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On sets of zero stationary harmonic measure

Eviatar B. Procaccia[a,*,1], Yuan Zhang[b,2,3]

[a] Technion - Israel Institute of Technology, Israel, Texas A&M University, United States of America
[b] Peking University, China

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Abstract

In this paper, we study properties of the stationary harmonic measure which are unique to the stationary case. We prove that any subset with an appropriate sub-linear horizontal growth has a non-zero stationary harmonic measure. On the other hand, we show that any subset with at least linear horizontal growth will have a 0 stationary harmonic measure at every point. This result is fundamental to any future study of stationary DLA. As an application we prove that any possible aggregation process with growth rates proportional to the stationary harmonic measure has non zero measure at all times.

1. Introduction

In this paper, we present conditions for an infinite subset in the upper half planar lattice to have non-zero stationary harmonic measure. Stationary harmonic measure is first introduced in [6], and plays a fundamental role in the study of diffusion limit aggregation (DLA) models on non-transitive graphs with absorbing boundary conditions. Roughly speaking, the stationary harmonic measure of a subset is the expected number of random walks hitting each of its points, when we drop infinitely many random walks from a horizontal line "infinitely high" and stop once they first hit the subset or the x-axis. The stationary harmonic measure is not a

* Corresponding author.
E-mail addresses: eviatarp@technion.ac.il (E.B. Procaccia), zhangyuan@math.pku.edu.cn (Y. Zhang).
URL: https://www.math.tamu.edu/~procaccia (E.B. Procaccia).

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probability measure and since this measure is defined on infinite sets it has different attributes than the classical harmonic measure, some of which was studied in this paper. The stationary harmonic measure can be used to construct stationary DLA, which plays an equivalent role as the harmonic measure in $\mathbb{Z}^d$ in the construction of the DLA model (see [1–3]). The analysis in this paper is crucial to the understanding of aggregation phenomenon under absorbing boundary conditions. Indeed recently the scaling limit of the classical DLA starting from a long line to a version of Stationary DLA was established [4,5]. An application of this paper is that if an aggregation process grows too fast it might stop growing but if the growth is moderate it will keep on growing forever.

For the precise discussions, we first set some notations defined in [6]. Let

$$\mathbb{H} = \{(x_1, x_2) \in \mathbb{Z}^2, x_2 \geq 0\}$$

be the upper half planar lattice (including x-axis), and $S_n, n \geq 0$ be a 2-dimensional simple random walk. For any $x \in \mathbb{Z}^2$, we will write $x = (x_1, x_2)$, with x_i denoting the ith coordinate of x. Then define the horizontal line of height n

$$L_n = \{(x, n),\ \ x \in \mathbb{Z}\}.$$

For any subset $A \subset \mathbb{Z}^2$ abbreviate the first hitting time

$$\bar{\tau}_A = \min\{n \geq 0,\ \ S_n \in A\}$$

and the first exit time

$$\tau_A = \min\{n \geq 1,\ \ S_n \in A\}.$$

For any subsets $A_1 \subset A_2$ and B and any $y \in \mathbb{Z}^2$, by definition one can easily check that

$$P_y\left(\tau_{A_1} < \tau_B\right) \leq P_y\left(\tau_{A_2} < \tau_B\right), \tag{1}$$

and that

$$P_y\left(\tau_B < \tau_{A_2}\right) \leq P_y\left(\tau_B < \tau_{A_1}\right). \tag{2}$$

Now we define the stationary harmonic measure on $\mathbb{H}$. For any $B \subset \mathbb{H}$, any edge $\vec{e} = x \to y$ with $x \in B$, $y \in \mathbb{H} \setminus B$ and any N, we define

$$\bar{\mathcal{H}}_{B,N}(\vec{e}) = \sum_{z \in L_N \setminus B} P_z\left(S_{\bar{\tau}_{B\cup L_0}} = x, S_{\bar{\tau}_{B\cup L_0}-1} = y\right). \tag{3}$$

Remark 1. As a paper on the nearest neighbor aggregation process, [6] concentrate mostly on the harmonic measure $\bar{\mathcal{H}}_B$ and $\bar{\mathcal{H}}_{B,N}$ of a subset B that intersects L_0 and that $B \cup L_0$ is connected. However, it is clear to see that the definition of $\bar{\mathcal{H}}_{B,N}(\vec{e})$ as well as the convergence in Proposition 1 is not related to connectivity and thus hold for any B.

By definition, $\bar{\mathcal{H}}_{B,N}(\vec{e}) > 0$ only if $y \in \partial^{out} B$ and $|x - y| = 1$. For all $x \in B$, we can also define

$$\bar{\mathcal{H}}_{B,N}(x) = \sum_{\vec{e} \text{ starting from } x} \bar{\mathcal{H}}_{B,N}(\vec{e}) = \sum_{z \in L_N \setminus B} P_z\left(S_{\bar{\tau}_{B\cup L_0}} = x\right). \tag{4}$$

And for each point $y \in \partial^{out} B$, we can also define

$$\hat{\mathcal{H}}_{B,N}(y) = \sum_{\substack{\vec{e} \text{ starting in } B \\ \text{ending at } y}} \bar{\mathcal{H}}_{B,N}(\vec{e}) = \sum_{z \in L_N \setminus B} P_z\left(\tau_B \leq \tau_{L_0}, S_{\bar{\tau}_{B\cup L_0}-1} = y\right). \tag{5}$$

In [6] we prove that,

Proposition 1 (*Proposition 1 in [6]*). *For any B and $\vec{e}$ above, there is a finite $\bar{\mathcal{H}}_B(\vec{e})$ such that*

$$\lim_{N\to\infty} \bar{\mathcal{H}}_{B,N}(\vec{e}) = \bar{\mathcal{H}}_B(\vec{e}). \tag{6}$$

And we call $\bar{\mathcal{H}}_B(\vec{e})$ the stationary harmonic measure of $\vec{e}$ with respect to B. We immediately have that the limits $\bar{\mathcal{H}}_B(x) = \lim_{N\to\infty} \bar{\mathcal{H}}_{B,N}(x)$ and $\hat{\mathcal{H}}_B(y) = \lim_{N\to\infty} \hat{\mathcal{H}}_{B,N}(y)$ also exist and we call them the stationary harmonic measure of x and y with respect to B.

Note that the stationary harmonic measure is not a probability measure. When using the stationary harmonic measure as growth rate, or more precisely, letting $\hat{\mathcal{H}}_B(y)$ be the Poisson intensity the state at site y changes from 0 to 1, we defined in [6] the (continuous time) DLA process in the upper half plane $\mathbb{H}$, starting from any finite initial configuration. For a finite subset B, it is shown in [6, Theorem 3] that there must be an $x \in B$ such that $\bar{\mathcal{H}}_B(x) > 0$. This implies that the continuous DLA model will keep growing from any configuration.

Meanwhile, such treatment of using stationary harmonic measure as Poisson intensities rather than probability distribution also opens the possibility to study DLA from an **infinite** initial configuration as an infinite interacting particle system. However, before possibly defining such an infinite growth model, one first has to ask which configuration in $\mathbb{H}$ can be "habitable" for our aggregation. In fact, for infinite B, it is possible for $\bar{\mathcal{H}}_B(\cdot)$ to be uniformly 0. Thus, for the possible DLA starting from such configuration, it will freeze forever without any growth. The intuitive reason for such phenomena is that when B is infinite, each point $x \in B$ may live in the shadow of other much higher points, which will block the random walk starting from "infinity" to visit the former first. In the following counterexample, we see that there can be a uniformly 0 harmonic measure even when the height of B is finite for each $x-$coordinate. We encourage the reader to check the subset here has zero stationary harmonic measure before reading the proof of the main results.

Counterexample 1. *Let*

$$B^0 = \bigcup_{n=-\infty}^{\infty} \{(n, k),\ k = 0, 1, \ldots, 2^{|n|}\}.$$

Then $\bar{\mathcal{H}}_{B^0}(x) = 0$ for all $x \in B^0$.

In this paper, we will concentrate on characterizing the infinite subsets with zero/nonzero stationary harmonic measure. Our first result is a much stronger statement than Counterexample 1: for any (infinite) $B \subset \mathbb{H}$, and any $x_1 \in \mathbb{Z}$, define

$$h_{x_1} = \sup\{x_2 \geq 0,\ x_1 \times [1, x_2] \subset B\}.$$

Definition 1. We say that B has **at least linear horizontal growth** if there are constants $c \in (0, \infty)$, and $M < \infty$ such that

$$h_{x_1} \geq |cx_1|$$

for all $|x_1| \geq M$.

Then we have

Theorem 1. *For any B which has at least linear horizontal growth and any $x \in B$*

$$\bar{\mathcal{H}}_B(x) = 0.$$

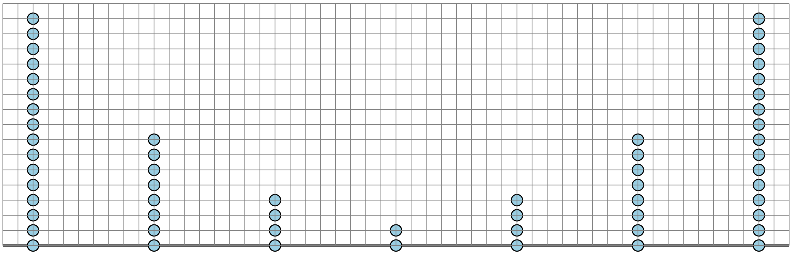

Fig. 1. Example of a big set with zero stationary harmonic measure at any point.

With Theorem 1, Counterexample 1 is immediate. On the other hand, we prove that for B's of which the spatial growth rate has (some) sub-linear upper bound, $\bar{\mathcal{H}}_B(\cdot)$ cannot be 0 everywhere:

Theorem 2. *For $B \subset \mathbb{H}$ if there exists an $\alpha > 1$ such that*

$$\left| B \setminus \{x \in \mathbb{H},\ x_2 \le |x_1|^{1/\alpha}\} \right| < \infty,$$

then there must be some $x \in B$ such that $\bar{\mathcal{H}}_B(x) > 0$.

Remark 2. The conditions of Theorem 1 are not essential as any linear stretching of Counterexample 1 (as can be seen in Fig. 1) will have uniformly zero stationary harmonic measure. There is also a gap between the conditions of Theorems 1 and 2. It would be interesting to obtain sharp conditions for sets of zero stationary harmonic measure.

Remark 3. Throughout this paper, we use c, C etc. to denote constants, while their exact values may vary from place to place.

2. Proof of Theorem 1

For any B with at least linear horizontal growth and any $x = (x_1, x_2) \in B$. We first introduce some notations which later will be helpful in the proof of this theorem. The notations are illustrated in Fig. 2. Recall Definition 1 and let

$$n_x = \max\{|x_1|, M, \lceil x_2/c \rceil\}$$

and

$$D_1 = [-n_x, n_x] \times [-\lceil cn_x \rceil, \lceil cn_x \rceil].$$

Then $x \in D_1$, and by Definition 1,

$$\hat{W}_c \setminus D_1 \subset B \setminus D_1 \tag{7}$$

where

$$\hat{W}_c = \{x \in \mathbb{H},\ x_2 < c|x_1|\}.$$

Moreover, it is not hard to check that for any $N > \lceil cn_x \rceil$ and $y \in L_N \setminus B$, a simple random walk starting from y hits x before hitting any other point in B only if it hits $l_{n_x} =$

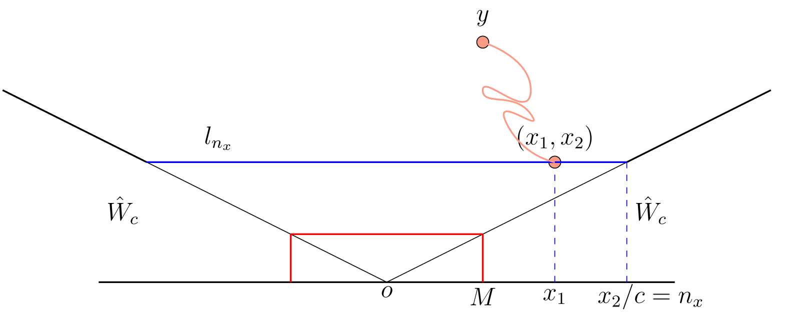

Fig. 2. Illustration of notations: all numbers are assumed to be integers.

$[-n_x, n_x] \times \lceil cn_x \rceil = L_{\lceil cn_x \rceil} \setminus \hat{W}_c$ before hitting $\hat{W}_c$. I.e.,

$$P_y(\tau_x = \tau_B) \le P_y(\tau_{l_{n_x}} < \tau_{\hat{W}_c}). \tag{8}$$

Thus by (7) and (8), and noting that a random walk starting from $y \in L_N \setminus B$ which first visit B through out fixed point x has to first hit l_{n_x} before hitting $\hat{W}_c$ (see also Fig. 2)

$$\begin{aligned}\bar{\mathcal{H}}_{B,N}(x) &= \sum_{y \in L_N \setminus B} P_y(\tau_x = \tau_B) \\ &\le \sum_{y \in L_N \setminus \hat{W}_c} P_y(\tau_{l_{n_x}} < \tau_{\hat{W}_c}) \\ &= \sum_{w \in l_{n_x}} \bar{\mathcal{H}}_{l_{n_x} \cup \hat{W}_c, N}(w).\end{aligned}$$

Then by the proof of Proposition 1 in [6] (which is based on the time reversal argument used in [2]), for any $w \in l_{n_x}$

$$\begin{aligned}&\bar{\mathcal{H}}_{l_{n_x} \cup \hat{W}_c, N}(w) \\ &= \sum_{y \in L_N \setminus \hat{W}_c} P_w\Big(\tau_{L_N} < \tau_{l_{n_x} \cup \hat{W}_c}, S_{\tau_{L_N}} = y\Big) E_y\left[\text{number of visits to } L_N \text{ in } [0, \tau_{l_{n_x} \cup \hat{W}_c})\right] \\ &\le \sum_{y \in L_N \setminus \hat{W}_c} P_w\Big(\tau_{L_N} < \tau_{\hat{W}_c}, S_{\tau_{L_N}} = y\Big) E_y\left[\text{number of visits to } L_N \text{ in } [0, \tau_{L_0})\right] \\ &= 4N \cdot P_w\Big(\tau_{L_N} < \tau_{\hat{W}_c}\Big).\end{aligned} \tag{9}$$

Now let $N_1 = \lceil cn_x \rceil$ and $N_2 = 2N_1$. For any $z \in l_{n_x}$ define the rectangular region

$$D_{1,z} = \big[z_1 - 4\lceil N_2/c \rceil, z_1 + 4\lceil N_2/c \rceil\big] \times [0, N_2].$$

Moreover, we define the four sides on the boundary of $D_{1,z}$

$$\begin{aligned}\partial^1 D_{1,z} &= \big[z_1 - 4\lceil N_2/c \rceil, z_1 + 4\lceil N_2/c \rceil\big] \times N_2, \\ \partial^2 D_{1,z} &= \big(z_1 + 4\lceil N_2/c \rceil\big) \times [0, N_2], \\ \partial^3 D_{1,z} &= \big[z_1 - 4\lceil N_2/c \rceil, z_1 + 4\lceil N_2/c \rceil\big] \times 0, \\ \partial^4 D_{1,z} &= -\big(z_1 + 4\lceil N_2/c \rceil\big) \times [0, N_2]\end{aligned}.$$

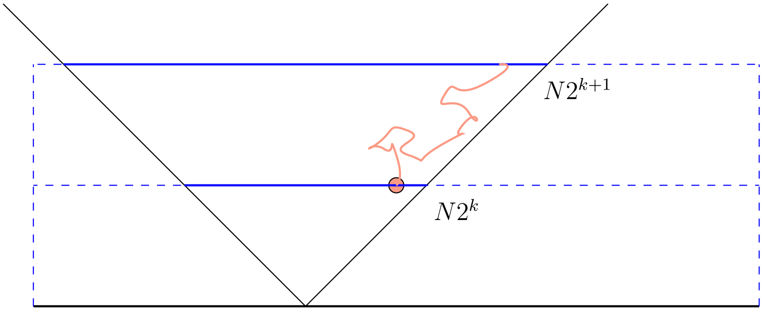

Fig. 3. Escaping probability for each step.

Note that if a random walk starting at z hits $\partial^2 D_{1,z} \cup \partial^3 D_{1,z} \cup \partial^4 D_{1,z}$ before hitting $\partial^1 D_{1,z}$, it must have already hit $\hat{W}_c$ before reaching L_{N_2}. Thus

$$P_z\Big(\tau_{L_{N_2}} < \tau_{\hat{W}_c}\Big) \le P_z\Big(\tau_{\partial^1 D_{1,z}} = \tau_{\partial D_{1,z}}\Big).$$

Then by translation invariance we have

$$P_z\Big(\tau_{\partial^1 D_{1,z}} = \tau_{\partial D_{1,z}}\Big) = P_0\Big(\tau_{\partial^1 D_{1,0}} = \tau_{\partial D_{1,0}}\Big).$$

And by symmetry

$$P_{(0,N_1)}\Big(\tau_{\partial^1 D_{1,0}} = \tau_{\partial D_{1,0}}\Big) = \frac{1}{2} - \frac{1}{2} \cdot P_0\Big(\tau_{\partial^2 D_{1,0}} \wedge \tau_{\partial^4 D_{1,0}} < \tau_{\partial^1 D_{1,0}} \wedge \tau_{\partial^3 D_{1,0}}\Big). \tag{10}$$

Note that the last term in (10) is the probability a random walk first reaches the two vertical sides of $D_{1,z}$ before the horizontal sides. By invariance principle, there is a constant $c > 0$ **independent of** N_1 such that

$$P_{(0,N_1)}\Big(\tau_{\partial^1 D_{1,0}} = \tau_{\partial D_{1,0}}\Big) \le \frac{1-c}{2}. \tag{11}$$

In general, define $N_k = 2^{k-1} N_1$ for all $k \ge 2$, and let

$$l_{N_k} = L_{N_k} \setminus \hat{W}_c,$$

$$D_{k,z} = \big[z_1 - 4\lceil N_{k+1}/c\rceil, z_1 + 4\lceil N_{k+1}/c\rceil\big] \times [0, N_{k+1}],$$

with $\partial^1 D_{k,z}$ - $\partial^4 D_{k,z}$ as its four sides defined as before. Using exactly the same argument as for $k = 1$, we have for any $z \in l_{N_k}$ (see Fig. 3),

$$P_z\Big(\tau_{L_{N_{k+1}}} < \tau_{\hat{W}_c}\Big) \le P_{(0,N_k)}\Big(\tau_{\partial^1 D_{k,0}} = \tau_{\partial D_{k,0}}\Big) \le \frac{1-c}{2}. \tag{12}$$

Noting that the upper bound in (12) is uniform for all $z \in L_{N_k} \setminus \hat{W}_c$, by strong Markov property we have for any $w \in l_{n_x}$

$$P_w\Big(\tau_{L_{N_k}} < \tau_{\hat{W}_c}\Big) \le \left(\frac{1-c}{2}\right)^{k-1} = 2^{-(k-1)(1+\gamma)} \tag{13}$$

where $\gamma = -\log_2(1-c) > 0$. Recalling that $N_k = 2^{k-1} N_1$, (10) and (13) give us

$$\bar{\mathcal{H}}_{l_{n_x} \cup \hat{W}_c, N_k}(w) \le 2^{k+1-(1+\gamma)(k-1)} N_1 = 2^{-\gamma(k-1)+2} N_1 \to 0 \tag{14}$$

as $k \to \infty$. Thus the proof of Theorem 1 is complete. □

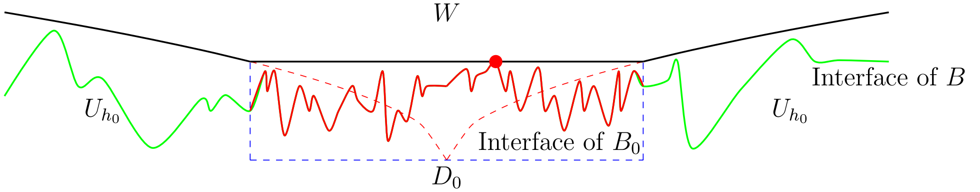

Fig. 4. Illustration of the subsets B, D_0, U_{h_0} and W.

3. Proof of Theorem 2

Recall that for B in Theorem 2 there exists an $\alpha > 1$ such that

$$|\hat{B}| = \left|B \setminus \{x \in \mathbb{H},\ x_2 \le |x_1|^{1/\alpha}\}\right| < \infty.$$

Let $\mathbb{Q}$ be all rational numbers. For technical reasons, consider

$$\mathbb{Q}_1 = \left\{\frac{\log(n_1)}{\log(n_2)},\ n_1, n_2 \in \mathbb{Z},\ n_1, n_2 \ge 2\right\}$$

and

$$\mathbb{Q}_2 = \left\{\frac{b - dx}{cx - a} \in \mathbb{R}, x \in \mathbb{Q}_1,\ a, b, c, d \in \mathbb{Z}\right\} \supset \mathbb{Q}_1$$

which are both countable by definition. Now one can without loss of generality assume that $\alpha \notin \mathbb{Q} \cup \mathbb{Q}_2$. Thus for any $a, b, c, d \in \mathbb{Z}$, and $\alpha' = (a\alpha + b)/(c\alpha + d)$, as long as $a^2 + b^2,\ c^2 + d^2 > 0$, we always have $\alpha' \notin \mathbb{Q} \cup \mathbb{Q}_1$, which implies that for all integers $m \ne n \ge 1$, $m^{\alpha'} \ne n$.

Then we define $\bar{B} = B \setminus \hat{B}$, and

$$h_0 = \begin{cases} \max\limits_{x \in \hat{B}}\{x_2\}, & \text{if } \hat{B} \ne \emptyset \\ \min\left\{h \in \mathbb{Z}:\ B \cap \left[-\lceil h^\alpha \rceil, \lceil h^\alpha \rceil\right] \times [0, h] \ne \emptyset\right\}, & \text{if } \hat{B} = \emptyset. \end{cases}$$

Note that if $\hat{B} \ne \emptyset$, for each $N > h_0$ and $|i| \le \lfloor N^\alpha \rfloor$, note that $|i| \le \lfloor N^\alpha \rfloor < N^\alpha$ by the definition of α, and that $N > h_0 = \max_{x \in \hat{B}}\{x_2\}$. We have

$$\{(i, N),\ |i| \le \lfloor N^\alpha \rfloor\} \cap B = \emptyset.$$

And if $\hat{B} = \emptyset$, h_0 is always finite since B is nonempty. Then define

$$D_0 = \left[-\lceil h_0^\alpha \rceil, \lceil h_0^\alpha \rceil\right] \times [0, h_0],$$

and

$$B_0 = B \cap D_0.$$

An illustration of the definitions above can be seen in Fig. 4. And we have $1 \le |B_0| < \infty$ and $\hat{B} \subset B_0$. Now to prove Theorem 2, we only need to show that there is a constant $c > 0$ (only a function of α) such that for all sufficiently large N,

$$\bar{\mathcal{H}}_{B,N}(B_0) \ge c. \tag{15}$$

Let $l_n = [-\lfloor n^\alpha \rfloor, \lfloor n^\alpha \rfloor] \times n$. We first prove that

Lemma 3.1. *There is a constant $c > 0$ such that for any sufficiently large N*

$$\bar{\mathcal{H}}_{B\cup l_{h_0},N}(l_{h_0}) \geq c.$$

Proof. Recall that

$$D_0 = \left[-\lceil h_0^\alpha \rceil, \lceil h_0^\alpha \rceil\right] \times [0, h_0].$$

Now define

$$U_{h_0} = \{x \in \mathbb{H},\ |x_1| \geq \lceil h_0^\alpha \rceil, x_2 \leq |x_1|^{1/\alpha}\}$$

and

$$\hat{W} = D_0 \cup U_{h_0},\ \ W = \mathbb{H} \setminus (D_0 \cup U_{h_0}).$$

Note that $(\pm\lceil h_0^\alpha \rceil, h_0) \in U_{h_0}$, which implies that $\hat{W}$ is connected, and that

$$W = \{x \in \mathbb{H},\ |x_2| > h_0,\ x_2 \geq |x_1|^{1/\alpha}\}.$$

Again, we recommend the reader to refer to Fig. 4 for an illustration of the subsets above.

At the same time, recall that

$$\bar{B} \subset \{x \in \mathbb{H}, x_2 \leq |x_1|^{1/\alpha}\}.$$

Combining this with the fact that $\hat{B} \subset D_0$ shown above, one has $B \cup l_{h_0} \subset \hat{W}$, which implies that

$$\bar{\mathcal{H}}_{B\cup l_{h_0},N}(l_{h_0}) \geq \bar{\mathcal{H}}_{\hat{W},N}(l_{h_0}) \geq \bar{\mathcal{H}}_{\hat{W},N}(\xi_0),$$

where $\xi_0 = (0, h_0)$. So for Lemma 3.1, it suffices to prove that, there is a constant $c > 0$, independent of N such that for all sufficiently large N,

$$\bar{\mathcal{H}}_{\hat{W},N}(\xi_0) \geq c. \tag{16}$$

The rest of this proof will concentrate on showing (16). First, by the existence of the limit [6, Proposition 1], the limit

$$\lim_{N\to\infty} \bar{\mathcal{H}}_{\hat{W},N}(\xi_0)$$

always exists. Thus in order to show (16), it suffices to show the inequality holds for a subsequence $N_k \uparrow \infty$, say $N_k = 2^k$, i.e., for all sufficiently large k,

$$\bar{\mathcal{H}}_{\hat{W},2^k}(\xi_0) \geq c. \tag{17}$$

Moreover, for each sufficiently large k, noting that all $x \in L_{2^k}$ such that $|x_1| > 2^{\alpha k}$ is in $\hat{W}$,

$$\begin{aligned} \bar{\mathcal{H}}_{\hat{W},2^k}(\xi_0) &= \sum_{z=(i,2^k),\ |i|\leq\lfloor 2^{\alpha k}\rfloor} P_z(\tau_{\xi_0} \leq \tau_{\hat{W}}) \\ &= \sum_{z\in l_{2^k}} P_{\xi_0}\left(\tau_{L_{2^k}} < \tau_{\hat{W}}, S_{\tau_{L_{2^k}}} = z\right) E_z[\#\text{ of visits to } l_{2^k} \text{ in } [0, \tau_{\hat{W}})]. \end{aligned} \tag{18}$$

Consider stopping times $\Gamma_k = \tau_{L_{2^k}}$. Intuitively speaking, we find a middle section $s \subset l_{2^k}$, whose formal definition will be presented in the later arguments (see (20) for details), and construct an event A such that

$$A \subset \left\{\Gamma_k < \tau_{\hat{W}},\ S_{\Gamma_k} \in s\right\}, P(A) \geq c2^{-k}.$$

At the same time, we show that for each $z \in s$,

$$E_z[\text{\# of visits to } l_{2^k} \text{ in } [0, \tau_{\hat{W}})] \geq c2^k.$$

In order to achieve this, one can put the trajectory between each L_{2^i} and $L_{2^{i+1}}$ within appropriately chosen linear wedges such that

(i) The slope of these wedges flatten out to 0, which make the success probability from L_{2^i} to $L_{2^{i+1}}$ close to $1/2$.
(ii) The flattening out rate is slower than $2^{i(\alpha-1)}$ so all the linear wedges are still confined within the middle section of $l_{2^{i+1}}$.

To carry out this outline, we first define several parameters needed later in the construction. Recalling that $\alpha > 1$, let $\beta = 4/(\alpha+3) \in (0, 1)$, and $\gamma = 2(\alpha-1)/(\alpha+3) \in (0, 2)$. Note that

$$\beta\alpha - 1 = \frac{4\alpha}{\alpha+3} - 1 = \frac{3\alpha-3}{\alpha+3} > \gamma > 0,$$

while at the same time

$$\beta + \gamma = \frac{2\alpha+2}{\alpha+3} =: \alpha_1 > 1.$$

Let

$$k_0 = \min\{k : \ 2^{\beta k} > 2h_0\} \vee 2\left\lceil \frac{\alpha+3}{\alpha-1} \right\rceil \vee \left\lceil \frac{3}{\alpha_1 - 1} \right\rceil.$$

For now, the restrictions above may look mysterious, but we will show the meaning of each of them along our proof.

The following lemma from calculus is used repeatedly in our arguments:

Lemma 3.2. *For all $x, y \geq 0$ and $\alpha > 1$,*

$$(x+y)^\alpha \geq x^\alpha + \alpha x^{\alpha-1} y. \tag{19}$$

Proof. Note that (19) is clearly true when $xy = 0$. Assuming $x, y > 0$, by mid value theorem and the fact that $x^{\alpha-1}$ is increasing, we have

$$(x+y)^\alpha = x^\alpha \left(1 + \frac{y}{x}\right)^\alpha \geq x^\alpha \left(1 + \alpha\frac{y}{x}\right) = x^\alpha + \alpha x^{\alpha-1} y. \quad \square$$

Lower bounds for the escaping probabilities: Now back to the proof of Lemma 3.1. Recalling that the definition of $\hat{W}$ and k_0 is independent of the choice of k in (17), consider the event

$$A_0 = \left\{\Gamma_{k_0} < \tau_{\hat{W}_0}, \ S_{\Gamma_{k_0}} = (0, 2^{k_0})\right\}.$$

One has that the probability of A_0 is also a positive number independent of k. I.e.,

$$P_{\xi_0}(A_0) = c > 0,$$

Now for $X_1 = (0, 2^{k_0})$, consider a new wedge

$$W_0 = \left\{x = (x_1, x_2) \in \mathbb{H}, \ x_2 - \lceil 2^{\beta k_0} \rceil \geq |x_1| \cdot 2^{-\gamma k_0}\right\}.$$

Note that $x_2 \geq \lceil 2^{\beta k_0} \rceil \geq 2h_0$, $x \notin D_0$. At the same time, by Lemma 3.2,

$$\begin{aligned} x_2^\alpha &\geq \left(\lceil 2^{\beta k_0}\rceil + |x_1| \cdot 2^{-\gamma k_0}\right)^\alpha \\ &\geq \lceil 2^{\beta k_0}\rceil^\alpha + \alpha \lceil 2^{\beta k_0}\rceil^{(\alpha-1)} |x_1| 2^{-\gamma k_0} \\ &\geq \lceil 2^{\beta k_0}\rceil^\alpha + \alpha 2^{[\beta(\alpha-1)-\gamma]k_0} |x_1|. \end{aligned}$$

Since $\beta(\alpha-1)-\gamma \geq \beta\alpha - 1 - \gamma > 0$, we have that $x_2^\alpha \geq \lceil 2^{\beta k_0}\rceil^\alpha + |x_1| > |x_1|$, which implies that $W_0 \subset W$.

For $k_1 = k_0 + 1$, and probability

$$p_0 = P_{\xi_0}\left(\Gamma_{k_1} < \tau_{W_0^c}\right),$$

one can see that

$$p_0 \leq P_{\xi_0}(\Gamma_{k_1} < \tau_{\hat{W}}).$$

Moreover, we have that

$$W_0 \cap L_{2^{k_1}} \subset s_1 := \{(x_1, 2^{k_1}),\ \ |x_1| \leq 2^{k_1(1+\gamma)}\}.$$

Then for each $y = (y_1, y_2) \in s_1$, we can always define a wedge

$$W_{1,y} = \left\{(x_1, x_2) \in \mathbb{H},\ \ x_2 - \lceil 2^{\beta k_1}\rceil \geq |x_1 - y_1| \cdot 2^{-\gamma k_1}\right\}.$$

The following lemma is technical and the proof is exported to the Appendix.

Lemma 3.3. *For every $y \in s_1$, $W_{1,y} \subset W$.*

Then for $k_2 = k_1 + 1$ define the probability

$$p_{1,y} = P_y\left(\Gamma_{k_2} \leq \tau_{W_{1,y}^c}\right).$$

By translation invariance, we have $p_{1,y} = p_1$ for all such y's. In general, for all $i \geq 1$ let $k_i = k_0 + i$. And for all

$$y \in s_i = \left\{(y_1, 2^{k_i}), |y_1| \leq \sum_{j=1}^{i} 2^{(1+\gamma)k_j}\right\} \tag{20}$$

we define wedge

$$W_{i,y} = \left\{(x_1, x_2) \in \mathbb{H},\ \ x_2 - \lceil 2^{\beta k_i}\rceil \geq |x_1 - y_1| \cdot 2^{-\gamma k_i}\right\}.$$

The following lemma is technical and the proof is exported to the Appendix.

Lemma 3.4. *For all $i \geq 1$ and $y \in s_i$, $W_{i,y} \subset W$.*

Also for each $y \in s_i$, and $z \in W_{i,y} \cap L_{2^{k_{i+1}}}$,

$$|z_1| \leq |y_1| + 2^{k_{i+1}+\gamma k_i} \leq \sum_{j=1}^{i+1} 2^{(1+\gamma)k_j}$$

which implies that

$$\left(\bigcup_{y \in s_i} W_{i,y}\right) \cap L_{2^{k_{i+1}}} \subset s_{i+1} = \left\{(y_1, 2^{k_{i+1}}), |y_1| \leq \sum_{j=1}^{i+1} 2^{(1+\gamma)k_j}\right\}.$$

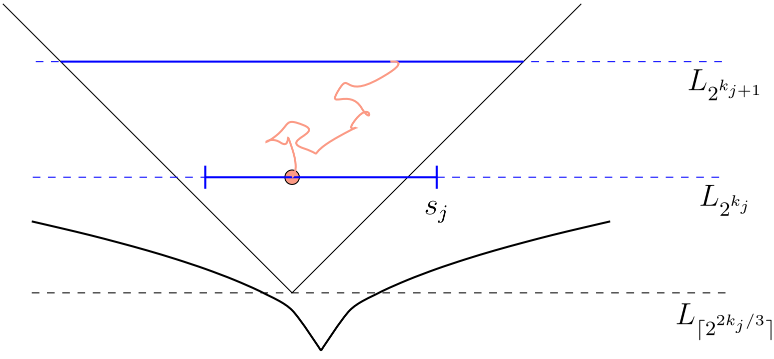

Fig. 5. Escaping probability from $L_{2^{k_j}}$ to $L_{2^{k_{j+1}}}$.

And for all $y \in s_i$ by translation invariance, define (see Fig. 5)

$$p_i = P_y(\Gamma_{k_{i+1}} \le \tau_{W^c_{i,y}}).$$

With the constructions above and strong Markov property, one can see that for each i

$$P_{\xi_0}\left(\Gamma_{k_i} < \tau_{\hat{W}}\right) \ge P_{\xi_0}\left(\tau_{s_i} < \tau_{\hat{W}}\right) \ge \prod_{j=0}^{i-1} p_j. \tag{21}$$

Now to find lower bounds for the success probability p_i, we need to the following simple lemma (easily proved with the invariance principle) showing that it is highly unlikely for a simple random walk starting from the middle of a very wide but short rectangular box to exit from the vertical sides:

Lemma 3.5. *For any integers $n, k \ge 1$, let rectangle*

$$R_{k,n} = [-nk, nk] \times [-k, k]$$

with

$$l^v_{k,n} = \{-nk, nk\} \times [-k, k]$$

as its two vertical sides and

$$l^h_{k,n} = [-nk, nk] \times \{-k, k\}$$

as its two horizontal sides. Then there is a $\delta \in (0, 1)$ such that for any $n, k \ge 1$ and any integer $x \in \{0\} \times [-k, k]$,

$$P_x\left(\tau_{l^v_{k,n}} < \tau_{l^h_{k,n}}\right) \le (1-\delta)^n.$$

With Lemma 3.5 we can bound from below the probabilities p_i. Recalling that by translation invariance, for each i, $y_{i,0} = (0, 2^{k_i})$ and

$$W_{i,0} = \left\{(x_1, x_2) \in \mathbb{H},\ x_2 - \lceil 2^{\beta k_i} \rceil \ge |x_1| \cdot 2^{-\gamma k_i}\right\},$$

we have

$$p_i = P_{y_{i,0}}\left(\Gamma_{k_{i+1}} \le \tau_{W^c_{i,0}}\right).$$

Then consider the rectangle

$$R_i = \left[-\lfloor 2^{\alpha_1 k_i}\rfloor, \lfloor 2^{\alpha_1 k_i}\rfloor\right] \times \left[2\lceil 2^{\beta k_i}\rceil, 2^{k_i+1}\right].$$

Recalling that $k_0 > 2(\alpha+3)/(\alpha-1) = 2/(1-\beta)$, we have

$$2\lceil 2^{\beta k_i}\rceil < 2^{\beta k_i+2} \leq 2^{k_i}$$

and $R_i \neq \emptyset$. We claim that $R_i \subset W_{i,0}$. To show this, it suffices to check that the two corners at the bottom are within $W_{i,0}$. I.e.,

$$\left(\lfloor 2^{\alpha_1 k_i}\rfloor, 2\lceil 2^{\beta k_i}\rceil\right) \in W_{i,0}.$$

To see this, note that by the definition of α,

$$2^{\gamma k_i}\left(2\lceil 2^{\beta k_i}\rceil - \lceil 2^{\beta k_i}\rceil\right) > 2^{(\beta+\gamma)k_i}$$

and that $\alpha_1 = \beta + \gamma$. Now let

$$\begin{aligned}
\text{top}_i &= \left[-\lfloor 2^{\alpha_1 k_i}\rfloor, \lfloor 2^{\alpha_1 k_i}\rfloor\right] \times 2^{k_i+1}\\
\text{bottom}_i &= \left[-\lfloor 2^{\alpha_1 k_i}\rfloor, \lfloor 2^{\alpha_1 k_i}\rfloor\right] \times 2\lceil 2^{\beta k_i}\rceil\\
\text{left}_i &= -\lfloor 2^{\alpha_1 k_i}\rfloor \times \left[2\lceil 2^{\beta k_i}\rceil, 2^{k_i+1}\right]\\
\text{right}_i &= \lfloor 2^{\alpha_1 k_i}\rfloor \times \left[2\lceil 2^{\beta k_i}\rceil, 2^{k_i+1}\right].
\end{aligned}$$

Note that

$$\frac{\lfloor 2^{\alpha_1 k_i}\rfloor}{2^{k_i+1}} > 2^{(\alpha_1-1)k_i-2} \uparrow +\infty,$$

and that $k_0 \geq \lceil 3/(\alpha_1-1)\rceil$, which implies $2^{(\alpha_1-1)k_i-2} \geq 2$ for all i. Let

$$m = -2\lceil 2^{\beta k_i}\rceil + 2^{k_i+1},\ n = \left\lfloor \frac{\lfloor 2^{\alpha_1 k_i}\rfloor}{m}\right\rfloor.$$

One can apply Lemma 3.5 to a translation of the box $R_{m,n}$ within R_i and have

$$P_{y_{i,0}}\left(\tau_{\text{left}_i\cup\text{right}_i} < \tau_{\text{top}_i\cup\text{bottom}_i}\right) \leq (1-\delta)^{2^{(\alpha_1-1)k_i-2}}. \tag{22}$$

Moreover, we have

$$P_{y_{i,0}}\left(\Gamma_{k_{i+1}} < \tau_{L_{2\lceil 2^{\beta k_i}\rceil}}\right) = \frac{2^{k_i} - 2\lceil 2^{\beta k_i}\rceil}{2^{k_i+1} - 2\lceil 2^{\beta k_i}\rceil} \geq \frac{1}{2} - 2^{(\beta-1)k_i+1}. \tag{23}$$

Now note that

$$\left\{\tau_{\text{top}_i} = \tau_{\partial^{in} R_i}\right\} = \left\{\Gamma_{k_{i+1}} < \tau_{L_{2\lceil 2^{\beta k_i}\rceil}}\right\} \setminus \left\{\tau_{\text{left}_i\cup\text{right}_i} < \tau_{\text{top}_i\cup\text{bottom}_i}\right\}.$$

We have by (22) and (23),

$$\begin{aligned}
p_i &\geq P_{y_{i,0}}\left(\tau_{\text{top}_i} = \tau_{\partial^{in} R_i}\right)\\
&\geq \frac{1}{2} - 2^{(\beta-1)k_i+1} - (1-\delta)^{2^{(\alpha_1-1)k_i-2}}.
\end{aligned} \tag{24}$$

Now recalling (21), we have

$$P_{\xi_0}\left(\Gamma_{k_i} < \tau_{\hat{W}}\right) \geq \prod_{j=0}^{i-1} p_j \geq 2^{-i}\prod_{j=0}^{i-1}\left[1 - 2^{(\beta-1)k_i+2} - 2(1-\delta)^{2^{(\alpha_1-1)k_i-2}}\right].$$

Noting that

$$\sum_{i=0}^{\infty} 2^{(\beta-1)k_i+2} + 2(1-\delta)^{2^{(\alpha_1-1)k_i-2}} < \infty,$$

there is a constant $c > 0$ such that for all $i \geq 0$,

$$P_{\xi_0}\left(\tau_{s_i} < \tau_{\hat{W}}\right) \geq c2^{-k_i}. \tag{25}$$

Lower bounds for the returning times: Now recall that

$$s_i = \left\{(y_1, 2^{k_i}), |y_1| \leq \sum_{j=1}^{i} 2^{(1+\gamma)k_j}\right\} \subset l_{2^{k_i}},$$

and that since $\gamma = 2(\alpha-1)/(\alpha+3)$, $1+\gamma = (3\alpha+1)/(\alpha+3) < \alpha$. For any $y = (y_1, 2^{k_i}) \in s_i$, by the upper bound found in (36) (see Appendix for details), we have

$$|y_1| + 2^{(1+\gamma)k_i+1} \leq 2^{(1+\gamma)k_i+2}. \tag{26}$$

Now note that

$$k_0 \geq 2\left\lceil \frac{\alpha+3}{\alpha-1} \right\rceil > \frac{(\alpha+3)(\alpha+2)}{\alpha^2-1} = \frac{\alpha+2}{\alpha-\gamma-1},$$

which implies that, $(k_i-1)\alpha > (1+\gamma)k_i + 2$ for all $i \geq 0$ and that

$$\left(2^{k_i-1}\right)^{\alpha} = 2^{(k_i-1)\alpha} > 2^{(1+\gamma)k_i+2}. \tag{27}$$

Combining (26) and (27) gives us for all $y \in s_i$,

$$N_y^i = y + \left[-\lfloor 2^{(1+\gamma)k_i+1}\rfloor, \lfloor 2^{(1+\gamma)k_i+1}\rfloor\right] \times \left[-2^{k_i-1}, 2^{k_i-1}\right] \subset W.$$

And thus

$$E_y\left[\#\text{ of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\hat{W}}]\right] \geq E_y\left[\#\text{ of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial^{in} N_y^i}]\right].$$

Now let $\Gamma_{i,1} = \Gamma_{k_i} = \tau_{L_{2^{k_i}}}$ and for each j

$$\Gamma_{i,j} = \inf\left\{n > \Gamma_{i,j-1},\ S_n \in L_{2^{k_i}}\right\}$$

be the jth time a random walk returns to $L_{2^{k_i}}$. We have

$$\begin{aligned} E_y\left[\text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial^{in} N_y^i}]\right] &= 1 + \sum_{j=1}^{\infty} P_y\left(\Gamma_{i,j} \leq \tau_{\partial^{in} N_y^i}\right) \\ &\geq 1 + \sum_{j=1}^{2^{k_i}} P_y\left(\Gamma_{i,j} \leq \tau_{\partial^{in} N_y^i}\right). \end{aligned}$$

Again we define

$$\begin{aligned}
\hat{\text{top}}_{y,i} &= \left[y_1 - \lfloor 2^{(1+\gamma)k_i+1}\rfloor, y_1 + \lfloor 2^{(1+\gamma)k_i+1}\rfloor\right] \times (2^{k_i} + 2^{k_i-1}) \\
\hat{\text{bottom}}_{y,i} &= \left[y_1 \lfloor 2^{(1+\gamma)k_i+1}\rfloor, y_1 + \lfloor 2^{(1+\gamma)k_i+1}\rfloor\right] \times (2^{k_i} - 2^{k_i-1}) \\
\hat{\text{left}}_{y,i} &= \left(y_1 - \lfloor 2^{(1+\gamma)k_i+1}\rfloor\right) \times [2^{k_i} - 2^{k_i-1}, 2^{k_i} + 2^{k_i-1}] \\
\hat{\text{right}}_{y,i} &= \left(y_1 + \lfloor 2^{(1+\gamma)k_i+1}\rfloor\right) \times [2^{k_i} - 2^{k_i-1}, 2^{k_i} + 2^{k_i-1}]
\end{aligned}$$

as the four sides of $\partial^{in} N_y^i$. Note that for any $1 \le j \le 2^{k_i}$,

$$P_y\left(\Gamma_{i,j} \le \tau_{\hat{\text{top}}_{y,i}} \wedge \tau_{\hat{\text{bottom}}_{y,i}}\right) \ge P_y\left(\Gamma_{i,j} \le \tau_{L_{2^{k_i}+2^{k_i-1}}} \wedge \tau_{L_{2^{k_i}-2^{k_i-1}}}\right) = \left(1-2^{-k_i}\right)^j.$$

Moreover,

$$\begin{aligned} P_y\left(\Gamma_{i,j} \le \tau_{\partial N_y^i}\right) &= P_y\left(\Gamma_{i,j} \le \tau_{\hat{\text{top}}_{y,i}} \wedge \tau_{\hat{\text{bottom}}_{y,i}}\right) \\ &\quad - P_y\left(\tau_{\hat{\text{left}}_{y,i}} \wedge \tau_{\hat{\text{right}}_{y,i}} < \Gamma_{i,j} \le \tau_{\hat{\text{top}}_{y,i}} \wedge \tau_{\hat{\text{bottom}}_{y,i}}\right) \\ &\ge \left(1-2^{-k_i}\right)^j - P_y\left(\tau_{\hat{\text{left}}_{y,i}} \wedge \tau_{\hat{\text{right}}_{y,i}} < \tau_{\hat{\text{top}}_{y,i}} \wedge \tau_{\hat{\text{bottom}}_{y,i}}\right). \end{aligned}$$

Note that

$$\frac{\lfloor 2^{(1+\gamma)k_i+1} \rfloor}{2^{k_i-1}} > 2^{\gamma k_i}.$$

Again by Lemma 3.5, we have

$$P_y\left(\tau_{\hat{\text{left}}_{y,i}} \wedge \tau_{\hat{\text{right}}_{y,i}} < \tau_{\hat{\text{top}}_{y,i}} \wedge \tau_{\hat{\text{bottom}}_{y,i}}\right) \le (1-\delta)^{2^{\gamma k_i}}.$$

Thus,

$$\begin{aligned} &E_y\left[\text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial^{in} N_y^i}]\right] \\ &\ge \sum_{j=1}^{2^{k_i}} P_y\left(\Gamma_{i,j} \le \tau_{\partial^{in} N_y^i}\right) \\ &\ge \left(\sum_{j=1}^{2^{k_i}} \left(1-2^{-k_i}\right)^j\right) - 2^{k_i}(1-\delta)^{2^{\gamma k_i}} \\ &\ge 2^{k_i}\left(1-2^{-k_i}\right)\left[1-\left(1-2^{-k_i}\right)^{2^{k_i}}\right] - C \\ &\ge c 2^{k_i} \end{aligned} \tag{28}$$

for some $c > 0$ independent of i and $y \in s_i$.

Now combining (18), (25) and (28)

$$\begin{aligned} \bar{\mathcal{H}}_{\hat{W},2^{k_i}}(\xi_0) &= \sum_{z \in l_{2^{k_i}}} P_{\xi_0}\left(\Gamma_{k_i} < \tau_{\hat{W}}, S_{\Gamma_{k_i}} = z\right) E_z\left[\text{number of visits to } l_{2^{k_i}} \text{ in } [0, \tau_{\hat{W}})\right] \\ &\ge \sum_{y \in s_i} P_{\xi_0}\left(\Gamma_{k_i} < \tau_{\hat{W}}, S_{\Gamma_{k_i}} = y\right) E_y\left[\text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial N_y^i}]\right] \\ &\ge P_{\xi_0}\left(\Gamma_{k_i} < \tau_{\hat{W}}\right) \inf_{y \in s_i} E_y\left[\text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial N_y^i}]\right] \\ &\ge c. \end{aligned} \tag{29}$$

And thus we have shown (17) and the proof of Lemma 3.1 is complete. □

Now back to finish the proof of Theorem 2, note that both l_{h_0} and $B_0 = B \cap D_0$ are finite and not depending on N. There is a $c > 0$ such that for any $z \in l_{h_0}$,

$$P_z(\bar{\tau}_{B_0} = \bar{\tau}_B) \geq c.$$

Thus by strong Markov property,

$$\begin{aligned}\bar{\mathcal{H}}_{B,2^{k_i}}(B_0) &= \sum_{z \in L_{2^{k_i}} \setminus B} P_z(\tau_{B_0} = \tau_B) \\ &\geq \bar{\mathcal{H}}_{B \cup l_{h_0}, 2^{k_i}}(l_{h_0}) \inf_{z \in l_{h_0}} P_z(\bar{\tau}_{B_0} = \bar{\tau}_B) \\ &\geq c.\end{aligned}$$

Then taking $i \to \infty$, Proposition 1 completes the proof of Theorem 2. □

4. Discussions

Now let us look back at the possible aggregation model. With Theorem 2, consider an interacting particle system first introduced in Proposition 3 of [6]: Let $\bar{\xi}_t$ defined on $\{0,1\}^{\mathbb{H}}$ with 1 standing for a site occupied while 0 for vacant, with transition rates as follows:

(i) For each occupied site $x = (x_1, x_2) \in \mathbb{H}$, if $x_2 > 0$ it will try to give birth to each of its nearest neighbors at a Poisson rate of $\sqrt{x_2}$. If $x_2 = 0$, it will try to give birth to each of its nearest neighbors at a Poisson rate of 1.

(ii) When x attempts to give birth to its nearest neighbors y already occupied, the birth is suppressed.

In [6] we prove that $\bar{\xi}_t$ with transition rates above is a well defined infinite interacting particle system. And let

$$B_t = \{x \in \mathbb{H},\ \bar{\xi}_t(x) = 1\}.$$

Moreover, recalling that in the proof of Lemma 6.2 in [6], for any $x \in \mathbb{H}$, $x \in L_0$ and $0 \leq t$, we define subset $I_{0,t}(x)$ as the collection of all possible offsprings of the particle at x when at time t, and let

$$\mathcal{I}_{t,T}(x) = \sup_{y \in I_{t,T}(x)} |x - y|.$$

When $B_0 = L_0$, for any $x' \in B_t$, one can easily check by definition there must be an $x \in L_0$ such that $x' \in I_{0,t}(x)$, which implies that

$$B_t = \bigcup_{x \in L_0} I_{0,t}(x).$$

Moreover, by (horizontal) translation invariance, we have $I_{0,t}(x)$ are identically distributed for all x.

Then by Theorem 6 in [6], we have for any $n \geq 1$

$$E\left[\mathcal{I}_{0,t}(x)^{2n}\right] < \infty,$$

which implies that

$$\sum_{x \in L_0} P\left(\mathcal{I}_{0,t}(x)^{2n} \geq |x_1|\right) < \infty.$$

Then by Borel–Cantelli Lemma, with probability one for all $x \in L_0$ sufficiently far away from 0, $\mathcal{I}_{0,t}(x) < |x_1|^{1/2n}$, which implies that

$$\left|B_t \setminus \{x \in \mathbb{H},\ x_2 \le |x_1|^{1/n}\}\right| < \infty. \tag{30}$$

Combining Theorem 2 and (30),

Corollary 1. *For any $t \ge 0$ and B_t defined above, there must be some $x \in B_t$ such that $\bar{\mathcal{H}}_{B_t}(x) > 0$.*

By Theorem 1 of [6], for any (infinite) $A \subset \mathbb{H}$, and any $x \in A$, $\bar{\mathcal{H}}_A(x) \le C\sqrt{x_2}$ for some uniform constant $C < \infty$. Thus, from any configuration, the transition rates of $\bar{\xi}_{Ct}$ are always larger than the stationary harmonic measure. Thus, when one defines an infinite Stationary DLA model in $\mathbb{H}$ (this question was recently addressed by Procaccia, Ye and Zhang, in [5]), it will be dominated by $\bar{\xi}_{Ct}$ and Corollary 1 shows that the infinite stationary DLA model starting from L_0 never hits an absorbing state and stop growing globally. The interesting thing here is, in order to show the model grows, we actually need to show it grows slowly .

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Proof of Lemma 3.3. By Lemma 3.2 and the fact that $\beta\alpha - 1 > \gamma$, for any $y \in s_1$ and any $x \in W_{1,y}$, if $x_1 = y_1$, then

$$x_2^\alpha \ge 2^{\beta\alpha k_1} > 2^{k_1(1+\gamma)} \ge |y_1| = |x_1|. \tag{31}$$

Otherwise, note that for any $a, b \ge 1$,

$$ab - (a + b - 1) = (a - 1)(b - 1) \ge 0. \tag{32}$$

Thus

$$\begin{aligned} x_2^\alpha &\ge \left(\lceil 2^{\beta k_1}\rceil + |x_1 - y_1| \cdot 2^{-\gamma k_1}\right)^\alpha \\ &\ge \lceil 2^{\beta k_1}\rceil^\alpha + \alpha\lceil 2^{\beta k_1}\rceil^{(\alpha-1)}|x_1 - y_1|2^{-\gamma k_1} \\ &\ge \lceil 2^{\beta k_1}\rceil^\alpha + \alpha 2^{[\beta(\alpha-1)-\gamma]k_1}|x_1 - y_1| \\ &\ge \lceil 2^{\beta k_1}\rceil^\alpha + \alpha 2^{[\beta(\alpha-1)-\gamma]k_1} + |x_1 - y_1| - 1 \\ &\ge \lceil 2^{\beta k_1}\rceil^\alpha + \alpha 2^{[\beta(\alpha-1)-\gamma]k_1} + |x_1| - |y_1| - 1. \end{aligned} \tag{33}$$

It is known in (31) that $2^{\beta\alpha k_1} \ge 2^{k_1(1+\gamma)} \ge |y_1|$. At the same time, since $\beta(\alpha - 1) - \gamma > 0$, $\alpha 2^{[\beta(\alpha-1)-\gamma]k_1} > 1$. Thus

$$x_2^\alpha \ge |x_1| + \left(\lceil 2^{\beta k_1}\rceil^\alpha - |y_1|\right) + \left(\alpha 2^{[\beta(\alpha-1)-\gamma]k_1} - 1\right) > |x_1| \tag{34}$$

which implies that $W_{1,y} \subset W$ for all $y \in s_1$. □

Proof of Lemma 3.4. For $x = (x_1, x_2) \in W_{i,y}$, one first assumes $x_1 \neq y_1$. Then

$$\begin{aligned} x_2^\alpha &\geq \left(\lceil 2^{\beta k_i} \rceil + |x_1 - y_1| \cdot 2^{-\gamma k_i}\right)^\alpha \\ &\geq \lceil 2^{\beta k_i} \rceil^\alpha + \alpha \lceil 2^{\beta k_i} \rceil^{(\alpha-1)} |x_1 - y_1| 2^{-\gamma k_1} \\ &\geq \lceil 2^{\beta k_i} \rceil^\alpha + \alpha 2^{[\beta(\alpha-1)-\gamma]k_i} |x_1 - y_1| \\ &\geq \lceil 2^{\beta k_i} \rceil^\alpha + \alpha 2^{[\beta(\alpha-1)-\gamma]k_i} + |x_1 - y_1| - 1 \\ &\geq \lceil 2^{\beta k_i} \rceil^\alpha + \alpha 2^{[\beta(\alpha-1)-\gamma]k_i} + |x_1| - |y_1| - 1. \end{aligned} \tag{35}$$

Similar to when $i = 1$, we have $\alpha 2^{[\beta(\alpha-1)-\gamma]k_i} > 1$ while at the same time,

$$|y_1| \leq \sum_{j=1}^{i} 2^{(1+\gamma)k_j} \leq \sum_{j=0}^{k_i} 2^{(1+\gamma)j} \leq \frac{2^{(1+\gamma)(k_i+1)} - 1}{2^{(1+\gamma)} - 1}.$$

Note that $\gamma > 0$, which implies that $2^{(1+\gamma)} - 1 \geq 2^\gamma$. Thus

$$|y_1| \leq \frac{2^{(1+\gamma)(k_i+1)} - 1}{2^{(1+\gamma)} - 1} \leq \frac{2^{(1+\gamma)(k_i+1)}}{2^\gamma} = 2^{(1+\gamma)k_i+1}. \tag{36}$$

Now recall that

$$\beta\alpha - 1 - \gamma = \frac{\alpha - 1}{\alpha + 3} > 0,$$

while

$$k_i > k_0 \geq \left\lceil \frac{\alpha + 3}{\alpha - 1} \right\rceil.$$

We have $(\beta\alpha)k_i \geq (1+\gamma)k_i + 1$, which implies $\lceil 2^{\beta k_i} \rceil^\alpha > |y_1|$ by the definition of α, and that $x_2^\alpha > |x_1|$. And if $x_1 = y_1$, one can also have

$$x_2^\alpha \geq \lceil 2^{\beta k_i} \rceil^\alpha > |y_1| = |x_1|.$$

Thus we have $W_{i,y} \subset W$. □

References

[1] I. Benjamini, A. Yadin, Upper bounds on the growth rate of diffusion limited aggregation, 2017, arXiv preprint arXiv:1705.06095.
[2] H. Kesten, Hitting probabilities of random walks on Z^d, Stochastic Process. Appl. 25 (1987) 165–184.
[3] H. Kesten, How long are the arms in DLA? J. Phys. A 20 (1) (1987) L29–L33.
[4] Y. Mu, E.B. Procaccia, Y. Zhang, Scaling limit of DLA on a long line segment, 2019, arXiv preprint arXiv:1912.02370.
[5] E.B. Procaccia, J. Ye, Y. Zhang, Stationary DLA is well defined, J. Stat. Phys. (2020) 1–23.
[6] E.B. Procaccia, Y. Zhang, Stationary harmonic measure and DLA in the upper half plane, J. Stat. Phys. 176 (4) (2019) 946–980.