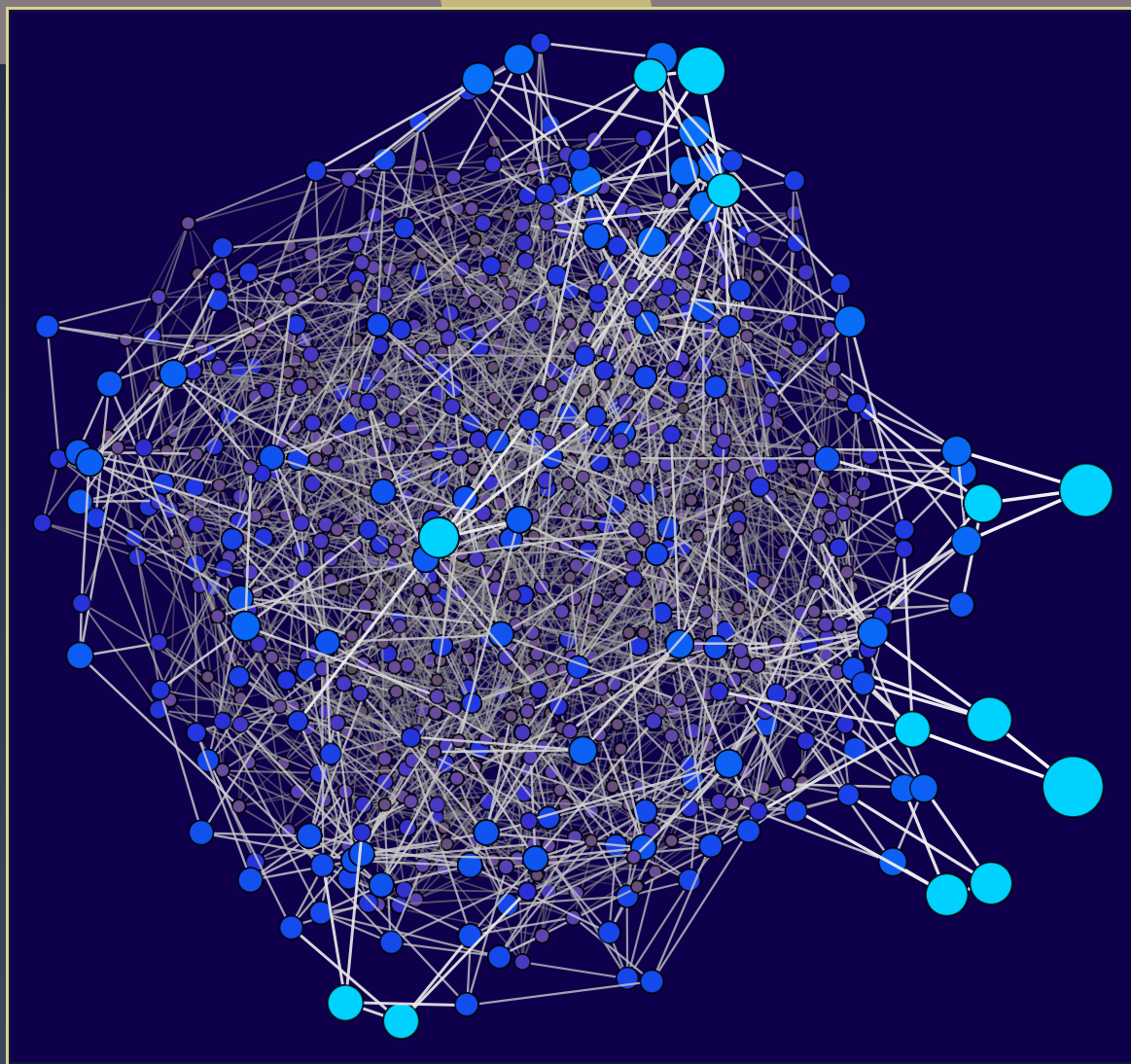


# ANTS XIV

## Proceedings of the Fourteenth Algorithmic Number Theory Symposium

Divisor class group arithmetic on  $C_{3,4}$  curves

Evan MacNeil, Michael J. Jacobson Jr., and Renate Scheidler



# Divisor class group arithmetic on $C_{3,4}$ curves

Evan MacNeil, Michael J. Jacobson Jr., and Renate Scheidler

We present novel explicit formulas for arithmetic in the divisor class group of a  $C_{3,4}$  curve. Our formulas handle all cases of inputs and outputs without having to fall back on a generic method. We also improve on the most commonly occurring case by reducing the number of required field inversions to one at the cost of a small number of additional field operations, resulting in running times that are between 11 and 21% faster than the prior state of the art depending on the field size, and even more for small field sizes when nontypical cases frequently arise.

## 1. Introduction

Computing in the divisor class group of an algebraic curve is a nontrivial component in computing  $L$ -series.  $L$ -series in turn are at the heart of the Sato–Tate conjecture and related conjectures. The Sato–Tate conjecture has been proved for elliptic curves with complex multiplication, but its analogues for other classes of algebraic curves remains open [14]. In order to test these conjectures for other curve families, it is desirable to have efficient algorithms to perform divisor class group arithmetic; see, for example, [7; 6; 13].

The  $C_{3,4}$  curves are a family of genus 3 plane curves. While they are rare among genus 3 curves, such special families of curves make interesting settings in which to study Sato–Tate-related conjectures. Fast explicit formulas exist to perform divisor class group arithmetic for genus 1 and genus 2 curves. However, the picture for genus 3 curves, and  $C_{3,4}$  curves in particular, is incomplete. Existing formulas for arithmetic on  $C_{3,4}$  curves were developed with cryptographic applications in mind, where the curves are defined over very large finite fields of characteristic greater than 3. A  $C_{3,4}$  curve over such a field is isomorphic to one given by a short-form equation (see Section 2), yielding faster arithmetic. Moreover, with very high probability, one will only encounter “typical” divisors (see Section 2) and many degenerate cases need not be considered. When these assumptions are violated, one may fall back on slower divisor addition algorithms that work on any algebraic curve.

---

*MSC2010:* 11R65, 14H45, 14Q05.

*Keywords:* computational number theory, computational algebraic geometry, divisor arithmetic,  $C_{3,4}$  curves, genus 3 nonhyperelliptic curves.

In [2], Arita specialized the algorithm for addition in the class group of a general  $C_{a,b}$  curve in [1] to the  $C_{3,4}$  case. He classified divisors of  $C_{3,4}$  curves into 19 types based on the forms of their Gröbner basis representations. The method allows addition of divisors of any type, although it handles this in a recursive manner that does not terminate for some curves over very small finite fields; Arita was predominantly interested in the cryptographic setting over a large finite field where this does not present a problem. However, number theoretic applications require extensive curve arithmetic over far smaller finite fields.

Other algorithms are less general but much faster. In [8], the most recent of these, Khuri-Makdisi, building upon the work of Flon et al. [4] and Abu Salem and Khuri-Makdisi [11] assumed a  $C_{3,4}$  curve defined by a short-form polynomial equation. In addition to restricting to disjoint divisors without multiple points, they assume that divisors being added or doubled are typical. They represent divisors by a pair of polynomials of minimal degree and obtain sums of divisors by computing kernels of maps between vector spaces. This yields the most efficient explicit formulas, describing the operation as an optimized sequence of field operations instead of via polynomial arithmetic or linear algebra, for the typical case. Thus, prior to our work herein, the state of the art for  $C_{3,4}$  curves was the addition and doubling procedures of [11] and the reduction method of [8]. Both of these are limited to typical divisors, and one had to resort to general arithmetic for all other cases.

Our contribution is to marry the methods of Salem and Khuri-Makdisi — who have the fastest explicit formulas to date — with the methods of Arita — whose formulas are the most general — in order to produce fast and fully general explicit formulas that cover all cases of  $C_{3,4}$  curve arithmetic. This approach is facilitated by the fact that Salem and Khuri-Makdisi’s representation of typical divisors resembles type 31 divisors from Arita’s classification. Our algorithms work in full generality: the curve may be defined over a field of any size and any characteristic, including 0, 2, and 3 (though our implementation only extends to finite fields), the curve equation may be in long or short form (see Section 2), divisors may be typical or atypical, nondisjoint, and have multiple points, and all our algorithms provably terminate.

We extend the approach of [11] for finding the kernel of the aforementioned map to computing its image as well and are thus able to handle atypical and nondisjoint divisors. We also improve on the state of the art of [8; 11] for typical divisors. Fully general algorithms for adding, doubling, and reducing divisors are presented in Sections 3, 4 and 5, respectively. These algorithms are used to develop fast explicit formulas in Section 6 that handle the most typical cases arising in  $C_{3,4}$  curve divisor arithmetic; specifically, adding/doubling disjoint typical divisors on a curve in short form over a field of characteristic greater than 3. The operation counts of these formulas are summarized in Table 1.1, where I, M, S, A refer to the number of field inversions, multiplications, squarings, and additions in the base field of the curve.<sup>1</sup> Our formulas improve on the prior state of the art by requiring only a single field inversion at the cost of a sufficiently small number of other field operations. Experiments confirm an overall running time speed-up by approximately 11–21% depending on the size of the field. Our algorithms are also used to produce explicit formulas for all atypical cases, including nondisjoint or atypical divisors and

<sup>1</sup>Arita did not distinguish between field multiplications and squarings, and neither Arita nor Flon et al. counted field additions in their work.

|                                 | Add |     |    |     | Double |     |    |     |
|---------------------------------|-----|-----|----|-----|--------|-----|----|-----|
|                                 | I   | M   | S  | A   | I      | M   | S  | A   |
| Arita [2]                       | 5   | 204 | –  | –   | 5      | 284 | –  | –   |
| Flon et al [4]                  | 2   | 148 | 15 | –   | 2      | 165 | 20 | –   |
| Khuri-Makdisi and Salem [8; 11] | 2   | 97  | 1  | 132 | 2      | 107 | 3  | 155 |
| <b>This work</b>                | 1   | 111 | 3  | 99  | 1      | 127 | 4  | 112 |

**Table 1.1.** Comparison of operation counts in prior work.

curves of arbitrary form and in any characteristic. These cases are so numerous that we choose instead to publish them in the form of Sage code on GitHub [9] and present their operation counts in Section 7.

By improving upon the typical case and completing the picture for the atypical cases, our results will have a significant impact on number theoretic computations heavy on arithmetic in the divisor class group of a  $C_{3,4}$  curve. As in [14] for example, one may wish to take a curve over  $\mathbb{Q}$ , reduce it modulo all primes up to some bound, and compute the order of the divisor class group of that reduced curve. The improvement in the typical case remains significant over all the computations, while the completion of the atypical cases becomes more significant over the smaller fields, where one frequently encounters these cases.

## 2. Preliminaries

Let  $K$  be a perfect field. A  $C_{3,4}$  curve is a nonsingular nonhyperelliptic projective curve  $C$  of genus 3 whose affine model is given by  $F(x, y) = 0$  where  $F \in K[x, y]$  is of the form

$$F(x, y) = y^3 + x^4 + c_8xy^2 + c_7x^2y + c_6x^3 + c_5y^2 + c_4xy + c_3x^2 + c_2y + c_1x + c_0.$$

We denote the unique point at infinity on  $C$  by  $P_\infty$ . When  $K$  has characteristic 0 or at least 5, the curve isomorphism  $(x, y) \mapsto (x - a/4, y - (c_8/3)x + (ac_8 - 4c_5)/3)$ ,  $a = (27c_6 - 9c_7c_8 + 2c_8^3)/27$ , over  $K$  transforms the polynomial  $F$  to the short form

$$F(x, y) = y^3 + x^4 + c_7x^2y + c_4xy + c_3x^2 + c_2y + c_1x + c_0.$$

Let  $\text{Div}_K^0(C)$  denote the group of degree zero divisors on  $C$  defined over  $K$ . Elements of  $\text{Div}_K^0(C)$  are of the form

$$D = \sum_{P \in C(\bar{K}) - \{P_\infty\}} \text{ord}_P(D)P - nP_\infty, \quad n = \sum_{P \in C(\bar{K}) - \{P_\infty\}} \text{ord}_P(D),$$

where the sum defining  $D$  is fixed under Galois automorphisms on  $\bar{K}$ . For brevity, we identify  $D$  with its finite part and refer to  $n = \deg(D)$  as its degree. A divisor  $D$  is *effective* if  $\text{ord}_P(D) \geq 0$  for all  $P \in C(\bar{K}) - \{P_\infty\}$  and *reduced* if in addition  $n$  is minimal among the degrees of all the divisors in the linear equivalence class of  $D$ . If  $D$  is reduced, then  $\deg(D) \leq 3$ . Every element of  $\text{Div}_K^0(C)$  is linearly equivalent to an effective divisor and to a unique reduced divisor in  $\text{Div}_K^0(C)$ .

For any two effective divisors  $D, D' \in \text{Div}_K^0(C)$ , define

$$\begin{aligned} \text{lcm}(D, D') &= \sum_{P \in C(\bar{K}) - \{P_\infty\}} \max\{\text{ord}_P(D), \text{ord}_P(D')\}(P - P_\infty), \\ \text{gcd}(D, D') &= \sum_{P \in C(\bar{K}) - \{P_\infty\}} \min\{\text{ord}_P(D), \text{ord}_P(D')\}(P - P_\infty). \end{aligned}$$

Then  $D + D' = \text{gcd}(D, D') + \text{lcm}(D, D')$ .

There is a canonical isomorphism from  $\text{Div}_K^0(C)$  to the group of fractional  $K[C]$ -ideals, written as  $D \mapsto I_D$ , with inverse  $I \mapsto \text{div}(I)$ .  $D$  is effective if and only if  $I_D$  is integral. If  $g_1, g_2, \dots \in K[C]$  are polynomials, then we write  $\text{div}(g_1, g_2, \dots)$  in place of  $\text{div}(\langle g_1, g_2, \dots \rangle)$ .

In [2], Arita described a monomial order on  $K[C]$  induced by the pole orders  $\text{ord}_{P_\infty}(x) = -3$  and  $\text{ord}_{P_\infty}(y) = -4$ . Every ideal  $I$  of  $K[C]$  has a unique reduced Gröbner basis with respect to this ordering that contains the *minimum polynomial* of  $I$ , i.e., the unique polynomial  $f_I$  in any Gröbner basis of  $I$  with the smallest leading monomial and leading coefficient 1. Under this isomorphism, we have the following correspondence between effective divisors and their associated  $K[C]$ -ideals:

| Divisors | $D + D'$     | $\text{lcm}(D, D')$ | $\text{gcd}(D, D')$ | $\bar{D}$       | $D \leq D'$            |
|----------|--------------|---------------------|---------------------|-----------------|------------------------|
| Ideals   | $I_D I_{D'}$ | $I_D \cap I_{D'}$   | $I_D + I_{D'}$      | $f_{I_D} : I_D$ | $I_D \supseteq I_{D'}$ |

Here,  $f_{I_D} : I_D$  is the unique  $K[C]$ -ideal satisfying  $I_D(f_{I_D} : I_D) = \langle f_{I_D} \rangle$ , the principal ideal generated by  $f_{I_D}$ . The corresponding divisor  $\bar{D} = \text{div}(f_{I_D} : I_D)$  is the *flip* of  $D$ ; it is equivalent to  $-D$  and is reduced. It follows that  $D$  is reduced if and only if  $D = \bar{\bar{D}}$ , and  $\bar{\bar{D}}$  is the *reduction* of  $D$ , i.e., the unique reduced divisor linearly equivalent to  $D$ . This gives rise to the following high-level algorithm for addition in the degree zero divisor class group of a  $C_{3,4}$  curve, found also in [2]. Given two reduced divisors  $D$  and  $D'$ , represented by the reduced Gröbner bases of their respective ideals  $I_D$  and  $I_{D'}$ , perform the following:

- (1) Compute the reduced Gröbner basis of  $J := I_D I_{D'}$ .
- (2) Compute the reduced Gröbner basis of  $J^* := f_J : J$ .
- (3) Compute the reduced Gröbner basis of  $J^{**} := f_{J^*} : J^*$ .

Then  $\text{div}(J^{**})$  is the unique reduced divisor equivalent to  $D + D'$ . In [8], Khuri-Makdisi showed how to combine the last two steps into a single efficient step.

Following [8], an effective divisor  $D$  is said to be *semitypical* if the reduced Gröbner basis of  $I_D$  consists of three polynomials, i.e.,  $I_D = \langle f, g, h \rangle$ . A divisor is *typical* if it is semitypical with  $h \in \langle f, g \rangle$ , where  $h$  is the generator with the largest pole order at infinity. A divisor that is not typical is called *atypical*. All typical divisors are semitypical, but atypical divisors may or may not be semitypical.

In [2], Arita classified all divisors of degree  $\leq 6$  into 19 types according to the leading monomials of their reduced Gröbner bases. Table 2.1 reproduces Arita's classification, along with a 20-th type corresponding to the zero divisor. Note that a divisor of degree  $d \leq 6$  is semitypical if and only if it is of type 31, 41, 51, or 61, and a type 31 divisor  $D$  is typical if and only if  $f_2$ , the coefficient of  $y$  in

| Deg | Type | Gröbner Basis   |
|-----|------|---|
| 0   | 0    | 1   |
| 1   | 11   | $x + f_0, \quad y + g_0$  |
| 2   | 21   | $y + f_1x + f_0, \quad x^2 + g_1x + g_0$  |
|     | 22   | $x + f_0, \quad y^2 + g_2y + g_0$   |
| 3   | 31   | $x^2 + f_2y + f_1x + f_0, \quad xy + g_2y + g_1x + g_0, \quad y^2 + h_2y + h_1x + h_0$  |
|     | 32   | $y + f_1x + f_0, \quad x^3 + g_3x^2 + g_1x + g_0$   |
|     | 33   | $x + f_0$   |
| 4   | 41   | $xy + f_3x^2 + f_2y + f_1x + f_0, \quad y^2 + g_3x^2 + g_2y + g_1x + g_0, \quad x^3 + h_3x^2 + h_2y + h_1x + h_0$   |
|     | 42   | $x^2 + f_1x + f_0, \quad xy + g_2y + g_1x + g_0$  |
|     | 43   | $x^2 + f_2y + f_1x + f_0, \quad y^2 + g_4xy + g_2y + g_1x + g_0$  |
|     | 44   | $y + f_1x + f_0$  |
| 5   | 51   | $y^2 + f_4xy + f_3x^2 + f_2y + f_1x + f_0, \quad x^3 + g_4xy + g_3x^2 + g_2y + g_1x + g_0, \quad x^2y + h_4xy + h_3x^2 + h_2y + h_1x + h_0$                             |
|     | 52   | $xy + f_3x^2 + f_2y + f_1x + f_0, \quad y^2 + g_3x^2 + g_2y + g_1x + g_0$   |
|     | 53   | $xy + f_3x^2 + f_2y + f_1x + f_0, \quad x^3 + g_5y^2 + g_3x^2 + g_2y + g_1x + g_0$  |
|     | 54   | $x^2 + f_2y + f_1x + f_0, \quad xy^2 + g_5y^2 + g_4xy + g_2y + g_1x + g_0$  |
| 6   | 61   | $x^3 + f_5y^2 + f_4xy + f_3x^2 + f_2y + f_1x + f_0, \quad x^2y + g_5y^2 + g_4xy + g_3x^2 + g_2y + g_1x + g_0, \quad xy^2 + h_5y^2 + h_4xy + h_3x^2 + h_2y + h_1x + h_0$ |
|     | 62   | $y^2 + f_4xy + f_3x^2 + f_2y + f_1x + f_0, \quad x^3 + g_4xy + g_3x^2 + g_2y + g_1x + g_0$  |
|     | 63   | $y^2 + f_4xy + f_3x^2 + f_2y + f_1x + f_0, \quad x^2y + g_6x^3 + g_4xy + g_3x^2 + g_2y + g_1x + g_0$  |
|     | 64   | $xy + f_3x^2 + f_2y + f_1x + f_0, \quad x^4 + g_6x^3 + g_5y^2 + g_3x^2 + g_2y + g_1x + g_0$   |
|     | 65   | $x^2 + f_2y + f_1x + f_0$   |

**Table 2.1.** Arita's classification of divisors into types.

$f_{I_D}$ , is nonzero (see [8, Proposition 2.12]). The types of  $\bar{D}$  and  $\bar{\bar{D}}$  are determined by the type of  $D$  as summarized in Table 2.2. Examples of computing the type of  $\bar{D}$  are found in Section 7.3 of [10]. A divisor is reduced if and only if it is of type 0, 11, 21, 22 or 31; in particular, all divisors of degree  $d \leq 2$  are reduced.

| Divisor         | Type |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|-----------------|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $D$             | 0    | 11 | 21 | 22 | 31 | 32 | 33 | 41 | 42 | 43 | 44 | 51 | 52 | 53 | 54 | 61 | 62 | 63 | 64 | 65 |
| $\bar{D}$       | 0    | 22 | 21 | 11 | 31 | 11 | 0  | 31 | 22 | 21 | 0  | 31 | 22 | 21 | 11 | 31 | 22 | 21 | 11 | 0  |
| $\bar{\bar{D}}$ | 0    | 11 | 21 | 22 | 31 | 22 | 0  | 31 | 11 | 21 | 0  | 31 | 11 | 21 | 22 | 31 | 11 | 21 | 22 | 0  |

**Table 2.2.** Divisor types and the type of their flip and double flip.



### 3. Addition

In this section, we describe how to add two distinct reduced divisors. Analogous to [11], we make use of certain Riemann–Roch spaces. For any nonzero function  $f \in K[C]$ , denote by  $\text{LM}(f)$  the leading monomial of  $f$ . Let  $m \in K[C]$  be a monomial and  $D$  an effective divisor in  $\text{Div}_K^0(C)$ . Define

$$W^m = \mathcal{L}(-\text{ord}_{P_\infty}(m)P_\infty) = \{f \in K[C] \mid \text{LM}(f) \leq m\},$$

$$W_D^m = \mathcal{L}(-\text{ord}_{P_\infty}(m)P_\infty - D) = \{f \in I_D \mid \text{LM}(f) \leq m\} = W^m \cap I_D.$$

Given a reduced Gröbner basis for  $I_D$ , it is easy to construct an echelon basis for  $W_D^m$  by taking monomial multiples of the basis elements and removing all those that result in duplicate leading monomials. Given an echelon basis for  $W_D^m$  with  $m$  sufficiently large, a reduced Gröbner basis for  $I_D$  can be obtained by removing any basis element whose leading monomial is divisible by that of another basis element.

Now let  $D, D'$  be distinct reduced divisors of respective degrees  $d = \deg(D)$  and  $d' = \deg(D')$ , with  $d \geq d'$ . Let  $m$  be the largest monomial appearing in the reduced Gröbner basis of any ideal  $I$  such that  $\text{div}(I)$  has degree  $d + d'$ . For example, if  $d + d' = 6$ , then the reduced Gröbner basis of an ideal of a type 64 divisor contains a polynomial with leading monomial  $m = x^4$ , and no other degree 6 divisor type has a larger monomial.

Put  $L = \text{lcm}(D, D')$  and  $G = \text{gcd}(D, D')$ . The divisors  $L$  and  $G$  arise from the kernel and image, respectively, of the matrix  $M$  in the diagram below. Here,  $\iota$  denotes inclusion and  $\pi$  is the natural projection:

$$W_L^m \xrightarrow{\ker M} W_D^m \xrightarrow{\quad \quad \quad} W^m \xrightarrow{\quad \quad \quad} \frac{W^m}{W_{D'}^m} \xrightarrow{\text{im } M} \frac{W_G^m}{W_{D'}^m}$$

$\begin{array}{c} \text{---} M \text{---} \\ \nearrow \quad \quad \searrow \\ \iota \quad \quad \quad \pi \end{array}$

A proof of this crucial result can be found in [10, Theorem 8.7]. This is a generalization of the addition procedure of [11], where the authors compute  $\ker M$  for  $m = x^2y$  only. This is sufficient when  $D$  and  $D'$  are disjoint (or equivalently,  $G = 0$ ) and typical, but their approach fails otherwise. A larger bounding monomial  $m$  can handle atypical divisor sums, and computing the image  $\text{im } M$  allows nondisjoint input divisors  $D, D'$ .

The kernel and image of  $M$  are obtained by first computing the reduced row echelon form of  $M$ , denoted  $\text{RREF}(M)$ , which in particular reveals the rank of  $M$  as well as the dimensions of its kernel and image. If  $M$  has full rank, which is typically the case, then  $G = 0$  and  $\ker M$  produces a reduced Gröbner basis for  $I_L = I_{D+D'}$ . If  $M$  has rank 0, then  $D' < D$ , in which case we find the divisor  $A$  such that  $D = D' + A$  and return  $\overline{2D'} + A$  via a call to the doubling algorithm in Section 4. Otherwise, we recursively compute the sum  $\overline{L} + G$ . In this recursive call, one of the input divisors has degree strictly less than  $d'$ , so this recursion terminates. Details of the algorithm and toy examples can be found in [10, Chapter 8].

#### 4. Doubling

Doubling a reduced divisor  $D$  is similar to adding two distinct reduced divisors. Here, we find a (not necessarily reduced) divisor  $A \neq D$  equivalent to  $D$  and compute the reduction  $\overline{A+D} = \overline{2D}$  using the addition algorithm from [Section 3](#). We describe an optimized approach for finding  $A$  that represents a significant improvement over the doubling method presented in [\[10, Chapter 9\]](#).

We begin with the most common case when  $D$  is a type 31 divisor. Let  $\{f, g, h\}$  be a reduced Gröbner basis of its associated ideal  $I_D$ .

**Lemma 4.1.** *Let  $D$  be of type 31. Then there exist polynomials*

$$\begin{aligned} r &= y + r_0, & s &= -(x + s_0), & t &= t_0, \\ r' &= x^2 + r'_2 y + r'_1 x + r'_0, & s' &= s'_0, & t' &= y + t'_0, \\ r'' &= r''_0, & s'' &= y + s''_0, & t'' &= x + t''_0 \end{aligned}$$

in  $K[C]$  such that  $rf + sg + th = 0$ ,  $r'f + s'g + t'h = F$  and  $r''f + s''g + t''h = 0$ .

*Proof.* Explicit formulas for  $r, s, t, r', s', t'$  are given in [Table 6.2](#). The polynomials  $r'' = h_1$ ,  $s'' = y - g_1 + h_2$  and  $t'' = -x - g_2$ , with  $g_1, g_2, h_1, h_2$  as given in [\(6-1\)](#), are easily verified to satisfy the third identity.  $\square$

The quantities  $r'', s'', t''$  are only auxiliary to the proof of [Proposition 4.2](#). Put

$$A = \text{div}(\tilde{f}, \tilde{g}, \tilde{h}) \quad \text{with } \tilde{f} = st' - ts', \quad \tilde{g} = tr' - rt', \quad \tilde{h} = rs' - sr'. \quad (4-1)$$

Then the leading monomials of  $\tilde{f}, \tilde{g}, \tilde{h}$  are  $xy, y^2, x^3$ , respectively, so  $A$  is of type 41 by [Table 2.1](#). It is easy to verify that  $\tilde{f}\tilde{g} = g\tilde{f}$  and  $\tilde{f}\tilde{h} = h\tilde{f}$  in  $K[C]$ . It follows that  $\tilde{f}I_D = \tilde{f}I_A$  and hence  $\text{div } \tilde{f} + A = \text{div } \tilde{f} + D$ , so  $A$  is equivalent to  $D$ .

The following proposition shows that  $A$  and  $D$  are typically disjoint. If not, we have  $D \not\subseteq A$ . Either way, we may add  $D$  and  $A$  using the addition algorithm from the previous section.

**Proposition 4.2.** *Let  $D$  be of type 31 and put  $G = \text{gcd}(D, A)$ . If  $D$  is typical, then  $G = 0$ , otherwise  $G$  has degree 1.*

*Proof.* We have  $\deg(G) \leq \deg(D) = 3$ . Suppose  $\deg(G) \geq 2$ . Then  $D - G$  and  $A - G$  are equivalent divisors of degree  $\leq 2$ . So these two divisors are reduced and hence equal, which is impossible since  $\deg(D) \neq \deg(A)$ . It follows that  $\deg(G) \leq 1$ .

Suppose  $\deg(G) = 1$ . Then  $\deg(D - G) = 2$ ,  $\deg(A - G) = 3$  and  $\overline{D - G} = \overline{A - G}$ , which by [Table 2.2](#) forces  $D - G$  to be of type 22 and  $A - G$  to be of type 32. Let  $x + a$  and  $x + b$  be the minimum polynomials of  $I_G$  and  $I_{D-G}$ , respectively. Then  $f = (x + a)(x + b) \in I_D$ . Appealing to the form of  $I_D$  characterized in [Table 2.1](#),  $f$  is the minimum polynomial of  $I_D$  and has a vanishing  $y$ -coefficient, so  $D$  is atypical.

Conversely, suppose that  $D$  is atypical. Referring to the quantities of [Lemma 4.1](#), we have  $t = -f_2 = 0$ . Put  $I = \langle r, s \rangle$ . Then  $I$  is a prime ideal of degree 1. From [\(4-1\)](#), we see that  $I_A \subseteq I$ . A simple symbolic



computation yields  $f = st''$ ,  $g = rt''$  and  $h = r''s - s''r$ , so  $I_D \subseteq I$ . It follows that  $I_G = I_A + I_D \subseteq I$ , so  $\text{div}(I) \leq G$ , which in turn implies  $\deg(G) \geq 1$ , and hence  $\deg(G) = 1$ .  $\square$

An optimization is possible when computing the kernel of  $M$  in

$$W_L^m \xrightarrow{\ker M} W_A^m \xrightarrow{\iota} W^m \xrightarrow{\pi} \frac{W^m}{W_D^m} \xrightarrow{\text{im } M} \frac{W_G^m}{W_D^m}.$$

$M$

The kernel consists of  $K[C]$ -linear combinations of  $\{\tilde{f}, \tilde{g}, \tilde{h}\}$  that belong to  $W_L^m$ . However, the following theorem shows that when  $D$  is typical, we may instead perform our computations on  $f, g, h$ . The latter have fewer monomials, so the resulting linear combinations are faster to generate.

**Theorem 4.3.** *Let  $D$  be of type 31,  $L = \text{lcm}(D, A)$  and  $G = \text{gcd}(D, A)$ . Let  $a, b, c \in K[C]$ . Then  $af + bg + ch \in I_{2D-G}$  if and only if  $a\tilde{f} + b\tilde{g} + c\tilde{h} \in I_L$ .*

*Proof.* We have  $2D - G + \text{div}(\tilde{f}) = L + D - A + \text{div}(\tilde{f}) = L + \text{div}(f)$ . Since  $f\tilde{g} = g\tilde{f}$  and  $f\tilde{h} = h\tilde{f}$ , the claim follows.  $\square$

If  $D$  is typical, then  $I_{2D-G} = I_{2D}$  by [Proposition 4.2](#).

Next, we provide analogous results for divisors  $D$  of types 11, 21, and 22. Here,  $I_D = \langle f, g \rangle$ .

**Theorem 4.4.** *Let  $D$  be of type 11, 21, or 22, and write  $I_D = \langle f, g \rangle$ . Then there exist nonzero polynomials  $\tilde{f}, \tilde{g} \in K[C]$  such that  $f\tilde{g} + g\tilde{f} = F$  and  $\tilde{f}\langle f, g \rangle = f\langle \tilde{f}, \tilde{g} \rangle$ . The divisor  $A = \text{div}(\tilde{f}, \tilde{g})$  is equivalent to  $D$  and  $\text{gcd}(A, D) = 0$ . Finally, for any  $a, b \in K[C]$ , we have  $af + bg \in I_{2D}$  if and only if  $a\tilde{f} + b\tilde{g} \in I_{A+D}$ .*

*Proof.* The first assertion follows from  $F \in \langle f, g \rangle$ . Since  $f\tilde{g} = -g\tilde{f}$  in  $K[C]$ , we have  $\tilde{f}\langle f, g \rangle = \langle f\tilde{f}, g\tilde{f} \rangle = \langle f\tilde{f}, f\tilde{g} \rangle = f\langle \tilde{f}, \tilde{g} \rangle$ , so  $\text{div}(\tilde{f}) + D = \text{div}(f) + A$ . This identity also yields the last assertion, provided that  $\text{gcd}(A, D) = 0$ .

Suppose first that  $D$  is of type 11. Then the leading monomials of  $f$  and  $g$  are  $x$  and  $y$ , respectively. A solution to  $f\tilde{g} + g\tilde{f} = F$  then requires that the leading monomials of  $\tilde{f}$  and  $\tilde{g}$  are  $y^2$  and  $x^3$ , respectively. Therefore  $A = \text{div}(\tilde{f}, \tilde{g})$  is a type 62 divisor. Suppose  $\text{gcd}(A, D) \neq 0$ . Then  $A - D$  would be a principal divisor of degree 5 which is impossible by [Table 2.1](#).

Likewise, suppose  $D$  is of type 21. Then  $A = \text{div}(\tilde{f}, \tilde{g})$  is of type 43. Suppose  $G = \text{gcd}(A, D) \neq 0$ . Since  $A - G \equiv D - G$ , we either have a degree 3 divisor that is equivalent to a degree 1 divisor, or a degree 2 divisor that is equivalent to 0, depending on the degree of  $G$ . Appealing to [Table 2.1](#), we see that both cases are impossible. The case when  $D$  is of type 22 is similar.  $\square$

Our addition and doubling routines call one another, but this process terminates. The doubling routine terminates on all inputs except atypical type 31 divisors ([Proposition 4.2](#)), in which case we must add  $\bar{L} + G$  where  $\deg G = 1$  and there is no need to subsequently double another type 31 divisor. Furthermore, the addition routine may call itself, but the degree of the smaller divisor strictly decreases, forcing it to eventually terminate.

## 5. Reduction

Reducing a divisor may be accomplished by flipping it twice, as was done in [2; 11]. However, in [8], it was shown that for typical degree 6 divisors, both flips can be combined into a single operation that is more efficient than even just the first flip. Below, we generalize this result to all typical and nonsemityypical divisors (of any degree). The remaining divisors, those that are semityypical but atypical, are addressed in [Theorem 5.2](#).

**Theorem 5.1.** *Let  $D$  be an effective divisor on  $C$  and let  $\{u, v\}$  be any generating set for  $I_D$  such that  $u$  is the minimum polynomial of  $I_D$ . Then there exist polynomials  $f, g \in K[C]$  such that  $fv = gu$  in  $K[C]$  and  $\bar{\bar{D}} = \text{div}(f, g)$ .*

*Proof.* Let  $f$  be the minimum polynomial of the colon ideal  $u : v$ . Then there exists  $g \in K[C]$  such that  $fv = gu$  in  $K[C]$ . The divisor  $A = \text{div}(f, g)$  is equivalent to  $D$  since  $uI_A = \langle fu, gu \rangle = \langle fu, fv \rangle = fI_D$ . The minimality of  $u$  and  $f$  implies that  $A$  is reduced and is hence the reduction of  $D$ .  $\square$

In particular, [Theorem 5.1](#) makes efficient reduction of all divisors listed in [Table 2.1](#) straightforward, except for atypical semityypical divisors, where  $I_D$  might be generated by no two of its Gröbner basis elements. Given  $I_D = \langle u, v \rangle$ , the type of  $\bar{\bar{D}}$  is first read from [Table 2.2](#). Then the leading monomials of  $f, g$ , with  $I_{\bar{\bar{D}}} = \langle f, g \rangle$ , are obtained from [Table 2.1](#). The coefficients of  $f, g$  are now easily computed by equating coefficients in the relation  $fv \equiv gu \pmod{F}$  and solving the resulting system of linear equations.

Reduction of atypical semityypical divisors is done via [Theorem 5.2](#) which represents an improvement for type 41 and 51 divisors over the method presented in [10, Section 10.1].

**Theorem 5.2.** *Let  $D$  be an atypical semityypical divisor, and write  $I_D = \langle f, g, h \rangle$ . Put  $I = \langle f, g \rangle$ . Then there exist  $K$ -rational points  $P, Q$  on  $C$  such that  $\text{div}(I) = D + (P - P_\infty)$  and  $\overline{\text{div}(I)} = Q - P_\infty$ .*

*Proof.* We have  $\deg \text{div}(I) = \dim_K(K[C]/I)$  and  $\deg D = \dim_K(K[C]/I_D)$ . Computing these dimensions for each atypical case using [Table 2.1](#) (the dimensions are determined by the leading coefficients of  $f$  and  $g$ ) yields  $\deg \text{div}(I) = \deg D + 1$  which establishes the existence of  $P$ .

Analogous to [Lemma 4.1](#), there exist polynomials  $r = x + r_0, s = y + s_1x + s_0 \in K[C]$  such that  $fs + gr = F$  when  $D$  is of type 51 and  $fs = gr$  otherwise. Since  $\text{div}(r, s)$  has degree 1, it is reduced and of the form  $Q - P_\infty$ . As in the proof of [Theorem 5.1](#), we see that  $I$  is equivalent to  $\langle r, s \rangle$ , which is hence the reduction of  $\text{div}(I)$ .  $\square$

**Corollary 5.3.**  $\bar{\bar{D}} = (Q - P_\infty) + \overline{P - P_\infty}$ .

*Proof.* By [Theorem 5.2](#),  $D = \text{div}(I) - (P - P_\infty)$  and  $\overline{\text{div}(I)} = Q - P_\infty$ . The reduced divisor equivalent to  $-(P - P_\infty)$  is  $\overline{P - P_\infty}$ . It follows that  $\bar{\bar{D}}$  is equivalent to  $(Q - P_\infty) + \overline{P - P_\infty}$ . Since  $\bar{\bar{D}}$  is reduced and both  $\bar{\bar{D}}$  and  $(Q - P_\infty) + \overline{P - P_\infty}$  have the same degree, they must both be reduced and therefore equal.  $\square$

Obtaining  $P$  amounts to finding polynomials  $p = x + p_0$  and  $q = y + q_1x + q_0$  such that  $hp, hq \in I$ . The polynomials  $r$  and  $s$  of [Theorem 5.2](#) determine  $Q$ .

## 6. Explicit formulas for typical divisors

Here, we derive explicit formulas handling the most typical cases in  $C_{3,4}$  arithmetic: adding disjoint type 31 divisors whose sum is typical, and doubling a typical type 31 divisor whose double is typical. If ever we detect that we are outside these cases, we may fall back on another series of explicit formulas.

Let  $D$  and  $D'$  be typical type 31 divisors, with respective associated ideals and Gröbner bases  $I_D = \langle f, g, h \rangle$  and  $\langle f', g', h' \rangle$ , where

$$\begin{aligned} f &= x^2 + f_2y + f_1x + f_0, & f' &= x^2 + f'_2y + f'_1x + f'_0, \\ g &= xy + g_2y + g_1x + g_0, & g' &= xy + g'_2y + g'_1x + g'_0, \\ h &= y^2 + h_2y + h_1x + h_0, & h' &= y^2 + h'_2y + h'_1x + h'_0. \end{aligned} \quad (6-1)$$

The optimal choice of monomial in the addition and doubling algorithms of [Section 3](#) and [Section 4](#) is  $m = x^2y$ . Bases for the vector spaces  $W_D^{x^2y}$  and  $W_{D'}^{x^2y}$  are  $\{f, g, h, xf, xg\}$  and  $\{f', g', h', xf', xg'\}$ , respectively. The matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix}$$

for adding  $D$  and  $D'$  is constructed by reducing the former basis modulo the latter; e.g., the reduction of  $f$  modulo  $\{f', g', h', xf', xg'\}$  is  $(f_2 - f'_2)y + (f_1 - f'_1)x + (f_0 - f'_0)$ , so  $a_1 = f_0 - f'_0$ ,  $a_6 = (f_1 - f'_1)$ , etc. Computing the first three columns requires only subtractions (counted as additions). The last two columns are given in terms of the first two by

$$\begin{pmatrix} a_4 & a_5 \\ a_9 & a_{10} \\ a_{14} & a_{15} \end{pmatrix} = \begin{pmatrix} 0 & -f'_0 & -g'_0 \\ 1 & -f'_1 & -g'_1 \\ 0 & -f'_2 & -g'_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_6 & a_7 \\ a_{11} & a_{12} \end{pmatrix}.$$

For doubling  $D$ , we construct the divisor  $A$  defined in [Section 4](#) using the polynomials defined in (4-1) and [Lemma 4.1](#). Then the left three columns of the matrix  $M$  used in the computation of  $D + A$  are the reductions of  $\tilde{f}, \tilde{g}, \tilde{h}$  modulo  $f, g, h$ . Let  $e_1 = -(f_1 + g_2)$  and  $e_2 = r'_2 - f_2$ . Then the left three columns of  $M$  are

$$\begin{pmatrix} t'_0s_0 + s'_0t_0 - g_0 & t'_0r_0 + t_0(f_0 - r'_0) - h_0 & f_0e_1 + g_0e_2 - s'_0r_0 - r'_0s_0 \\ t'_0 - g_1 & t_0(f_1 + f_1) - h_1 & f_1(e_1 + s_0) + g_1e_2 - r'_0 + f_0 \\ s_0 - g_2 & t'_0 - h_2 + r_0 - t_0e_2 & f_2(e_1 - g_2) + r'_2(g_2 - s_0) - s'_0 \end{pmatrix}.$$

The right two columns relate to the first three as above, with  $D$  in place of  $D'$ .

If the first column is zero, then  $D + D'$  (or  $D + A$ ) is atypical and we must fall back on other formulas. Otherwise, we assume  $a_1 \neq 0$  by swapping rows if necessary. Then elementary row operations convert

$M$  into row echelon form:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix} \longrightarrow \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 & b_7 \end{pmatrix}.$$

If  $b_1$  or  $b_5$  are zero, then  $D + D'$  (or  $D + A$ ) either contains points of multiplicity exceeding 1 or is atypical. To avoid an expensive inversion operation, we compute a scalar multiple of the reduced row echelon form  $\text{RREF}(M)$  and defer the necessary inversion until later:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 & b_7 \end{pmatrix} \longrightarrow \begin{pmatrix} Z & 0 & 0 & A_1 & A_2 \\ 0 & Z & 0 & B_1 & B_2 \\ 0 & 0 & Z & C_1 & C_2 \end{pmatrix}.$$

Now  $\ker(M) = \text{Span}_K\{U, V\}$ , where

$$U = Zxf - C_1h - B_1g - A_1f, \quad V = Zxg - C_2h - B_2g - A_2f.$$

Let

$$U = Zx^3 + U_5y^2 + \cdots + U_0 \quad \text{and} \quad V = Zx^2y + V_5x^2y + \cdots + V_0.$$

Formulas for the coefficients  $U_i, V_i$  are found in [Table 6.3](#), although note that the constant coefficients  $U_0$  and  $V_0$  are not needed and therefore not computed. Let  $u_0, \dots, u_5, v_0, \dots, v_5$  be the coefficients of  $u := U/Z$  and  $v := V/Z$ . To compute  $u_i, v_i$ , we will need the inverse of  $Z$ . However, we will also need the inverse of  $f_2'' = u_5^2 + u_4 - v_5$  later on. We compute both inverses at once with only a single inversion using a variation of Montgomery's Trick. Formulas for  $\zeta := Z^{-1}$  and  $\tau := (f_2'')^{-1}$  are found in [Table 6.3](#). We note that the intermediate value  $z_0$  is equal to  $Z^2 f_2''$ . If this is zero, then the sum is atypical and we fall back on other formulas. Once  $\zeta$  is known, we compute  $u_i = \zeta U_i$  and  $v_i = \zeta V_i$  for  $i = 1, \dots, 5$ .

Now  $I_{D+D'}$  (or  $I_{2D}$ ) is generated by  $\{u, v\}$ . We apply [Theorem 5.1](#) and find polynomials

$$f'' = x^2 + f_2''y + f_1''x + f_0'' \quad \text{and} \quad g'' = xy + g_3''x^2 + g_2''y + g_1''x + g_0''$$

satisfying

$$f''v \equiv g''u \pmod{F}.$$

We would then have to reduce  $g''$  modulo  $f''$  to eliminate the  $x^2$  term in  $g''$ . Since  $g_3'' = u_5$ , this means subtracting  $u_5$  times  $f''$  from  $g''$ . We avoid this by instead finding  $g'' = xy + g_2''y + g_1''x + g_0''$  such that  $f''v \equiv (g'' + u_5 f'')u \pmod{F}$ , thereby saving a multiplication and a few additions.

The third polynomial in the Gröbner basis of  $I_{D+D'}$  (or  $I_{2D}$ ) is

$$h'' = \tau((y + g_1'')f'' - (x + f_1'' - g_2'')g'').$$

Explicit formulas and operation counts for all the quantities above are given in [Tables 6.1, 6.2, and 6.3](#).

|   |         |
|---|---------|
| Addition  | 12M+17A |
| Input: $I_D = \langle f, g, h \rangle, I_{D'} = \langle f', g', h' \rangle$<br>$f = x^2 + f_2y + f_1x + f_0, f' = x^2 + f'_2y + f'_1x + f'_0$<br>$g = xy + g_2y + g_1x + g_0, g' = xy + g'_2y + g'_1x + g'_0$<br>$h = y^2 + h_2y + h_1x + h_0, h' = y^2 + h'_2y + h'_1x + h'_0$<br>Output: $M_{\text{add}} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix}$   |         |
| Compute elements $a_i$ of $M_{\text{add}}$  | 12M+17A |
| $a_1 = f_0 - f'_0$ $a_2 = g_0 - g'_0$ $a_3 = h_0 - h'_0$ $a_4 = -f'_0a_6 - g'_0a_{11}$<br>$a_5 = -f'_0a_7 - g'_0a_{12}$ $a_6 = f_1 - f'_1$ $a_7 = g_1 - g'_1$ $a_8 = h_1 - h'_1$<br>$a_9 = a_1 - f'_1a_6 - g'_1a_{11}$ $a_{10} = a_2 - f'_1a_7 - g'_1a_{12}$ $a_{11} = f_2 - f'_2$ $a_{12} = g_2 - g'_2$<br>$a_{13} = h_2 - h'_2$ $a_{14} = -f'_2a_6 - g'_2a_{11}$ $a_{15} = -f'_2a_7 - g'_2a_{12}$<br>If $a_1 = a_6 = a_{11} = 0$ , then abort.<br>If $a_1 = 0$ is zero but $a_6 \neq 0$ or $a_{11} \neq 0$ , then swap rows so $a_1 \neq 0$ . |         |

Table 6.1. Construction of matrix  $M$  — typical addition.

## 7. Implementation and testing

A Sage implementation of  $C_{3,4}$  curve arithmetic based on the algorithms in this paper is available at [9]. This implementation includes optimized addition and doubling subroutines `fast_add_31_31`, `fast_add_31_31_high_char`, `fast_double_31`, and `fast_double_31_high_char`. The high characteristic versions assume that the curve equation is given in short form and implement the formulas in Tables 6.1, 6.2, and 6.3. The other versions implement similar formulas with no assumptions on the coefficients  $c_5$ ,  $c_6$ , and  $c_8$ . The optimized subroutines assume the typical cases described in Section 6. When any of these assumptions are violated, an exception is thrown, and a less-optimized subroutine is called instead.

The less-optimized subroutines are nonetheless implemented via explicit formulas. These include addition subroutines for every pair of reduced divisor types (e.g., `add_31_21`), a doubling subroutine for every reduced divisor type (e.g., `double_31`), and a reduction subroutine for every unreduced divisor type (e.g., `reduce_61`).

Addition subroutines, given input divisors  $D$  and  $D'$ , compute  $L = \text{lcm}(D, D')$  and  $G = \text{gcd}(D, D')$  by computing the kernel and image of a matrix as described in Section 3. If  $G = 0$ , then the reduction of  $L$  is computed via the appropriate subroutine and  $\bar{L}$  is returned. Otherwise  $\bar{L}$  and  $G$  are added by calling another addition subroutine. The cost of evaluating  $D + D'$  depends on the type of  $L$ . Costs are given in Table 7.1(A) for the cases when  $G = 0$ . When  $G > 0$ , one or more recursive calls must be made. A full analysis of the cost in these cases was not done, due to the large number of subcases that can occur.

|   |            |
|---|------------|
| Doubling  | 28M+1S+41A |
| Input: $I_D = \langle f, g, h \rangle$<br>$f = x^2 + f_2y + f_1x + f_0$ , $g = xy + g_2y + g_1x + g_0$ , $h = y^2 + h_2y + h_1x + h_0$<br>Output: $M_{\text{doub}} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix}$   |            |
| Compute polynomials $r = y + r_0$ , $s = -(x + s_0)$ , $t = t_0$ such that $rf + sg + th = 0$   | 1A         |
| $r_0 = g_1$ $s_0 = f_1 - g_2$ $t_0 = -f_2$  |            |
| Compute polynomials $r' = x^2 + r'_2y + r'_1x + r'_0$ , $s' = s'_0$ , $t' = y + t'_0$ such that $r'f + s'g + t'h = F$   | 2M+1S+7A   |
| $r'_2 = c_7 - f_2$ $r'_1 = -f_1$ $t'_0 = -h_2 - f_2r'_2$<br>$s'_0 = c_4 - h_1 + f_1(f_2 - r'_2)$ $r'_0 = c_3 + f_1^2 - f_0$   |            |
| Compute reductions<br>$\bar{f} = \tilde{f}_2y + \tilde{f}_1x + \tilde{f}_0$ , $\bar{g} = \tilde{g}_2y + \tilde{g}_1x + \tilde{g}_0$ , $\bar{h} = \tilde{h}_2y + \tilde{h}_1x + \tilde{h}_0$   | 14M+25A    |
| $e_1 = -f_1 - g_2$ $e_2 = r'_2 - f_2$<br>$\tilde{f}_2 = s_0 - g_2$ $\tilde{f}_1 = t'_0 - g_1$<br>$\tilde{f}_0 = t'_0s_0 + s'_0t_0 - g_0$ $\tilde{g}_2 = t'_0 - h_2 + r_0 - t_0e_2$<br>$\tilde{g}_1 = t_0(f_1 + f_1) - h_1$ $\tilde{g}_0 = t'_0r_0 + t_0(f_0 - r'_0) - h_0$<br>$\tilde{h}_2 = f_2(e_1 - g_2) + r'_2(g_2 - s_0) - s'_0$<br>$\tilde{h}_1 = f_1(e_1 + s_0) + g_1e_2 - r'_0 + f_0$<br>$\tilde{h}_0 = f_0e_1 + g_0e_2 - s'_0r_0 - r'_0s_0$  |            |
| Compute matrix $M_{\text{doub}}$  | 12M+8A     |
| $a_1 = \tilde{f}_0$ $a_2 = \tilde{g}_0$ $a_3 = \tilde{h}_0$<br>$a_4 = -f_0a_6 - g_0a_{11}$ $a_5 = -f_0a_7 - g_0a_{12}$ $a_6 = \tilde{f}_1$<br>$a_7 = \tilde{g}_1$ $a_8 = \tilde{h}_1$ $a_9 = a_1 - f_1a_6 - g_1a_{11}$<br>$a_{10} = a_2 - f_1a_7 - g_1a_{12}$ $a_{11} = \tilde{f}_2$ $a_{12} = \tilde{g}_2$<br>$a_{13} = \tilde{h}_2$ $a_{14} = -f_2a_6 - g_2a_{11}$ $a_{15} = -f_2a_7 - g_2a_{12}$<br>If $a_1 = a_6 = a_{11}$ , then abort.<br>If $a_1 = 0$ but $a_6 \neq 0$ or $a_{11} \neq 0$ , then swap rows so $a_1 \neq 0$ . |            |

**Table 6.2.** Construction of matrix  $M$  — typical doubling.

Doubling subroutines, given an input divisor  $D$ , find generators for a divisor  $A$  equivalent to  $D$ , and compute  $G = \gcd(A, D)$  and  $2D - G$  as outlined in [Section 4](#). We recursively compute  $\overline{2D - G} + G$ . The cost depends on the type of  $2D - G$ , if  $G = 0$ , and if a recursive call must be made. [Table 7.1\(B\)](#)



|   |               |
|---|---------------|
| Computing $\ker M$  | 11+99M+3S+72A |
| <p>Input: <math>I_D = \langle f, g, h \rangle, M</math></p> $f = x^2 + f_2y + f_1x + f_0, \quad g = xy + g_2y + g_1x + g_0, \quad h = y^2 + h_2y + h_1x + h_0$ $M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix}$ <p>Output: <math>I_{D+D'} = \langle f'', g'', h'' \rangle</math> (or <math>I_{2D} = \langle f'', g'', h'' \rangle</math>)</p> $f'' = x^2 + f_2''y + f_1''x + f_0'', \quad g'' = xy + g_2''y + g_1''x + g_0'', \quad h'' = y^2 + h_2''y + h_1''x + h_0''$ |               |
| Compute row echelon form of $M$   | 21M+12A       |
| $d_1 = a_1a_{12} - a_2a_{11} \quad d_2 = a_6a_{12} - a_7a_{11}$ $b_1 = a_1a_7 - a_2a_6 \quad b_2 = a_1a_8 - a_3a_6 \quad b_3 = a_1a_9 - a_4a_6$ $b_4 = a_1a_{10} - a_5a_6 \quad b_5 = b_1a_{13} - d_1a_8 + d_2a_3 \quad b_6 = b_1a_{14} - d_1a_9 + d_2a_4$ $b_7 = b_1a_{15} - d_1a_{10} + d_2a_5$   |               |
| Compute $Z \cdot \text{RREF}(M)$  | 18M+6A        |
| $Y = a_1b_1 \quad Z = Yb_5$ $e_1 = b_3b_5 - b_2b_6 \quad e_2 = b_4b_5 - b_2b_7$ $A_1 = b_1(a_4b_5 - b_6a_3) - a_2e_1 \quad B_1 = a_1e_1 \quad C_1 = Yb_6$ $A_2 = b_1(a_5b_5 - b_7a_3) - a_2e_2 \quad B_2 = a_1e_2 \quad C_2 = Yb_7$   |               |
| Compute $\ker(M)$   | 18M+14A       |
| $U_1 = Zf_0 - C_1h_1 - B_1g_1 - A_1f_1 \quad U_2 = -C_1h_2 - B_1g_2 - A_1f_2$ $U_3 = Zf_1 - A_1 \quad U_4 = Zf_2 - B_1 \quad U_5 = -C_1$ $V_1 = Zg_0 - C_2h_1 - B_2g_1 - A_2f_1 \quad V_2 = -C_2h_2 - B_2g_2 - A_2f_2$ $V_3 = Zg_1 - A_2 \quad V_4 = Zg_2 - B_2 \quad V_5 = -C_2$   |               |
| Compute $\zeta = Z^{-1}, \tau = (f_2'')^{-1}$   | 11+5M+2S+3A   |
| $z_0 = U_5^2 + Z(U_4 - V_5) \quad z_1 = Zz_0 \quad z_2 = z_1^{-1} \quad z_3 = Zz_2 \quad \zeta = z_0z_2 \quad \tau = Z^2z_3$  |               |
| Compute $u_1, \dots, u_5, v_1, \dots, v_5$  | 10M           |
| $u_1 = \zeta U_1 \quad u_2 = \zeta U_2 \quad u_3 = \zeta U_3 \quad u_4 = \zeta U_4 \quad u_5 = \zeta U_5$ $v_1 = \zeta V_1 \quad v_2 = \zeta V_2 \quad v_3 = \zeta V_3 \quad v_4 = \zeta V_4 \quad v_5 = \zeta V_5$   |               |
| Compute $f'', g'', h''$   | 27M+1S+37A    |
| $r_0 = u_5(f_2'' + u_4 - c_7) + u_3 - v_4 \quad r_1 = f_2''(f_2'' - u_4)$ $g_0'' = u_5(c_3 - f_0'' - u_1 - f_1''u_3) - g_1''u_3 + f_1''v_3 + v_1$ $g_1'' = r_1 - u_5(u_3 + r_0) + v_3 \quad g_2'' = -u_4u_5 + v_4 - r_0 + \tau(u_4r_0 - u_5g_1'' - u_2)$ $f_0'' = -c_7(r_1 + g_2''u_5) + u_5(f_2''u_3 + f_1''u_4 - c_4 + u_2) + g_2''u_3 + g_1''u_4 - f_2''v_3 - f_1''v_4 + u_1 - v_2$ $f_1'' = r_0 + g_2'' \quad f_2'' = u_5^2 + u_4 - v_5$ $h_0'' = \tau(f_0''g_1'' - g_0''r_0) \quad h_1'' = \tau(g_1''g_2'' - g_0'') \quad h_2'' = g_1'' + \tau(f_0'' - g_2''r_0)$  |               |

Table 6.3. Computing  $\ker M$ .

contains the costs for the cases where  $G = 0$ . Here, “t” and “a” under the type column refer to typical and atypical divisors, respectively.

Our operation counts for the high characteristic formulas compare to the previous state of the art in [8] as follows:

|                   | Addition       | Doubling        |
|-------------------|----------------|-----------------|
| Khuri-Makdisi [8] | 2I+97M+1S+132A | 2I+107M+3S+155A |
| This work         | 1I+111M+3S+99A | 1I+127M+4S+112A |

These counts include a trade-off of one inversion for several multiplications. An inversion is generally considered to be as expensive as 80 multiplications, depending on implementation and environment details [3; 5]. Our formulas also significantly decrease the number of additions required, and the total number of field operations in both of our formulas is less than that of [8]. Over large fields such as those considered in [8], additions are generally considered to have negligible cost compared to multiplications and inversions, but in number theoretic computations such as [13] over smaller (typically word-sized) primes, this has been observed not to be the case.

To verify that our results represent an improvement over the previous state-of-the-art, we implemented the formulas from [11] and [8] in Sage and ran benchmark tests as follows. Given a prime  $p$ , choose a random  $C_{3,4}$  curve  $C$  over  $\mathbb{F}_p$  (with defining polynomial in short form) and two random divisors  $D_1$  and  $D_2$  on  $C$ . Details on random divisor generation are given in Section 12.2 of [10]. We counted how many terms in the Fibonacci-like sequence  $D_{i+2} = D_{i+1} + D_i$ ,  $i \geq 1$  (for addition) and the sequence  $D_{i+1} = 2D_i$ ,  $i \geq 1$  (for doubling) each algorithm is able to compute in 10 minutes. We chose to run these tests over the first 23 primes greater than  $2^{28}$ , as primes on this order are of interest in number theoretic applications (see [14], for example), and because degenerate cases are so rare that we can strictly compare our formulas to those of [11] and [8]. Our algorithm computed 126,310,162 additions as compared to 112,041,012 using the algorithm from [8], for a speedup of 12.74%. Similarly, our algorithm computed 120,827,482 doublings as compared to 108,489,487 for a speedup of 11.37%.

This benchmark was repeated over the first 11 primes larger than  $2^{255}$ , where we found a more significant speed-up, likely due to the increasing cost of inverting in large finite fields. Our algorithm computed 63,151,623 additions versus 52,185,141 using the algorithm from [8], for a speedup of 21.01%. Similarly, our algorithm computed 56,795,783 doublings as compared to 48,395,712 for a speedup of 17.36%.

We found the most significant speed-up over very small primes, where atypical cases are frequently encountered and our explicit formulas are much faster than generic arithmetic. Over the ten largest primes below  $2^8$ , we compared our formulas against those of [11] and [8], falling back on Sage’s generic ideal arithmetic for cases not handled by those papers. Our algorithm computed 53,670,222 additions as compared to 31,685,426 using the algorithm from [8], for a speedup of 69.38%, and 48,156,514 doublings as compared to 39,152,564 for a speedup of 23.00%.

It is important to acknowledge the role that the implementation environment plays in these results. The benchmarks were run in the Sage interpreter, which adds significant overhead to the calculations.

| Subroutine               | Op count |     |   |     | Type of $L$ |
|--------------------------|----------|-----|---|-----|-------------|
|                          | I        | M   | S | A   |             |
| add_11_11                | 1        | 3   | 0 | 4   | 21          |
| add_11_11                | 0        | 1   | 0 | 3   | 22          |
| add_21_11                | 1        | 13  | 0 | 14  | 31          |
| add_21_11                | 0        | 12  | 0 | 17  | 32          |
| add_21_21                | 2        | 68  | 1 | 58  | 41-t        |
| add_21_21                | 2        | 67  | 0 | 58  | 41-a        |
| add_21_21                | 1        | 27  | 0 | 19  | 42          |
| add_21_21                | 1        | 39  | 0 | 32  | 43          |
| add_21_21                | 0        | 12  | 0 | 9   | 44          |
| add_21_22                | 2        | 40  | 1 | 41  | 41-t        |
| add_21_22                | 2        | 39  | 0 | 41  | 41-a        |
| add_21_22                | 0        | 2   | 0 | 2   | 42          |
| add_22_11                | 1        | 5   | 0 | 5   | 31-a        |
| add_22_11                | 0        | 1   | 0 | 3   | 33          |
| add_22_22                | 1        | 11  | 0 | 17  | 43          |
| add_31_11                | 2        | 43  | 1 | 49  | 41-t        |
| add_31_11                | 2        | 22  | 0 | 49  | 41-a        |
| add_31_11                | 0        | 6   | 0 | 10  | 42          |
| add_31_11                | 1        | 16  | 0 | 32  | 43          |
| add_31_21                | 2        | 80  | 1 | 77  | 51-t        |
| add_31_21                | 2        | 78  | 1 | 74  | 51-a        |
| add_31_21                | 1        | 35  | 1 | 33  | 52          |
| add_31_21                | 1        | 57  | 1 | 51  | 53          |
| add_31_21                | 1        | 43  | 1 | 41  | 54          |
| add_31_22                | 2        | 69  | 0 | 64  | 51-t        |
| add_31_22                | 2        | 67  | 0 | 61  | 51-a        |
| add_31_22                | 1        | 24  | 0 | 20  | 52          |
| add_31_22                | 1        | 46  | 0 | 38  | 53          |
| add_31_22                | 1        | 36  | 0 | 29  | 54          |
| fast_add_31_31_high_char | 1        | 111 | 3 | 99  | 61-t        |
| fast_add_31_31           | 1        | 114 | 2 | 102 | 61-t        |
| add_31_31                | 2        | 127 | 0 | 110 | 61-a        |
| add_31_31                | 1        | 69  | 0 | 54  | 62          |
| add_31_31                | 1        | 85  | 0 | 67  | 63          |
| add_31_31                | 1        | 94  | 0 | 75  | 64          |
| add_31_31                | 0        | 32  | 0 | 28  | 65          |

(A) Addition

| Subroutine               | Op count |     |   |     | Type of $2D - G$ |
|--------------------------|----------|-----|---|-----|------------------|
|                          | I        | M   | S | A   |                  |
| double_11                | 1        | 15  | 1 | 20  | 21               |
| double_11                | 0        | 8   | 1 | 13  | 22               |
| double_21                | 2        | 86  | 1 | 85  | 41-t             |
| double_21                | 2        | 85  | 0 | 85  | 41-a             |
| double_21                | 1        | 50  | 0 | 47  | 42               |
| double_21                | 1        | 60  | 0 | 60  | 43               |
| double_21                | 0        | 7   | 0 | 12  | 44               |
| double_22                | 1        | 22  | 0 | 22  | 42               |
| double_22                | 1        | 25  | 0 | 29  | 43               |
| fast_double_31_high_char | 1        | 127 | 4 | 112 | 61               |
| fast_double_31           | 1        | 138 | 2 | 130 | 61               |
| double_31                | 2        | 159 | 0 | 156 | 61-t             |
| double_31                | 2        | 152 | 0 | 149 | 61-a             |
| double_31                | 1        | 94  | 0 | 90  | 62               |
| double_31                | 1        | 110 | 0 | 103 | 63               |
| double_31                | 1        | 119 | 0 | 111 | 64               |
| double_31                | 0        | 57  | 0 | 64  | 65               |

(B) Doubling

| Subroutine | Op count |    |   |    |
|------------|----------|----|---|----|
|            | I        | M  | S | A  |
| reduce_32  | 0        | 8  | 0 | 11 |
| reduce_33  | 0        | 0  | 0 | 0  |
| reduce_41t | 1        | 23 | 1 | 28 |
| reduce_41a | 1        | 22 | 0 | 28 |
| reduce_42  | 0        | 0  | 0 | 1  |
| reduce_43  | 0        | 6  | 0 | 11 |
| reduce_44  | 0        | 0  | 0 | 0  |
| reduce_51t | 1        | 24 | 0 | 32 |
| reduce_51a | 1        | 22 | 0 | 29 |
| reduce_52  | 0        | 1  | 0 | 3  |
| reduce_53  | 0        | 12 | 0 | 14 |
| reduce_54  | 0        | 7  | 0 | 10 |
| reduce_61t | 1        | 35 | 0 | 46 |
| reduce_61a | 1        | 28 | 0 | 39 |
| reduce_62  | 0        | 2  | 0 | 5  |
| reduce_63  | 0        | 8  | 0 | 13 |
| reduce_64  | 0        | 12 | 0 | 21 |
| reduce_65  | 0        | 0  | 0 | 0  |

(C) Reduction

Table 7.1. Operation counts for  $C_{3,4}$  arithmetic.

If implemented in a low level language, such as C/PARI, our improvements over [11; 8] may be more dramatic.

Correctness testing was accomplished by a combination of unit testing and random testing. Unit tests were constructed testing every branch of code in the addition, doubling, and reduction subroutines. These subroutines were also tested via hundreds of thousands of random inputs and the results were compared against Sage's vetted ideal arithmetic.

## 8. Conclusion

By generalizing the techniques of Abu Salem and Khuri-Makdisi [11] to atypical divisors as classified by Arita [2], we provided a fully general framework for efficient divisor arithmetic on  $C_{3,4}$  curves. Taken together with our additional improvements to the setting of typical divisors, we obtain speedups of between 11 and 21% depending on the field size, and even more for small fields were atypical cases arise more frequently.

There is room for further speed advances in  $C_{3,4}$  curve arithmetic, and work on this topic is ongoing. In our formulas for atypical divisors, addition/doubling and reduction are performed separately. Savings could be effected by combining these into a single optimized subroutine, as was done in Section 6 for the typical case. It is also possible to eliminate all inversions using an analogue of projective coordinates, but this would likely not help with number-theoretic computations where frequent equality tests of divisors are required.

Arithmetic on  $C_{3,4}$  curves continues to be significantly more expensive than arithmetic on genus 3 hyperelliptic curves. Preliminary results indicate that Shanks' NUCOMP algorithm [12] achieves significant savings in the latter setting, which raises the question whether a NUCOMP-like idea may be applied to  $C_{3,4}$  curve arithmetic as well.

## References

- [1] Seigo Arita, *Algorithms for computations in Jacobian group of  $C_{a,b}$  curve and their application to discrete-log-based public key cryptosystems*, Conference on the Mathematics of Public Key Cryptography (1999), 165–175.
- [2] Seigo Arita, *An addition algorithm in Jacobian of  $C_{3,4}$  curve*, IEICE Trans. Found. **E88-A** (2005), no. 6, 1589–1598.
- [3] Erik Dahmen, Katsuyuki Okeya, and Daniel Schepers, *Affine precomputation with sole inversion in elliptic curve cryptography*, Information Security and Privacy (Berlin, Heidelberg) (Josef Pieprzyk, Hossein Ghodosi, and Ed Dawson, eds.), Springer Berlin Heidelberg, 2007, pp. 245–258.
- [4] Stéphane Flon, Roger Oyono, and Christophe Ritzenthaler, *Fast addition on non-hyperelliptic genus 3 curves*, Algebraic geometry and its applications, Proceedings of the first SAGA conference, Ser. Number theory and its applications, vol. 4, World Sci. Publ., 2008, pp. 1–28.
- [5] Darrel Hankerson, Alfred Menezes, and Scott Vanstone, *Guide to elliptic curve cryptography*, Springer-Verlag, 2004.
- [6] David Harvey, Maike Massierer, and Andrew S. Sutherland, *Computing  $L$ -series of geometrically hyperelliptic curves of genus three*, LMS J. Comput. Math. **19** (2016), no. suppl. A, 220–234.
- [7] Kiran S. Kedlaya and Andrew V. Sutherland, *Computing  $L$ -series of hyperelliptic curves*, Algorithmic Number Theory, Lecture Notes in Comput. Sci., vol. 5011, Springer, Berlin, 2008, pp. 312–326.
- [8] Kamal Khuri-Makdisi, *On Jacobian group arithmetic for typical divisors on curves*, Research in Number Theory **4** (2018), no. 1.

- [9] Evan MacNeil, *c34-curves*, <https://github.com/emmacneil/c34-curves>, 2019.
- [10] Evan MacNeil, *Divisor class group arithmetic on  $C_{3,4}$  curves*, Master's thesis, University of Calgary, Canada, 2019, <https://prism.ucalgary.ca/handle/1880/111659>.
- [11] Fatima Abu Salem and Kamal Khuri-Makdisi, *Fast Jacobian group operations for  $C_{3,4}$  curves over a large finite field*, LMS J. Comput. Math. **10** (2007), 307–328.
- [12] Daniel Shanks, *On Gauss and composition I, II*, Proc. NATO ASI on Number Theory and Applications, Kluwer Academic Press, 1989, pp. 163–204.
- [13] Andrew V. Sutherland, *Fast Jacobian arithmetic for hyperelliptic curves of genus 3*, Proceedings of the Thirteenth Algorithmic Number Theory Symposium, The Open Book Ser., vol. 2, Math. Sci. Publ., Berkeley, CA, 2019, pp. 425–442.
- [14] Andrew V. Sutherland, *Sato-Tate distributions*, Analytic methods in arithmetic geometry, Contemp. Math., vol. 740, Amer. Math. Soc., Providence, RI, 2019, pp. 197–248.

Received 28 Feb 2020.

EVAN MACNEIL: [macneil.evan@ucalgary.ca](mailto:macneil.evan@ucalgary.ca)

*Department of Mathematics and Statistics, University of Calgary, Calgary AB, Canada*

MICHAEL J. JACOBSON JR.: [jacobs@ucalgary.ca](mailto:jacobs@ucalgary.ca)

*Department of Computer Science, University of Calgary, Calgary AB, Canada*

RENAME SCHEIDLER: [rscheidl@ucalgary.ca](mailto:rscheidl@ucalgary.ca)

*Department of Mathematics and Statistics, University of Calgary, Calgary AB, Canada*

VOLUME EDITORS

Stephen D. Galbraith  
Mathematics Department  
University of Auckland  
New Zealand

<https://orcid.org/0000-0001-7114-8377>

---

The cover image is based on an illustration from the article “Supersingular curves with small noninteger endomorphisms”, by Jonathan Love and Dan Boneh (see p. 9).

The contents of this work are copyrighted by MSP or the respective authors. All rights reserved.

Electronic copies can be obtained free of charge from <http://msp.org/obs/4> and printed copies can be ordered from MSP ([contact@msp.org](mailto:contact@msp.org)).

The Open Book Series is a trademark of Mathematical Sciences Publishers.

ISSN: 2329-9061 (print), 2329-907X (electronic)

ISBN: 978-1-935107-07-1 (print), 978-1-935107-08-8 (electronic)

First published 2020.

---



**MATHEMATICAL SCIENCES PUBLISHERS**

798 Evans Hall #3840, c/o University of California, Berkeley CA 94720-3840

[contact@msp.org](mailto:contact@msp.org)

<http://msp.org>



## Fourteenth Algorithmic Number Theory Symposium

The Algorithmic Number Theory Symposium (ANTS), held biennially since 1994, is the premier international forum for research in computational and algorithmic number theory. ANTS is devoted to algorithmic aspects of number theory, including elementary, algebraic, and analytic number theory, the geometry of numbers, arithmetic algebraic geometry, the theory of finite fields, and cryptography.

This volume is the proceedings of the fourteenth ANTS meeting, which took place 29 June to 4 July 2020 via video conference, the plans for holding it at the University of Auckland, New Zealand, having been disrupted by the COVID-19 pandemic. The volume contains revised and edited versions of 24 refereed papers and one invited paper presented at the conference.

## TABLE OF CONTENTS

|   |     |
|---|-----|
| Commitment schemes and diophantine equations — José Felipe Voloch   | 1   |
| Supersingular curves with small noninteger endomorphisms — Jonathan Love and Dan Boneh  | 7   |
| Cubic post-critically finite polynomials defined over $\mathbb{Q}$ — Jacqueline Anderson, Michelle Manes and Bella Tobin  | 23  |
| Faster computation of isogenies of large prime degree — Daniel J. Bernstein, Luca De Feo, Antonin Leroux and Benjamin Smith   | 39  |
| On the security of the multivariate ring learning with errors problem — Carl Bootland, Wouter Castryck and Frederik Vercauteren   | 57  |
| Two-cover descent on plane quartics with rational bitangents — Nils Bruin and Daniel Lewis  | 73  |
| Abelian surfaces with fixed 3-torsion — Frank Calegari, Shiva Chidambaram and David P. Roberts  | 91  |
| Lifting low-gonal curves for use in Tuitman's algorithm — Wouter Castryck and Floris Vermeulen  | 109 |
| Simultaneous diagonalization of incomplete matrices and applications — Jean-Sébastien Coron, Luca Notarnicola and Gabor Wiese   | 127 |
| Hypergeometric $L$ -functions in average polynomial time — Edgar Costa, Kiran S. Kedlaya and David Roe  | 143 |
| Genus 3 hyperelliptic curves with CM via Shimura reciprocity — Bogdan Adrian Dina and Sorina Ionica   | 161 |
| A canonical form for positive definite matrices — Mathieu Dutour Sikirić, Anna Haensch, John Voight and Wessel P.J. van Woerden   | 179 |
| Computing Igusa's local zeta function of univariates in deterministic polynomial-time — Ashish Dwivedi and Nitin Saxena   | 197 |
| Computing endomorphism rings of supersingular elliptic curves and connections to path-finding in isogeny graphs — Kirsten Eisenträger, Sean Hallgren, Chris Leonardi, Travis Morrison and Jennifer Park | 215 |
| New rank records for elliptic curves having rational torsion — Noam D. Elkies and Zev Klagsbrun   | 233 |
| The nearest-colattice algorithm: Time-approximation tradeoff for approx-CVP — Thomas Espitau and Paul Kirchner  | 251 |
| Cryptanalysis of the generalised Legendre pseudorandom function — Novak Kaluđerović, Thorsten Kleinjung and Dušan Kostić  | 267 |
| Counting Richelot isogenies between superspecial abelian surfaces — Toshiyuki Katsura and Katsuyuki Takashima   | 283 |
| Algorithms to enumerate superspecial Howe curves of genus 4 — Momonari Kudo, Shushi Harashita and Everett W. Howe   | 301 |
| Divisor class group arithmetic on $C_{3,4}$ curves — Evan MacNeil, Michael J. Jacobson Jr. and Renate Scheidler   | 317 |
| Reductions between short vector problems and simultaneous approximation — Daniel E. Martin  | 335 |
| Computation of paramodular forms — Gustavo Rama and Gonzalo Tornaría  | 353 |
| An algorithm and estimates for the Erdős–Selfridge function — Brianna Sorenson, Jonathan Sorenson and Jonathan Webster  | 371 |
| Totally $p$ -adic numbers of degree 3 — Emerald Stacy   | 387 |
| Counting points on superelliptic curves in average polynomial time — Andrew V. Sutherland   | 403 |