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Stationary DLA is Well Defined

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Abstract
In this paper, we construct an infinite stationary diffusion limited aggregation (SDLA) on the upper half planar lattice, growing from an infinite line, with local growth rate proportional to the stationary harmonic measure. This model was suggested by Itai Benjamini. The main issue is a known problem in DLA models, the long range effects of large arms. In this paper we overcome this difficulty via a multi-scale argument controlling the dynamical discrepancies created on all scales while running two coupled SDLA on different starting configurations.

Keywords Diffusion limited aggregation · Stationary harmonic measure · Interacting particle system

1 Introduction

Diffusion limited aggregation (DLA) is a set-valued process first defined by Witten and Sander [12] in order to study physical systems where the growth are governed by diffusion. DLA is defined recursively as a process on subsets of $\mathbb{Z}^2$. Starting from $A_0 = \{(0,0)\}$, at each time a new point a_{n+1} sampled from the harmonic probability measure on the outer

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vertex boundary of A_n is added to A_n. Intuitively, a_{n+1} is the first place that a random walk starting from infinity visits $\partial^{out} A_n$.

In many experiments and real world phenomena the aggregation grows from some initial boundary instead of a single point, e.g. ions diffusing in liquid until they connect a charged container floor (see [2] for numerous examples). Different aggregation processes, such as Eden and Internal DLA, with boundaries were studied in [1,3], and universal phenomena such as a.s. non existence of infinite trees were proved.

Motivated by a question from Itai Benjamini (through private communication), in this paper we construct an infinite stationary DLA (SDLA) on the upper half planar lattice, growing from an infinite line. Along the way we prove that this infinite stationary DLA can be seen as a limit of DLA in the upper half plane growing from a long finite line. This allows one to use the more symmetric and amenable model of SDLA to study local behavior of DLA. In addition SDLA admits new phenomena not observed in the full lattice DLA. One such interesting conjectured phenomenon, which results from the competition between different trees in the SDLA, is that eventually (and in finite time) every tree in the SDLA ceases to grow.

The main difficulties we encountered in this paper are also known problems in the study of classic DLA model. As has been pointed out in [7]:

"The difficulty comes from the fact that the dynamics is neither monotone nor local, and that it roughens the cluster."

In addition, the dynamics of SDLA is generally an infinite measure (the stationary harmonic measure) where the mass on each point is unbounded. In contrast, the regular harmonic measure for each finite set always sums up to 1.

As a result of the facts above, having the proposed growth dynamic of our aggregation (the stationary harmonic measure), it is not straightforward to assert that an Infinite Interacting Particle System (IPS) that corresponds to such dynamics will always exist. In [9, Theorem 5], though, a weaker result has been proved that when the initial aggregation is finite, a DLA in the upper half plane is well defined. However, part of the approach there (applying Poisson thinning on a "faster" interface model or truncate the space) is clearly inapplicable when we start from an infinite initial state.

In order to overcome the difficulties above, we consider an approach that couples a sequence of DLA's in the upper half plane, starting from increasingly longer line segments $[-n, n] \cap \mathbb{Z} \times \{0\}$. For each pair of "neighboring" copies starting from $[-n, n] \cap \mathbb{Z} \times \{0\}$ and $[-n-1, n+1] \cap \mathbb{Z} \times \{0\}$, a similar non-monotonicity as described in [7] indicates that there is no stochastic order between the two copies. So the key idea of this paper is a multi-scale argument that tracks the time-space evolution of discrepancies in this pair, and showing that, by any finite time, discrepancies are highly unlikely to reach a microscopic space scale around 0. See Sects. 5 and 6 for details. Thus, we show that such sequence has an a.s. limit which gives the SDLA model starting from the x-axis.

Through private communication, Paul Jung suggested that one may add some external growth rate to the stationary harmonic measure and consider a faster growing system where aggregation is determined by this new rate. The new system may have better regularity, and thus it is easier to show well-definition. Finally one let the external growth rate goes to zero and show that SDLA is well-defined.

1.1 Statement of Result

The main result we obtained in the paper is the well-definition of the (infinite) SDLA according to its transition rate given by the stationary harmonic measure, starting from the infinite initial configuration L_0:

Definition 1 An interacting particle system ζ_t is said to be an SDLA if the following conditions hold.

- $\{\zeta_t\}_{t\geq 0}$ is a Markov process on $\{0, 1\}^{\mathbb{H}}$.
- Starting position: $\mathbf{P}(\zeta_0 = L_0) = 1$.
- Transition rates: For any $t > 0$, for any $s \in [0, t]$ and $x, y \in \mathbb{H}$,

$$\lim_{\Delta s \to 0} \frac{\mathbf{P}\left(\zeta_{s+\Delta s}(x) = 1 | \zeta_s(x) = 0, \{\zeta_\xi\}_{\xi \leq s}\right)}{\Delta s} = \mathcal{H}_{L_0 \cup \zeta_s}(x) \text{ a.s.}$$

  and.

$$\lim_{\Delta s \to 0} \frac{\mathbf{P}\left(\zeta_{s+\Delta s}(x) = 1, \zeta_{s+\Delta s}(y) = 1, | \zeta_s(x) = 0, \zeta_s(y) = 0, \{\zeta_\xi\}_{\xi \leq s}\right)}{\Delta s} = 0 \text{ a.s.}$$

Theorem 1 *Let $t > 0$, then there is a well defined SDLA process $\{A_s^\infty\}_{s \leq t}$.*

Remark 1 The result remains true if one replace the initial state L_0 by any subset A_0 that can be seen as a connected forest of logarithmic horizontal growth rate. To be precise, A_0 can be written as $\cup_{n=-\infty}^{\infty} Tree_0^n$, where $Tree_0^n$ is connected for each n, with $Tree_0^n \cap L_0 = (n, 0)$ and moreover, there is some $C < \infty$ such that the diameter of all but finite number of the trees are no more than $C \log n$. In fact, the condition above is satisfied a.s. by A_t^∞ for each $t > 0$. We present the proof for $A_0 = L_0$ for simplicity but without loss of (much) generality.

Remark 2 In this paper we do not address the question of uniqueness. Since the transition rates are unbounded, the standard techniques for proving uniqueness in Interacting Particle Systems seem to fail.

Remark 3 Since the proofs developed in this paper, particularly for Theorem 5 and Lemma 5.1, are actually insensitive to the choices of different finite time t's, one may hereby, without loss of generality focus on the case $t = 1$ for simplicity.

A major tool one obtains for the study of SDLA is ergodicity of the process.

Theorem 2 *For every $t > 0$, A_t^∞ is ergodic with respect to shift in $\mathbb{Z} \setminus \{0\} \times \{0\}$.*

Remark 4 One of the main contributions of the proof of Theorem 1 is the representation of the SDLA as a local limit of SDLA processes starting from a finite line. This representation is used in the proof of Theorem 2.

1.2 Future Works and Open Problems

One of the main motivations of studying the SDLA from the x-axis is that it may serve as the local limit of the upper half of DLA (variant) from increasingly longer line segments, after an appropriate scaling of time. The convergence on the level of dynamics/harmonic measures has been seen in [11]. For the convergence of aggregation processes, one may first define:

Definition 2 Let the edge DLA on $\mathbb{Z}^2$ be a stochastic aggregation process defined as follows:

- At each step k a random walk "from infinity" is released and run until it hits the current aggregation EA_k
- The last step of the random walk is added to EA_k to form EA_{k+1}.

Now let $EA_t = EA_{N_t}$ be the continuous time version of EA_k, where N_t is a standard Poisson process independent to EA_k. And let EA_t^n be the process with initial state $EA_0^n = [-n, n] \cap \mathbb{Z} \times \{0\}$. Since it is very unlikely for random walk to "go around" a long line segment without hitting it, the following conjecture says the upper half of EA_t^n (with itself forms a process in the whole space) may behave similarly as the DLA restrict in the upper half plane after some appropriate scaling of time:

Conjecture 1 *There is a constant $c \in (0, \infty)$ such that $EA_{nt}^n \cap \mathbb{H}$ converges weakly to A_{ct}^∞ as $n \to \infty$.*

A proof of Conjecture 1 has recently been found by Yingxin Mu, Procaccia, and Zhang. The paper [8] is under preparation. At the same time, using arguments parallel to the proofs in this paper and in the upcoming paper [8], one may also have the the Corollary that the local limit of classic DLA's starting from a long (horizontal) line segment, after appropriate scaling of time, as a variant of the SDLA

Another open problem is the stabilization of the SDLA. To precisely state the conjecture, consider d_{chem} the chemical/internal distance on a graph. For each $x \in l_0 = (-\infty, \infty) \cap \mathbb{Z} \times \{0\}$, define

$$T_x(t) = \left\{ y \in A_t^\infty : d_{chem}(x, y) = \min_{z \in l_0} d_{chem}(z, y) \right\}$$

to be the branch in A_t^∞ rooted at x. The following conjecture predicts that all branches finally fall under the shadow of other branches and stop growing:

Conjecture 2 *Define*

$$T_x = \bigcup_{t \geq 0} T_x(t).$$

With probability one, $|T_x| < \infty$ for all $x \in l_0$.

2 Preliminaries

We first recall a number of notations and results from a previous paper by two of the authors [10]: Let $\mathbb{H} = \{(x, y) \in \mathbb{Z}^2, y \geq 0\}$ be the upper half plane (including the x-axis), and $(S_n)_{n \geq 0}$ be a 2-dimensional simple random walk. For any $x \in \mathbb{Z}^2$, we will write

$$x = (x_1, x_2)$$

with x_i denoting the ith coordinate of x, and $\|x\| = \|x\|_1 = |x_1| + |x_2|$. Then for each nonnegative integer n, define

$$L_n = \{(x, n), \ x \in \mathbb{Z}\}$$

to be the horizontal line of height n. For each subset $A \subset \mathbb{Z}^2$, we define the stopping times

$$\bar{\tau}_A = \min\{n \geq 0, \ S_n \in A\}$$

and

$$\tau_A = \min\{n \geq 1,\ S_n \in A\}.$$

For any subsets $A_1 \subset A_2$ and B and any $y \in \mathbb{Z}^2$, by definition one can easily check that

$$\begin{aligned} \mathbf{P}_y\left(\tau_{A_1} < \tau_B\right) &\leq \mathbf{P}_y\left(\tau_{A_2} < \tau_B\right), \\ \mathbf{P}_y\left(\bar{\tau}_{A_1} < \bar{\tau}_B\right) &\leq \mathbf{P}_y\left(\bar{\tau}_{A_2} < \bar{\tau}_B\right), \end{aligned} \tag{1}$$

and that

$$\begin{aligned} \mathbf{P}_y\left(\tau_B < \tau_{A_2}\right) &\leq \mathbf{P}_y\left(\tau_B < \tau_{A_1}\right), \\ \mathbf{P}_y\left(\bar{\tau}_B < \bar{\tau}_{A_2}\right) &\leq \mathbf{P}_y\left(\bar{\tau}_B < \bar{\tau}_{A_1}\right), \end{aligned} \tag{2}$$

where $\mathbf{P}_y(\cdot) = \mathbf{P}(\cdot|S_0 = y)$. In [10] we defined the stationary harmonic measure on $\mathbb{H}$ which will serve as the Poisson intensity in our continuous time DLA model. For any $B \subset \mathbb{H}$, any edge $\vec{e} = (x, y)$ with $x \in B$, $y \in \mathbb{H} \setminus B$ and any N, we define

$$\mathcal{H}_{B,N}(\vec{e}) = \sum_{z \in L_N \setminus B} \mathbf{P}_z\left(S_{\bar{\tau}_{B\cup L_0}} = x,\ S_{\bar{\tau}_{B\cup L_0}-1} = y\right). \tag{3}$$

By definition, a necessary condition for $\mathcal{H}_{B,N}(\vec{e}) > 0$ is $y \in \partial^{out} B$ and $|x - y| = 1$. For all $x \in B$, we can also define

$$\mathcal{H}_{B,N}(x) = \sum_{y:\ \vec{e}=(x,y)} \mathcal{H}_{B,N}(\vec{e}) = \sum_{z \in L_N \setminus B} \mathbf{P}_z\left(S_{\bar{\tau}_{B\cup L_0}} = x\right). \tag{4}$$

For each point $y \in \partial^{out} B$, we can also define

$$\hat{\mathcal{H}}_{B,N}(y) = \sum_{\vec{e}=(x,y),\ x \in B} \mathcal{H}_{B,N}(\vec{e}) = \sum_{z \in L_N \setminus B} \mathbf{P}_z\left(\tau_B \leq \tau_{L_0},\ S_{\bar{\tau}_{B\cup L_0}-1} = y\right). \tag{5}$$

By coupling and the strong Markov property, we showed in [10, Proposition 1] that $N \to \mathcal{H}_{A,N}(e)$ is bounded and monotone in N. Thus we proved that

Proposition 1 (Proposition 1, [10]). *For any B and $\vec{e}$ as above, there is a finite $\mathcal{H}_B(\vec{e})$ such that*

$$\lim_{N\to\infty} \mathcal{H}_{B,N}(\vec{e}) = \mathcal{H}_B(\vec{e}). \tag{6}$$

$\mathcal{H}_B(\vec{e})$ is called the stationary harmonic measure of $\vec{e}$ with respect to B. The limits $\mathcal{H}_B(x) = \lim_{N\to\infty} \mathcal{H}_{B,N}(x)$ and $\hat{\mathcal{H}}_B(y) = \lim_{N\to\infty} \hat{\mathcal{H}}_{B,N}(y)$ also exist [10] and are called the stationary harmonic measure of x and y with respect to B.

For any connected $B \subset \mathbb{H}$ such that $B \cap L_0 \neq \emptyset$, and any $x \in B$, $\mathcal{H}_B(x)$ was proved to have the following upper bound that depends only on the height of x:

Theorem 3 (Theorem 1, [10]). *There is some constant $C < \infty$ such that for each connected $B \subset \mathbb{H}$ with $L_0 \subset B$ and each $x = (x_1, x_2) \in B \setminus L_0$, and any N sufficiently larger than x_2*

$$\mathcal{H}_{B,N}(x) \leq C x_2^{1/2}. \tag{7}$$

Remark 5 The theorem above is sharp and the direction of inequality in (7) can be reversed when B is L_0 plus a vertical line. See Theorem 2, [10] for details.

Remark 6 It is easy to note that for any $B \subset \mathbb{H}$ such that $B \cap L_0 \neq \emptyset$ and any $x = (x_1, x_2) \in B \setminus L_0$, $\mathcal{H}_B(x) = \mathcal{H}_{B\cup L_0}(x)$. Thus one may without loss of generality assume that $L_0 \subset B$.

Remark 7 Unless specified otherwise, we use C to represent constant(s) that do not depend on subset B or point x, or n. However, their specific values may varies according to context.

With the upper bounds of the harmonic measure on the upper half plane, a pure growth model called the **interface process** was introduced in [10] which can be used as a dominating process for both the DLA model in $\mathbb{H}$ and the stationary DLA model that will be introduced in this paper. Consider an interacting particle system $\bar{\xi}_t$ defined on $\{0, 1\}^{\mathbb{H}}$, with 1 standing for an occupied site and 0 for a vacant site, with transition rates as follows:

(i) For each occupied site $x = (x_1, x_2) \in \mathbb{H}$, if $x_2 > 0$ it will try to give birth to each of its nearest neighbors at a Poisson rate of $\sqrt{x_2}$. If $x_2 = 0$, it will try to give birth to each of its nearest neighbors at a Poisson rate of 1.
(ii) If x attempts to give birth to a nearest neighbor y that is already occupied, the birth is suppressed.

We proved that an interacting particle system determined by the dynamic above is well-defined.

Proposition 2 (Proposition 3, [10]). *The interacting particle system* $\bar{\xi}_t \in \{0, 1\}^{\mathbb{H}}$ *satisfying (i) and (ii) is well defined.*

When the initial aggregation V_0 is the origin or finite, we defined the DLA process in $\mathbb{H}$ starting from V_0 (Theorem 5, [10]), according to the graphic representation (see [5] for introduction) of the interface process $\bar{\xi}_t$ and a procedure of Poisson thinning, see Page 30-31 of [10] for details. Note that under this construction, the DLA model with finite initial aggregation is contained in the interface process.

3 Coupling Construction

Now in order to prove Theorem 1, we construct a sequence of processes $\{A_t^n\}_{n=1}^{\infty}$, each of which is the DLA in $\mathbb{H}$ with initial aggregation $V_0^n = [-n, n] \times 0$, coupled together with the same interface process. To be precise, recall the graphic representation in [10]:

- For each $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{H}$ such that $\|x - y\| = 1$, we associate the edge $\vec{e} = (x, y)$ with an independent Poisson process $N_t^{x \to y}, t \geq 0$ with intensity $\lambda_{x \to y} = \sqrt{x_2} \vee 1$.
- For each $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{H}$ such that $\|x - y\| = 1$ let $\{U_i^{x \to y}\}_{i=1}^{\infty}$ be i.i.d. sequences of $U(0, 1)$ random variables independent of each other and of the Poisson processes.

At any time t when there is Poisson transition for edge $\vec{e} = (x, y)$, we draw the directed edge $(\vec{e}, t)$ in the phase space $\mathbb{H} \times [0, \infty)$. For any $x \in L_0$ and any fixed time t, recall that I_t^x is the set of all y's in $\mathbb{H}$ that are connected in the space-time block with x, by a directed path starting from x, going upwards vertically or following the directed edges, and ending at y in the graphic representation. In [10], it has been proved that for all $V_0 \subset \mathbb{H}$,

$$\bar{\xi}_t^{V_0} = \bigcup_{x \in V_0} I_t^x$$

distributed as the interface process with initial state V_0. Moreover, it was proven that for each $t < \infty$ and all $x \in \mathbb{H}$, $|I_t^x| < \infty$ with probability one, and there can be only a finite number of different paths emanating from x by time t, which may only have finite transitions involved. Now for all finite V_0, in [10] we look at the finite set of all the transitions involved in the evolution of $(\bar{\xi}_s^{V_0})_{s\in[0,t]}$, and order them according to the time of occurrence. The following thinning is applied in order to define a process $A_t = (V_t, E_t)$ starting at $A_0 = (V_0, \emptyset)$. Suppose a new transition involving in the evolution of $(\bar{\xi}_s^{V_0})_{s\in[0,t]}$ arrives at time t_i, and it is the jth overall Poisson transition on the edge $\vec{e} = (x, y)$, and one already knows $A_{t_i-} := \lim_{s\uparrow t_i} A_s$.

- If $x \notin V_{t_i-}$ or $y \in V_{t_i-}$, nothing happens.
- Otherwise:
  - If $U_j^{x\to y} \le \mathcal{H}_{V_{t_i-}}(\vec{e})/\lambda_{\vec{e}}$, then $V_{t_i} = V_{t_i-} \cup \{y\}$, $E_t = E_{t-} \cup \{\vec{e}\}$.
  - Otherwise, nothing happens.

Thus we define the process A_t up to all time t with V_t identically distributed as our DLA process starting from A_0. Now, for each n define A_t^n as the process with $A_0^n = ([-n, n]\times 0, \emptyset)$. We have coupled all A_t^n's using the same graphic representation and thinning factors. Now in order to prove Theorem 1, we first show the following theorem which states that for a finite space-time box, the discrepancy probabilities for our A^n's are summable.

Theorem 4 *For any compact subset $K \subset \mathbb{H}$ and any $T < \infty$, we have*

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\exists t \le T,\ s.t.\ A_t^n \cap K \ne A_t^{n+1} \cap K\right) < \infty. \tag{8}$$

Here for any $A = (V, E)$, we use the convention that $A \cap K = (V \cap K, \{\vec{e} = (x, y) \in E, \{x, y\} \cap K \ne \emptyset\})$.

Recalling Remark 3, we may from now on concentrate on $T = 1$. The proof of Theorem 4 is immediate once one proves that there exist constants $\alpha > 0$ and $C < \infty$ such that for all sufficiently large n

$$\mathbf{P}\left(\exists t \le 1,\ s.t.\ A_t^n \cap K \ne A_t^{n+1} \cap K\right) \le \frac{C}{n^{1+\alpha}}. \tag{9}$$

The same argument also implies

Corollary 1 *Let $A_t^{n,+}$ be the process with $A_0^{n,+} = ([-n, n+1]\times 0, \emptyset)$. Then for all sufficiently large n*

$$\mathbf{P}\left(\exists t \le 1,\ s.t.\ A_t^n \cap K \ne A_t^{n,+} \cap K\right) \le \frac{C}{n^{1+\alpha}}.$$

The same result holds for $A_t^{n,-}$ with $A_0^{n,-} = ([-n-1, n] \times 0, \emptyset)$.

Note that at $t = 0$, the initial aggregations A_0^n and A_0^{n+1} are different only by the two end points $(\pm(n+1), 0)$. Now we want to control the subset of the discrepancies so that they will not reach K by time 1. Intuitively, the idea we will follow in the detailed proof in the following sections can be summarized as follows:

(I) Note that the growth of A_t^n and A_t^{n+1} are both dominated by the interface process. So with very high probability none of A_1^n and A_1^{n+1} can reach height $\log(n)$.

(II) We can show that for a well coupled pair of A_t^n and A_t^{n+1}, the rate a new discrepancy is created can be bounded by the stationary harmonic measure of the existing discrepancy set. With a similar large deviation argument as Step (I), for any $\alpha > 0$, with very high probability the two processes will have fewer than n^α discrepancies by time 1.

(III) We can show that it is very unlikely for a newly created discrepancy to wander far away form the existing discrepancy set. So for all these discrepancies ever created till time 1, with very high probability none of them will go far enough from the boundary and ever find its way to K.

4 Logarithmic Growth of the Interface Process

In this section, we prove the logarithmic growth upper bound for A_t^n and A_t^{n+1} with $t \in [0, 1]$. Note that both are contained in the interface process $I_t^{[-n-1,n+1]\times 0}$. Thus it suffices to show that

Theorem 5 *For any $C < \infty$,*

$$\mathbf{P}\left(I_1^{[-n,n]\times 0} \not\subseteq [-n - \log n, n + \log n] \times [0, \log n]\right) < \frac{1}{n^C}$$

for all sufficiently large n.

Proof First note that

$$I_1^{[-n,n]\times 0} = \bigcup_{x\in[-n,n]\times 0} I_1^x.$$

By union bound, it suffices to show that for any $C < \infty$ and all sufficiently large k,

$$\mathbf{P}\left(\|I_1^0\|_2 \geq k\right) < \exp(-Ck), \tag{10}$$

where

$$\|A\|_2 = \max_{x\in A} \|x\|_2$$

for all finite $A \subset \mathbb{H}$.

Lemma 4.1 *For any $\tilde{c} \in (0, \infty)$,*

$$\mathbf{P}(\|I_1^0\|_2 > k) \leq \exp(-\tilde{c}k)$$

for all sufficiently large k.

Proof Under the event $\{\|I_1^0\|_2 > k\}$, by definition and the fact that I_1^0 is a nearest neighbor growth model, there has to exist a nearest neighbor sequence of points $0 = x_0, x_1, \ldots, x_m$ with $\|x_m\| \geq k$ such that for stopping times

$$\tau_i = \inf\{s \geq 0 : \ x_i \in I_s^0\},$$

we have that

$$0 = \tau_0 < \tau_1 < \cdots < \tau_m < 1.$$

Noting that $x_0, x_1, \ldots, x_m$ is a nearest neighbor path with $\|x_m\| \geq k$, which implies $m \geq k$ and we can take the first k steps of it. More precisely, there exists a nearest neighbor sequence of points $0 = x_0, x_1, \ldots, x_k$ such that for stopping times

$$\tau_i = \inf\{s \geq 0 : \ x_i \in I_s^0\},$$

we have that

$$0 = \tau_0 < \tau_1 < \cdots < \tau_k < 1.$$

Note that there are no more than $4 \times 3^{k-1}$ such different nearest neighbor sequences of points within $\mathbb{H}$ starting at 0. For each given path $0 = x_0, x_1, \ldots, x_k$, and each $1 \le i \le k$, define

$$\Delta_i = \min_{y:\|y-x_i\|=1} \inf \left\{ s > 0 : \; N^{y\to x_i}_{\tau_{i-1}+s} = N^{y\to x_i}_{\tau_{i-1}} + 1 \right\}.$$

By definition and the strong Markov property, Δ_i is an exponential random variable with rate $\hat{\lambda}_i = \sum_{y:\|y-x_i\|=1} \lambda_{y\to x_i} \le 4\sqrt{i+1}$, independent to $\mathcal{F}_{\tau_{i-1}}$. At the same time, note that by definition $\Delta_i \le \tau_i - \tau_{i-1}$, which implies that $\Delta_i \in \mathcal{F}_{\tau_i}$, and that $\{\Delta_i\}_{i=1}^k$ is a sequence of independent random variables. Let $\{T_i\}_{i=1}^k$ be independent exponential random variables with parameters $\lambda_i = 4\sqrt{i+1}$. Thus

$$\mathbf{P}(\tau_0 < \tau_1 < \cdots < \tau_k < 1) \le \mathbf{P}\left(\sum_{i=1}^k \Delta_i < 1\right) \le \mathbf{P}\left(\sum_{i=1}^k T_i < 1\right),$$

and

$$\mathbf{P}(\|I_1^0\|_2 > k) \le 4 \times 3^{k-1} \mathbf{P}\left(\sum_{i=1}^k T_i < 1\right). \tag{11}$$

For some constants $c_1, c_2 > 0$ (to be chosen later), define the event

$$G = \left\{ \left| \left\{ 1 \le i \le k : T_i \ge \frac{c_2}{\sqrt{i+1}} \right\} \right| > c_1 k \right\}.$$

Under the event G,

$$\sum_{i=1}^k T_i \ge \sum_{i:\, T_i \ge \frac{c_2}{\sqrt{i}}} T_i \ge c_1 k \frac{c_2}{\sqrt{k+1}} = \frac{1}{2} c_1 c_2 \sqrt{k} \ge 1, \tag{12}$$

where the last inequality holds for any sufficiently large k. Therefore,

$$\mathbf{P}\left(\sum_{i=1}^k T_i < 1\right) \le \mathbf{P}(G^c) \tag{13}$$

for all sufficiently large k depending on the choices of c_1 and c_2. Define

$$X_i = \mathbb{1}_{\left\{T_i \ge \frac{c_2}{\sqrt{i+1}}\right\}},$$

thus $\sum_{i=1}^k X_i$ is a binomial random variable with parameters k and

$$p = \mathbf{P}\left(T_i \ge \frac{c_2}{\sqrt{i+1}}\right) = e^{-4c_2},$$

which converges to 1 when $c_2 \to 0$. By the large deviation principle for the binomial distribution,

$$\mathbf{P}(G^c) = \mathbf{P}\left(\sum_{i=1}^k X_i < c_1 k\right) \le e^{-I(c_1,p)k}. \tag{14}$$

For p close enough to 1, we have $I(c_1, p) > \tilde{c} + \log(4)$ (see [4] for the exact rate function). The result follows from combining (11), (13), and (14) and choosing appropriate c_1 and c_2. □

Theorem 5 follows from Lemma 4.1 and union bound. □

5 Truncated Processes and Number of Discrepancies

In this section we complete Step (II) in the outline. Prior to that, we would like to use Theorem 5 to define a truncated version of coupled process (A_t^n, A_t^{n+1}). Define the stopping time

$$\Gamma = \inf\left\{t \geq 0 : \ V_t^n \cup V_t^{n+1} \not\subseteq [-n - \log n, n + \log n] \times [0, \log n]\right\}$$

to be the first time A_t^n or A_t^{n+1} grows outside the box $[-n - \log n, n + \log n] \times [0, \log n]$.

Remark 8 It is easy to see that V_t^n or V_t^{n+1} grows outside our box if and only if E_t^n or E_t^{n+1} does so.

Now we can define the **truncated processes**

$$(\hat{A}_t^n, \hat{A}_t^{n+1}) = \left(A_{t\wedge\Gamma}^n, A_{t\wedge\Gamma}^{n+1}\right).$$

I.e., we have the coupled processes stopped once either of them goes outside the box $[-n - \log n, n + \log n] \times [0, \log n]$. By definition, we have

$$(A_t^n, A_t^{n+1}) = (\hat{A}_t^n, \hat{A}_t^{n+1})$$

for all $t \in [0, \Gamma]$. At the same time, note that

$$V_t^n \cup V_t^{n+1} \subset \bigcup_{x \in [-n-1, n+1] \times 0} I_t^x$$

for all $t \geq 0$. Thus for all $C < \infty$ and all sufficiently large n,

$$\begin{aligned}
1 - \mathbf{P}&\left(A_t^n \equiv \hat{A}_t^n, A_t^{n+1} \equiv \hat{A}_t^{n+1}, \ \forall t \in [0, 1]\right) \\
&\leq \mathbf{P}\left(I_1^{[-n-1,n+1]\times\{0\}} \not\subseteq [-n - 1 - \log(n+1), n + 1 + \log(n+1)] \times [0, \log(n+1)]\right) \\
&< \frac{1}{n^C}. \qquad (15)
\end{aligned}$$

Thus in order to show Theorem 4, it suffices to prove that there exists constants $\alpha > 0$ and $C < \infty$ such that for all sufficiently large n

$$\mathbf{P}\left(\exists t \leq 1, \ s.t. \ \hat{A}_t^n \cap K \neq \hat{A}_t^{n+1} \cap K\right) \leq \frac{C}{n^{1+\alpha}}. \qquad (16)$$

Now we formally define the set of discrepancies for the coupled process $(\hat{A}_t^n, \hat{A}_t^{n+1})$. For any $t < \infty$, define

$$V_t^{D,n} = \left\{x \in \mathbb{H}, \ s.t. \ \exists s \leq t, \ x \in \hat{V}_s^n \triangle \hat{V}_s^{n+1}\right\}$$

as the set of **vertex discrepancies**, and

$$E_t^{D,n} = \left\{\vec{e} = (x, y), \ x, y \in \mathbb{H}, \ s.t. \ \exists s \leq t, \ \vec{e} \in \hat{E}_s^n \triangle \hat{E}_s^{n+1}\right\}$$

as the set of **edge discrepancies**, where $\triangle$ stands for the symmetric difference of sets. From their definitions, we list some basic properties of the sets of discrepancies as follows:

- $V_0^{D,n} = \{(\pm(n+1), 0)\}$, $E_0^{D,n} = \emptyset$.
- Both $V_t^{D,n}$ and $E_t^{D,n}$ are non-decreasing with respect to time.
- For any $x \in V_t^{D,n}$, either $x = (\pm(n+1), 0)$ or there has to be an edge $\vec{e}_x \in E_t^{D,n}$ ending at x.
- For any $\vec{e} = (a, x) \in E_t^{D,n}$, x has to be in $x \in V_t^{D,n}$.
- Whenever a new vertex is added in $V_t^{D,n}$, there has to be a new edge added to $E_t^{D,n}$. However, when a new edge is added to $E_t^{D,n}$, there may or may not be a a new vertex added in $V_t^{D,n}$.

From the observations above, it is immediate to see that $V_t^{D,n}$ is the same as the collection of all ending points in $E_t^{D,n}$, which also implies that $|V_t^{D,n}| \le |E_t^{D,n}| + 2$.

Moreover, for the event of interest, we have

$$\left\{\exists t \le 1,\ s.t.\ \hat{A}_t^n \cap K \ne \hat{A}_t^{n+1} \cap K\right\} = \left\{V_1^{D,n} \cap K \ne \emptyset\right\}. \tag{17}$$

As we outlined in the previous section, in order to prove the event in (17) has a super-linearly decaying probability as $n \to \infty$, we first control the growth of $|E_t^{D,n}|$. I.e., by time 1 there cannot be too many discrepancies created in the coupled systems. To be precise, we prove that

Lemma 5.1 *For any $\alpha > 0$, there is a $c > 0$ such that*

$$\mathbf{P}\left(|E_1^{D,n}| \ge n^{\alpha}\right) \le \exp(-n^c)$$

for all sufficiently large n.

Proof Note that $|E_0^{D,n}| = 0$. For $i = 1, 2, \ldots$, define the stopping time $\Delta_i = \inf\{t \ge 0,\ |E_t^{D,n}| = i\}$, with the convention $\inf \emptyset = \infty$. Given the configuration of $(\hat{A}_t^n, \hat{A}_t^{n+1})$, we first discuss the rate at which a new discrepancy is created. If $t > \Gamma$, each such rate is equal to zero by definition. Otherwise, each edge $\vec{e} = (x, y)$ in $\mathbb{H}$ can be classified according to the configuration as follows: define the indicator matrix

$$\mathbb{I}(\hat{A}_t^n, \hat{A}_t^{n+1})(\vec{e}) = \begin{pmatrix} \mathbb{1}_{x \in \hat{V}_t^n} & \mathbb{1}_{y \in \hat{V}_t^n} & \mathbb{1}_{\vec{e} \in \hat{E}_t^n} \\ \mathbb{1}_{x \in \hat{V}_t^{n+1}} & \mathbb{1}_{y \in \hat{V}_t^{n+1}} & \mathbb{1}_{\vec{e} \in \hat{E}_t^{n+1}} \end{pmatrix}.$$

By definition, the only edges that contribute to the increasing rate of $E_t^{D,n}$ are those with indicator matrices as one of the following:

$$\mathbb{I}_1 = \begin{pmatrix} 1\ 0\ 0 \\ 1\ 0\ 0 \end{pmatrix},\ \mathbb{I}_2 = \begin{pmatrix} 1\ 1\ 0 \\ 1\ 0\ 0 \end{pmatrix},$$
$$\mathbb{I}_3 = \begin{pmatrix} 1\ 0\ 0 \\ 0\ 0\ 0 \end{pmatrix},\ \mathbb{I}_4 = \begin{pmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \end{pmatrix},$$
$$\mathbb{I}_5 = \begin{pmatrix} 1\ 0\ 0 \\ 1\ 1\ 0 \end{pmatrix},\ \mathbb{I}_6 = \begin{pmatrix} 0\ 0\ 0 \\ 1\ 0\ 0 \end{pmatrix},$$
$$\mathbb{I}_7 = \begin{pmatrix} 0\ 1\ 0 \\ 1\ 0\ 0 \end{pmatrix},$$

and we will denote the collections of such edges $E_1, E_2, \ldots, E_7$.

Now the rate that a new edge is added to $E_t^{D,n}$ can be written as follows:

$$\lambda^D(\hat{A}_t^n, \hat{A}_t^{n+1}) = \sum_{\vec{e}\in E_1} \left|\mathcal{H}_{\hat{V}_t^n}(\vec{e}) - \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e})\right| + \sum_{\vec{e}\in E_2} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) + \sum_{\vec{e}\in E_3} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_4} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_5} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_6} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) + \sum_{\vec{e}\in E_7} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}). \tag{18}$$

For any $\vec{e} \in \cup_{i=2}^7 E_i$, note that at least one end point of $\vec{e}$ has to be within $\hat{V}_t^n \triangle \hat{V}_t^{n+1} \subset V_t^{D,n}$. Moreover, recall that for each point in $\mathbb{H}$, there can be no more than 4 directed edges emanating from it and 4 edges going towards it. Thus, $|\cup_{i=2}^7 E_i| \le 8|V_t^{D,n}| \le 8(|E_t^{D,n}|+2)$. Now recalling $t < \Gamma$, $\hat{A}_t^n \cup \hat{A}_t^{n+1} \subset [-n-\log n, n+\log n] \times [0, \log n]$, which implies that for each $\vec{e} \in \cup_{i=2}^7 E_i$, the corresponding harmonic measure in (18) is bounded from above by $2\sqrt{\log n}$. Thus

$$\sum_{\vec{e}\in E_2} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) + \sum_{\vec{e}\in E_3} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_4} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_5} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_6} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) + \sum_{\vec{e}\in E_7} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) \le 16(|E_t^{D,n}|+2)\sqrt{\log n}. \tag{19}$$

Now for each $\vec{e} = (x, y) \in E_1$, by definition x has to be in the inner boundary of $\hat{V}_t^n \cap \hat{V}_t^{n+1}$, while y is in the complement of $\hat{V}_t^n \cup \hat{V}_t^{n+1}$. Moreover, we have

$$\left|\mathcal{H}_{\hat{V}_t^n}(\vec{e}) - \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e})\right| \le \mathcal{H}_{\hat{V}_t^n\cap\hat{V}_t^{n+1}}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n\cup\hat{V}_t^{n+1}}(\vec{e}). \tag{20}$$

Using a similar method as in Section 5 of [10] and recalling the definition of stationary harmonic measure,

$$\begin{aligned}
&\mathcal{H}_{\hat{V}_t^n\cap\hat{V}_t^{n+1}}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n\cup\hat{V}_t^{n+1}}(\vec{e}) \\
&= \lim_{N\to\infty} \left(\mathcal{H}_{\hat{V}_t^n\cap\hat{V}_t^{n+1},N}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n\cup\hat{V}_t^{n+1},N}(\vec{e})\right) \\
&= \lim_{N\to\infty} \sum_{w\in L_N} \mathbf{P}_w\left(X_{\tau_{(\hat{V}_t^n\cap\hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n\cap\hat{V}_t^{n+1})\cup L_0}-1} = y\right) \\
&\quad - \lim_{N\to\infty} \sum_{w\in L_N} \mathbf{P}_w\left(X_{\tau_{(\hat{V}_t^n\cup\hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n\cup\hat{V}_t^{n+1})\cup L_0}-1} = y\right) \\
&= \lim_{N\to\infty} \sum_{w\in L_N} \mathbf{P}_w\left(X_{\tau_{(\hat{V}_t^n\cap\hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n\cap\hat{V}_t^{n+1})\cup L_0}-1} = y,\ X_{\tau_{(\hat{V}_t^n\cup\hat{V}_t^{n+1})\cup L_0}} \in \hat{V}_t^n \triangle \hat{V}_t^{n+1}\right) \\
&= \lim_{N\to\infty} \sum_{w\in L_N} \sum_{z\in\hat{V}_t^n\triangle\hat{V}_t^{n+1}} \mathbf{P}_w\left(X_{\tau_{(\hat{V}_t^n\cup\hat{V}_t^{n+1})\cup L_0}} = z\right) \mathbf{P}_z \\
&\quad \left(X_{\tau_{(\hat{V}_t^n\cap\hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n\cap\hat{V}_t^{n+1})\cup L_0}-1} = y\right).
\end{aligned}$$

Taking the summation over all $\vec{e} \in E_1$, and note that for all $z \in \hat{V}_t^n \triangle \hat{V}_t^{n+1}$,

$$\sum_{\vec{e}=x\to y\in E_1} \mathbf{P}_z\left(X_{\tau_{(\hat{V}_t^n\cap\hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n\cap\hat{V}_t^{n+1})\cup L_0}-1} = y\right) \le 1$$

since the summation above are over disjoint events. We have

$$\sum_{\vec{e}\in E_1} \mathcal{H}_{\hat{V}_t^n\cap\hat{V}_t^{n+1}}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n\cup\hat{V}_t^{n+1}}(\vec{e}) \le \mathcal{H}_{\hat{V}_t^n\cup\hat{V}_t^{n+1}}(\hat{V}_t^n \triangle \hat{V}_t^{n+1}).$$

Moreover, noting that by definition $\hat{V}_t^n \cup \hat{V}_t^{n+1}$ is connected in $\mathbb{H}$, and that

$$|\hat{V}_t^n \triangle \hat{V}_t^{n+1}| \le |V_t^{D,n}| \le |E_t^{D,n}| + 2,$$

by Theorem 3 we have,

$$\sum_{\vec{e}\in E_1} \mathcal{H}_{\hat{V}_t^n\cap\hat{V}_t^{n+1}}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n\cup\hat{V}_t^{n+1}}(\vec{e}) \le (|E_t^{D,n}| + 2)\sqrt{\log n}. \tag{21}$$

Now combining (19)–(21) and plugging them back to (18) gives us

$$\lambda^D(\hat{A}_t^n, \hat{A}_t^{n+1}) \le 17(|E_t^{D,n}| + 2)\sqrt{\log n}. \tag{22}$$

Recalling the definition of Δ_i, by Poisson thinning and the strong Markov property again we have

$$\mathbf{P}\left(|E_1^{D,n}| \ge n^\alpha\right) = \mathbf{P}\left(\sum_{i=0}^{n^\alpha-1} \Delta_i \le 1\right) \le \mathbf{P}\left(\sum_{i=0}^{n^\alpha-1} \sigma_i \le 1\right),$$

where $\{\sigma_i\}_{i=0}^{n^\alpha-1}$ is an independent sequence of exponential random variables with $\tilde{\lambda}_i = 17(i+2)\sqrt{\log n}$.

Thus, in order to prove Lemma 5.1, it suffices to prove the following result:

Lemma 5.2 *Let σ_i be defined as above. For all $\alpha < 1$, $\beta < \alpha$, and any $c_3 > 0$, for all n large enough*

$$\mathbf{P}\left(\sum_{i=0}^{n^\alpha-1} \sigma_i \le 1\right) < e^{-c_3 n^\beta}.$$

Proof For $\beta < \alpha$ defined in the lemma and some constants $c_1, c_2 > 0$ (to be chosen later) define the events for $j \in [1, n^\alpha/n^\beta] \cap \mathbb{N}$,

$$G_j = \left\{\left|\left\{(j-1)n^\beta \le i < jn^\beta : \sigma_i \ge \frac{c_2}{(i+2)\sqrt{\log n}}\right\}\right| > c_1 n^\beta\right\}.$$

Define $N_i = \mathbb{1}_{\left\{\sigma_i \ge \frac{c_2}{(i+2)\sqrt{\log n}}\right\}}$, thus $M_j = \sum_{i=(j-1)n^\beta}^{jn^\beta-1} N_i$ is a binomial random variable with parameters n^β and $p = \mathbf{P}\left(\sigma_i \ge \frac{c_2}{(i+2)\sqrt{\log n}}\right) = e^{-17c_2}$, which converges to 1 when $c_2 \to 0$. By the large deviation principle for binomial random variable,

$$\mathbf{P}(G_j^c) = \mathbf{P}\left(M_j \le c_1 n^\beta\right) \le e^{-I(c_1,p)n^\beta} \le e^{-c_3 n^\beta},$$

where the last inequality follows by taking p close enough to 1 such that $I(c_1, p) > c_3'$ (see [4] for the exact rate function). Since c_3' was arbitrary, for a slightly smaller c_3 we can obtain for large enough n,

$$\mathbf{P}\left(\bigcup_{j\in[1,\dots,n^\alpha/n^\beta]\cap\mathbb{N}} G_j^c\right) \le n^{\alpha-\beta} e^{-c_3' n^\beta} \le e^{-c_3 n^\beta}.$$

But under the event $\left\{\bigcap_{j\in[1,\ldots,n^\alpha/n^\beta]\cap\mathbb{N}} G_j\right\}$,

$$\sum_{i=1}^{n^\alpha}\sigma_i=\sum_{j=1}^{n^{\alpha-\beta}}\sum_{(j-1)n^\beta}^{jn^\beta-1}\sigma_i\geq\frac{c_2}{\sqrt{\log n}}\left(\frac{c_1n^\beta}{n^\beta+1}+\frac{c_1n^\beta}{2n^\beta+1}+\cdots+\frac{c_1n^\beta}{n^{\alpha-\beta}n^\beta+1}\right)$$
$$>\frac{1}{2}c_1c_2(\alpha-\beta)\sqrt{\log n}>1,$$

where the last two inequalities require taking a large enough n. □

Thus the proof of Lemma 5.1 completes. □

6 Locations of Discrepancies and Proof of Theorem 4

In the previous section, we have shown that, for any $\alpha>0$, by time 1 with stretch-exponentially high probability, there will be no more than n^α discrepancies. Now we show that it is highly unlikely that the first n^α possible discrepancies may ever reach our finite subset K.

To show this, note that now the truncated model $(\hat{A}_t^n,\hat{A}_t^{n+1})$ forms a finite state Markov process. In this section, it is more convenient to concentrate on the **embedded chain**

$$(\hat{A}_k^n,\hat{A}_k^{n+1}),\ k=0,1,2,\ldots$$

where all configuration $(\hat{A}_k^n,\hat{A}_k^{n+1})$ with

$$\hat{V}_k^n\cup\hat{V}_k^{n+1}\nsubseteq[-n-\log n,n+\log n]\times[0,\log n]$$

are absorbing states. It worth notice that the embedded chain here is a discrete time Markov chain.

Remark 9 Without causing further confusion, in this section we will use the parallel notations such as $(\hat{A}_k^n,\hat{A}_k^{n+1})$, $V_k^{D,n}$ and $E_k^{D,n}$ etc., for the embedded chain without more specification.

Recall the stopping times for the creation of new discrepancies:

$$\Delta_i=\inf\{k\geq 0,\ |E_k^{D,n}|=i\},$$

with the convention $\inf\emptyset=\infty$. In order to show Step (III), we only need to prove the lemma as follows:

Lemma 6.1 *There exists an $\alpha>0$ whose value will be specified later such that for any compact $K\subset\mathbb{H}$,*

$$\mathbf{P}\left(E_{\Delta_{n^\alpha}}^{D,n}\cap K\neq\emptyset\right)\leq n^{-1-\alpha}$$

for all sufficiently large n.

Proof We define

$$\vec{e}_i=\begin{cases}E_{\Delta_i}^{D,n}\setminus E_{\Delta_{i-1}}^{D,n}, & \text{if }\Delta_i<\infty\\ \emptyset & \text{otherwise}\end{cases}.$$

Note that $\vec{e}_i$ is either an empty set or a singleton with one edge. When it is a singleton, we do not distinguish between the singleton set and its unique element.

Now we are ready to introduce classifications on discrepancies as follows: Let $0 < \alpha < 1/5$.

- For any $i = 1$, we say $\vec{e}_1$ is **good** if either $\vec{e}_1 = \emptyset$ or

$$d(\vec{e}_1, (\pm(n+1), 0)) < n^{1-5\alpha}.$$

  Here $d(\cdot, \cdot)$ is defined as the minimum distance over all endpoints.
- For any $i \geq 1$, we say $\vec{e}_i$ is **good** if either $\vec{e}_i = \emptyset$ or

$$d(\vec{e}_i, E^{D,n}_{\Delta_{i-1}}) < n^{1-5\alpha}.$$

  Otherwise, we will say $\vec{e}_i$ is **bad**.
- If an $\vec{e}_i$ is bad, we call it **devastating** if and only if $\vec{e}_i$ intersects with $[-n^{1-3\alpha}, n^{1-3\alpha}] \times [0, \log n]$.

Moreover, one can also define

$$\kappa = \inf\{i \geq 1, \ s.t. \ \vec{e}_i \text{ is bad}\}.$$

By definition, one may see that for all sufficiently large n, $E^{D,n}_{\Delta_{n^\alpha}} \cap K \neq \emptyset$ only if either of the following two events happens:

- Event A: $\kappa < n^\alpha$, and $\vec{e}_\kappa$ is devastating.
- Event B: $\kappa < n^\alpha$, $\vec{e}_\kappa$ is bad but not devastating, and there is at least one bad event within $\kappa + 1, \kappa + 2, \ldots, n^\alpha$.

To see the above assertion, one can from the definition of A and B see that $(A \cup B)^c$ can also be written as the union of $C \cup D$, where the events are defined as follows:

- Event C: $\vec{e}_i$ are good for all $i = 1, 2, \ldots, n^\alpha$.
- Event D: $\kappa < n^\alpha$, $\vec{e}_\kappa$ is bad but not devastating, and there are no bad events within $\kappa + 1, \kappa + 2, \ldots, n^\alpha$.

Moreover, for each i, we define

$$l_i^+ = \min\left\{x_1 > 0 : \ s.t. \ \exists x_2 \text{ with } x = (x_1, x_2) \text{ a vertex for some edge within } E^{D,n}_{\Delta_i}\right\},$$

and

$$r_i^- = \max\left\{x_1 < 0 : \ s.t. \ \exists x_2 \text{ with } x = (x_1, x_2) \text{ a vertex for some edge within } E^{D,n}_{\Delta_i}\right\}.$$

Thus under event C or D,

$$l_i^+ \geq n^{1-3\alpha} - n^\alpha \times n^{1-5\alpha} \geq n^{1-3\alpha}/2,$$

and

$$r_i^- \leq -n^{1-3\alpha} + n^\alpha \times n^{1-5\alpha} \leq -n^{1-3\alpha}/2,$$

which implies no discrepancy may be within $[-n^{1-3\alpha}/2, n^{1-3\alpha}/2] \times [0, \log n] \supset K$ for all sufficiently large n.

Now we only need to find the desired upper bounds for the probabilities of events A and B. For any k, define the event

$$G_k = \{\vec{e}_i \text{ is good for } i = 1, \ldots, k-1\}.$$

6.1 Upper Bound on P(*A*)

For event A, by definition and the strong Markov property one has

$$\begin{aligned}\mathbf{P}(A) &= \sum_{k=1}^{n^\alpha} \mathbf{P}\left(G_k,\ \vec{e}_k \text{ is devastating}\right)\\ &= \sum_{k=1}^{n^\alpha}\sum_{j=0}^{\infty}\sum_{(\bar{A}_0,\tilde{A}_0)} \mathbf{P}\Big(G_k,\ \Delta_{k-1}<\infty,\ \Delta_k-\Delta_{k-1}>j,\ (\hat{A}^n_{\Delta_{k-1}+j}, \hat{A}^{n+1}_{\Delta_{k-1}+j}) = (\bar{A}_0, \tilde{A}_0)\Big)\\ &\quad \mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}\left(\Delta_1 = 1, \vec{e}_1 \text{ is devastating}\right), \end{aligned} \tag{23}$$

where $\mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}$ stands for the distribution of the truncated embedded process $(\hat{A}^n_k, \hat{A}^{n+1}_k)$ starting from initial condition $(\bar{A}_0, \tilde{A}_0)$.

At the same time, with similar calculation we have for any $k = 1, 2, \ldots, n^\alpha$,

$$\begin{aligned}&\mathbf{P}(G_k, \Delta_k < \infty)\\ &= \sum_{j=0}^{\infty}\sum_{(\bar{A}_0,\tilde{A}_0)} \mathbf{P}\Big(G_k, \Delta_{k-1}<\infty, \Delta_k-\Delta_{k-1}>j, (\hat{A}^n_{\Delta_{k-1}+j}, \hat{A}^{n+1}_{\Delta_{k-1}+j}) = (\bar{A}_0, \tilde{A}_0)\Big)\\ &\quad \mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}\left(\Delta_1 = 1\right) \le 1. \end{aligned} \tag{24}$$

Note that for any configuration $(\bar{A}_0, \tilde{A}_0)$ such that

$$\mathbf{P}\Big(G_k,\ \Delta_{k-1}<\infty,\ \Delta_k-\Delta_{k-1}>j,\ (\hat{A}^n_{\Delta_{k-1}+j}, \hat{A}^{n+1}_{\Delta_{k-1}+j}) = (\bar{A}_0, \tilde{A}_0)\Big) \neq 0,$$

one must have $|\bar{E}_0 \triangle \tilde{E}_0| \le k-1$. Now recalling the transition dynamic of the embedded chain, one has for all feasible $(\bar{A}_0, \tilde{A}_0)$ such that $\bar{V}_0 \cup \tilde{V}_0 \subset [-n-\log n, n+\log n]\times[0, \log n]$,

$$\mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}\left(\Delta_1 = 1\right) = \frac{\lambda^D(\bar{A}_0, \tilde{A}_0)}{\lambda^T(\bar{A}_0, \tilde{A}_0)},$$

where $\lambda^D(\cdot,\cdot)$ was defined in (18) and

$$\lambda^T(\bar{A}_0, \tilde{A}_0) = \sum_{\vec{e}} \max\{\mathcal{H}_{\bar{V}_0}(\vec{e}), \mathcal{H}_{\tilde{V}_0}(\vec{e})\}.$$

Otherwise $\mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}\left(\Delta_1 = 1\right) = 0$. Now for

$$\mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}\left(\Delta_1 = 1, \vec{e}_1 \text{ is devastating}\right),$$

recall that in (18) we have

$$\begin{aligned}\lambda^D(\bar{A}_0, \tilde{A}_0) &= \sum_{\vec{e}\in E_1}\left|\mathcal{H}_{\bar{V}_0}(\vec{e}) - \mathcal{H}_{\tilde{V}_0}(\vec{e})\right|\\ &\quad + \sum_{\vec{e}\in E_2}\mathcal{H}_{\bar{V}_0}(\vec{e}) + \sum_{\vec{e}\in E_3}\mathcal{H}_{\bar{V}_0}(\vec{e}) + \sum_{\vec{e}\in E_4}\mathcal{H}_{\bar{V}_0}(\vec{e})\\ &\quad + \sum_{\vec{e}\in E_5}\mathcal{H}_{\tilde{V}_0}(\vec{e}) + \sum_{\vec{e}\in E_6}\mathcal{H}_{\tilde{V}_0}(\vec{e}) + \sum_{\vec{e}\in E_7}\mathcal{H}_{\tilde{V}_0}(\vec{e}). \end{aligned}$$

For any $\vec{e} \in \cup_{i=2}^{7} E_i$, recall that at least one of the endpoints of $\vec{e}$ has to be in $\bar{V}_0 \triangle \tilde{V}_0$. Thus it is easy to see

$$d(\vec{e}, E^{D,n}_{\Delta_{k-1}}) = 0.$$

Combining this with the fact that for all feasible $(\bar{A}_0, \tilde{A}_0)$, $\bar{E}_0 \triangle \tilde{E}_0 \subset (-\infty, -n+2n^{1-4\alpha}) \cup (n-2n^{1-4\alpha}, \infty) \times [0, \log n]$, which is disjoint with $[-2n^{1-3\alpha}, 2n^{1-3\alpha}] \times [0, \log n]$, we have

$$\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(\Delta_1 = 1, \vec{e}_1 \text{ is devastating}) \le \frac{\sum_{\vec{e}=(x,y)\in E_1, |x_1| \le 2n^{1-3\alpha}} \left| \mathcal{H}_{\bar{V}_0}(\vec{e}) - \mathcal{H}_{\tilde{V}_0}(\vec{e}) \right|}{\lambda^T(\bar{A}_0, \tilde{A}_0)} \quad (25)$$

when $\bar{V}_0 \cup \tilde{V}_0 \subset [-n - \log n, n + \log n] \times [0, \log n]$ and equals to 0 otherwise. Thus for any configuration $(\bar{A}_0, \tilde{A}_0)$ such that

$$\mathbf{P}\left(G_k, \ \Delta_{k-1} < \infty, \ \Delta_k - \Delta_{k-1} > j, \ (\hat{A}^n_{\Delta_{k-1}+j}, \hat{A}^{n+1}_{\Delta_{k-1}+j}) = (\bar{A}_0, \tilde{A}_0)\right) \neq 0,$$

and that

$$\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(\Delta_1 = 1, \vec{e}_1 \text{ is devastating}) \neq 0,$$

we have

$$\frac{\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(\Delta_1 = 1, \vec{e}_1 \text{ is devastating})}{\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(\Delta_1 = 1)} \le \frac{\sum_{\vec{e}=(x,y)\in E_1, |x_1| \le 2n^{1-3\alpha}} \left| \mathcal{H}_{\bar{V}_0}(\vec{e}) - \mathcal{H}_{\tilde{V}_0}(\vec{e}) \right|}{\lambda^D(\bar{A}_0, \tilde{A}_0)}. \quad (26)$$

Now for the numerator of (26), again we have

$$\begin{aligned}
&\sum_{\vec{e}=(x,y)\in E_1, |x_1| \le 2n^{1-3\alpha}} \left| \mathcal{H}_{\bar{V}_0}(\vec{e}) - \mathcal{H}_{\tilde{V}_0}(\vec{e}) \right| \\
&\quad \le \sum_{\vec{e}=(x,y)\in E_1, |x_1| \le 2n^{1-3\alpha}} \left[\mathcal{H}_{\bar{V}_0 \cap \tilde{V}_0}(\vec{e}) - \mathcal{H}_{\bar{V}_0 \cup \tilde{V}_0}(\vec{e}) \right] \\
&\quad = \sum_{\vec{e}=(x,y)\in E_1, |x_1| \le 2n^{1-3\alpha}} \sum_{z \in \bar{V}_0 \triangle \tilde{V}_0} \mathcal{H}_{\bar{V}_0 \cup \tilde{V}_0}(z) \mathbf{P}_z\left(X_{\tau_{(\bar{V}_0 \cap \tilde{V}_0)\cup L_0} - 1} = y, X_{\tau_{(\bar{V}_0 \cap \tilde{V}_0)\cup L_0}} = x \right) \\
&\quad \le \mathcal{H}_{\bar{V}_0 \cup \tilde{V}_0}(\bar{V}_0 \triangle \tilde{V}_0) \sup_{z \in \bar{V}_0 \triangle \tilde{V}_0} \mathbf{P}_z\left(\tau_{Box} < \tau_{L_0}\right),
\end{aligned} \quad (27)$$

where

$$Box = [-2n^{1-3\alpha}, 2n^{1-3\alpha}] \times [0, \log n].$$

At the same time, note that for any feasible configuration $(\bar{A}_0, \tilde{A}_0)$,

$$\bar{V}_0 \triangle \tilde{V}_0 \subset Box_0 = [n - 2n^{1-4\alpha}, n + \log n] \cup [-n - \log n, -n + 2n^{1-4\alpha}] \times [0, \log n]$$

which implies that

$$\sup_{z \in \bar{V}_0 \triangle \tilde{V}_0} \mathbf{P}_z\left(\tau_{Box} < \tau_{L_0}\right) \le \sup_{z \in Box_0} \mathbf{P}_z\left(\tau_{Box} < \tau_{L_0}\right). \quad (28)$$

Moreover, for each edge $\vec{e} = (z, w)$ such that $z \in \bar{V}_0 \triangle \tilde{V}_0$ and $w \notin \bar{V}_0 \cup \tilde{V}_0$, by definition it has to belong to $E_3 \cup E_6$ and thus by (18)

$$\lambda^D(\bar{A}_0, \tilde{A}_0) \ge \mathcal{H}_{\bar{V}_0 \cup \tilde{V}_0}(\bar{V}_0 \triangle \tilde{V}_0). \quad (29)$$

Now combining (23)–(29) we have

$$\mathbf{P}(A) \leq n^{\alpha} \sup_{x\in Box_0} \mathbf{P}_x\left(\tau_{Box} < \tau_{L_0}\right). \tag{30}$$

Now we prove the following lemma:

Lemma 6.2 *For all $\alpha < 1/5$ and all sufficiently large n,*

$$\sup_{x\in Box_0} \mathbf{P}_x\left(\tau_{Box} < \tau_{L_0}\right) \leq n^{-1-2.5\alpha}.$$

Proof The proof of Lemma 6.2 follows a similar argument as in [11]. Note that for any $x \in Box_0$,

$$\mathbf{P}_x\left(\tau_{Box} < \tau_{L_0}\right) \leq \sum_{y\in\partial^{in} Box} \mathbf{P}_x(\tau_y < \tau_{L_0}).$$

Let $V_n = \{n/2\} \times [0, \infty)$, $V_n^1 = n/2 \times [0, n^4)$, and $V_n^2 = n/2 \times (n^4, \infty)$. By a similar argument as in [11] we have

$$\mathbf{P}_x\left(\tau_{V_n} < \tau_{L_0}\right) \leq n^{-1+\alpha/5}, \tag{31}$$

while

$$\mathbf{P}_x\left(\tau_{V_n} < \tau_{L_0}, \tau_{V_n} = \tau_{V_n^2}\right) \leq \frac{1}{n^3}.$$

Thus by the strong Markov property,

$$\begin{aligned} \mathbf{P}_x(\tau_y < \tau_{L_0}) &= \sum_{z\in V_n} \mathbf{P}_x\left(\tau_{V_n} < \tau_{L_0}, \tau_{V_n} = \tau_z\right) \mathbf{P}_z(\tau_y < \tau_{L_0}) \\ &\leq \frac{1}{n^3} + \sum_{z\in V_n^1} \mathbf{P}_x\left(\tau_{V_n} < \tau_{L_0}, \tau_{V_n} = \tau_z\right) \mathbf{P}_z(\tau_y < \tau_{L_0}). \end{aligned} \tag{32}$$

Moreover, for each $z \in V_n^1$, by reversibility of random walk ([6]), we have

$$\mathbf{P}_z(\tau_y < \tau_{L_0}) \leq \mathbf{P}_y(\tau_z < \tau_{L_0}) \mathbf{E}_z[\text{\# of visits to } z \text{ in } [0, \tau_{L_0})]. \tag{33}$$

For the first term in (33), the same argument for (31) implies that

$$\mathbf{P}_y(\tau_z < \tau_{L_0}) \leq \mathbf{P}_y(\tau_{V_n} < \tau_{L_0}) \leq n^{-1+\alpha/5}.$$

While for the second term in (33), by [11] there is a constant $C < \infty$ independent of n such that for all $z \in V_n^1$

$$\mathbf{E}_z[\text{\# of visits to } z \text{ in } [0, \tau_{L_0})] \leq C \log n.$$

Thus we have

$$\mathbf{P}_z(\tau_y < \tau_{L_0}) \leq Cn^{-1+\alpha/5} \log n. \tag{34}$$

Combining (31)–(34), we have for any $x \in Box_0$, $y \in \partial^{in} Box$,

$$\mathbf{P}_x(\tau_y < \tau_{L_0}) \leq Cn^{-2+2\alpha/5} \log n.$$

Finally, noting that $|\partial^{in} Box| \leq 5n^{1-3\alpha}$, we have

$$\sup_{x\in Box_0} \mathbf{P}_x\left(\tau_{Box} < \tau_{L_0}\right) \leq Cn^{-2+2\alpha/5} \log n \cdot n^{1-3\alpha} \leq n^{-1-2.5\alpha}$$

for all sufficiently large n. □

Combining (30) and Lemma 6.2, we have

$$\mathbf{P}(A) \le n^{\alpha} \sup_{x \in Box_0} \mathbf{P}_x \left(\tau_{Box} < \tau_{L_0}\right) \le n^{-1-1.5\alpha}. \tag{35}$$

6.2 Upper Bound on $\mathbf{P}(B)$

Now we find the upper bound for $\mathbf{P}(B)$. Recall that

- Event B: $\kappa < n^{\alpha}$, $\vec{e}_\kappa$ is bad but not devastating, and there is at least one bad event within $\kappa + 1, \kappa + 2, \ldots, n^{\alpha}$.

For any $k \ge 1$ define the event

$$B_k = \{\vec{e}_1, \ldots, \vec{e}_{k-1} \text{ are good}, \vec{e}_k \text{ is bad}\}.$$

By the Markov property, we have

$$\mathbf{P}(B) = \sum_{k=1}^{n^{\alpha}-1} \sum_{(\bar{A}_0, \tilde{A}_0)} \mathbf{P}\left(B_k, \vec{e}_k \text{ is not devastating}, (\hat{A}^n_{\Delta_k}, \hat{A}^{n-1}_{\Delta_k}) = (\bar{A}_0, \tilde{A}_0)\right) \left(\sum_{j=1}^{n^{\alpha}-k} \mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(B_j)\right). \tag{36}$$

Using the argument in Subsection 6.1, we have for all $k + j \le n^{\alpha}$ and any feasible configuration $(\bar{A}_0, \tilde{A}_0)$ such that

$$\mathbf{P}\left(B_k, \vec{e}_k \text{ is not devastating}, (\hat{A}^n_{\Delta_k}, \hat{A}^{n-1}_{\Delta_k}) = (\bar{A}_0, \tilde{A}_0)\right) \ne 0$$

and such that $\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(B_i) > 0$ for some $i \le n^{\alpha} - k$, we have

$$\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(B_j) \le \mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(G_j, \Delta_j < \infty)\mathbf{P}_{(0,\log n)}\left(\tau_{U_n} < \tau_{L_0}\right) \le \mathbf{P}_{(0,\log n)}\left(\tau_{U_n} < \tau_{L_0}\right),$$

where $U_n = \{-n^{1-5\alpha}/2, n^{1-5\alpha}/2\} \times [0, \infty)$. Again from [11], we have

$$\mathbf{P}_{(0,\log n)}\left(\tau_{U_n} < \tau_{L_0}\right) \le n^{-1+6\alpha}. \tag{37}$$

Thus by (36) and (37),

$$\mathbf{P}(B) \le n^{-1+7\alpha}\left(\sum_{k=1}^{n^{\alpha}-1} \mathbf{P}(B_k)\right). \tag{38}$$

Again using the same argument, we have for any $k \le n^{\alpha} - 1$,

$$\mathbf{P}(B_k) \le \mathbf{P}(G_k, \Delta_k < \infty)\mathbf{P}_{(0,\log n)}\left(\tau_{U_n} < \tau_{L_0}\right) \le n^{-1+6\alpha}$$

which implies that

$$\mathbf{P}(B) \le n^{-2+14\alpha}. \tag{39}$$

Letting $\alpha = 1/16$, Lemma 6.1 follows from Lemma 6.2 and (39). □

Proof of Theorem 4 At this point, Theorem 4 follows from the combination of Lemma 5.1 and Lemma 6.1. □

7 Proof of Theorem 1: Existence of the SDLA

Theorem 1 follows immediately once we show that the limiting process obtained by Theorem 4 has the desired property.

Lemma 7.1 *Fix a finite set K, $t > 0$ and some $\epsilon > 0$. $\exists N$ finite a.s., such that for all $n > N$, for all $0 \le s \le t$ and any $x \in K$,*

$$|\mathcal{H}_{L_0 \cup A_s^n}(x) - \mathcal{H}_{L_0 \cup A_s}(x)| < \epsilon. \tag{40}$$

Proof By [11, Lemma 2.6] and the sub-linear growth of the interface model proved in Theorem 5 and the fact we constructed all A_s^n to be subsets of the interface model, there exists some $m > 0$ such that for every $n \in \mathbb{N} \cup \{\infty\}$ and $x \in K$,

$$\left| \sum_{|y|<m^{1.1}} \mathbf{P}_{(y,m)} \left(S_{\tau_{L_0 \cup A_s^n}} = x \right) - \mathcal{H}_{L_0 \cup A_s^n}(x) \right| < \epsilon/2. \tag{41}$$

Let $K' \subset \mathbb{H}$ be a large finite subset such that

$$2m^{1.1} \max_{|y|<m^{1.1}} \mathbf{P}_{(y,m)}(\tau_{K'^c} < \tau_K) < \epsilon/2.$$

By Theorem 4 we know that there is some $N \in \mathbb{N}$ large enough such that for every $n > N$,

$$A_s^n \cap K' = A_s^N \cap K' = A_s \cap K'.$$

Thus

$$\left| \sum_{|y|<m^{1.1}} \mathbf{P}_{(y,m)} \left(S_{\tau_{L_0 \cup A_s^n}} = x \right) - \sum_{|y|<m^{1.1}} \mathbf{P}_{(y,m)} \left(S_{\tau_{L_0 \cup A_s}} = x \right) \right| < \epsilon/2.$$

Together with (41) we obtain (40). □

It remains to prove that $\{A_s\}_{s \le t}$ is Markov with the correct stationary harmonic measure as the transition rate.

Lemma 7.2 *For any $t > 0$, for any $s \in [0, t]$ and $x, y \in \mathbb{H}$,*

$$\lim_{\Delta s \to 0} \frac{\mathbf{P}\Big(A_{s+\Delta s}(x) = 1 \Big| A_s(x) = 0, \{A_\xi\}_{\xi \le s} \Big)}{\Delta s} = \mathcal{H}_{L_0 \cup A_s}(x) \text{ a.s.}$$

and.

$$\lim_{\Delta s \to 0} \frac{\mathbf{P}\Big(A_{s+\Delta s}(x) = 1,\, A_{s+\Delta s}(y) = 1 \Big| A_s(x) = 0,\, A_s(y) = 0,\, \{A_\xi\}_{\xi \le s} \Big)}{\Delta s} = 0 \text{ a.s.}$$

Proof Let $\epsilon > 0$ and G_n be the event that for all $s \le t$ and for all $x \in K$, $A_s^n(x) = A_s(x)$ and in addition,

$$|\mathcal{H}_{L_0 \cup A_s^n}(x) - \mathcal{H}_{L_0 \cup A_s}(x)| < \epsilon.$$

By Lemma 7.1 and Theorem 4, $\lim_{n\to\infty} \mathbf{P}(G_n^c) = 0$. Now uniformly for all $s < t$ and Δs small enough, there is an $n \in \mathbb{N}$ such that

$$\begin{aligned}
&\mathbf{P}\left(A_{s+\Delta s}(x) = 1 | A_s(x) = 0, \{A_\xi\}_{\xi\le s}\right) \\
&\quad \in \mathbf{P}\left(A_{s+\Delta s}(x) = 1 | A_s(x) = 0, \{A_\xi\}_{\xi\le s}, G_n\right) + (-\epsilon, \epsilon) \\
&\quad = \mathbf{P}\left(A^n_{s+\Delta s}(x) = 1 | A^n_s(x) = 0, \{A_\xi\}_{\xi\le s}, G_n\right) + (-\epsilon, \epsilon) \\
&\quad \in \mathbf{P}\left(A^n_{s+\Delta s}(x) = 1 | A^n_s(x) = 0, |\mathcal{H}_{L_0\cup A^n_s}(x) - \mathcal{H}_{L_0\cup A_s}(x)| < \epsilon, A_s\right) + (-2\epsilon, 2\epsilon) \\
&\quad \in (1 - e^{-\Delta s(\mathcal{H}_{L_0\cup A_s}(x)+\epsilon)}, 1 - e^{-\Delta s(\mathcal{H}_{L_0\cup A_s}(x)-\epsilon)}) + (-2\epsilon, 2\epsilon),
\end{aligned}$$

where we use the dominated convergence theorem for the first and second approximations. Now taking $\epsilon \to 0$ and then $\Delta s \to 0$ we obtain the first result. The second result follows a similar proof by noting that for A^n_s one can order distinct arrival times. □

Lemma 7.3 *$\{A_s\}_s$ is a Markov process i.e. For all finite $K \subset \mathbb{H}$ and any $J \in \{0,1\}^K$, $\mathbf{P}\Big(A_{s+t} \cap K = J \Big| \{A_\xi\}_{\xi\le s}\Big) = \mathbf{P}\Big(A_{s+t} \cap K = J \Big| A_s\Big)$.*

Proof This proof follows a similar scheme as the previous one. Let $\tilde{G}_n$ be the event that for all $\xi \le t+s$ and for all $x \in K$, $A^n_\xi(x) = A_\xi(x)$. By the proof of Theorem 4, with high probability the dynamics up to time $t+s$ inside K does not depend on the configuration in the complement of $K_m := [-m, m] \times [0, \infty)$ for large enough $m \in \mathbb{N}$ i.e. for any $\epsilon > 0$ we can find an $m \in \mathbb{N}$ such that

$$\begin{aligned}
&\mathbf{P}\Big(A_{s+t} \cap K = J \Big| \{A_\xi\}_{\xi\le s}\Big) \\
&\quad \in \mathbf{P}\Big(A_{s+t} \cap K = J \Big| \{A_\xi \cap K_m\}_{\xi\le s}\Big) + (-\epsilon, \epsilon) \\
&\quad \in \mathbf{P}\Big(A^n_{s+t} \cap K = J \Big| \{A^n_\xi \cap K_m\}_{\xi\le s}, \tilde{G}_n\Big) + (-2\epsilon, 2\epsilon) \\
&\quad \in \mathbf{P}\Big(A^n_{s+t} \cap K = J \Big| A^n_s \cap K_m, \tilde{G}_n\Big) + (-3\epsilon, 3\epsilon) \\
&\quad = \mathbf{P}\Big(A_{s+t} \cap K = J \Big| A_s \cap K_m, \tilde{G}_n\Big) + (-3\epsilon, 3\epsilon) \\
&\quad = \mathbf{P}\Big(A_{s+t} \cap K = J \Big| A_s \cap K_m\Big) + (-4\epsilon, 4\epsilon) \\
&\quad = \mathbf{P}\Big(A_{s+t} \cap K = J \Big| A_s\Big) + (-5\epsilon, 5\epsilon),
\end{aligned}$$

where the first equality uses the Markovity of $A^n_\cdot$, and the second equality uses the conditioning on $\tilde{G}_n$ for n large enough (and larger than m). □

Proof of Theorem 1 By Lemmas 7.2 and 7.3 we obtain that the almost sure limit $\{A_s\}_{s\le t} := \lim_{m\to\infty}\{A^m_s\}_{s\le t}$ obtained in Theorem 4 is a SDLA. □

8 Proof of Theroem 2: Ergodocity of the SDLA

Proof By Lemma 7.2 and the fact that the stationary harmonic measure is (well...) stationary, we obtain that A^∞_t is stationary with respect to the translation $\lambda_n(A^\infty_t) = A^\infty_t + n$, for any $n \in \mathbb{Z}$. It is enough to prove that A^∞_t is strongly mixing. Let $t > 0$ and K_1, K_2 be two finite subsets of $\mathbb{H}$ of horizontal distance

$$\min\{|x_1 - x_2| : \exists y_1, y_2 \in \mathbb{N} \cup \{0\},\ (x_1, y_1) \in K_1, (x_2, y_2) \in K_2\} > 4n$$

(n will be chosen big enough). Choose arbitrary $x_1, x_2 \in \mathbb{Z}$ such that $\exists y_1, y_2 \in \mathbb{N} \cup \{0\}$, satisfying $(x_1, y_1) \in K_1, (x_2, y_2) \in K_2$. We now consider two copies of A_t^n constructed according to Poisson thinning of the same interface model. $A_t^n(1)$ is centered around x_1 and $A_t^n(2)$ is centered around x_2 i.e. for $i \in \{1, 2\}$ the initial aggregation of $A_t^n(i)$ is $V_0^n(i) = [x_i - n, x_i + n] \times 0$. For $i \in \{1, 2\}$ and configurations $\xi_i \in \{0, 1\}^{K_i}$, define the events:

$$\mathscr{B}_i = \{A_t^\infty \cap K_i = \xi_i\}, \tag{42}$$

$$\mathscr{C}_i = \{A_t^n(i) \cap K_i = \xi_i\}, \tag{43}$$

$$\mathscr{D}_i = \{\max_{x \in A_t^n(i)} |x - x_i| < 3n/2\}. \tag{44}$$

Under the event $\mathscr{D}_1 \cap \mathscr{D}_2$, the events $\mathscr{C}_1$ and $\mathscr{C}_2$ are independent. This follows from the independence of Poisson processes on non intersecting domains. Moreover we know by Theorem 5 that

$$\lim_{n\to\infty} \mathbf{P}\left(\mathscr{D}_1^c \cup \mathscr{D}_2^c\right) = 0,$$

and by Theorem 4 that

$$\lim_{n\to\infty} \mathbf{P}\left(\mathscr{B}_1 \setminus \mathscr{C}_1 \cup \mathscr{B}_2 \setminus \mathscr{C}_2\right) = 0.$$

Thus

$$\lim_{n\to\infty} \mathbf{P}(\mathscr{B}_1 \cap \mathscr{B}_2) = \lim_{n\to\infty} \mathbf{P}(\mathscr{C}_1 \cap \mathscr{C}_2 | \mathscr{D}_1 \cap \mathscr{D}_2) = \lim_{n\to\infty} \mathbf{P}(\mathscr{C}_1 | \mathscr{D}_1 \cap \mathscr{D}_2) \cdot \mathbf{P}(\mathscr{C}_2 | \mathscr{D}_1 \cap \mathscr{D}_2) \tag{45}$$

$$= \lim_{n\to\infty} \mathbf{P}(\mathscr{B}_1) \cdot \mathbf{P}(\mathscr{B}_2) = \mathbf{P}(\mathscr{B}_1) \cdot \mathbf{P}(\mathscr{B}_2), \tag{46}$$

where in the last equality we used stationarity and abused notations to clarify that the limit is actually a constant sequence. □

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