Counterexamples to L^p collapsing estimates

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Abstract We show that certain L^2 space-time estimates for generalized density matrices which have been used by several authors in recent years to study equations of BBGKY or Hartree-Fock type, do not have non-trivial $L^p L^q$ generalizations.

1. Introduction and main results

In recent years, effective equations approximating the evolution of a large number of interacting bosons or fermions have been studied extensively. The best-known example is derivation of the cubic nonlinear Schrödinger equation in the celebrated work of Erdös, Schlein, and Yau [6, 7].

Since that work, a number of authors have studied the related Gross–Pitaevskii or Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchies, or the Hartree–Fock or Hartree–Fock–Bogoliubov equations, using harmonic analysis techniques and space-time L^2 estimates for a suitable trace density of solutions of the linear Schrödinger equation. We call such estimates "collapsing estimates" and list several instances, all in 3 space dimensions (thus, $x \in \mathbb{R}^3$, etc.).

If

$$G(t, x, y, z) = e^{\frac{it(\Delta_x + \Delta_y - \Delta_z)}{2}} G_0, \tag{1}$$

then

$$\left\| \nabla_x G(t, x, x, x) \right\|_{L^2(dt\, dx)} \lesssim \left\| \nabla_x \nabla_y \nabla_z G_0(x, y, z) \right\|_{L^2(dx\, dy\, dz)}. \tag{2}$$

For completeness, we mention how the above collapse (and estimate) occurs in applications. Consider solutions to the N-body linear Schrödinger equation

$$\begin{cases} \left(\frac{1}{i}\frac{\partial}{\partial t} - \sum_{i=1}^{N} \Delta_{x_j} + \frac{1}{N}\sum_{i<j} v_N(x_i - x_j)\right)\psi_N(t, x_1, \ldots, x_N) = 0, \\ \psi_N(0, x_1, \ldots x_N) = (\text{or} \sim)\phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_N), \end{cases}$$

where N is large, $x_k \in \mathbb{R}^3$, $v \in \mathcal{S}$, $v \geq 0$, $0 < \beta \leq 1$, and $v_N(x) = N^{3\beta} v(N^\beta x)$. The function ψ_N has $L^2(\mathbb{R}^{3N})$ norm 1 and is symmetric in the space variables. It describes the evolution of a large number of interacting bosons. The initial conditions represent a Bose–Einstein condensate. The problem is to approximate ψ_N with tensor

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products $\phi(t,x_1)\phi(t,x_2)\cdots\phi(t,x_N)$, where ϕ is normalized such that $\|\phi\|_{L^2(\mathbb{R}^3)} = 1$ and satisfies the cubic nonlinear Schrödinger (NLS) equation $\frac{1}{i}\frac{\partial}{\partial t}\phi - \Delta\phi + c|\phi|^2\phi = 0$. The approximation should hold as $N \to \infty$. The approach used in [6] and [7] is to average out most variables by taking a partial trace and look at the marginal density "matrices" in the remaining variables:

$$\gamma_N^{(k)}(t,\mathbf{x_k},\mathbf{y_k}) = \int \psi_N(t,\mathbf{x_k},\mathbf{x_{N-k}})\bar{\psi}_N(t,\mathbf{y_k},\mathbf{x_{N-k}})d\mathbf{x_{N-k}}.$$

Here, $\mathbf{x_k} = (x_1,\ldots,x_k)$ and $\mathbf{x_{N-k}} = (x_{k+1},\ldots,x_N)$. The $\gamma_N^{(k)}$ satisfy the BBGKY hierarchy:

$$\begin{aligned}
&\Big(\frac{1}{i}\frac{\partial}{\partial t} + \Delta_{x_1} - \Delta_{y_1}\Big)\gamma_N^{(1)}(t,x_1;y_1)\\
&\quad = -\frac{N-1}{N}\int v_N(x_1-x_2)\gamma_N^{(2)}(t,x_1,x_2;y_1,x_2)\,dx_2\\
&\qquad + \frac{N-1}{N}\int v_N(y_1-y_2)\gamma_N^{(2)}(t,x_1,y_2;y_1,y_2)\,dy_2.
\end{aligned}$$

There are similar equations relating $\gamma_N^{(k)}$ to $\gamma_N^{(k+1)}$. Formally, as $N \to \infty$, $\gamma_N^{(k)} \to \gamma^{(k)}$, which satisfies the Gross–Pitaevskii infinite hierarchy

$$\begin{aligned}
&\Big(\frac{1}{i}\frac{\partial}{\partial t} + \Delta_{x_1} - \Delta_{y_1}\Big)\gamma^{(1)}(t,x_1;y_1)\\
&\quad = -c\gamma^{(2)}(t,x_1,x_1;y_1,x_1) + c\gamma^{(2)}(t,x_1,y_1;y_1,y_1),\\
&\Big(\frac{1}{i}\frac{\partial}{\partial t} + \Delta_{x_1,x_2} - \Delta_{y_1,y_2}\Big)\gamma^{(2)}\\
&\quad = \text{ terms involving } \gamma^{(3)} \text{ with 3 collapsed variables,}\\
&\ldots
\end{aligned}$$

Naïvely, one expects $c = \int v$ and this is the case if $0 \le \beta < 1$, but c is the scattering length of v if $\beta = 1$. One solution to the above hierarchy is given by tensor products $\gamma^{(1)}(t,x_1;y_1) = \phi(t,x_1)\bar{\phi}(t,y_1)$, and similarly for higher $\gamma^{(k)}$. Estimates of the type (2) applied to $\gamma^{(k)}$ were introduced in [14] to simplify the original proof of [6] for the uniqueness of solutions to the hierarchy (see also [1, 3, 4]). The periodic case is treated in [10] and [13], as well as [5] (for the quintic NLS).

Another related example is as follows: If

$$\Lambda(t,x,y) = e^{\frac{it(\Delta_x+\Delta_y)}{2}}\Lambda_0, \tag{3}$$

then

$$\big\||\nabla|_x^{1/2}\Lambda(t,x,x)\big\|_{L^2(dt\,dx)} \lesssim \big\||\nabla|_x^{1/2}|\nabla|_y^{1/2}\Lambda_0(x,y)\big\|_{L^2(dx\,dy)}. \tag{4}$$

This estimate is useful for the Hartree–Fock–Bogoliubov equations (see [11, 12]). These equations are a coupled system of nonlinear Schrödinger–type equations for functions on $3+1$ variables and $6+1$ variables. Compared to the cubic NLS equation, they provide a "better" approximation for solutions to the system (1). The derivation requires Fock space techniques. The nonlinear terms in these equation contain factors such as $v_N(x-y)\Lambda(t,x,y)$ and, as $N \to \infty$, $v_N \to c\delta$.

Finally, if

$$\Gamma(t,x,y) = e^{\frac{it(\Delta_x - \Delta_y)}{2}}\Gamma_0, \tag{5}$$

then

$$\left\| |\nabla_x|^{\frac{1}{2}} \langle \nabla_x \rangle^{2\epsilon} \Gamma(t,x,x) \right\|_{L^2(dt\,dx)} \lesssim_\epsilon \left\| \langle \nabla_x \rangle^{\frac{1}{2}+\epsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\epsilon} \Gamma_0(x,y) \right\|_{L^2(dx\,dy)}. \tag{6}$$

Such estimates are relevant to both the Hartree–Fock–Bogoliubov equations mentioned above, and Hartree–Fock (see [2, Theorem 3.3]). The Hartree–Fock equations are effective equations approximating the evolution of a large number of fermions. The nonlinear terms in these equations include factors such as $\Gamma(t,x,x)$.

We also mention the approach of [8] and [9] which applies to equation (5) and allows a wide range of $L^p(dt)L^q(dx)$ estimates on the left-hand side, but the right-hand side of the inequality is estimated in a Schatten norm.

It is natural to ask whether one can replace the $L^2(dt)L^2(dx)$ norm on the left-hand side of estimates (2), (4), or (6) by an $L^p(dt)L^q(dx)$ norm while keeping the right-hand side in a Sobolev norm, which is useful for applications to PDEs. One can trivially make p or q bigger than 2 by putting more derivatives on the right-hand side, so the interesting question is if one can make p or q less than 2.

The main result of this note is that this is impossible.

We prove the following closely related results.

THEOREM 1.1

Let $n \geq 1$. Let Λ be given by (3), with $x, y \in \mathbb{R}^n$. Assume

$$\left\| |\nabla|_x^\alpha \Lambda(t,x,x) \right\|_{L^p(dt)L^q(dx)} \lesssim \left\| \Lambda_0(x,y) \right\|_{H^s(dx\,dy)} \tag{7}$$

for some $\alpha \geq 0$, $s \geq 0$. Then $p \geq 2$ and $q \geq 2$.

THEOREM 1.2

Let $n \geq 1$. Let Γ be given by (5), with $x, y \in \mathbb{R}^n$. Assume

$$\left\| |\nabla|_x^\alpha \Gamma(t,x,x) \right\|_{L^p(dt)L^q(dx)} \lesssim \left\| \Gamma_0(x,y) \right\|_{H^s(dx\,dy)} \tag{8}$$

for some $\alpha \geq 0$, $s \geq 0$. Then $p \geq 2$ and $q \geq 2$.

THEOREM 1.3

Let $n \geq 1$. Let G be given by (1), with $x, y, z \in \mathbb{R}^n$. Assume

$$\left\| |\nabla|_x^\alpha G(t,x,x,x) \right\|_{L^p(dt)L^q(dx)} \lesssim \left\| G_0(x,y,z) \right\|_{H^s(dx\,dy\,dz)} \tag{9}$$

for some $\alpha \geq 0$, $s \geq 0$. Then $p \geq 2$ and $q \geq 2$.

Notation. We write $A \lesssim B$ if $A \leq CB$ for some absolute constant C, $A \sim B$ if $A \lesssim B$ and $B \lesssim A$, $A \lesssim_\epsilon B$ if $A \leq C_\epsilon B$ for some constant C_ϵ depending on ϵ, where ϵ is an arbitrary positive number.

2. Proofs

2.1. Proof of Theorem 1.1

2.1.1. Necessity of $p \geq 2$. Let R be a large number (which will approach ∞ at the end of the proof). Let C be a fixed large number (depending on n). Let

$$F_0(x, y) = e^{-\frac{|x|^2+|y|^2}{2CR}}$$

so that

$$e^{\frac{it(\Delta x+\Delta y)}{2}} F_0 := F(t, x, y) = \frac{1}{(1+it/(CR))^n} e^{-\frac{|x|^2+|y|^2}{2(CR+it)}}. \tag{10}$$

We think of $F(t, x, y)$ as the basic "vertical tube" solution to the linear Schrödinger equation in $2n+1$ dimensions which is essentially 1 if $|x|, |y| \leq R^{1/2}$, $0 \leq t \leq R$. The rigorous statement is that C is chosen so that $\Re F(t, x, y) \geq \frac{1}{2}$ in the above range, where $\Re F(t, x, y)$ denotes the real part of F. Also, the Fourier transform (in space) of F is essentially supported at frequencies $|\xi|, |\eta| \leq R^{-1/2}$.

We choose the function $\Lambda(t, x, y)$ to be a sum of translates and modulations of $F(t, x, y)$ which are inclined at 45 degrees and are trained to reach the region $|x| \leq \frac{1}{100}$, $|y| \leq \frac{1}{100}$, $R - R^{\frac{1}{2}} < t < R$ with almost the same oscillation (and almost no cancellations). The summands will have Fourier transforms essentially supported in balls of radius $R^{-1/2}$ centered at unit vectors.

Explicitly, choose roughly $R^{n-\frac{1}{2}}$ points (x_k, y_k) which are spaced at distance $R^{1/2}$ from each other on the sphere $|(x, y)| = R$. For technical reasons, we choose only points for which all coordinates are $\geq \frac{R}{10n}$. Define $(\xi_k, \eta_k) = \frac{(x_k, y_k)}{R}$.

Choose the following initial conditions:

$$\Lambda_0(x, y) = \sum e^{i(x\cdot\xi_k + y\cdot\eta_k)} F_0(x + x_k, y + y_k).$$

The functions being summed are approximately orthogonal and each has L^2 norm $\sim R^{n/2}$:

$$\begin{aligned} &\int \left| F_0(x + x_k, y + y_k) F_0(x + x_l, y + y_l) \right| dx\, dy \\ &\qquad = \pi^n (CR)^n e^{-\frac{|(x_k, y_k) - (x_l, y_l)|^2}{4CR}}. \end{aligned} \tag{11}$$

Recalling that the sum has $\sim R^{n-\frac{1}{2}}$ terms, we derive

$$\|\Lambda_0\|_{L^2(dx\,dy)} \lesssim R^{n-\frac{1}{4}}.$$

The same type of upper bound holds for higher order derivatives (since $|(\xi_k, \eta_k)| = 1$); thus, for each fixed s,

$$\|\Lambda_0\|_{H^s(dx\,dy)} \lesssim R^{n-\frac{1}{4}}. \tag{12}$$

The solution looks like

$$\begin{aligned} \Lambda(t, x, y) &= \sum e^{-it\frac{(|\xi_k|^2+|\eta_k|^2)}{2}} e^{i(x\cdot\xi_k + y\cdot\eta_k)} F(t, x + x_k - t\xi_k, y + y_k - t\eta_k) \\ &= e^{-i\frac{t}{2}} \sum e^{i(x\cdot\xi_k + y\cdot\eta_k)} F(t, x + x_k - t\xi_k, y + y_k - t\eta_k), \end{aligned}$$

and

$$\left|\Lambda(t,x,y)\right| \geq \Re \sum e^{i(x\cdot\xi_k + y\cdot\eta_k)} F(t, x + x_k - t\xi_k, y + y_k - t\eta_k) \sim R^{n-\frac{1}{2}},$$

if $|(x,y)| \leq \frac{1}{100}$, $R - R^{\frac{1}{2}} < t < R$. Thus,

$$R^{\frac{1}{2p}} R^{n-\frac{1}{2}} \lesssim \left\|\Lambda(t,x,x)\right\|_{L^p(dt)L^q(dx)}, \tag{13}$$

so, recalling (12), if

$$\left\|\Lambda(t,x,x)\right\|_{L^p(dt)L^q(dx)} \lesssim \left\|\Lambda_0(x,y)\right\|_{H^s(dx\,dy)},$$

then $p \geq 2$.

Using the product rule and the lower bounds on the components of ξ_k, η_k, the same argument works for ordinary derivatives of order $\alpha = m \in \mathbb{N}$.

To justify the statement for fractional derivatives of noninteger order α, do a Littlewood–Paley decomposition in space $\Lambda(t,\cdot,\cdot) = P_{\leq 10}\Lambda(t,\cdot,\cdot) + P_{\geq 10}\Lambda(t,\cdot,\cdot)$, where $P_{\leq 10}$ localizes functions of $2n$ variables, smoothly at frequencies ≤ 10. Then $P_{\geq 10}\Lambda(t,\cdot,\cdot)$ is exponentially small as $R \to \infty$. This is true for the function F_0, and its translates by a unit vector in Fourier space.

A crude estimate is

$$\left\|P_{\geq 10}\Lambda(t,\cdot,\cdot)\right\|_{H^s} \lesssim_s e^{-\sqrt{R}}.$$

For our counterexample, we use $P_{\leq 10}\Lambda(t,\cdot,\cdot)$ instead of $\Lambda(t,\cdot,\cdot)$.

Thus, for R sufficiently large, $|\nabla^m P_{\leq 10}\Lambda(t,x,y)| \sim |\nabla^m \Lambda(t,x,y)| \sim R^{n-\frac{1}{2}}$ if $|(x,y)| \leq \frac{1}{100}$, $R - R^{\frac{1}{2}} < t < R$. The function $(P_{\leq 10}\Lambda)(t,x,x)$ is supported, in Fourier space, at frequencies $|\xi| \leq 20$. Denote, by abuse of notation, $P_{\leq 20}$ the operator localizing functions of n variables at frequencies $|\xi| \leq 20$. Let $m \in \mathbb{N}$, $m > \alpha$. Then the operator $\frac{\nabla^m}{|\nabla|^\alpha} P_{\leq 20}$ (defined in the obvious way on the Fourier transform side) is bounded on all L^p spaces, and

$$\begin{aligned} R^{\frac{1}{2p}} R^{n-\frac{1}{2}} &\lesssim \left\|\nabla^m (P_{\leq 10}\Lambda)(t,x,x)\right\|_{L^p(dt)L^q(dx)} \\ &= \left\|\frac{\nabla^m}{|\nabla|^\alpha} P_{\leq 20} |\nabla|^\alpha (P_{\leq 10}\Lambda)(t,x,x)\right\|_{L^p(dt)L^q(dx)} \\ &\lesssim \left\||\nabla|^\alpha (P_{\leq 10}\Lambda)(t,x,x)\right\|_{L^p(dt)L^q(dx)}, \end{aligned}$$

while

$$\|P_{\leq 10}\Lambda_0\|_{H^s(dx\,dy)} \lesssim C^n R^{n-\frac{1}{4}}.$$

Letting $R \to \infty$, we conclude $p \geq 2$ as before.

2.1.2. Necessity of $q \geq 2$. Let $F(t,x,y)$ be the basic vertical tube solution of height R (as in (10)). Let $m \gg 1$. Choose roughly $R^{mn-\frac{n}{2}}$ points x_k which are spaced at distance $\sim R^{\frac{1}{2}}$ in a large ball $B(0,R^m)$ of radius R^m in $\mathbb{R}^n$. Fix a unit vector $\xi \in S^{n-1}$.

We take initial conditions

$$\Lambda_0(x,y) = e^{i(x+y)\cdot\xi} \sum F_0(x + x_k, y + x_k).$$

Then

$$\Lambda(t,x,y) = e^{i(x+y)\cdot\xi} e^{-it} \sum F(t, x + x_k - t\xi, y + x_k - t\xi).$$

There are roughly $R^{mn-\frac{n}{2}}$ terms in the sum. The summands are essentially orthogonal (as in (11)), and each term has L^2 norm $\sim R^{n/2}$; thus,

$$\|\Lambda_0\|_{L^2(dx\,dy)} \sim R^{\frac{n}{4}+\frac{mn}{2}}.$$

On the other hand, each $F(t, x + x_k - t\xi, y + x_k - t\xi)$ is essentially 1 on a tube T_k of radius $R^{1/2}$ and length R in $2n + 1$ dimensions and rapidly decaying out of T_k. Note that at $t = 0$, T_k is centered at $(0, -x_k, -x_k)$. Moreover, these tubes T_k are in the same direction $(1, \xi, \xi)$ and hence disjoint. Therefore, $|\Lambda(t,x,y)| \gtrsim 1$ on the union of the tubes T_k. In particular, $|\Lambda(t,x,x)| \gtrsim 1$ for $0 \le t \le R$ and $x \in B(t\xi, R^m)$. We need only the previous estimate for $0 \le t \le 1$, where the claim is obvious. In addition, the Fourier transform of $\Lambda(t,x,x)$ is supported (essentially) in a $R^{-\frac{1}{2}}$ neighborhood of the point 2ξ, with $|\xi| = 1$, so $||\nabla|^\alpha \Lambda(t,x,x)| \gtrsim 1$ for $0 \le t \le 1$ and $x \in B(t\xi, R^m)$. Thus,

$$\left\| |\nabla|^\alpha \Lambda(t,x,x) \right\|_{L^p([0,1])L^q(dx)} \gtrsim R^{\frac{mn}{q}},$$

while $\|\Lambda_0\|_{H^s(dx\,dy)} \sim \|\Lambda_0\|_{L^2(dx\,dy)} \sim R^{\frac{n}{4}+\frac{mn}{2}}$ and $m \gg 1$, so $q \ge 2$ is necessary.

2.2. *Proof of Theorem 1.2*

The examples for Γ are similar to those for Λ and are included for completeness.

2.2.1. Necessity of $p \ge 2$. First we take the basic "vertical tube" solution. Let

$$F_0(x,y) = e^{-\frac{|x|^2+|y|^2}{2CR}}$$

so that

$$e^{\frac{it(\Delta_x - \Delta_y)}{2}} F_0 := F(t,x,y) = \frac{1}{(1 + (\frac{t}{CR})^2)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2(CR+it)}} e^{-\frac{|y|^2}{2(CR-it)}}. \tag{14}$$

The solution $F(t,x,y)$ is essentially 1 if $|x|, |y| \le R^{1/2}$, $0 \le t \le R$. More precisely, we choose a large constant $C = C(n)$ so that $\Re F(t,x,y) \ge \frac{1}{2}$ in the above range. Also, as before, the Fourier transform (in space) of F is essentially supported at frequencies $|\xi|, |\eta| \le R^{-1/2}$.

Pick roughly $R^{n-\frac{1}{2}}$ points (x_k, y_k) which are spaced at distance $\sim R^{1/2}$ from each other on the surface $\{(x,y) : |x| = |y|, \frac{R}{2} \le |x| \le R\}$. Define $(\xi_k, \eta_k) = \frac{1}{R}(x_k, y_k)$ so that $|\xi_k|^2 - |\eta_k|^2 = 0$ and $|(\xi_k, \eta_k)| \sim 1$.

Take the following initial conditions,

$$\Gamma_0(x,y) = \sum e^{i(x\cdot\xi_k - y\cdot\eta_k)} F_0(x + x_k, y + y_k),$$

so that the solution is

$$\begin{aligned}\Gamma(t,x,y) &= \sum e^{-it\frac{(|\xi_k|^2 - |\eta_k|^2)}{2}} e^{i(x\cdot\xi_k - y\cdot\eta_k)} F(t, x + x_k - t\xi_k, y + y_k - t\eta_k) \\ &= \sum e^{i(x\cdot\xi_k - y\cdot\eta_k)} F(t, x + x_k - t\xi_k, y + y_k - t\eta_k).\end{aligned}$$

Since the $\sim R^{n-\frac{1}{2}}$ terms in Γ_0 are essentially orthogonal and each has L^2 norm $\sim R^{n/2}$, we get

$$\|\Gamma_0\|_{L^2(dx\,dy)} \lesssim R^{n-\frac{1}{4}}.$$

Moreover, since $|(\xi_k, \eta_k)| \sim 1$, there also holds

$$\|\Gamma_0\|_{H^s(dx\,dy)} \lesssim R^{n-\frac{1}{4}}. \tag{15}$$

From the expression of Γ, we see that

$$\big|\Gamma(t,x,y)\big| \gtrsim R^{n-\frac{1}{2}} \quad \text{for } \big|(x,y)\big| \le \frac{1}{100}, R - R^{\frac{1}{2}} < t < R.$$

Therefore,

$$\big\|\Gamma(t,x,x)\big\|_{L^p(dt)L^q(dx)} \gtrsim R^{\frac{1}{2p}} R^{n-\frac{1}{2}};$$

so, recalling (15), if

$$\big\|\Gamma(t,x,x)\big\|_{L^p(dt)L^q(dx)} \lesssim \big\|\Gamma_0(x,y)\big\|_{H^s(dx\,dy)},$$

then $p \ge 2$. From a similar argument to the one in Section 2.1.1 (i.e., using only x_k, y_k for which all coordinates of ξ_k and $-\eta_k$ are $\ge \frac{1}{10n}$), $p \ge 2$ is also necessary for estimates of the form

$$\big\||\nabla|_x^\alpha \Gamma(t,x,x)\big\|_{L^p(dt)L^q(dx)} \lesssim \big\|\Gamma_0(x,y)\big\|_{H^s(dx\,dy)}.$$

2.2.2. Necessity of $q \ge 2$. Let $F(t,x,y)$ be the basic vertical tube solution of height R (as in (14)). Let $m \gg 1$. Choose roughly $R^{mn-\frac{n}{2}}$ points x_k which are spaced at distance $\sim R^{\frac{1}{2}}$ in a large ball $B(0,R^m)$ of radius R^m in $\mathbb{R}^n$. Fix a unit vector $\xi \in S^{n-1}$.

We take initial conditions

$$\Gamma_0(x,y) = e^{ix\cdot\xi} \sum F_0(x + x_k, y + x_k),$$

so that the solution is

$$\Gamma(t,x,y) = e^{ix\cdot\xi} \sum F(t, x + x_k - t\xi, y + x_k).$$

Note that $\Gamma(t,x,x) \gtrsim 1$ for $0 \le t \le 1$ and $|x| \le R^m$. Moreover, the Fourier transform of $\Gamma(t,x,x)$ is essentially supported in a $R^{-1/2}$ neighborhood of the point ξ with $|\xi| = 1$.

Then, the necessity of $q \ge 2$ follows from the same calculation as in Section 2.1.2.

2.3. Proof of Theorem 1.3

The examples for G are similar to those in previous subsections.

2.3.1. Necessity of $p \ge 2$. First we take the basic "vertical tube" solution. Let

$$F_0(x,y,z) = e^{-\frac{|x|^2+|y|^2+|z|^2}{2CR}}$$

so that

$$e^{\frac{it(\Delta_x+\Delta_y-\Delta_z)}{2}}F_0 := F(t,x,y,z) \tag{16}$$
$$= \frac{1}{(1+\frac{it}{CR})^n(1-\frac{it}{CR})^{\frac{n}{2}}} e^{-\frac{|x|^2+|y|^2}{2(CR+it)}} e^{-\frac{|z|^2}{2(CR-it)}}.$$

The solution $F(t,x,y,z)$ is essentially 1 if $|(x,y,z)| \le R^{1/2}$, $0 \le t \le R$. Also, the Fourier transform (in space) of F is essentially supported at frequencies $|(\xi,\eta,\zeta)| \le R^{-1/2}$.

Pick roughly $R^{\frac{3n-1}{2}}$ points (x_k,y_k,z_k) which are spaced at distance $\sim R^{1/2}$ from each other on the surface $\{(x,y,z) : |x|^2+|y|^2 = |z|^2, \frac{R}{2} \le |x|,|y| \le R\}$. Define $(\xi_k,\eta_k,\zeta_k) = \frac{1}{R}(x_k,y_k,z_k)$ so that

$$|\xi_k|^2+|\eta_k|^2 = |\zeta_k|^2 \qquad \text{and} \qquad \big|(\xi_k,\eta_k,\zeta_k)\big| \sim 1.$$

Take the following initial conditions,

$$G_0(x,y,z) = \sum e^{i(x\cdot\xi_k+y\cdot\eta_k-z\cdot\zeta_k)} F_0(x+x_k, y+y_k, z+z_k)$$

so that the solution is

$$G(t,x,y,z)$$
$$= \sum e^{i(x\cdot\xi_k+y\cdot\eta_k-z\cdot\zeta_k)} F(t, x+x_k-t\xi_k, y+y_k-t\eta_k, z+z_k-t\zeta_k)$$

since $|\xi_k|^2+|\eta_k|^2 = |\zeta_k|^2$.

Since the $\sim R^{\frac{3n-1}{2}}$ terms in G_0 are essentially orthogonal and each has L^2 norm $\sim R^{3n/4}$, we get

$$\|G_0\|_{L^2(dx\,dy\,dz)} \lesssim R^{\frac{3n}{2}-\frac{1}{4}}.$$

Moreover, since $|(\xi_k,\eta_k,\zeta_k)| \sim 1$, there also holds

$$\|G_0\|_{H^s(dx\,dy\,dz)} \lesssim R^{\frac{3n}{2}-\frac{1}{4}}. \tag{17}$$

From the expression of G, we see that

$$\big|G(t,x,y,z)\big| \gtrsim R^{\frac{3n-1}{2}} \quad \text{for } \big|(x,y,z)\big| \le \frac{1}{100}, R-R^{\frac{1}{2}} < t < R.$$

Therefore,

$$\big\|G(t,x,x,x)\big\|_{L^p(dt)L^q(dx)} \gtrsim R^{\frac{1}{2p}} R^{\frac{3n-1}{2}}.$$

Recalling (17), if

$$\big\|G(t,x,x,x)\big\|_{L^p(dt)L^q(dx)} \lesssim \big\|G_0(x,y,z)\big\|_{H^s(dx\,dy\,dz)},$$

then $p \ge 2$. From a similar argument as in Section 2.1.1, $p \ge 2$ is also necessary for estimates of the form

$$\big\||\nabla|_x^{\alpha} G(t,x,x,x)\big\|_{L^p(dt)L^q(dx)} \lesssim \big\|G_0(x,y,z)\big\|_{H^s(dx\,dy\,dz)}.$$

2.3.2. Necessity of $q \geq 2$. Let $F(t, x, y, z)$ be the basic vertical tube solution of height R (as in (16)). Let $m \gg 1$. Choose roughly $R^{mn-\frac{n}{2}}$ points x_k which are spaced at distance $\sim R^{\frac{1}{2}}$ in a large ball $B(0, R^m)$ of radius R^m in $\mathbb{R}^n$. Fix a unit vector $\xi \in S^{n-1}$.

We take initial conditions

$$G_0(x, y, z) = e^{i(x+y-z)\cdot\xi} \sum F_0(x + x_k, y + x_k, z + x_k)$$

so that the solution is

$$G(t, x, y) = e^{\frac{-it}{2}} e^{i(x+y-z)\cdot\xi} \sum F(t, x + x_k - t\xi, y + x_k - t\xi, z + x_k - t\xi).$$

There are roughly $R^{mn-\frac{n}{2}}$ terms in the sum. The summands are essentially orthogonal,and each term has L^2 norm $\sim R^{3n/4}$; thus,

$$\|G_0\|_{L^2(dx\,dy\,dz)} \sim R^{\frac{n}{2}+\frac{mn}{2}}.$$

On the other hand, each $F(t, x + x_k - t\xi, y + x_k - t\xi, z + x_k - t\xi)$ is essentially 1 on a tube T_k of radius $R^{1/2}$ and length R in $3n + 1$ dimensions and rapidly decaying out of T_k. Note that at $t = 0$, T_k is centered at $(0, -x_k, -x_k, -x_k)$. Moreover, these tubes T_k are in the same direction $(1, \xi, \xi, \xi)$ and hence disjoint. Therefore, $|G(t, x, y, z)| \gtrsim 1$ on the union of the tubes T_k. In particular, $|G(t, x, x, x)| \gtrsim 1$ for $0 \leq t \leq R$ and $x \in B(t\xi, R^m)$. Thus.

$$\big\|G(t, x, x, x)\big\|_{L^p([0,1])L^q(dx)} \gtrsim R^{\frac{mn}{q}}$$

(with a similar estimate for $|\nabla|^\alpha G(t, x, x, x)$), while $\|G_0\|_{H^s(dx\,dy)} \sim R^{\frac{n}{2}+\frac{mn}{2}}$ and $m \gg 1$, so $q \geq 2$ is necessary.

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