A census of exceptional Dehn fillings

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Abstract. This paper describes the complete list of all 205,822 exceptional Dehn fillings on the 1-cusped hyperbolic 3-manifolds that have ideal triangulations with at most 9 ideal tetrahedra. The data is consistent with the standard conjectures about Dehn filling and suggests some new ones.

1 Introduction

1.1 Dehn filling. Suppose M is a compact orientable 3-manifold with ∂M a torus. A *slope* on ∂M is an unoriented isotopy class of simple closed curve, or equivalently a primitive element of $H_1(\partial M;\mathbb{Z})$ modulo sign. The set of all slopes will be denoted $Sl(M)$, which can be viewed as the rational points in the projective line $P^1\big(H_1(\partial M;\mathbb{R})\big) \cong \mathbb{R}\mathrm{P}^1$. The Dehn fillings of M are parameterized by $\alpha \in Sl(M)$, with $M(\alpha)$ being the Dehn filling where α bounds a disk in the attached solid torus. When the interior of M admits a hyperbolic metric of finite volume, it is called a *1-cusped hyperbolic 3-manifold*. For such hyperbolic M, Thurston showed that all but finitely many $M(\alpha)$ are also hyperbolic [Thu]. The nonhyperbolic Dehn fillings are called *exceptional*, and the corresponding slopes the *exceptional slopes*. Understanding the possible exceptional fillings has been a major topic in the study of 3-manifolds over the past 40 years; see the surveys [Gor1, Gor2, Gor3, Gor4] for further background.

This paper gives a census of all exceptional Dehn fillings on a certain collection of 1-cusped hyperbolic 3-manifolds. Specifically, let $\mathscr{C}_t$ be the set of all orientable 1-cusped hyperbolic 3-manifolds that have ideal triangulations with at most t ideal tetrahedra. For $t \le 9$, the set $\mathscr{C}_t$ has been enumerated by [HW, CHW, Thi, Bur2]

and is included with SnapPy [CDGW], whose nomenclature for these manifolds (e.g. $m004$, $s011$, $v1002$, $t12345$, and $o9_{60000}$) I will use freely throughout. Each manifold M in $\mathcal{C}_t$ has a preferred basis for $H_1(\partial M;\mathbb{Z})$, and so I will denote slopes in $Sl(M)$ by elements in $\mathbb{Z}^2$. See Figure 1 for some basic statistics on the 59,107 manifolds in $\mathcal{C}_9$. The main result of this paper is:

> **1.2 Theorem.** *There are precisely 205,822 exceptional Dehn fillings on the manifolds in $\mathcal{C}_9$, that is, pairs (M,α) where $M(\alpha)$ is not hyperbolic, of the types listed in Table 2 and distributed as in Figure 3.*

The list of these exceptional (M,α) together with the precise topology of each $M(\alpha)$ is available at [Dun]. Here, in addition to describing the proof of Theorem 1.2 in Section 5, I will give summaries of this data as it relates to known results and open questions about Dehn filling in Sections 3 and 4.

1.3 Prior work. In the 1990s, Hodgson and Weeks studied the exceptional Dehn fillings on the 286 manifolds in $\mathcal{C}_5$; this work was never published but is referred to extensively in [Gor1] and provided many key examples in the subject. The series of papers [MP, MPR, Mar] classified all exceptional fillings on an important series of chain links with as many as 7 components; as noted in [Mar, §3], this determines the exceptional fillings on more than 95% of the 4,587 manifolds in $\mathcal{C}_7$. John Berge (personal communication) independently did a search for exceptional fillings on $\mathcal{C}_9$ using a new version of his program Heegaard [Ber], and found more than 99.3% of the exceptional fillings included in Theorem 1.2.

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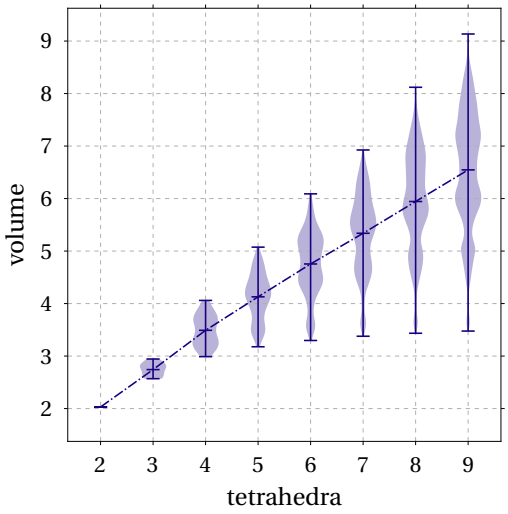
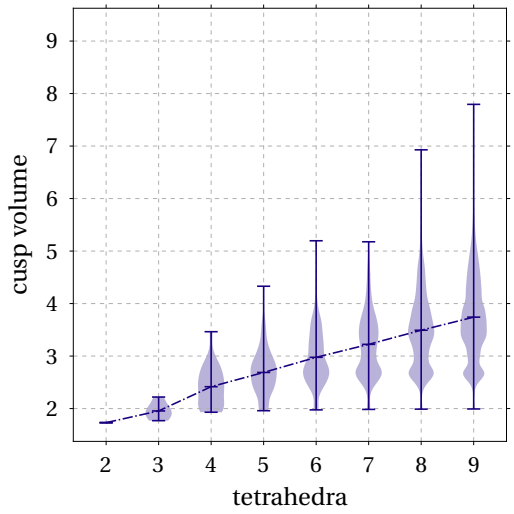
2 Background and conventions

I first review the different types of nonhyperbolic 3-manifolds to establish my conventions on the kinds of exceptional Dehn fillings one can study. Sources vary slightly on the latter point, and here I use a relatively fine-grained division, which is illustrated in Figure 4. The summary for experts, who may safely skip this section, is that here atoroidal means geometrically atoroidal, the manifold $S^2\times S^1$ is





tets	#manifolds
2	2
3	9
4	52
5	223
6	913
7	3,388
8	12,241
9	42,279
total	59,107

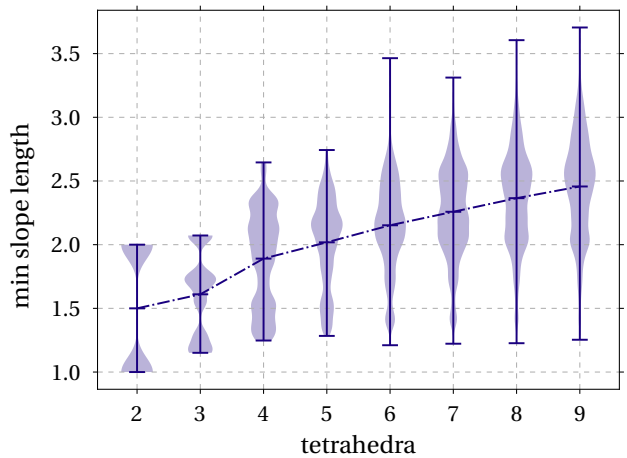


Figure 1. Some basic statistics on the manifolds in $\mathscr{C}_9$. The table in the lower left shows the number of manifolds in $\mathscr{C}_t \setminus \mathscr{C}_{t-1}$, that is, those whose minimal ideal triangulations have exactly t tetrahedra. In the top left, the hyperbolic volume is shown via a violin plot. Here, each "violin" shows the distribution of volume for $\mathscr{C}_t \setminus \mathscr{C}_{t-1}$, where the top and bottom horizontal bars are the min and max, the middle horizontal bar is the mean, and the violin "body" is a smoothed histogram of the volumes. The plot in the upper right shows the cusp volume, i.e. the volume of a maximal cusp neighborhood bounded by an embedded horotorus. The plot in the lower right shows the minimal slope length, that is, the shortest essential curve in the maximal horotorus. See Section 4.3 for the relevance of this cusp data.

	$b_1(M(\alpha)) = 0$		$b_1(M(\alpha)) > 0$
atoroidal			
S^3	1,267		
lens space	44,487		
finite π_1	13,446	$S^2 \times S^1$	242
Seifert fibered	71,111	Seifert fibered	118
connected sum	4,296	connected sum	169
toroidal			
Seifert fibered	1,730	Seifert fibered	159
graph manifold	63,325	graph manifold	3,043
hyperbolic piece	2,136	hyperbolic piece	74
		Sol torus bundle	219
totals	201,798		4,024

Table 2. Summary of the topological types of the 205,822 exceptional fillings from Theorem 1.2, broken down by betti number and whether the filling is toroidal. See Section 2 for precise definitions. Here, each filling is listed only once in the most restricted category possible from Figure 4, and there are no fillings that are both toroidal and connected sums.

$e(M)$	#manifolds
0	3
1	1,824
2	9,730
3	18,003
4	19,009
5	8,751
6	1,776
7	8
8	2
9	0
10	1

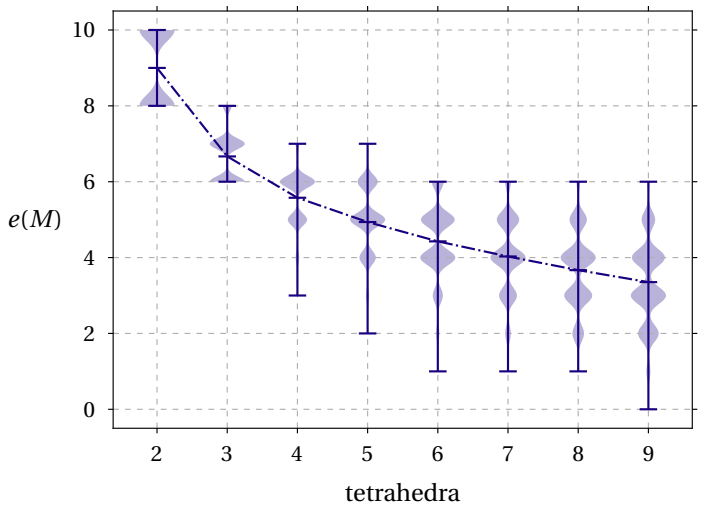


Figure 3. This figure describes how the exceptional fillings of Theorem 1.2 are distributed over the manifolds in $\mathscr{C}_9$. The number of exceptional fillings on M is denoted $e(M)$, and the table at left shows the number of manifolds in $\mathscr{C}_9$ having each possible value of $e(M)$, which is at most 10 by [LM]. At right is a violin plot of $e(M)$ as a function of the number of tetrahedra of M; see Figure 1 for more on violin plots.

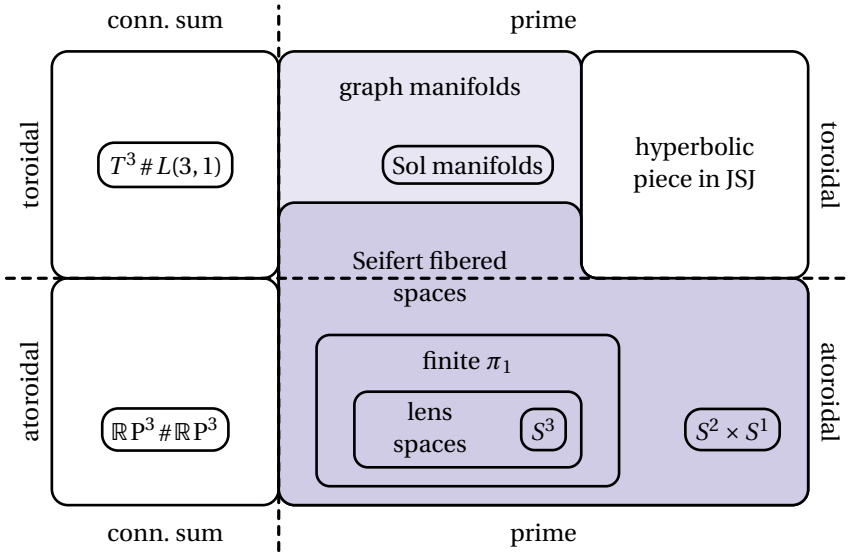


Figure 4. The different types of nonhyperbolic 3-manifolds, and hence different types of exceptional Dehn fillings. See Section 2 for definitions.

neither reducible nor an honorary lens space, and the term Seifert fibered will not include $\mathbb{R}P^3 \# \mathbb{R}P^3$. I will assume familiarity with basic 3-manifold topology, the Geometrization Theorem, and the resulting general structure of 3-manifolds, see e.g. [Hat, Sco, Bon] for details. Throughout, all 3-manifolds will be compact, orientable, and be either closed or have boundary that is a union of tori; the symbol M will always refer to such a manifold.

Our first two kinds of nonhyperbolic 3-manifolds are those containing certain spheres and tori. An embedded 2-sphere in M is *essential* if it does not bound a 3-ball. If there are no essential spheres then M is *irreducible*, and this includes all hyperbolic M. Those M containing *separating* essential spheres are called *connected sums*; here, I avoid the more common term reducible for this as for some authors reducible is the complement of irreducible and so includes $S^2 \times S^1$ whose only essential sphere is nonseparating. When M is not a connected sum it is *prime*. An M is *toroidal* when it contains an embedded essential torus T, that is, one where $\pi_1 T \to \pi_1 M$ is injective and T is not isotopic to a component of ∂M; this is sometimes called *geometrically toroidal*. When M is not toroidal it is *atoroidal*. All hyperbolic M are atoroidal.

When M has a foliation by circles it is *Seifert fibered* and called a *Seifert fibered space*; I will shortly revise this definition to exclude a particularly unusual such manifold. The Seifert fibered manifolds are exactly those admitting these six of the

eight possible geometries: S^3, $\mathbb{E}^3$, $\mathbb{H}^2\times\mathbb{R}$, $S^2\times\mathbb{R}$, Nil, and $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ and in particular are not hyperbolic. Those with spherical geometry are precisely the 3-manifolds with finite π_1, including S^3 itself and the lens spaces $L(p,q)$ which are all quotients of S^3 by a cyclic group. There are only two M with an $S^2\times\mathbb{R}$ geometry, namely $S^2\times S^1$ and $\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$, which are both rather special. First, $S^2\times S^1$ is the only closed 3-manifold with Heegaard genus one that is not a lens space, and also the only one whose fundamental group is infinite cyclic. Second, $\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$ is the unique Seifert fibered manifold that is a connected sum. I henceforth adopt the nonstandard convention that $\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$ is not Seifert fibered; this way, all Seifert fibered spaces are prime.

Any irreducible M has a collection of disjoint essential tori that cut it up into pieces that are either Seifert fibered or hyperbolic. The minimal such collection is unique up to isotopy and gives the *JSJ decomposition* of M. A *graph manifold* is one where all the pieces in the JSJ decomposition are Seifert fibered. So Seifert fibered manifolds are graph manifolds as are those that admit the Sol geometry; the latter are virtually torus bundles over the circle with Anosov monodromy. An irreducible M that is neither hyperbolic nor a graph manifold has a nontrivial JSJ decomposition where at least one piece is hyperbolic; such M have a *hyperbolic piece*.

Figure 4 summarizes all the different types of nonhyperbolic 3-manifolds. Of course, many manifolds satisfy several of these conditions, and in certain tables I will want each nonhyperbolic manifold to have a single type. In such cases, the type used will be the most restricted possible in Figure 4; for example, the type of $L(3,1)$ will be a lens space, even though it also has finite π_1, is Seifert fibered, and is a graph manifold. (This always makes sense because I set up the various definitions to minimize overlaps that are not containments.) This more restrictive convention is used in Tables 2 and Tables 6 only; Table 5 is correct with either convention.

3 Evidence for standard conjectures

A great deal has been proven about the possibilities for exceptional Dehn fillings; with regards to the gaps in our knowledge, the fillings of Theorem 1.2 are consistent with the standard conjectures as I now describe.

3.1 Knots in the 3-sphere. I start with the 1,267 manifolds in $\mathscr{C}_9$ that are exteriors of knots in S^3, which collectively have some 2,615 additional exceptional fillings.

(a) There are no Dehn fillings that are connected sums, consistent with the Cabling Conjecture [Gor3, §2.2].

(b) The Berge Conjecture [Gor3, §3.2] holds for the 178 nontrivial lens space fillings.

(c) All 1,143 nontrivial Seifert fibered fillings are along integral slopes and have the form $S^2(q_1, q_2, q_3)$ or $\mathbb{R}\mathrm{P}^2(q_1, q_2)$; compare [Gor3, §3.3]. In particular, all fillings with finite fundamental group are integral.

We now turn to considering all of the manifolds in $\mathscr{C}_9$.

3.2 Distances between exceptional slopes. A key topological invariant of a pair of slopes α and β on a torus is their geometric intersection number $\Delta(\alpha,\beta)$. When α and β are exceptional slopes for a particular M, then $\Delta(\alpha,\beta) \le 8$ by [LM]. Gordon conjectured there are only four possible M with exceptional slopes where $\Delta \ge 5$ [Gor1, Conjecture 3.4], and this holds for $\mathscr{C}_9$. Much of the work on exceptional fillings has focused on understanding the maximum possible $\Delta(\alpha,\beta)$ where $M(\alpha)$ and $M(\beta)$ are particular types of exceptional fillings. I summarize what is observed for $\mathscr{C}_9$ and how it relates to the known upper bounds on $\Delta(\alpha,\beta)$ in Table 5. In all cases, the maximum value of $\Delta(\alpha,\beta)$ for $\mathscr{C}_9$ is the same as that already found in the literature, compare with [Gor1, Gor2, Gor3, Gor4] and also page 971 and Section A.2 of [MP]. I think it very likely that all possible maximum values of $\Delta(\alpha,\beta)$ have been observed at this point.

3.3 Atoroidal Seifert fibered and finite π_1 fillings. There are two cases of M in $\mathscr{C}_9$ with slopes α and β with $\Delta(\alpha,\beta) = 4$ where $M(\alpha)$ is an atoroidal Seifert fibered space with $\pi_1\big(M(\alpha)\big)$ infinite and $\pi_1\big(M(\beta)\big)$ is finite and noncyclic. These are already contained in [MP], but this aspect is not highlighted there and so is worth describing here. The first example is $m007$ where $m007(-2,1)$ is the Seifert fibered space $S^2\big((2,1),(3,1),(9,-7)\big)$ and $m007(2,1)$ is $S^2\big((2,1),(3,2),(3,-1)\big)$ which has nonabelian fundamental group of order 120; this example is $M3_2$ with slopes -4 and 0 in Table A.3 of [MP]. The second is $m034$ with $m034(2,1) = S^2\big((2,1),(3,1),(11,-9)\big)$ and $m034(-2,1) = S^2\big((2,1),(3,2),(5,-3)\big)$ where the latter has nonabelian fundamental group of order 2,040; it is the example described in Table A.8 [MP], with $r/s = 2$ and slopes -4 and 0. Here, my conventions for describing Seifert fibered spaces follow Regina [BBP$^+$].

3.4 Many exceptional fillings. For a 1-cusped manifold M, let $e(M)$ denote the number of exceptional fillings. The distribution of $e(M)$ is shown in Figure 3. There are only 11 manifolds in $\mathscr{C}_9$ where $e(M) \ge 7$, namely $m003$, $m004$, $m006$, $m007$, $m009$, $m016$, $m017$, $m023$, $m035$, $m038$, and $m039$. According to Gordon [Gor1, pages 136–7], these 11 were first noticed by Hodgson when he examined the 286 manifolds in $\mathscr{C}_5$. Gordon writes there that "In view of this data it is tempting to believe that these eleven manifolds are the only ones with $e(M) \ge 7$", and it was later shown [LM] that one always has $e(M) \le 10$. These 11 are also the only manifolds with $e(M) \ge 7$ among all Dehn fillings on the magic manifold [MP]. In light of the

	S^3	lens	finite	#sum	$S^2 \times S^1$	SFS(A)	tor
S^3	$-\infty$	1	1	$-\infty$	$-\infty$	1	2
lens space	[CGLS]	1	2	1	1	2	3
finite π_1	§3(c)	[BZ1]	3	1	1	4	5
conn. sum	§3(a)	[BZ2]	[BGZ1]	1	$-\infty$	2	3
$S^2 \times S^1$	[Gab2]	[CGLS]	[BGZ1]	[GL3]	$-\infty$	1	2
Seifert (ator.)	§3(c)			[BCSZ]		6	7
toroidal	[GL2]	[Lee2]		[Oh, Wu]	[Lee1]	[BGZ2]	8

Table 5. This table lists the maximum value of $\Delta(\alpha,\beta)$ where $M \in \mathscr{C}_9$ and the fillings $M(\alpha)$ and $M(\beta)$ have the indicated types; here it is implicit that $\alpha \neq \beta$ and $-\infty$ is used when no such pair exists. As these values form a symmetric matrix, only the entries on or above the diagonal are given. Below the diagonal is a reference for the strongest proven upper bound on $\Delta(\alpha,\beta)$ for that pair of types, with references for the diagonal being, in order, [GL1, CGLS, BZ3, GL3, Gab1, LM, Gor]. Cases where the value for $\mathscr{C}_9$ is *strictly smaller* than the best proven upper bound are indicated by the shaded boxes. Thus in the unshaded cases the maximum possible value of $\Delta(\alpha,\beta)$ for *all* 3-manifolds has been established exactly by the indicated reference.

additional data here, it is safe to promote this temptation to a conjecture.

3.5 Connected sums. The connected sums in this census are all built of quite simple pieces. Specifically, the summands all have finite π_1 or are $S^2 \times S^1$; there are only two summands in all but three cases: the filling $o9_{39343}(1,0)$ is $\mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3$ and both $o9_{41447}(1,0)$ and $o9_{43255}(1,0)$ are the manifold $L(3,1)\#\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$. While there are infinite families with two connected sum fillings [EMW], it is an open question whether there is a manifold with three such fillings, see [HM, §4]. In $\mathscr{C}_9$ there are only 14 manifolds with two distinct Dehn fillings that are connected sums, and none with more than two. Another question from [HM, §4] is when there are two such fillings, must both have at least one summand that is $\mathbb{R}\mathrm{P}^3 = L(2,1)$, $L(3,1)$, or $L(4,1)$? The answer is yes for the 14 such manifolds in $\mathscr{C}_9$.

Type of $M(\alpha)$	$\min(\ell(\alpha))$	$\max(\ell(\alpha))$
S^3	1.000	3.323
$S^2 \times S^1$	1.288	3.328
connected sum	1.398	3.707
lens space	1.189	3.928
Sol torus bundle	2.288	4.185
finite π_1	1.520	4.443
Seifert (toroidal)	1.906	4.583
Seifert (atoroidal)	1.935	4.841
graph manifold	2.178	5.318
hyperbolic piece	3.520	6.000

Table 6. This table gives minimum and maximum lengths of each type of exceptional slope in Theorem 1.2 as measured in the torus bounding a maximal cusp. Lengths have been rounded to three decimal places, and here type refers to the most restricted category possible from Figure 4, which is why the $\min(\ell(\alpha))$ for lens spaces is bigger than that for S^3. Compare with [HP, Table 1].

4 New observations

Here are some interesting patterns that I couldn't find in the existing literature. I encourage you to download the complete data at [Dun] and find others that I have missed.

4.1 Finite nonabelian fillings. The maximum number of fillings on M in $\mathcal{C}_9$ where the fundamental group is finite and nonabelian is three. There are only four such M, namely $m011$, $s757$, $v2702$, and $v2797$. I conjecture that these are the only four manifolds with this property.

4.2 Toroidal fillings. The maximum number of toroidal fillings on M in $\mathcal{C}_9$ is 4, and there only 27 such M, namely $s772$, $s778$, $s911$, $v2640$, $t08282$, $t11538$, $t12033$, $t12035$, $t12036$, $t12041$, $t12043$, $t12045$, $t12050$, $t12548$, $t12648$, $o9_{35259}$, $o9_{36732}$, $o9_{37030}$, $o9_{38039}$, $o9_{39094}$, $o9_{40054}$, $o9_{41000}$, $o9_{41004}$, $o9_{41006}$, $o9_{41007}$, $o9_{41008}$, $o9_{43799}$. Are there are infinitely many such examples? Perhaps we should expect there to be since the previous list includes manifolds with 6, 7, 8, and 9 ideal tetrahedra. None of the examples with 4 toroidal Dehn fillings is the exterior of a knot in S^3, consistent with a conjecture of [EM, Page 60].

M	α	$M(\alpha)$	$\lvert\pi_1(M(\alpha))\rvert$	$\ell(\alpha)$
$t05002$	$(-1,1)$	$S^2\big((2,1),(3,2),(3,-1)\big)$	120	4.004139
$o9_{28194}$	$(-1,1)$	$S^2\big((2,1),(3,2),(4,-1)\big)$	528	4.017192
$o9_{35417}$	$(1,1)$	$S^2\big((2,1),(2,1),(8,3)\big)$	352	4.021047
$o9_{35418}$	$(1,1)$	$S^2\big((2,1),(2,1),(11,-3)\big)$	352	4.021047
$v3479$	$(1,1)$	$S^2\big((2,1),(3,2),(5,-1)\big)$	3,480	4.028619
$o9_{36221}$	$(1,1)$	$S^2\big((2,1),(2,1),(10,3)\big)$	520	4.033399
$o9_{36224}$	$(-1,1)$	$S^2\big((2,1),(2,1),(13,-3)\big)$	520	4.033399
$v2420$	$(-1,1)$	$S^2\big((2,1),(3,2),(4,-1)\big)$	528	4.060890
$m342$	$(1,1)$	$S^2\big((2,1),(3,2),(3,-1)\big)$	120	4.067597
$m011$	$(2,1)$	$S^2\big((2,1),(3,2),(5,-3)\big)$	2,040	4.085768
$o9_{41134}$	$(-1,1)$	$S^2\big((2,1),(3,2),(4,-1)\big)$	528	4.184451
$o9_{12592}$	$(-1,1)$	$S^2\big((2,1),(3,2),(5,-3)\big)$	2,040	4.195283
$s954$	$(1,1)$	$S^2\big((2,1),(3,2),(4,-1)\big)$	528	4.207000
$s546$	$(-1,1)$	$S^2\big((2,1),(3,2),(5,-3)\big)$	2,040	4.442966

Table 7. All pairs (M,α) with $M\in\mathcal{C}_9$ and $\pi_1\big(M(\alpha)\big)$ finite where $\ell(\alpha)>4$.

4.3 Lengths of exceptional slopes. Hoffman and Purcell [HP] studied the length of exceptional slopes α in the horotorus cutting off a maximal cusp for M. By the 6-Theorem, the length $\ell(\alpha)$ of such α is at most 6. Table 6 details the longest exceptional slopes of each type observed in $\mathcal{C}_9$; compare with Table 1 of [HP]. The new feature is slopes of length more than 4 yielding manifolds with finite fundamental group; these are listed in Table 7. Can some of these be made into an infinite family of finite exceptional slopes with $\ell(\alpha)\to 5$ analogous to Proposition 4.2 of [HP]?

The referee kindly pointed out that one can use a covering trick to create even longer slopes for some types starting with the examples in Table 7. Specifically, the extreme example for lens spaces comes from $M=o9_{18855}$ and the slope $\alpha=(1,1)$ where $M(\alpha)=L(39,16)$ and $\ell(\alpha)\approx 3.92794$. The core curve of the Dehn filling turns out to generate $H_1(M(\alpha);\mathbb{Z})\cong\mathbb{Z}/39\mathbb{Z}$, and consequently one can take a 39-fold cyclic cover of M to get the exterior of a knot in S^3 whose meridian μ also has $\ell(\mu)\approx 3.92794$. As discussed in [HP], it is conjectured that for an S^3 filling one always has $\ell(\mu)\le 4$, and there are several families of such where $\ell(\mu)\to 4$ from below.

One can apply the same trick to $s546(-1,1)$ from Table 7 to produce a hyperbolic knot in the Poincaré homology sphere where the meridian has length about 4.442966; specifically, take the 17-fold cyclic cover of $s546$ corresponding to the kernel of the map $\pi_1\big(s546(-1,1)\big)\to H_1\big(s546(-1,1);\mathbb{Z}\big)\cong\mathbb{Z}/17\mathbb{Z}$.

4.4 A cabling conjecture for $S^2\times S^1$. As per Table 5, there are no known hyperbolic knot exteriors in $S^2\times S^1$ with a Dehn filling that is a connected sum. Thus, as in

Section 3(a) for the case of S^3, I conjecture that none exist, i.e. that no hyperbolic knot in $S^2 \times S^1$ has a Dehn surgery yielding a connected sum.

5 Outline of the proof of Theorem 1.2

I turn now to the proof of Theorem 1.2. Initially, I found a candidate $\mathscr{E}$ for the list of all exceptional fillings as a byproduct of another project. However, the proof of the correctness of $\mathscr{E}$ follows the approach of [MPR]. The list $\mathscr{E}$, related data, and the code used in the proof can all be obtained from [Dun]; to run the code, which requires using several software packages together in consort, the Docker image [ComTop] may be helpful.

Proof of Theorem 1.2. The set $\mathscr{E}$ consists of 205,822 pairs (M,α) where $M \in \mathscr{C}_9$ and $\alpha \in Sl(M)$. There are two things to show: that every $M(\alpha)$ is not hyperbolic of the type claimed in Table 2 and that all other fillings on $M \in \mathscr{C}_9$ are hyperbolic.

For the latter task, for each $M \in \mathscr{C}_9$ I found an embedded cusp neighborhood so that I could measure the lengths of slopes in its horotorus boundary; this was done rigorously in SnapPy [CDGW] running inside [Sage] using the approach of [HIKMOT] and [DHL, §3.6]. By the 6-Theorem of [Ago, Lac], it suffices to examine all slopes $\beta \in Sl(M)$ where $\ell(\beta) \leq 6$. For the cusp neighborhoods I used, overall there were some 355,128 such slopes. For the 149,306 pairs (M,β) that were not in $\mathscr{E}$, I checked that $M(\beta)$ was hyperbolic using the method of [HIKMOT] as reimplemented in SnapPy. As in the proof of Theorem 5.2 of [HIKMOT], it was sometimes necessarily to search around for a triangulation that could be used to certify the existence of a hyperbolic structure. This completes the proof that filling along any slope not in $\mathscr{E}$ yields a hyperbolic manifold.

In the other direction, to show that each $M(\alpha)$ in $\mathscr{E}$ is not hyperbolic, I primarily used Regina [BBP$^+$], specifically its combinatorial recognition methods [Bur1, §4]. These work when the input triangulation has the very particular form associated to a standard triangulation of a Seifert fibered space or graph manifold. Of course, there are many triangulations of such manifolds which do not have this structure, so I generated many different 1-vertex triangulations of each $M(\alpha)$ and fed them into Regina until it succeeded in recognizing the topology. This worked for all but 2,890 of the $M(\alpha)$. Of those remaining, in 680 cases the Recognizer program of [Mat, MT] showed that they were graph manifolds. (Currently, Regina can only identify graph manifolds where the graph in question is either a segment with two or three vertices or a loop with one vertex. These 680 all have slightly more complicated graphs, for example a loop with either two or three vertices.) For each of the remaining 2,210 manifolds, Regina found at least one essential normal torus. Cutting along a

suitable collection of such essential tori gave pieces that always included a cusped hyperbolic 3-manifold with an ideal triangulation with at most 6 ideal tetrahedra; in particular, each of these 2,210 manifolds is nonhyperbolic with a non-trivial JSJ decomposition with a hyperbolic piece. Thus every $M(\alpha)$ in $\mathscr{E}$ is not hyperbolic. This completes the proof that $\mathscr{E}$ is precisely the list of exceptional fillings on the manifolds in $\mathscr{C}_9$.

To prove the correctness of Table 2, the hard part is ensuring that nothing listed as a (proper) graph manifold is actually Seifert fibered. Everything else can be read off from the Seifert/graph descriptions found in the previous step, though I double-checked much of it in other ways. For example, I used Magma [BCP] to give an independent check that the 59,200 spherical manifolds had the claimed type of fundamental group (this could also be done with GAP [GAP]). I also had Regina compute directly which manifolds are toroidal using normal surface techniques and this matched what follows from the Seifert/graph descriptions. As mentioned, Regina identifies structure in the given triangulation, which might well be a graph manifold structure that can be simplified after the fact. For example, it will sometimes return graph manifolds where one of the nodes is a solid torus. In such instances, additional triangulations were examined until a more concise description was found. To certify a graph description as minimal, I just checked that all Seifert pieces have incompressible boundary (i.e. no solid tori) and that no two Seifert pieces are glued together so that the fibers match up; here, I took care to consider the possibility of switching the Seifert fibration for the exceptional piece which is both $D^2\big((2,1),(2,1)\big)$ and the twisted circle bundle over the Möbius band. □

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