GENERICALLY FREE REPRESENTATIONS III: EXTREMELY BAD CHARACTERISTIC

SKIP GARIBALDI AND ROBERT M. GURALNICK

Abstract. In parts I and II, we determined which faithful irreducible representations V of a simple linear algebraic group G are generically free for $\mathrm{Lie}(G)$, i.e., which V have an open subset consisting of vectors whose stabilizer in $\mathrm{Lie}(G)$ is zero, with some assumptions on the characteristic of the field. This paper settles the remaining cases, which are of a different nature because $\mathrm{Lie}(G)$ has a more complicated structure and there need not exist general dimension bounds of the sort that exist in good characteristic.

Let G be a simple algebraic group over an algebraically closed field k. In case $k = \mathbb{C}$, it has been known for more than 40 years which irreducible representations V of G are *generically free*, i.e., have the property that the stabilizer in G of a generic $v \in V$ is the trivial group scheme. Recent applications of this to the theory of essential dimension have motivated the desire to extend these results to arbitrary k. We did this in previous papers — [GaGu 17], [GuL], and parts I [GaGu I] and II [GaGu II] — except for a handful of cases that we address here, completing the solution to the problem. In particular we prove the following, which was announced at the end of part I.

Theorem A. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a faithful irreducible representation of a simple algebraic group over an algebraically closed field k.*

1. *G_v is finite étale for generic $v \in V$ if and only if $\dim V > \dim G$ and $(G, \mathrm{char}\, k, V)$ does not appear in Table 1.*
2. *G acts generically freely on V if and only if $\dim V > \dim G$ and $(G, \mathrm{char}\, k, V)$ appears in neither Table 1 nor Table 3.*

We say that ρ is *faithful* if $\ker \rho$ is the trivial group scheme. This hypothesis is harmless in the sense that (i) every generically free representation is faithful and (ii) every irreducible representation $\rho : G \to \mathrm{GL}(V)$ canonically gives a faithful irreducible representation $G/(\ker \rho) \to \mathrm{GL}(V)$.

The hypothesis that ρ is faithful in Theorem A excludes those representations that factor through a special isogeny of G. The hypothesis that ρ is faithful also excludes those representations that occur as the Frobenius twist of some other representation, since in that case $\ker \rho$ contains the first Frobenius kernel. Nonetheless, we do consider such representations in detail in this paper. (To apply Theorem A in the latter case, note that the first Frobenius kernel G_1 is contained in $\ker \rho$. One obtains from ρ a representation of the group G/G_1, which is isomorphic to G [J, I.9.5], and can check whether that representation is faithful.)

Recall that an algebraic group H is *finite étale* if $\mathrm{Lie}(H) = 0$.

2010 *Mathematics Subject Classification.* 20G05 (primary); 17B10 (secondary).

G	char k	rep'n	$\dim V$	$\dim \mathfrak{g}_v$	G	char k	high weight	$\dim V$	$\dim \mathfrak{g}_v$
SL_8/μ_4	2	$\wedge^4$	70	3	Sp_8	3	0100	40	2
SL_9/μ_3	3	$\wedge^3$	84	2	Sp_4	5	11	12	1
Spin_{16}/μ_2	2	half-spin	128	4	SL_4	p odd	$01p^e$, $e \ge 1$	24	1
					SL_4/μ_2	2	012^e, $e \ge 2$	24	1

TABLE 1. Irreducible and faithful representations V of simple G with $\dim V > \dim G$ that are not generically free for $\mathfrak{g}$, up to graph automorphism. For each, the stabilizer $\mathfrak{g}_v$ of a generic $v \in V$ is a toral subalgebra. The weights on the right side are numbered as in Table 2.

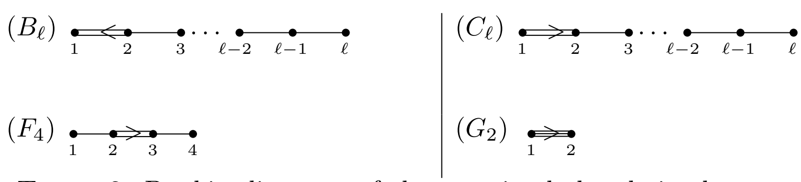

TABLE 2. Dynkin diagrams of the non-simply-laced simple root systems, with simple roots numbered as in [Lü].

G	char k	V	$\dim V$	G	char k	V	$\dim V$
A_1	$\ne 2,3$	S^3	4	A_2	$\ne 2,3$	S^3	10
A_1	$\ne 2,3$	S^4	5	A_3	$\ne 2$	$L(2\omega_2)$	$20-\varepsilon$
A_8	$\ne 3$	$\wedge^3$	84	A_7	$\ne 2$	$\wedge^4$	70
A_3	3	$L(\omega_1+\omega_2)$	16	A_ℓ	$p \ne 0$	$L(\omega_1+p^i\omega_\ell)$, $L(\omega_1+p^i\omega_1)$	$(\ell+1)^2$
B_ℓ $(\ell \ge 2)$	$\ne 2$	$L(2\omega_\ell)$	$2\ell^2+3\ell-\varepsilon$	C_4	$\ne 2$	"spin"	$42-\varepsilon$
D_ℓ $(\ell \ge 4)$	$\ne 2$	$L(2\omega_\ell)$	$2\ell^2+\ell-1-\varepsilon$	D_8	$\ne 2$	half-spin	128

TABLE 3. Irreducible faithful representations V of simple G with $\dim V > \dim G$ such that G_v is finite étale and $\ne 1$ for generic $v \in V$, up to graph automorphism, adapted from [GuL]. The symbol ε denotes 0 or 1 depending on the value of char k.

The remaining cases of the theorem that need to be covered in this paper are those where char k is *special*[1] for G, meaning that G has type G_2 and char $k = 3$ or G has type B_n $(n \ge 2)$, C_n $(n \ge 2)$, or F_4 and char $k = 2$. These are the cases where the Dynkin diagram of G has a multiple bond of valence char k. Equivalently, these are the cases where G possesses so called "special" isogenies, which are neither central nor the Frobenius, cf. [BoTi, §3].

In a future work, we combine Theorem A with the results of [GuL] to prove the existence of a stabilizer in general position for every action of a simple algebraic group on an irreducible representation.

A special case of Theorem A is the following:

[1]This choice of vocabulary imitates [S]; we have written instead the more illuminating "extremely bad characteristic" in the title. The hypothesis "char k special" is properly more restrictive than "char k very bad", in that 2 is very bad but not special (i.e., not extremely bad) for type G_2.

Theorem B. *Let G be a simple linear algebraic group over an algebraically closed field k such that* $\mathrm{char}\, k$ *is special, and let $\rho: G \to \mathrm{GL}(V)$ be an irreducible and faithful representation. Then V is generically free for $\mathfrak{g}$ if and only if $\dim V > \dim G$.*

Large, possibly reducible representations. Regardless of whether ρ is faithful, the stabilizer $\mathfrak{g}_v$ of a generic $v \in V$ contains $\ker \mathrm{d}\rho$. We say that $\mathfrak{g}$ acts *virtually freely* on V if $\mathfrak{g}_v = \ker \mathrm{d}\rho$; this is the natural generalization of the notion of "generically free" to include the case where ρ need not be faithful.

In part I, [GaGu I], we proved a general bound when G is simple and $\mathrm{char}\, k$ is not special: if $V^{[\mathfrak{g},\mathfrak{g}]} = 0$ and $\dim V$ is big enough, then $\mathfrak{g}$ acts virtually freely on V. However, Example 9.3 shows that such a result does not hold when $\mathrm{char}\, k$ is special. Rather than producing a possibly complicated statement that encompasses both cases, we give instead the separate statements Proposition 6.1, Corollary 7.10, and Proposition 8.4 for the cases where $\mathrm{char}\, k$ is special. Note that these results have no requirements that G acts irreducibly or faithfully. Roughly speaking, for $\mathfrak{n}$ the Lie algebra of the kernel of the very special isogeny as in §1, we give results for those V on which $\mathfrak{n}$ acts as zero ($\mathfrak{n}V = 0$) or without fixed points ($V^{\mathfrak{n}} = 0$).

Irreducible representations. Recall that every irreducible representation V of G has a highest weight λ. Write λ as a sum $\lambda = \sum_\omega c_\omega \omega$ where the sum runs over the fundamental dominant weights ω. One says that λ is *restricted* when $p := \mathrm{char}\, k \neq 0$ if $0 \le c_\omega < p$ for all ω. (In case $\mathrm{char}\, k = 0$, all dominant weights are, by definition, restricted.)

Our next result is a variation on Theorem B, where we add the hypothesis that the highest weight of ρ is restricted and drop the hypothesis that ρ is faithful.

Theorem C. *Let G be a simple linear algebraic group over an algebraically closed field k. Let $\rho: G \to \mathrm{GL}(V)$ be an irreducible representation for G with a restricted highest weight.*

(1) *If $\mathfrak{g}$ does* <u>*not*</u> *act virtually freely on V, then $\dim V < \dim G$ or $\mathfrak{g}_v$ is a toral subalgebra for generic $v \in V$.*
(2) *Suppose* $\mathrm{char}\, k$ *is special for G and ρ is not the trivial representation. Then $\mathfrak{g}$ acts virtually freely on V if and only if $\dim V > \dim G$, except for those cases where $(G, \mathrm{char}\, k, V)$ appears in Table 4.*

G	$\mathrm{char}\, k$	V	$\dim V$	$\dim \ker \mathrm{d}\rho$
Sp_6	2	spin	8	14
Sp_8 (but not PSp_8)	2	spin	16	27
Sp_{10}	2	spin	32	44
Sp_{12} or PSp_{12}	2	spin	64	65

TABLE 4. The nontrivial restricted irreducible representations of simple G with $\dim V \le \dim G$ that are virtually free for $\mathfrak{g}$.

We remark that, in the setting of Theorem C and on the level of abstract groups, $G_v(k)$ is always finite when $\dim V > \dim G$ by [GuL].

The organization of the paper is as follows. We first (§1) recall properties of $\mathfrak{g}$ and the irreducible representations of G, focusing on the case of special characteristic.

A short section (§2) then recalls results used to constrain the size of $\mathfrak{g}_v$ for generic $v \in V$. Section 3 studies the case where G has type $A_1 \times \cdots \times A_1$, which arises in handling types B and C. Sections 4 and 5 prove some results on generic stabilizers by leveraging the Lie algebras of the long and short root subgroups. The next several sections are devoted to groups by type, each under the assumption that $\operatorname{char} k$ is special: F_4 and G_2 in §6, B_n in §7, and C_n in §8. In each section, we prove that, under various hypotheses on the representation V, if $\dim V$ is large enough, then $\mathfrak{g}$ acts virtually freely on V. We prove Theorem C(2) for each group in its section. The results based on $\dim V$ are far from uniform, so we provide in §9 an example to show that the uniform result from part I is false as stated if one drops the hypothesis that $\operatorname{char} k$ is not special. We prove Theorem C(1) in a short section 10. To prove Theorem B, it remains to treat the case where the highest weight is not restricted, which we do in §11. Finally, we prove Theorem A in §12.

We assume throughout that the field k is algebraically closed, as in the first line of the paper. This hypothesis is used, for example, in Lemma 2.5 and various results from parts I and II.

Acknowledgements. The referees deserve thanks for their detailed and helpful comments. We thank Brian Conrad for his advice on group schemes. Guralnick was partially supported by NSF grants DMS-1600056 and DMS-1901595.

1. Structure of $\mathfrak{g}$ and V

In this section, we assume that G is a semisimple linear algebraic group over k, and put $\mathfrak{g} := \operatorname{Lie}(G)$.

Structure of $\mathfrak{g}$. We refer to [Hi], [Ho], or [P, §1] for properties of $\mathfrak{g} := \operatorname{Lie}(G)$ when G is simple. For example, when G is simply connected, we have: (1) $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ *is a reducible G-module if and only if* $\operatorname{char} k$ *is special*, and (2) $\mathfrak{g}$ *has a unique proper maximal G-submodule*, which we denote by $\mathfrak{m}$. Statement (2) can be seen by direct computation as in [Ho] or because $\mathfrak{g}$ is a Weyl module of G in the sense of [J], the one whose highest weight is the highest root.

Irreducible representations of G when $\operatorname{char} k \neq 0$. Fix a pinning for G, which includes the data of a maximal torus T and a choice of simple roots Δ. Then irreducible representations $\rho\colon G \to \operatorname{GL}(V)$ (up to equivalence) are in bijection with the set of dominant weights $\lambda \in T^*$, i.e., those λ such that $\langle \lambda, \delta^\vee \rangle \geq 0$ for all $\delta \in \Delta$.

Suppose now that $p := \operatorname{char} k \neq 0$. Write $\lambda = \lambda_0 + p^r \lambda_1$ for some $r \geq 1$, where $\lambda_0 = \sum_\omega c_\omega \omega$ and $0 \leq c_\omega < p^r$ for all ω. If λ_0 and $p^{r-1}\lambda_1$ belong to T^* (e.g., if G is simply connected), then $L(\lambda) \cong L(\lambda_0) \otimes L(p^{r-1}\lambda_1)^{[p]}$ [J, II.3.16], the tensor product of $L(\lambda_0)$ and a Frobenius twist of $L(p^{r-1}\lambda_1)$. As a representation of $\mathfrak{g}$ (forgetting about the action of $G(k)$), this is the direct sum of $\dim L(\lambda_1)$ copies of $L(\lambda_0)$.

The case where $\operatorname{char} k$ is special. For the remainder of this section, suppose that G is simple and simply connected and $\operatorname{char} k$ is special for G.

There is a very special isogeny π that sends G to a simply connected group whose root system is inverse to the root system of G, see [CGP, §7.1] or [S, §10] for a concrete description. We put $N := \ker \pi$ and $\mathfrak{n} := \operatorname{Lie}(N) = \ker d\pi$. As N is normal in G, it follows that the subspace

$$V^{\mathfrak{n}} := \{ v \in V \mid d\rho(x)v = 0 \text{ for all } x \in \mathfrak{n} \}$$

is a G-invariant submodule of V for every representation $\rho\colon G \to \mathrm{GL}(V)$.

Put $\mathfrak{g}_{\mathrm{short}}$ (resp., $\mathfrak{g}_{\mathrm{long}}$) for the subspace of $\mathfrak{g}$ spanned by the subalgebras $\mathfrak{g}_\alpha$ for α a short (resp., long) root. Put $\mathfrak{t} := \mathrm{Lie}(T)$ and $\mathfrak{t}_{\mathrm{short}}$ for the subspace spanned by $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for α a short root.

Examining the tables in [Hi] and [Ho] and the description of $\mathfrak{g}$ in [CGP, §7.1], we find that $\mathfrak{n} = \mathfrak{t}_{\mathrm{short}} \oplus \mathfrak{g}_{\mathrm{short}}$ as a T-module and the following:

Lemma 1.1. *Let G be a simple and simply connected split algebraic group over a field k whose characteristic is special for G. If L is a nonzero and proper G-invariant subspace of $\mathfrak{g}$, then L is one of $\mathfrak{z}(\mathfrak{g})$, $\mathfrak{n}$, or $\mathfrak{m}$. In particular, every G-invariant subspace of $\mathfrak{g}$ is contained in $\mathfrak{z}(\mathfrak{g})$ or contains $\mathfrak{n}$.*

Remark 1.2. The subspace $\mathfrak{g}_< := \mathfrak{t} \oplus \mathfrak{g}_{\mathrm{short}}$ is a Lie p-subalgebra of $\mathfrak{g}$; it is the Lie algebra of the subgroup $G_<$ of G generated by T and the short root subgroups, see §5. As a representation of T, $\mathfrak{g}/\mathfrak{g}_<$ has weights the long roots and $\mathfrak{g}/\mathfrak{m}$ has weights the long roots and possibly zero (with some multiplicity), so trivially $\mathfrak{m} \subseteq \mathfrak{g}_<$. Note that $\mathfrak{g}_<$ need not be G-invariant.

Remark 1.3. By definition of $\mathfrak{m}$, it contains $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{n}$. It turns out that $\dim \mathfrak{m}/\mathfrak{n} \le 1$, compare Lemma 4.1. Beyond this, much variation is possible. For example, for $G = \mathrm{Sp}_{2\ell}$, $\mathfrak{n}$ does not contain $\mathfrak{z}(\mathfrak{g})$ for odd $\ell \ge 3$ and $\mathfrak{g}_< = \mathfrak{m}$ for $\ell \ge 2$.

Irreducible representations of G when $\mathrm{char}\, k$ is special. Now suppose that $\mathrm{char}\, k$ is special for G, so in particular Δ has two root lengths. Write a dominant weight λ as $\lambda = \sum c_\delta \omega_\delta$, where $c_\delta \ge 0$ and ω_δ is the fundamental weight dual to $\delta^\vee$ for $\delta \in \Delta$. We write $\lambda = \lambda_s + \lambda_\ell$ where $\lambda_s = \sum_{\delta \text{ short}} c_\delta \omega_\delta$ and $\lambda_\ell = \sum_{\delta \text{ long}} c_\delta \omega_\delta$, i.e., $\langle \lambda_s, \delta^\vee \rangle = 0$ for δ long and $\langle \lambda_\ell, \delta^\vee \rangle = 0$ for δ short. Steinberg [S] shows that, when G is simply connected, $L(\lambda) \cong L(\lambda_\ell) \otimes L(\lambda_s)$ and that furthermore $L(\lambda_\ell)$ factors through the very special isogeny.

Suppose now that λ is restricted. Then $L(\lambda_s)$ is irreducible as a representation of $\mathfrak{n}$ [S, p. 52], so Lemma 1.1 shows that the kernel of this representation is contained in $\mathfrak{z}(\mathfrak{g})$ if $\lambda_s \ne 0$. Similarly, as an $\mathfrak{n}$-module, $L(\lambda)$ is a direct sum of $\dim L(\lambda_\ell)$ copies of $L(\lambda_s)$, and again the kernel of the representation is contained in $\mathfrak{z}(\mathfrak{g})$ if $\lambda_s \ne 0$.

In summary, for λ restricted and G simply connected, we have either (1) $\lambda_s = 0$ and $\ker \mathrm{d}\rho \supseteq \mathfrak{n}$, or (2) $\lambda_s \ne 0$ and $\ker \mathrm{d}\rho \subseteq \mathfrak{z}(\mathfrak{g})$.

2. Lemmas for computing $\mathfrak{g}_v$

Choose a representation $\rho\colon G \to \mathrm{GL}(V)$. For $x \in \mathfrak{g}$, put

$$V^x := \{v \in V \mid \mathrm{d}\rho(x)v = 0\}$$

and x^G for the G-conjugacy class $\mathrm{Ad}(G)x$ of x. Recall the following from part I:

Lemma 2.1. *For $x \in \mathfrak{g}$,*

$$x^G \cap \mathfrak{g}_v = \emptyset \quad \text{for generic } v \in V \tag{2.2}$$

is implied by:

$$\dim x^G + \dim V^x < \dim V, \tag{2.3}$$

which is implied by:

(2.4) *There exist $e > 0$ and $x_1, \ldots x_e \in x^G$ such that the subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ generated by $x_1, \ldots, x_e$ has $V^{\mathfrak{s}} = 0$ and $e \cdot \dim x^G < \dim V$.*

We use this in combination with Lemma 2.5 below.

For us, $\mathfrak{g} = \mathrm{Lie}(G)$ and $\operatorname{char} k = p \neq 0$, so the Frobenius morphism on G induces a p-operation $x \mapsto x^{[p]}$ on $\mathfrak{g}$, see [SF] for properties. When G is a sub-group-scheme of GL_n and $x \in \mathfrak{g}$, the element $x^{[p]}$ is the p-th power of x with respect to the typical, associative multiplication for n-by-n matrices, see [DG, §II.7, p. 274].

An element $x \in \mathfrak{g}$ is *nilpotent* if $x^{[p]^n} = 0$ for some $n > 0$, *toral* if $x^{[p]} = x$, and *semisimple* if x is contained in the Lie p-subalgebra of $\mathfrak{g}$ generated by $x^{[p]}$, cf. [SF, §2.3]. We recall from part I:

Lemma 2.5. *Suppose G is semisimple over an algebraically closed field k of characteristic $p > 0$, and let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$.*

1. *If (2.2) holds for every toral or nilpotent $x \in \mathfrak{g} \setminus \mathfrak{h}$, then $\mathfrak{g}_v \subseteq \mathfrak{h}$ for generic $v \in V$.*
2. *If $\mathfrak{h}$ consists of semisimple elements and (2.2) holds for every $x \in \mathfrak{g} \setminus \mathfrak{h}$ with $x^{[p]} \in \{0, x\}$, then $\mathfrak{g}_v \subseteq \mathfrak{h}$ for generic v in V.* □

One typical application of part (2) of the lemma is when $\mathfrak{h} = \mathfrak{z}(\mathfrak{g})$.

3. A Heisenberg Lie algebra

Let $G = \mathrm{Spin}_{2n+1}$ for some $n \geq 2$ over a field k (always assumed algebraically closed) of characteristic 2. The short root subalgebras of $\mathfrak{g}$ generate a "Heisenberg" Lie algebra $\mathfrak{h}$ of dimension $2n+1$ such that $[\mathfrak{h}, \mathfrak{h}]$ is the 1-dimensional center $\mathfrak{z}(\mathfrak{h})$. The algebra $\mathfrak{h}$ is the image of $\mathfrak{sl}_2^{\times n}$ under a central isogeny $\mathrm{SL}_2^{\times n} \to \mathrm{SL}_2^{\times n}/Z$ where Z is isomorphic to $\mu_2^{\times(n-1)}$, and the quotient $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ is the image of $\mathfrak{sl}_2^{\times n} \to \mathfrak{pgl}_2^{\times n}$.

For $G = \mathrm{Sp}_{2n}$ for some $n \geq 2$ over the same k, the very special isogeny $\pi : \mathrm{Sp}_{2n} \to \mathrm{Spin}_{2n+1}$ has $\mathrm{d}\pi(\mathfrak{sp}_{2n}) = \mathfrak{h}$, so we may identify $\mathfrak{h}$ with $\mathfrak{g}/\ker \mathrm{d}\pi$.

Lemma 3.1. *Suppose $\rho \colon G \to \mathrm{GL}(V)$ is a representation of $G = \mathrm{Spin}_{2n+1}$ or Sp_{2n}. In the latter case, assume additionally that $\mathrm{d}\rho$ vanishes on $\ker \mathrm{d}\pi$.*

1. *If $4n + \dim V^{\mathfrak{z}(\mathfrak{h})} < \dim V$, then $\dim x^G + \dim V^x < \dim V$ for all nonzero $x \in \mathfrak{h}$.*
2. *If $V^{\mathfrak{h}} = 0$ and $4n^2 < \dim V$, then $\dim x^G + \dim V^x < \dim V$ for all noncentral $x \in \mathfrak{h}$.*

Proof. For x nonzero central, $\dim x^G + \dim V^x = \dim V^{\mathfrak{z}(\mathfrak{h})}$, verifying (1), so suppose x is noncentral.

In case (1), there is a $g \in G(k)$ so that $[x, x^g]$ is nonzero central in $\mathfrak{h}$, so $\dim V^x \leq \frac{1}{2}(\dim V + \dim V^{\mathfrak{z}(\mathfrak{h})})$. As $\dim x^G < 2n+1$, the claimed inequality follows.

In case (2), the Weyl group of G acts transitively on groups of a given length, so $2n$ conjugates of x generate $\mathfrak{h}$, and therefore to prove the claim it suffices to note that $2n \cdot \dim x^G < \dim V$ (Lemma 2.1). □

4. Subgroup generated by the long roots

Suppose G is simple and simply connected and $\operatorname{char} k$ is special for G. Fix a maximal torus T in G. The long root subgroups of G (relative to T) generate a subgroup $G_>$ that is also simply connected and the type of $(G_>, G)$ is one of (A_2, G_2), (D_4, F_4), (D_n, B_n) for $n \geq 2$, or (A_1^n, C_n) for $n \geq 2$, [CGP, Prop. 7.1.7]. We put $\mathfrak{g}_> := \mathrm{Lie}(G_>)$, which as a vector space is a direct sum $\mathfrak{t} \oplus \mathfrak{g}_{\mathrm{long}}$.

Lemma 4.1. *In the notation of the preceding paragraph,*

(1) *The composition* $\mathfrak{g}_> \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ *is onto.*
(2) $\mathfrak{z}(\mathfrak{g}_>) + \mathfrak{n} = \mathfrak{m}$*, the unique maximal* G*-invariant subspace of* $\mathfrak{g}$*.*
(3) $\mathfrak{g}_> \cap \mathfrak{m} = \mathfrak{z}(\mathfrak{g}_>)$*.*

Proof. (1) is obvious because $\mathfrak{g} = \mathfrak{g}_> \oplus \mathfrak{g}_{\text{short}}$ as T-modules and $\mathfrak{g}_{\text{short}} \subseteq \mathfrak{n}$.

For (3), as $\mathfrak{m}$ and $\mathfrak{g}_>$ are T-invariant and $\mathfrak{g}_{\text{long}} \cap \mathfrak{m} = 0$, we have $\mathfrak{m} \cap \mathfrak{g}_> \subseteq \mathfrak{t}$. For $z \in \mathfrak{g}_> \cap \mathfrak{m}$ and x_α a root element in $\mathfrak{g}_>$, $[z, x_\alpha] \in \mathfrak{m}$, so $z \in \mathfrak{z}(\mathfrak{g}_>)$, i.e., (3) holds.

Finally, we have $\mathfrak{g} = \mathfrak{g}_> \oplus \mathfrak{g}_{\text{short}} = \mathfrak{g}_> + \mathfrak{n}$, where the second sum is not direct. Hence $\mathfrak{m} = (\mathfrak{g}_> \cap \mathfrak{m}) + \mathfrak{n}$, and (3) implies (2). □

Long Root Proposition 4.2. *Let* G *be a simple and simply connected algebraic group such that* $\operatorname{char} k$ *is special for* G*. Suppose that* $\rho\colon G \to \mathrm{GL}(V)$ *vanishes on the kernel of the very special isogeny. If* V *has a subquotient* W *such that* $W^{\mathfrak{g}} = 0$ *and*

$$\dim W > \begin{cases} 64 & \text{if } G = F_4 \\ 20 & \text{if } G = G_2 \\ 30 & \text{if } G = \mathrm{Spin}_7 \\ 4n^2 & \text{if } G = \mathrm{Spin}_{2n+1} \text{ with } n \geq 4 \text{ or } G = \mathrm{Sp}_{2n} \text{ with } n \geq 2, \end{cases}$$

then $\mathfrak{g}_v \subseteq \mathfrak{m}$ *for generic* $v \in V$*.*

Proof. We verify, for $x \in \mathfrak{g}_> \setminus \mathfrak{z}(\mathfrak{g}_>)$ such that $x^{[p]} \in \{0, x\}$, that

$$\dim x^{G_>} + \dim W^x < \dim W. \tag{4.3}$$

For $G = \mathrm{Sp}_{2n}$ with $n \geq 2$, we apply Lemma 3.1(2). Otherwise, note that $G_>$ is simple and not Sp_{2n} for any $n \geq 1$, so $[\mathfrak{g}_>, \mathfrak{g}_>] = \mathfrak{g}_>$. Therefore, $W^{[\mathfrak{g}_>, \mathfrak{g}_>]} = W^{\mathfrak{g}_>} = W^{\mathfrak{g}} = 0$. For $G = G_2$, F_4, or Spin_{2n+1} for $n \geq 3$, we have (4.3) by [GaGu I, Th. 12.2], where in case Spin_7 we use the identity $\mathrm{Spin}_6 = \mathrm{SL}_4$.

Because (4.3) holds, we deduce that $\dim x^{G_>} + \dim V^x < \dim V$ (elementary, see [GaGu I, Example 2.1]). As $\mathfrak{g}_> \cap \mathfrak{m} = \mathfrak{z}(\mathfrak{g}_>)$ consists of semisimple elements, it follows that $(\mathfrak{g}_>)_v \subseteq \mathfrak{m}$ for generic $v \in V$, whence $\mathfrak{g}_v \subseteq \mathfrak{m}$. □

5. Subgroup generated by the short roots

Continue the notation of the preceding section. In particular, G is assumed simply connected and $\operatorname{char} k$ is special for G. The root subgroups in G corresponding to short roots generate a subgroup $G_<$, and the type of $(G_<, G)$ is (A_2, G_2), (D_4, F_4), (A_1^n, B_n) for $n \geq 2$, or (D_n, C_n) for $n \geq 2$ [CGP, Prop. 7.1.7]. We put $\mathfrak{g}_< := \mathrm{Lie}(G_<)$, compare Remark 1.2.

Lemma 5.1. *For* G *simply connected of type* G_2*,* F_4*, or* C_n *with* $n \geq 3$ *such that* $\operatorname{char} k$ *is special, we have* $[\mathfrak{g}_<, \mathfrak{g}_<] = \mathfrak{n}$*.*

Proof. Put $\widetilde{G_<}$ for the simply connected cover of $G_<$. Because $G_<$ is simple and not Sp_{2n} for any $n \geq 1$, $[\mathfrak{g}_<, \mathfrak{g}_<]$ is the image of $\mathrm{Lie}(\widetilde{G_<})$ in $\mathfrak{g}_<$ [GaGu I, Lemma 3.1], and in particular it is the subalgebra generated by the root subalgebras of $\mathfrak{g}$ corresponding to short roots, which is $\mathfrak{n}$. □

Short Root Proposition 5.2. *Let* G *be a simple and simply connected algebraic group such that* $\operatorname{char} k$ *is special for* G*, and let* V *be a representation of* G*. If* V

has a subquotient W such that $W^{\mathfrak{n}} = 0$ and

$$\dim W > \begin{cases} 64 & \text{if } G = F_4 \\ 20 & \text{if } G = G_2 \\ 30 & \text{if } G = \mathrm{Sp}_6 \\ 4n^2 & \text{if } G = \mathrm{Spin}_{2n+1} \text{ with } n \geq 2 \text{ or } G = \mathrm{Sp}_{2n} \text{ with } n \geq 4, \end{cases}$$

then $\mathfrak{n}_v \subseteq \mathfrak{z}(\mathfrak{g}_<)$ for generic $v \in V$.

For $G = \mathrm{Spin}_{2n+1}$, $\mathfrak{n} = \mathfrak{h}$ and $\mathfrak{z}(\mathfrak{g}_<) = \mathrm{Lie}(Z(\mathrm{SL}_2^{\times n} / \mu_2^{\times(n-1)})) = \mathfrak{z}(\mathfrak{g})$.

Proof. Let $x \in \mathfrak{n}$ satisfy $x^{[p]} \in \{0, x\}$. If $G = \mathrm{Spin}_{2n+1}$ for $n \geq 2$, we apply Lemma 3.1(2) to see that

$$\dim x^{G_<} + \dim W^x < \dim W \tag{5.3}$$

if x is not central in $\mathfrak{n}$. In the other cases, $W^{[\mathfrak{g}_<, \mathfrak{g}_<]} = W^{\mathfrak{n}} = 0$, and we apply [GaGu I, Th. 12.2] to find (5.3) if x is not central in $\mathfrak{g}_<$.

Then $\dim x^{G_<} + \dim V^x < \dim V$. So, for generic $v \in V$, it follows that $\mathfrak{n}_v \subseteq \mathfrak{z}(\mathfrak{g}_<)$. □

6. Type F_4 or G_2

Suppose G has type F_4 or G_2 and $\mathrm{char}\, k = 2$ or 3 respectively. The maximal ideal $\mathfrak{m}$ equals the kernel $\mathfrak{n}$ of the very special isogeny; it is the unique nonzero and proper ideal of $\mathfrak{g}$. Both $\mathfrak{n}$ and $\mathfrak{g}/\mathfrak{n}$, as Lie algebras, are the simple quotient $\tilde{\mathfrak{g}}/\mathfrak{z}(\tilde{\mathfrak{g}})$, where $\tilde{\mathfrak{g}} = \mathfrak{spin}_8$ or $\mathfrak{sl}_3$ respectively.

The arguments used in the previous two sections can be extended slightly to give a result that will be sufficient to handle the cases where $G = F_4$ or G_2.

Proposition 6.1. *Let G be a simple algebraic group of type F_4 or G_2 over a field k such that $\mathrm{char}\, k = 2$ or 3 respectively. Let $\rho \colon G \to \mathrm{GL}(V)$ be a representation of G. If V has a G-subquotient W with $W^{\mathfrak{n}} = 0$ and $\dim W > 240$ or 48 respectively, then for generic $v \in V$, $\mathfrak{g}_v = 0$.*

Proof. We will first verify that, for $x \in \mathfrak{g} \setminus \mathfrak{n}$ with $x^{[p]} \in \{0, x\}$, we have

$$\dim x^G + \dim W^x < \dim W. \tag{6.2}$$

Suppose first that $x \in \mathfrak{g} \setminus \mathfrak{n}$ is a long root element, and in particular there is a maximal torus T so that x is a root element in the long root subalgebra $\mathfrak{g}_>$. Then e $G_>$-conjugates of x suffice to generate $\mathfrak{g}_>$, where $(G, e) = (F_4, 4)$ or $(G_2, 3)$ by [GaGu I, Prop. 10.4, 6.4]. As a representation of $G_>$, $\mathfrak{g}$ is a sum of $\mathfrak{g}_>$ and three inequivalent 8-dimensional representations (for type F_4) or two inequivalent 3-dimensional representations (type G_2), so $e+1$ G-conjugates of x will generate $\mathfrak{g}$. As

$$(e+1) \dim x^G \leq (e+1)(\dim G - \mathrm{rank}\, G) < \dim W,$$

(6.2) follows.

If $x \in \mathfrak{g} \setminus \mathfrak{n}$ is nilpotent, then as in [GaGu I, Remark 11.3] there is a long root element y in the closure of x^G. By the previous paragraph, $e+1$ G-conjugates of y generate $\mathfrak{g}$, and as in [GaGu I, Lemma 4.3(1)] the same holds for x. Again (6.2) follows.

If $x \in \mathfrak{g} \setminus \mathfrak{n}$ is toral, then it can be expressed as $\sum c_\alpha h_\alpha$ where the sum ranges over the simple roots α and $h_\alpha \in \mathrm{Lie}(T)$ corresponds to the coroot $\alpha^\vee$. As $x \notin \mathfrak{n}$,

$c_\beta \neq 0$ for some long simple root β. Arguing as in [GaGu I, Example 4.1] we deduce that a long root element x_β lies in the closure of $x^{\mathbb{G}_m G}$ and again we have verified (6.2).

As the nilpotent and toral elements of $\mathfrak{g}_v$ lie in $\mathfrak{n}$ for generic $v \in V$, so does all of $\mathfrak{g}_v$. Finally we apply the Short Root Proposition 5.2, to see that $\mathfrak{g}_v = 0$. $\square$

Restricted irreducible representations. Let $G = F_4$ and $\operatorname{char} k = 2$ or $G = G_2$ and $\operatorname{char} k = 3$, and suppose $\rho \colon G \to \mathrm{GL}(V)$ is an irreducible representation with restricted highest weight λ. Here we prove Theorem C(2) in this case.

If $\dim V \leq \dim G$, then by A.50 and A.49 in [Lü], V is either the natural module (of dimension 26 or 7, respectively) or the irreducible quotient $\mathfrak{g}/\mathfrak{n}$ of the adjoint representation. For ρ the natural module, $\ker \mathrm{d}\rho = 0$ and a generic vector has stabilizer of type D_4 or A_2 respectively (of dimension 28 or 8 respectively). Note that this stabilizer has dimension larger than $\dim \mathfrak{g}/\mathfrak{n}$, so it meets $\mathfrak{n}$, the image of $\mathfrak{g}$ under the very special isogeny. It follows that composing the natural representation with the very special isogeny gives a representation with $\ker \mathrm{d}\rho = \mathfrak{n}$ that is not virtually free; this is $\mathfrak{g}/\mathfrak{n}$.

If $\dim V > 240$ or 48 respectively, then V is virtually free by Proposition 6.1. Table A.50 in [Lü] shows that we have considered all restricted irreducible representations of F_4, so the proof of Theorem C(2) is complete in that case.

For G_2, there are two remaining possibilities for ρ, according to Table A.49. The first, with highest weight $2\omega_2$ (numbered as in Table 2), has dimension 27 and $\ker \mathrm{d}\rho = 0$. It factors through the representation of SO_7 on the irreducible component of $\mathrm{S}^2(k^7)$. As that representation is generically free for $\mathfrak{so}_7$ by [GaGu I, Lemma 13.1], so is $\mathrm{d}\rho$. Alternatively, one can verify that this $\mathrm{d}\rho$ is virtually free using a computer.

The last possibility for ρ, with highest weight $2\omega_1$, is obtained by composing the representation in the preceding paragraph with the very special isogeny. This representation is virtually free by the considerations in the previous paragraph, or by Prop. 4.2, completing the proof of Theorem C(2) for G of type G_2.

7. Type B_n with $n \geq 2$

For $G = \mathrm{Spin}_{2n+1}$ for some $n \geq 2$ over a field k of characteristic 2, the Lie algebra $\mathfrak{g}$ is uniserial where the subalgebra $\mathfrak{n}$ is the Heisenberg Lie algebra $\mathfrak{h}$ from §3.

Any representation of G is a direct sum $V_1 \oplus V_2$ where $V_1^{\mathfrak{z}(\mathfrak{g})} = 0$ and $\mathfrak{z}(\mathfrak{g})$ acts trivially on V_2; these are just the eigenspaces of $\mathfrak{z}(\mathfrak{g})$.

Representations with $V^{\mathfrak{z}(\mathfrak{g})} = 0$.

Example 7.1 (spin representation)**.** The spin representation $V := L(\omega_1)$ of G (where we number the weights of G as in Table 2) is generically free if and only if $n \geq 7$ [GaGu 17], if and only if $\dim V > \dim G$. We remark that one can check with a computer that for $n = 2, 3, 4, 5, 6$, a sum of $4, 4, 3, 2, 2$ copies of V is generically free for $\mathfrak{spin}_{2n+1}$.

Example 7.2. If V is an irreducible representation of G and $V^{\mathfrak{z}(\mathfrak{g})} = 0$, then $V \cong L(\omega_1) \otimes W$ for some irreducible representation W. This follows from the discussion in section 1 because δ_1 is the only short simple root.

Example 7.3. Suppose $x \in \mathfrak{so}_{2n+1}$ has $x^{[2]} = 0$ and rank $r > 0$. The largest possible conjugacy classes for x have a 4-dimensional indecomposable summand $W_2(2)$ or a 3-dimensional indecomposable summand $D(2)$ (following the notation in [He] or [LiS, §5.6]), and the centralizer in SO_{2n+1} of one of these largest classes has dimension

$$\sum_{i=1}^{r} 2(i-1) + \sum_{i=r+1}^{2n+1-r} (i-1) = \binom{2n+1-r}{2} + \binom{r}{2}.$$

Consequently, $\dim x^{\mathrm{SO}_{2n+1}} \le r(2n+1-r)$. (Compare [GaGu I, Example 10.5] for SO_{2n}.)

Lemma 7.4. *Let $G = \mathrm{Spin}_{2n+1}$ for some $n \ge 2$ over a field k of characteristic 2, and suppose that V is a representation of G such that $V^{\mathfrak{z}(\mathfrak{g})} = 0$. Then:*

1. *For noncentral $x \in \mathfrak{g}$, $\dim V^x \le \frac{3}{4} \dim V$.*
2. *If $\dim V > 4n^2 + 4n$, then (a) $\dim x^G + \dim V^x < \dim V$ for all noncentral $x \in \mathfrak{g}$ such that $x^{[2]} \in \{0, x\}$ and (b) V is generically free for $\mathfrak{g}$.*
3. *If $n \ge 7$, then V is generically free for $\mathfrak{g}$.*

Proof. We first prove (1). By passing to orbit closures, it suffices to prove this in case x is nilpotent. The crux case is where V is the spin representation, where the claim holds if $n = 2$ (because $\dim V = 4$) and if $n \ge 3$ by [GaGu 17, Prop. 2.1(i)].

If V is irreducible, then it is $L(\omega_1) \otimes W$ for some irreducible W. In this case, $\dim V^x \le (\dim L(\omega_1)^x)(\dim W)$ as in [GaGu II, Lemma 11.1], proving the claim. Finally, for a composition series $0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$ of V, we have $(V_i/V_{i-1})^{\mathfrak{z}(\mathfrak{g})} = 0$ because $\mathfrak{z}(\mathfrak{g})$ acts semisimply on V and $\dim(V')^x \le \frac{3}{4} \dim V$ for $V' := \oplus V_i/V_{i-1}$, so also for V, proving that $\dim V^x \le \frac{3}{4} \dim V$.

To prove (2), fix noncentral $x \in \mathfrak{g}$ such that $x^{[2]} \in \{0, x\}$. If $x^{[2]} = 0$, then the image $\bar{x} \in \mathfrak{so}_{2n+1}$ of x — as a $(2n+1)$-by-$(2n+1)$ matrix — has rank $r > 0$ so $\dim \bar{x}^{\mathrm{SO}_{2n+1}} \le r(2n-r+1)$, whence $\dim x^{\mathrm{Spin}_{2n+1}} \le r(2n-r+1)$ because the map $\mathfrak{spin}_{2n+1} \to \mathfrak{so}_{2n+1}$ is injective on nilpotents. (Indeed, if x and $x+z$ are square-zero and z is central in $\mathfrak{spin}_{2n+1}$, then $0 = (x+z)^{[2]} = z^{[2]}$, so $z = 0$.) If $x^{[2]} = x$, then $\bar{x}$ has *even* rank r, and the centralizer of x has type $D_{r/2} \times B_{n-r/2}$ for some r; we find the same formula for $\dim x^G$. The upper bound on $\dim x^G$ is maximized at $r = n + 1/2$, so $\dim x^G \le n^2 + n$. Thus, (a) holds, and (b) follows as in §2.

Suppose now that $n \ge 7$. If V is the spin representation, then it is generically free (Example 7.1). If V is irreducible but not the spin representation, i.e., $V \cong (\text{spin}) \otimes W$ for some nontrivial W, then $\dim V \ge 2n2^n > 4n^2 + 4n$, and again V is generically free. For general V, each irreducible representation in its composition series is generically free (by the preceding), so V is as well. □

Note the following corollary.

Corollary 7.5. *Let $G = \mathrm{Spin}_{2n+1}$ for some $n \ge 2$ over a field k of characteristic 2, and suppose that V is an irreducible representation of G such that $V^{\mathfrak{z}(\mathfrak{g})} = 0$. If $\dim V > \dim G$, then V is generically free.*

Proof. Write $V = L(\omega_1) \otimes W$ as in Example 7.2. If W is trivial, then the claim is from Example 7.1. If $n \ge 7$, then the claim is Lemma 7.4. So suppose W is nontrivial and $2 \le n < 7$. As $\dim W \ge 2n$, we have $\dim V > 4n^2 + 4n$ unless $n = 2$ or 3 and $\dim W = 2n$.

So suppose $n = 2$ or 3 and W is the orthogonal module or a nontrivial Frobenius twist of it. In the latter case, V is a direct sum of $2n$ copies of the spin module and so is generically free (Example 7.1). In the former case, one can verify with a computer that V is generically free, as was recorded in [GaGu II, Examples 5.2, 5.3]. □

Note that the only irreducible modules with $V^{\mathfrak{z}(\mathfrak{g})} = 0$ and $\dim V \leq \dim G$ are the spin modules for $n \leq 6$.

Representations killed by $\mathfrak{z}(\mathfrak{g})$. We have dealt with those representations V of G such that $V^{\mathfrak{z}(\mathfrak{g})} = 0$. If $\mathfrak{z}(\mathfrak{g})V = 0$, then the highest weight of each of the composition factors of V lies in the root lattice. We have:

Lemma 7.6. *In a root system of type B_n ($n \geq 2$), for λ in the root lattice and α a short root, $\langle \alpha^\vee, \lambda \rangle$ is an even integer.*

Proof. Because the Weyl group acts transitively on short roots, we may assume that α is the short simple root. Because the expression $\langle \alpha^\vee, \lambda \rangle$ is linear in λ, we may assume that λ is a simple root. Then the claim follows from looking at the Cartan matrix. □

We note for later use:

Lemma 7.7. *If $\operatorname{char} k = 2$ and $\rho \colon \mathrm{SO}_{2n+1} \to \mathrm{GL}(V)$ is an irreducible representation for some $n \geq 2$, then ρ is not faithful and the composition of $\mathfrak{spin}_{2n+1} \to \mathfrak{so}_{2n+1}$ with $\mathrm{d}\rho$ vanishes on $\mathfrak{n}$.*

Proof. The highest weight λ of ρ is in the root lattice (because SO_{2n+1} is adjoint), so Lemma 7.6 shows that λ vanishes on coroots corresponding to short roots. Thus by [S, Th. 11.1] the composition $\mathfrak{spin}_{2n+1} \to \mathfrak{so}_{2n+1} \to \mathfrak{gl}(V)$ vanishes on the ideal $\mathfrak{n}$ of $\mathfrak{spin}_{2n+1}$ for $\mathfrak{n}$ as in §1, which has nonzero image in $\mathfrak{so}_{2n+1}$. □

Remark 7.8. Lemma 7.6 can be viewed, by way of the duality between the root systems of types B and C, as equivalent to the statement that for type C every long root is 2 times a weight. From this perspective, Lemma 7.7 is the analogue for type B of the well-known fact that, when $\operatorname{char} k = 2$, Cartan subalgebras in $\mathfrak{sp}_{2n}$ are properly larger than maximal toral subalgebras.

By the way, Lemmas 7.6 and 7.7 also apply to type A_1, mutatis mutandis. See [GaGu I, Example 3.3] for the version of Lemma 7.7.

The very special isogeny $G = \mathrm{Spin}_{2n+1} \to \mathrm{Sp}_{2n}$ is another way of viewing the trivial statement that the alternating bilinear form on the natural module of G is G-invariant. It factors through $\mathrm{Spin}_{2n+1} \to \mathrm{SO}_{2n+1}$, and the image $\mathfrak{g}/\mathfrak{n}$ of $\mathfrak{g}$ in $\mathfrak{sp}_{2n}$ is isomorphic to the derived subalgebra of $\mathfrak{so}_{2n}$, which is a simple G-module (i.e., $\mathfrak{n} = \mathfrak{m}$) if n is odd and has a 1-dimensional center if n is even (i.e., $\mathfrak{n}$ has codimension 1 in $\mathfrak{m}$).

Lemma 7.9. *Let $\rho : \mathrm{Spin}_{2n+1} \to \mathrm{GL}(V)$ be a representation over a field of characteristic 2, for some $n \geq 3$. If ρ factors through the very special isogeny, $V^{\mathfrak{spin}_{2n+1}} = 0$, and $\dim V > 4n^2$, then $\mathfrak{spin}_{2n+1}$ and $\mathfrak{so}_{2n+1}$ act virtually freely on V.*

Proof. By hypothesis, ρ factors through $\mathrm{Spin}_{2n+1} \to \mathrm{SO}_{2n+1} \to \mathrm{Sp}_{2n}$. The image of $\mathfrak{so}_{2n+1}$ in $\mathfrak{sp}_{2n}$ is $\mathfrak{so}_{2n}$, the unique maximal Sp_{2n}-invariant ideal in $\mathfrak{sp}_{2n}$ and the

Lie algebra of a subgroup $\mathrm{SO}_{2n} \subset \mathrm{Sp}_{2n}$. It suffices to verify that $\mathfrak{so}_{2n}$ acts virtually freely on V.

The image of $\mathfrak{spin}_{2n+1}$ in $\mathfrak{so}_{2n}$ is $[\mathfrak{so}_{2n}, \mathfrak{so}_{2n}]$, so $V^{[\mathfrak{so}_{2n}, \mathfrak{so}_{2n}]} = 0$. Applying now [GaGu I, Th. A] gives the claim. ☐

Corollary 7.10. *Let V be a representation of $G := \mathrm{Spin}_{2n+1}$ for some $n \geq 2$ and assume $\mathrm{char}\, k = 2$. If $V^{\mathfrak{g}} = 0$ and $\dim V > 8n^2 + 4n$, then a generic $v \in V$ has $\mathfrak{g}_v \subseteq \mathfrak{m}$ and $\mathfrak{g}$ acts virtually freely on V.*

Proof. Write $V = V_1 \oplus V_2$ as at the start of this section. If $\dim V_1 > 4n^2 + 4n$, then V_1 and so V is generically free. Otherwise $\dim V_2 > 4n^2$ and the group of type D has generic stabilizer contained in its center by [GaGu I]. ☐

One can do better by intersecting the generic stabilizers for V_1 and V_2.

Example 7.11 (natural representation)**.** Here we treat the natural module, $V := L(\omega_n)$ for $n \geq 2$. We have $\dim V = 2n < \dim G$. As in the proof of Lemma 7.9, the image of $\mathfrak{so}_{2n+1}$ in $\mathfrak{gl}(V)$ is a copy of $\mathfrak{so}_{2n}$ which acts on V with generic stabilizer $\mathfrak{so}_{2n-1}$, hence $\mathfrak{so}_{2n+1}$ acts on V with kernel $\mathfrak{n}$ and generic stabilizer $\mathfrak{n} + \mathfrak{so}_{2n-1}$.

Note that $V^{\mathfrak{g}} = V^{\mathfrak{so}_{2n}} = 0$. For $W := \oplus^c V$ with $c > 2n$, Lemma 7.9 says that W is virtually free. (Compare [GaGu II, Example 10.3] for the case $\mathrm{char}\, k \neq 2$.)

Example 7.12 (adjoint representation)**.** Here we treat $V := L(\omega_{n-1})$ for $n \geq 3$, the irreducible quotient of the Weyl module $\mathfrak{spin}_{2n+1}$. As in §4, the long root subalgebra $\mathfrak{g}_>$ is $\mathfrak{spin}_{2n}$, and $\mathfrak{spin}_{2n+1} = \mathfrak{n} + \mathfrak{spin}_{2n}$. This V is the irreducible quotient of the $\mathfrak{spin}_{2n}$-module $\mathfrak{spin}_{2n}$. By [GaGu I, Example 3.4], the stabilizer in $\mathfrak{spin}_{2n}$ of a generic vector $v \in V$ is $\mathrm{Lie}(T)$ for T a maximal torus depending on v and we conclude that $\mathfrak{g}_v = \mathfrak{n} + \mathrm{Lie}(T)$. (Alternatively, this representation factors through the very special isogeny and one can find $\mathfrak{g}_v$ by pulling back the stabilizer in $\mathfrak{sp}_{2n}$ described in Example 8.7.)

Restricted irreducible representations. Let V be a restricted irreducible representation of a group G of type B_n for some $n \geq 2$ over a field k of characteristic 2; we prove Theorem C(2) for this G. The highest weight $\lambda = \sum_{i=1}^n c_i \omega_i$ of V has $c_i \in \{0, 1\}$ for all i. If $\lambda = 0$, equivalently $\ker \mathrm{d}\rho = \mathfrak{g}$, then there is nothing to do.

If $c_1 \neq 0$, then we are in the case of Example 7.2 and Corollary 7.5, so $\ker \rho = 0$ and there is nothing more to do.

So assume that $c_1 = 0$. Thus, the highest weight λ belongs to the root lattice and the representation ρ factors through not just SO_{2n+1} but Sp_{2n}. To prove Theorem C(2), it suffices to show that (1) $\mathfrak{spin}_{2n+1}$ does not act virtually freely when $\dim V \leq \dim G$ and (2) $\mathfrak{so}_{2n+1}$ does act virtually freely when $\dim V > \dim G$.

Assume first that $n > 11$. Applying Lemma 7.9, we may assume that $\dim V \leq 4n^2$, so $\dim V < n^3$. By [Lü], V is either the $2n$-dimensional representation $L(\omega_n)$ as in Example 7.11 or it is $L(\omega_{n-1})$ as in Example 7.12.

Next consider $4 \leq n \leq 11$. Applying again Lemma 7.9, we may assume that $\dim V \leq 4n^2$. Examining the tables in [Lü], we find that the only possibilities for V that have not yet been considered are the cases where $n = 4$ or 5, $\lambda = \omega_{n-2}$, and $\dim V = 48$ or 100 respectively. In both cases, $\mathfrak{n}$ is in the kernel of the representation and computer calculations with Magma as in [GaGu II] produce a vector in V with stabilizer $\mathfrak{n}$. Note that $\dim V > \dim G$ in both cases.

If $n = 2$, the only restricted irreducible module is the orthogonal 4-dimensional module and clearly the result holds.

If $n = 3$, the only irreducible restricted modules with $c_1 = 0$ are the orthogonal module (Example 7.11), the module with high weight ω_2 of dimension 14 (Example 7.12), and the module with high weight $\omega_2 + \omega_3$ of dimension 64. We verify with a computer that Theorem C(2) holds in the latter case.

Example 7.13 (Spin_9)**.** For later reference, we examine more carefully the spin representation V of $G = \mathrm{Spin}_9$ when $\mathrm{char}\, k = 2$. The stabilizer G_v of a generic vector $v \in V$ is isomorphic to Spin_7 [GaGu 17] and is contained in a subgroup Spin_8 in G (generated by long root subgroups as in §4) in such a way that the composition $\mathrm{Spin}_7 \subset \mathrm{Spin}_8 \subset \mathrm{Spin}_9 \to \mathrm{SO}_9$ is injective. (See for example [V] for a discussion of the first inclusion in the case $k = \mathbb{R}$.) In particular, $\mathfrak{z}(\mathfrak{spin}_7) \neq \mathfrak{z}(\mathfrak{spin}_9)$.

Now $\mathfrak{z}(\mathfrak{spin}_9) \subset \mathfrak{z}(\mathfrak{spin}_8)$, where the terms have dimensions 1 and 2 respectively. We claim that furthermore $\mathfrak{z}(\mathfrak{spin}_7) \subset \mathfrak{z}(\mathfrak{spin}_8)$ so $\mathfrak{z}(\mathfrak{spin}_7) + \mathfrak{z}(\mathfrak{spin}_9) = \mathfrak{z}(\mathfrak{spin}_8)$. To see this, restrict V to Spin_8 to find a direct sum $V_1 \oplus V_2$ of inequivalent 8-dimensional irreducible representations. One V_i restricts to be the spin representation of Spin_7 and the other is uniserial with composition factors of dimension 1, 6, 1, corresponding to the inclusion $\mathrm{SO}_7 \subset \mathrm{SO}_8$. From this we can read off the action of the central μ_2 of Spin_7 on V and we find that it is central in Spin_8, proving the claim.

We have $\mathfrak{spin}_8 \cap \mathfrak{m} = \mathfrak{z}(\mathfrak{spin}_8)$ by Lemma 4.1 and $\mathfrak{spin}_8 \cap \mathfrak{n} = \mathfrak{z}(\mathfrak{spin}_9)$. Using that $\mathfrak{g}_v = \mathfrak{spin}_7$, we conclude that $\mathfrak{n}_v = 0$ and $\mathfrak{m}_v = \mathfrak{z}(\mathfrak{spin}_7)$.

8. Type C_n with $n \geq 3$

Alternating bilinear forms. The simply connected group Sp_{2n} of type C_n can be viewed as the automorphism group of a nondegenerate alternating bilinear form. We now recall some facts about this correspondence.

Example 8.1. Suppose b is a nondegenerate alternating bilinear form on a finite-dimensional vector space V over a field F (of any characteristic). This gives an "adjoint" involution $\sigma : \mathrm{End}_F(V) \to \mathrm{End}_F(V)$ such that $b(Tv, v') = b(v, \sigma(T)v')$ for all $T \in \mathrm{End}_F(V)$ and $v, v' \in V$. If $x \in \mathrm{End}_F(V)$ is such that $\sigma(x) = \pm x$, then we find an equation $b(xv, v') = \pm b(v, xv')$. By taking $v \in \ker x$ or $(\mathrm{im}\, x)^\perp$ and allowing v' to vary over V we find each of the containments between $\ker x$ and $(\mathrm{im}\, x)^\perp$, i.e., $\ker x = (\mathrm{im}\, x)^\perp$. If additionally $x^2 = x$ (i.e., x is a projection), then V is an orthogonal direct sum $(\ker x) \oplus (\mathrm{im}\, x)$.

If $x \in \mathrm{End}_F(V)$ is such that $\sigma(x)x = 0$, then $b(xv, xv') = 0$ for all $v, v' \in V$ and we find that $\mathrm{im}\, x \subseteq (\mathrm{im}\, x)^\perp$, i.e., $\mathrm{im}\, x$ is totally singular.

We may view Sp_{2n} as the subgroup of GL_{2n} preserving the bilinear form $b(v, v') := v^\top J v'$ where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. The Lie algebra $\mathfrak{sp}_{2n}$ of Sp_{2n} consists of matrices $\left(\begin{smallmatrix} A & B \\ C & -A^\top \end{smallmatrix}\right)$ for $A, B, C \in \mathfrak{gl}_n$ such that $B^\top = B$ and $C^\top = C$, compare [KMRT, §25.A]. In the notation of Example 8.1, $\sigma(g) = -Jg^\top J$ and $\mathfrak{sp}_{2n}$ consists of those $x \in \mathfrak{gl}_{2n}$ such that $\sigma(x) + x = 0$. (In particular, $\ker x = (\mathrm{im}\, x)^\perp$ for $x \in \mathfrak{sp}_{2n}$. If moreover $\mathrm{char}\, k = 2$ and $x^{[2]} = 0$, then $\sigma(x)x = x^{[2]} = 0$, so $\mathrm{im}\, x$ is totally singular.)

The group GSp_{2n} of similarities of b is the sub-group-scheme of GL_{2n} generated by Sp_{2n} and the scalar transformations. Its Lie algebra consists of those $x \in \mathfrak{gl}_{2n}$ such that $\sigma(x) + x \in kI_{2n}$, i.e., x of the form $\left(\begin{smallmatrix} A & B \\ C & \mu I_n - A^\top \end{smallmatrix}\right)$ for $A, B, C \in \mathfrak{gl}_n$ and $\mu \in k$, where $B^\top = B$ and $C^\top = C$. Then $\mathfrak{sp}_{2n} \subset \mathfrak{gsp}_{2n}$, the quotient $\mathrm{GSp}_{2n} / \mathbb{G}_m \cong \mathrm{Sp}_{2n} / \mu_2$ is the adjoint group PSp_{2n}, and the natural map $\mathfrak{gsp}_{2n} \to \mathfrak{psp}_{2n}$ is surjective.

The preceding two paragraphs apply to any field k; we now explicitly assume $\operatorname{char} k = 2$ and describe the toral elements in $\mathfrak{gsp}_{2n}$, i.e., those x such that $x^2 = x$. As such an x is a projection, it gives a decomposition of k^{2n} as a direct sum of vector spaces $(\ker x) \oplus (\operatorname{im} x)$. If x belongs to $\mathfrak{sp}_{2n}$, then this is an orthogonal decomposition as in Example 8.1 where the restrictions of b to $\ker x$ and $\operatorname{im} x$ are nondegenerate. Otherwise, $\sigma(x) + x = I_{2n}$, so

$$b(xv, xv') = b((I_{2n} - \sigma(x))v, xv') = b(v, xv') - b(v, x^2 v') = 0 \quad (v, v' \in V).$$

That is, $\operatorname{im} x$ is totally isotropic. Analogously, $1 - x \in \mathfrak{gsp}_{2n}$ is toral and so $\ker x = \operatorname{im}(1-x)$ is also totally isotropic. In sum, we find that $\ker x$ and $\operatorname{im} x$ are maximal totally isotropic subspaces.

Dimension bounds. For the remainder of this section, we set $G = \mathrm{Sp}_{2n}$ with $n \geq 3$ over a field k of characteristic 2. We first make some remarks about nilpotent elements of square 0.

Lemma 8.2. *Let $x \in \mathfrak{sp}_{2n}$ for $n \geq 3$ have $x^{[2]} = 0$ and rank $r > 0$. Then $\dim x^{\mathrm{Sp}_{2n}} \leq r(2n+1) - r^2$.*

Proof #1. Fix a totally singular r-dimensional subspace W of the natural module V and let P be the maximal parabolic subgroup of Sp_{2n} stabilizing W. Let C be the set of elements y in $\mathfrak{g}$ with $y^{[2]} = 0$ and $\operatorname{im} y \subseteq W$. As above, $\ker y = (\operatorname{im} y)^{\perp}$ and in particular $\ker y \supseteq W^{\perp}$. So we see that C is the center of the nilpotent radical of $\operatorname{Lie}(P)$ and can be identified with the space of r-by-r symmetric matrices. In particular, $\dim C = r(r+1)/2$.

Now let $x \in \mathfrak{g}$ with $x^{[2]} = 0$ and $\operatorname{im} x = W$. Consider the map $f : G \times C \to \mathfrak{g}$ given by $f(g, y) = \operatorname{Ad}(g)y$. Every fiber has dimension at least $\dim P$, since $f(P \times C) = C$. Thus the dimension of the image of f is at most $\dim C + \dim G/P = r(r+1) + 2r(n-r) = r(2n+1) - r^2$. Since x^G is contained in the image of f, we obtain the same inequality for $\dim x^G$.

We remark that, by Richardson, the Levi subgroup of P has a dense orbit on C, whence the largest such class has dimension precisely $r(2n+1) - r^2$. Note also that the fact that $\operatorname{im} x$ is totally singular shows that x is a sum of commuting root elements. If r is odd, then we can always take x to be a sum of commuting long root elements. If r is even, then we can always take x to be either a commuting sum of long root elements (the larger class) or a sum of commuting short root elements. Note that the smaller class is contained in the closure of the larger class. It is well known (cf. [LiS, Chap. 4]) that if $x \in \mathfrak{g}$ has even rank and $x^{[2]} = 0$, then x is an element of $\mathfrak{so}_{2n}$. □

Proof #2. There are two possibilities for the conjugacy class of x in case r is even, see [He, 4.4] or [LiS, p. 70]. We focus on the larger class; regardless of the parity of r we may assume that the restriction of the natural module to x includes a 2-dimensional summand denoted by $V(2)$ in [LiS]. For this x, the function denoted by χ in the references amounts to $1 \mapsto 0$ and $2 \mapsto 1$. The formulas in these references now give that the centralizer of x in Sp_{2n} has dimension

$$\sum_{i=1}^{r} (2i-1) + \sum_{i=r+1}^{2n-r} i = 2n - r + \binom{2n-r}{2} + \binom{r}{2}. \qquad \square$$

The Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2n}$ has derived subalgebra $\mathfrak{m} = \mathfrak{so}_{2n}$ of codimension $2n$; it is the unique maximal G-invariant ideal in $\mathfrak{g}$. The subalgebra $\mathfrak{n} = \ker d\pi$ has codimension 1 in $\mathfrak{m}$ and is $[\mathfrak{m}, \mathfrak{m}]$. The quotient $\mathfrak{g}/\mathfrak{n}$ is the Heisenberg Lie algebra from section 3. The unique maximal ideal $\mathfrak{m}$ also contains $\mathfrak{z}(\mathfrak{g})$ (dimension 1). The center $\mathfrak{z}(\mathfrak{g})$ is contained in $\mathfrak{n}$ if and only if n is even.

Lemma 8.3. *Let $G = \mathrm{Sp}_{2n}$ for some $n \geq 3$ over a field k of characteristic 2, and let $x \in \mathfrak{sp}_{2n}$ be noncentral.*

1. *If x is toral, then $\max\{4, \lceil n/s \rceil\}$ conjugates of x suffice to generate a subalgebra containing $\mathfrak{n}$, where $2s$ is the dimension of the smallest eigenspace.*
2. *If $x^{[2]} = 0$ and x has even rank $2s > 0$ or odd rank $2s + 1 \geq 3$, then $\max\{4, \lceil n/s \rceil\}$ conjugates of x generate a subalgebra containing $\mathfrak{n}$.*
3. *If $x^{[2]} = 0$ and x has rank 1, then $\max\{8, 2n\}$ conjugates of x generate a subalgebra containing $\mathfrak{n}$.*

Proof. If x is toral, or is nilpotent with even rank, then x is in $\mathfrak{m} = \mathfrak{so}_{2n}$ (cf. the remarks in proof #1 of Lemma 8.2), and the claim is [GaGu I, Prop. 10.4].

If $x^{[2]} = 0$ and x rank $2s + 1 \geq 3$, the description of the classes given in the proof of Lemma 8.2 shows that the closure of $x^{\mathrm{Sp}_{2n}}$ contains a y of rank $2s$ such that $y^{[2]} = 0$. As $\max\{4, \lceil n/s \rceil\}$ conjugates of y suffice to generate a subalgebra containing $\mathfrak{n}$, the same holds for x.

If $x^{[2]} = 0$ and x has rank 1, we choose y conjugate to x such that $(x + y)^{[2]} = 0$ and $x + y$ has rank 2. By (2), $\max\{4, n\}$ conjugates of $x + y$ generate a subalgebra containing $\mathfrak{n}$, whence (3) holds. □

Proposition 8.4. *Let (V, ρ) be a representation of $G = \mathrm{Sp}_{2n}$ or PSp_{2n} with $n \geq 3$ over a field k of characteristic 2. If $V^{\mathfrak{n}} = 0$ and*

$$\dim V > \begin{cases} 48 & \text{if } G = \mathrm{Sp}_6 \text{ or } \mathrm{PSp}_6 \text{ (i.e., } n = 3\text{)} \\ 72 & \text{if } G = \mathrm{Sp}_8; \\ 80 & \text{if } G = \mathrm{PSp}_8; \text{ and} \\ 6n^2 - 6n & \text{if } n \geq 5, \end{cases}$$

then $\mathfrak{g}$ acts virtually freely on V.

In the statement, in case $G = \mathrm{PSp}_{2n}$, ρ induces a representation of Sp_{2n} and so it still makes sense to speak of the action of $\mathfrak{n} \subset \mathfrak{sp}_{2n}$ on V.

Proof. For noncentral $x \in \mathfrak{g}$ such that $x^{[2]} \in \{0, x\}$, we check that x does not meet $\mathfrak{g}_v$ for generic $v \in V$, and therefore that $\mathfrak{g}_v \subseteq \mathfrak{z}(\mathfrak{g})$.

Case $G = \mathrm{Sp}_{2n}$ for $n \geq 4$. Suppose first that $x^{[2]} = x$, or $x^{[2]} = 0$ and x has even rank. Then x belongs to $\mathfrak{m} = \mathfrak{so}_{2n}$ and is not itself central in $\mathfrak{m}$. The Short Root Proposition 5.2 gives that for $\dim V > 4n^2$, x does not meet $\mathfrak{m}_v$ for generic $v \in V$.

We may assume that $x^{[2]} = 0$ and x has odd rank $r = 2s + 1$, where $\dim x^G$ is bounded as in Lemma 8.2. We find an $e > 0$ such that e conjugates of x generate a subalgebra of $\mathfrak{g}$ containing the image of $\mathfrak{n}$ and such that $e \cdot \dim x^G$ is at most the right side of the inequality in the statement. This verifies by Lemmas 2.1 and 2.5(2) that x does not meet $\mathfrak{g}_v$ for generic $v \in V$.

If $r = 1$, then applying Lemma 8.3(3) gives that $e \cdot \dim x^G \leq 4n \max\{4, n\}$.

If $r = 3$, then $\dim x^G \leq 6(n - 1)$. Applying Lemma 8.3(2) gives $e \dim x^G \leq 6(n - 1) \max\{4, n\}$.

Suppose $r \geq 5$ (so also $n \geq 5$). If $s \geq n/4$, then 4 conjugates suffice to generate a subalgebra containing $\mathfrak{n}$. As $\dim x^G \leq 2r(n-s)$, it suffices to bound $8r(n-s)$. This is maximized at $r = n + \frac{1}{2}$ and so we obtain a bound of $4n^2 + 4n$.

If $s < n/4$, then it suffices to bound $(n/s+1)(2r)(n-s) = (n^2-s^2)(2r/s) \leq 5n^2 - 5s^2 \leq 5n^2 - 20$.

Case $G = \mathrm{Sp}_6$. As in the $n \geq 4$ case, we may reduce to the case where $x^{[2]} = 0$ and x has odd rank.

If x has rank 3, then $\dim x^G \leq 12$ and we get a bound of 48. If x is a long root element, then 6 conjugates suffice to generate $\mathfrak{sp}_6$ (by reducing to the rank 2 case and using generation by 3 conjugates). Then $6 \dim x^G = 36$ and the result holds.

Case $G = \mathrm{PSp}_{2n}$ *for* $n \geq 3$. For $\mathrm{GSp}_{2n} := (\mathrm{Sp}_{2n} \times \mathbb{G}_m)/\mu_2$, the group of similarities of the bilinear form, we have a natural surjection $\mathrm{GSp}_{2n} \to \mathrm{PSp}_{2n}$ whose differential $\mathfrak{gsp}_{2n} \to \mathfrak{psp}_{2n}$ is surjective and has central kernel k, the scalar matrices. Let $x \in \mathfrak{gsp}_{2n}$ be noncentral such that $x^{[2]} \in \{0, x\}$. If x belongs to $\mathfrak{sp}_{2n}$, we have already verified that x cannot lie in $(\mathfrak{sp}_{2n})_v$ for generic $v \in V$. So assume that x is toral and does not belong to $\mathfrak{sp}_{2n}$, i.e., is the projection on a maximal totally isotropic subspace as in the paragraph preceding the statement. Up to conjugacy we may assume that x is $\left(\begin{smallmatrix} 0_n & 0_n \\ 0_n & I_n \end{smallmatrix}\right)$. The nilpotent linear transformation $y := \left(\begin{smallmatrix} 0_n & I_n \\ 0_n & 0_n \end{smallmatrix}\right)$ belongs to $\mathfrak{sp}_{2n}$, has rank n and satisfies $y^{[2]} = 0$. Note that $x + ty$ is conjugate to x for any scalar t. It follows that y is in the closure of $x^{\mathbb{G}_m G}$.

Lemma 8.3 gives that 4 conjugates of y suffice to generate a subalgebra of $\mathfrak{sp}_{2n}$ containing $\mathfrak{n}$. Take $M = \mathfrak{n} = [\mathfrak{so}_{2n}, \mathfrak{so}_{2n}]$ and $N = \mathfrak{z}(\mathfrak{n})$, so M/N is the irreducible representation $L(\omega_{n-1})$ of G whose highest weight is the highest short root. As

$$\dim \mathfrak{gsp}_{2n}/M = 2n + 1 < 2n^2 - n - 1 \leq \dim M/N,$$

[GaGu I, Lemma 4.3(3)] says that 4 conjugates of x also suffice to generate a subalgebra of $\mathfrak{sp}_{2n}$ containing $\mathfrak{n}$. On the other hand, $\dim x^{\mathrm{GSp}_{2n}} = n^2 + n$, giving $e \cdot \dim x^{\mathrm{GSp}_{2n}} \leq 4n^2 + 4n$. Therefore, it suffices to take the bound for PSp_{2n} to be the maximum of $4n^2 + 4n$ and the bound for Sp_{2n}. □

Restricted irreducible representations. Let V be a restricted irreducible representation of an algebraic group G of type C_n with $n \geq 3$ over k of characteristic 2; we aim to prove Theorem C(2) for this G. The highest weight $\lambda = \sum_{i=1}^n c_i \omega_i$, numbered as in Table 2, has $c_i \in \{0, 1\}$ for all i. If $\lambda = 0$, equivalently $\ker d\rho = \mathfrak{g}$, then there is nothing to do.

Example 8.5 ("spin" representation)**.** Put $G' := \mathrm{Spin}_{2n+1}$. Composing the very special isogeny $\mathrm{Sp}_{2n} \to G'$ with the (injective) spin representation of G' of dimension 2^n yields the irreducible representation $(V, \rho) = L(\omega_1)$ of Sp_{2n}. For $n \geq 7$, $\dim V > \dim G$ and the stabilizer $(\mathfrak{g}')_v$ of a generic $v \in V$ is zero [GaGu 17, Th. 1.1], so $(\mathfrak{sp}_{2n})_v = \ker d\rho$.

For $3 \leq n < 7$, $\dim V < \dim \mathrm{Sp}_{2n}$ and one can check using a computer that there exist vectors in V whose stabilizer is $\ker d\rho$, therefore, $\mathfrak{sp}_{2n}$ acts virtually freely on V. Alternatively, for $n = 3, 5, 6$ and generic $v \in V$, $(\mathrm{Spin}_{2n+1})_v$ is G_2, $\mathrm{SL}_5 \rtimes \mathbb{Z}/2$, $(\mathrm{SL}_3 \times \mathrm{SL}_3) \rtimes \mathbb{Z}/2$ respectively [GaGu 17]; each of these has a Lie algebra that is a direct sum of simples, and so it can not meet the solvable ideal that is the image of $\mathfrak{sp}_{2n}$ in $\mathfrak{spin}_{2n+1}$.

For n even, the representation factors through PSp_{2n}. For $n \geq 8$ and $n = 6$, $\mathfrak{psp}_{2n}$ acts virtually freely as in the preceding two paragraphs. For $n = 4$, the image of $\mathfrak{psp}_8$ in $\mathfrak{spin}_9$ is the maximal proper Spin_9-submodule, for which a generic vector v in the spin representation has a 1-dimensional stabilizer (Example 7.13), i.e., $\dim(\mathfrak{psp}_8)_v/\ker d\rho = 1$, i.e., $\mathfrak{psp}_8$ does not act virtually freely.

Therefore, we have addressed the cases where $\lambda_s = 0$, i.e., $\lambda \in \{0, \omega_1\}$, so we now assume that $\lambda_s \neq 0$, whence, $V^{\mathfrak{n}} = 0$. As this excludes all the representations in Table 4, our task is to prove that (1) $\mathfrak{sp}_{2n}$ does not act virtually freely if $\dim V \leq 2n^2 + n$ and (2) $\mathfrak{psp}_{2n}$ acts virtually freely if $\dim V > 2n^2 + n$.

Example 8.6 (natural representation)**.** Here we treat $V := L(\omega_n)$, the natural representation of Sp_{2n}, which has $\dim V = 2n < \dim \mathrm{Sp}_{2n}$. In this case Sp_{2n} is transitive on all nonzero vectors and so we see the generic stabilizer is the derived subalgebra of the maximal parabolic subalgebra that is the stabilizer of a 1-dimensional space.

Example 8.7 ("$\wedge^2$" representation)**.** Let $V := L(\omega_{n-1})$, which has $\dim V < \dim G$. We will show that the generic stabilizer in Sp_{2n} is $A_1 \times \cdots \times A_1$ (more precisely the stabilizer of n pairwise orthogonal two-dimensional non-degenerate subspaces), so V is virtually free for neither $\mathfrak{sp}_{2n}$ nor for $\mathfrak{psp}_{2n}$.

Let M be the natural module for Sp_{2n}, of dimension $2n$. Let $W = \wedge^2 M$ which we can identify with the set of skew adjoint operators on M, i.e., the linear maps $T\colon M \to M$ with $(Tv, v) = 0$ for all $v \in M$. If n is odd, then $W \cong k \oplus V$. If n is even, then W is uniserial with 1-dimensional socle and head with V as the nontrivial composition factor. The unique submodule W_0 of codimension 1 in each case consists of those elements T with reduced trace equal to 0.

It follows as in [GoGu] or [GaGu 15, Example 8.5] that a generic element T of W (or W_0) is semisimple and has n distinct eigenvalues each of multiplicity 2. It follows that, as a group scheme, the stabilizer of such an element is as given and so the result holds for n odd.

Assume that $n \geq 4$ is even. Note that a generic element T of W also has the property that the $n(n-1)/2$ differences of the eigenvalues of T are distinct. It follows easily that the stabilizer of such an element in $V = W_0/W^{\mathrm{Sp}_{2n}}$ is the same as in W_0 and the result follows.

If $n > 11$ and $\dim V \leq 6n^2 - 6n$, then V is $L(\omega_n)$ or $L(\omega_{n-1})$ by [Lü], so we may assume $3 \leq n \leq 11$. Therefore we are reduced to considering Tables A.32–A.40 in [Lü], one table for each value of n. In those tables, the representations V not already handled by Prop. 8.4 or Examples 8.5, 8.6, 8.7 are: $L(\omega_{n-2})$ ("$\wedge^3$") for Sp_{2n} with $n = 4, 5$ and the representation $L(\omega_1 + \omega_3)$ of PSp_6 of dimension 48. A computer check verifies that $\mathfrak{g}$ acts virtually freely on these representations.

As we have verified Theorem C(2) for groups of type C (in this section), B (in section 7), and F_4 and G_2 (in section 6), the proof is complete.

9. A large representation that is not virtually free

In this section, G is a simple and simply connected algebraic group over k and $\mathrm{char}\, k$ is special for G. The main theorem of part I includes a statement of the following type: *If* $\mathrm{char}\, k$ *is not special and* V *is a* G*-module with a subquotient* X *such that* $X^{[\mathfrak{g},\mathfrak{g}]} = 0$ *and* $\dim X$ *is big enough, then* $\mathfrak{g}$ *acts virtually freely on* V. Example 9.3 below shows that such a result does not hold verbatim when $\mathrm{char}\, k$ is special.

As in §1, we put $\mathfrak{n}$ for the kernel of the (differential of the) very special isogeny in $\mathfrak{g} = \mathrm{Lie}(G)$.

Lemma 9.1. *For any representation V of any Lie algebra $\mathfrak{d}$, we have: $(V/V^{\mathfrak{d}})^{\mathfrak{d}} \subseteq V^{[\mathfrak{d},\mathfrak{d}]}/V^{\mathfrak{d}}$. In particular, if $\mathfrak{d} = [\mathfrak{d},\mathfrak{d}]$, then $(V/V^{\mathfrak{d}})^{\mathfrak{d}} = 0$.*

Proof. Suppose for some $v \in V$ that $zv \in V^{\mathfrak{d}}$ for all $z \in \mathfrak{d}$. For $x, y \in \mathfrak{d}$, we have $[x, y]v = x(yv) - y(xv) = 0$, i.e., $v \in V^{[\mathfrak{d},\mathfrak{d}]}$. □

Lemma 9.2. *For G simple and simply connected but not Sp_4 over a field k such that $\mathrm{char}\, k$ is special, we have:*

1. *$\mathfrak{n}$ is not virtually free as an $\mathfrak{n}$-module.*
2. *$\mathfrak{n}/\mathfrak{n}^{[\mathfrak{g},\mathfrak{g}]} \neq 0$ and $(\mathfrak{n}/\mathfrak{n}^{[\mathfrak{g},\mathfrak{g}]})^{[\mathfrak{g},\mathfrak{g}]} = 0$.*

We ignore what happens in the omitted case of Sp_4.

Proof. We first verify (1). If G has type G_2 or F_4, then $\mathfrak{n}$ is the simple quotient of $\mathfrak{sl}_3$ or $\mathfrak{spin}_8$ by its center. Thus $\mathfrak{n}$ acts on $\mathfrak{n}$ with trivial kernel and the stabilizer of a generic element of $\mathfrak{n}$ is the image of a maximal torus in $\mathfrak{sl}_3$ or $\mathfrak{spin}_8$, of dimension 1 or 2.

If G has type C_n with $n \geq 3$, then $\mathfrak{n} = [\mathfrak{so}_{2n}, \mathfrak{so}_{2n}]$, of codimension 1 in $[\mathfrak{g},\mathfrak{g}] = \mathfrak{so}_{2n}$. Then $\mathfrak{n}^{[\mathfrak{g},\mathfrak{g}]} = \mathfrak{n} \cap \mathfrak{z}(\mathfrak{so}_{2n})$ and the quotient $\mathfrak{n}/\mathfrak{n}^{[\mathfrak{g},\mathfrak{g}]}$ is the irreducible representation of $\mathfrak{so}_{2n}$ with highest weight the highest root. Since the quotient is not virtually free for $\mathfrak{n}$ as in [GaGu I, Example 3.4] (using that $G \neq \mathrm{Sp}_4$), neither is $\mathfrak{n}$ itself. (Note that this also gives (2) in this case.)

If G has type B_n with $n \geq 2$, then $\mathfrak{n} = \mathfrak{h}$, the Heisenberg algebra from §3. For a generic $h \in \mathfrak{h}$, the map $x \mapsto [x, h] \in \mathfrak{z}(\mathfrak{h})$ is a nonzero linear map, so has kernel of codimension 1, completing the proof of (1).

Now consider (2). In the remaining cases where G has type B_n with $n \geq 3$, G_2, or F_4, the algebra $\mathfrak{g}$ is perfect (immediately giving the second claim by Lemma 9.1), so $\mathfrak{n}^{[\mathfrak{g},\mathfrak{g}]} = \mathfrak{n}^{\mathfrak{g}} = \mathfrak{n} \cap \mathfrak{z}(\mathfrak{g})$, which is zero for type G_2 and F_4 and has codimension $2n$ in $\mathfrak{n}$ for type B_n. □

Example 9.3. Let G be as in Lemma 9.2 and let U be a representation of G on which $\mathfrak{g}$ acts with kernel $\mathfrak{n}$. For G of type B_n with $n \geq 3$, we take U to be the natural irreducible representation of dimension $2n$. For type G_2 and F_4, we take $U = \mathfrak{g}/\mathfrak{n}$. For type C_n with $n \geq 3$, $\mathfrak{g}/\mathfrak{n}$ is the Heisenberg subalgebra in $\mathfrak{spin}_{2n+1}$ and we take U to be the spin representation. Note that for types B_n, F_4, and G_2, the algebra $\mathfrak{g}$ is perfect and U is irreducible so $U^{[\mathfrak{g},\mathfrak{g}]} = 0$. For type C_n, $[\mathfrak{g},\mathfrak{g}]$ maps to the center of $\mathfrak{spin}_{2n+1}$, and again we have $U^{[\mathfrak{g},\mathfrak{g}]} = 0$.

Now take W to be a finite sum of enough copies of U so that $\mathfrak{g}$ acts virtually freely on W, compare Example 7.11 for type B_n and Example 7.1 for type C_n. For types F_4 and G_2, take $W = U \oplus U$.

Let $V = \mathfrak{n} \oplus \bigoplus^m W$ for W as in the preceding paragraph. The subquotient $X = V/V^{[\mathfrak{g},\mathfrak{g}]} = \mathfrak{n}/\mathfrak{n}^{[\mathfrak{g},\mathfrak{g}]} \oplus \bigoplus^m W$ has, by Lemma 9.2, $X^{[\mathfrak{g},\mathfrak{g}]} = 0$ and $\dim X > m \dim W$ can be made arbitrarily large by increasing m. On the other hand, for a generic vector $v = (n, w_1, \ldots, w_m) \in V$, the stabilizer $\mathfrak{g}_v$ equals $\mathfrak{n}_n$, so by Lemma 9.2 $\mathfrak{g}$ does not act virtually freely on V.

10. Proof of Theorem C(1)

We now prove claim (1) of Theorem C. Recall that $\rho\colon G \to \mathrm{GL}(V)$ is assumed irreducible with restricted highest weight and $\mathfrak{g}$ does not act virtually free on V.

If $\operatorname{char} k$ is special, we apply part (2) of Theorem C to see that $\dim V \le \dim G$. If $\operatorname{char} k$ is not special, then the kernel of $\mathrm{d}\rho$ is a central toral subalgebra, so we can assume ρ is faithful, and apply Theorem A from [GaGu II] (quoted here as the case of Theorem A where $\operatorname{char} k$ is not special), which says that $\dim V \le \dim G$ or $(G, \operatorname{char} k, V)$ belongs to Table 1, in which case $\mathfrak{g}_v$ is a toral subalgebra.

It remains to treat the case where $\dim G = \dim V$. The tables in [Lü] show that V is the adjoint representation and $\operatorname{char} k$ is not special, so $\mathfrak{g}_v$ is a toral subalgebra for generic v. □

11. Proof of Theorem B

The goal of this section is to complete the proof of Theorem B, so we adopt its hypotheses. In particular, $\operatorname{char} k$ is assumed to be special for G and ρ is faithful. Note that $V^{\mathfrak{n}} = 0$; otherwise, as V is irreducible, $V^{\mathfrak{n}} = V$ and $\mathrm{d}\rho$ would not be faithful.

Write the highest weight λ of ρ as $\lambda_0 + p\lambda_1$ where λ_0, λ_1 are dominant, $p = \operatorname{char} k$, and λ_0 is restricted. Assume $\lambda_1 \ne 0$ for otherwise we are done by Theorem C, because the representations in Table 4 are not faithful. As G acts faithfully, it follows as in [GaGu II, Lemma 1.1] that $\lambda_0 \ne 0$ and $\dim V > \dim G$. We will verify that $\mathfrak{g}$ acts generically freely on V.

As $\lambda \in T^*$ and $p\lambda_1$ is in the root lattice hence in T^*, it follows that λ_0 is also in T^*. Therefore, the representations $L(\lambda_0)$ and $L(p\lambda_1)$ are representations of G. The representation $L(\lambda_0) \otimes L(p\lambda_1)$ is a representation of G that is irreducible (because its restriction to the simply connected cover of G is the representation $L(\lambda_0) \otimes L(\lambda_1)^{[p]}$) and has highest weight λ, so it is equivalent to V.

We will repeatedly use below that for any representation X of G and any $e > 0$, the representation $X \otimes X^{[p]^e}$ is virtually free for $\mathfrak{g}$, see [GaGu II, Lemma 10.2].

Type C. Suppose first that G has type C_n for some $n \ge 3$. The smallest nontrivial irreducible representation of Sp_{2n} is the natural representation of dimension $2n$. If $\dim L(\lambda_0) = \dim L(p\lambda_1) = 2n$, then $G = \mathrm{Sp}_{2n}$ and $L(p\lambda_1)$ is a Frobenius twist of $L(\lambda_0)$, so V is generically free as in the preceding paragraph. Otherwise, at least one of $\dim L(\lambda_0)$, $\dim L(p\lambda_1)$ is greater than $2n$. The second smallest nontrivial irreducible representation of Sp_{2n} is $L(\omega_{n-1})$ of dimension

$$s(n) = \begin{cases} 2n^2 - n - 1 & \text{if } n \text{ is odd;} \\ 2n^2 - n - 2 & \text{if } n \text{ is even.} \end{cases}$$

So $\dim V \ge 2n\, s(n)$, the values of which are as follows:

n	3	4	≥ 5
$2n\, s(n)$	84	208	$> 6n^2 + 6n$

Therefore, G acts generically freely by Prop. 8.4.

G simply connected. We are reduced to considering G of type B_n for $n \ge 2$, F_4, or G_2. By Lemma 7.7, G is simply connected, and therefore $V \cong L(\lambda_0) \otimes L(\lambda_1)^{[p]}$. As a representation of $\mathfrak{g}$, this is a sum of $\dim L(\lambda_1)$ copies of $L(\lambda_0)$, and in particular $L(\lambda_0)$ is itself faithful. Put m for the dimension of the smallest nontrivial irreducible

representation of G, which is $2n$ (type B_n), 26 (type F_4), or 7 (type G_2). Then V contains the $\mathfrak{g}$-submodule $X := \oplus^m L(\lambda_0)$ on which $\mathfrak{g}$ acts faithfully, and we will show that $\mathfrak{g}$ acts generically freely on X.

If $\dim L(\lambda_0) = m$, then X is isomorphic to $L(\lambda_0) \otimes L(\lambda_0)^{[p]}$ as $\mathfrak{g}$-modules, and we are done, so assume $\dim L(\lambda_0) > m$. For m' for the dimension of the second smallest nontrivial irreducible representation, we have $\dim L(\lambda_0) \geq m'$, whence $\dim X \geq mm'$.

If G has type F_4 or G_2, then $m' = 246$ or 27 respectively, and Prop. 6.1 shows that $\mathfrak{g}$ acts generically freely on X.

So suppose $G = \mathrm{Spin}_{2n+1}$ for some $n \geq 2$. The smallest faithful irreducible representation of G is the spin representation $L(\omega_1)$ of dimension 2^n, so $\dim V \geq m2^n = n2^{n+1}$. If $n \geq 4$, then $\dim V > 4n^2+4n$, and we are done by Lemma 7.4. For $n = 2, 3$, a sum of $2n$ copies of the spin representation is generically free (Example 7.1), so we may assume that $\lambda_0 \neq \omega_1$. The next smallest faithful representation of Spin_{2n+1} is $L(\omega_1 + \omega_n)$ of dimension 16 for $n = 2$ or 48 for $n = 3$. Therefore, $\dim X \geq 2n \cdot \dim L(\omega_1 + \omega_n) > 4n^2 + 4n$ and again we are done by Lemma 7.4, completing the proof of Theorem B. □

12. Proof of Theorem A

Theorem A now follows quickly from what has gone before. We repeat the argument given at the end of part I for the convenience of the reader.

The stabilizer G_v of a generic $v \in V$ is finite étale if and only if the stabilizer $\mathfrak{g}_v$ of a generic $v \in V$ is zero, i.e., if and only if $\mathfrak{g}$ acts generically freely on V. By Theorem A in [GaGu II] (for which the case where $\mathrm{char}\, k$ is special is Th. B in this paper), this occurs if and only if $\dim V > \dim G$ and $(G, \mathrm{char}\, k, V)$ does not appear in Table 1, proving Theorem A(1).

For Theorem A(2), we must enumerate in Table 3 those representations V such that $\dim V > \dim G$, V does not appear in Table 1, and the group of points $G_v(k)$ is not trivial. Those V with the latter property are enumerated in [GuL], completing the proof of Theorem A(2). □

References

[BoTi] Armand Borel and Jacques Tits, *Homomorphismes "abstraits" de groupes algébriques simples*, Ann. of Math. (2) **97** (1973), 499–571.

[CGP] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, 2nd ed., Cambridge University Press, 2015.

[DG] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson, Paris, 1970.

[GaGu 15] S. Garibaldi and R.M. Guralnick, *Simple groups stabilizing polynomials*, Forum of Mathematics: Pi **3** (2015), e3 (41 pages).

[GaGu 17] ———, *Spinors and essential dimension*, Compositio Math. **153** (2017), 535–556, with an appendix by A. Premet.

[GaGu I] ———, *Generically free representations I: Large representations*, to appear in *Algebra & Number Theory*, arxiv:1711.05502v3.

[GaGu II] ———, *Generically free representations II: Irreducible representations*, February 2019, arxiv:1711.06400v3.

[GoGu] D. Goldstein and R. Guralnick, *Alternating forms and self-adjoint operators*, J. Algebra **308** (2007), no. 1, 330–349.

[GuL] R.M. Guralnick and R. Lawther, *Generic stabilizers in actions of simple algebraic groups I: modules and first Grassmannian varieties*, arxiv:1904.13375v2, August 2019.

[He] W.H. Hesselink, *Nilpotency in classical groups over a field of characteristic 2*, Math. Zeit. **166** (1979), 165–181.

[Hi] G. Hiss, *Die adjungierten Darstellungen der Chevalley-Gruppen*, Arch. Math. (Basel) **42** (1984), 408–416.

[Ho] G.M.D. Hogeweij, *Almost-classical Lie algebras. I, II*, Nederl. Akad. Wetensch. Indag. Math. **44** (1982), no. 4, 441–460.

[J] J.C. Jantzen, *Representations of algebraic groups*, second ed., Math. Surveys and Monographs, vol. 107, Amer. Math. Soc., 2003.

[KMRT] M.-A. Knus, A.S. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Colloquium Publications, vol. 44, Amer. Math. Soc., 1998.

[LiS] M. Liebeck and G. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Math. Surveys and Monographs, no. 180, Amer. Math. Soc., 2012.

[Lü] F. Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169.

[P] R. Pink, *Compact subgroups of linear algebraic groups*, J. Algebra **206** (1998), no. 2, 438–504.

[S] R. Steinberg, *Representations of algebraic groups*, Nagoya Math. J. **22** (1963), 33–56, [= Collected Papers, pp. 149–172].

[SF] H. Strade and R. Farnsteiner, *Modular Lie algebras and their representations*, Monographs and textbooks in pure and applied math., vol. 116, Marcel Dekker, New York, 1988.

[V] V.S. Varadarajan, Spin(7)*-subgroups of* SO(8) *and* Spin(8), Expo. Math. **19** (2001), 163–177.