma \curvearrowright \Gamma/\Gamma_0$, that is, $c(gh, k\Gamma_0) = c(g, hk\Gamma_0)c(h, k\Gamma_0)$, for all $g, h \in \Gamma$ and $k\Gamma_0 \in \Gamma/\Gamma_0$.

**Definition 3.4.** Let $\sigma_n : \Gamma_0 \to \mathrm{Sym}(X_n)$ be an asymptotic homomorphism. For every $n$, we define the *induced asymptotic homomorphism* $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\sigma_n) : \Gamma \to \mathrm{Sym}(\Gamma/\Gamma_0 \times X_n)$ by letting

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\sigma_n)(g)(h\Gamma_0, x) = (gh\Gamma_0, \sigma_n(c(g, h\Gamma_0))x), \text{ for every } g \in \Gamma, h\Gamma_0 \in \Gamma/\Gamma_0 \text{ and } x \in X_n.$$

The fact that $\widetilde{\sigma}_n := \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\sigma_n)$ is an asymptotic homomorphism follows by calculating that

$$\mathrm{d_H}(\widetilde{\sigma}_n(gh), \widetilde{\sigma}_n(g)\widetilde{\sigma}_n(h)) = \frac{1}{|\Gamma/\Gamma_0|} \sum_{k\Gamma_0 \in \Gamma/\Gamma_0} \mathrm{d_H}(\sigma_n(c(gh, k\Gamma_0)), \sigma_n(c(g, hk\Gamma_0))\sigma_n(c(h, k\Gamma_0))$$

and using the cocycle formula.

*Proof of Lemma 3.3.* Assume that $\Gamma_0$ is not very flexibly P-stable. Then there are an asymptotic homomorphism $\sigma_n : \Gamma_0 \to \mathrm{Sym}(X_n)$, a finite set $F \subset \Gamma_0$ and $\delta > 0$ such that for any sets $Y_n \supset X_n$ and homomorphisms $\tau_n : \Gamma_0 \to \mathrm{Sym}(Y_n)$ we have $\max\{\mathrm{d_H}(\sigma_n(g), \tau_n(g)_{|X_n}) \mid g \in F\} \geq \delta$, for all $n$.

Let $\widetilde{X}_n := \Gamma/\Gamma_0 \times X_n$ and denote by $\widetilde{\sigma}_n := \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\sigma_n) : \Gamma \to \mathrm{Sym}(\widetilde{X}_n)$ the induced asymptotic homomorphism. Consider a sequence of sets $Y_n \supset \widetilde{X}_n$ and homomorphisms $\tau_n : \Gamma \to \mathrm{Sym}(Y_n)$. If $g \in \Gamma_0$, then $\widetilde{\sigma}_n(g)$ leaves $e\Gamma_0 \times X_n$ invariant and $\widetilde{\sigma}_n(g)(e\Gamma_0, x) = (e\Gamma_0, \sigma_n(g)x)$, for every $x \in X_n$. Thus, the restriction of $\widetilde{\sigma_{n|\Gamma_0}}$ to $e\Gamma_0 \times X_n$ can be identified to $\sigma_n$. Since $\tau_{n|\Gamma_0}$ is a homomorphism, it follows that $\max\{\mathrm{d_H}(\widetilde{\sigma}_n(g)_{|e\Gamma_0\times X_n}, \tau_n(g)_{|e\Gamma_0\times X_n}) \mid g \in F\} \geq \delta$. Thus,

$$\max\{\mathrm{d_H}(\widetilde{\sigma}_n(g), \tau_n(g)_{|\widetilde{X}_n}) \mid g \in F\} \geq \frac{\delta}{[\Gamma : \Gamma_0]} > 0, \text{ for all } n,$$

which implies that $\Gamma$ is not very flexibly P-stable. This proves the main assertion.

For the moreover assertion, assume the setting above and let $g \in \Gamma \setminus \{e\}$. Then we have that

sigma_n

$$|\{\tilde{x} \in \widetilde{X}_n \mid \widetilde{\sigma}_n(g)\tilde{x} = \tilde{x}\}| = \sum_{h\Gamma_0 \in \Gamma/\Gamma_0, gh\Gamma_0 = h\Gamma_0} |\{x \in X_n \mid \sigma_n(c(g,h\Gamma_0))x = x\}|. \tag{3.1}$$

If $h\Gamma_0 \in \Gamma/\Gamma_0$ is such that $gh\Gamma_0 = h\Gamma_0$, then we have $c(g,h\Gamma_0) = s(h\Gamma_0)^{-1}gs(h\Gamma_0) \neq e$. Thus, if $\sigma_n : \Gamma_0 \to \mathrm{Sym}(X_n)$ is a sofic approximation of $\Gamma_0$, then using (3.1) it follows that $\widetilde{\sigma}_n : \Gamma \to \mathrm{Sym}(\widetilde{X}_n)$ is a sofic approximation of $\Gamma$, and repeating the above argument implies the moreover assertion. ■

## 4. Permutation groups almost commuting with the regular representation

The main goal of this section is to prove the following result. This implies that any group of permutations of a finite group $G$ that "almost commutes" with the left regular representation of $G$ must arise from the right regular representation of $G$. More generally, we get precise structural information about any permutation group of a set containing $G$ whose restriction to $G$ almost commutes with the left regular representation of $G$. This generalization will be crucial later on in allowing us to prove that certain product groups are not very flexibly P-stable.

almost

**Theorem 4.1.** *Let $G$ be a finite group, $S$ be a set of generators and put $\kappa := \kappa(G,S)$. Denote by $\alpha, \beta : G \to \mathrm{Sym}(X)$ the left and right multiplication actions of $G$ on $X := G$ given by $\alpha(g)x = gx$ and $\beta(g)x = xg^{-1}$, for every $g \in G$ and $x \in X$. Let $Y$ be a set containing $X$ and $K < \mathrm{Sym}(Y)$ be a subgroup. Let $\varepsilon \in (0, \frac{\kappa^4}{200})$ and assume that*

$$|\{x \in X \cap k^{-1}X \mid \alpha(g)kx \neq k\alpha(g)x\}| \leq \varepsilon \cdot |X|, \text{ for all } g \in S \text{ and } k \in K.$$

*Then $K_0 = \{k \in K \mid |X \cap kX| \geq \frac{|X|}{2}\}$ is a subgroup of $K$. Moreover, we can find a homomorphism $\delta : K_0 \to G$, a $K_0$-invariant set $X_1 \subset Y$, a $\beta(\delta(K_0))$-invariant set $X_2 \subset X$, and a bijection $\varphi : X_1 \to X_2$ such that*

*(1) $|X \setminus X_2| < \frac{4160}{\kappa^4} \cdot \varepsilon \cdot |X|$,*
*(2) $|\{x \in X_1 \mid \varphi(x) \neq x\}| \leq \frac{2048}{\kappa^4} \cdot \varepsilon \cdot |X|$, and*
*(3) $\varphi \circ k_{|X_1} = \beta(\delta(k)) \circ \varphi$, for all $k \in K_0$.*

The proof of Theorem 4.1 relies on the following two lemmas.

commutant

**Lemma 4.2.** [Th10] *Let $G$ be a finite group, $S$ be a set of generators and put $\kappa := \kappa(G,S)$. Denote by $\alpha, \beta : G \to \mathrm{Sym}(G)$ the left and right multiplication actions of $G$ on itself. Then for every $\varphi \in \mathrm{Sym}(G)$, there exists $h \in G$ such that $\kappa^2 \cdot \mathrm{d_H}(\varphi, \beta(h)) \leq 4 \cdot \max_{g \in S} \mathrm{d_H}(\alpha(g) \circ \varphi, \varphi \circ \alpha(g))$.*

After proving Lemma 4.2, we realized that it also follows from the proof of [Th10, Theorem 2.2]. Nevertheless, we include a self-contained proof for completeness.

*Proof.* Let $\varphi \in \mathrm{Sym}(G)$ and put $\varepsilon = \max_{g \in S} \mathrm{d_H}(\alpha(g) \circ \varphi, \varphi \circ \alpha(g))$. Consider the unitary representation of $G$ on $\mathrm{B}(\ell^2(G))$ given by $g \cdot T = \alpha(g)T\alpha(g)^*$, where we view $\mathrm{Sym}(G)$ as a subgroup of $\mathrm{U}(\ell^2(G))$ and endow $\mathrm{B}(\ell^2(G))$ with the normalized Hilbert-Schmidt norm. Lemma 2.5 implies that

$$\kappa \cdot \max_{g \in G} \|\alpha(g) \circ \varphi - \varphi \circ \alpha(g)\|_{\mathrm{HS}} \leq 2 \cdot \max_{g \in S} \|\alpha(g) \circ \varphi - \varphi \circ \alpha(g)\|_{\mathrm{HS}}.$$

Recalling that $\|\sigma - \tau\|_{HS} = \sqrt{2 \cdot \mathrm{d_H}(\sigma,\tau)}$, for all $\sigma, \tau \in \mathrm{Sym}(G)$, the last inequality rewrites as $\mathrm{d_H}(\alpha(g) \circ \varphi, \varphi \circ \alpha(g)) \leq \frac{4\varepsilon}{\kappa^2}$, for all $g \in G$. Equivalently, we have $|\{x \in G \mid \varphi(gx) \neq g\varphi(x)\}| \leq \frac{4\varepsilon}{\kappa^2} \cdot |G|$, for every $g \in G$, and hence

$$\sum_{x\in G} |\{g \in G \mid \varphi(gx) \neq g\varphi(x)\}| = \sum_{g\in G} |\{x \in G \mid \varphi(gx) \neq g\varphi(x)\}| \leq \frac{4\varepsilon}{\kappa^2} \cdot |G|^2.$$

Thus, there exists $x \in G$ such that $|\{g \in G \mid \varphi(gx) \neq g\varphi(x)\}| \leq \frac{4\varepsilon}{\kappa^2} \cdot |G|$. Hence, $h = \varphi(x)^{-1}x \in G$ satisfies $\mathrm{d_H}(\varphi, \beta(h)) \leq \frac{4\varepsilon}{\kappa^2}$. ■

**Lemma 4.3.** *Let $X$ be a finite set, $K$ a group and $\alpha_1, \alpha_2 : K \to Sym(X)$ homomorphisms. Assume that $\mathrm{d_H}(\alpha_1(k), \alpha_2(k)) \leq \varepsilon$, for all $k \in K$, for some $\varepsilon > 0$.*

*Then there exist an $\alpha_1(K)$-invariant set $X_1 \subset X$, an $\alpha_2(K)$-invariant set $X_2 \subset X$, and a bijection $\varphi : X_1 \to X_2$ such that $|X \setminus X_1| = |X \setminus X_2| \leq 16\varepsilon \cdot |X|$, $|\{x_1 \in X_1 \mid \varphi(x_1) \neq x_1\}| \leq 16\varepsilon \cdot |X|$, and*

$$\varphi \circ \alpha_1(k)_{|X_1} = \alpha_2(k) \circ \varphi, \text{ for all } k \in K.$$

*Moreover, if $\varepsilon < \frac{1}{16}$ and $\alpha_1$ is transitive, then $\alpha_1$ and $\alpha_2$ are conjugate.*

*Proof.* We follow closely the proofs of [Hj03, Lemma 2.5] and [Io06, Theorem 1.3]. We start by defining $V = \frac{1}{|K|}\sum_{k\in K} \alpha_2(k)^{-1} \circ \alpha_1(k) \in \mathrm{B}(\ell^2(X))$. Then $\alpha_2(k)^{-1} V \alpha_1(k) = V$, for every $k \in K$. Thus, the matrix coefficients $V_{x_1,x_2} = \langle V\delta_{x_1}, \delta_{x_2}\rangle$ satisfy

$$V_{x_1,x_2} = V_{\alpha_1(k)x_1,\alpha_2(k)x_2}, \text{ for all } x_1, x_2 \in X \text{ and } k \in K. \tag{4.1}$$

Since $\|\alpha_1(k)^{-1} \circ \alpha_2(k) - \mathrm{Id}\|_{\mathrm{HS}} = \sqrt{2 \cdot \mathrm{d_H}(\alpha_1(k), \alpha_2(k))} \leq \sqrt{2\varepsilon}$, for every $k \in K$, we deduce that $\|V - \mathrm{Id}\|_{\mathrm{HS}} \leq \sqrt{2\varepsilon}$. Equivalently, we have

$$\frac{1}{|X|}\Big(\sum_{x_1\in X} |V_{x_1,x_1} - 1|^2 + \sum_{x_1,x_2\in X, x_1\neq x_2} |V_{x_1,x_2}|^2\Big) \leq 2\varepsilon. \tag{4.2}$$

Let $A$ be the set of $x_1 \in X$ for which there exists a unique $x_2 = \varphi(x_1) \in X$ such that $|V_{x_1,x_2}| > \frac{1}{2}$. Then equation (4.1) implies that $A$ is $\alpha_1(K)$-invariant and

$$\varphi(\alpha_1(k)x_1) = \alpha_2(k)\varphi(x_1), \text{ for all } x_1 \in A \text{ and } k \in K. \tag{4.3}$$

Moreover, $A$ contains the set $X_0$ of $x_1 \in X$ such that $|V_{x_1,x_1} - 1|^2 + \sum_{x_2\in X, x_2\neq x_1} |V_{x_1,x_2}|^2 < \frac{1}{4}$. On the other hand, (4.2) implies that $\frac{|X\setminus X_0|}{4|X|} \leq 2\varepsilon$. Thus, $|X \setminus A| \leq |X \setminus X_0| \leq 8\varepsilon\, |X|$. Similarly, the set $B$ of $x_2 \in X$ for which there is a unique $x_1 \in X$ with $|V_{x_1,x_2}| > \frac{1}{2}$ satisfies $|X \setminus B| \leq 8\varepsilon\, |X|$.

Define $X_1 = \{x_1 \in A \mid \varphi(x_1) \in B\}$ and $X_2 = \varphi(X_1)$. Then the restriction of $\varphi$ to $X_1$ is one-to-one. Since $B$ is $\alpha_2(K)$-invariant, (4.3) gives that $X_1$ is $\alpha_1(K)$-invariant and $X_2$ is $\alpha_2(K)$-invariant. Since $\varphi(x_1) = x_1$ for all $x_1 \in X_0$, we get that $X_0 \cap B \subset X_1$. Thus, $|X \setminus X_1| \leq |X \setminus X_0| + |X \setminus B| \leq 16\varepsilon\, |X|$ and $|\{x_1 \in X_1 \mid \varphi(x_1) \neq x_1\}| \leq |X_1 \setminus (X_0 \cap B)| \leq |X \setminus (X_0 \cap B)| \leq 16\varepsilon\, |X|$.

If $\varepsilon < \frac{1}{16}$, then $|X \setminus X_1| \leq 16\, \varepsilon\, |X| < |X|$, and thus $X_1$ is non-empty. Since $X_1$ is $\alpha_1(K)$-invariant, if $\alpha_1$ is transitive, we get that $X_1 = X$ and the moreover assertion follows. ■

**Proof of Theorem 4.1.** We will first show that $K_0$ is a subgroup of $K$. The proof of this assertion is inspired by the proof of [GTD15, Theorem 2.4]. Note that $K_0$ is clearly closed under inverses. If $g \in S$ and $k \in K$, then

$$\begin{aligned}\alpha(g)(X \cap kX) \setminus (X \cap kX) &= \alpha(g)(\{x \in X \cap kX \mid \alpha(g)x \notin X \cap kX\}) \\ &= \alpha(g)k(\{x \in X \cap k^{-1}X \mid \alpha(g)kx \notin X \cap kX\}) \\ &\subset \alpha(g)k(\{x \in X \cap k^{-1}X \mid \alpha(g)kx \neq k\alpha(g)x\}),\end{aligned}$$

and thus $|\alpha(g)(X \cap kX) \setminus (X \cap kX)| \leq \varepsilon \cdot |X|$.

Therefore, if $k\in K_0$, then for every $g\in S$ we have $|\alpha(g)(X\cap kX)\triangle(X\cap kX)|\leq 2\varepsilon\cdot|X|\leq 4\varepsilon\cdot|X\cap kX|$. By applying Lemma 2.4 to $X\cap kX\subset X$ we deduce that

$$\kappa^2\cdot|X\cap kX|\cdot|X\setminus kX|\leq\max_{g\in S}|\alpha(g)(X\cap kX)\triangle(X\cap kX)|\cdot|X|\leq 4\varepsilon\cdot|X\cap kX|\cdot|X|.$$

Hence, $|X\setminus kX|\leq\frac{4\varepsilon}{\kappa^2}\cdot|X|$ and thus

invariance

$$|X\triangle kX|\leq\frac{8\varepsilon}{\kappa^2}\cdot|X|,\text{ for every } k\in K_0. \tag{4.4}$$

If $k,k'\in K_0$, then $|X\triangle k'kX|\leq|X\triangle k'X|+|k'X\triangle k'kX|=|X\triangle k'X|+|X\triangle kX|\leq\frac{16\varepsilon}{\kappa^2}\cdot|X|$, thus $|X\cap k'kX|\geq(1-\frac{8\varepsilon}{\kappa^2})\cdot|X|\geq|X|/2$ since $\kappa\leq 2$ and hence $\varepsilon<\frac{\kappa^4}{200}<\frac{\kappa^2}{16}$. This shows that $kk'\in K_0$ and therefore $K_0$ is a subgroup of $K$.

Secondly, we will prove the existence of a map $\delta:K_0\to G$ such that

unif

$$|\{x\in X\mid kx\neq\beta(\delta(k))x\}|\leq\frac{64\varepsilon}{\kappa^4}\cdot|X|,\text{ for every } k\in K_0. \tag{4.5}$$

To see this, let $k\in K_0$. Let $\tilde{k}\in\mathrm{Sym}(X)$ such that $\tilde{k}x=kx$, for every $x\in X\cap k^{-1}X$. If $g\in S$, then since $\tilde{k}\alpha(g)x=k\alpha(g)x$, for all $x\in X\cap\alpha(g)^{-1}k^{-1}X$, by using the hypothesis, we get that

$$|\{x\in X\mid\alpha(g)\tilde{k}x\neq\tilde{k}\alpha(g)x\}|\leq\varepsilon\cdot|X|+|X\setminus(k^{-1}X\cap\alpha(g)^{-1}k^{-1}X)|\leq\varepsilon\cdot|X|+2\cdot|X\setminus k^{-1}X|.$$

In combination with (4.4), this gives that

$$|\{x\in X\mid\alpha(g)\tilde{k}x\neq\tilde{k}\alpha(g)x\}|\leq(1+\frac{8}{\kappa^2})\varepsilon\cdot|X|,\text{ for every } g\in S.$$

Now, Lemma 4.2 gives $\delta(k)\in G$ such that $|\{x\in X\mid\tilde{k}x\neq\beta(\delta(k))x\}|\leq\frac{4}{\kappa^2}(1+\frac{8}{\kappa^2})\varepsilon\cdot|X|$. Together with (4.4) we get that

$$\begin{aligned}|\{x\in X\mid kx\neq\beta(\delta(k))x\}|&\leq|\{x\in X\mid\tilde{k}x\neq\beta(\delta(k))x\}|+|X\setminus k^{-1}X|\\&\leq\frac{4}{\kappa^2}(1+\frac{8}{\kappa^2})\varepsilon\cdot|X|+\frac{4}{\kappa^2}\varepsilon\cdot|X|.\end{aligned}$$

Since $\kappa\leq 2$, we have that $\frac{8}{\kappa^2}\leq\frac{32}{\kappa^4}$ and (4.5) follows.

Thirdly, we claim that $\delta:K_0\to G$ is a homomorphism. Denote $X_k=\{x\in X\mid kx=\beta(\delta(k))x\}$ for $k\in K$. Given $k',k\in K_0$, we have that $\beta(\delta(k'k))x=k'kx=k'\beta(\delta(k))x=\beta(\delta(k'))\beta(\delta(k))x$, for every $x\in X_{k'k}\cap X_k\cap\beta(\delta(k))^{-1}X_{k'}$. Thus, by using (4.5) we get that

$$|\{x\in X\mid\beta(\delta(k'k))x\neq\beta(\delta(k'))\beta(\delta(k))x\}|\leq\frac{192\varepsilon}{\kappa^4}\cdot|X|.$$

Since $\varepsilon<\frac{\kappa^4}{200}$, we get that there exists $x\in X$ such that $\beta(\delta(k'k))x=\beta(\delta(k'))\beta(\delta(k))x$. Equivalently, $x\delta(k'k)^{-1}=x\delta(k)^{-1}\delta(k')^{-1}$, and thus $\delta(k'k)=\delta(k')\delta(k)$, which proves that $\delta$ is a homomorphism.

Finally, we will derive the rest of the conclusion by applying Lemma 4.3. First, note that equation (4.4) together with Lemma 2.1 provides a $K_0$-invariant set $X_0\subset Y$ such that $|X_0\triangle X|\leq\frac{16\varepsilon}{\kappa^2}\cdot|X|$. We put $Z=X_0\cup X$ and define homomorphisms $\alpha_1,\alpha_2:K_0\to\mathrm{Sym}(Z)$ by letting for every $k\in K_0$

$$\alpha_1(k)_{|X_0}=k_{|X_0},\alpha_1(k)_{|Z\setminus X_0}=\mathrm{Id}_{Z\setminus X_0}\text{ and}$$

$$\alpha_2(k)_{|X}=\beta(\delta(k))_{|X},\ \alpha_2(k)_{|Z\setminus X}=\mathrm{Id}_{Z\setminus X}.$$

Since $\{x\in Z\mid\alpha_1(k)x\neq\alpha_2(k)x\}\subset(X_0\triangle X)\cup\{x\in X_0\cap X\mid kx\neq\beta(\delta(k))x\}$, (4.5) implies that

$$|\{x\in Z\mid\alpha_1(k)x\neq\alpha_2(k)x\}|\leq\frac{16\varepsilon}{\kappa^2}\cdot|X|+\frac{64\varepsilon}{\kappa^4}\cdot|X|\leq\frac{128\varepsilon}{\kappa^4}\cdot|X|,\text{ for every } k\in K_0.$$

By applying Lemma 4.3, we find an $\alpha_1(K_0)$-invariant set $Z_1 \subset Z$, an $\alpha_2(K_0)$-invariant set $Z_2 \subset Z$, and a bijection $\varphi : Z_1 \to Z_2$ such that

- $|Z \setminus Z_1| = |Z \setminus Z_2| \leq \frac{16 \cdot 128\varepsilon}{\kappa^4} \cdot |X|$,
- $|\{z \in Z \mid \varphi(z) \neq z\}| \leq \frac{16 \cdot 128\varepsilon}{\kappa^4} \cdot |X|$ and
- $\varphi \circ \alpha_1(k)_{|Z_1} = \alpha_2(k) \circ \varphi$, for all $k \in K_0$.

Then $Z_1 \cap X_0$ is $\alpha_1(K_0)$-invariant and $Z_2 \cap X$ is $\alpha_2(K_0)$-invariant. Thus, $X_1 = (Z_1 \cap X_0) \cap \varphi^{-1}(Z_2 \cap X)$ is $\alpha_1(K_0)$-invariant and $X_2 = \varphi(X_1)$ is $\alpha_2(K_0)$-invariant. Since $X_1 \subset X_0$ and $X_2 \subset X$, we get that $X_1$ is $K_0$-invariant, $X_2$ is $\beta(\delta(K_0))$-invariant, and $\varphi \circ k_{|X_1} = \beta(\delta(k)) \circ \varphi_{|X_1}$, for all $k \in K_0$. This proves condition *(3)* for $\varphi_{|X_1}$.

In order to complete the proof, it remains to establish conditions *(1)* and *(2)*. First, we note that

$$|\{x \in X_1 \mid \varphi(x) \neq x\}| \leq |\{z \in Z \mid \varphi(z) \neq z\}| \leq \frac{2048\varepsilon}{\kappa^4} \cdot |X|.$$

Second, since $X_2 = \varphi(Z_1 \cap X_0) \cap (Z_2 \cap X)$, we have $|X \setminus X_2| \leq |Z \setminus X_2| \leq |Z \setminus \varphi(Z_1 \cap X_0)| + |Z \setminus (Z_2 \cap X)|$. Since $|Z \setminus \varphi(Z_1 \cap X_0)| = |Z \setminus (Z_1 \cap X_0)|$, we altogether derive that

$$\begin{aligned} |X \setminus X_2| &\leq |Z \setminus Z_1| + |Z \setminus X_0| + |Z \setminus Z_2| + |Z \setminus X| \\ &= 2|Z \setminus Z_1| + |X_0 \triangle X| \\ &\leq \frac{32 \cdot 128\varepsilon}{\kappa^4} \cdot |X| + \frac{16\varepsilon}{\kappa^2} \cdot |X|. \end{aligned}$$

Since $\frac{16}{\kappa^2} \leq \frac{64}{\kappa^4}$, we have that $\frac{32 \cdot 128}{\kappa^4} + \frac{16}{\kappa^2} \leq \frac{4160}{\kappa^4}$, which finishes the proof. ■

## 5. A rigidity result for asymptotic homomorphisms

In this section we prove the following consequence of Theorem 4.1. For an informal description of this result, see the comments on the proof of Theorem A in the end of the introduction. conjugation

**Theorem 5.1.** *Let $\Gamma$ and $\Lambda$ be finitely generated groups. Assume that $\Gamma$ has property ($\tau$) with respect to a sequence of finite index normal subgroups $\{\Gamma_n\}_{n=1}^{\infty}$. For every $n$, denote $X_n = \Gamma/\Gamma_n$, let $p_n : \Gamma \to X_n$ be the quotient homomorphism and $q_n : \Lambda \to X_n$ be a homomorphism.*

*Assume that $\sigma_n : (\Gamma * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(X_n)$, $n \in \mathbb{N}$, is an asymptotic homomorphism such that*

1. *For every $n \in \mathbb{N}$, we have $\sigma_n(g, h)x = p_n(g)xq_n(h)^{-1}$, for all $g \in \Gamma, h \in \Lambda, x \in X_n$.*
2. *For every $n \in \mathbb{N}$, there exist a set $Y_n$ which contains $X_n$ and a homomorphism $\tau_n : (\Gamma * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(Y_n)$ such that $\lim\limits_{n\to\infty} \mathrm{d_H}(\sigma_n(g), \tau_n(g)_{|X_n}) = 0$, for all $g \in (\Gamma * \mathbb{Z}) \times \Lambda$.*

*Then* $\lim\limits_{n\to\infty} \Big( \max\{\mathrm{d_H}(\sigma_n(t, e) \circ \sigma_n(e, h), \sigma_n(e, h) \circ \sigma_n(t, e)) \mid h \in \Lambda\} \Big) = 0$, *for every $t \in \mathbb{Z}$.*

*Moreover, there exists a homomorphism $\sigma_n' : (\Gamma * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(X_n)$ such that*

(a) *$\sigma'_{n|\Gamma\times\Lambda} = \sigma_{n|\Gamma\times\Lambda}$, for every $n \in \mathbb{N}$, and*
(b) *$\lim\limits_{n\to\infty} \mathrm{d_H}(\sigma_n(g), \sigma_n'(g)) = 0$, for every $g \in (\Gamma * \mathbb{Z}) \times \Lambda$.*

*Proof.* Let $S$ and $T$ be finite sets of generators for $\Gamma$ and $\Lambda$, respectively. For $n \in \mathbb{N}$, we denote by $\beta_n : \Gamma/\Gamma_n \to \mathrm{Sym}(X_n)$ the homomorphism given by $\beta_n(g)x = xg^{-1}$, for every $g \in \Gamma/\Gamma_n$ and $x \in X_n$. For ease of notation, we will write $g$ and $h$ instead of $(g, e)$ and $(e, h)$, for $g \in \Gamma * \mathbb{Z}$ and $h \in \Lambda$.

In the first part of the proof we will use Theorem 4.1 to prove the following:

**Claim.** For every $n$ large enough, there exist a $\tau_n(\Lambda)$-invariant set $X_n' \subset Y_n$, a subgroup $L_n < X_n$, a $\beta_n(L_n)$-invariant set $X_n'' \subset X_n$ and a bijection $\varphi_n : X_n' \to X_n''$ such that

(1) $\lim\limits_{n\to\infty} \frac{|X_n'|}{|X_n|} = \lim\limits_{n\to\infty} \frac{|X_n''|}{|X_n|} = 1$,
(2) $\lim\limits_{n\to\infty} \frac{1}{|X_n|}\, |\{x \in X_n' \mid \varphi_n(x) \neq x\}| = 0$, and
(3) $\varphi_n \circ \tau_n(\Lambda)_{|X_n'} \circ \varphi_n^{-1} = \beta_n(L_n)_{|X_n''}$.

*Proof of the claim.* For $n \geq 1$, we put $\varepsilon_n = 2 \cdot \max\{\mathrm{d_H}(\sigma_n(g), \tau_n(g)_{|X_n}) \mid g \in S \cup T\}$ and

$$K_n = \{k \in \tau_n(\Lambda) \mid |X_n \cap kX_n| \geq \frac{|X_n|}{2}\}.$$

If $k \in \tau_n(\Lambda)$ and $g \in S$, then $k, \tau_n(g) \in \mathrm{Sym}(Y_n)$ commute and thus

$\{x \in X_n \cap k^{-1}X_n \mid \sigma_n(g)kx \neq k\sigma_n(g)x\} \subset \{x \in X_n \cap k^{-1}X_n \mid \sigma_n(g)kx \neq \tau_n(g)kx \;\text{ or }\; \sigma_n(g)x \neq \tau_n(g)x\}.$

Therefore, for all $k \in \tau_n(\Lambda)$ and $g \in S$ we have

S
$$|\{x \in X_n \cap k^{-1}X_n \mid \sigma_n(g)kx \neq k\sigma_n(g)x\}| \leq 2 \cdot \mathrm{d_H}(\sigma_n(g), \tau_n(g)_{|X_n})\, |X_n| \leq \varepsilon_n \cdot |X_n|. \tag{5.1}$$

Moreover, if $g \in T$, then $X_n \setminus \tau_n(g)^{-1}X_n = \{x \in X_n \mid \tau_n(g)x \notin X_n\} \subset \{x \in X_n \mid \tau_n(g)x \neq \sigma_n(g)x\}$, and therefore

T
$$|X_n \cap \tau_n(g)X_n| = |X_n| - |X_n \setminus \tau_n(g)^{-1}X_n| \geq (1 - \varepsilon_n) \cdot |X_n|, \text{ for every } g \in T. \tag{5.2}$$

Since $\Gamma$ has property $(\tau)$ with respect to $\{\Gamma_n\}$ we have $\kappa := \inf_n \kappa(X_n, p_n(S)) > 0$. Since $\lim\limits_{n\to\infty} \varepsilon_n = 0$, we have $\varepsilon_n < \min\{\frac{\kappa^4}{200}, \frac{1}{2}\}$ for every $n$ large enough. By (5.1), we can apply Theorem 4.1 to deduce that $K_n$ is a subgroup of $\tau_n(\Lambda)$ and there exist a $K_n$-invariant set $X_n' \subset Y_n$, a subgroup $L_n < X_n$, a $\beta_n(L_n)$-invariant subset $X_n'' \subset X_n$ and a bijection $\varphi_n : X_n' \to X_n''$ such that

- $|X_n'| = |X_n''| > (1 - \frac{4160\, \varepsilon_n}{\kappa^4}) \cdot |X_n|$,
- $|\{x \in X_n' \mid \varphi_n(x) \neq x\}| \leq \frac{2048\, \varepsilon_n}{\kappa^4} \cdot |X_n|$,
- $\varphi_n \circ K_{n|X_n'} \circ \varphi_n^{-1} = \beta_n(L_n)_{|X_n''}$.

Since $\varepsilon_n < \frac{1}{2}$, (5.2) guarantees that $\tau_n(T) \subset K_n$. Since $K_n$ is a subgroup of $\tau_n(\Lambda)$ and $T$ generates $\Lambda$, we derive that $K_n = \tau_n(\Lambda)$. Since $\lim\limits_{n\to\infty} \varepsilon_n = 0$, the claim follows. $\square$

Secondly, we claim that

claimunu
$$\sigma_n(\Lambda) \subset \beta_n(L_n), \text{ for every } n \text{ large enough.} \tag{5.3}$$

To see this, let $h \in T$. Then $\sigma_n(h) = \beta_n(q_n(h))$ and thus $\lim\limits_{n\to\infty} \mathrm{d_H}(\beta_n(q_n(h)), \tau_n(h)_{|X_n}) = 0$. On the other hand, conditions (1)-(3) from above imply that we can find a sequence $h_n \in L_n$ such that $\lim\limits_{n\to\infty} \mathrm{d_H}(\tau_n(h)_{|X_n}, \beta_n(h_n)) = 0$. Thus, we derive that $\lim\limits_{n\to\infty} \mathrm{d_H}(\beta_n(q_n(h)), \beta_n(h_n)) = 0$. Since $\mathrm{d_H}(\beta(k), \beta(k')) = \delta_{k,k'}$, for all $k, k' \in X_n$, we get that $q_n(h) = h_n \in L_n$, for large enough $n$. Since this holds for every $h \in T$, and $T$ is finite and generates $\Lambda$, the claim made in (5.3) follows.

Thirdly, we claim that if $g \in \Gamma * \mathbb{Z}$, then $\sigma_n(g)$ asymptotically commutes with $\beta_n(L_n)$:

claimdoi
$$\lim_{n\to\infty} \Big( \max\{\mathrm{d_H}(\sigma_n(g) \circ \beta_n(h), \beta_n(h) \circ \sigma_n(g)) \mid h \in L_n\} \Big) = 0 \tag{5.4}$$

To see this, let $l_n \in L_n$, for every $n$. Condition (3) implies that $\beta_n(l_n)_{|X_n''} = \varphi_n \circ \tau_n(k_n)_{|X_n'} \circ \varphi_n^{-1}$, for some $k_n \in \Lambda$. By combining (1) and (2) it follows that $\lim\limits_{n\to\infty} \mathrm{d_H}(\beta_n(l_n), \tau_n(k_n)_{|X_n}) = 0$. On the other hand, we have $\lim\limits_{n\to\infty} \mathrm{d_H}(\sigma_n(g), \tau_n(g)_{|X_n}) = 0$. Since $\tau_n(g)$ and $\tau_n(k_n)$ commute, we get that $\lim\limits_{n\to\infty} \mathrm{d_H}(\sigma_n(g) \circ \beta_n(l_n), \beta_n(l_n) \circ \sigma_n(g)) = 0$. As this holds for any sequence $l_n \in L_n$, (5.4) follows.

It is now clear that the combination of (5.3) and (5.4) gives that

claimtrei

$$\lim_{n\to\infty}\Big(\max\{\mathrm{d_H}(\sigma_n(g)\circ\sigma_n(h),\sigma_n(h)\circ\sigma_n(g))\mid h\in\Lambda\}\Big)=0,\text{ for every } g\in\Gamma*\mathbb{Z}. \tag{5.5}$$

Taking $g\in\mathbb{Z}$, this proves the main assertion. If $g\in\mathbb{Z}$ is a generator, then (5.5) together with Lemma 5.2 below implies the existence of $\sigma_n'(g)\in\mathrm{Sym}(X_n)$ which commutes with $\sigma_n(\Lambda)$ and satisfies that $\lim\limits_{n\to\infty}\mathrm{d_H}(\sigma_n'(g),\sigma_n(g))=0$. This implies the moreover assertion. ■

In order to complete the proof of Theorem 5.1, it remains to prove the following lemma. commutant2

**Lemma 5.2.** *Let $G$ be a finite group, $X$ a finite set, $\alpha: G\to\mathrm{Sym}(X)$ a homomorphism and $\varphi\in\mathrm{Sym}(X)$. Then there exists $\psi\in\mathrm{Sym}(X)$ which commutes with $\alpha(G)$ such that*

$$\mathrm{d_H}(\varphi,\psi)\le 32\cdot\max_{g\in G}\mathrm{d_H}(\alpha(g)\circ\varphi,\varphi\circ\alpha(g)).$$

*Proof.* Put $\varepsilon=\max_{g\in G}\mathrm{d_H}(\alpha(g)\circ\varphi,\varphi\circ\alpha(g))$. Then $\mathrm{d_H}(\varphi^{-1}\circ\alpha(g)\circ\varphi,\alpha(g))\le\varepsilon$, for any $g\in G$. By applying Lemma 4.3 to the homomorphisms $\varphi^{-1}\circ\alpha\circ\varphi,\alpha: G\to\mathrm{Sym}(X)$ we obtain an $\alpha(G)$-invariant set $X_1\subset X$, an $\varphi^{-1}\alpha(G)\varphi$-invariant set $X_2\subset X$ and a bijection $\sigma: X_1\to X_2$ such that $|X\setminus X_1|\le 16\varepsilon\cdot|X|$, $|\{x\in X_1\mid \sigma(x)\neq x\}|\le 16\varepsilon\cdot|X|$ and

$$\varphi^{-1}\circ\alpha(g)\circ\varphi\circ\sigma=\sigma\circ\alpha(g)_{|X_1},\text{ for all } g\in G.$$

Thus, $X_3=\varphi(X_2)$ is $\alpha(G)$-invariant and the bijection $\tau=\varphi\circ\sigma: X_1\to X_3$ satisfies

conjugat

$$\alpha(g)\circ\tau=\tau\circ\alpha(g)_{|X_1},\text{ for every } g\in G. \tag{5.6}$$

Next, we say that two actions $\beta: G\to\mathrm{Sym}(Y)$ and $\gamma: G\to\mathrm{Sym}(Z)$ are conjugate if there exists a bijection $\rho: Y\to Z$ such that $\rho\circ\beta(g)=\gamma(g)\circ\rho$, for every $g\in G$. Let $\mathrm{Sub}_\sim(G)$ be the set of equivalence classes $[H]$ of subgroups $H$ of $G$ modulo inner conjugacy. For a subgroup $H<G$, denote by $\zeta(\beta)([H])$ the number of disjoint $\beta(G)$-orbits $\beta(G)y$, with $y\in Y$, such that the restriction of $\beta$ to $\beta(G)y$ is conjugate to the action $G\curvearrowright G/H$. Then the conjugacy class of an action $\beta: G\to\mathrm{Sym}(Y)$ is completely determined by the map $\zeta(\beta):\mathrm{Sub}_\sim(G)\to\mathbb{N}$.

Finally, (5.6) implies that the restrictions of $\alpha$ to $X_1$ and $X_3$ are conjugate, hence $\zeta(\alpha_{|X_1})=\zeta(\alpha_{|X_3})$. This implies that $\zeta(\alpha_{|X\setminus X_1})=\zeta(\alpha_{|X\setminus X_3})$ and so restrictions of $\alpha$ to $X\setminus X_1$ and $X\setminus X_3$ are conjugate. In combination with 5.6, we derive that there exists $\psi\in\mathrm{Sym}(X)$ which commutes with $\alpha(G)$ (i.e., a self-conjugacy of $\alpha$) such that $\psi_{|X_1}=\tau$. Hence,

$$|\{x\in X\mid\psi(x)\neq\varphi(x)\}|\le|X\setminus X_1|+|\{x\in X_1\mid\sigma(x)\neq x\}|\le 32\varepsilon\cdot|X|$$

and the conclusion follows. ■

## 6. Construction of asymptotic homomorphisms

This section is devoted to the construction of asymptotic homomorphisms. In the next two sections, we will combine this construction with Theorem 5.1 to deduce our main results. tech2

**Lemma 6.1.** *Let $\Gamma$ and $\Lambda$ be finitely generated groups. Let $\{\Gamma_n\}_{n=1}^\infty$ be a sequence of finite index normal subgroups of $\Gamma$, put $X_n=\Gamma/\Gamma_n$ and denote by $p_n:\Gamma\to X_n$ the quotient homomorphism. Assume that there exists a sequence of homomorphisms $q_n:\Lambda\to X_n$ such that $\Lambda$ does not have property ($\tau$) with respect to the sequence $\{\ker(q_n)\}_{n=1}^\infty$. Let $t=\pm1$ be a generator of $\mathbb{Z}$.*

*Then there exists an asymptotic homomorphism $\sigma_n:(\Gamma*\mathbb{Z})\times\Lambda\to\mathrm{Sym}(X_n)$ such that*

*(1) $\sigma_n(g,h)x=p_n(g)xq_n(h)^{-1}$, for all $g\in\Gamma,h\in\Lambda,x\in X_n$, and*
*(2) $\max\{\mathrm{d_H}(\sigma_n(t,e)\circ\sigma_n(e,h),\sigma_n(e,h)\circ\sigma_n(t,e))\mid h\in\Lambda\}\ge\frac{1}{126}$, for infinitely many $n\ge 1$.*

*Proof.* The first part of the proof is devoted to the construction of $\sigma_n$. Let $\sigma_n:\Gamma\times\Lambda\to\mathrm{Sym}(X_n)$ be given by *(1)*. In order to extend $\sigma_n$ to an asymptotic homomorphism of $(\Gamma*\mathbb{Z})\times\Lambda$ we will define $\sigma_n(t,e)\in\mathrm{Sym}(X_n)$ such that $\lim\limits_{n\to\infty}\mathrm{d_H}(\sigma_n(t,e)\circ\sigma_n(e,h),\sigma_n(e,h)\circ\sigma_n(t,e))=0$, for any $h\in\Lambda$.

To this end, let $T\subset\Lambda$ be a finite generating set. Since $\Lambda$ does not have property $(\tau)$ with respect to $\{\ker(q_n)\}_{n=1}^{\infty}$ we have that $\inf_n\kappa(q_n(\Lambda),q_n(T))=0$. Thus, after passing to a subsequence, we may assume that $\lim\limits_{n\to\infty}\kappa(q_n(\Lambda),q_n(T))=0$.

Lemma 2.8 then implies that for every $n$ large enough there exists a set $C_n\subset q_n(\Lambda)$ such that

$$\frac{1}{7}\leq\frac{|C_n|}{|q_n(\Lambda)|}\leq\frac{1}{6}\ \text{and}\ \lim_{n\to\infty}\frac{|C_nq_n(h)\triangle C_n|}{|q_n(\Lambda)|}=0,\ \text{for every}\ h\in\Lambda. \tag{6.1}$$

Let $Z_n\subset X_n$ be a set of representatives for the left cosets of $q_n(\Lambda)$. We define $B_n=Z_n\cdot C_n\subset X_n$, and claim that $B_n$ satisfies the following:

(a) $\frac{1}{7}\leq\frac{|B_n|}{|X_n|}\leq\frac{1}{6}$,

(b) $\lim\limits_{n\to\infty}\frac{|B_nq_n(h)\triangle B_n|}{|X_n|}=0$, for every $h\in\Lambda$, and

(c) $\frac{1}{|q_n(\Lambda)|}\sum_{h\in q_n(\Lambda)}|B_nh\cap Y|=\frac{|B_n|\cdot|Y|}{|X_n|}$, for every $Y\subset X_n$.

Indeed, (a) and (b) follow from (6.1). To verify (c), note that if $x\in X_n$, then there is a unique $z\in Z_n$ such that $x^{-1}z\in q_n(\Lambda)$ and thus we have that

$$|x^{-1}B_n\cap q_n(\Lambda)|=|\{h\in q_n(\Lambda)\mid h\in x^{-1}Z_n\cdot C_n\}|=|\{h\in q_n(\Lambda)\mid h\in(x^{-1}z)C_n\}|=|C_n|.$$

Therefore, for every subset $Y\subset X_n$, we deduce that

$$\sum_{h\in q_n(\Lambda)}|B_nh\cap Y|=\sum_{h\in q_n(\Lambda),x\in X_n}\mathbf{1}_{B_nh}(x)\mathbf{1}_Y(x)=\sum_{x\in Y}|x^{-1}B_n\cap q_n(\Lambda)|=|C_n|\cdot|Y|$$

Since $|B_n|=\frac{|X_n|}{|q_n(\Lambda)|}\cdot|C_n|$, condition (c) is also satisfied.

Let $n$ large enough. Since $\sum_{g\in X_n}|B_n\setminus g^{-1}B_n|=|B_n|\cdot(|X_n|-|B_n|)$, we can find $g_n\in X_n$ such that $A_n=B_n\setminus g_n^{-1}B_n$ satisfies $\frac{|A_n|}{|X_n|}\geq\frac{|X_n|-|B_n|}{|X_n|}\cdot\frac{|B_n|}{|X_n|}\geq\frac{5}{6}\cdot\frac{1}{7}=\frac{5}{42}$. Moreover, $A_n\cap g_nA_n=\emptyset$ and since $A_n\triangle A_nq_n(h)\subset(B_n\triangle B_nq_n(h))\cup g_n^{-1}(B_n\triangle B_nq_n(h))$, by (b) we get that

$$\lim_{n\to\infty}\frac{|A_nq_n(h)\triangle A_n|}{|X_n|}=0,\ \text{for every}\ h\in\Lambda. \tag{6.2}$$

We are now ready to define $\sigma_n(t,e)\in\mathrm{Sym}(X_n)$ by letting

$$\sigma_n(t,e)x=\begin{cases}g_nx,\ \text{if}\ x\in A_n,\\ g_n^{-1}x,\ \text{if}\ x\in g_nA_n,\\ x,\ \text{otherwise.}\end{cases}$$

Then Lemma 2.2 implies that for every $h\in\Lambda$, $\mathrm{d_H}(\sigma_n(t,e)\circ\sigma_n(e,h),\sigma_n(e,h)\circ\sigma_n(t,e))$ is equal to

$$\begin{cases}\frac{2|A_n\setminus A_nq_n(h)|+|(A_n\cup g_nA_n)\setminus(A_n\cup g_nA_n)q_n(h)|}{|X_n|},\ \text{if}\ g_n^2\neq e\\ \frac{2|(A_n\cup g_nA_n)\setminus(A_n\cup g_nA_n)q_n(h)|}{|X_n|},\ \text{if}\ g_n^2=e.\end{cases} \tag{6.3}$$

Since $(A_n\cup g_nA_n)\setminus(A_n\cup g_nA_n)q_n(h)\subset(A_n\setminus A_nq_n(h))\cup g_n(A_n\setminus A_nq_n(h))$, (6.2) implies that for all $h\in\Lambda$, $\lim\limits_{n\to\infty}\mathrm{d_H}(\sigma_n(t,e)\circ\sigma_n(e,h),\sigma_n(e,h)\circ\sigma_n(t,e))=0$. This ends the first part of the proof.

In the second part of the proof we will prove condition *(2)* from the conclusion. Let $n$ large enough. By using (6.2), (6.3), that $\frac{|A_n|}{|X_n|} \geq \frac{5}{42}$ and that $A_n \subset B_n$, for all $h \in \Lambda$, we get that

$$\begin{aligned}
&\mathrm{d_H}(\sigma_n(t,e) \circ \sigma_n(e,h), \sigma_n(e,h) \circ \sigma_n(t,e)) \\
&\geq \frac{|(A_n \cup g_n A_n) \setminus (A_n \cup g_n A_n) q_n(h)|}{|X_n|} \\
&\geq \frac{|A_n|}{|X_n|} - \frac{|(A_n \cup g_n A_n) \cap (A_n q_n(h) \cup g_n A_n q_n(h))|}{|X_n|} \\
&\geq \frac{5}{42} - \frac{|(B_n \cup g_n B_n) \cap (B_n q_n(h) \cup g_n B_n q_n(h))|}{|X_n|}
\end{aligned} \tag{6.4}$$

Since $|(B_n \cup g_n B_n) \cap (B_n h \cup g_n B_n h)| \leq 2|B_n h \cap B_n| + |B_n h \cap g_n B_n| + |B_n h \cap g_n^{-1} B_n|$, by using condition (c) we derive that

$$\frac{1}{|q_n(\Lambda)|} \sum_{h \in q_n(\Lambda)} |(B_n \cup g_n B_n) \cap (B_n h \cup g_n B_n h)| \leq 4 \cdot \frac{|B_n|^2}{|X_n|}.$$

Thus, there exists $h_n \in \Lambda$ such that $|(B_n \cup g_n B_n) \cap (B_n q_n(h_n) \cup g_n B_n q_n(h_n))| \leq 4 \cdot \frac{|B_n|^2}{|X_n|}$. By combining this with (6.4) and the inequality $\frac{|B_n|}{|X_n|} \leq \frac{1}{6}$ from (a), it follows that $h_n$ satisfies

$$\mathrm{d_H}(\sigma_n(t,e) \circ \sigma_n(e,h_n), \sigma_n(e,h_n) \circ \sigma_n(t,e)) \geq \frac{5}{42} - 4 \cdot \frac{|B_n|^2}{|X_n|^2} \geq \frac{5}{42} - \frac{4}{36} = \frac{1}{126} \tag{6.5}$$

This proves condition *(2)* and finishes the proof. ∎

## 7. Proofs of Theorem A, Corollary B and Corollary 4.2

The proof of Theorem A relies on the following result that combines Theorem 5.1 and Lemma 6.1.

**Theorem 7.1.** *Let $\Gamma$ and $\Lambda$ be finitely generated groups. Assume that $\Gamma$ has property ($\tau$) with respect to a sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of finite index normal subgroups. Suppose that there exist homomorphisms $q_n : \Lambda \to \Gamma/\Gamma_n$ such that $\Lambda$ does not have property ($\tau$) with respect to the sequence $\{\ker(q_n)\}_{n=1}^{\infty}$. Then $\Sigma \times \Lambda$ is not very flexibly* P*-stable, for any finitely generated group $\Sigma$ which factors onto $\Gamma * \mathbb{Z}$.*

*Proof.* Assume by contradiction that $\Sigma \times \Lambda$ is very flexibly P-stable. Let $\pi : \Sigma \to \Gamma * \mathbb{Z}$ be an onto homomorphism, and denote still by $\pi$ the product homomorphism $\pi \times \mathrm{Id}_\Lambda : \Sigma \times \Lambda \to (\Gamma * \mathbb{Z}) \times \Lambda$. Let $t = \pm 1$ be a generator of $\mathbb{Z}$. Denote $X_n = \Gamma/\Gamma_n$ and let $p_n : \Gamma \to X_n$ be the quotient homomorphism. By Lemma 6.1, there exists an asymptotic homomorphism $\sigma_n : (\Gamma * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(X_n)$ such that

(1) $\sigma_n(g,h)x = p_n(g) x q_n(h)^{-1}$, for all $g \in \Sigma, h \in \Lambda, x \in X_n$.
(2) $\max\{\mathrm{d_H}(\sigma_n(t,e) \circ \sigma_n(e,h), \sigma_n(e,h) \circ \sigma_n(t,e)) \mid h \in \Lambda\} \geq \frac{1}{126}$, for infinitely many $n \geq 1$.

Then $\sigma_n \circ \pi : \Sigma \times \Lambda \to \mathrm{Sym}(X_n)$ is an asymptotic homomorphism. Thus, since $\Sigma \times \Lambda$ is assumed very flexibly P-stable, for every $n \in \mathbb{N}$, we can find a set $Y_n \supset X_n$ together with a homomorphism $\tau_n : \Sigma \times \Lambda \to \mathrm{Sym}(Y_n)$ such that $\lim_{n\to\infty} \mathrm{d_H}(\sigma_n(\pi(g)), \tau_n(g)_{|X_n}) = 0$, for every $g \in \Sigma \times \Lambda$.

Since $\Gamma$ is finitely generated and $\pi$ is onto, we can find a finitely generated subgroup $\Delta < \Sigma$ and $\tilde{t} \in \Sigma$ such that $\pi(\Delta) = \Gamma$ and $\pi(\tilde{t}) = t$. Let $\rho : \Delta * \mathbb{Z} \to \Sigma$ the homomorphism given by $\rho_{|\Delta} = \mathrm{Id}_\Delta$ and $\rho(t) = \tilde{t}$. Denote still by $\rho$ the product homomorphism $\rho \times \mathrm{Id}_\Lambda : (\Delta * \mathbb{Z}) \times \Lambda \to \Sigma \times \Lambda$. Then $\alpha_n := \sigma_n \circ \pi \circ \rho : (\Delta * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(X_n)$ is an asymptotic homomorphism which satisfies that $\lim_{n\to\infty} \mathrm{d_H}(\alpha_n(g), \tau_n(\pi(\rho(g)))_{|X_n}) = 0$, for every $g \in (\Delta * \mathbb{Z}) \times \Lambda$.

Now, note that (1) gives that $\alpha_n(g,h)x = (p_n \circ \pi)(g)xq_n(h)^{-1}$, for all $g \in \Delta, h \in \Lambda$ and $x \in X_n$. Since $\Gamma$ has property $(\tau)$ with respect to $\{\Gamma_n\}_{n=1}^{\infty}$, $\Delta$ has property $(\tau)$ with respect to $\{\ker(p_n \circ \pi)\}_{n=1}^{\infty}$. Since $p_n \circ \pi : \Delta \to X_n$ is an onto homomorphism and $\tau_n \circ \pi \circ \rho : (\Delta * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(Y_n)$ is a homomorphism, for all $n \in \mathbb{N}$, applying Theorem 5.1 to $\alpha_n : (\Delta * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(X_n)$ gives that

$$\lim_{n\to\infty} \big(\max\{\mathrm{d_H}(\alpha_n(t,e) \circ \alpha_n(e,h), \alpha_n(e,h) \circ \alpha_n(t,e)) \mid h \in \Lambda\}\big) = 0.$$

However, since $\alpha_n(t,e) = \sigma_n(t,e)$ and $\alpha_n(e,h) = \sigma_n(e,h)$, for every $h \in \Lambda$, this contradicts (2). ■

**Proof of Theorem A.** Let $\Sigma$ and $\Lambda$ be finitely generated groups such that $\Sigma$ admits a non-abelian free quotient and $\Lambda$ does not have property $(\tau)$. Our goal is to prove that $\Sigma \times \Lambda$ is not very flexibly P-stable. By Lemma 3.3 it suffices to find a finite index subgroup $\Sigma_0 < \Sigma$ such that $\Sigma_0 \times \Lambda$ is not very flexibly P-stable.

Let us first prove the conclusion in the case when $\Lambda$ admits an infinite cyclic quotient, since this requires less technology than the general case. Let $\rho : \Lambda \to \mathbb{Z}$ be an onto homomorphism. Since $\Sigma$ factors onto $\mathbb{F}_2$, it has a finite index subgroup $\Sigma_0$ which factors onto $\mathbb{F}_3$. Towards showing that $\Sigma_0 \times \Lambda$ is not very flexibly P-stable, recall that $\Gamma = \mathbb{F}_2$ can be realized as a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, by letting for instance $\Gamma = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle$. Since $\mathrm{SL}_2(\mathbb{Z})$ has the Selberg property [LW93] (i.e., property $(\tau)$ with respect to its congruence subgroups), $\Gamma$ has property $(\tau)$ with respect to $\{\Gamma_n\}_{n=1}^{\infty}$, where $\Gamma_n = \Gamma \cap \big(\ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\frac{\mathbb{Z}}{n\mathbb{Z}})\big)$. Let $p_n : \Gamma \to \Gamma/\Gamma_n$ be the quotient homomorphism. Let $\eta : \mathbb{Z} \to \Gamma$ the homomorphism given by $\eta(1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and denote $q_n = p_n \circ \eta \circ \rho : \Lambda \to \Gamma/\Gamma_n$. Since $q_n$ factors through $\rho : \Lambda \to \mathbb{Z}$, for every $n$, and $\lim_{n\to\infty} |q_n(\Lambda)| = +\infty$, it follows that $\Lambda$ does not have property $(\tau)$ with respect to $\{\ker(q_n)\}_{n=1}^{\infty}$. Since $\Sigma_0$ factors onto $\mathbb{F}_3 = \Gamma * \mathbb{Z}$, Theorem 7.1 implies that $\Sigma_0 \times \Lambda$ is not very flexibly P-stable.

In order to establish the general case we will use a theorem of Kassabov [Ka05, Theorem 2] which provides an integer $L \geq 2$ and onto homomorphisms $\pi_n : \mathbb{F}_L \to \mathrm{Sym}(n)$, for every $n \in \mathbb{N}$, such that $\inf_n \kappa(\mathrm{Sym}(n), \pi_n(S)) > 0$, where $S \subset \mathbb{F}_L$ is a free generating set. In other words, $\Gamma = \mathbb{F}_L$ has property $(\tau)$ with respect to $\{\ker(\pi_n)\}_{n=1}^{\infty}$. Since $\Sigma$ factors onto $\mathbb{F}_2$, it has a finite index subgroup $\Sigma_0$ which factors onto $\mathbb{F}_{L+1}$. We will show that $\Sigma_0 \times \Lambda$ is not very flexibly P-stable.

To this end, note that since $\Lambda$ does not have property $(\tau)$, there exists a sequence $\{\Lambda_n\}_{n=1}^{\infty}$ of finite index normal subgroups such that $\lim_{n\to\infty} \kappa(\Lambda/\Lambda_n, \delta_n(T)) = 0$, where $T \subset \Lambda$ is a finite generating set and $\delta_n : \Lambda \to \Lambda/\Lambda_n$ denotes the quotient homomorphism. For every $n \in \mathbb{N}$, put $G_n = \mathrm{Sym}(\Lambda/\Lambda_n)$ and let $i_n : \Lambda/\Lambda_n \to G_n$ be the embedding given by left multiplication action of $\Lambda/\Lambda_n$ on itself. We denote $q_n = i_n \circ \delta_n : \Lambda \to G_n$. Finally, we put $k_n = |\Lambda/\Lambda_n|$ and let $p_n : \Gamma \to G_n$ be the onto homomorphism obtained by composing $\pi_{k_n} : \Gamma \to \mathrm{Sym}(n_k)$ with an isomorphism $\mathrm{Sym}(k_n) \cong G_n$. By construction, $\Gamma$ has property $(\tau)$ with respect to $\{\ker(p_n)\}_{n=1}$, while $\Lambda$ does not have property $(\tau)$ with respect to $\{\ker(q_n)\}_{n=1}^{\infty}$ (as $\ker(p_n) = \ker(\pi_{k_n})$ and $\ker(q_n) = \Lambda_n$, for every $n \in \mathbb{N}$). Since $\Sigma_0$ factors onto $\mathbb{F}_{L+1} = \Gamma * \mathbb{Z}$, Theorem 7.1 implies that $\Sigma_0 \times \Lambda$ is not very flexibly stable. ■

**Proof of Corollary B.** Since $\mathbb{Z}^d$ and $\mathbb{F}_n$ do not have property $(\tau)$ for any integers $d, n \geq 1$, parts (1) and (2) follow from Theorem A. Let $m, n$ be integers such that $|m| = |n| \geq 2$. Then the Baumslag-Solitar group $\mathrm{BS}(m,n) = \langle a, t | ta^m t^{-1} = a^n \rangle$ has a finite index subgroup isomorphic to $\mathbb{F}_k \times \mathbb{Z}$ for some $k \geq 2$ (see., e.g., [Le05, Proposition 2.6]). Since $\mathbb{F}_k \times \mathbb{Z}$ is not very flexbily P-stable by part (1), the same is true for $\mathrm{BS}(m,n)$ by Lemma 3.3. This proves part (3). To prove part (4), let $n \geq 3$ be an integer. Recall that the pure braid group $\mathrm{PB}_n$ has infinite center, $\mathrm{Z}(\mathrm{PB}_n) \cong \mathbb{Z}$, and admits a non-trivial splitting $\mathrm{PB}_n \cong \mathrm{PB}_n/\mathrm{Z}(\mathrm{PB}_n) \times \mathrm{Z}(\mathrm{PB}_n)$ (see [FM11, Chapter 9]). Since $\mathrm{PB}_m$ factors onto $\mathrm{PB}_{m-1}$, for any $m \geq 3$, and $\mathrm{PB}_3 \cong \mathbb{F}_2 \times \mathbb{Z}$, we get that $\mathrm{PB}_n$ factors onto $\mathbb{F}_2$. Thus,

$\mathrm{PB}_n/\mathrm{Z}(\mathrm{PB}_n)$ factors onto $\mathbb{F}_2$. Applying Theorem A implies that $\mathrm{PB}_n$ is not very flexibly P-stable. Since $\mathrm{PB}_n$ is a finite index subgroup of $\mathrm{B}_n$, the same holds for $\mathrm{B}_n$ by Lemma 3.3. ■

**Proof of Corollary C.** By part (1) of Corollary B, $\mathbb{F}_2\times\mathbb{Z}$ is not P-stable. Let $a_1, a_2$ be generators of $\mathbb{F}_2$ and $b$ be a generator of $\mathbb{Z}$. As $\mathbb{F}_2\times\mathbb{Z}$ is not P-stable, we can find an asymptotic homomorphism $\sigma_n : \mathbb{F}_2\times\mathbb{Z}\to \mathrm{Sym}(X_n)$ and $\delta > 0$ such that for any homomorphism $\tau_n : \mathbb{F}_2\times\mathbb{Z}\to\mathrm{Sym}(X_n)$

nothm

$$\max\{\mathrm{d_H}(\sigma_n(g), \tau_n(g)) \mid g \in \{a_1, a_2, b\}\} \geq \delta, \text{ for every } n. \tag{7.1}$$

Then $\sigma = (\sigma_n) : \mathbb{F}_2\times\mathbb{Z} \to \prod_{\mathcal{U}}\mathrm{Sym}(X_n)$ is a homomorphism, hence $\Sigma = \sigma(\mathbb{F}_2)$ and $\Lambda = \sigma(\mathbb{Z})$ are commuting subgroups of $\prod_{\mathcal{U}}\mathrm{Sym}(X_n)$. Assume by contradiction that there exist commuting subgroups $\Sigma_n, \Lambda_n$ of $\mathrm{Sym}(X_n)$ such that $\Sigma \subset \prod_{\mathcal{U}}\Sigma_n$ and $\Lambda\subset\prod_{\mathcal{U}}\Lambda_n$. Then for every $n$ we can find $\rho_n(a_1), \rho_n(a_2) \in \Sigma_n$ and $\rho_n(b)\in\Lambda_n$ such that $\lim\limits_{n\to\mathcal{U}} \mathrm{d_H}(\sigma_n(g), \rho_n(g)) = 0$, for every $g \in \{a_1, a_2, b\}$. Since $\Sigma_n$ and $\Lambda_n$ commute, there exists a homomorphism $\tau_n : \mathbb{F}_2\times\mathbb{Z}\to\mathrm{Sym}(X_n)$ such that $\tau_n(g) = \rho_n(g)$, for every $g\in\{a_1, a_2, b\}$. This contradicts (7.1) and finishes the proof. ■

## 8. Proof of Theorem D

By the moreover assertion of Lemma 3.3, it suffices to prove that $\Sigma\times\Lambda$ is not weakly very flexibly P-stable, where $\Sigma = \mathbb{F}_m$ and $\Lambda = \mathbb{Z}^d$ or $\Lambda = \mathbb{F}_k$, for $m, k\geq 2$ and $d\geq 1$. Since any subgroup of index 2 of $\mathbb{F}_m$ is isomorphic to $\mathbb{F}_{2m-1}$, by Lemma 3.3 we may assume that $m\geq 3$. Let $\Gamma = \mathbb{F}_{m-1}$, so that $\Sigma = \Gamma*\mathbb{Z}$. We view $\Sigma$ as a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and denote by $\pi_r : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/r\mathbb{Z})$ the quotient homomorphism, for prime $r$.

By Corollary B, $\Sigma\times\Lambda$ is not very flexibly P-stable. However, the asymptotic homomorphism constructed in the proof of Corollary B which witnesses that $\Sigma\times\Lambda$ is not very flexibly P-stable is not a sofic approximation of $\Sigma\times\Lambda$. Therefore, we cannot conclude that $\Sigma\times\Lambda$ is not weakly very flexibly P-stable. Instead, we first use a variation of our construction to build an asymptotic homomorphism $\sigma_n : \Sigma\times\Lambda\to\mathrm{Sym}(X_n)$ whose restriction to $\Lambda$ is a sofic approximation of $\Lambda$. We then exploit the fact that if $\pi_n : \Sigma \to \mathrm{Sym}(Y_n)$ is a sofic approximation of $\Sigma$, then $\widetilde{\sigma}_n : \Sigma\times\Lambda\to\mathrm{Sym}(Y_n\times X_n)$ given by $\widetilde{\sigma}_n(g,h)(x,y) = (\pi_n(g)x, \sigma_n(g,h)y)$, for every $g\in\Sigma, h\in\Lambda, x\in Y_n, y\in X_n$ is a sofic approximation of $\Sigma\times\Lambda$. In the rest of the proof, we implement this idea by treating two cases:

**Case 1.** $\Lambda = \mathbb{Z}^d$, for some $d\geq 1$.

Fix $n\in\mathbb{N}$ and let $r_{n,0}, r_{n,1}, .., r_{n,d}$ be $d+1$ distinct primes greater than $n$. Define $X_n = \prod_{i=1}^d \mathrm{SL}_2(\mathbb{Z}/r_{n,i}\mathbb{Z})$ and homomorphisms $p_n : \Gamma\to X_n, q_n : \Lambda \to X_n$ by letting for $g\in\Gamma$ and $(h_1, ..., h_d)\in\Lambda$

$$p_n(g) = (\pi_{r_{n,1}}(g), ..., \pi_{r_{n,d}}(g)) \text{ and } q_n(h_1, ..., h_d) = \left(\begin{pmatrix}1 & h_1\\ 0 & 1\end{pmatrix}, ..., \begin{pmatrix}1 & h_d\\ 0 & 1\end{pmatrix}\right).$$

Since $\Gamma$ is a non-amenable subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we get that $p_n : \Gamma\to X_n$ is onto for $n$ large enough. Since $\Lambda$ is abelian it does not have property ($\tau$) with respect to $\{\ker(q_n)\}_{n=1}^\infty$. Thus, Lemma 6.1 provides an asymptotic homomorphism $\sigma_n : \Sigma\times\Lambda = (\Gamma*\mathbb{Z})\times\Lambda\to\mathrm{Sym}(X_n)$ such that

(1) $\sigma_n(g,h)x = p_n(g)xq_n(h)^{-1}$, for all $g\in\Gamma, h\in\Lambda, x\in X_n$. unos
(2) $\max\{\mathrm{d_H}(\sigma_n(t,e)\circ\sigma_n(e,h), \sigma_n(e,h)\circ\sigma_n(t,e)) \mid h\in\Lambda\}\geq\frac{1}{126}$, for infinitely many $n\geq 1$, doss

where $t = \pm 1$ is a generator of $\mathbb{Z}$.

Let $\widetilde{X}_n = \mathrm{SL}_2(\mathbb{Z}/r_{n,0}\mathbb{Z})\times X_n$ and define homomorphisms $\widetilde{p}_n : \Gamma\to\widetilde{X}_n$ and $\widetilde{q}_n : \Lambda\to\widetilde{X}_n$ by letting

pnqn

$$\widetilde{p}_n(g) = (\pi_{r_{n,0}}(g), p_n(g)) \text{ and } \widetilde{q}_n(h) = (e, q_n(h)), \text{ for every } g\in\Gamma, h\in\Lambda. \tag{8.1}$$

Further, define $\widetilde{\sigma}_n : \Sigma \times \Lambda \to \mathrm{Sym}(\widetilde{X}_n)$ by letting

$$\widetilde{\sigma}_n(g,h)(x,y) = (\pi_{r_{n,0}}(g)x, \sigma_n(g,h)y), \text{ for every } g \in \Sigma, h \in \Lambda, x \in \mathrm{SL}_2(\mathbb{Z}/r_{n,0}\mathbb{Z}), y \in X_n. \tag{8.2}$$

Then $(\widetilde{\sigma}_n)_{n\in\mathbb{N}}$ is an asymptotic homomorphism of $\Sigma \times \Lambda$ and conditions (1) and (2) above rewrite as:

(i) $\widetilde{\sigma}_n(g,h)x = \widetilde{p}_n(g)x\widetilde{q}_n(h)^{-1}$, for all $g \in \Gamma, h \in \Lambda, x \in \widetilde{X}_n$.

(ii) $\max\{\mathrm{d_H}(\widetilde{\sigma}_n(t,e) \circ \widetilde{\sigma}_n(e,h), \widetilde{\sigma}_n(e,h) \circ \widetilde{\sigma}_n(t,e)) \mid h \in \Lambda\} \geq \frac{1}{126}$, for infinitely many $n \geq 1$.

Since $\Gamma$ is a non-amenable subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we get that $\widetilde{p}_n : \Gamma \to \widetilde{X}_n$ is onto for $n$ large enough. Moreover, a theorem of Bourgain and Varjú [BV10, Theorem 1] implies that $\Gamma$ has property $(\tau)$ with respect to $\{\ker(\widetilde{p}_n)\}_{n=1}^{\infty}$. By combining this fact with conditions (i) and (ii) above, we can apply Theorem 5.1 to conclude that there is no sequence of homomorphisms $\tau_n : \Sigma \times \Lambda \to \mathrm{Sym}(Y_n)$, for any sets $Y_n \supset \widetilde{X}_n$, such that $\lim_{n\to\infty} \mathrm{d_H}(\widetilde{\sigma}_n(g), \tau_n(g)_{|\widetilde{X}_n}) = 0$, for every $g \in \Sigma \times \Lambda$.

Thus, in order to deduce that $\Sigma \times \Lambda$ is not weakly very flexibly P-stable, it suffices to argue that $(\widetilde{\sigma}_n)_{n\in\mathbb{N}}$ is a sofic approximation of $\Sigma \times \Lambda$. To see this, let $(g,h) \in (\Sigma \times \Lambda) \setminus \{(e,e)\}$. If $g \neq e$, then as $\lim_{n\to\infty} r_{n,0} = +\infty$, we get that $\pi_{r_{n,0}}(g) \neq e$, for $n$ large enough. By using the definition (8.2) of $\widetilde{\sigma}_n$, we get that $\mathrm{d_H}(\widetilde{\sigma}_n(g,h), \mathrm{Id}_{\widetilde{X}_n}) = 1$, for $n$ large enough. If $g = e$, then $h \neq e$ and since $\lim_{n\to\infty} r_{n,i} = +\infty$, for all $1 \leq i \leq d$, we get that $\widetilde{q}_n(h) \neq e$, for $n$ large enough. By using the definition (8.2) of $\widetilde{\sigma}_n$, we get that $\mathrm{d_H}(\widetilde{\sigma}_n(e,h), \mathrm{Id}_{\widetilde{X}_n}) = 1$, for $n$ large enough. Since $\widetilde{\sigma}_n(e,e) = \mathrm{Id}_{\widetilde{X}_n}$, for all $n \in \mathbb{N}$, this proves that $(\widetilde{\sigma}_n)_{n\in\mathbb{N}}$ is a sofic approximation of $\Sigma \times \Lambda$, finishing the proof of **Case 1**.

**Case 2.** $\Lambda = \mathbb{F}_k$, for some $k \geq 2$.

View $\Lambda$ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let $\rho : \Lambda \to \mathrm{SL}_2(\mathbb{Z})$ be a homomorphism such that $\rho(\Lambda) \cong \mathbb{Z}$. For instance, if $a_1, ..., a_k \in \Lambda$ are generators, we can let $\rho(a_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\rho(a_2) = ... = \rho(a_k) = e$.

Fix $n \in \mathbb{N}$ and let $r_{n,0}, r_{n,1}, r_{n,2}$ be 3 distinct primes greater than $n$. Define $X_n = \prod_{i=1}^{2} \mathrm{SL}_2(\mathbb{Z}/r_{n,i}\mathbb{Z})$ and homomorphisms $p_n : \Gamma \to X_n, q_n : \Lambda \to X_n$ by letting for $g \in \Gamma$ and $h \in \Lambda$

$$p_n(g) = (\pi_{r_{n,1}}(g), \pi_{r_{n,2}}(g)) \text{ and } q_n(h) = (\pi_{r_{n,1}}(\rho(h)), \pi_{r_{n,2}}(h)).$$

Since $\Gamma$ is a non-amenable subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we get that $p_n : \Gamma \to X_n$ is onto for $n$ large enough. Since the image of $\rho$ is infinite abelian and $\lim_{n\to\infty} r_{n,1} = +\infty$, $\Lambda$ does not have property $(\tau)$ with respect to $\{\ker(\pi_{r_{n,1}} \circ \rho)\}_{n=1}^{\infty}$. Since $\ker(q_n) \subset \ker(\pi_{r_{n,1}} \circ \rho)$, for every $n \in \mathbb{N}$, it follows that $\Lambda$ does not have property $(\tau)$ with respect to $\{\ker(q_n)\}_{n=1}^{\infty}$. Applying Lemma 6.1 provides an asymptotic homomorphism $\sigma_n : \Sigma \times \Lambda = (\Gamma * \mathbb{Z}) \times \Lambda \to \mathrm{Sym}(X_n)$ which satisfies conditions (1) and (2) from above.

Next, let $\widetilde{X}_n = \mathrm{SL}_2(\mathbb{Z}/r_{n,0}\mathbb{Z}) \times X_n$ and define the homomorphisms $\widetilde{p}_n : \Gamma \to \widetilde{X}_n$, $\widetilde{q}_n : \Lambda \to \widetilde{X}_n$ and the asymptotic homomorphism $\widetilde{\sigma}_n : \Sigma \times \Lambda \to \mathrm{Sym}(\widetilde{X}_n)$ by the same formulae as in the proof of **Case 1**. Then $(\widetilde{\sigma}_n)_{n\in\mathbb{N}}$ satisfies conditions (i) and (ii) from above.

Moreover, if $h \in \Lambda \setminus \{e\}$, then since $\lim_{n\to\infty} r_{n,2} = +\infty$, we get that $\pi_{r_{n,2}}(h) \neq e$, for $n$ large enough. This implies that $\mathrm{d_H}(\widetilde{\sigma}_n(e,h), \mathrm{Id}_{\widetilde{X}_n}) = 1$, for $n$ large enough. By repeating verbatim the rest of the argument from the proof of **Case 1**, it follows that $(\widetilde{\sigma}_n)$ is a sofic approximation of $\Sigma \times \Lambda$ and that $\Sigma \times \Lambda$ is not weakly very flexibly P-stable. This finishes the proof of **Case 2**.

Finally, the proof of Corollary B shows that any group from parts (1)-(3) in its statement has a finite index subgroups which is isomorphic to either $\mathbb{F}_m \times \mathbb{Z}^d$ or to $\mathbb{F}_m \times \mathbb{F}_k$, for some $m, k \geq 2$ and $d \geq 1$. Thus, any group from Corollary B, parts (1)-(3), is not weakly very flexibly P-stable. ■

## References


[AE10] M. Abért and G. Elek: *Dynamical properties of profinite actions*, Erg. Th. Dynam. Sys., **32** (2012), 1805-1835.

[AP14] G. Arzhantseva and L. Paunescu: *Almost commuting permutations are near commuting permutations*, J. Funct. Anal., **269**(3):745-757, 2015.

[AP17] G. Arzhantseva and L. Paunescu: *Constraint metric approximations and equations in groups*, J. Algebra **516** (2018), 329-351.

[BL18] O. Becker and A. Lubotzky: *Group stability and property (T)*, J. Funct. Anal. **278** (2020), no. 1, 108298.

[BLT18] O. Becker, A. Lubotzky and A. Thom: *Stability and invariant random subgroups*, Duke Math. J. **168** (2019), no. 12, 2207-2234.

[BB19] L. Bowen and P. Burton: *Flexible stability and nonsoficity*, preprint arXiv:1906.02172, to appear in Trans. Amer. Math. Soc.

[BM18] O. Becker and J. Mosheiff: *Abelian groups and polynomially stable*, preprint arXiv:1811.00578.

[Bo17] L. Bowen: *Examples in the entropy theory of countable group actions*, preprint arXiv:1704.06349, to appear in Erg. Th. Dynam. Sys.

[BP78] B. Baumslag and S. J. Pride: *Groups with two more generators than relators*, J. London Math. Soc. (2) **17** (1978), no. 3, 425-426.

[BV10] J. Bourgain and P. Varjú: *Expansion in $SL_d(\mathbb{Z}/q\mathbb{Z})$, q arbitrary*, Invent. Math. **188** (2012), no. 1, 151-173.

[CGLT17] M. De Chiffre, L. Glebsky, A. Lubotzky and A. Thom: *Stability, cohomology vanishing, and non-approximable groups*, preprint arXiv:1711.10238.

[ES04] G. Elek and E. Szabó: *Hyperlinearity, essentially free actions and $L^2$-invariants. The sofic property*, Math. Ann. **332** (2005), 421-441.

[ESS18] S. Eilers, T. Shulman and A.P.W. Sørensen: *C*-stability of discrete groups*, preprint arXiv:1808.06793.

[FM11] B. Farb and D. Margalit: *A primer on mapping class groups*, Princeton Mathematical Series, October 2011.

[Gl10] L. Glebsky: *Almost commuting matrices with respect to normalized Hilbert-Schmidt norm*, preprint arXiv:1002.3082.

[GR09] L. Glebsky and L. M. Rivera: *Almost solutions of equations in permutations*, Taiwanese J. Math. **13** (2009), no. 2A, 493-500.

[GTD15] D. Gaboriau and R. Tucker-Drob: *Approximations of standard equivalence relations and Bernoulli percolation at $p_u$.*, C. R. Math. Acad. Sci. Paris **354** (2016), no. 11, 1114-1118.

[Ha76] P. Halmos: *Some unsolved problems of unknown depth about operators on Hilbert space*, Proc. Roy. Soc. Edinburgh Sect. A **76** (1976/77), no. 1, 67-76.

[Hj03] G. Hjorth: *A converse to Dye's theorem*, Trans. Amer. Math. Soc. **357** (2005), no. 8, 3083-3103.

[HL08] D. Hadwin and W. Li: *A note on approximate liftings*, Oper. Matrices, 3(1):125-143, 2009.

[HS17] D. Hadwin and T. Shulman: *Stability of group relations under small Hilbert-Schmidt perturbations*, J. Funct. Anal. **275** (2018), no. 4, 761-792.

[Ka05] M. Kassabov: *Symmetric groups and expander graphs*, Invent. Math. **170** (2007), no. 2, 327-354.

[Io06] A. Ioana: *Orbit inequivalent actions for groups containing a copy of $\mathbb{F}_2$*, Invent. Math., **185** (2011), 55-73.

[IS19] A. Ioana and P. Spaas: $\mathrm{II}_1$ *factors with exotic central sequence algebras*, preprint arXiv:1904.06816, to appear in J. Inst. Math. Jussieu.

[La07] M. Lackenby: *Detecting large groups*, J. Algebra **324** (2010) 2636-2657.

[Le05] G. Levitt: *On the automorphism group of generalized Baumslag-Solitar groups*, Geom. Topol. **11** (2007), 473-515.

[LLM19] N. Lazarovich, A. Levit and Y. Minsky: *Surface groups are flexibly stable*, preprint arXiv:1901.07182.

[LL19a] A. Levit and A. Lubotzky: *Infinitely presented stable groups and invariant random subgroups of metabelian groups*, preprint arXiv:1909.11842.

[LL19b] A. Levit and A. Lubotzky: *Uncountably many permutation stable groups*, preprint arXiv:1910.11722.

[LO18] A. Lubotzky and I. Oppenheim: *Non p-norm approximated Groups*, preprint arXiv:1807.06790, to appear in Journal d'Analyse Mathematique.

[Lu94] A. Lubotzky: *Discrete Groups, Expanding Graphs and Invariant Measures.* With an appendix by Jonathan D. Rogawski, Progress in Mathematics, vol. 125. Birkhäuser Verlag, Basel, xii+195 pp (1994).

[LZ89] A. Lubotzky and R.J. Zimmer: *Variants of Kazhdan's property for subgroups of semisimple groups*, Israel J. Math. **66** (1989), no. 1-3, 289-299.

[LZ03] A. Lubotzky and A. Zuk: *On property* ($\tau$), preprint 2003, available at `http://www.ma.huji.ac.il/alexlub/`

[LW93] A. Lubotzky and B. Weiss: *Groups and expanders*, in Expanding graphs (Princeton, NJ, 1992), 95-109, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 10, Amer. Math. Soc., Providence, RI, 1993.

[OW80] D. Ornstein and B. Weiss: *Ergodic theory of amenable group actions.* I. The Rohlin lemma, Bull. Amer. Math. Soc. (N.S.) 2 (1980), no. 1, 161-164.

[Ro69] P. Rosenthal: *Research problems: are almost commuting matrices near commuting matrices?*, Amer. Math. Monthly **76**(8) (1969) 925-926.

[Th10] S. Thomas: *On the number of universal sofic groups*, Proc. Amer. Math. Soc. **138** (2010), no. 7, 2585-2590.

[Th17] A. Thom: *Finitary approximations of groups and their applications*, Proc. Int. Cong. of Math. 2018 Rio de Janeiro, Vol. 2 (1775-1796).

[Vo83] D. Voiculescu: *Asymptotically commuting finite rank unitary operators without commuting approximants*, Acta Sci. Math. (Szeged) **45**(1-4) (1983) 429-431.

[Zh19] T. Zheng: *On rigid stabilizers and invariant random subgroups of groups of homeomorphisms*, preprint arXiv:1901.04428.

Department of Mathematics, University of California San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA

*Email address*: aioana@ucsd.edu