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Bifurcation and pattern formation in diffusive Klausmeier-Gray-Scott model of water-plant interaction ☆

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ABSTRACT

A reaction-diffusion model describing water and plant interaction proposed by Klausmeier is studied. The existence of non-constant steady state solutions is shown through bifurcation methods, and the existence of large amplitude spatial patterned solutions is proved using associated shadow system. It is rigorously shown that non-homogeneous patterned vegetation states exist when the rainfall is at a lower level in which homogeneous vegetation state cannot survive. Even when the rainfall is very low, slow plant diffusion and fast water diffusion can support a vegetation state with vegetation concentrating on a small area. This provides an example of diffusion-induced persistence that non-constant steady states may exist in a reaction-diffusion system when there are no positive constant steady states.

1. Introduction

Spatial vegetation patterns are distinctive feature of landscapes found in many semiarid regions [7,53], and their appearance often serves as an early-warning indicator of critical ecosystem transition such as desertification [38,39]. Mathematical models have been established to study the generation of vegetation pattern formation, and it has been theorized that the interplay of water source distribution and plant growth leads to self-organization of the spatial patterns of vegetation. Mathematically theory of reaction-diffusion system, Turing diffusion-induced instability [51] and its variants have been proposed as possible models to generate complex spatiotemporal patterns.

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One relatively simple nonlinear partial differential equation model was proposed in Klausmeier [16] and its nondimensionalized form is

$$\begin{cases} W_t = v\dfrac{\partial W}{\partial x_1} + a - WB^2 - W, \\ B_t = \Delta B + WB^2 - mB. \end{cases} \tag{1.1}$$

Here $W(x,t)$ and $B(x,t)$ are water and plant biomass density respectively, and $x \in \mathbb{R}^n$ for $n = 1, 2$ or 3. In this model there are three parameters: a is the water input or rainfall rate; m measures plant losses; and v is the rate at which water flows downhill. The plant moves diffusively and the water flows down hill (so water diffusion is ignored). The primary finding using (1.1) is the formation of banded stripe vegetation patterns caused by the downhill water flow [16,41–46,52]. Other similar models for vegetation patterns in semiarid ecosystems have been proposed and analyzed in, for example, [1,4–8,50,53–56].

In this paper, we consider the pattern formation in a diffusive Klausmeier model where the plants grow on flat land instead of hill:

$$\begin{cases} W_t = d_1\Delta W + a - WB^2 - W, & in \quad \Omega,\ t > 0, \\ B_t = d_2\Delta B + WB^2 - mB, & in \quad \Omega,\ t > 0, \\ \dfrac{\partial W}{\partial \nu} = \dfrac{\partial B}{\partial \nu} = 0, & on \quad \partial\Omega,\ t > 0, \\ W(x,0) = W_0(x) \geq 0, B(x,0) = B_0(x) \geq 0, & in \quad \Omega. \end{cases} \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and ν denotes the unit outer normal to $\partial\Omega$; a no-flux boundary condition is imposed so that the water-plant ecosystem is closed; and $d_1, d_2 > 0$ are the water diffusion coefficient and plant diffusion coefficient, respectively. Note that the kinetic system in (1.2) also arises from an autocatalytic chemical reaction model first proposed by Gray and Scott [9–11,32], so system (1.2) is also called diffusive Klausmeier-Gray-Scott model [40,52]. A weakly nonlinear stability analysis of positive equilibrium point of the model (1.2) was performed in [14], and parameter regions corresponding to bare-soil and vegetative patterns were identified. In [32], numerical simulations reveal a surprising variety of irregular spatiotemporal patterns for the Gray-Scott model, and some of them resemble the steady irregular patterns observed in thin gel reactor experiments and others consist of spots that grow until they reach a critical size. Spike layer spatial patterns in the diffusive Gray-Scott system have been considered in [17,18,57,58]. In the studies of Gray-Scott model, the input a is often set as a constant 1. We emphasize the effect of water input (rainfall) a on the spatial pattern formation in the current study.

In this paper, we give a theoretical analysis of the Klausmeier-Gray-Scott model (1.2) to explain the existence of spatial vegetation patterns. Our main findings on the dynamics of system (1.2) are

1. The model (1.2) has two positive constant positive steady states $E_\pm$ for any $a > 2m$, and the one with low plant biomass E_- is always unstable. Moreover, there exists a function $\tilde{d}_2(a)$ for $a > 0$ such that (1.2) only has constant steady states when $d_2 > \tilde{d}_2(a)$.
2. The high plant biomass constant positive steady state E_+ is linearly stable when $d_2 > \dfrac{m}{2}d_1$ and any $a > 2m$, or $d_2 < \dfrac{m}{2}d_1$ and $a > a^*(> 2m)$ (a^* is defined in Section 2). This indicates that the spatially uniform high biomass steady state is most likely to be achieved in these parameter regimes.
3. The high plant biomass constant positive steady state E_+ could lose its stability when the rain fall is in an intermediate rage ($2m < a < a^*$) and the plant diffusion coefficient is small ($d_2 < \dfrac{m}{2}d_1$), and non-constant steady states emanate from the spatially uniform high biomass steady state E_+ through a symmetry-breaking bifurcation. Numerical bifurcation diagram shows that the bifurcation branch of these non-constant steady state extends to an extinction rainfall threshold $a_* < 2m$, so patterned solutions exist under a lower rainfall $a < 2m$ which cannot support a uniform vegetation state.

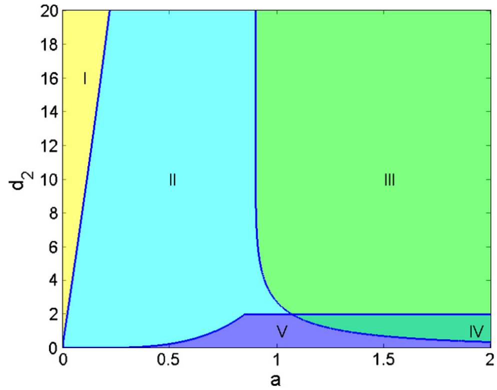


Fig. 1. Parameter regimes for the existence and stability of constant steady states and the existence/nonexistence of nonconstant steady states of (1.2). Here $m = 0.45$ and $d_1 = 80$.

4. Non-constant steady state solutions with spiky plant biomass profile exist even when the rainfall is very low, so slow plant diffusion rate d_2 and fast water diffusion rate d_1 can support patterned vegetation states with vegetation concentrating on a small area.

The parameter regimes for the existence and stability of constant steady states and the existence/nonexistence of nonconstant steady states of (1.2) on the $a - d_2$ plane are shown in Fig. 1. The positive $a - d_2$ quadrant is divided into five subregions $\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ and V defined as follows:

$$\mathrm{I} = \left\{(a, d_2) : 0 < a < 2m, d_2 > \tilde{d}_2(a)\right\}; \quad \mathrm{II} = \mathbb{R}_+^2 \backslash (\mathrm{I} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{V});$$
$$\mathrm{III} = \left\{(a, d_2) : a > 2m, d_2 > \max\{\hat{d}_2(a), d_2^*\}\right\}; \quad \mathrm{IV} = \left\{(a, d_2) : a > 2m, d_2^* > d_2 > \hat{d}_2(a)\right\};$$
$$\mathrm{V} = \left\{(a, d_2) : 0 < a < 2m, \min\{K_1 a^{4/n}, d_2^*\} > d_2\right\} \bigcup \left\{(a, d_2) : a > 2m, \min\{\hat{d}_2(a), d_2^*\} > d_2\right\}.$$

Here the curves $d_2 = d_2^*$, $d_2 = \tilde{d}_2(a)$ and $d_2 = \hat{d}_2(a)$ are defined in Propositions 3.2, 2.3 and Remark 2.6 respectively, and the constant K_1 is defined in Theorem 3.3. In the subregion I there is only the bare-soil state $E_0 = (a, 0)$; in the subregion $\mathrm{III} \cup \mathrm{IV}$, the high plant biomass constant positive steady state E_+ is linearly stable; in the subregion IV, E_+ and E_0 are both locally asymptotically stable, and there exist other non-constant steady states; and in the subregion V, E_+ is unstable and there exist other non-constant steady states. The existence or nonexistence of non-constant steady states in the subregion II is not known. Note that a substantial area of the subregion V satisfies $a < 2m$, which shows the existence of non-constant positive steady states while there is no positive constant steady states.

Our results show that pattern formation in reaction-diffusion system such as (1.2) is not just the result of symmetry-breaking bifurcations, as patterns exist far away from bifurcation points. We use bifurcation theory, singular perturbation methods, and numerical simulations to show that the small amplitude patterns generated from bifurcations when the rainfall is ample connect to highly localized patterns existing only with low rainfall level. In between these two pattern formation regimes, the spatial pattern transits from spots to labyrinth and to gaps. Such transition has been hypothesized and simulated in modeling effort by [38], and here we provide a more theoretical justification for the case of Klausmeier-Gray-Scott model (1.2).

Our results also show that constant positive steady states of (1.2) only exist when $a \geq 2m$, but non-constant positive steady states still exist for $a < 2m$. The case of $a < 2m$ provides an example of "diffusion-induced persistence" as all solutions in the system tend to the bare-soil state when diffusion is absent, but some solutions in the system persist and converge to a non-constant positive steady state when diffusion is present. This is different from Turing's "diffusion-induced instability" which requires the existence of a constant positive steady state to perturb from. Note Turing instability can be found (see Section 2 and [14]) in Klausmeier-Gray-Scott model (1.2), but some patterned solutions found here are beyond Turing realm.

The rest of the paper is organized as follows. For the Klausmeier-Gray-Scott model (1.2), we consider the symmetry-breaking bifurcation with parameter a and the existence of non-constant steady states in Section 2. In Section 3, by using the associated shadow system, we show the existence of non-constant positive steady states of (1.2) when the rainfall is at a lower level in which homogeneous grassland cannot survive and the water diffusion rate is large. Throughout this paper, $\mathbb{N}_0$ is the set of all nonnegative integers, $\mathbb{C}$ is the set of all complex numbers, and $X_{\mathbb{C}} = X \oplus iX = \{x_1 + ix_2 : x_1, x_2 \in X\}$ is the complexification of a linear space X.

2. Bifurcations

In this section, we consider the existence of non-constant positive steady state solutions of (1.2) via bifurcation analysis using the water input a as a bifurcation parameter.

2.1. A priori estimates and nonexistence of solutions

The non-negative steady state solutions of (1.2) satisfy the following semilinear elliptic equations:

$$\begin{cases} d_1\Delta W + a - WB^2 - W = 0, & in \quad \Omega, \\ d_2\Delta B + WB^2 - mB = 0, & in \quad \Omega, \\ \dfrac{\partial W}{\partial \nu} = \dfrac{\partial B}{\partial \nu} = 0, & on \quad \partial\Omega. \end{cases} \tag{2.1}$$

Clearly, system (2.1) has a trivial solution $(W, B) = (a, 0)$, which means a bare-soil state. If $a \geq 2m$, system (2.1) admits two positive constant solutions $(W_\pm(a), B_\pm(a))$ with

$$W_\pm(a) = \frac{a \mp \sqrt{a^2 - 4m^2}}{2}, \quad B_\pm(a) = \frac{a \pm \sqrt{a^2 - 4m^2}}{2m}. \tag{2.2}$$

In this subsection, we give some *a priori* estimates for the non-negative solutions and the nonexistence of nonconstant positive solutions of system (2.1). Firstly, we recall the following maximum principle (see Lemma 2.3 in [19] or Proposition 2.2 in [23]).

Lemma 2.1. *Let Ω be a bounded Lipschitz domain, and $g \in C(\bar{\Omega} \times \mathbb{R})$. If $w \in W^{1,2}(\Omega)$ is a weak solution of the inequalities*

$$\Delta w + g(x, w) \geq 0, \quad in \ \ \Omega, \qquad \frac{\partial w}{\partial \nu} \geq 0 \ \ on \ \ \partial\Omega,$$

and if there is a constant K such that $g(x, w) < 0$ for $w > K$, then $w \leq K$ a.e. in Ω.

We now have the following *a priori* estimates for the non-negative solutions of (2.1).

Proposition 2.2. *Suppose d_1, d_2, a, m are all positive constants. Let (W, B) be any nonnegative solution to (2.1). Then either (W, B) is the constant solution $(a, 0)$ or a positive solution satisfying*

$$0 < W(x) \leq a \quad and \quad 0 < B(x) \leq \left(\frac{d_1}{d_2} + \frac{1}{m}\right) a, \quad x \in \bar{\Omega}. \tag{2.3}$$

Proof. If there exists $x_0 \in \Omega$ such that $W(x_0) = 0$, then $\Delta W(x_0) > 0$ which results in a contraction from the first equation of (2.1). If there exists $x_0 \in \Omega$ such that $B(x_0) = 0$, then from the strong maximum principle $B(x) \equiv 0$ on $\bar{\Omega}$. Thus, $W(x)$ satisfies

$$\begin{cases} d_1\Delta W + a - W = 0, & in \quad \Omega, \\ \dfrac{\partial W}{\partial \nu} = 0, & on \quad \partial\Omega. \end{cases}$$

Then $W(x) \equiv a$. Therefore, $(W(x), B(x))$ is either the constant solution $(a, 0)$ or it satisfies $W(x) > 0$ and $B(x) > 0$.

From Lemma 2.1, we have $W(x) \leq a$ for any $x \in \bar{\Omega}$. Let $U = d_1W + d_2B$. Adding the two equations in (2.1) we have

$$-\Delta U = a - W - mB, \; x \in \Omega, \quad \frac{\partial U}{\partial \nu} = 0, \; x \in \partial\Omega.$$

Let $x_1 \in \bar{\Omega}$ be a maximum point of U, then it follows from Lemma 2.1 that $mB(x_1) \leq a$. Hence we have

$$d_2B(x) \leq U(x) \leq U(x_1) = d_1W(x_1) + d_2B(x_1) \leq ad_1 + \frac{ad_2}{m}, \; x \in \bar{\Omega}.$$

This yields the upper bound of $B(x)$ in (2.3). □

Now we show that system (2.1) admits positive non-constant solutions only if the plant diffusion rate d_2 is somewhat small.

Proposition 2.3. *For any fixed $d_1, a, m > 0$, there exists $\tilde{d_2} = \tilde{d_2}(d_1, a, m, \Omega)$ defined by*

$$\tilde{d_2}(d_1, a, m, \Omega) = \frac{M + \sqrt{M^2 + 4d_1mM}}{m}, \quad M = \frac{a^2}{\mu_1}\Big(\frac{a^2}{2m^2\sqrt{d_1\mu_1}} + 1\Big), \tag{2.4}$$

where μ_1 is the smallest positive eigenvalue of $-\Delta$ on Ω with Neumann boundary condition such that when $d_2 > \tilde{d_2}$, the only nonnegative solutions to (2.1) are the constant ones $(a, 0)$ and $(W_\pm, B_\pm)$.

Proof. Let (W, B) be a positive solution of (2.1). Denote the averages of W and B over Ω by

$$\bar{W} = \frac{1}{|\Omega|}\int_\Omega W(x)dx, \qquad \bar{B} = \frac{1}{|\Omega|}\int_\Omega B(x)dx.$$

Define $\phi = W - \bar{W}$ and $\psi = B - \bar{B}$. Then $\int_\Omega \phi dx = \int_\Omega \psi dx = 0$. Adding the two equations in (2.1) and integrating over Ω, we find that $\bar{W} + m\bar{B} = a$, which implies

$$\bar{B} \leq \frac{a}{m}. \tag{2.5}$$

Multiplying the equation of W in (2.1) by ϕ, and using (2.5), the *a priori* estimates in Proposition 2.2 and the Cauchy-Schwarz inequality, we obtain that

$$
\begin{aligned}
&d_1 \int_\Omega |\nabla\phi|^2 dx = \int_\Omega (a - WB^2 - W)\phi dx \\
=& \int_\Omega [((a-W)-(a-\bar{W})) - (WB^2 - W\bar{B}^2 + W\bar{B}^2 - \bar{W}\bar{B}^2)]\phi dx \\
=& \int_\Omega (-1-\bar{B}^2)\phi^2 dx - \int_\Omega W(B+\bar{B})\phi\psi dx \le a^2\Big(\frac{d_1}{d_2}+\frac{2}{m}\Big)\int_\Omega |\phi\psi| dx - \int_\Omega \phi^2 dx \\
\le& a^2\Big(\frac{d_1}{d_2}+\frac{2}{m}\Big)\left(\int_\Omega \phi^2 dx\right)^{1/2}\left(\int_\Omega \psi^2 dx\right)^{1/2} - \int_\Omega \phi^2 dx \\
\le& \frac{a^4}{4}\Big(\frac{d_1}{d_2}+\frac{2}{m}\Big)^2 \int_\Omega \psi^2 dx.
\end{aligned}
\tag{2.6}
$$

Combining with the Poincaré inequality $\mu_1 \int_\Omega \psi^2 dx \le \int_\Omega |\nabla\psi|^2 dx$, we have

$$
\int_\Omega |\nabla\phi|^2 dx \le \frac{a^4}{4d_1\mu_1}\Big(\frac{d_1}{d_2}+\frac{2}{m}\Big)^2 \int_\Omega |\nabla\psi|^2 dx. \tag{2.7}
$$

Similarly multiplying the equation of B in (2.1) by ψ, and using similar estimates and (2.7), we obtain that

$$
\begin{aligned}
&d_2 \int_\Omega |\nabla\psi|^2 dx = \int_\Omega (WB^2 - mB)\psi dx \\
=& \int_\Omega [(WB^2 - W\bar{B}^2 + W\bar{B}^2 - \bar{W}\bar{B}^2) - m(B-\bar{B})]\psi dx \\
=& \int_\Omega \bar{B}^2\phi\psi dx + \int_\Omega (W(B+\bar{B})-m)\psi^2 dx \le \frac{a^2}{m^2}\int_\Omega |\phi\psi| dx + a^2\Big(\frac{d_1}{d_2}+\frac{2}{m}\Big)\int_\Omega \psi^2 dx \\
\le& \frac{a^2}{m^2}\left(\int_\Omega \phi^2 dx\right)^{1/2}\left(\int_\Omega \psi^2 dx\right)^{1/2} + a^2\Big(\frac{d_1}{d_2}+\frac{2}{m}\Big)\int_\Omega \psi^2 dx \\
\le& \frac{a^2}{m^2\mu_1}\left(\int_\Omega |\nabla\phi|^2 dx\right)^{1/2}\left(\int_\Omega |\nabla\psi|^2 dx\right)^{1/2} + \frac{a^2}{\mu_1}\Big(\frac{d_1}{d_2}+\frac{2}{m}\Big)\int_\Omega |\nabla\psi|^2 dx \\
=& \frac{a^2}{\mu_1}\Big(\frac{d_1}{d_2}+\frac{2}{m}\Big)\Big(\frac{a^2}{2m^2\sqrt{d_1\mu_1}}+1\Big)\int_\Omega |\nabla\psi|^2 dx.
\end{aligned}
\tag{2.8}
$$

Now (2.8) implies that when

$$
d_2 > \frac{a^2}{\mu_1}\Big(\frac{d_1}{d_2}+\frac{2}{m}\Big)\Big(\frac{a^2}{2m^2\sqrt{d_1\mu_1}}+1\Big), \tag{2.9}
$$

we have $\int_\Omega |\nabla\psi|^2 dx = 0$ and thus $\int_\Omega |\nabla\phi|^2 dx = 0$ from (2.7). Hence, $\nabla\phi \equiv \nabla\psi \equiv 0$ for all $x \in \overline{\Omega}$. Therefore, (W, B) must be a constant solution. Finally we can derive $\tilde{d_2}$ in (2.4) from (2.9), and this verifies the assertion. □

Remark 2.4. It is noted that (2.9) implies that for fixed $d_1, d_2, m > 0$ and Ω, (2.1) has non-constant solutions only if $a > a_1$, where

$$a_1 = \sqrt{\frac{-M_1M_2\mu_1 + \sqrt{M_1^2M_2^2\mu_1^2 + 4\mu_1 d_2 M_2}}{2M_2}}, \quad M_1 = 2m^2\sqrt{\frac{d_1}{\mu_1}}, \quad M_2 = \frac{1}{2m^2\sqrt{d_1\mu_1}}\Big(\frac{d_1}{d_2} + \frac{2}{m}\Big).$$

2.2. Stability of constant steady states

In this subsection, we consider the stability of positive constant solutions of (2.1), based on the Turing instability mechanism. It is easy to show that the bare-soil state $(a, 0)$ is always a locally asymptotically stable steady state of (1.2) for any parameter values. So we focus on the stability of the positive constant steady state $(W_\pm(a), B_\pm(a))$ defined in (2.2). The Jacobian matrix of the corresponding kinetic system at a positive constant steady state (W, B) is

$$J = \begin{pmatrix} -1 - B^2 & -2m \\ B^2 & m \end{pmatrix}. \tag{2.10}$$

Then the corresponding characteristic equation is $\lambda^2 - T_0\lambda + D_0 = 0$, with the trace of J being $T_0 = m - (1 + B^2)$, and the determinant of J being $D_0 = m(B^2 - 1)$. Note that $B_-(a) < 1$ and $B_+(a) > 1$, which means that the positive steady state $(W_-(a), B_-(a))$ is always an unstable saddle whenever it exists. The stability of $(W_+(a), B_+(a))$ with respect to the ODE dynamics can be determined by the sign of T_0 at $(W_+(a), B_+(a))$. Direct computations show that $(W_+(a), B_+(a))$ is locally asymptotically stable if $0 < m < 2$.

In this section, we always assume $a \geq 2m$ and $0 < m < 2$ so the positive constant steady states exist. Define the real-valued Sobolev spaces

$$\begin{aligned} X &= \Big\{(W, B) \in W^{2,q}(\Omega) \times W^{2,q}(\Omega) : \frac{\partial W}{\partial \nu} = \frac{\partial B}{\partial \nu} = 0 \quad on \ \partial\Omega\Big\}, \\ Y &= L^q(\Omega) \times L^q(\Omega), \end{aligned}$$

where $q > n$, and a nonlinear mapping G by

$$G(a, W, B) := \begin{pmatrix} d_1\Delta W + a - WB^2 - W \\ d_2\Delta B + WB^2 - mB \end{pmatrix}. \tag{2.11}$$

Then $G : \mathbb{R}^+ \times X \to Y$ is Fréchet differentiable, and at the constant steady state $(W_+(a), B_+(a))$, the linearized operator is

$$\mathcal{L}(a) := \begin{pmatrix} -1 - B_+^2(a) + d_1\Delta & -2m \\ B_+^2(a) & m + d_2\Delta \end{pmatrix}, \tag{2.12}$$

with the domain $\mathcal{D}(\mathcal{L}(a)) = X_{\mathbb{C}}$.

The eigenvalue problem

$$-\Delta\varphi = \mu\varphi \quad in\ \Omega, \quad \frac{\partial\varphi}{\partial\nu} = 0 \quad on\ \partial\Omega \tag{2.13}$$

has eigenvalues μ_k satisfying $0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \cdots \to +\infty$ with corresponding eigenfunctions $\varphi_k(x)$ for $k \in \mathbb{N}_0$. Let

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \varphi_k(x)$$

be an eigenfunction of $\mathcal{L}(a)$ corresponding to an eigenvalue $\lambda(a)$, *i.e.* $\mathcal{L}(a)(\phi,\psi)^T = \lambda(a)(\phi,\psi)^T$. Then from the Fourier theory, there exists $k \in \mathbb{N}_0$ and $(\alpha_k, \beta_k) \in \mathbb{R}^2\backslash\{(0,0)\}$, such that $\mathcal{L}_k(a)(\alpha_k,\beta_k)^T = \lambda(a)(\alpha_k,\beta_k)^T$, where

$$\mathcal{L}_k(a) := \begin{pmatrix} -1 - B_+^2 - d_1\mu_k & -2m \\ B_+^2 & m - d_2\mu_k \end{pmatrix}. \tag{2.14}$$

Then the characteristic equation of $\mathcal{L}_k(a)$ is

$$\lambda^2 - T_k(a)\lambda + D_k(a) = 0, \tag{2.15}$$

and the eigenvalues $\lambda(a)$ of $\mathcal{L}_k(a)$ are given by

$$\lambda(a) = \frac{T_k(a) \pm \sqrt{T_k^2(a) - 4D_k(a)}}{2},$$

where

$$\begin{aligned} T_k(a) &= -(d_1+d_2)\mu_k + m - 1 - B_+^2(a), \\ D_k(a) &= d_1 d_2 \mu_k^2 + [d_2(1+B_+^2(a)) - d_1 m]\mu_k + m(B_+^2(a) - 1). \end{aligned} \tag{2.16}$$

For the diffusion-induced instability (Turing instability) to occur at $(W_+(a), B_+(a))$, the constant steady state $(W_+(a), B_+(a))$ is stable with respect to the kinetic ordinary differentiate equation system while is unstable with respect to the reaction-diffusion system (1.2). We show the following stability/instability result to identify the parameter regime where the Turing instability occurs.

Theorem 2.5. *Assume d_1, d_2, a, m are positive constants and $0 < m < 2$. Then*

(i) *If $\frac{d_1}{d_2} < \frac{2}{m}$, then $(W_+(a), B_+(a))$ is locally asymptotically stable for any $a > 2m$;*
(ii) *If $\frac{d_1}{d_2} > \frac{2}{m}$, then there exists a unique $a^* := a^*\left(\frac{d_1}{d_2}\right)$ such that $(W_+(a), B_+(a))$ is locally asymptotically stable for $a > a^*$, and it is possibly unstable when $2m < a < a^*$.*

Proof. Since $0 < m < 2$ and $a \ge 2m$, the constant steady state $(W_+(a), B_+(a))$ is locally asymptotically stable with respect to the kinetic ordinary differentiate equation system. Hence $T_0(a) < 0$ and $D_0(a) > 0$. It is clear that $T_k(a) = -(d_1 + d_2)\mu_k + T_0(a) < 0$ for any $k \in \mathbb{N}$. So for the Turing instability to occur, it is necessary that $D_k(a) < 0$ for some $k \in \mathbb{N}$ [13,60]. Then a necessary condition for the instability of $(W_+(a), B_+(a))$ with respect to system (1.2) is

$$\begin{cases} d_2(1+B_+^2(a)) - d_1 m < 0, \\ [d_2(1+B_+^2(a)) - d_1 m]^2 - 4d_1 d_2 m (B_+^2(a) - 1) > 0, \end{cases} \tag{2.17}$$

which is equivalent to

$$\frac{d_1}{d_2} > \frac{3B_+^2(a) - 1 + 2B_+(a)\sqrt{2(B_+^2(a)-1)}}{m} := G_0(B_+(a)), \tag{2.18}$$

where

$$G_0(B) := \frac{3B^2 - 1 + 2B\sqrt{2(B^2-1)}}{m}, \quad \textit{for } B > 1. \tag{2.19}$$

Then $\frac{\partial G_0(B_+(a))}{\partial a} = G_0'(B_+)B_+'(a)$. It is easy to calculate that

$$G_0'(B) = \frac{6B\sqrt{B^2-1} + 2\sqrt{2}(2B^2-1)}{m\sqrt{B^2-1}} > 0, \quad B > 1,$$
$$B_+'(a) = \frac{a + \sqrt{a^2-4m^2}}{2m\sqrt{a^2-4m^2}} > 0, \quad a > 2m.$$

Thus, $\frac{\partial G_0(B_+(a))}{\partial a} > 0$ for all $a > 2m$, which implies that $G_0(B_+(a))$ is strictly increasing in a. Therefore we have

$$\min_{a \in [2m, +\infty)} G_0(B_+(a)) = G_0(B_+(2m)) = \frac{2}{m}. \tag{2.20}$$

From (2.18) and (2.20), we conclude that when $\frac{d_1}{d_2} < \frac{2}{m}$, (2.17) cannot hold, thus $(W_+(a), B_+(a))$ is locally asymptotically stable for any $a > 2m$. This proves part (i).

To prove (ii), from (2.16), we define a function

$$D(a,p) := d_1 d_2 p^2 + [d_2(1+B_+^2(a)) - d_1 m]p + m(B_+^2(a) - 1). \tag{2.21}$$

Then $D(a,p) = \frac{1}{2m^2}K(a,p)$ where

$$K(a,p) := 2d_1 d_2 m^2 p^2 - [2m^3 d_1 - d_2(a^2 + a\sqrt{a^2-4m^2})]p + m(a^2 - 4m^2 + a\sqrt{a^2-4m^2}).$$

Solving $K(a,p) = 0$, we have

$$a^2(p) = \frac{m^2(2m + md_1 p - d_1 d_2 p^2)^2}{(m^2 - d_2^2 p^2)(d_1 p + 1)},$$

which implies $p < p^* := \frac{m}{d_2}$. Then when $p < p^*$ we have

$$a(p) = \frac{m(2m + md_1 p - d_1 d_2 p^2)}{\sqrt{(m^2 - d_2^2 p^2)(d_1 p + 1)}}, \tag{2.22}$$

and

$$a'(p) = \frac{mp(d_1 d_2^2 p^2 + 2md_1 d_2 p + m(2d_2 - md_1))(d_1 d_2 p + 2d_2 - md_1)}{2\sqrt{(m^2 - d_2^2 p^2)^3 (d_1 p + 1)^3}}.$$

When $\dfrac{d_1}{d_2} > \dfrac{2}{m}$, define

$$p_1 := \frac{-md_1 + \sqrt{2md_1(md_1 - d_2)}}{d_1 d_2}, \quad p_2 := \frac{md_1 - 2d_2}{d_1 d_2}.$$

Then $0 < p_1 < p_2 < p^*$, $a(0) = a(p_2) = 2m$, $a'(p) > 0$ when $0 < p < p_1$, $a'(p) < 0$ when $p_1 < p < p_2$ and $a'(p_1) = a'(p_2) = 0$. This implies that $\max\limits_{p \in [0,p_2]} a(p) = a(p_1)$. Define

$$a^* = a(p_1) := a^*\left(\frac{d_1}{d_2}\right), \tag{2.23}$$

where $a^*(r)$ is defined by, for $r > 2/m$,

$$a^*(r) = \frac{m^2 r(4 - 4mr + 3\sqrt{2mr(mr-1)})}{\sqrt{\sqrt{2mr(mr-1)}(2mr - \sqrt{2mr(mr-1)})(\sqrt{2mr(mr-1)} + 1 - mr)}}.$$

Indeed the function $a^*(r)$ is the inverse function of $H(B)$ defined in (2.19), and direct computation shows that $(a^*)'(r) > 0$ when $r > 2/m$ and $a^*(2/m) = 2m$. Note that the inequality (2.18) is equivalent to $2m < a < a^*(d_1/d_2)$. This implies $(W_+(a), B_+(a))$ is linearly stable (and locally asymptotically stable) for $a > a^*(d_1/d_2)$, and it is possibly unstable when $2m < a < a^*(d_1/d_2)$. □

Remark 2.6. For fixed $d_1, m > 0$ and $a > 2m$, (2.18) also implies that $(W_+(a), B_+(a))$ is linearly stable (and locally asymptotically stable) when $d_2 > \hat{d}_2(d_1, a, m)$, and it is unstable when $0 < d_2 < \hat{d}_2(d_1, a, m)$, where

$$\hat{d}_2(d_1, a, m) = \frac{d_1}{G_0(B_+(a))}, \tag{2.24}$$

$G_0(B)$ and $B_+(a)$ are defined in (2.19) and (2.2).

2.3. Global steady state bifurcation

From the last subsection, $(W_+(a), B_+(a))$ may be unstable if $d_1/d_2 > 2/m$ and $2m < a < a^*$. In this subsection, by applying the well-known Crandall-Rabinowitz bifurcation theorem [3] and its global bifurcation version [49], we obtain a global bifurcation diagram for the steady state solutions of model (1.2) when $d_1/d_2 > 2/m$ and $2m < a < a^*$.

Define the sets

$$\begin{aligned} \Gamma =& \{(a, W, B) \in \mathbb{R}^+ \times X : (a, W, B) \textit{ satisfies } (2.1), W > 0, B > 0, W \not\equiv \text{const}, B \not\equiv \text{const}\}, \\ Z_0 =& \{(W, B) \in X : W + W_+ > 0, B + B_+ > 0\}. \end{aligned}$$

Then we have the following result on the global bifurcation for the steady state solutions of model (1.2).

Theorem 2.7. *Assume $d_2 > 0, 0 < m < 2$ and $d_1 > \dfrac{2}{m} d_2$. Let μ_j be an eigenvalue of (2.13) with the corresponding eigenfunction φ_j such that*

(i) *μ_j is a simple eigenvalue;*
(ii) $0 < \mu_j < \dfrac{m}{d_2}$.

Let $a(p)$ and a^* be defined as in (2.22) and (2.23) respectively. Define $a_j^S := a(\mu_j)$. Then

1. *$a = a_j^S$ is a bifurcation point for system (2.1) where a steady state bifurcation occurs from the curve of trivial steady states $\Gamma_0^+ = \{(a, W_+(a), B_+(a)) : a \geq 2m\}$;*
2. *There exists a connect component Γ_j of the closure of Γ such that near $(a, W, B) = (a_j^S, W_+(a_j^S), B_+(a_j^S))$, Γ_j can be parameterized as $\Gamma_j = \{(a_j(s), W_j(s), B_j(s)) : s \in (0, \varepsilon)\}$, with $a_j(0) = a_j^S, W_j(s) = W_+(a_j^S)+s(d_2\mu_j-m)\varphi_j+s\varphi_{1,j}(s), B_j(s) = B_+(a_j^S)+sB_+^2(a_j^S)\varphi_j+s\varphi_{2,j}(s), \varphi_{1,j}(0) = \varphi_{2,j}(0) = 0$, $\varphi_{1,j}(s)$ and $\varphi_{2,j}(s)$ are differentiable functions defined as $\varphi_{1,j}, \varphi_{2,j} : [0, \varepsilon) \to Z_1$, where $Z_1 = \{(W, B) \in X : \int_\Omega [(d_2\mu_j - m)W + B_+^2(a_j^S)B]\varphi_j dx = 0\}$ is a subspace of X complement to $\mathrm{Span}\{(d_2\mu_j - m, B_+^2(a_j^S))\varphi_j\}$;*
3. *Either Γ_j is unbounded and its projection onto a-axis is (a_j^S, ∞), or Γ_j is bounded and it contains another point $(a_k^S, W_+(a_k^S), B_+(a_k^S))$ with $k \neq j$ or $(2m, a/2, a/(2m))$.*

Proof. Setting $\tilde{W} = W - W_+, \tilde{B} = B - B_+$ and neglecting the tildes, we can rewrite system (2.1) as

$$\begin{cases} d_1\Delta W + a - (W + W_+)(B + B_+)^2 - (W + W_+) = 0, & in \quad \Omega, \\ d_2\Delta B + (W + W_+)(B + B_+)^2 - m(B + B_+) = 0, & in \quad \Omega, \\ \dfrac{\partial W}{\partial \nu} = \dfrac{\partial B}{\partial \nu} = 0, & on \quad \partial\Omega, \end{cases} \tag{2.25}$$

and the positive constant solution (W_+, B_+) of model (2.1) is translated to $(0, 0)$ solution of (2.25). Define a nonlinear mapping $F : \mathbb{R}^+ \times Z_0 \to Y$ by

$$F(a, W, B) := \begin{pmatrix} d_1\Delta W + a - (W + W_+)(B + B_+)^2 - (W + W_+) \\ d_2\Delta B + (W + W_+)(B + B_+)^2 - m(B + B_+) \end{pmatrix}. \tag{2.26}$$

Then the nonlinear map F is infinitely differential in W, B, and $F(a, 0, 0) = 0$ for all $a \geq 2m$. At a bifurcation point $(a, W, B) = (a_0, 0, 0)$,

$$F_{(W,B)}(a_0, 0, 0)[\phi, \psi] := \begin{pmatrix} d_1\Delta\phi - (1 + B_+^2(a_0))\phi - 2m\psi \\ d_2\Delta\psi + B_+^2(a_0)\phi + m\psi \end{pmatrix}. \tag{2.27}$$

Recall that μ_k is the k-th eigenvalue of (2.13) with the corresponding eigenfunction φ_k. Then $F_{(W,B)}(a_0, 0, 0)[\phi, \psi] = 0$ has a nontrivial solution if and only if

$$D_k(a_0) = d_1 d_2 \mu_k^2 + [d_2(1 + B_+^2(a_0)) - d_1 m]\mu_k + m(B_+^2(a_0) - 1) = 0, \tag{2.28}$$

for some $k \in \mathbb{N}$. Follow the proof of Theorem 2.5, define $a(p)$ and a^* as in (2.22) and (2.23) respectively, then the function $a(p) : [0, p_2] \to [2m, a^*]$ is monotone increasing on the interval $[0, p_1]$, and is monotone decreasing on the interval $[p_1, p_2]$. On the other hand, we can also solve p from (2.21) to obtain

$$p_\pm(a) = \frac{[d_1 m - d_2(1 + B_+^2(a))] \pm \sqrt{[d_1 m - d_2(1 + B_+^2(a))]^2 - 4md_1 d_2(B_+^2(a) - 1)}}{2d_1 d_2}. \tag{2.29}$$

Then $p_\pm(a)$ are well defined for $2m \leq a \leq a^*$, and the function $p_-(a)$ ($p_+(a)$) is monotone increasing (decreasing) on the interval $[2m, a^*]$, and $p_-(a^*) = p_+(a^*) = p_1$.

Since μ_j satisfies (i) and (ii), from the proof of Theorem 2.5, $F_{(W,B)}(a, 0, 0)[\phi, \psi] = 0$ has a nontrivial solution when $a = a_j^S := a(\mu_j)$. Furthermore, direct calculations show that the null space $\mathcal{N}(F_{(W,B)}(a_j^S, 0, 0)) = \mathrm{Span}\{(\phi_0, \psi_0)\}$, where $(\phi_0, \psi_0) = (d_2\mu_j - m, B_+^2(a_j^S))\varphi_j$. This implies that $\dim\mathcal{N}(F_{(W,B)}(a_j^S, 0, 0)) = 1$.

Next we show that codim$\mathcal{R}(F_{(W,B)}(a_j^S,0,0)) = 1$ where $\mathcal{R}(F_{(W,B)}(a_j^S,0,0))$ is the range space. Suppose there exists a $(\phi,\psi) \in Z_0$ such that

$$F_{(W,B)}(a_j^S,0,0)[\phi,\psi] := \begin{pmatrix} d_1\Delta\phi - (1+B_+^2(a_j^S))\phi - 2m\psi \\ d_2\Delta\psi + B_+^2(a_j^S)\phi + m\psi \end{pmatrix} = \begin{pmatrix} \sigma \\ \tau \end{pmatrix}, \tag{2.30}$$

where $y_1 := (\sigma,\tau) \in Y$. Note that the conjugate operator of $F_{(W,B)}(a_j^S,0,0)$

$$F^*_{(W,B)}(a_j^S,0,0)[\phi,\psi] := \begin{pmatrix} d_1\Delta\phi - (1+B_+^2(a_j^S))\phi + B_+^2(a_j^S)\psi \\ d_2\Delta\psi - 2m\phi + m\psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.31}$$

has a nontrivial solution $y_2 := (m - d_2\mu_j, 2m)\varphi_j$. Then according to the Fredholm alternative, problem (2.30) has a solution (ϕ,ψ) if and only if $\langle y_1,y_2\rangle = 0$, where $\langle\cdot,\cdot\rangle$ is the complex-valued L^2 inner product on the Hilbert space $L^2(\Omega)\times L^2(\Omega)$, which is defined as $\langle\Phi_1,\Phi_2\rangle = \int_\Omega(\bar{\phi_1}\phi_2 + \bar{\psi_1}\psi_2)dx$, with $\Phi_i = (\phi_i,\psi_i) \in L^2(\Omega)\times L^2(\Omega)$, $i = 1,2$. Then we have $\mathcal{R}(F_{(W,B)}(a_j^S,0,0)) = \{(\sigma,\tau) \in Y : l(\sigma,\tau) = 0\}$, where $l : Y \to \mathbb{R}$ is a linear function in Y^* defined by $l(\sigma,\tau) = \int_\Omega[(m - d_2\mu_j)\sigma + 2m\tau]\varphi_j dx$. Therefore, $F_{(W,B)}(a_j^S,0,0)$ is a Fredholm operator with index 0, and dim$\mathcal{N}(F_{(W,B)}(a_j^S,0,0)) =$ codim$\mathcal{R}(F_{(W,B)}(a_j^S,0,0)) = 1$.

Finally we prove the transversality condition: $F_{(a,W,B)}(a_j^S,0,0)[\phi,\psi] \notin \mathcal{R}(F_{(W,B)}(a_j^S,0,0))$, where $(\phi,\psi) \in \mathcal{N}(F_{(W,B)}(a_j^S,0,0))$ and $(\phi,\psi) \neq (0,0)$. Note that

$$F_{(a,W,B)}(a_j^S,0,0)[\phi,\psi] := \begin{pmatrix} -2B_+(a_j^S)B'_+(a_j^S)\phi \\ 2B_+(a_j^S)B'_+(a_j^S)\phi \end{pmatrix}, \tag{2.32}$$

and $B'_+(a_j^S) = \dfrac{a_j^S + \sqrt{(a_j^S)^2 - 4m^2}}{2m\sqrt{(a_j^S)^2-4m^2}} > 0$. Then

$$\begin{aligned} l(F_{(a,W,B)}(a_j^S,0,0)[\phi_0,\psi_0]) &= 2B_+(a_j^S)B'_+(a_j^S)\int_\Omega(d_2\mu_j - m)\varphi_j\phi_0 dx \\ &= 2B_+(a_j^S)B'_+(a_j^S)\int_\Omega(d_2\mu_j - m)^2\varphi_j^2 dx > 0, \end{aligned} \tag{2.33}$$

as $\mu_j < m/d_2$. Therefore, $F_{(a,W,B)}(a_j^S,0,0)[\phi,\psi] \notin \mathcal{R}(F_{(W,B)}(a_j^S,0,0))$.

Now from the local bifurcation theorem in [3], near the bifurcation point $(a_j^S,0,0)$ the set of positive solutions of (2.25) can be parameterized as $\Gamma'_j = \{(a_j(s),W_j(s),B_j(s)) : s \in (0,\varepsilon)\}$, with $a_j(0) = a_j^S, W_j(s) = s(d_2\mu_j - m)\varphi_j + s\varphi_{1,j}(s), B_j(s) = sB_+^2(a_j^S)\varphi_j + s\varphi_{2,j}(s)$, $\varphi_{1,j}(0) = \varphi_{2,j}(0) = 0$, $\varphi_{1,j}(s)$ and $\varphi_{2,j}(s)$ are differentiable functions defined by $\varphi_{1,j},\varphi_{2,j} : [0,\varepsilon) \to Z_1$, where $Z_1 = \{(W,B) \in Z_0 : \int_\Omega[(d_2\mu_j - m)W + B_+^2(a_j^S)B]\varphi_j dx = 0\}$ is a subspace of Z_0 complement to Span$\{(\phi_0,\psi_0)\}$.

Moreover from the global bifurcation theorem in [49, Theorem 4.3], there exists a connect component Γ_j of $\overline{\Gamma}$ containing Γ'_j such that $(a_j^S,0,0) \in \Gamma_j$ and two possibilities may occur:

(i) Γ_j is not compact in $\mathbb{R}^+\times Z_0$;
(ii) there exists another bifurcation point $(a_k^S,0,0) \in \Gamma_j$ with $k \neq j$.

If case (i) occurs, Γ_j is either unbounded in $\mathbb{R}^+\times Z_0$, or Γ_j contains a boundary point of $\mathbb{R}^+\times Z_0$. We prove the latter cannot occur. From Remark 2.4, Γ_j does not contain a point such that $a = 0$. Suppose Γ_j contains a point $(a,W,B) \in \mathbb{R}^+\times\partial Z_0$. Then there exists $x_0 \in \overline{\Omega}$ such that $W(x_0)+W_+(a) = 0$ or $B(x_0)+B_+(a) = 0$.

If $W(x_0) + W_+(a) = 0$ and $x_0 \in \Omega$, then x_0 is a local minimum of W, but $d_1\Delta W(x_0) = -a < 0$ which is a contradiction. If $W(x_0) + W_+(a) = 0$ and $x_0 \in \partial\Omega$, again we reach a contradiction by using the Hopf boundary lemma. Thus we must have $B(x_0) + B_+(a) = 0$. By applying maximum principle and Hopf boundary lemma again, we conclude that $B(x) + B_+(a) \equiv 0$ for $x \in \overline{\Omega}$. Hence (a, W, B) is the bare-soil state. But at the constant steady state $(a, W, B) = (a, -W_+(a), -B_+(a))$, the linearized operator is

$$F_{(W,B)}(a, -W_+(a), -B_+(a)) = \begin{pmatrix} d_1\Delta - 1 & 0 \\ 0 & d_2\Delta - m \end{pmatrix}. \tag{2.34}$$

Then it is easy to see that all eigenvalues of $F_{(W,B)}(a, -W_+(a), -B_+(a))$ are negative, so it cannot be a bifurcation point such that $(a, -W_+(a), -B_+(a)) \in \Gamma_j$. Therefore Γ_j does not contain a boundary point of $\mathbb{R}^+ \times Z_0$. Hence Γ_j is unbounded in $\mathbb{R}^+ \times Z_0$. From Proposition 2.2 and standard elliptic estimates, Γ_j is bounded in Z_0 for any bounded a-interval. From Remark 2.4, the projection of Γ_j onto a-axis is contained in (a_1, ∞). Thus the projection of Γ_j onto a-axis must contain (a_j^S, ∞).

If (ii) occurs, we note that the branch of trivial solutions $(a, 0, 0)$ is only defined for $a \geq 2m$ not all $a > 0$. Hence in case (ii), Γ_j may contain another bifurcation point $(a_k^S, W_+(a_k^S), B_+(a_k^S))$ with $k \neq j$, but it is also possible it contains the end point $(2m, a/2, a/(2m))$ at $a = 2m$. This completes the proof. □

We make some further remarks on the set of nonconstant solutions of (2.1).

Remark 2.8.

1. Since the bifurcation point $a_j^S = a(\mu_j)$ satisfies $0 < \mu_j < m/d_2$, the number of bifurcation points is finite so the number of connected components Γ_j emanating from the branch of constant solutions $\Gamma_0^+ = \{(a, W_+(a), B_+(a) : a \geq 2m\}$ is also finite. It is possible that $\Gamma_j = \Gamma_k$ for some $j \neq k$ as they can connect to each other through secondary bifurcations not occurring on Γ_0^+.
2. Theorem 2.7 shows that the branch Γ_j of nonconstant steady state solutions of (1.2) could be bounded and it may connect back to the branch of constant solutions. This kind of bounded bifurcating branches is called “loops” or “mushroom” [22,28,33].
3. The branch Γ_j of nonconstant steady state solutions of (1.2) may connect to the other branch of constant solutions $\Gamma_0^- = \{(a, W_-(a), B_-(a) : a \geq 2m\}$ and not Γ_0^+ directly. In that case, Γ_j connects to Γ_0^-, then Γ_0^- connects to Γ_0^+ at the saddle-node bifurcation point $(a, W, B) = (2m, a/2, a/(2m))$, which is the second alternative in Theorem 2.7 (iii).
4. The bifurcation direction of Γ_j at $a = a_j^S$ ($a_j'(0)$ and $a_j''(0)$) can be calculated following the calculation in [13,47]. We include that in Appendix A, and a numerical example is given below.

We demonstrate our theoretical results to the Klausmeier-Gray-Scott model (1.2) with $m = 0.45, d_1 = 80, d_2 = 1$. Then we can calculate that $p_1 = 0.1775$, $p_2 = 0.4250$, $p^* = 0.45$, $a^* = 1.3313$ and the graphs of $D(a, p) = 0$ and bifurcation points are shown in Fig. 2.

For the one-dimensional domain $\Omega = (0, 10\pi)$, the eigenvalues of (2.13) are $\lambda_k = k^2/100$ for $k \in \mathbb{N}$, and the steady state bifurcation points $a = a_j^S$ marked in Fig. 2 left panel are

$$\begin{aligned} &a_1^S = 0.9334 < a_6^S = 1.0661 < a_2^S = 1.0836 < a_3^S = 1.2446 \\ &< a_5^S = 1.2860 < a_4^S = 1.3283. \end{aligned}$$

The steady state bifurcations for $\Omega_1 = (0, 10\pi)$ are always pitchfork bifurcation, *i.e.* $a_j'(0) = 0$; and the bifurcation is supercritical one if $a''(0) > 0$ and subcritical one if $a''(0) < 0$. By using Maple and the algorithm in Appendix A, we find that $a_4''(0) = -2527.1042 < 0$ at $a = a_4^S$. This implies that the bifurcation

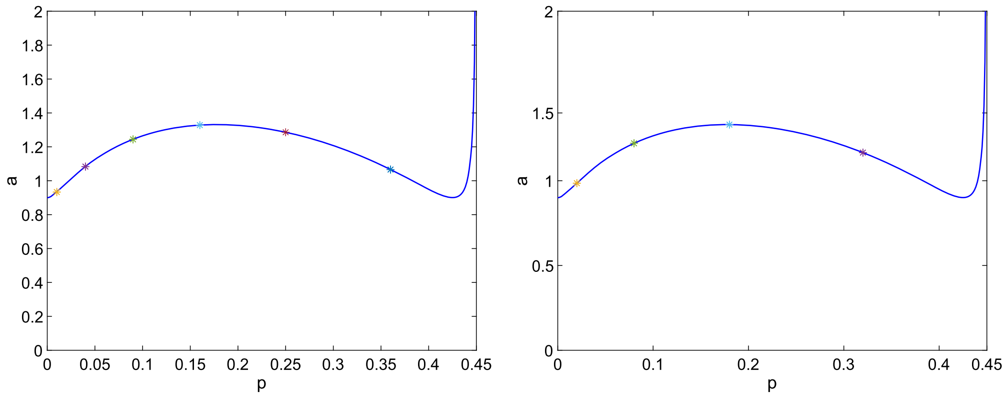


Fig. 2. Graph of $D(a, p) = 0$. Parameters: $m = 0.45, d_1 = 80, d_2 = 1$. (Left) $\Omega = (0, 10\pi)$; (Right) $\Omega = (0, 10\pi) \times (0, 10\pi)$.

at the most unstable mode, *i.e.* the pitchfork bifurcation at $(a_4^S, W_+(a_4^S), B_+(a_4^S))$, is subcritical and the bifurcating non-constant steady state solutions are linearly stable. For the two-dimensional domain $\Omega_2 = (0, 10\pi) \times (0, 10\pi)$, the eigenvalues of (2.13) are $\lambda_{k_1,k_2} = (k_1^2 + k_2^2)/100$ for $k_1, k_2 \in \mathbb{N}$, then only valid bifurcation points corresponding to simple eigenvalues are (marked by stars in Fig. 2 right panel):

$$a_{1,1}^S = 0.9858 < a_{4,4}^S = 1.1660 < a_{2,2}^S = 1.2208 < a_{3,3}^S = 1.3312.$$

For Ω_2, there are many other non-simple eigenvalues where bifurcations can also happen, but $a_{3,3}^S$ indeed is the largest one among all bifurcation points. In Fig. 3, numerical bifurcation diagrams of (1.2) from a_4^S for Ω_1 and $a_{3,3}^S$ for Ω_2 are shown.

3. Existence of patterns with small rain fall

In this section, we show the existence of non-constant solutions of (2.1) for large d_1 and small rain fall a.

3.1. The shadow system

To show the existence of non-constant solutions of (2.1) for large d_1, we introduce the shadow system of (1.2). The shadow system of (1.2) is obtained by formally letting $d_1 \to \infty$ (see [12,15,31]). From the first equation of (1.2) and the Neumann boundary condition we obtain

$$\frac{1}{|\Omega|}\frac{\partial}{\partial t}\int_\Omega W dx = \frac{1}{|\Omega|}\int_\Omega (a - WB^2 - W)dx. \tag{3.1}$$

If $d_1 \to +\infty$, then $W(x, t) \to \xi(t)$ in the first equation of (1.2) because of the boundary condition, so that (3.1) is written as

$$\xi_t = a - \frac{\xi}{|\Omega|}\int_\Omega B^2 dx - \xi. \tag{3.2}$$

Hence the shadow system of model (1.2) is in form

$$\begin{cases} B_t = d_2 \Delta B + \xi B^2 - mB, & in \quad \Omega, \ t > 0, \\ \xi_t = a - \dfrac{\xi}{|\Omega|} \displaystyle\int_\Omega B^2 dx - \xi, & t > 0, \\ \dfrac{\partial B}{\partial \nu} = 0, & on \quad \partial\Omega, \ \ t > 0. \end{cases} \tag{3.3}$$

It is easy to see that if $u = u_{d_2,m}(x)$ is a solution of the scalar equation

$$\begin{cases} d_2 \Delta u - mu + u^2 = 0, & in \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & on \quad \partial\Omega, \end{cases} \tag{3.4}$$

then the shadow system (3.3) has two positive steady state solutions $(B_a^{\pm}(x), \xi_a^{\pm})$ for any $a \geq 2||u||_2/|\Omega|^{1/2}$, where $|\Omega|$ is the Lebesgue measure of Ω,

$$\xi_a^{\pm} = \frac{a \pm \sqrt{a^2 - \frac{4}{|\Omega|}\int_\Omega u^2(x)dx}}{2}, \quad B_a^{\pm}(x) = \frac{u(x)}{\xi_a^{\pm}}. \tag{3.5}$$

So we have the following results regarding the set S of positive steady state solutions of the shadow system (3.3) with parameter a.

Proposition 3.1.

(i) *For any positive solution $u(x)$ of (3.4), there is a subset S_u of S (the set of positive steady state solutions of (3.3)) in form of*

$$S_u = \{(a, B_a^+, \xi_a^+) : a \geq 2||u||_2/|\Omega|^{1/2}\} \cup \{(a, B_a^-, \xi_a^-) : a \geq 2||u||_2/|\Omega|^{1/2}\}.$$

(ii) *For any positive solution $u(x)$ of (3.4), $||u||_2 \leq m|\Omega|^{1/2}$, and the equality holds only when $u(x) \equiv m$ for $x \in \overline{\Omega}$. In particular, the projection of S_u onto a-axis contains $[2m, \infty)$, and the projection is precisely $[2m, \infty)$ only for the branch of positive constant steady states of (3.3).*

Proof. Part (i) is clear from the definition in (3.5). Part (ii) follows from integrating (3.4) and Cauchy-Schwarz inequality:

$$m \int_\Omega u dx = \int_\Omega u^2 dx \geq \frac{1}{|\Omega|} \left(\int_\Omega u dx \right)^2.$$

Apparently the equality holds only when u is a constant. □

We recall some existence and multiplicity results for the non-constant solutions of the nonlinear Schrödinger equation (3.4).

Proposition 3.2. *Suppose d_2, m are positive constants, $\Omega \subset \mathbb{R}^n$ $(1 \leq n \leq 5)$ is a bounded domain with smooth boundary $\partial\Omega$, and μ_j $(j \geq 1)$ are the positive eigenvalues of (2.13). Then*

(i) *There exist $d_2^{**} > d_2^* > 0$ such that when $0 < d_2 < d_2^*$, (3.4) has a non-constant positive least energy solution $u(x, d_2)$ satisfying*

$$\frac{m}{6}\int_\Omega u^2 dx \le J(u) \le C_0 d_2^{n/2}, \tag{3.6}$$

where $C_0 > 0$ depends only on Ω and m, and the energy function is defined by

$$J(v) = \int_\Omega \Big(\frac{d_2}{2}|\nabla v|^2 + \frac{m}{2}v^2 - \frac{1}{3}v_+^3\Big)dx, \quad \textit{for} \quad v \in W^{1,2}(\Omega), \tag{3.7}$$

where $v_+ := \max\{v, 0\}$; On the other hand (3.4) has no positive non-constant solution for $d_2 > d_2^{**}$.

(ii) *Let $d_2^j = m/\mu_j$ for $j \ge 1$. Then each $(d_2, u) = (d_2^j, m)$ is a bifurcation point where non-constant positive solutions of (3.4) bifurcate from the constant solution $u = m$. If in addition, μ_j is an eigenvalue with odd algebraic multiplicity, then there is a continuum Σ_j of positive non-constant solutions of (3.4) such that $(d_2^j, m) \in \overline{\Sigma_j}$, and either the projection of Σ_j onto d_2-axis contains $(0, d_2^j)$, or $\overline{\Sigma_j}$ contains another bifurcation point (d_2^k, m) with $k \ne j$.*

(iii) *If $n = 1$ and $\Omega = (0, l\pi)$, then $d_2^j = ml^2/j^2$, each $\overline{\Sigma_j}$ is a curve with only one degenerate point at $(d_2, u) = (d_2^j, m)$ and the projection of Σ_j onto d_2-axis is $(0, d_2^j]$. In particular, (3.4) has exactly $2j$ non-constant positive solutions if $d_2^{j+1} \le d_2 < d_2^j$ and all of them are unstable, and each solution (d_2, u) on Σ_j satisfies that $u(x) - m$ changes sign exactly j times. Moreover the bifurcation from (d_2^1, m) is a supercritical pitchfork one, and each solution on Σ_1 is non-degenerate with Morse index is two. Here the Morse index is the number of the strictly positive eigenvalues.*

Proof. (i) The existence of a non-constant positive solution $u(x, d_2)$ for small d_2 satisfying the energy bound follows from Theorem 2 in [20]. Here the exponent $p = 2 < (n+2)/(n-2)$ for $3 \le n \le 5$ and there is no restriction on the exponent when $n = 1, 2$. By integrating (3.4), we have $\int_\Omega (-d_2|\nabla u|^2 - mu^2 + u^3)dx = 0$ which implies that

$$\frac{m}{6}\int_\Omega u^2 dx \le J(u) = \frac{1}{6}\int_\Omega (d_2|\nabla u|^2 + mu^2)dx \le C_0 d_2^{n/2}. \tag{3.8}$$

Moreover this solution can be chosen as the least energy positive solution which has the smallest $J(u)$ among all positive solutions of (3.4) (see [29]). The nonexistence of positive non-constant solution for large d_2 follows from part (ii) of Theorem 3 in [21].

(ii) The fact that $d_2 = d_2^j$ is a bifurcation point follows from Theorem 11.4 of [37], as (3.4) has a variational formulation with energy function J defined in (3.7). The global bifurcation conclusion follows from Theorem 1.3 in [36], as (3.4) has no positive non-constant solution for $d_2 > d_2^*$ from part (i) and all positive non-constant solutions of (3.4) are bounded by a constant $C >$ which only depends on m and Ω (Theorem 3 in [21]).

(iii) The properties of Σ_j follow from Theorems 2.5 and 2.7 of [48], and the fact that each solution (d_2, u) on Σ_j satisfies that $u(x) - m$ changes sign exactly j times is proved in [35]. The results for solutions on Σ_1 are from Theorems C of [27]. □

Note that [25–27] also have results on the structure of the solution set of (3.4) for $\Omega = B^n$, the unit ball in $\mathbb{R}^n$, and these results can also be applied to the shadow system (3.3) on a ball similar to the way below. Now combining Propositions 3.1 and 3.2, we obtain the following existence and multiplicity results of positive steady state solutions of shadow system (3.3). The proof is obvious from the correspondence between the solution u of (3.4) and the ones of (3.3) defined in (3.5).

Theorem 3.3. *Suppose d_2, m, a are positive constants, $\Omega \subset \mathbb{R}^n$ $(1 \le n \le 5)$ is a bounded domain with smooth boundary, and μ_j $(j \ge 1)$ are the positive eigenvalues of* (2.13).

(i) *When $0 < d_2 < d_2^*$ and $a > \tilde{a} = \sqrt{\dfrac{24C_0 d_2^{n/2}}{m|\Omega|}}$, the shadow system* (3.3) *has two distinct positive steady state solutions $(B_a^\pm(x), \xi_a^\pm)$ defined as in* (3.5)*, where $u(x)$ is the positive least energy solution of* (3.4) *in Proposition 3.2 part* (i)*; On the other hand, when $d_2 > d_2^{**}$ (defined in Proposition 3.2 part* (i)*), for any $a > 0$, the system* (3.3) *only has three constant nonnegative steady state solutions $(0, a)$ and $(B_a^\pm, \xi_a^\pm)$ with $u(x) = m$.*

(ii) *Let Σ_j be the continuum of positive steady state solutions of* (3.4) *defined in part* (ii) *of Proposition 3.2. Then the set of positive steady state solutions of* (3.3) *contains a connected component in the form of*

$$\tilde{\Sigma}_j = \{(d_2, a, B_a^\pm(x), \xi_a^\pm) : (d_2, u) \in \Sigma_j, a \ge 2||u||_2/|\Omega|^{1/2}\},$$

where $(B_a^\pm(x), \xi_a^\pm)$ is defined as in (3.5).

(iii) *If $n = 1$ and $\Omega = (0, l\pi)$, for $d_2^{j+1} \le d_2 < d_2^j$ where $d_2^j = ml^2/j^2$ and $a > 2m$, system* (3.3) *has exactly $4j + 2$ positive solutions in which $4j$ of them are non-constant solutions and the other two are constant ones. For fixed d_2, the $4j$ non-constant solutions also exist for some $a < 2m$.*

Proof. (i) When $d_2 < d_2^*$, from (3.6), the positive least energy solution $u(x)$ of (3.4) satisfies $||u||_2 \le \sqrt{\dfrac{6C_0 d_2^{n/2}}{m|\Omega|}}$. Then for $a > \tilde{a} = \sqrt{\dfrac{24C_0 d_2^{n/2}}{m|\Omega|}} \ge \dfrac{2||u||_2}{|\Omega|^{1/2}}$, (3.3) has two distinct positive solutions $(B_a^\pm(x), \xi_a^\pm)$ defined as in (3.5) from Proposition 3.1. Note that $a > \tilde{a}$ is equivalent to $d_2 < K_1 a^{4/n}$ where $K_1 = m|\Omega|/(24C_0)$.

(ii) and (iii) follow from Proposition 3.2 part (ii) and (iii). In part (iii), the existence parameter interval for a is $(2||u||_2/|\Omega|^{1/2}, \infty)$ from Proposition 3.1 part (i). Each of these $4j$ intervals contains at least $[2m, \infty)$ from part (ii) of Proposition 3.1 part (ii). □

The linear stability of the non-constant solution $(B_a^\pm(x), \xi_a^\pm)$ of (3.3) can be determined when d_2 is small by the methods in [2,57,59]. For that purpose, we set $d_2 = \epsilon^2$. Linearizing (3.3) at a steady state $(B_\epsilon(x), \xi_\epsilon) = (u_\epsilon(x)/\xi_\epsilon, \xi_\epsilon)$ of (3.3), we obtain

$$\begin{cases} \psi_t = \epsilon^2 \Delta \psi - m\psi + 2u_\epsilon \psi + \xi_\epsilon^{-2} u_\epsilon^2 \eta, & in \quad \Omega,\ t > 0, \\ \eta_t = -\dfrac{2}{|\Omega|} \displaystyle\int_\Omega u_\epsilon \psi dx - \dfrac{1}{|\Omega| \xi_\epsilon^2} \int_\Omega u_\epsilon^2 dx \eta - \eta, & t > 0, \\ \dfrac{\partial \psi}{\partial \nu} = 0, & on \quad \partial\Omega,\ \ t > 0. \end{cases} \tag{3.9}$$

Define the linearized operator

$$\mathcal{L}_{\infty,\epsilon} := \begin{pmatrix} \epsilon^2 \Delta - m + 2u_\epsilon & \xi_\epsilon^{-2} u_\epsilon^2 \\ -\dfrac{2}{|\Omega|} \displaystyle\int_\Omega u_\epsilon \cdot dx & -\dfrac{1}{|\Omega| \xi_\epsilon^2} \displaystyle\int_\Omega u_\epsilon^2 dx - 1 \end{pmatrix},$$

in the space $X_1 := W_\nu^{2,q}(\Omega) \times \mathbb{R}^+$ where $W_\nu^{2,q}(\Omega) = \left\{ u \in W^{2,q}(\Omega) : \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}$. The linear stability of $(B_\epsilon(x), \xi_\epsilon)$ with respect to (3.3) is determined by the eigenvalue problem $\mathcal{L}_{\infty,\epsilon}(\psi, \eta)^T = \lambda(\psi, \eta)^T$, that is

$$\begin{cases} \epsilon^2\Delta\psi - m\psi + 2u_\epsilon\psi + \xi_\epsilon^{-2}u_\epsilon^2\eta = \lambda\psi, & in \quad \Omega, \\ -\dfrac{2}{|\Omega|}\displaystyle\int_\Omega u_\epsilon\psi dx - \dfrac{1}{|\Omega|\xi_\epsilon^2}\int_\Omega u_\epsilon^2 dx\eta - \eta = \lambda\eta, & \\ \dfrac{\partial\psi}{\partial\nu} = 0, & on \quad \partial\Omega, \end{cases} \tag{3.10}$$

or equivalently, the following nonlocal eigenvalue problem:

$$\begin{cases} \epsilon^2\Delta\psi - m\psi + 2u_\epsilon\psi - \dfrac{2u_\epsilon^2\int_\Omega u_\epsilon\psi dx}{\int_\Omega u_\epsilon^2 dx + |\Omega|\xi_\epsilon^2(1+\lambda)} = \lambda\psi, & in \quad \Omega, \\ \dfrac{\partial\psi}{\partial\nu} = 0, & on \quad \partial\Omega. \end{cases}$$

We consider the linear stability of the positive steady state solution $(B_\epsilon(x), \xi_\epsilon)$ of (3.3) corresponding to a solution u_ϵ of (3.4) with small ϵ. For that purpose, we recall the following result regarding a spike layer solution u_ϵ of (3.4) (see Theorem A in [57]).

Lemma 3.4. *Suppose $\Omega \subset \mathbb{R}^n$ $(1 \le n \le 5)$ is a bounded domain with smooth boundary $\partial\Omega$. Let $P_0 \in \partial\Omega$ be a nondegenerate critical point of the mean curvature function $H(P)$ for $P \in \partial\Omega$. Then for ϵ sufficiently small, problem (3.4) with $d_2 = \epsilon^2$ has a solution u_ϵ such that u_ϵ has only one local maximum point P_ϵ and $P_\epsilon \in \partial\Omega$. Moreover, $P_\epsilon \to P_0$ as $\epsilon \to 0$ and $u_\epsilon(y) := u_\epsilon(\epsilon y + P_\epsilon) \to w(y)$ as $\epsilon \to 0$ uniformly for $y \in \Omega_{\epsilon,P_\epsilon} := \{y : \epsilon y + P_\epsilon \in \bar{\Omega}\}$, where w is the unique solution of the following problem:*

$$\begin{cases} \Delta w - mw + w^2 = 0, \quad w > 0 & in \quad \mathbb{R}^n, \\ w(0) = \max\limits_{y\in\mathbb{R}^n} w(y), \quad w(y) \to 0 \quad as \quad |y| \to \infty. \end{cases} \tag{3.11}$$

The solution u_ϵ of (3.4) in Lemma 3.4 is a spike-layer solution which concentrates near a non-degenerate critical point of the mean curvature function of the boundary. In particular, the least energy solution defined in Proposition 3.2 is a spike layer solution which concentrates at the maximum point of the mean curvature function $H(P)$ [30]. The stability of a spike layer solution with respect to (3.4) is determined by the linearized operator $\mathcal{L}_\epsilon : W^{2,q}_\nu(\Omega) \to L^q(\Omega)$ defined as

$$\mathcal{L}_\epsilon = \epsilon^2\Delta - m + 2u_\epsilon. \tag{3.12}$$

Then we have the following result on the spectrum set $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ (see [2, Theorem 4.6] or [57, Theorem 3.1]).

Lemma 3.5. *Let u_ϵ be the positive solution of (3.4) in Lemma 3.4.*

1. *$\sigma(\mathcal{L}_\epsilon)$ consists of a sequence of real-valued eigenvalues $\tilde{\lambda}_{j,\epsilon}$ satisfying*

$$\tilde{\lambda}_{1,\epsilon} > \tilde{\lambda}_{2,\epsilon} \ge \tilde{\lambda}_{3,\epsilon} \ge \cdots \ge \tilde{\lambda}_{j,\epsilon} \ge \cdots \to -\infty.$$

   (i) *As $\epsilon \to 0$, $\tilde{\lambda}_{1,\epsilon} \to \lambda_1(\mathcal{L}_0) > 0$, where $\lambda_1(\mathcal{L}_0)$ is the principal eigenvalue of $\mathcal{L}_0 = \Delta - m + 2w$ on $W^{2,p}(\mathbb{R}^n)$ and w is the unique positive solution of (3.11), and $\tilde{\lambda}_{j,\epsilon} \le -m$ for $j \ge n+1$.*
   (ii) *As $\epsilon \to 0$, $\tilde{\lambda}_{j,\epsilon} = \epsilon^2\gamma\eta_j + o(\epsilon^2)$, $2 \le j \le n$, where γ is a positive constant and η_j is the $(j-1)$-th eigenvalue of the Hessian of the mean curvature function of the boundary manifold $D^2H(P)$.*

Next we have the following result which connects the stability of the positive solution of the shadow system (3.10) to the one of (3.4) from Theorem 4.1 in [57].

Lemma 3.6. *Suppose that $\Omega \subset \mathbb{R}^n$ $(2 \le n \le 4)$ is a bounded domain with smooth boundary $\partial\Omega$. Let $\lambda_{j,\epsilon}$ be the eigenvalues of $\mathcal{L}_{\infty,\epsilon}$, and let $\tilde{\lambda}_{j,\epsilon}$ be the eigenvalues of $\mathcal{L}_\epsilon$. Then $\lambda_{j,\epsilon} = (1+o(1))\tilde{\lambda}_{j+1,\epsilon}$ for $j = 1, 2, \cdots, n$, and $\mathrm{Re}(\lambda_{j,\epsilon}) < -c_0 < 0$ for $j > n$ where $c_0 > 0$.*

Now from Lemma 3.5 and Lemma 3.6, we can conclude the following results about the stability of the non-constant solutions of (3.3) corresponding to spike layer solution of (3.4).

Theorem 3.7. *Suppose that $\Omega \subset \mathbb{R}^n$ $(2 \le n \le 4)$ is a bounded domain with smooth boundary $\partial\Omega$. Let $P_0 \in \partial\Omega$ be a nondegenerate critical point of the mean curvature function $H(P)$, and let η_j, $2 \le j \le n$ be the $(j-1)$-th eigenvalue of the Hessian of the mean curvature function of the boundary manifold $D^2H(P)$. Let $u_\epsilon(x)$ be a positive solution of (3.4) concentrating near P_0 as in Lemma 3.4, and let $(B_\epsilon(x), \xi_\epsilon) = (u_\epsilon(x)/\xi_\epsilon, \xi_\epsilon)$ be the corresponding non-constant steady state solution of (3.3). Then for sufficiently small $\epsilon > 0$, or equivalently sufficiently small $d_2 > 0$,*

(i) *If $\eta_j < 0$ for all $2 \le j \le n$, then $(B_\epsilon(x), \xi_\epsilon)$ of (3.3) is linearly stable.*
(ii) *If $\eta_j > 0$ for some $2 \le j \le n$, then $(B_\epsilon(x), \xi_\epsilon)$ of (3.3) is unstable.*

Part (i) Theorem 3.7 implies that when $u_\epsilon(x)$ is the least energy solution of (3.4), the solution concentrates near the maximum point of the mean curvature function on the boundary, and $(B_\epsilon(x), \xi_\epsilon)$ is a linearly stable steady state of (3.3) as $\eta_j < 0$ for all $2 \le j \le n$ in that case.

3.2. Solutions of the original system for large d_1

In this subsection, we return to the original reaction-diffusion system (1.2) and show the existence of non-constant steady states when d_1 is sufficiently large by using the results on the shadow system (3.3) and implicit function theorem (see for example [24,34]).

Theorem 3.8. *Suppose that $\Omega \subset \mathbb{R}^n$ $(2 \le n \le 4)$ is a bounded domain with smooth boundary $\partial\Omega$. Let $P_0 \in \partial\Omega$ be a nondegenerate critical point of the mean curvature function $H(P)$. Then there exists a positive $\tilde{d}_2 < d_2^*$ (defined in Proposition 3.2 part (i)), such that for $0 < d_2 < \tilde{d}_2$ there exists a constant $s(d_2) > 0$ such that (1.2) has a nonconstant positive steady state solution such that the plant biomass concentrates near P_0 when $d_1 > 1/s(d_2)$ and $0 < d_2 < \tilde{d}_2$. Moreover if P_0 is the maximum point of $H(P)$, then the corresponding nonconstant positive steady state solution is linearly stable with respect to (1.2).*

Proof. Define

$$Y_1 := \Big\{ u \in W_\nu^{2,q}(\Omega) : \int_\Omega u(x)dx = 0 \Big\},$$

and the projection operator $\mathrm{P} : W_\nu^{2,q}(\Omega) \to Y_1$ by

$$\mathrm{P}u(x) = u(x) - \frac{1}{|\Omega|}\int_\Omega u(s)ds. \tag{3.13}$$

We consider the following equation:

$$\begin{cases} \Delta\phi + s\mathrm{P}[a - (\xi+\phi)B^2 - (\xi+\phi)] = 0, & in \;\; \Omega, \\ d_2\Delta B + (\xi+\phi)B^2 - mB = 0, & in \;\; \Omega, \\ a - \dfrac{\xi+\phi}{|\Omega|}\displaystyle\int_\Omega B^2 dx - (\xi+\phi) = 0, & \\ \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial B}{\partial\nu} = 0, & on \;\; \partial\Omega. \end{cases} \tag{3.14}$$

Define an operator $\mathcal{X}(d_2, s, \phi, B, \xi)$ by

$$\mathcal{X}(d_2, s, \phi, B, \xi) := \begin{pmatrix} \Delta\phi + s\mathrm{P}[a - (\xi+\phi)B^2 - (\xi+\phi)] \\ d_2\Delta B + (\xi+\phi)B^2 - mB \\ a - \frac{\xi+\phi}{|\Omega|}\int_\Omega B^2 dx - (\xi+\phi) \end{pmatrix}. \tag{3.15}$$

Then $\mathcal{X}(d_2, s, \phi, B, \xi)$ is an analytic mapping from the open set $\{(d_2, s, \phi, B, \xi) : \xi > 0, \xi+\phi > 0\}$ of $\mathbb{R}^+ \times \mathbb{R}^+ \times Y_1 \times W_\nu^{2,q}(\Omega) \times \mathbb{R}^+$ into $L^q(\Omega) \times L^q(\Omega) \times \mathbb{R}$.

Let $(B(x, d_2), \xi(d_2))$ be a solution to the shadow system (3.3) when $d_2 < \tilde{d}_2$ so that Theorem 3.7 holds. Fixing $d_2 < \tilde{d}_2$, by the definition of $\mathcal{X}$, we have $\mathcal{X}(d_2, 0, 0, B(x, d_2), \xi(d_2)) = 0$ and the Fréchet derivative of $\mathcal{X}(d_2, s, \phi, B, \xi)$ at $(d_2, 0, 0, B(x, d_2), \xi(d_2))$ is given by

$$\mathcal{X}_{(\phi,B,\xi)}(d_2, 0, 0, B(x, d_2), \xi(d_2)) = \begin{pmatrix} \Delta & \mathbf{0} \\ \mathcal{B} & \mathcal{L}_\infty \end{pmatrix}, \tag{3.16}$$

where $\mathbf{0} = (0, 0)$,

$$\mathcal{L}_\infty = \begin{pmatrix} d_2\Delta + (2\xi(d_2)B(x, d_2) - m) & B^2(x, d_2) \\ -\frac{2\xi(d_2)}{|\Omega|}\int_\Omega B(x, d_2) \;\cdot dx & -\frac{1}{|\Omega|}\int_\Omega B^2(x, d_2)dx - 1 \end{pmatrix}, \tag{3.17}$$

and

$$\mathcal{B} = \begin{pmatrix} B^2(x, d_2) \\ -\frac{1}{|\Omega|}\int_\Omega B^2(x, d_2)dx - 1 \end{pmatrix}. \tag{3.18}$$

As Δ is an isomorphism from Y_1 to $L^q(\Omega)$ under homogeneous Neumann boundary condition, and from Theorem 3.8, $\mathcal{L}_\infty$ is nondegenerate, then $\mathcal{X}_{(\phi,B,\xi)}(d_2, 0, 0, B(x, d_2), \xi(d_2))$ is nondegenerate. Consequently, by the implicit function theorem there exists a one-parameter of solutions $(\phi_s(x), B_s(x), \xi_s)$ of (3.14) for $s \in (0, s(d_2))$ for some $s(d_2) > 0$. Notice that if $(\phi_s(x), B_s(x), \xi_s)$ satisfies (3.14) with $s > 0$, then $(B_s(x), \xi_s + \phi_s(x))$ is a solution of (1.2) with $d_1 = 1/s$. Therefore there exists a family of non-constant steady states $(\xi_s + \phi_s(x), B_s(x))$ of (1.2) for $d_1 = 1/s$ with $s \in (0, s(d_2))$. The stability of the solution follows from Theorem 1.4 in [57]. □

From Theorem 3.3 and Theorem 3.8, we have the following results regarding the pattern formation in the reaction-diffusion Klausmeier-Gray-Scott system (1.2).

Corollary 3.9. *For any positive $\delta < d_2^*$, when $0 < d_2 < \delta$ and $a > \tilde{a} = \sqrt{\dfrac{24C_0\delta^{n/2}}{m|\Omega|}}$, there exists $d_1^\delta > 0$ such that (1.2) has a non-constant positive steady state solution provided $d_1 > d_1^\delta$. In particular such a solution could exist for arbitrarily small rainfall value $a = O(\delta^{n/4}) \ll 2m$.*

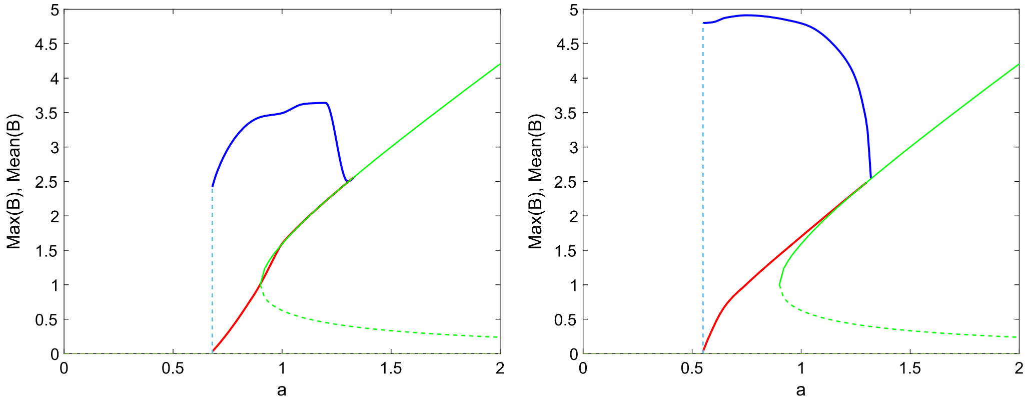
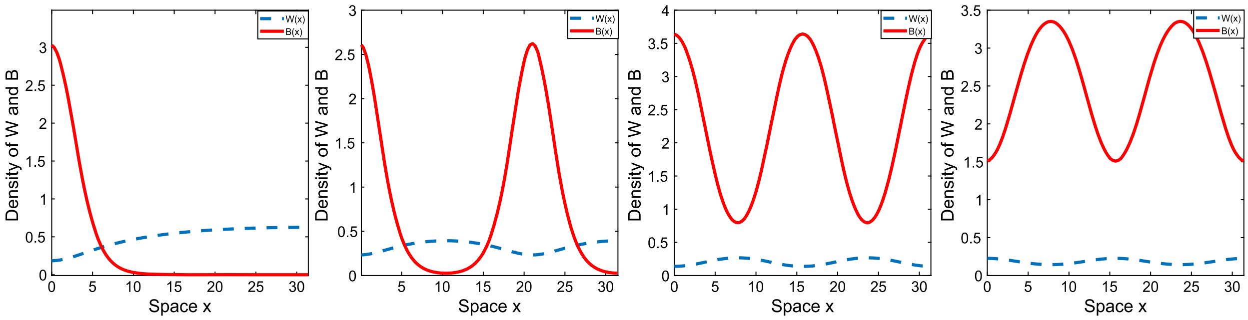


Fig. 3. Bifurcation diagrams of (2.1) when $m = 0.45$, $d_1 = 80$, $d_2 = 1$. (Left) $\Omega_1 = (0, 10\pi)$; (Right) $\Omega_2 = (0, 10\pi) \times (0, 10\pi)$. The horizontal axis is a (rainfall). Green curve: constant plant density; blue curve: maximum value of patterned steady state plant biomass; and red curve: mean value of patterned steady state plant biomass. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)



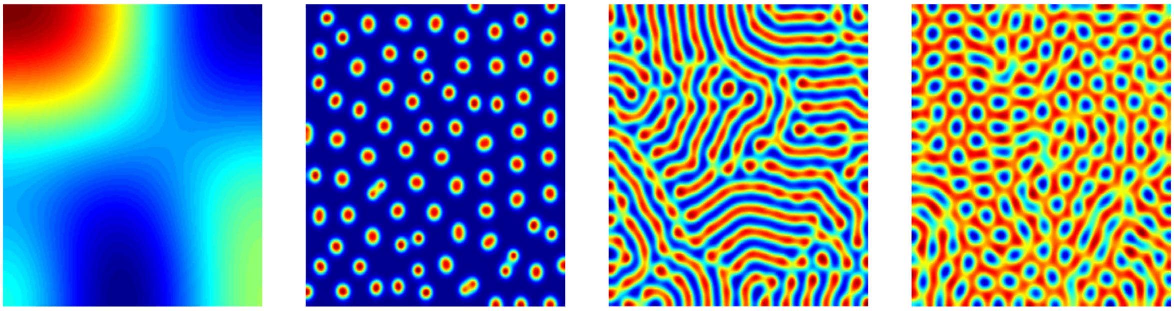
(a) $a = 0.66$ (b) $a = 0.7$ (c) $a = 1.2$ (d) $a = 1.3$

Fig. 4. Patterned plant distribution in (1.2) with $m = 0.45$, $d_1 = 80$, $d_2 = 1$ and $\Omega = (0, 10\pi)$.

Fig. 3 shows the bifurcation diagram of steady state solutions of (1.2) for both $\Omega_1 = (0, 10\pi)$ and $\Omega_2 = (0, 10\pi) \times (0, 10\pi)$. Indeed the diagram shows that the transcritical bifurcation branch of the stable non-constant steady state solutions emerging from the constant ones continues to the left to a threshold value $a_* < 2m = 0.9$, so that for $a \in (a_*, 2m)$, only non-constant positive steady state solutions exist not the constant ones. This verifies the assertion in Corollary 3.9 as in Fig. 3, the water diffusion coefficient d_1 is large and the plant diffusion coefficient d_2 is small. When the rainfall a is near the extinction threshold a_*, the total biomass approaches to 0 but the maximum value of the patterned solution approaches to a very high level, which indicates the concentration of plant biomass.

Fig. 4 shows the profile of the spatial patterns for varying rainfall a in a one-dimensional domain $\Omega_1 = (0, 10\pi)$. In Fig. 4 (a) and (b), the rainfall $a = 0.66$ and $a = 0.7$ are smaller than smallest rainfall $a = 0.9$ supporting a uniform steady state, which implies that patterned vegetation states could exist with much smaller amount of rainfall. Fig. 4 (a) slows a spike layer solution for the plant concentrating on one of the end points which corresponds to the least energy solution discussed above. When the rainfall increases, the number of plant concentration areas (patches) also increases (see Fig. 4 (b), (c) and (d)).

Fig. 5 shows the profile of the spatial patterns for varying rainfall a in a two-dimensional domain $\Omega_2 = (0, 10\pi) \times (0, 10\pi)$. Again when the rainfall a is near the threshold ($a = 0.502$ in Fig. 5 (a)), slow plant diffusion and fast water diffusion can support a vegetation state with vegetation concentrating on a small area, and the solution is a quarter spike concentrating at a corner of the square. When the rainfall increases, the spatial pattern becomes to spots, labyrinth and gaps (see Fig. 5 (b), (c) and (d)).



(a) $a = 0.502$ (b) $a = 0.6$ (c) $a = 1.2$ (d) $a = 1.3$

Fig. 5. Patterned plant distribution in (1.2) with $m = 0.45$, $d_1 = 80$, $d_2 = 1$ and $\Omega_2 = (0, 10\pi) \times (0, 10\pi)$. Here, blue color area is bare soil ($B = 0$) and red color area is high vegetation concentration ($B > 0$).

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Appendix A. Direction of the pitchfork bifurcation

Setting $u = W - \frac{m}{B_+}, v = B - B_+$, system (1.2) can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + a - \Big(u + \dfrac{m}{B_+}\Big)(v + B_+)^2 - \Big(u + \dfrac{m}{B_+}\Big), & in \quad \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + \Big(u + \dfrac{m}{B_+}\Big)(v + B_+)^2 - m(v + B_+), & in \quad \Omega, \ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & on \quad \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & in \quad \Omega. \end{cases} \tag{A.1}$$

Then the positive constant steady state $(W, B) = (\frac{m}{B_+}, B_+)$ of model (1.2) turns to the one $(u, v) = (0, 0)$ of model (A.1). Here,

$$B_+ = B_+(a) := \frac{a + \sqrt{a^2 - 4m^2}}{2m}. \tag{A.2}$$

The corresponding steady state system of (A.1) is

$$\begin{cases} -d_1 \Delta u = a - \Big(u + \dfrac{m}{B_+}\Big)(v + B_+)^2 - \Big(u + \dfrac{m}{B_+}\Big), & in \quad \Omega, \\ -d_2 \Delta v = \Big(u + \dfrac{m}{B_+}\Big)(v + B_+)^2 - m(v + B_+), & in \quad \Omega, \\ u_x(x, t) = v_x(x, t) = 0, & on \quad \partial\Omega. \end{cases} \tag{A.3}$$

According to [13,47], we have the following lemma.

Lemma A.1. *Suppose that the conditions of Theorem 2.7 are satisfied at $a = a_0 := a_j^S$. Then the steady state bifurcation of model (2.1) is always pitchfork bifurcation, i.e. $a'(0) = 0$; the bifurcation are supercritical bifurcations if $a''(0) > 0$ and subcritical bifurcations if $a''(0) < 0$.*

Proof. For system (1.2), define a mapping $G(a,u,v)$ by

$$G(a,u,v) := \begin{pmatrix} d_1 u_{xx} + f(a,u,v) \\ d_2 v_{xx} + g(a,u,v) \end{pmatrix}, \tag{A.4}$$

with

$$f(a,u,v) = a - \Big(u + \frac{m}{B_+}\Big)(v+B_+)^2 - \Big(u + \frac{m}{B_+}\Big),$$
$$g(a,u,v) = \Big(u + \frac{m}{B_+}\Big)(v+B_+)^2 - m(v+B_+).$$

Then $G : \mathbb{R}^+ \times Z_0 \to Y$ is Fréchet differentiable, and at a constant steady state (W_+, B_+), the linearized operator at $(a_0, 0, 0)$ is

$$\mathcal{L}(a_0) := \begin{pmatrix} -1 - B_+^2(a_0) + d_1\Delta & -2m \\ B_+^2(a_0) & m + d_2\Delta \end{pmatrix}.$$

Assume

$$\begin{aligned} q = (\alpha, \beta) &:= \Big(1, \frac{B_+^2(a_0)}{\mu_j d_2 - m}\Big)^T, \\ q^* = (\alpha^*, \beta^*) &:= \Big(\frac{m - \mu_j d_2}{2m}, 1\Big)^T. \end{aligned} \tag{A.5}$$

Then for $j \in \mathbb{N}$, we have $\mathcal{L}_j(a_0)(\alpha,\beta)^T \varphi_j(x) = 0$, and $\mathcal{L}_j^*(a_0)(\alpha^*,\beta^*)^T \varphi_j(x) = 0$, where

$$\mathcal{L}_j(a_0) := \begin{pmatrix} -1 - B_+^2(a_0) - d_1\mu_j & -2m \\ B_+^2(a_0) & m - d_2\mu_j \end{pmatrix}, \tag{A.6}$$

and $\mathcal{L}_j^*(a_0)$ is the adjoint operator of $\mathcal{L}_j(a_0)$. According to Theorem 2.7, near the bifurcation point $(a_j^S, 0, 0)$ the set of positive solutions of (A.3) can be parameterized as

$$\Gamma_j' = \{(a_j(s), u(s), v(s)) : s \in (0, \varepsilon)\},$$

with $a_j(0) = a_j^S, u(s) = s\alpha\varphi_j + s\varphi_{1,j}(s), v(s) = s\beta\varphi_j + s\varphi_{2,j}(s), \varphi_{1,j}(0) = \varphi_{2,j}(0) = 0$, $\varphi_{1,j}(s)$ and $\varphi_{2,j}(s)$ are differentiable functions defined by $\varphi_{1,j}, \varphi_{2,j} : [0,\varepsilon) \to Z_1$, where $Z_1 = \{(u,v) \in Z_0 : \int_\Omega [(d_2\mu_j - m)u + B_+^2(a_j^S)v]\varphi_j dx = 0\}$ is a subspace of Z_0 complement to $\operatorname{span}\{(\phi_0, \psi_0)\}$.

From [47],

$$a'(0) = -\frac{\langle \zeta, G_{(u,v),(u,v)}[q,q]\rangle}{2\langle \zeta, G_{a(u,v)}[q,q]},$$

where $\zeta \in Y^*$ satisfying $\mathcal{N}(\zeta) = \mathcal{R}(\mathcal{L}(a_0))$ and the function ζ is given by

$$\langle \zeta, (p_1, p_2)\rangle = \int_\Omega (d_1^{-1}\alpha^* p_1 + d_2^{-1}\beta^* p_2)\varphi_j(x)dx,$$

for $(p_1, p_2) \in Y$. Then

$$\langle \zeta, G_{(u,v),(u,v)}[q,q] \rangle = \int_\Omega G_{(u,v),(u,v)}[q,q] \cdot p dx, \tag{A.7}$$

where $p = (d_1^{-1}\alpha^*, d_2^{-1}\beta^*)\varphi_j(x)$. Thus,

$$a'(0) = -\frac{\int_\Omega G_{(u,v),(u,v)}[q,q] \cdot p dx}{2\int_\Omega G_{a(u,v)}[q] \cdot p dx}.$$

Direct computations show that

$$\int_\Omega G_{(u,v),(u,v)}[q,q] \cdot p dx = \int_\Omega k_j \varphi_j^3(x) dx,$$
$$\int_\Omega G_{a(u,v)}[q] \cdot p dx = \int_\Omega r_j \varphi_j^2(x) dx,$$

where

$$\begin{aligned} k_j &= d_1^{-1}\alpha^*(f_{uu}\alpha^2 + 2f_{uv}\alpha\beta + f_{vv}\beta^2) + d_2^{-1}\beta^*(g_{uu}\alpha^2 + 2g_{uv}\alpha\beta + g_{vv}\beta^2), \\ r_j &= d_1^{-1}\alpha^*(f_{au}\alpha + f_{av}\beta) + d_2^{-1}\beta^*(g_{au}\alpha + g_{av}\beta). \end{aligned} \tag{A.8}$$

Here, all the partial derivatives of f and g are calculated at $(a_0, 0, 0)$. Hence, $a'(0) = 0$ and the bifurcation is a pitchfork bifurcation.

Thus, the sign of $a''(0)$ is needed to determine the direction of the pitchfork bifurcation. According to [47], $a''(0)$ is given by

$$a''(0) = -\frac{\langle \zeta, G_{(u,v),(u,v),(u,v)}[q,q,q] \rangle + 3\langle \zeta, G_{(u,v),(u,v)}[q,\theta] \rangle}{2\langle \zeta, G_{a(u,v)}[q] \rangle}, \tag{A.9}$$

where θ is the solution of

$$G_{(u,v),(u,v)}[q,q] + G_{(u,v)}[\theta] = 0. \tag{A.10}$$

Similarly to (A.7), we have

$$a''(0) = -\frac{\int_\Omega G_{(u,v),(u,v),(u,v)}[q,q,q] \cdot p dx + 3\int_\Omega G_{(u,v),(u,v)}[q,\theta] \cdot p dx}{3\int_\Omega G_{a(u,v)}[q] \cdot p dx}. \tag{A.11}$$

Direct calculation shows that [13]

$$\int_\Omega G_{(u,v),(u,v),(u,v)}[q,q,q] \cdot p dx = \int_\Omega s_j \varphi_j^4(x) dx,$$
$$\int_\Omega G_{(u,v),(u,v)}[q,\theta] \cdot p dx = \int_\Omega t_j^1 \varphi_j^2(x) dx + \int_\Omega t_j^2 \varphi_j^4(x) dx,$$

where

$$\begin{aligned} s_j =& d_1^{-1}\alpha^*(f_{uuu}\alpha^3 + 3f_{uuv}\alpha^2\beta + 3f_{uvv}\alpha\beta^2 + f_{vvv}\beta^3) \\ &+ d_2^{-1}\beta^*(g_{uuu}\alpha^3 + 3g_{uuv}\alpha^2\beta + 3g_{uvv}\alpha\beta^2 + g_{vvv}\beta^3), \end{aligned}$$

$$\begin{aligned} t_j^1 =& d_1^{-1}\alpha^*[(f_{uu}\alpha + f_{uv}\beta)\Theta_0^1 + (f_{uv}\alpha + f_{vv}\beta)\Theta_0^2] \\ &+ d_2^{-1}\beta^*[(g_{uu}\alpha + g_{uv}\beta)\Theta_0^1 + (g_{uv}\alpha + g_{vv}\beta)\Theta_0^2], \\ t_j^2 =& d_1^{-1}\alpha^*[(f_{uu}\alpha + f_{uv}\beta)\Theta_j^1 + (f_{uv}\alpha + f_{vv}\beta)\Theta_j^2] \\ &+ d_2^{-1}\beta^*[(g_{uu}\alpha + g_{uv}\beta)\Theta_j^1 + (g_{uv}\alpha + g_{vv}\beta)\Theta_j^2], \end{aligned} \tag{A.12}$$

with

$$\Theta_0^1 = \theta_0^1 - \theta_{2j}^1, \Theta_0^2 = \theta_0^2 - \theta_{2j}^2, \Theta_j^1 = 2\theta_{2j}^1, \Theta_j^2 = 2\theta_{2j}^2.$$

Here,

$$\theta = \sum_{m=0}^{\infty} \begin{pmatrix} \theta_m^1 \\ \theta_m^2 \end{pmatrix} \varphi_m(x)$$

satisfies the equation (A.10) and $(\theta_m^1, \theta_m^2) = (0, 0)$ for all odd m, and

$$\begin{pmatrix} \theta_0^1 \\ \theta_0^2 \end{pmatrix} = \frac{1}{2D_0} \begin{pmatrix} g_v(f_{uu}\alpha^2 + 2f_{uv}\alpha\beta + f_{vv}\beta^2) - f_v(g_{uu}\alpha^2 + 2g_{uv}\alpha\beta + g_{vv}\beta^2) \\ f_u(g_{uu}\alpha^2 + 2g_{uv}\alpha\beta + g_{vv}\beta^2) - g_u(f_{uu}\alpha^2 + 2f_{uv}\alpha\beta + f_{vv}\beta^2) \end{pmatrix},$$

$$\begin{pmatrix} \theta_{2j}^1 \\ \theta_{2j}^2 \end{pmatrix} = \frac{1}{2D_{2j}} \begin{pmatrix} (g_v - \frac{4d_2 j^2}{l^2})(f_{uu}\alpha^2 + 2f_{uv}\alpha\beta + f_{vv}\beta^2) - f_v(g_{uu}\alpha^2 + 2g_{uv}\alpha\beta + g_{vv}\beta^2) \\ (f_u - \frac{4d_1 j^2}{l^2})(g_{uu}\alpha^2 + 2g_{uv}\alpha\beta + g_{vv}\beta^2) - g_u(f_{uu}\alpha^2 + 2f_{uv}\alpha\beta + f_{vv}\beta^2) \end{pmatrix},$$

where D_0, D_{2j} are the determinants of $\mathcal{L}_0(a_0)$ and $\mathcal{L}_{2j}(a_0)$. Hence,

$$a''(0) = -\frac{s_j + 4t_j^1 + 3t_j^2}{4r_j}. \tag{A.13}$$

Note that $B_+(a_0) = \frac{a_0 + \sqrt{a_0^2 - 4m^2}}{2m}$, $B'_+(a_0) = \frac{a_0 + \sqrt{a_0^2 - 4m^2}}{2m\sqrt{a_0^2 - 4m^2}}$, and

$$\begin{aligned} &f_u = -B_+^2(a_0) - 1, && f_v = -2m, && g_u = B_+^2(a_0), && g_v = m, \\ &f_{au} = -2B_+(a_0)B'_+(a_0), && f_{av} = 0, && g_{au} = 2B_+(a_0)B'_+(a_0), && g_{av} = 0, \\ &f_{uu} = 0, && f_{uv} = -2B_+(a_0), && f_{vv} = -\frac{2m}{B_+(a_0)}, && \\ &g_{uu} = 0, && g_{uv} = 2B_+(a_0), && g_{vv} = \frac{2m}{B_+(a_0)}, && \\ &f_{uuu} = 0, && f_{uuv} = 0, && f_{uvv} = -2, && f_{vvv} = 0, \\ &g_{uuu} = 0, && g_{uuv} = 0, && g_{uvv} = 2, && g_{vvv} = 0. \end{aligned}$$

Substituting r_j, s_j, t_j^1 and t_j^2 into (A.13), we have

$$\operatorname{sgn}\{a''(0)\} = -\operatorname{sgn}\Big\{3\beta^2 + \beta B_+(a_0)(4\Theta_0^1 + 3\Theta_j^1) + \Big(B_+(a_0) + \frac{\beta m}{B_+(a_0)}\Big)(4\Theta_0^2 + 3\Theta_j^2)\Big\}, \tag{A.14}$$

where

$$4\Theta_0^1 + 3\Theta_j^1 = \Big(4B_+(a_0)\alpha\beta + \frac{2m}{B_+(a_0)}\beta^2\Big)\Big(\frac{2m}{D_0} + \frac{m + \frac{4d_2 j^2}{l^2}}{D_{2j}}\Big),$$

$$4\Theta_0^2 + 3\Theta_j^2 = \Big(4B_+(a_0)\alpha\beta + \frac{2m}{B_+(a_0)}\beta^2\Big)\Big(-\frac{2}{D_0} - \frac{1 + \frac{4d_1 j^2}{l^2}}{D_{2j}}\Big). \quad \square$$

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