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RESIDUAL FINITENESS FOR CENTRAL PUSHOUTS

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Abstract. We prove that pushouts $A *_C B$ of residually finite-dimensional (RFD) C^*-algebras over central subalgebras are always residually finite-dimensional provided the fibers A_p and B_p, $p \in \operatorname{spec} C$ are RFD, recovering and generalizing results by Korchagin and Courtney-Shulman. This then allows us to prove that certain central pushouts of amenable groups have RFD group C^*-algebras. Along the way, we discuss the problem of when, given a central group embedding $H \leq G$, the resulting C^*-algebra morphism is a continuous field: this is always the case for amenable G but not in general.

Introduction

Residual finiteness properties have elicited considerable interest, both in the operator algebra literature and in group theory. On the operator-algebraic side one typically considers *residually finite-dimensional* (henceforth RFD) C^*-algebras, i.e. those whose elements are separated by representations on finite-dimensional Hilbert spaces. The group-theoretic analogue is the concept of a *residually finite* (or RF) group, i.e. one whose finite-index normal subgroups intersect trivially (i.e. having "enough" finite quotients).

The literature on RFD C^*-algebras is rather substantial, as is that on RF groups; so much so, in fact, that it would be impossible to do it justice. For samplings (the best a short note such as this one can do) we direct the reader to, say, [2, 3, 11, 14, 15, 18, 20, 21, 25] (for RFD C^*-algebras) and [4–6] or [22, §6.5], [24, Chapters 6, 14, 15], [13, Chapter 2] (for RF groups), and references therein.

We are concerned here with the types of "permanence" properties for residual finiteness as studied, say, in [3, 5, 11, 14, 20, 21, 25], to the effect that various types of pushouts (also known as amalgamated free products) of RFD or RF objects are again such. Specifically, the main result of [20] is that pushouts of separable commutative C^*-algebras are RFD. More generally, the main theorem of [14] proves this for central pushouts $A *_C B$ with A and B separable and *strongly* RFD, i.e. such that all of their quotients are RFD.

The present note is motivated in part by the desire to recover these results without the separability and strong RFD-ness assumptions.

The preliminary Section 1 recalls some background and sets conventions.

In the short Section 2 we prove Lemma 2.1, stating that central pushouts of RFD C^*-algebras with RFD fibers are RFD.

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Section 3 is devoted to the problem of when (or whether), given a central group embedding $H \leq G$, the resulting embedding $C^*(H) \to C^*(G)$ is a continuous field over spec $C^*(H)$ in the sense of Lemma 1.1 below. This holds for amenable groups (Lemma 3.2), but not, for instance, for property-(T) groups (Lemma 3.4).

Section 4 centers around the variant of [25, Theorem 6.9] obtained in Lemma 4.1. The latter proves that (the full group C^*-algebra of) $G_1 *_H G_2$ is RFD provided G_i are amenable residually finite (RF) and $H \leq G_i$ is a common central subgroup such that the quotients G_i/H are RF. The former result, on the other hand, assumes that $G_1 *_H G_2$ itself is RF.

Although Lemma 4.1 is formally stronger for that reason, we nevertheless show in Lemma 4.6 that its hypotheses imply the residual finiteness of the pushout $G_1 *_H G_2$. [25, Theorem 6.9] and Lemma 4.1 are thus equivalent, albeit non-obviously.

1. Preliminaries

1.1. **Fields of C^*-algebras.** All C^*-algebras and pushouts are assumed unital. Pushouts over $\mathbb{C}$ are undecorated, i.e. $A * B$ denotes what in the literature (e.g. [15]) is sometimes referred to as $A *_{\mathbb{C}} B$. Given a C^* morphism $C \to A$ with $C = C(X)$ commutative, we denote by A_p, the *fiber* of A at the point p of the spectrum X of C:

$$A_p := A/\langle \ker p \rangle,$$

where $p \in X$ is regarded as a character $p : C(X) \to \mathbb{C}$ and angled brackets denote the ideal generated by the respective set. Similarly, for $a \in A$ we denote by a_p its image through the surjection $A \to A_p$.

We will refer to a central C^* morphism $C \to A$ as a C-*algebra* A (e.g. [19, Definition 1.5] or [9, Définition 2.6]). Following standard practice (see [9, Définition 3.1] for instance), we have

Definition 1.1. A C-algebra A constitutes a *continuous field of C^*-algebras* if for every A the norm function

$$\operatorname{spec} C \ni p \mapsto \|a\|_{A_p}$$

is continuous.

The function is known to always be *upper* semicontinuous, for example by [23, Proposition 1.2].

1.2. **Group filtrations.** Following [5, §2.2], we need the following notion applicable to a residually finite group G.

Definition 1.2. A *filtration* on G is a family of finite-index normal subgroups $N_\alpha \trianglelefteq G$ with trivial intersection.

Let $H \leq G$ be a subgroup. A filtration $\{N_\alpha\}$ of G is *an H-filtration* provided

$$\bigcap_\alpha HN_\alpha = H.$$

2. Pushouts over central subalgebras

The main result of this section is

Theorem 2.1. *Let A and B be two C^*-algebras, $C \subseteq A, B$ central C^*-embeddings, and assume all corresponding fibers A_p and B_p are RFD for*

$$p \in X := \text{spec } C.$$

Then, $M := A *_C B$ is RFD.

Proof. Because C is central in A and B it is central in M. Note the isomorphism

$$M_p \cong A_p *_{C_p} B_p \cong A_p * B_p.$$

For $m \in M$ we have

$$\|m\| = \sup_{p \in X} \|m_p\|_{M_p} \tag{2-1}$$

by [9, Proposition 2.8] and hence we can approximate the norm of m arbitrarily well with the norms of its images through representations of $M_p \cong A_p * B_p$ as p ranges over X. By the RFD-ness assumption on A_p and B_p, their coproduct $A_p * B_p$ is again RFD by [15, Theorem 3.2]. This finishes the proof. ☐

Remark 2.2. Note that in fact, in the proof above one does not need the precise norm estimate (2-1): all that is needed is that every $0 \neq m \in M$ be non-zero in some quotient $M \to M_p$, $p \in X$.

Remark 2.3. The conclusion of Lemma 2.1 cannot hold without the RFD-fiber assumption, as shown in [25, Proposition 8.3]. That result relies on the construction in [1] of an amenable RF group Γ with a central subgroup $N < \Gamma$ such that Γ/N is not RF, and in that context proves that the pushout $C^*(\Gamma) *_{C^*(N)} C^*(\Gamma)$ is not RFD.

We can see directly, in this case, that the fiber

$$C^*(\Gamma)_p \cong C^*(\Gamma/N)$$

at the point $p \in \text{spec } C^*(N)$ corresponding to the trivial morphism $N \to \{1\}$ is not RFD. Indeed, since Γ/N is finitely generated (finitely presented, even) by [1], the RFD-ness of $C^*(\Gamma/N)$ would entail the residual finiteness of Γ/N (e.g. by [25, Proposition 2.4]), yielding a contradiction.

3. Fibers over dense sets

It will be useful, in working with group C^*-algebras, to allow the points $p \in X$ from Lemma 2.1 to range only over a dense subset $Y \subseteq X := \text{spec } C$ rather than the entire spectrum. We cannot do this in full generality, as (2-1) does not hold for the supremum over only a dense subset $Y \subseteq X$:

Example 3.1. Let $I = [0, 1]$, $M = C_b(I)$ (all bounded functions) and $C = C(I) \subset M$ (continuous functions), both equipped with the supremum norm. The indicator function m of $\{0\} \subset I$ is non-zero, even though its image in every fiber

$$M_p,\ p \in Y := (0, 1] \subset I$$

vanishes. For that reason, we do *not* have

$$\|m\| = \sup_{p \in Y} \|m_p\|_{M_p}.$$

What went wrong in the above example is that M was not a continuous field over C in the sense of Lemma 1.1. One might hope that the pathology in Lemma 3.1 is at least avoidable for *group* C^*-algebras; this is not the case, as Lemma 3.4 below shows. Nevertheless, for amenable groups one can do better (see also [10, Theorem 3.7] for a broader discussion of the continuity of pushout fields).

Theorem 3.2. *Let G_i, $i = 1, 2$ be two amenable groups and $H \le G_i$ a common central subgroup. Then,*

$$C^*(G_1 *_H G_2) \cong C^*(G_1) *_{C^*(H)} C^*(G_2)$$

is a continuous field of C^*-algebras over $\widehat{H} = \operatorname{spec} C^*(H)$ in the sense of Lemma 1.1.

Proof. Since, as noted in §1.1, for $C := C^*(H)$ the map

$$C^*(G_1) *_C C^*(G_2) =: M \ni m \mapsto \|m_p\|_{M_p}$$

is always upper semicontinuous, it remains to prove that for every dense subset

$$Y \subseteq \widehat{H}$$

of the spectrum of and every $m \in M$ we have

$$\|m\| = \sup_{p \in Y} \|m_p\|_{M_p}.$$

First, fix an arbitrary unitary representation ρ_2 of G_2 where H acts by scalars, with central character $p_0 \in \widehat{H}$. Let

$$Y \ni p_\lambda \to p_0$$

be a net of characters from Y converging to p_0 and set

$$q_\lambda := p_\lambda p_0^{-1} \to 1 \in \widehat{H}$$

be the corresponding "error".

Because

$$G_i/H, \ i = 1, 2$$

are amenable homogeneous spaces of G_i respectively in the sense of [8, §2], the trivial representation $\mathbf{1}_{G_2}$ is weakly contained in the induced representation $\operatorname{Ind}_H^{G_2} \mathbf{1}_H$. Since $q_\lambda \to \mathbf{1}_H$ and induction is continuous [7, Theorem F.3.5], we have

$$\operatorname{Ind}_H^{G_2} q_\lambda \to \operatorname{Ind}_H^{G_2} \mathbf{1}_H \to \mathbf{1}_{G_2}$$

in the Fell topology, and hence

$$\rho_2 \otimes \operatorname{Ind}_H^{G_2} q_\lambda \to \rho_2 \otimes \mathbf{1}_{G_2} \cong \rho_2. \tag{3-1}$$

Because H acts by p_0 in ρ_2, it acts via $p_\lambda = q_\lambda p_0$ in the left hand side of (3-1). In short, we can Fell-approximate ρ_2 with unitary representations where H acts by characters from Y.

Now fix $m \in M$. According to [9, Proposition 2.8], the norm of m is achieved in some unitary representation ρ of $G := G_1 *_H G_2$ where H acts by scalars, with central character $p_0 \in \widehat{H}$. Working only with non-degenerate (indeed, unital) representations isomorphic to their own $\aleph_0$-multiples, [16, Lemma 2.4] implies that it will suffice to approximate ρ arbitrarily well in the Fell topology [7, Appendix F.2] with unitary representations where H operates with central characters belonging to the dense subset $Y \subseteq \widehat{H}$.

In turn, in order to achieve the above it is enough to approximate the restrictions $\rho|_{G_i}$ with representations where H acts via elements of Y. This, however, is what the first part of the proof does. $\square$

Over the course of the proof of Lemma 3.2 we have obtained

Proposition 3.3. *Let $H \leq G$ be a central subgroup of an amenable group. Then, for every dense subset*

$$Y \subseteq \widehat{H} = \text{spec } C^*(H)$$

the canonical map

$$C^*(G) \to \prod_{p \in Y} C^*(G)_p \tag{3-2}$$

is one-to-one.

In other words, a variant of [9, Proposition 2.8] with only a *dense* subset of the spectrum rather the entirety of it.

To prepare the ground for Lemma 3.4, note that by [7, Theorem F.4.4] (3-2) is an embedding if and only if the G-representations where H acts by characters in Y form a dense set in the Fell topology. This cannot possibly happen if

- G has the Kazhdan property (T) (e.g. [7, §1.1]), and hence the trivial representation is isolated in the unitary dual;
- the dense subset $Y \subset \widehat{H}$ does not contain the trivial element.

To construct such examples all we need is a property-(T) group G with an *infinite* central subgroup H, whereupon we can simply take

$$Y = \widehat{H} \setminus \{1\}.$$

Example 3.4. Let G be the universal cover $\widetilde{Sp_4(\mathbb{R})}$ of the 4×4 real symplectic group. It is shown in [17, Theorem 6.8] that G has property (T) (indeed, even the stronger property (T*); note that the authors of that paper denote Sp_4 by Sp_2).

The fundamental group of $Sp_4(\mathbb{R})$ is $\mathbb{Z}$, so we can simply take that copy of $\mathbb{Z}$ as the infinite central subgroup $H < G$.

Finally, if a *discrete* group is desired then one can simply take the preimage through

$$\widetilde{Sp_4(\mathbb{R})} \to Sp_4(\mathbb{R})$$

of any lattice in the latter (it will again have property (T) by [7, Theorem 1.7.1]). See also [7, §1.7] for a discussion of property (T) permanence under passage to universal covering groups.

4. Central pushouts of RF groups

The present section attempts to prove a slightly more general version of [25, Theorem 6.9]. The difference is that we only assume that G_i are individually RF rather than assuming that $G_1 *_H G_2$ is.

Theorem 4.1. *Let G_i, $i = 1, 2$ be two amenable groups and $H \leq G_i$ a common central subgroup such that*

- *each G_i is RF;*
- *each G_i/H is RF.*

Then, $G_1 *_H G_2$ is RFD.

Proof. We will obtain the result as an application of Lemma 3.2, with

$$A = C^*(G_1),\ B = C^*(G_2) \text{ and } C = C^*(H).$$

According to that result, what we have to argue is that the fibers A_p and B_p are RFD for a dense set of points p in the spectrum

$$\widehat{H} = \text{Pontryagin dual of } H = \operatorname{spec} C^*(H).$$

This is precisely what Lemma 4.5 does, finishing the proof. □

Lemma 4.2. *Let $H \le G$ be a central inclusion such that G and G/H are both RF. Then, the characters $p : H \to \mathbb{S}^1$ whose kernels*

$$N = \ker p$$

give rise to RF quotients G/N form a dense subset of the Pontryagin dual $\widehat{H}$.

Proof. Since G is RF the normal finite-index subgroups $G_\alpha \trianglelefteq G$ have trivial intersection. Additionally, since G/H is RF, G has an H filtration in the sense of Lemma 1.2 (e.g. [25, Proposition 3.3]) and hence

$$\bigcap_\alpha HG_\alpha = H. \tag{4-1}$$

Since

$$H_\alpha := G_\alpha \cap H \le H$$

have trivial intersection, the union of the duals

$$\widehat{H/H_\alpha} \subseteq \widehat{H}$$

is a dense subgroup. We claim that any $p \in \widehat{H/H_\alpha}$ will meet the requirements in the statement; proving this will achieve the desired conclusion, so it is the task we turn to next.

Fix such a character $p : H \to \mathbb{S}^1$, factoring through some

$$H \to H/H_{\alpha_0},$$

and let $N \le H$ be its kernel. Then, by its very construction, N will contain $H_{\alpha_0} = H \cap G_{\alpha_0}$. This means that

$$NG_{\alpha_0} \cap H \subseteq N$$

(which is then an equality), and hence the intersection

$$\bigcap_\alpha NG_\alpha \subseteq H$$

(where the latter inclusion uses (4-1)) cannot possibly contain N *strictly*. In conclusion we have

$$\bigcap_\alpha NG_\alpha = N,$$

meaning that the filtration $\{G_\alpha\}$ is compatible with N (i.e. an N-filtration) and hence G/N is RF. □

Remark 4.3. The RF requirements in Lemma 4.1 are both crucial:

On the one hand, [25, §8] recalls the example given by Abels in [1] of a central inclusion $\mathbb{Z} < G$ into an RF amenable group such that $G/\mathbb{Z}$ is not RF.

On the other hand, [12] contains an example (attributed there to C. Kanta Gupta) of a non-residually finite amenable group G whose quotient by an order-two normal subgroup $K \trianglelefteq G$ is residually finite. Centrality is easy to arrange, since the centralizer of K in G will have finite index in the latter.

Although Lemma 4.2 ensures that the quotients G/N for

$$N = \ker(p : H \to \mathbb{S}^1)$$

are RF, we have yet to prove that the resulting fiber algebras $C^*(G)/\langle h - p(h),\ h \in H\rangle$ are RFD.

Lemma 4.4. *Let $H \le G$ be a central subgroup, $p \in \widehat{H}$ a character of finite order, and $N = \ker p$ its kernel in H. If G/N is RFD then so is the fiber C^*-algebra*

$$C^*(G)_p := C^*(G)/\langle h - p(h),\ h \in H\rangle \tag{4-2}$$

corresponding to p is RFD.

Proof. Note that $C^*(G)_p$ is precisely the fiber of $C^*(G/N)$ at the character $\overline{p}$ induced by p on the quotient (finite cyclic) group H/N: indeed, denoting images of elements $g \in G$ in G/N by $\overline{g}$, we have

$$C^*(G)_p = C^*(G)/\langle h - p(h),\ h \in H\rangle = C^*(G/N)/\left\langle \overline{h} - \overline{p}\left(\overline{h}\right),\ \overline{h} \in H/N\right\rangle = C^*(G/N)_{\overline{p}}$$

because the kernel of $C^*(G) \to C^*(G/N)$ is generated by $h - 1$, $h \in N$, which are already contained in the ideal we are modding out in (4-2).

For that reason, we may as well assume that

- N is trivial, and hence
- H is finite cyclic;
- G is RFD.

But now note that $C^*(G)_p$ is a fiber of the RFD C^*-algebra over the *finite-dimensional* central subalgebra $C^*(H) \le C^*(G)$. In general, a C^*-algebra A will break up as a product of the fibers A_p over a finite-dimensional central C^*-subalgebra $C \le A$, by simply cutting A with the minimal projections in C.

In particular, under our assumptions the fiber $C^*(G)_p$ is a Cartesian factor of $C^*(G)$, and hence the RFD-ness of the latter entails that of the former. □

Corollary 4.5. *Under the hypotheses of Lemma 4.2 the characters $p : H \to \mathbb{S}^1$ for which the fiber (4-2) is RFD form a dense subset of $\widehat{H}$.*

Proof. Indeed, the characters p in the proof of Lemma 4.2 are of finite order, and hence Lemma 4.4 applies. □

4.1. **Recovering residual finiteness for a pushout.** Recall that [25, Theorem 6.9] assumes $G_1 *_C G_2$ is RF, whereas Lemma 4.1 only requires that G_i, $i = 1, 2$ be RF individually (along with G_i/H). We argue here that that distinction is only apparent:

Theorem 4.6. *Let $H \le G_i$, $i = 1, 2$ be a common central subgroup such that G_i and G_i/H are all RF. Then, $G_1 *_H G_2$ is RF.*

Proof. Note first that the case $G_1 = G_2$ is clear: indeed, the assumptions that G and G/H are RF then show that G has an H-filtration. The two copies of that filtration in the two copies of G are then (H, H, id)-compatible in the sense of [5, §2.2], and hence $G *_H G$ is RF by [5, Proposition 2].

It thus remains to reduce the problem to the case $G_1 = G_2$. To do this, consider the *tensor product*

$$G := G_1 \otimes_H G_2$$

defined by identifying the two copies of H in $G_1 \times G_2$; in other words, $G_1 \otimes_H G_2$ is $G_1 *_H G_2$ modulo the relations making the elements of

$$G_1 \text{ and } G_2 \ \leq\ G_1 *_H G_2$$

commute.

We then have

$$G/H \cong (G_1/H) \times (G_2/H),$$

which is thus RF by assumption. On the other hand, if we show that $G *_H G$ itself is RF then so is

$$G_1 *_H G_2 \leq G *_H G$$

(where the inclusions $G_i \leq G = G_1 \otimes_H G_2$ are the obvious ones).

To summarize, we have thus far

- observed that the conclusion holds when $G_1 = G_2$ (equal to a common group G, say);
- reduced the problem to its instance for $G = G_1 \otimes_H G_2$, modulo
- the desired hypothesis that that G is RF.

In conclusion, all that remains to be proven is that under our hypotheses $G := G_1 \otimes_H G_2$ is indeed RF; we relegate this to Lemma 4.7. ☐

Lemma 4.7. *Let $H \leq G_i$, $i = 1, 2$ be a common central subgroup such that G_i and G_i/H are all RF. Then, $g := G_1 \otimes_H G_2$ is RF.*

Proof. If $\{G_{i,\alpha}\}$ are H-filtrations of G_i respectively for $i = 1, 2$ then the images G_α of

$$G_{1,\alpha} \times G_{2,\alpha} \leq G_1 \times G_2$$

through the surjection

$$G_1 \times G_2 \to G = G_1 \otimes_H G_2$$

identifying the two copies of H will form an H-filtration for G. ☐

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