Electronic Journal of Probability

Electron. J. Probab. **25** (2020), article no. 145, 1–39.
ISSN: 1083-6489 https://doi.org/10.1214/20-EJP545

On covering monotonic paths with simple random walk[*][†]

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Abstract

In this paper we study the probability that a d dimensional simple random walk (or the first L steps of it) covers each point in a nearest neighbor path connecting 0 and the boundary of an L_1 ball. We show that among all such paths, the one that maximizes the covering probability is the monotonic increasing one that stays within distance 1 from the diagonal. As a result, we can obtain an exponential upper bound on the decaying rate of covering probability of any such path when $d \geq 4$. The main tool is a general combinatorial inequality, that is interesting in its own right.

Keywords: random walk; covering; monotonic paths.
MSC2020 subject classifications: 60C05; 60G50.
Submitted to EJP on April 2, 2020, final version accepted on November 5, 2020.

1 Introduction

Cover times of graphs by a simple random walk is a well studied subject [8]. However there is not much literature on the basic question of subgraphs covering probabilities. Such questions are useful for geometric studies of random walk traces [9], entropic calculations such as those appearing in Wulff constructions [1] and percolation questions such as for random interlacements [2, 12, 14].

In this paper, we study the probability that a finite subset, especially the trace of a nearest neighbor path in $\mathbb{Z}^d$ is completely covered by the trace of a d dimensional simple random walk.

For any finite subset $A \subset \mathbb{Z}^d$ and a d dimensional simple random walk $\{X_n\}_{n=0}^{\infty}$ starting at 0, we say that A is completely covered by the first L steps of the random walk if

$$A \subseteq \mathrm{Trace}(X_0, X_1, \cdots, X_L) := \{x \in \mathbb{Z}^d : \exists 0 \leq i \leq L, X_i = x\}.$$

[*]Research supported by NSF grant 1407558.
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For simplicity we state our first result for $d = 2$. For an integer $l_0 \geq 0$ and the subspace of reflection $l : x = y + l_0$, define $\varphi_l : \mathbb{Z}^2 \to \mathbb{Z}^2$ as the reflection mapping around l. I.e., for any $(x, y) \in \mathbb{Z}^2$,

$$\varphi_l(x, y) = (l_0 + y, x - l_0).$$

Suppose two disjoint finite sets $A_0, B_0 \subset \mathbb{Z}^2 \cap \{(x, y) : x \leq y + l_0\}$ both stay on the left of l. We then have the following theorem which states that the covering probability cannot get larger when we reflect one of them to the other side of the line:

Theorem 1.1. *For any integer $L \geq 0$,*

$$P\left(A_0 \cup B_0 \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^L\big)\right) \geq P\left(A_0 \cup \varphi_l(B_0) \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^L\big)\right).$$

Remark 1.2. By taking the union over all the L's, one can immediately see the theorem also holds for $L = \infty$.

Remark 1.3. One would think (like the authors first did) that Theorem 1.1 should follow from repeated use of the reflection principle. Two problems arise when one explores this idea. The first is that reflecting a path does not conserve the hitting order within the sets, which makes it hard to determine the times of reflection. The second is that even if we consider the sets before and after reflection with the same hitting order we can get a contradiction to the monotonicity of cover probabilities with the specified order. See Figure 1 for an example. Here the numbers associated with each vertex represent a specified hitting order. One may see that, after the reflection, it is now harder for a random walk starting from vertex $1'$ to reach vertex 2 without firsting hitting vertices 3, 4, and 5. An anonymous referee suggested to use Reimer's inequality to de-correlate excursions around l. Here we present a purely combinatorial argument not relying on strong probability tools.

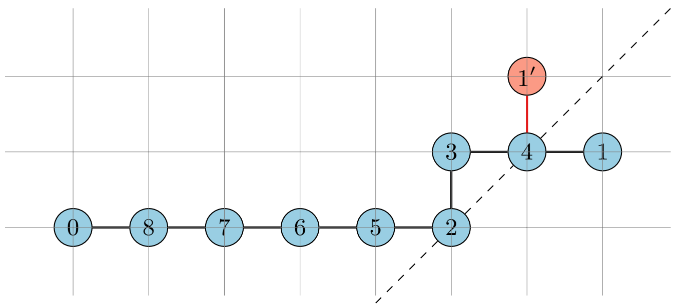

Figure 1: A counter example to monotonicity for every order.

With Theorem 1.1, we could consider the problem of covering a nearest neighbor path in $\mathbb{Z}^d$. For any integer $N \geq 1$, let $\partial B_1(0, N)$ be the boundary of the L_1 ball in $\mathbb{Z}^d$ with radius N. We say that a nearest neighbor path

$$\mathcal{P} = \big(P_0, P_1, \cdots, P_K\big)$$

connects 0 and $\partial B_1(0, N)$ if $P_0 = 0$ and $\inf\{n : \|P_n\|_1 = N\} = K$. And we say that a path $\mathcal{P}$ is covered by the first L steps of $\{X_n\}_{n=0}^\infty$ if

$$\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}(X_0, X_1, \cdots, X_L).$$

Then we are able to use Theorem 1.1 to show that the covering probability of any such path can be bounded by that of the diagonal. Let

$$\overset{\nearrow}{\mathcal{P}} = \Big(\mathrm{arc}_1[0 : d-1], \mathrm{arc}_2[0 : d-1], \cdots, \mathrm{arc}_{[N/d]}[0 : d-1], \mathrm{arc}_{[N/d]+1}[0 : N - d[N/d]]\Big)$$

be the staircase path spiraling around the d-dimensional diagonal, where

$$\mathrm{arc}_1[0:d-1] = \left(0, e_1, e_1+e_2, \cdots, \sum_{i=1}^{d-1} e_i\right)$$

and $\mathrm{arc}_k = (k-1)\sum_{i=1}^d e_i + \mathrm{arc}_1$.

Remark 1.4. It can be useful to note that $\mathrm{arc}_1[0:d-1]$ forms a nearest neighbor path in $\mathbb{Z}^d$ from $(0,0,\cdots,0)$ to $(1,1,\cdots,1,0)$, which "jumps" exactly $d-1$ steps, and that $\mathrm{arc}_k[0:d-1]$ is $\mathrm{arc}_1[0:d-1]$ shifted by $(k-1)\sum_{i=1}^d e_i$. One may also note that $\mathrm{arc}_k[0:d-1]$'s are connected and together form a nearest neighbor spiral around the diagonal.

Theorem 1.5. *For each integers $L \geq N \geq 1$, let $\mathcal{P}$ be any nearest neighbor path in $\mathbb{Z}^d$ connecting 0 and $\partial B_1(0,N)$. Let $X_n, n \geq 0$ be a d dimensional simple random walk starting at 0. Then*

$$P\big(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}(X_0,\cdots,X_L)\big) \leq P\big(\overset{\nearrow}{\mathcal{P}} \subseteq \mathrm{Trace}(X_0,\cdots,X_L)\big).$$

The following main theorem gives an upper bound of the covering probability over all nearest neighbor paths connecting 0 and $\partial B_1(0,N)$.

Theorem 1.6. *Let $d \geq 4$ and let $\{X_n\}_{n=0}^\infty$ be a d dimensional simple random walk starting at 0. Then there is a $P_d \in (0,1)$ such that for any nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_K)$ connecting 0 and $\partial B_1(0,N)$, we always have*

$$P\left(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^\infty\big)\right) \leq P_d^{[N/d]}.$$

Here P_d is equal to the probability that $\{X_n\}_{n=0}^\infty$ ever returns to the d dimensional diagonal line.

Note that in Theorem 1.6 the upper bound is not very sharp since we only look at returning to the exact diagonal line for $[N/d]$ times, which may cover at most $1/d$ of the total points in $\overset{\nearrow}{\mathcal{P}}$. Although for any fixed d, we still have an exponential decay with respect to N, when $d \to \infty$, such exponential decaying speed, which is lower bounded by $\left(\frac{1}{2d}\right)^{1/d}$, goes to one. Fortunately, in Appendix A we are able to show that $\lim_{d\to\infty} 2dP_d = 1$, and then further find an upper bound on the asymptotic of the probability that a d dimensional simple random walk starting from some point in Trace$\big(\overset{\nearrow}{\mathcal{P}}\big)$ will ever return to Trace$\big(\overset{\nearrow}{\mathcal{P}}\big)$. Note that we now need to return at least N times to cover all the points in $\overset{\nearrow}{\mathcal{P}}$. We state this result as an additional theorem which is stronger than Theorem 1.6. However, the proof of Theorem 1.7 is much more elaborate and is left in the Appendix.

Theorem 1.7. *There is a $C \in (0,\infty)$ such that for any $d \geq 4$ and any nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_K) \subset \mathbb{Z}^d$ connecting 0 and $\partial B_1(0,N)$ and $\{X_n\}_{n=0}^\infty$ which is a d dimensional simple random walk starting at 0, we always have*

$$P\left(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^\infty\big)\right) \leq \left(\frac{C}{d}\right)^N.$$

The proof of Theorem 1.7 can be found at the end of Section A.1.

Remark 1.8. Actually, any $C > 3/2$ will serve as a good upper bound for sufficiently large d. See Remark A.5 in Appendix A for details.

Remark 1.9. Note that we do not present a proof of Theorem 1.7 for $d = 3$. With Theorem 1.5 at hand, it is possible to prove some upper bounds by considering returns to an infinite transient subset of the diagonal. However this yields non sharp bounds and requires extra techniques. We consider this case in [13].

Remark 1.10. Note that the probability to cover a space filling curve in $B_1(0, N)$ decays asymptotically slower that c^{N^d}. Sznitman [15, Section 2] showed that the probability a random walk path covers $B_1(0, N)$ completely can be bounded below by $ce^{-cN^{d-1}\log N}$.

A natural generalization of Theorem 1.7 is to try applying the same reflection process in this paper but also consider the repetition of visits rather than just looking at the trace of the path. In other words, consider the probability that the random walk's local time along a certain path is larger than a sequence of given values. Note that the event the random walk covers a path is equivalent to the event that the random walk's local time along this path ≥ 1. However, it is shown that once we consider local time, the diagonal line (with repetition) no longer maximizes the covering probability. See Section 6 for details.

For the minimizer of covering probabilities over the family of monotonic nearest neighbor paths starting at 0, we also conjecture that the cover probability is minimized when the path goes straightly along a coordinate axis. I.e.,

Conjecture 1.11. *For each integers $L \geq N \geq 1$, let $\mathcal{P}$ be any nearest neighbor monotonic path in $\mathbb{Z}^d$ with length N. Let $X_n, n \geq 0$ be a d dimensional simple random walk starting at 0. Then*

$$P\big(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}(X_0, \cdots, X_L)\big) \geq P\big(\vec{\mathcal{P}} \subseteq \mathrm{Trace}(X_0, \cdots, X_L)\big)$$

where

$$\vec{\mathcal{P}} = \Big((0, 0, \cdots, 0), (1, 0, \cdots, 0), \cdots, (N-1, 0, \cdots, 0)\Big).$$

Remark 1.12. Note that the constants we get in Theorem 1.6 are not sharp. In fact, the upper bound we obtain for the covering of the diagonal path is of order $(1/2d)^N$. If we use the same argument as in Theorem 1.6 for the straight line we will get a bound of $[1/2(d-1)]^N$, since a return to the straight line is equivalent to a $d-1$ dimensional random walk returning to the origin. Thus we get that the bound we obtain is larger for the path that we conjecture minimizes the cover probability.

The structure of this paper is as follows: in Section 2 we prove a combinatorial inequality, which can be found later equivalent to finding a one-to-one mapping between nearest neighbor trajectories. In Section 3 we use this combinatorial inequality to prove Theorem 1.1. With Theorem 1.1, we construct a finite sequence of paths with non-decreasing covering probabilities in Section 4 to show that the covering probability is maximized by the path that goes along the diagonal, see Theorem 1.5 and 4.1. The proof of Theorem 1.6 is completed is Section 5, while in Section 6 we discuss the two conjectures and show numerical simulations. In Appendix A we prove that $\lim_{d\to\infty} 2dP_d = 1$ and then show that the probability a simple random walk returns to $\overset{\nearrow}{\mathcal{P}}$ also has an upper bound of $O(d^{-1})$, which implies Theorem 1.7. In Appendix B we prove that the monotonicity fails when considering covering probability with repetitions.

2 Combinatorial inequalities

In this section, we discuss a combinatorial inequality problem, which can be found equivalent to finding a one-to-one mapping between nearest neighbor trajectories. For $n \in \mathbb{Z}^+$ and Ω a set of n integer numbers, say $\Omega = \{1, 2, \cdots, n\}$ and any $A \subset \Omega$, abbreviate $-A = \{-x : x \in A\}$ and $A^c = \Omega \setminus A$.

For any $m \in \mathbb{Z}^+$, consider a **collection of arcs** which is a "vector" of subsets

$$\vec{V} = V_1 \otimes V_2 \otimes \cdots \otimes V_m,\ V_k \subseteq \Omega,$$

where each V_k is called an arc, and an m dimensional vector $\vec{D} = (\delta_1, \cdots, \delta_m) \in \{-1, 1\}^m$ which is called a **configuration**. Then we can introduce the **inner product**

$$\vec{D} \cdot \vec{V} = \bigcup_{k=1}^{m} \delta_k V_k \subseteq -\Omega \cup \Omega. \tag{2.1}$$

Moreover, for any subset $A \subseteq \Omega$, we denote $-A^c \cup A \subset -\Omega \cup \Omega$ as the **reflection** induced by A. I.e., the reflection induced by A is when we keep A and reflect the rest to the negative. We say a configuration $\vec{D}$ of $\vec{V}$ **covers** the reflection A if

$$-A^c \cup A \subseteq \vec{D} \cdot \vec{V},$$

and let

$$\mathcal{C}\left(\vec{V}, A\right) = \left\{\vec{D} :\ -A^c \cup A \subseteq \vec{D} \cdot \vec{V}\right\}$$

be the subset of all such configurations.

In the simple random walk covering problem we wish to prove that the covering probability of a set is higher if it resides above some line than if some subsets of it are reflected below the line. The arcs will stand for a random walk path's excursions around a given line and $\vec{D} \cdot \vec{V}$ will specify which excursions are reflected. The next Lemma will conclude that there are more ways to reflect the random walk excursions to cover a set if non of its subsets are reflected.

Lemma 2.1. *For any $m, n \in \mathbb{Z}^+$, and any collection of arcs $\vec{V} = V_1 \otimes V_2 \otimes \cdots \otimes V_m$*

$$\left|\mathcal{C}\left(\vec{V}, \Omega\right)\right| \geq \left|\mathcal{C}\left(\vec{V}, A\right)\right| \tag{2.2}$$

for all $A \subseteq \Omega = \{1, 2, \cdots, n\}$.

Before proving the lemma we set some notations. For any n and $m = m_0 + 1$, we can separate the last arc V_{m_0+1} and the rest of the arcs and look at the truncated system at m_0. I.e.,

$$\vec{V}[1 : m_0] = V_1 \otimes V_2 \otimes \cdots \otimes V_{m_0}$$

and

$$\vec{D}[1 : m_0] = (\delta_1, \cdots, \delta_{m_0}).$$

We have for any A

$$\mathcal{C}\left(\vec{V}, A\right) = \left\{\vec{D} : -A^c \cup A \subseteq \vec{D}[1 : m_0] \cdot \vec{V}[1 : m_0]\right\} \cup \mathcal{P}_m\left(\vec{V}, A\right) \tag{2.3}$$

where

$$\mathcal{P}_m\left(\vec{V}, A\right) = \left\{\vec{D} :\ -A^c \cup A \not\subseteq \vec{D}[1 : m_0] \cdot \vec{V}[1 : m_0],\ -A^c \cup A \subseteq \vec{D} \cdot \vec{V}\right\}.$$

Noting that

$$\left|\left\{\vec{D} :\ -A^c \cup A \subseteq \vec{D}[1 : m_0] \cdot \vec{V}[1 : m_0]\right\}\right| = 2\left|\mathcal{C}\left(\vec{V}[1 : m_0], A\right)\right|,$$

and that the two sets in (2.3) are disjoint,

$$\left|\mathcal{C}\left(\vec{V}, A\right)\right| = 2\left|\mathcal{C}\left(\vec{V}[1 : m_0], A\right)\right| + \left|\mathcal{P}_m\left(\vec{V}, A\right)\right|. \tag{2.4}$$

In order to study the cardinality of $\mathcal{P}$, we first show that

Lemma 2.2. *Recall* $\Omega = \{1, 2, \cdots, n\}$*. For* $m = m_0 + 1$*, any* $\vec{V} = V_1 \otimes V_2 \otimes \cdots \otimes V_m$*, any* $A \subset \Omega$*, and any two different* $\vec{D}_1$ *and* $\vec{D}_2$ *in* $\mathcal{P}_m\left(\vec{V}, A\right)$*, we must have*

$$\vec{D}_1[1 : m_0] \neq \vec{D}_2[1 : m_0].$$

Proof. The proof is straightforward. Suppose $\vec{D}_1[1 : m_0] = \vec{D}_2[1 : m_0] = D'$, then their $m_0 + 1$st coordinates must be different. Thus

$$V_{m_0+1} \cup \vec{D}_1[1 : m_0] \cdot \vec{V}[1 : m_0] \supseteq -A^c \cup A$$

and

$$-V_{m_0+1} \cup \vec{D}_1[1 : m_0] \cdot \vec{V}[1 : m_0] \supseteq -A^c \cup A.$$

The first inclusion above implies that

$$\left(\vec{D}_1[1 : m_0] \cdot \vec{V}[1 : m_0]\right)^c \cap \Big(- A^c \cup A\Big) \subseteq V_{m_0+1}$$

while the second implies that

$$\left(\vec{D}_1[1 : m_0] \cdot \vec{V}[1 : m_0]\right)^c \cap \Big(- A^c \cup A\Big) \subseteq -V_{m_0+1}.$$

Combining the two inclusion gives us

$$-A^c \cup A \subseteq \vec{D}_1[1 : m_0] \cdot \vec{V}[1 : m_0]$$

which contradicts with the definition of $\mathcal{P}_m\left(\vec{V}, A\right)$. Thus the proof is complete. □

Now we can prove the main result of this section.

Proof of Lemma 2.1. First we give an explanation of how the inductive arguments work in this proof: In the inductive basis we have proved Lemma 2.1 holds for all $n = 1, m \geq 1$ and all $n \geq 1, m = 1$. To see why it is now sufficient to prove the desired result for $n = n_0, m = m_0 + 1$, note that once the inequality has been shown for $n = n_0, m = m_0 + 1$, the same argument immediately gives us the that the inequality holds for $n = n_0, m = m_0 + 2$, and thus for n_0 and all m. Then by the inductive basis we have Lemma 2.1 for $n = n_0 + 1, m = m_0 = 1$. Repeat this argument one may verify the lemma for all n and m.

Note that the reflection induced by Ω is Ω itself while reflection induced by Ø is $-\Omega$. By symmetry one can immediately see that $\vec{D} \in \mathcal{C}\left(\vec{V}, \Omega\right)$ if and only if $-\vec{D} \in \mathcal{C}\left(\vec{V}, Ø\right)$. And thus

$$\left|\mathcal{C}\left(\vec{V}, \Omega\right)\right| = \left|\mathcal{C}\left(\vec{V}, Ø\right)\right|.$$

So we will concentrate on the case when $A \neq Ø$ and prove the inequality by induction on m and n. To show the basis of induction, it is easy to see that for any n and $m = 1$

$$\left|\mathcal{C}\left(\vec{V}, \Omega\right)\right| = 0 = \left|\mathcal{C}\left(\vec{V}, A\right)\right|$$

if $V_1 \neq \Omega$ and

$$\left|\mathcal{C}\left(\vec{V}, \Omega\right)\right| = 1 > 0 = \left|\mathcal{C}\left(\vec{V}, A\right)\right|$$

if $V_1 = \Omega$. Then for any m and $n = 1$, by definition we must have $V_i = \{1\}$ or Ø for each i, and we always have $A = \Omega$. Thus

$$\left|\mathcal{C}\left(\vec{V}, \Omega\right)\right| = \left|\mathcal{C}\left(\vec{V}, A\right)\right| = 2^{n_e(\vec{V})}\left(2^{m - n_e(\vec{V})} - 1\right),$$

where $n_e(\vec{V})$ is the number of empty sets in $V_1, \cdots, V_m$. With the method of induction, suppose the desired inequality is true for all $n < n_0$ and all $n = n_0, m \le m_0$. Then for $n = n_0, m = m_0 + 1$, with Lemma 2.2, we now know that there is one-to-one mapping between each configuration in $\mathcal{P}_m\left(\vec{V}, A\right)$ and its first m_0 coordinates. Note that

$$\mathcal{P}_m\left(\vec{V}, A\right) = \left(\mathcal{P}_m\left(\vec{V}, A\right) \cap \{\delta_{m_0+1} = 1\}\right) \cup \left(\mathcal{P}_m\left(\vec{V}, A\right) \cap \{\delta_{m_0+1} = -1\}\right).$$

Define $\mathcal{C}_1\left(\vec{V}[1:m_0], A\right) = \mathcal{P}_m\left(\vec{V}, A\right) \cap \{\delta_{m_0+1} = 1\}$, and $\mathcal{C}_2\left(\vec{V}[1:m_0], A\right) = \mathcal{P}_m\left(\vec{V}, A\right) \cap \{\delta_{m_0+1} = -1\}$. We have

$$\left|\mathcal{P}_m\left(\vec{V}, A\right)\right| = \left|\mathcal{C}_1\left(\vec{V}[1:m_0], A\right)\right| + \left|\mathcal{C}_2\left(\vec{V}[1:m_0], A\right)\right|. \tag{2.5}$$

Note that one may also write

$$\mathcal{C}_1\left(\vec{V}[1:m_0], A\right) = \left\{\vec{D}' \in \{-1,1\}^{m_0}, \emptyset \neq \left(\vec{D}' \cdot \vec{V}[1:m_0]\right)^c \cap \left(-A^c \cup A\right) \subseteq V_{m_0+1}\right\}$$

and

$$\mathcal{C}_2\left(\vec{V}[1:m_0], A\right) = \left\{\vec{D}' \in \{-1,1\}^{m_0}, \emptyset \neq \left(\vec{D}' \cdot \vec{V}[1:m_0]\right)^c \cap \left(-A^c \cup A\right) \subseteq -V_{m_0+1}\right\},$$

and that $\mathcal{C}\left(\vec{V}[1:m_0], A\right)$, $\mathcal{C}_1\left(\vec{V}[1:m_0], A\right)$ and $\mathcal{C}_2\left(\vec{V}[1:m_0], A\right)$ are disjoint.

Moreover, we consider a new ambient environment $\Omega' = V^c_{m_0+1}$. Within Ω', one may consider the arc $V'_k = V_k \cap V^c_{m_0+1} \subseteq \Omega'$ for each k and

$$\vec{V}' = \left(V_1 \cap V^c_{m_0+1}\right) \otimes \left(V_2 \cap V^c_{m_0+1}\right) \otimes \cdots \otimes \left(V_{m_0} \cap V^c_{m_0+1}\right).$$

Moreover, let $A' = A \cap V^c_{m_0+1} \subset \Omega'$. Then we can similarly define

$$\mathcal{C}\left(\vec{V}', A'\right) = \left\{\vec{D}' \in \{-1,1\}^{m_0} : -\left(\Omega' \cap A'^c\right) \cup A' \subseteq \vec{D}' \cdot \vec{V}'\right\}.$$

We claim that

$$\mathcal{C}\left(\vec{V}[1:m_0], A\right) \cup \mathcal{C}_1\left(\vec{V}[1:m_0], A\right) \cup \mathcal{C}_2\left(\vec{V}[1:m_0], A\right) \subseteq \mathcal{C}\left(\vec{V}', A'\right). \tag{2.6}$$

In other words, in order to be in one of the 3 disjoint subsets above, we must guarantee that all points in $\Omega' = V^c_{m_0+1}$ under reflection of A are covered by the configuration $\vec{D}'$ of $\vec{V}[1:m_0]$. To verify (2.6), one can note that for any

$$\vec{D}' \in \mathcal{C}\left(\vec{V}[1:m_0], A\right) \cup \mathcal{C}_1\left(\vec{V}[1:m_0], A\right) \cup \mathcal{C}_2\left(\vec{V}[1:m_0], A\right)$$

we have

$$\vec{D}' \cdot \vec{V}[1:m_0] \supseteq \left(-A^c \cup A\right) \cap \left(-V^c_{m_0+1} \cup V^c_{m_0+1}\right)$$

which implies

$$\begin{aligned}&\left(\vec{D}' \cdot \vec{V}[1:m_0]\right) \cap \left(-V^c_{m_0+1} \cup V^c_{m_0+1}\right)\\ &\quad\supseteq \left(-A^c \cup A\right) \cap \left(-V^c_{m_0+1} \cup V^c_{m_0+1}\right).\end{aligned} \tag{2.7}$$

In (2.7) we have the right hand side equals to

$$A' \cup -\left(A^c \cap V^c_{m_0+1}\right) = -\left(\Omega' \cap A'^c\right) \cup A',$$

and the left hand side equals to

$$\bigcup_{k=1}^{m_0}\left(\delta_k V_k \cap \Big(-V^c_{m_0+1}\cup V^c_{m_0+1}\Big)\right).$$

Noting that for each k

$$\delta_k V_k \cap \Big(-V^c_{m_0+1}\cup V^c_{m_0+1}\Big) = \delta_k\big(V_k \cap V^c_{m_0+1}\big),$$

we have

$$\bigcup_{k=1}^{m_0}\left(\delta_k V_k \cap \Big(-V^c_{m_0+1}\cup V^c_{m_0+1}\Big)\right) = \vec{D}'\cdot\vec{V}'$$

which shows that $\vec{D}'$ is also in $\mathcal{C}\left(\vec{V}', A'\right)$ and thus verifies (2.6).

Specifically, when $A=\Omega$, note that for any $\vec{D}'\in\mathcal{C}\left(\vec{V}',\Omega'\right)$, $\Omega'=\vec{D}'\cdot\vec{V}'\subseteq\vec{D}'\cdot\vec{V}[1:m_0]$. Thus

$$\vec{D}'\cdot\vec{V}[1:m_0]\cup V_{m_0}=\Omega$$

which implies that

$$\mathcal{C}\left(\vec{V}[1:m_0],\Omega\right)\cup\mathcal{C}_1\left(\vec{V}[1:m_0],\Omega\right)\cup\mathcal{C}_2\left(\vec{V}[1:m_0],\Omega\right)=\mathcal{C}\left(\vec{V}',\Omega'\right). \tag{2.8}$$

Combining (2.4)-(2.8) and the induction hypothesis, we have

$$\begin{aligned}
\left|\mathcal{C}\left(\vec{V},\Omega\right)\right| &= \left|\mathcal{C}\left(\vec{V}[1:m_0],\Omega\right)\right| + \left|\mathcal{C}\left(\vec{V}[1:m_0],\Omega\right)\right| + \left|\mathcal{C}_1\left(\vec{V}[1:m_0],\Omega\right)\right| + \left|\mathcal{C}_2\left(\vec{V}[1:m_0],\Omega\right)\right| \\
&= \left|\mathcal{C}\left(\vec{V}[1:m_0],\Omega\right)\right| + \left|\mathcal{C}\left(\vec{V}',\Omega'\right)\right| \\
&\geq \left|\mathcal{C}\left(\vec{V}[1:m_0],A\right)\right| + \left|\mathcal{C}\left(\vec{V}',A\right)\right| \\
&\geq \left|\mathcal{C}\left(\vec{V}[1:m_0],A\right)\right| + \left|\mathcal{C}\left(\vec{V}[1:m_0],A\right)\right| + \left|\mathcal{C}_1\left(\vec{V}[1:m_0],A\right)\right| + \left|\mathcal{C}_2\left(\vec{V}[1:m_0],A\right)\right| \\
&= \left|\mathcal{C}\left(\vec{V},A\right)\right|.
\end{aligned}$$

And thus the proof of Lemma 2.1 is complete. □

3 Proof of Theorem 1.1

With the combinatorial inequality above, we can study the covering probability of simple random walks. Let $\mathcal{N}_L$ be the set of all nearest neighbor paths starting at 0 of length $L+1$ and consider 2 subsets of $\mathcal{N}_L$ as follows:

$$\mathcal{N}_{L,1}=\left\{\vec{x}\in\mathcal{N}_L, A_0\cup B_0\subseteq \text{Trace}\big(\vec{x}\big)\right\},$$

and

$$\mathcal{N}_{L,2}=\left\{\vec{x}\in\mathcal{N}_L, A_0\cup \varphi_l(B_0)\subseteq \text{Trace}\big(\vec{x}\big)\right\}.$$

For the simple random walk $\{X_n\}_{n=0}^\infty$ starting at 0, it is easy to see that for each $\vec{x}=(x_0,x_1,\cdots,x_L)\in\mathcal{N}_L$,

$$P(X_n=x_n,\ n=1,2,\cdots,L)=\left(\frac{1}{2d}\right)^L.$$

Thus in order to prove Theorem 1.1, it suffices to show that

$$|\mathcal{N}_{L,1}|\geq|\mathcal{N}_{L,2}|. \tag{3.1}$$

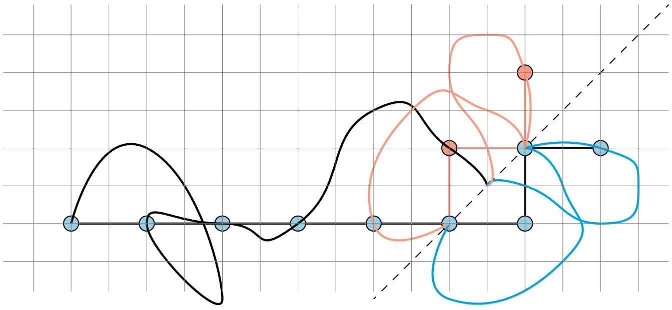

Figure 2: Combinatorial covering argument.

To prove (3.1), we need to first partition $\mathcal{N}_L$ into disjoint subsets, each of which serves as an equivalence class under the equivalence relation on $\mathcal{N}_L$ described below.

For each $\vec{x} = (x_0, x_1, \cdots, x_L) \in \mathcal{N}_L$, let $T_0 = 0$, $T_1 = \inf\{n : x_n \in l\}$, and

$$T_n = \inf\{m > T_{n-1} : x_m \in l\}$$

for each integer $n \in [2, L]$ to be the time of the nth visit to l. Here we use the convention that $\inf\{\emptyset\} = \infty$ and let $T_{L+1} = \infty$. Then for each $n = 0, 1, \cdots, L$, define the nth arc of $\vec{x}$ as

$$\text{arc}(\vec{x}, n) = \vec{x}[T_n : T_{n+1}) = \left(x_{T_n}, x_{T_n+1}, \cdots, x_{T_{n+1}-1}\right),$$

where we use the convention that $\vec{x}[k : \infty) = \vec{x}[k : L]$, and $\vec{x}[0 : 0) = \vec{x}[\infty : \infty) = \emptyset$. I.e., the nth arc is the piece of the path between the (possible) nth and $n+1$st visit to l. Since $\vec{x}$ is a nearest neighbor path, all points in each arc must be on the same side of l. Then for any $D = (\delta_1, \delta_2, \cdots, \delta_L) \in \{-1, 1\}^L$ we can define a mapping $\varphi_{l,D}$ on $\mathcal{N}_L$ so that for any $\vec{x}$

$$\varphi_{l,D}(\vec{x}) = \left(\text{arc}(\vec{x}, 0), \varphi_l^{\frac{1-\delta_1}{2}}\left(\text{arc}(\vec{x}, 1)\right), \varphi_l^{\frac{1-\delta_2}{2}}\left(\text{arc}(\vec{x}, 2)\right), \cdots, \varphi_l^{\frac{1-\delta_L}{2}}\left(\text{arc}(\vec{x}, L)\right)\right).$$

In words, we keep the part of the path the same until it first visits l. Then for the nth arc, we keep it unchanged if $\delta_n = 1$ and reflect it around l if $\delta_n = -1$. By definition, it is easy to see that $\varphi_{l,D}(\vec{x}) \in \mathcal{N}_L$. And since

$$\varphi_{l,D} \circ \varphi_{l,D}(\vec{x}) \equiv \vec{x},$$

$\varphi_{l,D}$ forms a bijection on $\mathcal{N}_L$. Now we can introduce the equivalence relation on $\mathcal{N}_L$ previously mentioned. For each two $\vec{x}, \vec{y} \in \mathcal{N}_L$, we say $\vec{x} \sim \vec{y}$ if there exist a $D \in \{-1, 1\}^L$ such that (see Figure 2)

$$\varphi_{l,D}(\vec{x}) = \vec{y}.$$

Then one can immediately check that if $\vec{x} \sim \vec{y}$ and D is the configuration such that $\varphi_{l,D}(\vec{x}) = \vec{y}$, then

$$\vec{x} = \varphi_{l,D}(\vec{y}) \Rightarrow \vec{y} \sim \vec{x} \tag{3.2}$$

and for $D_0 \equiv 1$,

$$\vec{x} = \varphi_{l,D_0}(\vec{x}) \Rightarrow \vec{x} \sim \vec{x}. \tag{3.3}$$

Moreover, if

$$\varphi_{l,D_1}(\vec{x}) = \vec{y}, \; \varphi_{l,D_2}(\vec{y}) = \vec{z},$$

let

$$D_3 = \left(2 \times \mathbb{1}_{(\delta_{1,1}=\delta_{2,1})} - 1, 2 \times \mathbb{1}_{(\delta_{1,2}=\delta_{2,2})} - 1, \cdots, 2 \times \mathbb{1}_{(\delta_{1,L}=\delta_{2,L})} - 1\right)$$

we have

$$\varphi_{l,D_3}(\vec{x}) = \vec{z} \Rightarrow \vec{x} \sim \vec{z}. \tag{3.4}$$

Combining (3.2)-(3.4), we have that $\sim$ forms a equivalence relation on $\mathcal{N}_L$, where each path in $\mathcal{N}_L$ belongs to one equivalence class. Thus all the equivalence classes are disjoint from each other and there has to be a finite number of them, forming a partition of $\mathcal{N}_L$. We denote these equivalence classes as

$$\mathcal{C}_{L,1}, \mathcal{C}_{L,2}, \cdots, \mathcal{C}_{L,J} \tag{3.5}$$

where each of them can be represented by its specific element $\vec{x}_{k,+},\ k = 1, \cdots, J$, which is the unique path in each class that always stays on the left of l.

Then for each k, let $n_{k,1} < n_{k,2} < \cdots < n_{k,m_k}$ be all the n's such that

$$\left|\text{Trace}\left(\text{arc}\left(\vec{x}_{k,+}, n\right)\right)\right| > 1.$$

Note that the only case when we have $\left|\text{Trace}\left(\text{arc}\left(\vec{x}_{k,+}, n\right)\right)\right| = 0$ is when $T_n = T_{n+1} = \infty$ and the only case when it equals to 1 is when $T_n = L$. Then for any $\vec{x} \in \mathcal{C}_{L,k}$ and any D_1, D_2 such that

$$\varphi_{l,D_1}(\vec{x}_{k,+}) = \varphi_{l,D_2}(\vec{x}_{k,+}) = \vec{x},$$

we must have $\delta_{1,n_{k,i}} = \delta_{2,n_{k,i}}$ for all i's. So we have a well defined onto mapping f between $\mathcal{C}_{L,k}$ and $\{-1,1\}^{m_k}$ where each $\vec{x}$ such that

$$\vec{x} = \varphi_{l,D}(\vec{x}_{k,+})$$

for some D is mapped to

$$f(\vec{x}) = (\delta_{2,n_{k,1}}, \delta_{2,n_{k,2}}, \cdots, \delta_{2,n_{k,m_k}}).$$

Moreover, for any two configurations D_1 and D_2 such that $\delta_{1,n_{k,i}} = \delta_{2,n_{k,i}}$ for all i's, we also must have

$$\varphi_{l,D_1}(\vec{x}_{k,+}) = \varphi_{l,D_2}(\vec{x}_{k,+})$$

The reason of that is for all n not in $\{n_{k,i}\}_{i=1}^{m_k}$,

$$\left|\text{Trace}\left(\text{arc}\left(\vec{x}_{k,+}, n\right)\right)\right| \le 1$$

which means those arcs are either empty or just one point x_{T_n} right on the diagonal, which does not change at all under any possible reflection. Thus we have proved that the mapping f is a bijection between $\mathcal{C}_{L,k}$ and $\{-1,1\}^{m_k}$.

At this point we have the tools we need and can go back to compare the two covering probabilities. Noting that the equivalence classes in (3.5) form a partition of $\mathcal{N}_L$, it suffices to show that for each $k \le J$

$$|\mathcal{N}_{L,1} \cap \mathcal{C}_{L,k}| \ge |\mathcal{N}_{L,2} \cap \mathcal{C}_{L,k}|, \tag{3.6}$$

for each class $\mathcal{C}_{L,k}$. First, if

$$(A_0 \cup B_0) \cap l \nsubseteq \text{Trace}(\vec{x}_{k,+}[T_1, \cdots, T_L])$$

then one can immediately see

$$|\mathcal{N}_{L,1} \cap \mathcal{C}_{L,k}| = |\mathcal{N}_{L,2} \cap \mathcal{C}_{L,k}| = 0.$$

Otherwise, let $\Omega_k = (A_0 \cup B_0) \cap \text{Trace}(\vec{x}_{k,+}[0, T_1])^c \cap l^c$ and $A_k = A_0 \cap \text{Trace}(\vec{x}_{k,+}[0, T_1])^c \cap l^c$. And let $n_k = |\Omega_k|$. We can also list all points in Ω_k as $\omega_1, \cdots, \omega_{n_k}$ and all points in $\varphi_l(\Omega_k)$ as $\omega_{-1}, \cdots, \omega_{-n_k}$, where $\varphi_l(\omega_j) = \omega_{-j}$ for all j. Then it is easy to check that

$$\mathcal{N}_{L,1} \cap \mathcal{C}_{L,k} = \{\vec{x} \in \mathcal{C}_{L,k}, \Omega_k \subseteq \text{Trace}(\vec{x})\} \tag{3.7}$$

since all other points in $A_0 \cup B_0$ are guaranteed to be visited by $\vec{x}_{k,+}[0, T_1]$ or $\vec{x}_{k,+}[T_1, \cdots, T_L]$ which are both shared over all paths in this equivalence class. And we also have

$$\mathcal{N}_{L,2} \cap \mathcal{C}_{L,k} \subseteq \left\{\vec{x} \in \mathcal{C}_{L,k}, A_k \cup \varphi_l\big(\Omega_k \cap A_k^c\big) \subseteq \text{Trace}(\vec{x})\right\} \tag{3.8}$$

since $A_k \subseteq A_0$, $\Omega_k \cap A_k^c \subseteq B_0$. Finally, define

$$V_{k,i} = \left\{j : \omega_j \in \Omega_k \cap \text{Trace}\left(\text{arc}\big(\vec{x}_{k,+}, n_{k,i}\big)\right)\right\}, \ i = 1, 2, \cdots, m_k$$

and $\vec{V}_k = V_{k,1} \otimes V_{k,2} \otimes \cdots \otimes V_{k,m_k}$. Then by the constructions above we have for any $\omega_j \in \Omega_k$, $\omega_j \in \text{Trace}(\vec{x})$ if and only if there exists some i such that $j \in V_{k,i}$ and $f(\vec{x})[i] = 1$. And similarly $\varphi_l(\omega_j) \in \text{Trace}(\vec{x})$ if and only if there exists some i so that $j \in V_{k,i}$ and $f(\vec{x})[i] = -1$. In combination, for any $\omega_j \in \Omega_k \cup \varphi_l(\Omega_k)$, $\omega_j \in \text{Trace}(\vec{x})$ if and only if there exists some i such that $j \in f(\vec{x})[i]V_{k,i}$. Then taking the intersections and letting $\bar{\Omega}_k = \{1, 2, \cdots, n_k\}$ and

$$\bar{A}_k = \{j : \omega_j \in A_k\},$$

we have

$$\{\vec{x} \in \mathcal{C}_{L,k}, \Omega_k \subseteq \text{Trace}(\vec{x})\} = \left\{\vec{x} \in \mathcal{C}_{L,k}, \ \bar{\Omega}_k \subseteq f(\vec{x}) \cdot \vec{V}_k\right\} \tag{3.9}$$

and

$$\begin{aligned} &\left\{\vec{x} \in \mathcal{C}_{L,k}, A_k \cup \varphi_l\big(\Omega_k \cap A_k^c\big) \subseteq \text{Trace}(\vec{x})\right\} \\ &\quad = \left\{\vec{x} \in \mathcal{C}_{L,k}, \ -(\bar{\Omega}_k \cap \bar{A}_k^c) \cup \bar{A}_k \subseteq f(\vec{x}) \cdot \vec{V}_k\right\}. \end{aligned} \tag{3.10}$$

Noting that the mapping f is a bijection between $\mathcal{C}_{L,k}$ and $\{-1, 1\}^{m_k}$,

$$\left|\left\{\vec{x} \in \mathcal{C}_{L,k}, \ \bar{\Omega}_k \subseteq f(\vec{x}) \cdot \vec{V}_k\right\}\right| = |\mathcal{C}(\vec{V}_k, \bar{\Omega}_k)| \tag{3.11}$$

and

$$\left|\left\{\vec{x} \in \mathcal{C}_{L,k}, \ -(\bar{\Omega}_k \cap \bar{A}_k^c) \cup \bar{A}_k \subseteq f(\vec{x}) \cdot \vec{V}_k\right\}\right| = |\mathcal{C}(\vec{V}_k, \bar{A}_k)|. \tag{3.12}$$

Apply Lemma 2.1 on $\bar{\Omega}_k$, $\vec{V}_k$ and $\bar{A}_k$. The proof of this theorem is complete. □

Remark 3.1. With exactly the same argument, we can also have the same reflection theorem on reflecting over a line $y = x + n$, $n \geq 1$ or $x = -y + n$.

4 Path maximizing covering probability

We first consider the simpler (but essential important) case when $d = 2$. We first outline the idea of the proof as follows:

- To apply Theorem 1.1 specifically on covering the trace of a nearest neighbor path in $\mathcal{P} \subset \mathbb{Z}^2$ connecting 0 and $\partial B_1(0, N)$, we can assume without loss of generality that the last point of this path, $P_K \in \partial B_1(0, N)$ is in the first quadrant.
- For each such path with at least one point (x, y) that is not a "neighbor" of the diagonal, i.e. $|x-y| \geq 2$, we can always reflect it as follows: (1) Consider $l : \ x = y+1$ or $y = x + 1$ be the axis of reflection. Then l divides $\mathbb{Z}^2$ into 2 parts. (2) Let A_0 be the collection of points in our path that are in the same half as point 0, and let the remaining point in our path be $\varphi_l(B_0)$. Then by Theorem 1.1, one may replace $\varphi_l(B_0)$ with B and always increase the probability of covering.

- Then note that, after the reflection, $A_0 \cup B_0$ is the trace of another nearest neighbor path, and we can reduce the total difference
$$\sum |x_i - y_i|$$
by at least one in each step. After a finite number of steps, we will end up with a nearest neighbor path that stays within $\{|x-y| \leq 1\}$.
- Finally, among all those such paths that of distance no more than one from the diagonal, applying Theorem 1.1 for reflection over $x = y$, we can show that the covering probability is maximized when we move all the "one step corners" to the same side of the diagonal, which itself gives us a monotonic path that stays within distance one above or below the diagonal. Thus we have the theorem as follows.

Theorem 4.1. *For each integers $L \geq N \geq 1$, let $\mathcal{P}$ be any nearest neighbor path in $\mathbb{Z}^2$connecting 0 and $\partial B_1(0,N)$. Let $X_n, n \geq 0$ be a 2 dimensional simple random walk starting at 0. Then*

$$P\big(\text{Trace}(\mathcal{P}) \subseteq \text{Trace}(X_0, \cdots, X_L)\big) \leq P\big(\overset{\nearrow}{\mathcal{P}} \subseteq \text{Trace}(X_0, \cdots, X_L)\big)$$

where
$$\overset{\nearrow}{\mathcal{P}} = \Big((0,0),(0,1),(1,1),(1,2),\cdots,([N/2], N-[N/2])\Big).$$

Proof. As outlined above, we first show that

Lemma 4.2. *For each integers $L \geq N \geq 1$, let*

$$\mathcal{P} = \Big((x_0,y_0),\cdots,(x_K,y_K)\Big)$$

be any nearest neighbor path in $\mathbb{Z}^2$ connecting 0 and $\partial B_1(0,N)$ with length $K+1 \geq N+1$ where there is an $i \leq K$ such that $|x_i - y_i| \geq 2$ and where $x_K \geq 0, y_K \geq 0$. $X_n,\ n \geq 0$ be a 2 dimensional simple random walk starting at 0. Then there exists a nearest neighbor path $\mathcal{P}_1$ staying within $\{|x-y| \leq 1\}$ such that

$$P\big(\text{Trace}(\mathcal{P}) \in \text{Trace}(X_0,\cdots,X_L)\big) \leq P\big(\text{Trace}(\mathcal{P}_1) \in \text{Trace}(X_0,\cdots,X_L)\big).$$

Proof. For any path $\mathcal{Q}$ with lenght $K+1$, define its **total difference** as

$$D_T(\mathcal{Q}) = \sum_{(x_i,y_i)\in \text{Trace}(\mathcal{Q})} |x_i - y_i|. \tag{4.1}$$

For each such path in this lemma, without loss of generality we can always assume there is some i such that $x_i - y_i \geq 2$. Otherwise, by definition one must have an i such that $y_i - i_i \geq 2$. Then applying reflection over $x = y$, we are back to the first case. Consider the line of reflection $l : x = y + 1$ (otherwise consider $l : y = x + 1$). It is easy to see that there is at least one point along this path on the right side of l. I.e.

$$B_0' = \text{Trace}(\mathcal{P}) \cap \{x > y + 1\} \neq \emptyset.$$

Define $A_0 = \text{Trace}(\mathcal{P}) \cap B_0'^c$, $B_0 = \varphi_l(B_0') \cap A_0^c$, and $\hat{B}_0' = \varphi_l(B_0) \subseteq B_0'$. Thus we have $\text{Trace}(\mathcal{P}) = A_0 \cup B_0'$ and

$$P\big(A_0 \cup B_0' \subseteq \text{Trace}(X_0,\cdots,X_L)\big) \leq P\big(A_0 \cup \hat{B}_0' \subseteq \text{Trace}(X_0,\cdots,X_L)\big). \tag{4.2}$$

Applying Theorem 1.1 on A_0 and B_0 gives us

$$P\big(A_0 \cup \hat{B}_0' \subseteq \text{Trace}(X_0,\cdots,X_L)\big) \leq P\big(A_0 \cup B_0 \subseteq \text{Trace}(X_0,\cdots,X_L)\big). \tag{4.3}$$

Then noting that $\mathcal{P}$ is a nearest neighbor path starting at 0 with length $K+1$, let $C_{K,i}$ be the equivalence class it belongs to under the relation $\sim$, and let $\mathcal{P}' = \vec{x}_i$ be the representing element of $C_{K,i}$ where all arcs are reflected to the left of l. Then it is easy to see that

$$\text{Trace}(\mathcal{P}') = A_0 \cup \varphi_l(B_0') = A_0 \cup B_0. \tag{4.4}$$

Combine (4.2)-(4.4),

$$P\big(\mathcal{P} \subseteq \text{Trace}(X_0, \cdots, X_L)\big) \le P\big(\mathcal{P}' \subseteq \text{Trace}(X_0, \cdots, X_L)\big). \tag{4.5}$$

Then note that for any j such that $x_j - y_j \ge 2$,

$$\varphi_l(x_j, y_j) = (y_i + 1, x_j - 1) \in \mathbb{Z}^2$$

while

$$\|(x_j, y_j)\|_1 \ge \|(y_i + 1, x_j - 1)\|_1$$

for all $x_i \ge y_i + 2$. Since (x_K, y_K) is in the first quadrant, if in addition we also have $x_K \ge y_K + 1$, then $\varphi_l(x_K, y_K)$ remains in the first quadrant with $\|\varphi_l(x_K, y_K)\|_1 = N$. Thus the new nearest neighbor path $\mathcal{P}'$ is also one connecting 0 and $\partial B_1(0, N)$. Moreover,

$$D_T(\mathcal{P}) = \sum_{j:(x_j,y_j)\in B_0'} (x_j - y_j) + \sum_{j:(x_j,y_j)\in A_0} |x_j - y_j|,$$

while

$$D_T(\mathcal{P}') = \sum_{j:(x_j,y_j)\in \hat{B}_0'} (x_j - y_j - 2) + \sum_{j:(x_j,y_j)\in A_0} |x_j - y_j|,$$

which shows that $D_T(\mathcal{P}') \le D_T(\mathcal{P}) - 2$. Then if there is a point (x_j', y_j') in the new path $\mathcal{P}'$ with $|x_j' - y_j'| \ge 2$, we can repeat the previous process and the covering probability is non-decreasing. Noting that for each time we decrease the total difference by at least 2 while $D_T(\mathcal{P})$ is a finite number, after repeating a finite number of times, we must end up with a nearest neighbor path where no point satisfies $|x-y| \ge 2$. Thus, we find a nearest neighbor path $\mathcal{P}_1$ connecting 0 and $\partial B_1(0, N)$ staying within $\{|x-y| \le 1\}$ with a higher covering probability. □

With Lemma 4.2, note that for any nearest neighbor path connecting 0 and $\partial B_1(0, N)$ staying within $\{|x-y| \le 1\}$, we can always look at the part of it after its last visit to 0 and it has a higher covering probability. And note that such part has to contain a self-avoiding path from 0 to $\partial B_1(0, N)$. Letting $N_0 = N - [N/2]$ and $\bar{\Omega} = \{1, 2, \cdots, N_0\}$, we have the following lemma whose proof is elementary and is omitted here:

Lemma 4.3. *For each nearest neighbor path $\mathcal{P}_1$ connecting 0 and $\partial B_1(0, N)$ staying within $\{|x-y| \le 1\}$, we must have that*

$$(j, j) \in \text{Trace}(\mathcal{P}_1),\ \forall j = 1, 2, \cdots, [N/2]$$

and that

$$(j-1, j) \text{ or } (j, j-1) \in \text{Trace}(\mathcal{P}_1),\ \forall j = 1, 2, \cdots, N_0.$$

This lemma guarantees that such self-avoiding path has to be monotonic as well. Otherwise, suppose the path contains any decreasing edge, say $(j, j) \to (j, j-1)$. Then vertex (j, j) has to be visited more than once, which contradict with the self-avoiding condition. Thus it is sufficient to show that $\overset{\nearrow}{\mathcal{P}}$ has the highest covering probability over all nearest neighbor monotonic paths $\mathcal{P}_1$ connecting 0 and $\partial B_1(0, N)$ that stay within

$\{|x-y|\le 1\}$. We can show this by specifying what the trace of each such path looks like. With the Lemma 4.3, for each such $\mathcal{P}_1$ define

$$\bar{A}=\big\{j:(j-1,j)\in \mathrm{Trace}(\mathcal{P}_1)\big\}\subseteq \bar{\Omega}$$

and

$$\bar{B}=\big\{j:(j,j-1)\in \mathrm{Trace}(\mathcal{P}_1)\big\}\subseteq \bar{\Omega}.$$

Then we have $\bar{A}\cup\bar{B}=\bar{\Omega}$, so that

$$\mathcal{P}_1\supseteq\{(j,j),\ j=0,1,\cdots[N/2]\}\cup\{(j-1,j),\ j\in\bar{A}\}\cup\{(j,j-1),\ j\in\bar{\Omega}\cap\bar{A}^c\}.$$

Define

$$A_0=\{(j,j),\ j=0,1,\cdots[N/2]\}\cup\{(j-1,j),\ j\in\bar{A}\},\ B_0=\varphi_{l^0}(\{(j,j-1),\ j\in\bar{\Omega}\cap\bar{A}^c\}),$$

where l^0 is the line $x=y$. Then

$$P\big(\mathrm{Trace}(\mathcal{P}_1)\subseteq \mathrm{Trace}(X_0,\cdots,X_L)\big)\le P\big(A_0\cup\varphi_{l^0}(B_0)\subseteq \mathrm{Trace}(X_0,\cdots,X_L)\big).$$

And note that

$$A_0\cup B_0=\Big((0,0),(0,1),(1,1),(1,2),\cdots,([N/2],N-[N/2])\Big)=\overset{\nearrow}{\mathcal{P}},$$

which itself gives a monotonic nearest neighbor path connecting 0 and $\partial B_1(0,N)$. So Theorem 1.1 finishes the proof. □

For fixed N, the inequality in Theorem 4.1 becomes equality when $L=\infty$ since the 2 dimensional simple random walk is recurrent and both probabilities go to one. However, we can easily generalize the same result to higher dimensions. This will similarly give us Theorem 1.5.

Proof of Theorem 1.5. This theorem can be proved by reflecting only on two coordinates in $\mathbb{Z}^d$ while keeping all the others unchanged. For any $n\ge 0$, we look at, without loss of generality, the subspace $l: a_1=a_2+l_0,\ l_0\ge 0$ when $d\ge 3$, and define reflection φ_l over l as follows: for each point $(a_1,\cdots,a_d)\in\mathbb{Z}^d$,

$$\varphi_l(a_1,\cdots,a_d)=(a_2+l_0,a_1-l_0,a_3,\cdots,a_d).$$

Then for all paths in $\mathcal{N}_L$ (all nearest neighbor paths starting at 0 of length $L+1$), we can again define $T_0=0$, $T_1=\inf\{n:x_n\in l\}$, and

$$T_n=\inf\{n\ge T_{n-1}:x_n\in l\}$$

for each integer $n\in[2,L]$ to be the time of the nth visit to subpsace l, and divide $\mathcal{N}_L$ into a partition of equivalence classes under $\varphi_{l,D}$ for all $D\in\{-1,1\}^L$. Then for each pair of disjoint finite subsets $A_0,B_0\subseteq\{x\le y+l_0\}$, let

$$\mathcal{N}_{L,1}=\big\{\vec{x}\in\mathcal{N}_L,A_0\cup B_0\subseteq \mathrm{Trace}(\vec{x})\big\},$$

and

$$\mathcal{N}_{L,2}=\big\{\vec{x}\in\mathcal{N}_L,A_0\cup\varphi_l(B_0)\subseteq \mathrm{Trace}(\vec{x})\big\}.$$

For each equivalence class $\mathcal{C}_{L,k}$ as above, the exact same argument as in the proof of Theorem 1.1 guarantees that

$$|\mathcal{N}_{L,1}\cap\mathcal{C}_{L,k}|\ge|\mathcal{N}_{L,2}\cap\mathcal{C}_{L,k}|.$$

So again we have

$$P\left(A_0 \cup B_0 \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^L\big)\right) \\ \geq P\left(A_0 \cup \varphi_l(B_0) \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^L\big)\right). \tag{4.6}$$

Then apply (4.6) to any nearest neighbor path connecting 0 and $\partial B_1(0,N)$

$$\mathcal{P} = (P_0, P_1, P_2, \cdots, P_K)$$

where $K \geq N$. And without loss of generality we can also assume that $P_L \in (\mathbb{Z}^+ \cup \{0\})^d$. Let the subspace of reflection be $l : a_1 = a_2 + 1$,

$$A_0 = \mathrm{Trace}(\mathcal{P}) \cap \{a_1 \leq a_2 + 1\},\ B_0' = \mathrm{Trace}(\mathcal{P}) \cap \{a_1 > a_2 + 1\}.$$

and

$$B_0 = \varphi_l(B_0') \cap A_0^c,\ \ \hat{B}_0' = \varphi_l(B_0).$$

Without loss of generality we can assume B_0' is not empty, note that $\mathrm{Trace}(\mathcal{P}) = A_0 \cup B_0'$, and that similar to the proof of Theorem 4.1, we can again let $\mathcal{P}'$ be the representing element in the equivalence class under $\sim$ that contains $\mathcal{P}$, which is another nearest neighbor path connecting 0 and $\partial B_1(0,N)$ where all the arcs are reflected to the same side of l as 0. Then $\mathrm{Trace}(\mathcal{P}') = A_0 \cup B_0$, and $\mathrm{Trace}(\mathcal{P}) = A_0 \cup B_0' \supseteq A_0 \cup \hat{B}_0'$. By (4.6) we have

$$P\big(\mathrm{Trace}(\mathcal{P}) \in \mathrm{Trace}(X_0, \cdots, X_L)\big) \leq P\big(\mathcal{P}' \in \mathrm{Trace}(X_0, \cdots, X_L)\big). \tag{4.7}$$

Moreover, define

$$D_T(\mathcal{P}) = \sum_{P_n \in \mathrm{Trace}(\mathcal{P})} \sum_{i,j \leq d} |p_{n,i} - p_{n,j}|$$

be the total difference of $\mathcal{P}$. Then note that for each n

$$\sum_{i,j \leq d} |p_{n,i} - p_{n,j}| = |p_{n,1} - p_{n,2}| + f_n(p_{n,1}) + f_n(p_{n,2}) + \sum_{3 \leq i,j \leq d} |p_{n,i} - p_{n,j}|$$

where

$$f_n(p) = \sum_{i=3}^{d} |p - p_{n,i}|$$

which is a convex function of p. Thus, we rewrite

$$D_T(\mathcal{P}) = \sum_{P_n \in A_0} \sum_{i,j \leq d} |p_{n,i} - p_{n,j}| + \sum_{P_n \in B_0'} \sum_{i,j \leq d} |p_{n,i} - p_{n,j}|$$

and

$$D_T(\mathcal{P}') = \sum_{P_n' \in A_0} \sum_{i,j \leq d} |p_{n,i}' - p_{n,j}'| + \sum_{P_n' \in B_0} \sum_{i,j \leq d} |p_{n,i}' - p_{n,j}'|$$

For each n such that $P_n' = (p_{n,1}, \cdots, p_{n,d}) \in A_0$, we always have

$$\sum_{i,j \leq d} |p_{n,i} - p_{n,j}| = \sum_{i,j \leq d} |p_{n,i}' - p_{n,j}'|.$$

Otherwise, we must have $P_n' \in B_0$ and there must always be a $P_n = \varphi_l(P_n') \in \hat{B}_0' \subseteq B_0'$, which implies that

$$\sum_{i,j \leq d} |p_{n,i}' - p_{n,j}'| = |p_{n,1}' - p_{n,2}'| + f_n(p_{n,1}') + f_n(p_{n,2}') + \sum_{3 \leq i,j \leq d} |p_{n,i} - p_{n,j}|.$$

And since $P_n \in B_0'$, $p_{n,1} \geq p_{n,2} + 2$, so that for $p'_{n,1} = p_{n,2} + 1$ and $p'_{n,2} = p_{n,1} - 1$, we must have

$$\max\{p_{n,1}, p_{n,2}\} > \max\{p'_{n,1}, p'_{n,2}\}, \ \min\{p_{n,1}, p_{n,2}\} < \min\{p'_{n,1}, p'_{n,2}\}, \tag{4.8}$$

which implies that $|p'_{n,1} - p'_{n,2}| < |p_{n,1} - p_{n,2}|$. Then combining (4.8), and that $p'_{n,1} + p'_{n,2} = p_{n,1} + p_{n,2}$ with the fact that $f_n(p)$ is convex, we also have

$$f_n(p'_{n,1}) + f_n(p'_{n,2}) \leq f_n(p_{n,1}) + f_n(p_{n,2})$$

which further implies that $D_T(\mathcal{P}) \geq D_T(\mathcal{P}') + 1$. Again, noting that $D_T(\mathcal{P})$ itself is finite, so after at most a finite number of iterations, we will end up with a path $\mathcal{P}_1$ connecting 0 and within region

$$R = \left\{(a_1, a_2, \cdots, a_d) \in \mathbb{Z}^d, \ \max_{i,j \leq d} |a_i - a_j| \leq 1\right\}.$$

At the same time, note that we have assumed $P_L \in (\mathbb{Z}^+ \cup \{0\})^d$. So by (4.8) (and its parallel versions for other pairs of coordinates), the end point of $\mathcal{P}_1$ remains in $(\mathbb{Z}^+ \cup \{0\})^d$ and thus has the same L_1 norm as P_L, which is N. Thus, $\mathcal{P}_1$ remains a path connecting 0 and $\partial B_1(0, N)$.

Moreover, it is easy to see that for each point $\vec{a}_0 = (a_{1,0}, a_{2,0}, \cdots, a_{d,0})$ in region R and each subspace $l : a_i = a_j$, $\vec{a}'_0 = \varphi_l(\vec{a}_0)$ must satisfy

$$a'_{k,0} = \begin{cases} a_{k,0}, & \text{if } k \neq i, j \\ a_{j,0}, & \text{if } k = i \\ a_{i,0}, & \text{if } k = j. \end{cases} \tag{4.9}$$

Similar to the argument in the proof of Theorem 4.1, one may apply reflection over $a_2 = a_1$ towards 0, which reflects points in $\{a_2 = a_1 + 1\}$ to $\{a_1 = a_2 + 1\}$. And then similar reflections can be applied over $a_3 = a_1, \ \cdots$ and $a_d = a_1$. We will have a sequence of paths $\mathcal{P}_{2,i}, \ i = 2, \cdots d$ in R with covering probabilities that never decrease. Moreover, by the definition of our reflections, for each $n \leq K$ and $2 \leq j \leq d$ let $p_{2,i,n,j}$ be the jth coordinate of the nth vertex in $\mathcal{P}_{2,i}$. We have that $\{p_{2,i,n,1}\}_{i=2}^d$ is nondecreasing while $\{p_{2,i,n,j}\}_{i=2}^d$ is nonincreasing, and that

$$p_{2,j,n,1} \geq p_{2,j,n,j}, \ \forall 2 \leq j \leq d.$$

Thus for $\mathcal{P}_2 = \mathcal{P}_{2,d}$, we must have

$$p_{2,n,1} \geq \max_{2 \leq j \leq d} p_{2,n,j} \tag{4.10}$$

for all $n \leq K$. Then we reflect $\mathcal{P}_2$ over $a_3 = a_2$, $a_4 = a_2, \ \cdots$, and $a_d = a_2$ which also gives us a sequence of paths $\mathcal{P}_{3,i}, \ i = 3, \cdots d$ in R with covering probabilities that never decrease. Letting $\mathcal{P}_3 = \mathcal{P}_{3,d}$, similarly we must have

$$p_{3,n,2} \geq \max_{3 \leq j \leq d} p_{3,n,j}. \tag{4.11}$$

Moreover recalling the formulas of reflections within R in (4.9), all reflections over $a_i = a_2$, $i \geq 3$ will not change $\max_{2 \leq j \leq d} p_{2,n,j}$ for any n. Thus, we still have (4.10) holds for $\mathcal{P}_3$. Repeating this process and we will have a sequence $\mathcal{P}_4, \mathcal{P}_5, \cdots, \mathcal{P}_d$ with covering probabilities that never decrease, where each of them stays within R. And finally for $\mathcal{P}_d$, we must have

$$p_{d,n,i} \geq \max_{i+1 \leq j \leq d} p_{d,n,j}, \tag{4.12}$$

for all $i \leq d - 1, \ n \leq N$. Noting again that $\mathcal{P}_d$ is a nearest neighbor path, then

$$\text{Trace}(\mathcal{P}_d) \supseteq \overset{\nearrow}{\mathcal{P}}$$

and the proof of this Theorem is complete. □

5 Proof of Theorem 1.6

With Theorem 1.5, the proof of Theorem 1.6 follows immediately from the fact that the simple random walk on $\mathbb{R}^d,\ d \geq 4$ returns to the one dimensional line $x_1 = x_2 = \cdots = x_d$ with probability less than 1. Note that for any nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_N)$ and $\{X_n\}_{n=0}^{\infty}$ connecting 0 and $\partial B_1(0, N)$ which is a d dimensional simple random walk, letting

$$\mathcal{Q} = \left\{0, \sum_{i=1}^{d} e_i, 2\sum_{i=1}^{d} e_i, \cdots, [N/d] \sum_{i=1}^{d} e_i\right\}$$

be the points in $\overset{\nearrow}{\mathcal{P}}$ on the diagonal, we always have by Theorem 1.5,

$$\begin{aligned} P\left(\mathrm{Trace}(\mathcal{P}) \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^{\infty}\big)\right) &\leq P\left(\overset{\nearrow}{\mathcal{P}} \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^{\infty}\big)\right) \\ &\leq P\left(\mathcal{Q} \subseteq \mathrm{Trace}\big(\{X_n\}_{n=0}^{\infty}\big)\right). \end{aligned}$$

Moreover, let $\{\tau_n\}_{n=1}^{\infty}$ be the sequence of stopping times of all visits to the diagonal line $\ell : x_1 = x_2 = \cdots = x_d$. Then

$$P\left(\mathcal{Q} \subseteq \text{Trace}\big(\{X_n\}_{n=0}^{\infty}\big)\right) \leq P(\tau_{[N/d]} < \infty). \tag{5.1}$$

To bound the probability on the right hand side of (5.1), define a new Markov process $\{Y_n\}_{n=0}^{\infty} \in \mathbb{Z}^{d-1}$, where

$$Y_n = (X_{n,1} - X_{n,2}, X_{n,2} - X_{n,3}, \cdots, X_{n,d-1} - X_{n,d}).$$

Note that we can also write $\tau_n = \inf\{n > \tau_{n-1}, Y_n = 0\}$ and that Y_n itself is a $d-1$ dimensional random walk with generator

$$\begin{aligned} \mathcal{L}f(y) =& \frac{1}{2d}\left[(\sum_{i=1}^{d-2} f(y + e_i - e_{i+1}) + f(y - e_i + e_{i+1})\right] \\ &+ \frac{1}{2d}\left[f(y + e_1) + f(y - e_1) + f(y + e_{d-1}) + f(y - e_{d-1})\right] - f(y) \end{aligned}$$

for function f on $\mathbb{Z}^{d-1}$. With $d - 1 \geq 3$, we have $P(\tau_n < \infty | \tau_{n-1} < \infty) = P_d = 1 - G_Y(0)^{-1} < 1$. And thus the proof of Theorem 1.6 complete. □

6 Discussions

In this section we discuss the conjectures and show numerical simulations.

6.1 Covering probabilities with repetitions

In the proof of Theorem 1.5, note that each time we apply Theorem 1.1 and get a new path $\mathcal{P}'$ with higher covering probability, we always have

$$\mathrm{Trace}(\mathcal{P}) = A_0 \cup B_0'$$

and

$$\mathrm{Trace}(\mathcal{P}') = A_0 \cup B_0$$

where $B_0 = \varphi_l(B_0') \cap A_0^c \subseteq \varphi_l(B_0')$. This, together with the fact that A_0 is disjoint with both B_0 and B_0', implies that

$$|\mathrm{Trace}(\mathcal{P})| = |A_0| + |B_0'| \geq |A_0| + |B_0| = |\mathrm{Trace}(\mathcal{P}')|.$$

In words, although the length of the path remains the same after reflection, the size of its trace may decrease. In fact, for any simple path connecting 0 and $\partial B_1(0,N)$, at the end of our sequence of reflections, we will always end up with a (generally non-simple) path whose trace is of size $N+1$.

One natural approach towards a sharper upper bound is taking the repetitions of visits in a non-simple path into consideration. For any path

$$\mathcal{P} = \big(P_0, P_1, \cdots, P_K\big)$$

starting at 0 which may not be simple, and any point $P \in \mathrm{Trace}(\mathcal{P})$, we can define the first visit to P as $T_{1,P} = \inf\{n : P_n = P\}$ and

$$T_{k,P} = \inf\{n > T_{k-1} : P_n = P\}$$

to be the kth visit, with convention $\inf\{\emptyset\} = \infty$. Then we can define the repetition of $P \in \mathrm{Trace}(\mathcal{P})$ in the path $\mathcal{P}$ as

$$n_{P,\mathcal{P}} = \sup\{k : T_{k,P} < \infty\} \tag{6.1}$$

and denote the collection of all such repetitions as $N_{\mathcal{P}} = \{n_{P,\mathcal{P}} : P \in \mathrm{Trace}(\mathcal{P})\}$. It is to easy to see that $n_{P,\mathcal{P}} \equiv 1$ for all $P \in \mathrm{Trace}(\mathcal{P})$ when $\mathcal{P}$ is a simple path, and that

$$\sum_{P \in \mathrm{Trace}(\mathcal{P})} n_{P,\mathcal{P}} = K + 1.$$

For d dimensional simple random walk $\{X_n\}_{n=0}^{\infty}$ starting at 0 and any point $P \in \mathbb{Z}^d$, we can again define the stopping times $\tau_{0,P} = 0$, $\tau_{1,P} = \inf\{n : X_n = P\}$ and

$$\tau_{n,P} = \inf\{n > \tau_{n-1,P} : X_n = P\} \tag{6.2}$$

with convention $\inf\{\emptyset\} = \infty$. Note that for any integer $m > 0$, the **local time** of random walk $\{X_n\}_{n=0}^{\infty}$ at point P and time m can be define as

$$\xi(m, P) = \max\{n : \ \tau_{n,P} \le m\}.$$

Then we have

Definition 6.1. *For each nearest neighbor path $\mathcal{P}$, and d dimensional simple random walk $\{X_n\}_{n=0}^{\infty}$, we say that $\{X_n\}_{n=0}^{L}$ covers $\mathcal{P}$ **up to its repetitions** if*

$$\xi(L, p) \ge n_{P,\mathcal{P}}, \forall P \in \mathrm{Trace}(\mathcal{P}).$$

And we denote such event by $\mathrm{Trace}(\mathcal{P}) \otimes N_{\mathcal{P}} \subseteq \{X_n\}_{n=0}^{L}$.

Our **hope** was, for any nearest neighbor path $\mathcal{P}$ and subspace of reflection like $l : x_i = x_j + l_0$, the probability of a simple random walk $\{X_n\}_{n=0}^{L}$ starting at 0 covering $\mathcal{P}$ up to its repetitions **may** be upper bounded by that of covering the path $\mathcal{P}'$ up to its repetitions, where $\mathcal{P}'$ is the representing element in the equivalence class in $\mathcal{N}_K$ containing $\mathcal{P}$ under the reflection φ_l. In words, $\mathcal{P}'$ is the path we get by making all the arcs in $\mathcal{P}$ reflected to the same side as 0.

Note that although $\mathcal{P}'$ may not be simple and the size of its trace could decrease, this will at the same time increase the repetition on those points which are symmetric to the disappeared ones correspondingly. In fact, under Definition 6.1, the total number of points our random walk needs to (re-)visit is always

$$\sum_{P \in \mathrm{Trace}(\mathcal{P}')} n_{P,\mathcal{P}'} = K + 1.$$

So if our previous guess were true, then we will be able to follow the same process as in Section 3 and 4 and end with the same path along the diagonal, but this time with a higher probability of being covered up to its repetitions.

Unfortunately, here we present the counterexample and numerical simulations showing that Theorem 4.1 and 1.5 no longer holds for of certain non-monotonic paths. The idea of constructing those examples can be seen in the following preliminary model: Let l be the line of reflection and suppose we have one point x on the same side of l as 0. Then suppose there is a equivalence class $\mathcal{C}_{L,k}$ with its representative element $\vec{x}_k$ having $2n$ arcs each visiting x once. Then we look at the covering probability of $\{x, \varphi_l(x)\} \otimes (n, n)$ and that of its reflection $\{x\} \otimes 2n$. For the first one, we only need to choose n of $2n$ arcs in $\vec{x}_k$ and reflect them to the other side while keeping the rest unchanged. So we have $\binom{2n}{n}$ configurations. However, for the second covering probability which one may hope to be higher, the only configuration that may give us the covering up to this repetition is $\vec{x}_k$ itself. Thus, at least in this equivalence class, the number of configurations covering $\{x, \varphi_l(x)\} \otimes (n, n)$ is larger than that of configurations covering $\{x\} \otimes 2n$.

With this idea in mind we give the following counterexample on actual 3 dimensional paths which shows precisely and rigorously that the covering probability is not increased after reflection.

Counterexample 1. Consider the following points $o = (0, 0, 0),\ y = (1, 0, 0),\ z = (0, 1, 0),\ w = (1, 1, 0) \in \mathbb{Z}^3$, and paths

$$\mathcal{P} = (o, y, w, z)$$

and

$$\mathcal{P}' = (o, y, w, y)$$

which is the representative element of the equivalence class containing $\mathcal{P}$, under reflection over $l : x_2 = x_1$. Let $\{X_n\}_{n=0}^{\infty}$ be a simple random walk starting at 0. Moreover, we use the notation $A = \mathbb{Z}^3 \setminus \{y, z, w\}$ and define stopping times $\tau_a = \inf\{n \ge 1 :\ X_n = a\}$ for all $a \in \mathbb{Z}^3$, and $\tau_A = \inf\{n \ge 1 :\ X_n \notin A\}$. Thus we have

Proposition 6.2. For the paths $\mathcal{P}$ and $\mathcal{P}'$ defined above,

$$\begin{aligned} &P_o\left(\mathrm{Trace}(\mathcal{P}) \otimes N_{\mathcal{P}} \subseteq \{X_n\}_{n=0}^{\infty}\right) \\ &\quad = 2P_o(\tau_y = \tau_A)[P_o(\tau_y < \tau_w) + P_o(\tau_w < \tau_y)]P_o(\tau_y < \infty) \\ &\quad + 2P_o(\tau_w = \tau_A)P_o(\tau_y < \tau_z)P_o(\tau_y < \infty) \approx 0.08 \end{aligned} \tag{6.3}$$

which is larger than

$$\begin{aligned} &P_o\left(\mathrm{Trace}(\mathcal{P}') \otimes N_{\mathcal{P}'} \subseteq \{X_n\}_{n=0}^{\infty}\right) \\ &\quad = P_o(\tau_y < \tau_w)[P_o(\tau_0 < \tau_y) + P_o(\tau_y < \tau_0)]P_o(\tau_y < \infty) \\ &\quad + P_o(\tau_w < \tau_y)P_o(\tau_y < \infty)P_o(\tau_0 < \infty) \approx 0.065. \end{aligned} \tag{6.4}$$

The proof of Proposition 6.2 is basically a standard application of Green's functions for finite subsets. So we leave the detailed calculations in Appendix B. For anyone who believes in law of larger numbers, we recommend them to look at the following numerical simulation which shows the empirical probabilities (which almost exactly agree with Proposition 6.2) of covering both paths with half a million independent paths of 3-dimensional simple random walks run up to $L = 40000$.

For a finite length $\{X_n\}_{n=0}^{L}$ with $L < \infty$, although it is harder to calculate the exact covering probabilities theoretically, the following simulations on $L = 4000, 400$ and 40 show that the inequality in Proposition 6.2 remains robust for fairly small L (see Figure 5).

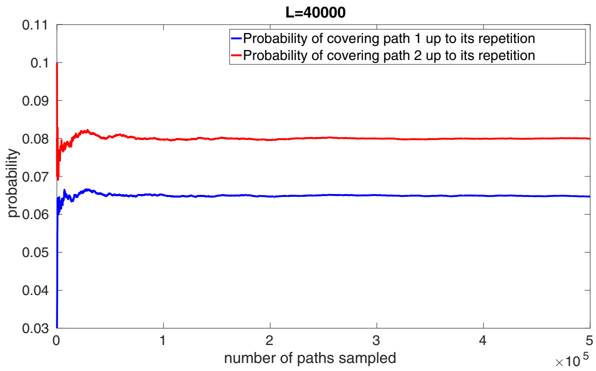

Figure 3: covering probabilities of path 1=$\mathcal{P}$ and path 2=$\mathcal{P}'$, $L = 40000$

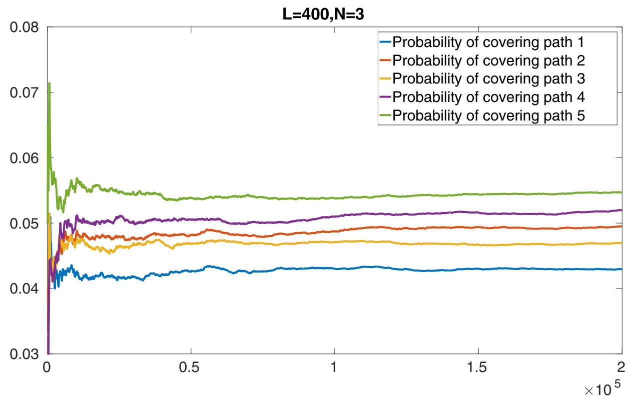

Figure 4: covering probabilities of monotonic paths starting at 0 of length 4

6.2 Monotonic path minimizing covering probability

In Conjecture 1.11, we conjecture that when concentrating on monotonic paths, the covering probability is minimized when the path takes a straight line along some axis. The intuition is, while all monotonic paths connecting 0 and $\partial B_1(0, N)$ have the same L_1 distance, the L_2 distance is maximized along the straight line, which makes it the most difficult to cover. This conjecture is supported for small N. In the following example, we have $d = 3$ and $N = 3$. By symmetry of simple random walk, one can easily see that for each monotonic path of length $N + 1 = 4$, starting at 0, the covering probability must equal to that of one of the following five:

$$\begin{aligned}
\text{path1} &: (0,0,0) \to (1,0,0) \to (2,0,0) \to (3,0,0)\\
\text{path2} &: (0,0,0) \to (1,0,0) \to (2,0,0) \to (2,1,0)\\
\text{path3} &: (0,0,0) \to (1,0,0) \to (1,1,0) \to (1,2,0)\\
\text{path4} &: (0,0,0) \to (1,0,0) \to (1,1,0) \to (2,1,0)\\
\text{path5} &: (0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1).
\end{aligned}$$

The following simulation (see Figure 4) shows that when $L = 400$, the covering probability of path 1 is the smallest of them all. It should be easy to use the same calculation in Proposition 6.2 to show the rigorous result when $L = \infty$.

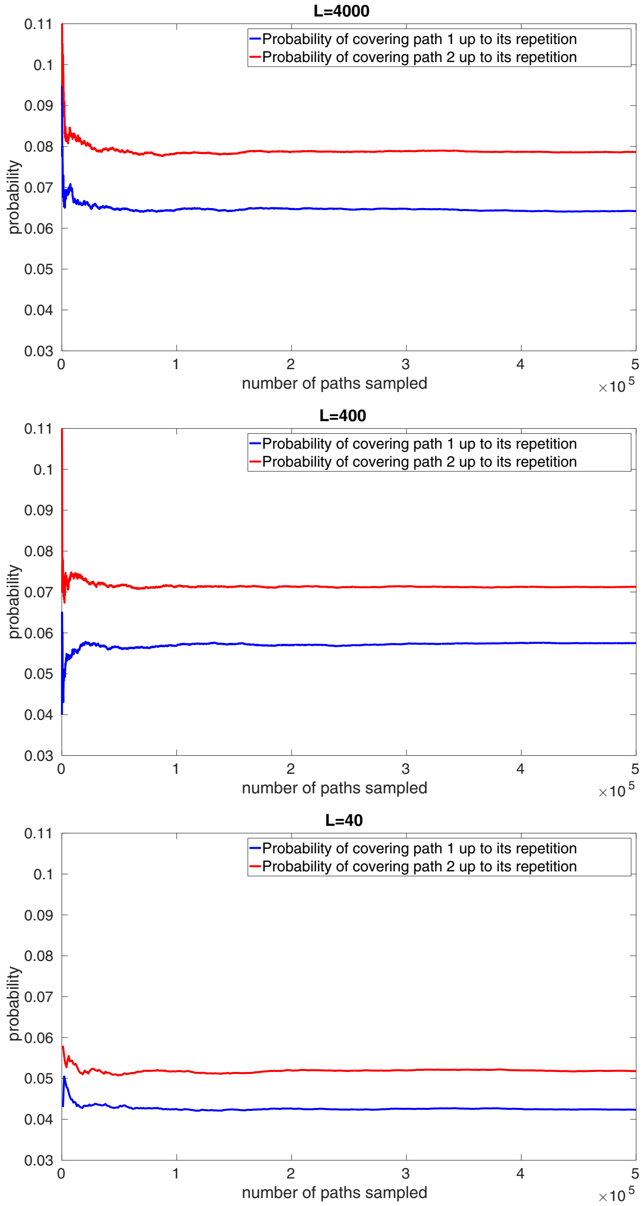

Figure 5: covering probabilities of path 1=$\mathcal{P}$ and path 2=$\mathcal{P}'$, $L = 4000, 400, 40$

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A

A.1 Introduction

In this appendix, we find the asymptotic behavior of the returning probability (to the diagonal line and the path $\overset{\nearrow}{\mathcal{P}}$) of a d dimensional simple random walk as $d \to \infty$. For asymptotics of the return probability to the origin, the result is stated in [10]. To be precise, for a d dimensional simple random walk $\{X_{d,n}\}_{n=0}^{\infty}$ starting at 0 and any $x \in \mathbb{Z}^d$, define the stopping time

$$\tau_{d,x} = \inf\{n \geq 1,\ X_{d,n} = x\}.$$

Then the returning probability is defined by

$$p_d = P(\tau_{d,0} < \infty). \tag{A.1}$$

In [10], it is stated that $\lim_{d\to\infty} 2dp_d = 1$. However, we believe the proof in [10] is not completely rigorous. Rigorous proof of the asymptotic above can be found in [6], and then independently in [4]. Moreover, using the same method, one may also show the higher order asymptotic of p_d, which is stated in [3].

In this appendix, we apply a similar method on non-simple random walks. Particularly, for a specific $d-1$ dimensional one defined by

$$\hat{X}_{d-1,n} = \left(X_{d,n}^1 - X_{d,n}^2, X_{d,n}^2 - X_{d,n}^3, \cdots, X_{d,n}^{d-1} - X_{d,n}^d\right)$$

where $X_{d,n}^i$ is the ith coordinate of $X_{d,n}$, we can show the same asymptotic for $\hat{X}_{d-1,n}$, which also gives us the asymptotic of the probability that a d dimensional simple random walk ever returns to the diagonal line. To make the statement precise, consider the diagonal line in $\mathbb{Z}^d$

$$l_d = \{(n, n, \cdots, n) \in \mathbb{Z}^d, n \in \mathbb{Z}\} \subset \mathbb{Z}^d.$$

Define the stopping time

$$\tau_{d,l_d} = \inf\{n \geq 1,\ X_{d,n} \in l_d\},$$

and let

$$P_d = P(\tau_{d,l_d} < \infty)$$

be the returning probability to l_d.

Theorem A.1. *For P_d defined above, we have*

$$\lim_{d\to\infty} 2dP_d = 1. \tag{A.2}$$

With Theorem A.1, we further look at the probability that a d dimensional simple random walk starting from some point in $\text{Trace}(\overset{\nearrow}{\mathcal{P}})$ will ever return to $\text{Trace}(\overset{\nearrow}{\mathcal{P}})$. Note that for each point

$$x = (x^{(1)}, x^{(2)}, \cdots, x^{(d)}) \in \text{Trace}(\overset{\nearrow}{\mathcal{P}})$$

we must have either $\forall 1 \leq i,j \leq d,\ x^{(i)} = x^{(j)}$ or there must be some $1 \leq k < d$ and $0 \leq n \leq \lfloor N/d \rfloor$ such that

$$x^{(i)} = \begin{cases} n+1 & i \leq k \\ n & i > k. \end{cases}$$

Thus when looking at $\hat{x} = (x^{(1)} - x^{(2)}, x^{(2)} - x^{(3)} \cdots, x^{(d-1)} - x^{(d)})$ we must have either $\hat{x} = 0$ or $\hat{x} = e_{d-1,i}$ for some $i = 1, 2, \cdots, d$. In this appendix, we will use the notation $e_{d-1,0} = 0$ and let $D_{d-1} = \{e_{d-1,i} : i = 0, 1, \cdots, d-1\} \subset \mathbb{Z}^d$. One can immediately see that when simple random walk $X_{d,n}$ starting from some point in $\text{Trace}(\overset{\nearrow}{\mathcal{P}})$ returns to $\text{Trace}(\overset{\nearrow}{\mathcal{P}})$, we must have that the corresponding non simple random walk $\hat{X}_{d-1,n}$ starting from D_{d-1} returns to D_{d-1}. Thus for any simple random walk $X_{d,n}$ starting at 0, define the stopping times $T_{d,0} = 0$

$$T_{d,1} = \inf\left\{n \geq 1 :\ X_{d,n} \in \text{Trace}(\overset{\nearrow}{\mathcal{P}})\right\},$$

and

$$T_{d,k} = \inf\left\{n \geq T_{d,k-1} :\ X_{d,n} \in \text{Trace}(\overset{\nearrow}{\mathcal{P}})\right\}$$

for all $k \geq 2$ with the convention $\inf\{n \geq \infty\} = \infty$. And for $\hat{X}_{d-1,n}$ also starting at 0, and any $0 \leq i,j \leq d-1$, define the stopping time

$$T_{d-1}^{(i,j)} = \inf\{n \geq 1 : \hat{X}_{d-1,n} = e_{d-1,j} - e_{d-1,i}\}.$$

Then it is easy to see that for any $k \geq 0$

$$P(T_{d,k+1} < \infty | T_{d,k} < \infty) \leq \sup_{0\leq i\leq d-1} P\left(\inf_{0\leq j\leq d-1}\left\{T_{d-1}^{(i,j)}\right\} < \infty\right). \tag{A.3}$$

With basically similar but more complicated technique as in the proof of Theorem A.1 we have

Theorem A.2. *There is a $C < \infty$ such that for all $d \geq 4$,*

$$\sup_{0\leq i\leq d-1} P\left(\inf_{0\leq j\leq d-1}\left\{T_{d-1}^{(i,j)}\right\} < \infty\right) \leq \frac{C}{d}. \tag{A.4}$$

With Theorem A.2, the proof of Theorem 1.7 is imminent.

Proof of Theorem 1.7. With (A.3) and (A.4), we can immediately have

$$P\big(\overset{\nearrow}{\mathcal{P}} \subset \mathrm{Trace}(X_0, X_1 \cdots)\big) \le P(T_{d,N} < \infty)$$

while

$$P_x(T_{d,N} < \infty) \le \left[\sup_{0\le i\le d-1} P\left(\inf_{0\le j\le d}\left\{T_{d-1}^{(i,j)}\right\} < \infty\right)\right]^N \le \left(\frac{C}{d}\right)^N.$$

And the proof of Theorem 1.7 is complete. □

A.2 Useful facts from calculus

In this section, we list some very basic but useful facts from calculus that we are going to use later in the proof.
① For any function $f(x) \in C(\mathbb{R})$ and any $a \in \mathbb{R}$,

$$\int_a^{a+2\pi} f\big(\cos(x), \sin(x)\big)dx = \int_0^{2\pi} f\big(\cos(x), \sin(x)\big)dx. \tag{A.5}$$

② for any nonnegative integers m, n and

$$C_{m,n} = \int_0^{2\pi} \cos^m(x)\sin^n(x)dx$$

we have $C_{m,n} = 0$ if at least one of m and n is odd.
③ There is a $c > 0$ such that $1 - \cos(x) \ge cx^2$ for all $x \in [-3\pi/2, 3\pi/2]$.
④ There is some $c_1 > 0$ such that within $[-c_1, c_1]$,

$$e^x \le 1 + x + x^2.$$

⑤ For any $x > 0$, $\log(1 + x) \le x$.
⑥ With ②, we can also have that for any $n \in \mathbb{Z}^+$, integers $k_1, k_2, \cdots, k_n \ge 0$ and any $a_1, a_2, \cdots, a_n \in R$, suppose

$$K = \sum_{i=1}^n k_n$$

is a odd number. We always have

$$\int_{-\pi}^{\pi} \prod_{i=1}^n \cos^{k_i}(\theta - a_i)d\theta = 0.$$

A.3 Returning probability to the diagonal line

In this section we prove Theorem A.1. Recalling that

$$\hat{X}_{d-1,n} = \left(X_{d,n}^1 - X_{d,n}^2, X_{d,n}^2 - X_{d,n}^3, \cdots, X_{d,n}^d - X_{d,n}^{d-1}\right)$$

we have $X_{d,n} \in l_d$ if and only if $X_{d,n}^1 = X_{d,n}^2 = \cdots = X_{d,n}^d$, which in turns is equivalent to $\hat{X}_{d-1,n} = 0$. And for the new process $\hat{X}_{d-1,n}$, one can easily check that it also forms a

$d-1$ dimensional random walk with transition probability

$$\begin{aligned}
&P(\hat{X}_{d-1,1} = \pm e_{d-1,1}) = \frac{1}{2d} \\
&P(\hat{X}_{d-1,1} = \pm(e_{d-1,1} - e_{d-1,2})) = \frac{1}{2d} \\
&P(\hat{X}_{d-1,1} = \pm(e_{d-1,2} - e_{d-1,3})) = \frac{1}{2d} \\
&\qquad\qquad \vdots \\
&P(\hat{X}_{d-1,1} = \pm(e_{d-1,d-2} - e_{d-1,d-1})) = \frac{1}{2d} \\
&P(\hat{X}_{d-1,1} = \pm e_{d-1,d-1}) = \frac{1}{2d}
\end{aligned}$$

so that $\hat{X}_{d-1,n}$ also forms a finite range symmetric random walk. Moreover, the characteristic function of the increment of $\hat{X}_{d-1,n}$ is given by

$$\hat{\phi}_{d-1}(\theta) = \frac{1}{d}\left(\cos(\theta_1) + \sum_{i=1}^{d-2}\cos(\theta_{i+1} - \theta_i) + \cos(\theta_{d-1})\right). \tag{A.6}$$

And we also have

$$\hat{\tau}_{d-1,0} = \inf\{n > 1 : \hat{X}_{d-1,n} = 0\}$$

together with

$$P(\hat{\tau}_{d-1,0} < \infty) = 1 - \hat{G}_{d-1}^{-1}(0)$$

where $\hat{G}_{d-1}(\cdot)$ is the Green's function for $\hat{X}_{d-1,n}$. I.e.,

$$\hat{G}_{d-1}(0) = \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \frac{1}{1 - \hat{\phi}_{d-1}(\theta)} d\theta. \tag{A.7}$$

Then we only need to show that

$$\lim_{d\to\infty} 2d[\hat{G}_{d-1}(0) - 1] = 1. \tag{A.8}$$

Moreover, using exactly the same embedded random walk argument as in Lemma 1 of [11] on $X_{d,n}$ and τ_{d,l_d}, one can immediately have $P_{d+1} \le P_d$, which is also equivalent to $\hat{G}_d(0) \le \hat{G}_{d-1}(0)$. So in order to show (A.8), we can without loss of generality concentrate on even d's.

Since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \frac{x^4}{1-x}$$

for all $x \ne 1$, we have

$$\begin{aligned}
\hat{G}_{d-1}(0) = 1 &+ \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \hat{\phi}_{d-1}(\theta) d\theta \\
&+ \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \hat{\phi}_{d-1}^2(\theta) d\theta \\
&+ \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \hat{\phi}_{d-1}^3(\theta) d\theta \\
&+ \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \frac{\hat{\phi}_{d-1}^4(\theta)}{1 - \hat{\phi}_{d-1}(\theta)} d\theta.
\end{aligned} \tag{A.9}$$

Note that by ① and ②, for any $i = 1, \cdots, d-2$,

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_{i+1} - \theta_i) d\theta \\
&= \left(\frac{1}{2\pi}\right)^{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos(\theta_{i+1} - \theta_i) d\theta_{i+1} d\theta_i \\
&= \left(\frac{1}{2\pi}\right)^{2} \int_{-\pi}^{\pi} \int_{-\pi-\theta_i}^{\pi-\theta_i} \cos(\theta) d\theta d\theta_i \\
&= \left(\frac{1}{2\pi}\right)^{2} \int_{-\pi}^{\pi} 0 d\theta_i = 0,
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos^2(\theta_{i+1} - \theta_i) d\theta \\
&= \left(\frac{1}{2\pi}\right)^{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos^2(\theta_{i+1} - \theta_i) d\theta_{i+1} d\theta_i \\
&= \left(\frac{1}{2\pi}\right)^{2} \int_{-\pi}^{\pi} \int_{-\pi-\theta_i}^{\pi-\theta_i} \cos^2(\theta) d\theta d\theta_i \\
&= \left(\frac{1}{2\pi}\right)^{2} \int_{-\pi}^{\pi} \pi d\theta_i = \frac{1}{2},
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos^3(\theta_{i+1} - \theta_i) d\theta \\
&= \left(\frac{1}{2\pi}\right)^{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos^3(\theta_{i+1} - \theta_i) d\theta_{i+1} d\theta_i \\
&= \left(\frac{1}{2\pi}\right)^{2} \int_{-\pi}^{\pi} \int_{-\pi-\theta_i}^{\pi-\theta_i} \cos^3(\theta) d\theta d\theta_i \\
&= \left(\frac{1}{2\pi}\right)^{2} \int_{-\pi}^{\pi} 0 d\theta_i = 0.
\end{aligned}$$

And by ⑥, for any $1 \le i < j \le d-2$,

$$\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_{i+1} - \theta_i) \cos(\theta_{j+1} - \theta_j) d\theta = 0$$

since there has to be one index within $\{i, i+1, j, j+1\}$ with multiplicity 1. Similar, one also has for each $1 \le i \le d-2$

$$\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_{i+1} - \theta_i) \cos(\theta_1) d\theta = \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_{i+1} - \theta_i) \cos(\theta_d) d\theta = 0.$$

Similarly, by ⑥, for any $1 \le i < j \le d-2$,

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_{i+1} - \theta_i) \cos(\theta_{j+1} - \theta_j) \cos(\theta_1) d\theta \\
&= \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_{i+1} - \theta_i) \cos(\theta_{j+1} - \theta_j) \cos(\theta_d) d\theta \\
&= \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos^2(\theta_{i+1} - \theta_i) \cos(\theta_{j+1} - \theta_j) d\theta \\
&= \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_{i+1} - \theta_i) \cos^2(\theta_{j+1} - \theta_j) d\theta = 0,
\end{aligned}$$

while for all $1 \le i \le d-2$

$$\begin{aligned} &\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos^2(\theta_{i+1}-\theta_i)\cos(\theta_1)d\theta \\ =&\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos^2(\theta_{i+1}-\theta_i)\cos(\theta_d)d\theta = 0. \end{aligned}$$

Also, for $1 \le i < j < k \le d-2$

$$\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_{i+1}-\theta_i)\cos(\theta_{j+1}-\theta_j)\cos(\theta_{k+1}-\theta_k)d\theta = 0.$$

Thus one can see that the first 3 integration terms in (A.9) satisfy the followings:

$$\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \hat{\phi}_{d-1}(\theta)d\theta = \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \hat{\phi}^3_{d-1}(\theta)d\theta = 0$$

and

$$\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \hat{\phi}^2_{d-1}(\theta)d\theta = \frac{1}{2d}.$$

And we have

$$\hat{G}_{d-1}(0) = 1 + \frac{1}{2d} + \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \frac{\hat{\phi}^4_{d-1}(\theta)}{1-\hat{\phi}_{d-1}(\theta)}d\theta. \tag{A.10}$$

And we only need to show that for sufficiently large even d

$$\hat{\mathcal{E}}_{d-1} = \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \frac{\hat{\phi}^4_{d-1}(\theta)}{1-\hat{\phi}_{d-1}(\theta)}d\theta = o(d^{-1}). \tag{A.11}$$

To show (A.11), we rewrite the integral above into the expectation of some function of a sequence of i.i.d. random variables. Let $\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_{d-1}$ be i.i.d. uniform random variables on $[-\pi, \pi]$, we can define

$$\hat{Y}_{d-1} = \frac{1}{d}\left(\cos(\hat{X}_1) + \sum_{i=1}^{d-2}\cos(\hat{X}_{i+1}-\hat{X}_i) + \cos(\hat{X}_{d-1})\right) \in [-1,1]$$

and

$$\hat{Z}_{d-1} = \begin{cases} \dfrac{\hat{Y}^4_{d-1}}{1-\hat{Y}_{d-1}}, & \hat{Y}_{d-1} < 1 \\ 0, & \hat{Y}_{d-1} = 1. \end{cases}$$

Then according to our construction and the definition of $\hat{\mathcal{E}}_d$, we have

$$\hat{\mathcal{E}}_{d-1} = E[\hat{Z}_{d-1}]. \tag{A.12}$$

Again let event $\hat{A}_{d-1} = \{|\hat{Y}_{d-1}| \le d^{-0.4}\}$, then for any $d \ge 6$,

$$E[\hat{Z}_{d-1}] \le \frac{d^{-1.6}}{1-d^{-0.4}}P(\hat{A}_{d-1}) + E[\hat{Z}_{d-1}\mathbb{1}_{\hat{A}^c_{d-1}}] \le 2d^{-1.6} + E[\hat{Z}_{d-1}\mathbb{1}_{\hat{A}^c_{d-1}}].$$

Then let

$$\hat{B}_{d-1} = \left\{\sqrt{\hat{X}^2_1 + \hat{X}^2_2 + \cdots + \hat{X}^2_{d-1}} \le \frac{1}{d}\right\}.$$

We can further have

$$\begin{aligned} E[\hat{Z}_{d-1}] &\leq 2d^{-1.6} + E[\hat{Z}_{d-1}\mathbb{1}_{\hat{A}^c_{d-1}\cap\hat{B}^c_{d-1}}] + E[\hat{Z}_{d-1}\mathbb{1}_{A^c_{d-1}\cap\hat{B}_{d-1}}] \\ &\leq 2d^{-1.6} + P(\hat{A}^c_{d-1}) \max_{\omega\in\hat{B}^c_{d-1}} \{\hat{Z}_{d-1}(\omega)\} + E[\hat{Z}_{d-1}\mathbb{1}_{\hat{B}_{d-1}}]. \end{aligned} \tag{A.13}$$

To control the third term in (A.13), note that for any $d \geq 3$, and any $i = 1, 2, \cdots, d-2$, within the event $\hat{B}_{d-1}$,

$$|X_i - X_j| \leq \frac{2}{d} \leq \pi.$$

Thus within the event $\hat{B}_{d-1} \cap \{\hat{Y}_{d-1} \neq 1\}$, we have by ③

$$\hat{Z}_{d-1} \leq \frac{1}{1-\hat{Y}_{d-1}} \leq \frac{d}{c\left(\hat{X}_1^2 + \sum_{i=1}^{d-2} |\hat{X}_{i+1} - \hat{X}_i|^2 + \hat{X}_{d-1}^2\right)}. \tag{A.14}$$

Moreover, for any $(x_1, x_2, \cdots, x_{d-1}) \in \mathbb{R}^d$, we have

$$x_1^2 + \sum_{i=1}^{d-2} |x_{i+1} - x_i|^2 + x_{d-1}^2 \geq \sigma(d-1,1)^2 \left(\sum_{i=1}^{d-1} x_i^2\right)$$

where $\sigma(d-1,1)$ is the smallest singular value of $d-1$ by $d-1$ Jordan block with $\lambda = 1$. In [5] it has been proved that

$$\sigma(d-1,1) \geq \frac{1}{d-1} \geq \frac{1}{d}.$$

Thus we have

$$x_1^2 + \sum_{i=1}^{d-2} |x_{i+1} - x_i|^2 + x_{d-1}^2 \geq d^{-2} \left(\sum_{i=1}^{d-1} x_i^2\right). \tag{A.15}$$

Combining (A.14) and (A.15) gives us

$$\hat{Z}_{d-1} \leq \frac{d^3}{c\sum_{i=1}^{d-1} \hat{X}_i^2} \tag{A.16}$$

which implies that

$$E[\hat{Z}_{d-1}\mathbb{1}_{\hat{B}_{d-1}}] \leq \left(\frac{1}{2\pi}\right)^{d-1} \int_{B_{2,d-1}(0,1/d)} \frac{d^3}{c\sum_{i=1}^{d-1} x_i^2} dx_1 dx_2 \cdots dx_{d-1}, \tag{A.17}$$

where $B_{2,d-1}(0,1/d)$ is the L_2 ball in $\mathbb{R}^{d-1}$ centered at 0 with radius $1/d$. For the integral in (A.17), use the d dimensional spherical coordinates

$$\begin{aligned} x_1 &= r\cos(\theta_1) \\ x_2 &= r\sin(\theta_1)\cos(\theta_2) \\ x_3 &= r\sin(\theta_1)\sin(\theta_2)\cos(\theta_3) \\ &. \\ &. \\ &. \\ x_{d-2} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{d-3})\cos(\theta_{d-2}) \\ x_{d-1} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{d-3})\sin(\theta_{d-2}) \end{aligned}$$

where $r \geq 0$, $\theta_i \in [0, \pi]$ for $i = 1, 2, \cdots, d-3$, and $\theta_{d-2} \in [0, 2\pi]$. Then we have

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^{d-1} \int_{B_{2,d-1}(0,1/d)} \frac{d^3}{c\sum_{i=1}^{d-1} x_i^2} dx_1 dx_2 \cdots dx_{d-1} \\
&= \frac{d^3}{c(2\pi)^{d-1}} \int_{(0,1/d)\times[0,\pi]^{d-3}\times[0,2\pi]} r^{d-4} \prod_{i=1}^{d-3} \sin^{d-2-i}(\theta_i) dr d\theta_1 d\theta_2 \cdots d\theta_{d-2} \\
&\leq \frac{d^3}{c(2\pi)^{d-1}} \int_{(0,1/d)\times[0,\pi]^{d-3}\times[0,2\pi]} r^{d-4} dr d\theta_1 d\theta_2 \cdots d\theta_{d-2} \\
&\leq \frac{d^3}{c2^{d-1}} \int_0^{1/d} r^{d-4} dr = \frac{1}{c2^{d-1}} \cdot \frac{d^3}{d-3} \cdot d^{3-d} = o(d^{-1}).
\end{aligned} \tag{A.18}$$

Combining (A.17) and (A.18), we have

$$E[Z_d \mathbb{1}_{B_d}] \leq \frac{1}{c2^{d-1}} \cdot \frac{d^3}{d-3} \cdot d^{3-d} = o(d^{-1}). \tag{A.19}$$

Then for the second term $P(\hat{A}_{d-1}^c) \max_{\omega \in \hat{B}_{d-1}^c} \{\hat{Z}_{d-1}(\omega)\}$, we first control the probability $P(\hat{A}_{d-1}^c)$ for sufficiently large even number $d = 2n$. Note that

$$\hat{Y}_{d-1} \leq \frac{2}{d} + \frac{1}{d} \sum_{i=1}^{d-2} \cos(\hat{X}_{i+1} - \hat{X}_i)$$

and that

$$\hat{Y}_{d-1} \geq -\frac{2}{d} + \frac{1}{d} \sum_{i=1}^{d-2} \cos(\hat{X}_{i+1} - \hat{X}_i).$$

So we have for sufficiently large even number $d = 2n$

$$P(\hat{Y}_{d-1} \geq d^{-0.4}) \leq P\left(\frac{1}{d} \sum_{i=1}^{d-2} \cos(\hat{X}_{i+1} - \hat{X}_i) \geq \frac{d^{-0.4}}{2}\right)$$

and

$$P(\hat{Y}_{d-1} \leq -d^{-0.4}) \leq P\left(\frac{1}{d} \sum_{i=1}^{d-2} \cos(\hat{X}_{i+1} - \hat{X}_i) \leq \frac{d^{-0.4}}{2}\right).$$

Moreover note that for $d = 2n$ we have

$$\frac{1}{d} \sum_{i=1}^{d-2} \cos(\hat{X}_{i+1} - \hat{X}_i) = \frac{n-1}{2n} (\hat{Y}_{1,d-1} + \hat{Y}_{2,d-1})$$

where

$$\hat{Y}_{1,d-1} = \frac{1}{n-1} \sum_{i=1}^{n-1} \cos(\hat{X}_{2i+1} - \hat{X}_{2i})$$

and

$$\hat{Y}_{2,d-1} = \frac{1}{n-1} \sum_{i=1}^{n-1} \cos(\hat{X}_{2i} - \hat{X}_{2i+1}).$$

Noting that $\hat{Y}_{1,d-1}$ and $\hat{Y}_{2,d-1}$ are again sampled means of i.i.d. random variables with expectation 0 and variance $1/2$. Although now we have $\hat{Y}_{1,d-1}$ and $\hat{Y}_{2,d-1}$ are correlated, we can still have the upper bound

$$P\left(\frac{1}{d} \sum_{i=1}^{d-2} \cos(\hat{X}_{i+1} - \hat{X}_i) \geq \frac{d^{-0.4}}{2}\right) \leq P(\hat{Y}_{1,d-1} \geq \frac{d^{-0.4}}{2}) + P(\hat{Y}_{2,d-1} \geq \frac{d^{-0.4}}{2})$$

and

$$P\left(\frac{1}{d}\sum_{i=1}^{d-2}\cos(\hat{X}_{i+1}-\hat{X}_i)\le -\frac{d^{-0.4}}{2}\right)\le P(\hat{Y}_{1,d-1}\le -\frac{d^{-0.4}}{2})+P(\hat{Y}_{2,d-1}\le -\frac{d^{-0.4}}{2}).$$

Apply Cramér's Theorem on $\hat{Y}_{1,d-1}$ and $\hat{Y}_{2,d-1}$, we have that there is some $u, U\in(0,\infty)$ (actually we can use $u=1/16$ and $U=2$) such that

$$P(\hat{A}^c_{d-1})\le U\exp(-ud^{0.2}). \tag{A.20}$$

Lastly for $\max_{\omega\in\hat{B}^c_{d-1}}\{\hat{Z}_{d-1}(\omega)\}$, note that the range of $\hat{X}_{i+1}-\hat{X}_i$ is $[-2\pi,2\pi]$ which is no longer a subset of $[-3\pi/2,3\pi/2]$, we will not be able to use ③ directly to find an upper bound. to overcome this issue, we have the following lemma:

Lemma A.3. *For any d, consider the following two subsets of $\mathbb{R}^{d-1}$:*

$$D_1(d)=[-\pi,\pi]^{d-1}\cap\left\{\frac{1-\cos(x_1)}{d}+\sum_{i=1}^{d-2}\frac{1-\cos(x_{i+1}-x_i)}{d}+\frac{1-\cos(x_{d-1})}{d}\le d^{-7}\right\}$$

and

$$D_2(d)=\left\{(x_1,\cdots,x_{d-1}):\ |x_i|\le\frac{i}{\sqrt{c}d^3},\ \forall i=1,2,\cdots,d-1\right\}$$

where c is the constant in ③. Then there is some $d_0<\infty$ such that for all $d\ge d_0$, $D_1(d)\subseteq D_2(d)$.

Proof. Let d_0 be a positive integer such that $\sqrt{c}d_0^2>1$. Then for any $d\ge d_0$ and any $(x_1,\cdots,x_{d-1})\in D_1(d)$, by the definition of $D_1(d)$ and the fact that $x_1\in[-\pi,\pi]$, we must have

$$\frac{cx_1^2}{d}\le\frac{1-\cos(x_1)}{d}\le d^{-7}$$

which implies that

$$|x_1|\le\frac{1}{\sqrt{c}d^3}. \tag{A.21}$$

Now suppose there is a $(x_1,\cdots,x_{d-1})\in D_1(d)\cap D_2(d)^c$. Let $k=\inf\{i:\ |x_i|>\frac{i}{\sqrt{c}d^3}\}$. Then (A.21) ensures that $k>1$. Then for x_{k-1},

$$|x_{k-1}|\le\frac{k-1}{\sqrt{c}d^3}\le\frac{d}{\sqrt{c}d^3}\le\frac{1}{\sqrt{c}d_0^2}<1.$$

Thus we must have $|x_{k-1}-x_k|\le 3\pi/2$, which gives that

$$\frac{1-\cos(x_k-x_{k-1})}{d}\ge\frac{c}{d}|x_k-x_{k-1}|^2\ge\frac{c}{d}(|x_k|-|x_{k-1}|)^2>\frac{1}{d^7}.$$

And now we have a contradiction. □

Moreover, we have that for any $(x_1,\cdots,x_{d-1})\in D_2(d)$,

$$x_1^2+\cdots+x_{d-1}^2\le\frac{\sum_{i=1}^{d-1}i^2}{cd^6}=O(d^{-3})=o(d^{-2}).$$

Thus there is another $d_1<\infty$ such that for all $d\ge d_1$,

$$D_1(d)\subset D_2(d)\subset B_{2,d}(0,1/d). \tag{A.22}$$

This means for any $d \ge d_1$, and any $(x_1, \cdots, x_{d-1}) \in B_{2,d}(0, 1/d)^c$,

$$\frac{1}{1 - \frac{1}{d}\left(\cos(x_1) + \sum_{i=1}^{d-2} \cos(x_{i+1} - x_i) + \cos(x_{d-1})\right)} \le d^7, \tag{A.23}$$

which gives us

$$\max_{\omega \in \hat{B}_{d-1}^c} \{\hat{Z}_{d-1}(\omega)\} \le d^7. \tag{A.24}$$

Thus combine (A.20) and (A.24) we finally have

$$P(\hat{A}_{d-1}^c) \max_{\omega \in \hat{B}_{d-1}^c} \{\hat{Z}_{d-1}(\omega)\} \le U d^7 \exp(-u d^{0.2}) = o(d^{-1}) \tag{A.25}$$

and the proof of Theorem A.1 is complete. □

A.4 Proof of Theorem A.2

With the asymptotic of the return of $\{\hat{X}_{d-1,n}\}_{n=0}^\infty$ obtained in the previous section, we are able to use a similar but more complicate argument to show the same asymptotic for the probability that $\{\hat{X}_{d-1,n}\}_{n=0}^\infty$ returns to the set D_{d-1}. First using again exactly the same embedded random walk argument as in Lemma 1 of [11], it is easy to note that each time $\{\hat{X}_{d-1,n}\}_{n=0}^\infty$ returns to D_{d-1} is also a time when $\{\hat{X}_{d-2,n}\}_{n=0}^\infty$, the embedded Markov chain tracking the changes of the first $d-2$ coordinates of $\{\hat{X}_{d-1,n}\}_{n=0}^\infty$, which is also a $d-2$ dimensional version of the non simple random walk of interest, returns to D_{d-2}. This implies

$$\sup_{0 \le i \le d-1} P\left(\inf_{0 \le j \le d-1} \left\{T_{d-1}^{(i,j)}\right\} < \infty\right) \le \sup_{0 \le i \le d-2} P\left(\inf_{0 \le j \le d-2} \left\{T_{d-2}^{(i,j)}\right\} < \infty\right)$$

and we can without loss of generality again concentrate on even numbers of d's. Then for each i, one can immediately have

$$P\left(\inf_{0 \le j \le d-1} \left\{T_{d-1}^{(i,j)}\right\} < \infty\right) \le \sum_{j=0}^{d-1} P\big(T_{d-1}^{(i,j)} < \infty\big).$$

Thus, in order to prove Theorem A.2, it is sufficient to show that for all sufficiently large even d's, there is a $C < \infty$ such that for any $0 \le i \le d-1$

$$\sum_{j=0}^{d-1} P\big(T_{d-1}^{(i,j)} < \infty\big) < \frac{C}{d}. \tag{A.26}$$

Then for any $0 \le i \ne j \le d-1$, by strong Markov property

$$P\big(T_{d-1}^{(i,j)} < \infty\big) = \frac{\hat{G}_{d-1}(e_j - e_i)}{\hat{G}_{d-1}(0)}.$$

and

$$P\big(T_{d-1}^{(i,i)} < \infty\big) = \frac{\hat{G}_{d-1}(0) - 1}{\hat{G}_{d-1}(0)}.$$

In Theorem A.1, we have already proved that $\hat{G}_{d-1}(0) = 1 + (2d)^{-1} + o(d^{-1})$. Thus now it is sufficient to show that for any i

$$\sum_{j=0}^{d-1} \hat{G}_{d-1}(e_j - e_i) - 1 \le \frac{C}{d}. \tag{A.27}$$

For any $0 \le i, j \le d-1$, we have

$$\hat{G}_{d-1}(e_j - e_i) = \hat{G}_{d-1}(e_i - e_j) = \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \frac{\cos(\theta_j - \theta_i)}{1 - \hat{\phi}_{d-1}(\theta)} d\theta. \tag{A.28}$$

Thus, we will concentrate on controlling

$$G_{d-1}^{(i,j)} = \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \frac{\cos(\theta_j - \theta_i)}{1 - \hat{\phi}_{d-1}(\theta)} d\theta$$

with $0 \le i < j \le d-1$. Using the same technique as in the proof of Theorem A.1, and noting that

$$\int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i) d\theta = 0$$

we first have

$$\begin{aligned} G_{d-1}^{(i,j)} = & \left(\frac{1}{2\pi}\right)^{d-1} \sum_{p=1}^{5} \left(\int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i) \hat{\phi}_{d-1}^{p}(\theta) d\theta \right) \\ & + \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \frac{\cos(\theta_j - \theta_i) \hat{\phi}_{d-1}^{6}(\theta)}{1 - \hat{\phi}_{d-1}(\theta)} d\theta \end{aligned} \tag{A.29}$$

and we call

$$\hat{\mathcal{E}}_d^{(i,j)} = \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \frac{\cos(\theta_j - \theta_i) \hat{\phi}_{d-1}^{6}(\theta)}{1 - \hat{\phi}_{d-1}(\theta)} d\theta \tag{A.30}$$

to be the tail term. For any $0 \le i \ne j \le d-1$, let $d(i,j)$ be their distance up to mod(d). I.e.,

$$d(i,j) = \min\{|j-i|, d - |j-i|\} \ge 1.$$

The reason we want to have the distance under mod(d) is that our non simple random walk $\{\hat{X}_{d-1,n}\}_{n=0}^{\infty}$ has some "periodic boundary condition" where we need one transition to move from $e_{d-1,d-1}$ to $e_{d-1,0}$. Then we have the following lemma which implies that for all but a finite number of (i,j)'s, the tail term $\hat{\mathcal{E}}_d^{(i,j)}$ is actually all we get for $G_{d-1}^{(i,j)}$.

Lemma A.4. *For any $k \in \mathbb{Z}^+$ and any $0 \le i \ne j \le d-1$ such that $d(i,j) > k$,*

$$\int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i) \hat{\phi}_{d-1}^{k}(\theta) d\theta = 0. \tag{A.31}$$

Proof. By symmetry we can without generality assume that $j > i$. Recalling that

$$\hat{\phi}_{d-1}(\theta) = \frac{1}{d} \left(\cos(\theta_1) + \sum_{i=1}^{d-2} \cos(\theta_{i+1} - \theta_i) + \cos(\theta_{d-1}) \right),$$

we have

$$\hat{\phi}_{d-1}^{k}(\theta) = \frac{1}{d^k} \sum_{0 \le i_1, i_2, \cdots, i_k \le d-1} \prod_{h=1}^{k} \cos(\theta_{i_h} - \theta_{i_h+1}),$$

where we use the convention that $\theta_0 = \theta_d = 0$. For each term in the summation above, it is easy to see that there is some nonnegative integers $k_0, \cdots, k_{d-1}$ with $\sum_{h=0}^{d-1} k_h = k$ such that we can rewrite the term as

$$\prod_{h=0}^{d-1} \cos^{k_h}(\theta_h - \theta_{h+1}). \tag{A.32}$$

Thus, it is sufficient to show that for any nonnegative integers $k_0, \cdots, k_{d-1}$ with $\sum_{h=0}^{d-1} k_h = k$

$$\int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i) \left(\prod_{h=0}^{d-1} \cos^{k_h}(\theta_h - \theta_{h+1}) \right) d\theta = 0. \tag{A.33}$$

First, if $i = 0$ then we have $j > k$ and $d - j > k$. Thus we can separate the product in (A.32) as

$$\prod_{h=0}^{d-1} \cos^{k_h}(\theta_h - \theta_{h+1}) = \Pi[0 : j-1] \cdot \Pi[j : d-1]$$

where

$$\Pi[0 : j-1] = \prod_{h=0}^{j-1} \cos^{k_h}(\theta_h - \theta_{h+1}),\ \Pi[j : d-1] = \prod_{h=j}^{d-1} \cos^{k_h}(\theta_h - \theta_{h+1}).$$

Thus $\Pi[0 : j-1]$ is a product of j terms while $\Pi[j : d-1]$ is a product of $d-j$ terms. Note that

$$\begin{aligned}
&\cos(\theta_j) \left(\prod_{h=0}^{d-1} \cos^{k_h}(\theta_h - \theta_{h+1}) \right) \\
&\quad = \cos(\theta_j) \cos^{k_{j-1}}(\theta_j - \theta_{j-1}) \cos^{k_j}(\theta_j - \theta_{j+1}) \left(\prod_{h \in \{0,\cdots,d-1\} \setminus \{j-1,j\}} \cos^{k_h}(\theta_h - \theta_{h+1}) \right).
\end{aligned}$$

If $k_{j-1} + k_j$ is an even number, integrate over θ_j and ⑥ gives us (A.33). If $k_{j-1} + k_j$ is odd, without loss of generality we can assume k_{j-1} is odd. Noting that

$$\sum_{h=0}^{j-1} k_h \le k < j,$$

by the pigeon hole principle we must have at least one of those k_h's to be zero, which is even. Thus, let $h_0 = \sup_{h \le j-1}\{k_h \text{ is even}\}$. Then $h_0 \in [0, j-2]$, where we use the standard convention that $\sup\{\emptyset\} = -\infty$ and $\inf\{\emptyset\} = \infty$. By definition k_{h_0+1} is odd, and thus

$$\begin{aligned}
&\cos(\theta_j) \left(\prod_{h=0}^{d-1} \cos^{k_h}(\theta_h - \theta_{h+1}) \right) \\
&= \cos^{k_{h_0}}(\theta_{h_0+1} - \theta_{h_0}) \cos^{k_{h_0+1}}(\theta_{h_0+1} - \theta_{h_0+2}) \left(\cos(\theta_j) \prod_{h \in \{0,\cdots,d-1\} \setminus \{h_0, h_0+1\}} \cos^{k_h}(\theta_h - \theta_{h+1}) \right).
\end{aligned}$$

Note that $k_{h_0} + k_{h_0+1}$ is odd, so we integrate over θ_{h_0+1} and ⑥ again gives us (A.33).

Symmetrically, if we have k_j is odd, then we can look at $h_1 = \inf_{h \ge j}\{k_h \text{ is even}\}$ and have $h_1 \in [j+1, d-1]$. This in turn implies that $k_{h_1} + k_{h_1-1}$ is odd, so we integrate over θ_{h_1} and use ⑥. We use the same argument in the following discussions.

Similarly if $i > 0$, with $d(i,j) > k$ implying $j - i > k$ as well as $d + i - j > k$, we can also have

$$\prod_{h=0}^{d-1} \cos^{k_h}(\theta_h - \theta_{h+1}) = \Pi[0 : i-1] \cdot \Pi[i : j-1] \cdot \Pi[j : d-1]$$

where

$$\Pi[0:i-1]=\prod_{h=0}^{i-1}\cos^{k_h}(\theta_h-\theta_{h+1})$$
$$\Pi[i:j-1]=\prod_{h=i}^{j-1}\cos^{k_h}(\theta_h-\theta_{h+1})$$
$$\Pi[j:d-1]=\prod_{h=j}^{d-1}\cos^{k_h}(\theta_h-\theta_{h+1}).$$

And again note that

$$\cos(\theta_j-\theta_i)\left(\prod_{h=0}^{d-1}\cos^{k_h}(\theta_h-\theta_{h+1})\right)$$
$$=\cos(\theta_j-\theta_i)\cos^{k_{j-1}}(\theta_j-\theta_{j-1})\cos^{k_j}(\theta_j-\theta_{j+1})\left(\prod_{h\in\{0,\cdots,d-1\}\setminus\{j-1,j\}}\cos^{k_h}(\theta_h-\theta_{h+1})\right)$$

and that

$$\cos(\theta_j-\theta_i)\left(\prod_{h=0}^{d-1}\cos^{k_h}(\theta_h-\theta_{h+1})\right)$$
$$=\cos(\theta_i-\theta_j)\cos^{k_{i-1}}(\theta_i-\theta_{i-1})\cos^{k_i}(\theta_i-\theta_{i+1})\left(\prod_{h\in\{0,\cdots,d-1\}\setminus\{i-1,i\}}\cos^{k_h}(\theta_h-\theta_{h+1})\right).$$

So if either $k_{i-1}+k_i$ or $k_{j-1}+k_j$ is a even number,⑥ again gives us (A.33).

Now suppose both $k_{i-1}+k_i$ and $k_{j-1}+k_j$ are odd. If either k_i or k_{j-1} is odd, we can without loss of generality assume the odd one is k_{j-1}. Note that

$$\sum_{h=i}^{j-1}k_h\le k<j-i.$$

Let $h_0=\sup_{h\le j-1}\{k_h \text{ is even}\}$. Then $h_0\in[i,j-2]$. Then again we have that $k_{h_0}+k_{h_0+1}$ is odd, so we can integrate over θ_{h_0+1} and ⑥ again gives us (A.33).

Otherwise, we must have both k_{i-1} and k_j are odd numbers. Again note that

$$\sum_{h=0}^{i-1}k_h+\sum_{h=j}^{d-1}k_h\le k<d+i-j.$$

At least one of the k_h's above must be 0, and let's say again without loss of generality it is in $[0,i-1]$. Once more let $h_0=\sup_{h\le i-1}\{k_h \text{ is even}\}$. Then $h_0\in[0,i-2]$, and $k_{h_0}+k_{h_0+1}$ is odd so we can once again integrate over θ_{h_0+1} to use ⑥ to gives us (A.33). Combining all the possible situations together, the proof of this lemma is complete. □

With Lemma A.4, one can immediately see that for any $0\le i\le d-1$ and any j such that $d(i,j)\ge 6$,

$$G_{d-1}^{(i,j)}=\hat{\mathcal{E}}_d^{(i,j)}=\left(\frac{1}{2\pi}\right)^{d-1}\int_{[-\pi,\pi]^{d-1}}\frac{\cos(\theta_j-\theta_i)\hat{\phi}_{d-1}^6(\theta)}{1-\hat{\phi}_{d-1}(\theta)}d\theta$$

which immediately implies that

$$\left|G_{d-1}^{(i,j)}\right| \le \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \frac{\hat{\phi}_{d-1}^6(\theta)}{1-\hat{\phi}_{d-1}(\theta)} d\theta. \tag{A.34}$$

Then recalling that in the proof Theorem A.1 we have $\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_{d-1}$ be i.i.d. uniform random variables on $[-\pi,\pi]$ and

$$\hat{Y}_{d-1} = \frac{1}{d}\left(\cos(\hat{X}_1) + \sum_{i=1}^{d-2} \cos(\hat{X}_{i+1} - \hat{X}_i) + \cos(\hat{X}_{d-1})\right) \in [-1,1].$$

And we define

$$\bar{Z}_{d-1} = \begin{cases} \dfrac{\hat{Y}_{d-1}^6}{1-\hat{Y}_{d-1}}, & \hat{Y}_{d-1} < 1 \\ 0, & \hat{Y}_{d-1} = 1. \end{cases}$$

Then again we have for any i,j

$$\hat{\mathcal{E}}_d^{(i,j)} \le E[\bar{Z}_{d-1}]. \tag{A.35}$$

Recall the event $\hat{A}_{d-1} = \{|\hat{Y}_{d-1}| \le d^{-0.4}\}$, then for any $d \ge 6$,

$$E[\bar{Z}_{d-1}] \le \frac{d^{-2.4}}{1-d^{-0.4}} P(\hat{A}_{d-1}) + E[\bar{Z}_{d-1}\mathbb{1}_{\hat{A}_{d-1}^c}] \le 2d^{-1.6} + E[\bar{Z}_{d-1}\mathbb{1}_{\hat{A}_{d-1}^c}].$$

Then recall

$$\hat{B}_{d-1} = \left\{\sqrt{\hat{X}_1^2 + \hat{X}_2^2 + \cdots + \hat{X}_{d-1}^2} \le \frac{1}{d}\right\}.$$

We can similarly have

$$\begin{aligned} E[\bar{Z}_{d-1}] &\le 2d^{-2.4} + E[\bar{Z}_{d-1}\mathbb{1}_{\hat{A}_{d-1}^c \cap \hat{B}_{d-1}^c}] + E[\bar{Z}_{d-1}\mathbb{1}_{A_{d-1}^c \cap \hat{B}_{d-1}}] \\ &\le 2d^{-2.4} + P(\hat{A}_{d-1}^c) \max_{\omega \in \hat{B}_{d-1}^c} \{\bar{Z}_{d-1}(\omega)\} + E[\bar{Z}_{d-1}\mathbb{1}_{\hat{B}_{d-1}}]. \end{aligned} \tag{A.36}$$

Noting that in (A.14)-(A.23) and (A.24), we find upper bounds for $\hat{Z}_{d-1}\mathbb{1}_{\hat{B}_{d-1}}$ and $\max_{\omega \in \hat{B}_{d-1}^c}\{\bar{Z}_{d-1}(\omega)\}$ using $1/(1-\hat{Y}_{d-1})$ which is also an upper bound for the smaller corresponding terms with $\bar{Z}_{d-1}$. Thus (A.19) and (A.25) give us that the second and third terms of (A.36) are also $o(d^{-2.4})$. Which implies there is a $C_1 < \infty$ such that for sufficiently large even number d,

$$\left|G_{d-1}^{(i,j)}\right| \le C_1 d^{-2.4}$$

whenever $d(i,j) \ge 6$, and that

$$\hat{\mathcal{E}}_d^{(i,j)} \le C_1 d^{-2.4}$$

for all $0 \le i,j \le d-1$. Combining the observation here with Lemma A.4, we have for sufficiently large d any i

$$\begin{aligned} \sum_{j=0}^{d-1} G_{d-1}(e_j - e_i) - 1 &= \sum_{j=0}^{d-1} \hat{\mathcal{E}}_d^{(i,j)} + \sum_{j:d(i,j)\le 5} \sum_{p=1}^{5} \left(\frac{1}{2\pi}\right)^{d-1} \left(\int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i)\hat{\phi}_{d-1}^p(\theta) d\theta\right) \\ &\le C_1 d^{-1.4} + \sum_{j:d(i,j)\le 5} \sum_{p=1}^{5} \left(\frac{1}{2\pi}\right)^{d-1} \left(\int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i)\hat{\phi}_{d-1}^p(\theta) d\theta\right). \end{aligned} \tag{A.37}$$

Note that for any $n > 0$, $|\{j : d(i,j) = n\}| \leq 2$. So the second term in (A.37) is just a finite summation of no more than 55 terms. When $p = 1$, if $d(i,j) = 0$,

$$\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i)\hat{\phi}_{d-1}(\theta)d\theta = \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \hat{\phi}_{d-1}(\theta)d\theta = 0.$$

And if $d(i,j) = 1$,

$$\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i)\hat{\phi}_{d-1}(\theta)d\theta = \frac{1}{4\pi^2 d} \int_{[-\pi,\pi]^2} \cos^2(\theta_j - \theta_i)d\theta_j d\theta_i = \frac{1}{2d}.$$

For $d(i,j) \geq 2$, by Lemma A.4

$$\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i)\hat{\phi}_{d-1}(\theta)d\theta = 0.$$

And for $p \geq 2$ and any i, j

$$\begin{aligned} \left|\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \cos(\theta_j - \theta_i)\hat{\phi}^p_{d-1}(\theta)d\theta\right| &= \left|\left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \hat{\phi}^2_{d-1}(\theta)\cos(\theta_j - \theta_i)\hat{\phi}^{p-2}_{d-1}(\theta)d\theta\right| \\ &\leq \left(\frac{1}{2\pi}\right)^{d-1} \int_{[-\pi,\pi]^{d-1}} \hat{\phi}^2_{d-1}(\theta)d\theta = \frac{1}{2d}. \end{aligned}$$

Thus we have shown that all terms in this finite summation is either 0 or $O(d^{-1})$. Take $C = 28$ and the proof of Theorem 1.7 is complete. □

Remark A.5. It is clear that the upper bound $C = 28$ we find here is not precise since here we only want the right order and are actually having very weak upper bounds for those 55 terms in the summation. Actually, any $C > 3/2$ will be a good upper bound for sufficiently large d. Among the 55 terms in the summation, one can easily see that the term $j = i$, $p = 2$ and the two terms with $d(i,j) = 1$, $p = 1$ are the only ones $\sim d^{-1}$ and each of them is $1/(2d) + o(d^{-1})$. All the other terms are either 0 or $o(d^{-1})$. The calculation is trivial calculus but very tedious, especially for someone who is reading (or writing) this not too short paper.

B

In this appendix we prove that the monotonicity fails when considering covering probability with repetitions.

Proof of Proposition 6.2. To show the first part of Equation (6.3) and (6.4), note that $\{\text{Trace}(\mathcal{P}) \otimes N_{\mathcal{P}} \subseteq \{X_n\}_{n=0}^\infty\}$ is a subset of event $\{\tau_A < \infty\}$. Thus by strong Markov property and symmetry of simple random walk

$$\begin{aligned} &P\left(\text{Trace}(\mathcal{P}) \otimes N_{\mathcal{P}} \subseteq \{X_n\}_{n=0}^\infty\right) \\ &= P_w(\tau_y < \infty, \tau_z < \infty)P(\tau_w = \tau_A) + P_z(\tau_y < \infty, \tau_w < \infty)P(\tau_z = \tau_A) \\ &\ + P_y(\tau_z < \infty, \tau_w < \infty)P(\tau_y = \tau_A) \\ &= 2P_o(\tau_y = \tau_A)[P_o(\tau_y < \tau_w) + P_o(\tau_w < \tau_y)]P_o(\tau_y < \infty) \\ &\ + 2P_o(\tau_w = \tau_A)P_o(\tau_y < \tau_z)P_o(\tau_w < \infty). \end{aligned} \tag{B.1}$$

Similarly, note that $\{\text{Trace}(\mathcal{P}') \otimes N_{\mathcal{P}'} \subseteq \{X_n\}_{n=0}^\infty\}$ is a subset of $\{\tau_{A_1} < \infty\}$, where $A_1 = \mathbb{Z}^3 \setminus \{y, w\}$

$$\begin{aligned} &P\left(\text{Trace}(\mathcal{P}') \otimes N_{\mathcal{P}'} \subseteq \{X_n\}_{n=0}^\infty\right) \\ &= P_y(\tau_y < \infty, \tau_w < \infty)P(\tau_y = \tau_{A_1}) + P_w(\tau_{2,y} < \infty)P(\tau_w = \tau_{A_1}) \\ &= P_o(\tau_y < \tau_w)[P_o(\tau_o < \tau_y) + P_o(\tau_y < \tau_o)]P_o(\tau_y < \infty) \\ &\ + P_o(\tau_w < \tau_y)P_o(\tau_y < \infty)P_o(\tau_0 < \infty) \end{aligned} \tag{B.2}$$

where $\tau_{2,y}$ is defined in (6.2). To calculate the probability we have in (B.1) and (B.2), one may first note that

$$P_o(\tau_y < \infty) = \frac{G(y)}{G(o)},\ P_o(\tau_w < \infty) = \frac{G(w)}{G(o)},$$

where $G(\cdot)$ is the Green's function of 3-dimensional simple random walk. I.e.,

$$G(x) = \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} \frac{1}{1-\phi(\theta)} e^{-iy\cdot\theta} d\theta$$

with

$$\phi(\theta) = \frac{1}{3}[\cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3)].$$

Thus

$$G(o) = \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} \frac{1}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta \approx 1.5153, \tag{B.3}$$

$$G(y) = \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} \frac{\cos(\theta_1)}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta \approx 0.5153, \tag{B.4}$$

$$G(2y) = \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} \frac{\cos(2\theta_1)}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta \approx 0.2563, \tag{B.5}$$

$$G(w) = \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} \frac{\cos(\theta_1+\theta_2)}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta \approx 0.3301, \tag{B.6}$$

$$P_o(\tau_y < \infty) = \frac{\int_{[-\pi,\pi]^3} \frac{\cos(\theta_1)}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta}{\int_{[-\pi,\pi]^3} \frac{1}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta} \approx 0.3401, \tag{B.7}$$

$$P_o(\tau_o < \infty) = P_o(\tau_y < \infty) \approx 0.3401, \tag{B.8}$$

$$P_o(\tau_{2y} < \infty) = \frac{\int_{[-\pi,\pi]^3} \frac{\cos(2\theta_1)}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta}{\int_{[-\pi,\pi]^3} \frac{1}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta} \approx 0.1691, \tag{B.9}$$

and

$$P_o(\tau_w < \infty) = \frac{\int_{[-\pi,\pi]^3} \frac{\cos(\theta_1+\theta_2)}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta}{\int_{[-\pi,\pi]^3} \frac{1}{1-\frac{1}{3}[\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)]} d\theta} \approx 0.2178. \tag{B.10}$$

Then for $A_2 = \mathbb{Z}^3 \setminus \{z\}$, we have

$$P_o(\tau_y < \tau_z) = P_o(\tau_y < \tau_{A_2}) = \frac{G_{A_2}(o,y)}{G_{A_2}(y,y)}$$

where $G_{A_2}(\cdot)$ is the Green's function for set A_2, see Section 4.6 of [7] for reference. Then by Proposition 4.6.2 of [7],

$$G_{A_2}(o,y) = G(y) - P_o(\tau_y < \infty)G(w),\ G_{A_2}(y,y) = G(o) - P_o(\tau_w < \infty)G(w),$$

which gives

$$P_o(\tau_y < \tau_z) = P_o(\tau_z < \tau_y) = \frac{G(y) - P_o(\tau_y < \infty)G(w)}{G(o) - P_o(\tau_w < \infty)G(w)} \approx 0.2792. \tag{B.11}$$

Similarly, for $A_3 = \mathbb{Z}^3 \setminus \{y, z\}$ we have

$$P_o(\tau_w = \tau_A) = P_o(\tau_w < \tau_{A_3}) = \frac{G_{A_3}(o,w)}{G_{A_3}(w,w)},$$

where

$$G_{A_3}(o,w) = G(w) - [P_o(\tau_y < \tau_z) + P_o(\tau_z < \tau_y)]G(y)$$

and

$$G_{A_3}(w,w) = G(o) - [P_o(\tau_y < \tau_z) + P_o(\tau_z < \tau_y)]G(y)$$

which gives

$$P_o(\tau_w = \tau_A) = \frac{G(w) - 2P_o(\tau_y < \tau_z)G(y)}{G(o) - 2P_o(\tau_y < \tau_z)G(y)} \approx 0.0344. \tag{B.12}$$

Then for $A_4 = \mathbb{Z}^3 \setminus \{w\}$,

$$P_o(\tau_y < \tau_w) = P_o(\tau_y < \tau_{A_4}) = \frac{G_{A_4}(o,y)}{G_{A_4}(y,y)}$$

where

$$G_{A_4}(o,y) = G(y) - P_o(\tau_w < \infty)G(y), \ G_{A_4}(y,y) = G(o) - P_o(\tau_y < \infty)G(y).$$

Thus

$$P_o(\tau_y < \tau_w) = P_o(\tau_z < \tau_w) = \frac{G(y) - P_o(\tau_w < \infty)G(y)}{G(o) - P_o(\tau_y < \infty)G(y)} \approx 0.3008. \tag{B.13}$$

And for $A_5 = \mathbb{Z}^3 \setminus \{y\}$,

$$P_o(\tau_w < \tau_y) = P_o(\tau_w < \tau_{A_5}) = \frac{G_{A_5}(o,w)}{G_{A_5}(w,w)},$$

where

$$G_{A_5}(o,w) = G(w) - P_o(\tau_y < \infty)G(y), \ G_{A_4}(w,w) = G(o) - P_o(\tau_y < \infty)G(y).$$

Thus

$$P_o(\tau_w < \tau_y) = P_o(\tau_w < \tau_z) = \frac{G(w) - P_o(\tau_y < \infty)G(y)}{G(o) - P_o(\tau_y < \infty)G(y)} \approx 0.1155. \tag{B.14}$$

And for $A_6 = \mathbb{Z}^3 \setminus \{z, w\}$,

$$P_o(\tau_y = \tau_A) = P_o(\tau_y < \tau_{A_6}) = \frac{G_{A_6}(o,y)}{G_{A_6}(y,y)},$$

where

$$G_{A_6}(o,y) = G(y) - P_o(\tau_w < \tau_y)G(y) - P_o(\tau_y < \tau_w)G(w)$$

and

$$G_{A_6}(y,y) = G(o) - P_o(\tau_w < \tau_y)G(w) - P_o(\tau_y < \tau_w)G(y).$$

Thus we have

$$P_o(\tau_y = \tau_A) = \frac{G(y) - P_o(\tau_w < \tau_y)G(y) - P_o(\tau_y < \tau_w)G(w)}{G(o) - P_o(\tau_w < \tau_y)G(w) - P_o(\tau_y < \tau_w)G(y)} \approx 0.2696 \tag{B.15}$$

which by symmetry also equals to $P_o(\tau_z = \tau_A)$. Finally for $P_o(\tau_o < \tau_y)$ and $P_o(\tau_o < \tau_y)$, using one step argument at time 0,

$$P_o(\tau_o < \tau_y) = \frac{2}{3}P_o(\tau_y < \tau_w) + \frac{1}{6}P_o(\tau_y < \tau_{2y})$$

and

$$P_o(\tau_y < \tau_o) = \frac{1}{6} + \frac{2}{3}P_o(\tau_w < \tau_y) + \frac{1}{6}P_o(\tau_{2y} < \tau_y).$$

So again for $A_7 = \mathbb{Z}^3 \setminus \{2y\}$, we have

$$P_o(\tau_y < \tau_{2y}) = \frac{G_{A_7}(o,y)}{G_{A_7}(y,y)},$$

where

$$G_{A_7}(o,y) = G(y) - P_o(\tau_{2y} < \infty)G(y)$$

and

$$G_{A_7}(y,y) = G(o) - P_o(\tau_y < \infty)G(y).$$

Thus

$$P_o(\tau_o < \tau_y) = \frac{2}{3}P_o(\tau_y < \tau_w) + \frac{1}{6}\frac{G(y) - P_o(\tau_{2y} < \infty)G(y)}{G(o) - P_o(\tau_y < \infty)G(y)} \approx 0.2538. \tag{B.16}$$

And for $P_o(\tau_{2y} < \tau_y)$, recalling that $A_5 = \mathbb{Z}^3 \setminus \{y\}$ we have

$$P_o(\tau_{2y} < \tau_y) = \frac{G_{A_5}(o,2y)}{G_{A_5}(2y,2y)},$$

where

$$G_{A_5}(o,2y) = G(2y) - P_o(\tau_y < \infty)G(y),\ G_{A_5}(2y,2y) = G(o) - P_o(\tau_y < \infty)G(y).$$

Thus

$$P_o(\tau_y < \tau_o) = \frac{1}{6} + \frac{2}{3}P_o(\tau_w < \tau_y) + \frac{1}{6}\frac{G(2y) - P_o(\tau_y < \infty)G(y)}{G(o) - P_o(\tau_y < \infty)G(y)} \approx 0.2538. \tag{B.17}$$

At this point, we finally have all the variables needed calculated, apply (B.3-B.17) to (B.1) and (B.2), the proof of Proposition 6.2 is complete. □