SUBCONVEXITY BOUNDS FOR TWISTED L-FUNCTIONS

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Abstract. We prove hybrid subconvexity bounds for a wide class of twisted L-functions $L(s, f\times\chi)$ at the central point, including a new instance of the Weyl subconvexity bound.

1. Introduction

The subconvexity problem is an important problem in the theory of L-functions concerned with bounding an automorphic L-function $L(s)$ on the critical line $\Re(s) = \frac{1}{2}$. A bound which can readily be obtained using essentially only the functional equation of the L-function is called the convexity bound and is given by

$$L(\tfrac{1}{2} + it) \ll \mathfrak{q}^{\kappa+\epsilon}, \tag{1.1}$$

where $\mathfrak{q}$ is the analytic conductor of the L-function [20, Section 5.1], $\kappa = \frac{1}{4}$ and $\epsilon > 0$ is arbitrarily small. The subconvexity problem is to establish (1.1) with some number κ strictly less than $\frac{1}{4}$. The best possible expectation, essentially, is that we may take $\kappa = 0$, and this is the famous Lindelöf conjecture. The subconvexity bound for $GL(1)$ L-functions is classical, and for $GL(2)$ automorphic L-functions it was completely resolved by Michel and Venkatesh [24]. However it is of great interest to not only break the convexity bound, but to reach certain milestones called the Burgess exponent ($\kappa = \frac{3}{16}$) and the Weyl exponent ($\kappa = \frac{1}{6}$). These are named after classical results in $GL(1)$, but also occur in higher rank. It is unknown why the Burgess and Weyl exponents occur in so many unrelated situations, but in all cases they seem to represent natural barriers which can require deep ideas to overcome.

We mention a few examples in $GL(1)$ and $GL(2)$ at the central point $s = \frac{1}{2}$. For Dirichlet L-functions of modulus q, the analytic conductor is equal to q. Burgess [10] proved the bound $L(\frac{1}{2}, \chi) \ll q^{\frac{3}{16}+\epsilon}$, Conrey and Iwaniec [13] proved the Weyl bound $q^{\frac{1}{6}+\epsilon}$ for real characters, Petrow and Young [29, 28] proved the Weyl bound for all Dirichlet characters, and Milićević [25] proved a sub-Weyl bound in the case of Dirichlet characters of prime power modulus p^n with n large. Turning now to $GL(2)$, let f be a holomorphic Hecke cusp form or a Hecke Maass cusp form for the full modular group, with associated L-function $L(f, s)$. The Weyl subconvexity bound was proven for $L(\frac{1}{2}, f)$ in the spectral and weight aspects by Ivić [18] and Peng [27] respectively. If f is a newform of level q, the analytic conductor of $L(s, f)$ in the level aspect is q. For f with trivial nebentypus, Duke, Friedlander and Iwaniec [14] proved the first subconvexity bound $L(\frac{1}{2}, f) \ll q^{\frac{1}{4}-\frac{1}{192}+\epsilon}$. This has seen some recent improvements. The current best result for q an odd prime is $\kappa = \frac{1}{5}$ conditionally on the Ramanujan conjecture, by Kiral and Young [22] and Blomer and Khan [6]. Unconditionally it is $\kappa = 0.2073$ by Blomer, Humphries, Khan, Milinovich [5]. For newforms with primitive nebentypus, the best result is due to Blomer and Khan [7], with exponent $\kappa = \frac{1}{4} - \frac{1}{128}$.

It is highly frustrating that the subconvexity exponent in the level aspect does not attain even the Burgess milestone, despite better results in other $GL(2)$ aspects. There are however exceptions for special forms which arise as twists. For twists of a form f of *fixed* level by a primitive Dirichlet character χ of modulus q, the Burgess bound for $L(\frac{1}{2}, f \times \chi)$ in the q aspect is known by the results of Bykovskiĭ [12] and Blomer, Harcos, and Michel [4, 2], and the Weyl bound is known when χ is real

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by work of Petrow and Young [30]. The Weyl bound was also established for twists by characters of prime power moduli by Blomer and Milićević [9] and Munshi and Singh [26]. Outstanding results also exist for twists of forms whose level is not fixed. Conrey and Iwaniec [13] proved the Weyl bound for $L(1/2, f \times \chi)$ for newforms f of level q and trivial nebentypus, twisted by real primitive characters χ of modulus q (see also [31]). Petrow and Young [29] generalized this result to allow any primitive character χ of modulus q as long as f is a newform of level q and nebentypus $\overline{\chi}^2$. The works of Iwaniec and Conrey and of Petrow and Young are particularly significant because they also lead to the Weyl subconvexity bound for Dirichlet L-functions.

The goal of this paper is to prove subconvexity bounds for a wide class of twisted L-functions, including a new instance of the Weyl subconvexity bound. We restrict to prime level and modulus in order to minimize technical details and concentrate more on the main ideas.

Theorem 1.1. *Let $\epsilon > 0$. Let f be a holomorphic newform with weight $k \geq 2$, prime level q and trivial nebentypus. Let χ be a primitive Dirichlet character of prime modulus p. Suppose that $p^{\epsilon} \leq q \leq p^{2+\epsilon}$ and $(q, p) = 1$. We have*

$$L(\tfrac{1}{2}, f \times \chi) \ll_{k,\epsilon} p^{\epsilon}(q^{\frac{1}{2}} + p^{\frac{1}{2}}).$$

By Atkin-Lehner theory [20, Proposition 14.20], $L(s, f \times \chi)$ is the L-function attached to a newform of level qp^2 and nebentypus χ^2. Thus the result above gives the convexity bound or better in all cases, a subconvexity bound when $q \leq p^{2-\delta}$ for some $\delta > 0$, the Burgess subconvexity bound or better for $p^{\frac{2}{3}-\epsilon} \leq q \leq p^{\frac{6}{5}+\epsilon}$, and the Weyl subconvexity bound for $p^{1-\epsilon} \leq q \leq p^{1+\epsilon}$, which is the best our result can do. This last bound may be viewed as complementary to the work of Conrey and Iwaniec [13] and Petrow and Young [29]. They obtained the Weyl bound when the level of the form and modulus of the Dirichlet character coincide, while we do the same when the level and modulus are of the same size, but coprime.

Theorem 1.1 is a type of hybrid subconvexity bound, since it can yield subconvexity in both parameters p and q simultaneously. Such hybrid bounds are of great interest and have been studied by Blomer and Harcos [2, 3], Aggarwal, Jo, and Nowland [1], and Chen and Hou [17]. Of these results, the last one yields subconvexity for the widest range: for q as large as $p^{\frac{3}{2}-\delta}$, but it falls short of any milestone subconvexity bound. Our result works in an even wider range, and gives much stronger bounds when q and p are about the same size. Moreover our proof is much simpler and completely avoids the shifted convolution problem, which was central in the aforementioned works on hybrid subconvexity. Theorem 1.1 follows immediately by establishing the following second moment estimate. This an analogue of a bound of Holowinsky and Templier [16, Theorem 1] on the first moment of Rankin-Selberg $GL(2) \times GL(2)$ L-functions in the level aspect.

Theorem 1.2. *Keep the notation in the statement of Theorem 1.1. Let $B_k^{\star}(q)$ denote a basis of holomorphic newforms of level q and weight $k \geq 2$, with trivial nebentypus. We have*

$$\sum_{f \in B_k^{*}(q)} |L(\tfrac{1}{2}, f \times \chi)|^2 \ll_{k,\epsilon} p^{\epsilon}(q + p) \tag{1.2}$$

for any $\epsilon > 0$.

For comparison, the large sieve [20, Theorem 7.26] would give a bound of $O(p^{\epsilon}(q + q^{\frac{1}{2}}p))$ for the second moment. In connection to this, we mention a further application that our work may have. In [11, Theorem 1.2], Buttcane and Khan proved the first moment bound

$$\sum_{f \in B_{2k}^{*}(q)} L(\tfrac{1}{2}, \mathrm{sym}^2 g \times f) \ll_{k,\epsilon} q^{1+\epsilon}, \tag{1.3}$$

where $g \in B_k^{*}(q)$ and k is large enough in terms of ϵ. Liu, Masri, and Young [23, page 16] observed that if (1.3) can be extended to small k, which may be possible using current technology, then the large sieve estimate for the left hand side of (1.2) would imply a hybrid subconvexity bound for $L(\frac{1}{2}, f \times \Theta_{\chi})$ that is uniform in *all* ranges of q and p, where Θ_{χ} is the theta series associated to an

ideal class group character of $\mathbb{Q}(\sqrt{-p})$. We note that using our Theorem 1.2 instead of the large sieve would give a stronger subconvexity bound for $L(\frac{1}{2}, f \times \Theta_\chi)$.

Throughout, we follow the ϵ-convention: that is, ϵ will always be positive number which can be taken as small as we like, but may differ from one occurence to another. All implied constants may depend on ϵ and k.

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2. Preliminaries

2.1. **Poisson summation.** One of the basic tools we will use is the Poisson summation formula. We state it here.

Lemma 2.1. *Let ψ be a function in the Schwartz class. We have*

$$\sum_{\substack{n\in\mathbb{Z}\\ n\equiv a \bmod r}} \psi(n) = \frac{1}{r}\sum_{m\in\mathbb{Z}} e\Big(\frac{am}{r}\Big)\hat{\psi}\Big(\frac{m}{r}\Big),$$

where $\hat{\psi}(\xi) = \int_{-\infty}^{\infty} \psi(t)e(-t\xi)dt$ and $e(x) = e^{2\pi i x}$.

2.2. **Modular forms and L-functions.** Let $S_k(q)$ denote the space of holomorphic cusp forms of prime level q, weight k, and trivial nebentypus. Let $S_k^*(q) \subset S_k(q)$ denote the space of newforms. Every $f \in S_k(q)$ has a Fourier series expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e(nz)$$

for $\Im(z) > 0$. Let $B_k(q)$ denote an orthogonal basis of $S_k(q)$ which contains a basis $B_k^\star(q)$ of $S_k^*(q)$, normalized so that $a_f(1) = 1$ for every $f \in B_k^*(q)$.

Let χ be primitive Dirichlet character of prime modulus p. For $f \in B_k(q)$, define

$$(f \times \chi)(z) = \sum_{n=1}^{\infty} \chi(n) a_f(n) n^{\frac{k-1}{2}} e(nz).$$

For $f \in B_k^\star(q)$, this is a newform of level qp^2, weight k, and nebentypus χ^2. The associated L-function is entire and for $\Re(s) > 1$ equals

$$L(s, f \times \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a_f(n)}{n^s}.$$

This satisfies the functional equation

$$\Lambda(s, f \times \chi) := \Big(\frac{pq^{\frac{1}{2}}}{2\pi}\Big)^s \Gamma(s + \tfrac{k-1}{2}) L(s, f \times \chi) = \varepsilon \Lambda(1 - s, f \times \overline{\chi}), \tag{2.1}$$

for some complex number ε of modulus 1, which depends on q, p, and χ. Thus the analytic conductor at $s = \frac{1}{2}$ is kp^2q. These facts may be found in [20, Section 14]. Let $L(s, \mathrm{sym}^2 f)$ denote the symmetric square L-function attached to $f \in B_k^*(q)$. This is defined in [20, Section 5.12], but all we need are the bounds

$$(kq)^{-\epsilon} \ll L(1, \mathrm{sym}^2 f) \ll (kq)^{\epsilon} \tag{2.2}$$

given in [15]. In particular, $L(1, \mathrm{sym}^2 f)$ is positive.

2.3. **Approximate functional equation.** We will need the following consequence of the approximate functional equation [20, Theorem 5.3].

Lemma 2.2. *We have the bound*

$$|L(\tfrac{1}{2}, f\times\chi)|^2 \ll \Big|\sum_{n\geq 1}\frac{a_f(n)\chi(n)}{n^{\frac{1}{2}}}V\Big(\frac{n}{q^{\frac{1}{2}}p}\Big)\Big|^2 + \Big|\sum_{n\geq 1}\frac{a_f(n)\overline{\chi}(n)}{n^{\frac{1}{2}}}V\Big(\frac{n}{q^{\frac{1}{2}}p}\Big)\Big|^2, \tag{2.3}$$

where we define the real function

$$V(x) = \frac{1}{2\pi i}\int_{(2)}(2\pi x)^{-u}\frac{\Gamma(\frac{k}{2}+u)}{\Gamma(\frac{k}{2})}\frac{du}{u}$$

for $x > 0$.

By a standard argument of shifting contours, we may restrict the sum above to $n \leq q^{\frac{1}{2}}p^{1+\epsilon}$, up to an error of $O(p^{-100})$. Further, by splitting the n-sum in (2.3) into $O(\log q)$ sums over dyadic intervals and applying the Cauchy-Schwarz inequality, for the purposes of Theorem 1.2 it suffices to prove

$$\frac{1}{N}\sum_{f\in B_k^*(q)}\Big|\sum_{n\geq 1}\chi(n)a_f(n)V\Big(\frac{n}{N}\Big)\Big|^2 \ll p^{\epsilon}(q+p) \tag{2.4}$$

for any

$$1 \leq N \leq q^{\frac{1}{2}}p^{1+\epsilon}$$

and any smooth function V compactly supported on the positive reals with $\|V^{(j)}\|_{\infty} \ll_j (p^{\epsilon})^{j+1}$ for all $j \geq 0$.

2.4. **Petersson trace formula.** The Petersson trace formula [21, Proposition 2.1] states that

$$\sum_{f\in B_k(q)}\frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f,f\rangle}a_f(n_1)\overline{a_f(n_2)} = \delta_{(n_1=n_2)} + 2\pi i^{-k}\sum_{c\geq 1}\frac{S(n_1,n_2,cq)}{cq}J_{k-1}\Big(\frac{4\pi\sqrt{n_1n_2}}{cq}\Big), \tag{2.5}$$

where $\langle f,f\rangle$ is the Petersson inner product, $S(n_1,n_2,cq)$ is a Kloosterman sum, $J_{k-1}(x)$ is a J-Bessel function, and δ_P equals 1 if the statement P is true, and 0 otherwise.

By [21, equations (2.3, 2.24)], we have

$$\frac{2\pi^2}{q(k-1)L(1,\mathrm{sym}^2 f)} = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f,f\rangle}$$

for $f \in B_k^*(q)$. This combined with the bound (2.2) means that for (2.4), it suffices to prove

$$\frac{1}{N}\sum_{f\in B_k^*(q)}\frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f,f\rangle}\Big|\sum_{n\geq 1}\chi(n)a_f(n)V\Big(\frac{n}{N}\Big)\Big|^2 \ll p^{\epsilon}\Big(1+\frac{p}{q}\Big).$$

By positivity, we may enlarge $B_k^*(q)$ to $B_k(q)$, expand out the square $|\sum_{n\geq 1}\dots|^2$, and apply the Petersson trace formula (2.5). After doing so, we need to prove

$$\Big|\frac{1}{N}\sum_{\substack{n_1,n_2\geq 1\\ n_1=n_2}}V\Big(\frac{n_1}{N}\Big)V\Big(\frac{n_2}{N}\Big)\Big| + \Big|\frac{1}{N}\sum_{n_1,n_2,c\geq 1}\frac{S(n_1,n_2,cq)\chi(n_1)\overline{\chi}(n_2)}{cq}V\Big(\frac{n_1}{N}\Big)V\Big(\frac{n_2}{N}\Big)J_{k-1}\Big(\frac{4\pi\sqrt{n_1n_2}}{cq}\Big)\Big| \ll p^{\epsilon}\Big(1+\frac{p}{q}\Big). \tag{2.6}$$

The contribution of the 'diagonal' $\sum_{n_1=n_2}$ is obviously $O(p^{\epsilon})$. For the off-diagonal we first note the following.

Lemma 2.3. *For $x \le p^\epsilon$ we have*

$$J_{k-1}(4\pi x) = xW_1(x)$$

where $W_1(x)$ is a smooth function satisfying $\|W_1^{(j)}\|_\infty \ll_j (p^\epsilon)^{j+1}$ for all $j \ge 0$ and x in this range.
For $x > p^\epsilon$, we have

$$J_{k-1}(4\pi x) = \Re\Big(\frac{e(2x)}{\sqrt{x}} W_2(x)\Big), \tag{2.7}$$

where $\Re$ denotes the real part and W_2 is a smooth function satisfying

$$x^j W_2^{(j)}(x) \ll_{j,\epsilon} (p^\epsilon)^{j+1}$$

for all $j \ge 0$ and x in this range.

Proof. When $x \le p^{-\epsilon}$, use the power series [19, page 82],

$$J_{k-1}(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (x/2)^{k-1+2\ell}}{\ell!(k+\ell-1)!}.$$

When $x > p^{-\epsilon}$, use [21, section 4]. □

2.5. **Conclusion of the set-up.** We now apply Lemma 2.3 to (2.6). The extra factors $\frac{\sqrt{n_1 n_2}}{cq}$ or $(\frac{\sqrt{n_1 n_2}}{cq})^{-1/2}$ can be written as $\frac{N}{cq}$ and $(\frac{N}{cq})^{-1/2}$ after redefining $V(\frac{n_1}{N})$ and $V(\frac{n_2}{N})$. By considering the real and imaginary parts of W_2 separately, we can assume that it is a real function. Thus it suffices to prove, for each sign $\pm$, that

$$\mathcal{S}_1 + \mathcal{S}_2^{\pm} \ll p^\epsilon\Big(1 + \frac{p}{q}\Big),$$

where

$$\mathcal{S}_1 = \frac{1}{C_1^2 q^2} \sum_{C_1 \le c \le 2C_1} \Big| \sum_{n_1,n_2 \ge 1} S(n_1,n_2,cq)\chi(n_1)\overline{\chi}(n_2) V\Big(\frac{n_1}{N}, \frac{n_2}{N}\Big)\Big|, \tag{2.8}$$

$$\mathcal{S}_2^{\pm} = \frac{1}{N^{\frac{3}{2}} q^{\frac{1}{2}} C_2^{\frac{1}{2}}} \sum_{C_2 \le c \le 2C_2} \Big| \sum_{n_1,n_2 \ge 1} S(n_1,n_2,cq)\chi(n_1)\overline{\chi}(n_2) e\Big(\frac{\pm 2\sqrt{n_1 n_2}}{cq}\Big) V\Big(\frac{n_1}{N}, \frac{n_2}{N}\Big)\Big|, \tag{2.9}$$

for any

$$C_1 \ge \frac{N}{p^\epsilon q}, \qquad 1 \le C_2 \le \frac{N}{p^\epsilon q}, \tag{2.10}$$

and where we set

$$V(x_1,x_2) := V_i(x_1,x_2) = V(x_1)V(x_2)W_i\Big(\frac{N\sqrt{x_1 x_2}}{cq}\Big)$$

for $i = 1, 2$. We drop the subscript from V because all we need is that V is compactly supported on $\mathbb{R}^+ \times \mathbb{R}^+$ and satisfies

$$\frac{\partial^i}{\partial x_1^i}\frac{\partial^j}{\partial x_2^j} V(x_1,x_2) \ll_j (p^\epsilon)^{1+i+j}$$

for all $i, j \ge 0$.

3. Proof of Theorem 1.2

The goal now is to prove the required bounds for $\mathcal{S}_1$ and $\mathcal{S}_2^{\pm}$.

3.1. **Poisson summation in n_1.** By separating n_1 into residue classes modulo cpq, and applying Poisson summation, we get that

$$\mathcal{S}_1 = \frac{N}{C_1^3 q^3 p} \sum_{C_1 \le c \le 2C_1} \Big| \sum_{n_2 \ge 1} \sum_{m_1 \in \mathbb{Z}} \sum_{a \bmod cpq} S(a, n_2, cq)\chi(a)\overline{\chi}(n_2) e\Big(\frac{am_1}{cpq}\Big) \mathcal{V}_1\Big(m_1, \frac{n_2}{N}\Big)\Big|, \tag{3.1}$$

$$\mathcal{S}_2^{\pm} = \frac{1}{N^{\frac{1}{2}} C_2^{\frac{3}{2}} q^{\frac{3}{2}} p} \sum_{C_2 \le c \le 2C_2} \Big| \sum_{n_2 \ge 1} \sum_{m_1 \in \mathbb{Z}} \sum_{a \bmod cpq} S(a, n_2, cq)\chi(a)\overline{\chi}(n_2) e\Big(\frac{am_1}{cpq}\Big) \mathcal{V}_2^{\pm}\Big(m_1, \frac{n_2}{N}\Big)\Big|, \tag{3.2}$$

where

$$\mathcal{V}_1(m_1, y) = \int_{-\infty}^{\infty} e\Big(-\frac{xm_1N}{cpq}\Big) V(x, y)dx, \tag{3.3}$$

$$\mathcal{V}_2^{\pm}(m_1, y) = \int_{-\infty}^{\infty} e\Big(\frac{\pm 2\sqrt{xyN^2}}{cq}\Big) e\Big(-\frac{xm_1N}{cpq}\Big) V(x, y)dx. \tag{3.4}$$

Lemma 3.1. *Suppose $c \asymp C_1$ in $\mathcal{V}_1(m_1, y)$ and $c \asymp C_2$ in $\mathcal{V}_2(m_1, y)$. Suppose $m_1 \neq 0$. We have*

$$\mathcal{V}_1(m_1, y) = W(y)\delta_{(|m_1| \le \frac{C_1 q p^{1+\epsilon}}{N})} + O(p^{-100}),$$

$$\mathcal{V}_2^{+}(m_1, y) = \Big(\frac{cq}{N}\Big)^{\frac{1}{2}} e\Big(\frac{ypN}{cqm_1}\Big) W(y)\delta_{(p^{1-\epsilon} \le m_1 \le p^{1+\epsilon})} + O(p^{-100}),$$

$$\mathcal{V}_2^{-}(m_1, y) = \Big(\frac{cq}{N}\Big)^{\frac{1}{2}} e\Big(\frac{ypN}{cqm_1}\Big) W(y)\delta_{(-p^{1+\epsilon} \le m_1 \le -p^{1-\epsilon})} + O(p^{-100}),$$

where $W(y)$ is a smooth compactly supported function on the positive reals, depending on $\pm, c, p, q, N, m_1$, with $\|W^{(j)}\|_\infty \ll_j (p^\epsilon)^j$ for all $j \ge 0$.

Proof. For $\mathcal{V}_1(m_1, y)$, we are simplying renaming this function to $W(y)$. It's clear from the expression (3.3) that W satisfies the required properties. Further, by integrating by parts multiple times in (3.3), integrating $e(-\frac{xm_1N}{cpq})$ and differentiating the rest of the integrand, we get that (3.3) is $O(p^{-100})$ unless

$$|m_1| \le \frac{C_1 q p^{1+\epsilon}}{N}.$$

For $\mathcal{V}_2^{\pm}(m_1, y)$, we apply a stationary phase approximation. Let

$$h(x) = h_{\pm}(x) = 2\pi\Big(\frac{\pm 2\sqrt{xyN^2}}{cq} - \frac{xm_1N}{cpq}\Big)$$

be the phase of the integrand.

When $|m_1| < p^{1-\epsilon}$ or $|m_1| > p^{1+\epsilon}$, we have $|h'(x)| \gg \frac{N}{cq} \gg p^\epsilon$. We can apply [8, Lemma 8.1] with the parameters $R = Y = \frac{N}{cq}$, $X = Q = U = p^\epsilon$ to see that the integral in (3.4) is $O(p^{-100})$.

Now suppose $p^{1-\epsilon} \le |m_1| \le p^{1+\epsilon}$. The stationary point x_0 satisfies $h'(x_0) = 0$, which implies

$$m_1\sqrt{x_0} = \pm p\sqrt{y}. \tag{3.5}$$

In the + case, there is a unique solution when $p^{1-\epsilon} \le m_1 \le p^{1+\epsilon}$. In the − case, there is a unique solution when $-p^{1+\epsilon} \le m_1 \le -p^{1-\epsilon}$. We show the details for $\mathcal{V}_2^{+}$ and $m_1 > 0$ only. From (3.5), we get that

$$x_0 = \frac{p^2 y}{m_1^2}.$$

We have

$$h(x_0) = 2\pi\frac{ypN}{cqm_1}$$

and

$$h''(x_0) = \frac{-m_1^3}{2yp^3}\frac{N}{cq}.$$

Applying [8, Proposition 8.2] with the parameters $Q = 1$, $X = V = V_1 = p^\epsilon$, $Y = \frac{N}{cq}$, $\delta = \frac{1}{9}$, we get that

$$\mathcal{V}_+(m_1, y) = \Big(\frac{cq}{N}\Big)^{\frac{1}{2}} e\Big(\frac{ypN}{cqm_1}\Big) W(y)\delta_{(p^{1-\epsilon}\leq m_1\leq p^{1+\epsilon})} + O(p^{-100}),$$

where $W(y)$ is as described in the statement of the lemma. The δ function indicates the range of m_1, but anyway this information is contained in $W(y)$. □

We now evaluate the arithmetic part in (3.1) and (3.2). We first observe that we can restrict to $(c, p) = 1$ in (2.8) and (2.9), up to an admissible error. This is because in (2.9) we have $c \ll \frac{N}{p^\epsilon q} \ll \frac{p^{1+\epsilon}}{q^{\frac{1}{2}}}$ and so c is too small to be divisible by p. This uses the assumption $q \geq p^\epsilon$. In (2.8), we can trivially bound the terms with $p|c$, using Weil's bound for the Kloosterman sum, to see that their contribution is $O(\frac{p^{1/2+\epsilon}}{q^{1/2}})$, which is dominated by the required bound in (2.6). Having restricted to $(c, p) = 1$, we can write the complete sum mod cpq in terms of complete sums mod cq and mod p. We get, writing $\sum^*$ to denote a sum over primitive residue classes and a bar over a residue class to mean the inverse of that residue class,

$$\begin{aligned}\sum_{a \bmod cpq} S(a, n_2, cq)\chi(a)e\Big(\frac{am_1}{cpq}\Big) &= \sum_{\substack{a_1 \bmod cq\\ a_2 \bmod p}} S(a_1, n_2, cq)\chi(a_2)e\Big(\frac{a_1m_1\overline{p}}{cq}\Big)e\Big(\frac{a_2m_1\overline{cq}}{p}\Big)\\ &= \sum_{\substack{a_1 \bmod cq\\ a_2 \bmod p}} \sum_{b \bmod cq}^{*} e\Big(\frac{a_1\overline{b} + n_2b}{cq}\Big)\chi(a_2)e\Big(\frac{a_1m_1\overline{p}}{cq}\Big)e\Big(\frac{a_2m_1\overline{cq}}{p}\Big)\\ &= cq\tau(\chi)\delta_{(m_1,cq)=1}e\Big(\frac{-n_2p\overline{m_1}}{cq}\Big)\overline{\chi}(m_1)\chi(cq), \qquad (3.6)\end{aligned}$$

where $\tau(\chi)$ is the Gauss sum. The final equality uses [20, equation (3.12)] and the orthogonality of additive characters [20, page 44].

After applying (3.6) and Lemma 3.1 to (3.1) and (3.2), and using that $|\tau(\chi)| = p^{\frac{1}{2}}$, we see that it suffices to prove

$$\frac{N}{C_1^2q^2p^{\frac{1}{2}}} \sum_{C_1\leq c\leq 2C_1} \sum_{\substack{|m_1|\leq \frac{C_1qp^{1+\epsilon}}{N}\\ (m_1,cq)=1}} \Big|\sum_{n_2\geq 1} e\Big(\frac{-n_2p\overline{m_1}}{cq}\Big)\overline{\chi}(n_2)W\Big(\frac{n_2}{N}\Big)\Big| \ll p^\epsilon\Big(1 + \frac{p}{q}\Big), \qquad (3.7)$$

$$\frac{1}{Np^{\frac{1}{2}}} \sum_{C_2\leq c\leq 2C_2} \sum_{\substack{p^{1-\epsilon}\leq |m_1|\leq p^{1+\epsilon}\\ (m_1,cq)=1}} \Big|\sum_{n_2\geq 1} e\Big(\frac{-n_2p\overline{m_1}}{cq}\Big)e\Big(\frac{n_2p}{cqm_1}\Big)\overline{\chi}(n_2)W\Big(\frac{n_2}{N}\Big)\Big| \ll p^\epsilon\Big(1 + \frac{p}{q}\Big). \qquad (3.8)$$

3.2. **Reciprocity.** This is a key step. Using reciprocity, or the Chinese remainder theorem, for $(m_1, cq) = 1$, we have

$$e\Big(\frac{n_2p}{cqm_1}\Big) = e\Big(\frac{n_2p\overline{m_1}}{cq}\Big)e\Big(\frac{n_2p\overline{cq}}{m_1}\Big).$$

Thus (3.7) and (3.8) become

$$\frac{N}{C_1^2q^2p^{\frac{1}{2}}}\sum_{C_1\le c\le 2C_1}\sum_{\substack{|m_1|\le\frac{C_1qp^{1+\epsilon}}{N}\\(m_1,cq)=1}}\Big|\sum_{n_2\ge 1}e\Big(\frac{n_2p\overline{cq}}{m_1}\Big)e\Big(\frac{-n_2p}{cqm_1}\Big)\overline{\chi}(n_2)W\Big(\frac{n_2}{N}\Big)\Big|\ll p^{\epsilon}\Big(1+\frac{p}{q}\Big), \tag{3.9}$$

$$\frac{1}{Np^{\frac{1}{2}}}\sum_{C_2\le c\le 2C_2}\sum_{\substack{p^{1-\epsilon}\le|m_1|\le p^{1+\epsilon}\\(m_1,cq)=1}}\Big|\sum_{n_2\ge 1}e\Big(\frac{n_2p\overline{cq}}{m_1}\Big)\overline{\chi}(n_2)W\Big(\frac{n_2}{N}\Big)\Big|\ll p^{\epsilon}\Big(1+\frac{p}{q}\Big). \tag{3.10}$$

3.3. **Poisson summation in n_2.**

3.3.1. *The case of large c.* We first consider the left hand side (3.9). Observe that we can add the condition $(m_1,p)=1$ to the sum, because the total contribution of the terms with $p|m_1$ is trivially $O(\frac{p^{1/2+\epsilon}}{q^{1/2}})$. The following notation will assume that m_1 is positive, with the case of negative m_1 being entirely similar. Define

$$\mathcal{W}(m_2)=\int_{-\infty}^{\infty}e\Big(\frac{-ym_2N}{m_1p}\Big)e\Big(\frac{-yNp}{cqm_1}\Big)W(y)dy, \tag{3.11}$$

and keep in mind that W is compactly supported. By separating n_2 into residue classes modulo m_1p and appying Poisson summation, we get that

$$\begin{aligned}
&\sum_{n_2\ge 1}e\Big(\frac{n_2p\overline{cq}}{m_1}\Big)\overline{\chi}(n_2)e\Big(\frac{-n_2p}{cqm_1}\Big)W\Big(\frac{n_2}{N}\Big)\\
&=\frac{N}{m_1p}\sum_{b \bmod m_1p}\sum_{m_2\in\mathbb{Z}}e\Big(\frac{bp\overline{cq}}{m_1}\Big)\overline{\chi}(b)e\Big(\frac{bm_2}{m_1p}\Big)\mathcal{W}(m_2)\\
&=\frac{N}{m_1p}\sum_{\substack{b_1 \bmod m_1\\ b_2 \bmod p}}\sum_{m_2\in\mathbb{Z}}e\Big(\frac{b_1p\overline{cq}}{m_1}\Big)\overline{\chi}(b_2)e\Big(\frac{b_1m_2\overline{p}}{m_1}\Big)e\Big(\frac{b_2m_2\overline{m_1}}{p}\Big)\mathcal{W}(m_2)\\
&=\frac{N\tau(\overline{\chi})}{p}\sum_{\substack{m_2\in\mathbb{Z}\\ cm_2\equiv p^2\overline{q} \bmod m_1}}\overline{\chi}(m_1)\chi(m_2)\mathcal{W}(m_2).
\end{aligned} \tag{3.12}$$

By integrating by parts multiple times in (3.11), we find that $\mathcal{W}(m_2)\ll p^{-100}$ unless

$$\Big|\frac{m_2N}{m_1p}+\frac{Np}{cqm_1}\Big|\le p^{\epsilon},$$

in which case $\mathcal{W}(m_2)\ll 1$. This implies

$$cm_2=\Big\lfloor\frac{-p^2}{q}\Big\rfloor+\ell$$

for

$$|\ell|\le\frac{cm_1p^{1+\epsilon}}{N}.$$

Since ℓ has only $O(\ell^{\epsilon})$ divisors, once we apply (3.12) to (3.9) and bound everything absolutely, we may collapse the the sum over c and m_2 to a sum over ℓ only. The congruence condition in (3.12) translates to a congruence condition on ℓ. Thus we see that to establish (3.9), it suffices to prove that for any α_{m_1}, we have

$$\frac{N^2}{C_1^2q^2p}\sum_{1\le m_1\le\frac{C_1pq^{1+\epsilon}}{N}}\sum_{\substack{|\ell|\le\frac{C_1m_1p^{1+\epsilon}}{N}\\ \ell\equiv\alpha_{m_1} \bmod m_1}}1\ll p^{\epsilon}\Big(\frac{p}{q}+1\Big).$$

This follows since the ℓ-sum is $O(\frac{C_1 p^{1+\epsilon}}{N}+1)$.

3.3.2. *The case of small c.* We now consider the left hand side (3.10), assuming $m_1>0$. Again, a trivial bound for the terms with $p|m_1$ allows us to restrict to $(m_1,p)=1$. The details of Poisson summation in n_2 are the same as in subsection 3.3.1, except that the transform function will equal

$$\mathcal{W}(m_2)=\int_{-\infty}^{\infty} e\Big(\frac{-ym_2N}{m_1p}\Big)W(y)dy,$$

from which it follows using integration by parts that we can restrict to

$$|m_2|\leq p^{\epsilon}\frac{m_1p}{N}\leq\frac{p^{2+\epsilon}}{N}.$$

The same congruence condition as in (3.12) will hold here. Thus to establish (3.10), it suffices to prove that for any α_{m_1},

$$\frac{1}{p}\sum_{p^{1-\epsilon}\leq m_1\leq p^{1+\epsilon}}\ \sum_{\substack{C_2\leq c<2C_2\\ |m_2|\leq\frac{p^{2+\epsilon}}{N}\\ cm_2\equiv\alpha_{m_1}\bmod m_1}} 1\ll p^{\epsilon}\Big(\frac{p}{q}+1\Big).$$

This again is immediate, keeping in mind (2.10).

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