ARITHMETIC REPRESENTATIONS OF FUNDAMENTAL GROUPS II: FINITENESS

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Abstract. Let X be a smooth curve over a finitely generated field k, and let ℓ be a prime different from the characteristic of k. We analyze the dynamics of the Galois action on the deformation rings of mod ℓ representations of the geometric fundamental group of X. Using this analysis, we prove several finiteness results for function fields over algebraically closed fields in arbitrary characteristic, and a weak variant of the Frey-Mazur conjecture for function fields in characteristic zero.

For example, we show that if X is a normal, connected variety over $\mathbb{C}$, the (typically infinite) set of representations of $\pi_1(X^{\mathrm{an}})$ into $GL_n(\overline{\mathbb{Q}_\ell})$, which come from geometry, has no limit points. As a corollary, we deduce that if L is a finite extension of $\mathbb{Q}_\ell$, then the set of representations of $\pi_1(X^{\mathrm{an}})$ into $GL_n(L)$, which arise from geometry, is finite.

Contents

1. Introduction

The purpose of this paper is to study the representations of the étale fundamental group $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ of a variety X over a finitely generated field k, via an analysis of the Galois action on $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$. This work was begun in [Lit18], which studied integral aspects of such representations. In this paper we focus on finiteness results for representations of these groups. We are motivated by the Shafarevich conjecture [Fal83, Satz 6], the Fontaine-Mazur finiteness conjectures [FM95, Conjectures 2a and 2b], and the Frey-Mazur conjecture (see the question at the end of the introduction of [Maz78] for the original question, and the introduction of [BT16] for a corrected statement), though we prove nothing new about these conjectures.

Part of the goal of this work is to give *anabelian* approaches to function field analogues of standard conjectures about representations of Galois groups of number fields, in the hope that these techniques can be transported to the number field setting. In particular, most of our main results have purely group-theoretic statements.

1.1. **Main results.** Let k be a finitely generated field, and X/k a curve (a smooth, separated, geometrically connected k-scheme of dimension 1). Choose an algebraic closure $\bar{k}$ of k, and let $\bar{x}$ be a geometric point of X.

Definition 1.1.1. Let ℓ be a prime different from the characteristic of k, and let L be an ℓ-adic field (an algebraic extension of $\mathbb{Q}_\ell$ or the completion thereof). We say that a continuous representation

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(L)$$

is *arithmetic* if there exists a finite extension k' of k and a continuous representation

$$\tilde{\rho} : \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \to GL_n(L)$$

such that ρ is isomorphic to a subquotient of $\tilde{\rho}|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$.

The main examples of arithmetic representations are those *arising from geometry* (see Definition 3.1.6 for a precise definition, which is substantially less restrictive but a bit more complicated than the situation below):

Example 1.1.2. Suppose

$$f : Y \to X_{\bar{k}}$$

is a smooth proper morphism of varieties over $\bar{k}$. Then for any $i \geq 0$, any subquotient of the monodromy representation

$$\pi_1(X_{\bar{k}}, \bar{x}) \to GL((R^i f_* \underline{\mathbb{Q}_\ell})_{\bar{x}} \otimes L)$$

is arithmetic. (See Proposition 3.1.9 for a proof of a generalization of this fact.)

The purpose of this paper is to study arithmetic representations, and to apply this study to the understanding the representations which arise from geometry.

1.1.1. *Results on finiteness.* If ρ is a representation of a group into $GL_n(R)$ for some ring R, we denote by $\mathrm{ch}(\rho)$ the characteristic polynomial of ρ — that is, the function

$$\mathrm{ch}(\rho) : G \to R[t]$$
$$g \mapsto \det(tI - \rho(g)).$$

Our first main result is:

Theorem 1.1.3. *Let $c \in \mathbb{R}$ be such that $0 < c \leq 1$. Let*

$$\bar{\rho} : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r})$$

be a representation. Then the set of isomorphism classes of semisimple arithmetic representations

$$\rho : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{C}_\ell)$$

with

$$\mathrm{ch}(\rho) \equiv \mathrm{ch}(\bar{\rho}) \bmod \ell^c$$

is finite.

Here $\mathbb{C}_\ell := \widehat{\overline{\mathbb{Q}_\ell}}$ is the completion of the algebraic closure of $\mathbb{Q}_\ell$. Note that the equality

$$\mathrm{ch}(\rho) \equiv \mathrm{ch}(\bar{\rho}) \bmod \ell^c$$

takes place in $\mathscr{O}_{\mathbb{C}_\ell}/(\ell^c)[t]$, which contains $\mathbb{F}_{\ell^r}[t]$.

Remark 1.1.4. A slight weakening of Theorem 1.1.3 admits a more pithy statement. Namely, Theorem 1.1.3 implies that the set of semisimple arithmetic representations of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{C}_\ell)$ has no limit points. That is, if $\{\rho_i\}$ is a sequence of semisimple arithmetic representations with $\mathrm{ch}(\rho_i) \to \mathrm{ch}(\rho)$ uniformly, then the sequence $\{\rho_i\}$ is eventually constant.

As a corollary, we have the following finiteness result:

Corollary 1.1.5. *Suppose L is finite over $\mathbb{Q}_\ell$, with residue field $\mathbb{F}_{\ell^r}$, ring of integers $\mathscr{O}_L$, and maximal ideal $\mathfrak{m}_L$. Then*

1. *If $\bar{\rho} : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r})$ is a continuous representation, the set of isomorphism classes of semisimple arithmetic representations (resp. representations which arise from geometry)*
$$\tilde{\rho} : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(L)$$
with $\mathrm{ch}(\tilde{\rho}) \equiv \mathrm{ch}(\rho) \bmod \mathfrak{m}_L$ is finite.
2. *If $\mathrm{char}(k) = 0$, the set of isomorphism classes of semisimple arithmetic representations (resp. representations which arise from geometry)*
$$\rho : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(L)$$
is finite.
3. *If $\mathrm{char}(k) > 0$, the set of isomorphism classes of semisimple tame arithmetic representations (resp. tame representations which come from geometry)*
$$\rho : \pi_1^{tame}(X_{\bar{k}}, \bar{x}) \to GL_n(L)$$
is finite.

Remark 1.1.6. Theorem 1.1.3 and the parts of Corollary 1.1.5 about semisimple arithmetic representations are purely group-theoretic statements about the structure of the arithmetic fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$.

Remark 1.1.7. Note that if L is replaced by $\overline{\mathbb{Q}_\ell}$ in Corollary 1.1.5 above, the statement is false, as may be seen by taking $X = \mathbb{G}_m$; in this case $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \simeq \widehat{\mathbb{Z}}$, and the arithmetic representations (resp. representations which arise from geometry) $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to \overline{\mathbb{Q}_\ell}^\times$ are precisely the characters of finite order, of which there are infinitely many.

Remark 1.1.8. Corollary 1.1.5(3) is false without the tameness assumption. For example, one may take $X = \mathbb{A}^1_k$; then $\mathrm{Hom}(\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}), \mathbb{F}_p)$ is not finitely generated, and hence there are infinitely many representations $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_p(\mathbb{Q}_\ell)$ with finite image. Representations with finite image are always arithmetic (and always arise from geometry). One may, however, replace the tame fundamental group with any topologically finitely-generated quotient of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$.

Remark 1.1.9. One may prove higher-dimensional analogues of Theorem 1.1.3 and Corollary 1.1.5 by reduction to the case of curves, via a Lefschetz argument.

Moreover, one may deduce results for varieties over arbitrary fields by a standard spreading-out and specialization argument. For example, Corollary 1.1.5 implies that if X is a connected, normal variety over $\mathbb{C}$, the set of representations of $\pi_1(X^{\text{an}})$ into $GL_n(\mathbb{Q}_\ell)$, which arise from geometry, is finite.

Remark 1.1.10. One may deduce a stronger form of Corollary 1.1.5(3) from the results of this paper; namely, that the set of semisimple arithmetic representations with bounded Swan conductor is in fact finite. We do not give a proof here.

1.1.2. *A weak analogue of the Frey-Mazur conjecture.* Suppose now that

$$\bar{\rho} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r})$$

is absolutely irreducible. Then Theorem 1.1.3 implies (by [Car94, Théorème 1]) that given a semisimple arithmetic representation

$$\tilde{\rho} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Z}_\ell})$$

lifting $\bar{\rho}$ (note that $\tilde{\rho} \otimes \overline{\mathbb{Q}_\ell}$ is necessarily absolutely irreducible), there exists $d \in \mathbb{Q}_{>0}$ such that for any semisimple arithmetic $\tilde{\rho}'$ with

$$\tilde{\rho}' \simeq \tilde{\rho} \bmod \ell^d,$$

we have that $\tilde{\rho}' \simeq \tilde{\rho}$. (Here $\overline{\mathbb{Z}_\ell}$ is the valuation ring of $\overline{\mathbb{Q}_\ell}$.) That is, there is a ball of radius ℓ^{-d} around $\tilde{\rho}$ such that $\tilde{\rho}$ is the unique semisimple arithmetic lift of $\bar{\rho}$ within this ball. However, the proof of Theorem 1.1.3 gives no way to effectively compute such a constant d.

Our final main result gives a method to effectively compute such a constant $d > 0$, in terms of cohomological invariants of $\tilde{\rho}$, as long as $\text{char}(k) = 0$ and $\tilde{\rho}$ arises from geometry. This is a weak version of the Frey-Mazur conjecture for function fields (see e.g. [Maz78, BT16]), which asserts that a monodromy representation should be determined by its mod ℓ^d reduction, for ℓ^d large in terms of the geometric invariants of X and the dimension of $\tilde{\rho}$, if $\tilde{\rho}$ arises from the Tate module of an Abelian X-scheme.

Theorem 1.1.11. *Let X be a smooth, geometrically connected curve over a finitely generated field k of characteristic zero, and let $\bar{x}$ be a geometric point of X. Let*

$$\rho : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

be a representation which arises from geometry, lifting an absolutely irreducible residual representation $\bar{\rho}$. Then there exists an explicit constant $N = N(X_{\bar{k}}, c(\rho), \ell)$ such that any semisimple arithmetic representation

$$\tilde{\rho} : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

with

$$\text{ch}(\tilde{\rho}) \equiv \text{ch}(\rho) \bmod \ell^N$$

satisfies $\rho \simeq \tilde{\rho}$. Here $c(\rho)$ is defined as in Definition 5.1.7.

Remark 1.1.12. The statement of Theorem 1.1.11 is not purely group-theoretic, as ρ is required to arise from geometry. However, the proof only requires ρ to be pure and *geometric in the sense of Fontaine-Mazur* — see Remark 5.1.6 for details.

In general, the constant $c(\rho)$ appearing in Theorem 1.1.11 may be bounded independently of ℓ, assuming the Tate conjecture; in particular, assuming the Tate conjecture, if $\{\rho_\ell\}$ is a compatible system of representations of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ arising from geometry, we have $N(c(\rho_\ell), \ell) \to 0$ as $\ell \to \infty$. See Remark 5.4.1 for details.

Without the Tate conjecture, we do not know how to estimate the constant $c(\rho)$ appearing in Theorem 1.1.11 above; however, if ρ has finite image, we may bound it using a result of Serre [Ser13, Lettre à Ken Ribet, p. 60]. In this case, we have the following more uniform result, which is a strengthening of the main theorem of [Lit18]:

Theorem 1.1.13. *Let $X, k, \bar{x}$ be as in Theorem 1.1.11. Let*

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}})$$

be a representation which factors through a finite quotient G of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$. Then there exists a sequence of constants $N_G(\ell) > 0$ with $N_G(\ell) \to 0$ as $\ell \to \infty$ such that if

$$\tilde{\rho} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

is semisimple arithmetic with

$$\mathrm{ch}(\rho) \equiv \mathrm{ch}(\tilde{\rho}) \bmod \ell^{N_G(\ell)},$$

we have $\rho \otimes \overline{\mathbb{Q}_\ell} \simeq \tilde{\rho}$.

In other words, Theorem 1.1.3 implies that there is a ball around $\rho \otimes \overline{\mathbb{Q}_\ell}$ in which it is the unique semisimple arithmetic representation of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$. Theorem 1.1.13 implies that the radius of this ball tends to 1 as $\ell \to \infty$. A closely related result was proven by Cadoret and Moonen [CM18, Theorem B], simultaneously with Theorem 1.1.13.

In particular, for $\ell \gg 0$, if L is a finite extension of $\mathbb{Q}_\ell$ with residue field $\mathbb{F}_{\ell^r}$, there is a unique semisimple arithmetic lift of any representation

$$\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to G \to GL_n(\mathbb{F}_{\ell^r})$$

to $GL_n(L)$, namely the obvious one which factors through G. For example, applying the result in the case ρ is trivial, we have the following simple consequences:

Corollary 1.1.14. *There exists $\ell_0(X) \gg 0$, independent of n, such that for $\ell > \ell_0(X)$, the unique semisimple arithmetic representation*

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}) \to GL_n(\mathbb{Z}_\ell),$$

which is trivial mod ℓ, is the trivial representation.

Here semisimplicity means that $\rho \otimes \mathbb{Q}_\ell$ is semisimple.

Corollary 1.1.15. *Let X be a smooth connected curve over $\mathbb{C}$. Then there exists an integer $N = N(X) > 0$ such that if*

$$\rho : \pi_1(X^{an}) \to GL_n(\overline{\mathbb{Z}})$$

(1) arises from geometry, and
(2) is trivial mod M for some integer $M > N$,

then ρ is trivial.

This latter corollary is immediate from the fact that representations arising from geometry are semisimple arithmetic.

1.2. Overview of the proof.

1.2.1. *Sketch proof of Theorem 1.1.3.* The proof proceeds in two steps. First, we show that any semisimple arithmetic representation into $GL_n(\mathbb{C}_\ell)$ is in fact defined over $\overline{\mathbb{Q}_\ell}$ (Theorem 3.2.3). This is the only place in the paper in which Lafforgue's work is used; we require as input from Lafforgue the fact that if k is finite of characteristic different from ℓ and

$$\rho : \pi_1^{\text{ét}}(X, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

is irreducible when restricted to $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$, then $\rho \otimes \rho^\vee$ is pure of weight zero. If one only wishes to prove the parts of Corollary 1.1.5 about representations arising from geometry, one may avoid this input at the cost of a slightly more complicated argument.

We reduce to the case of finite fields by a specialization argument. In this case, we show via a dynamical argument (Corollary 4.1.5), that for c as in the Theorem, there exists $k(c)/k$ finite such that any ρ satisfying the condition of Theorem 1.1.3 is invariant under the action of $G_{k(c)}$ (Corollary 4.1.6). Now suppose there were infinitely many such semisimple ρ. The condition $\mathrm{ch}(\rho) \equiv \mathrm{ch}(\bar{\rho}) \mod \ell^c$ defines an affinoid subdomain of the rigid generic fiber of the space of pseudorepresentations lifting $\mathrm{ch}(\bar{\rho})$; the infinitude of ρ satisfying the given condition would imply that this subdomain contains infinitely many $G_{k(c)}$-fixed points. Thus the space of $G_{k(c)}$-fixed points would be a *positive-dimensional* rigid space, and hence would have a $\mathbb{C}_\ell$-point not defined over $\overline{\mathbb{Q}_\ell}$. This contradicts the result of the previous paragraph.

1.2.2. *Sketch proof of Theorem 1.1.11.* The proof is a variant on the proof of Theorem 1.2 of [Lit18], replacing the use of the pro-unipotent fundamental group in that paper with the use of deformation rings. As in the statement of the theorem, let

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

be a representation which arises from geometry, lifting an absolutely irreducible residual representation

$$\bar{\rho} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r}).$$

Let S_ρ be the deformation ring of ρ and $R_{\bar{\rho}}$ the deformation ring of $\bar{\rho}$.

We define a weight filtration $W^\bullet$ on S_ρ and, for $\alpha \in \mathbb{Z}_\ell^\times$ sufficiently close to 1, we construct elements $\sigma_\alpha \in G_k$ which act on $\mathrm{gr}_W^{-i} S_\rho$ via $\alpha^i \cdot \mathrm{Id}$ (Theorem 5.1.8). Using this analysis, we construct a G_k-stable closed ball U in the rigid generic fiber $E_{\bar{\rho}}$ of $R_{\bar{\rho}}$, containing the point of $E_{\bar{\rho}}$ corresponding to ρ, such that the span of the σ_α eigenvectors in $\mathscr{O}_U$ is dense.

The σ_α-eigenvectors in $\mathscr{O}_U$ are convergent power-series vanishing at the point of $E_{\bar{\rho}}$ corresponding to ρ; we estimate their coefficients in terms of α. Using this estimate, we may estimate the radius of a ball U' in which ρ is the unique common zero of these σ_α-eigenvectors, and hence the unique σ_α-periodic point of U'. We thus conclude that it is the unique arithmetic representation contained in U'.

1.3. **Comparison to previous work.** We first discuss predecessors to Theorem 1.1.3 and Corollary 1.1.5. Deligne showed [Del87] that if X is a normal complex algebraic variety, there are finitely many representations

$$\pi_1^{\text{top}}(X, x) \to GL_n(\mathbb{Q})$$

underlying a polarizable variation of Hodge structure, and hence finitely many such representations which arise from geometry; Corollary 1.1.5 is analogous, but replaces $\mathbb{Q}$ with an ℓ-adic field. Even over $\mathbb{C}$, Corollary 1.1.5 does not follow from Deligne's work [Del87], because ℓ-adic fields contain number fields of arbitrary degree. Corollary 1.1.5 does not imply Deligne's result about variations of Hodge structure, of course, but it does immediately imply the corollary that there are finitely many representations

$$\pi_1^{\text{top}}(X, x) \to GL_n(\mathbb{Q})$$

which *arise from geometry*.

Work of Deligne, Drinfel'd, and Lafforgue implies (via automorphic methods) that if X is a variety over a finite field $\mathbb{F}_q$, the set of semisimple $\overline{\mathbb{Q}_\ell}$-representations of its Weil group $W(X)$, with fixed rank and *bounded wild ramification at infinity*, is finite up to twist by characters of $W(\mathbb{F}_q)$ [EK12, Theorem 2.1]. Perhaps Theorem 1.1.3 should not be surprising in light this result; indeed, Theorem 1.1.3 may be deduced from Deligne's result by combining Theorem 3.2.3 of this paper (to reduce to proving an analogous statement for $\overline{\mathbb{Q}}_\ell$-representations) with the dynamical arguments of Section 4 (to bound the field of definition of a representation satisfying the hypotheses of the theorem). We provide a slightly different argument for the sake of self-containedness. As in the case of the previous work mentioned in this paragraph, this paper relies heavily on Lafforgue's work.

Theorems 1.1.11 and 1.1.13 are weak variants of the Frey-Mazur conjecture for function fields; the proofs are a (somewhat involved) variant of the proof of [Lit18, Theorem 1.2]. There has been much recent work on the function field Frey-Mazur conjecture for representations arising from families of elliptic curves; see e.g. [BT16]. Theorem 1.1.13 also has complex-analytic analogues (with rather different uniformities) in the case of monodromy representations arising from families of Abelian varieties, in e.g. [Nad89, HT06].

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2. Preliminaries on deformation rings

We now begin preparations for the proof of Theorem 1.1.3, which will proceed by analyzing the dynamics of the Galois action on framed deformation rings of residual representations of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$, and on moduli of pseudorepresentations. We first recall the definitions of the objects in question.

2.1. **Basics of deformation rings.** Let ℓ be a prime. Let G be a profinite group satisfying Mazur's condition (Φ_ℓ) [Maz89, 1.1]:

For each open subgroup of finite index $H \subset G$, the set of continuous homomorphisms $\mathrm{Hom}_{\mathrm{cont}}(H, \mathbb{F}_\ell)$ is finite. (Φ_ℓ)

Let $\bar{\rho} : G \to GL_n(\mathbb{F}_{\ell^r})$ be a continuous representation. Write $\Lambda = W(\mathbb{F}_{\ell^r})$ for the Witt ring of $\mathbb{F}_{\ell^r}$, and let $\mathscr{C}_\Lambda$ be the category of local Artinian Λ-algebras with residue field $\mathbb{F}_{\ell^r}$. Recall that the framed deformation functor

$$D_{\bar{\rho}}^{\square} : \mathscr{C}_\Lambda \to \mathrm{Sets}$$

assigns to an object A of $\mathscr{C}_\Lambda$ the set of

$$\{\rho : G \to GL_n(A) \mid \rho \otimes_A \mathbb{F}_{\ell^r} = \bar{\rho}\}.$$

In other words, $D_{\bar{\rho}}^{\square}$ parametrizes lifts of $\bar{\rho}$ with a fixed basis.

We let $D_{\bar{\rho}} : \mathscr{C}_\Lambda \to \mathrm{Sets}$ be the deformation functor which assigns to an object A of $\mathscr{C}_\Lambda$ the set

$$\{(\rho : G \to GL_n(A) \mid \rho \otimes_A \mathbb{F}_{\ell^r} = \bar{\rho})\}/\sim$$

where here $\rho \sim \rho'$ if there exists an element

$$\gamma \in \ker(GL_n(A) \to GL_n(\mathbb{F}_{\ell^r}))$$

such that $\rho^\gamma = \rho'$. This is Mazur's notion of 'strict equivalence' [Maz89, 1.1].

There is an evident map $D_{\bar\rho}^{\square} \to D_{\bar\rho}$, given by forgetting the framing.

As G satisfies Mazur's finiteness condition (Φ_ℓ), $D_{\bar\rho}^{\square}$ is pro-representable by a local Noetherian Λ-algebra $R_{\bar\rho}^{\square}$ with residue field $\mathbb{F}_{\ell^r}$ [Maz89, Proof of Proposition 1]. In general the functor $D_{\bar\rho}$ is not pro-representable, though it is if $\bar\rho$ is absolutely irreducible; in this case we call the pro-representing object $R_{\bar\rho}$. The groups

$$T_i = H^i_{\mathrm{cont}}(G, \bar\rho \otimes \bar\rho^\vee)$$

form a tangent-obstruction theory for $D_{\bar\rho}$ [Maz89, 1.6].

In particular, if $T_2 = 0$, $D_{\bar\rho}$ is formally smooth; as the forgetful natural transformation $D_{\bar\rho}^{\square} \to D_{\bar\rho}$ is formally smooth [Böc13, Corollary 1.4.6], this implies $D_{\bar\rho}^{\square}$ is formally smooth as well. In this situation, $R_{\bar\rho}^{\square}$ is, by the Cohen structure theorem, non-canonically isomorphic to a power series ring over Λ; if it exists, $R_{\bar\rho}$ is non-canonically isomorphic to a power series ring over Λ as well, with tangent space canonically isomorphic to T_1.

Finally, we recall from [Tay91, 1.1], [Che14, definition on page 3] the definition of a pseudorepresentation, which formalizes the algebraic properties of the determinant of a representation $\det(\rho) : R[[G]] \to R$, where

$$\rho : G \to GL_n(R)$$

is a representation.

Definition 2.1.1 (Pseudorepresentations)**.** Let A be a commutative ring and R a not-necessarily commutative A-algebra. Let $A-\mathrm{alg}$ be the category of commutative A-algebras, and let

$$\underline{R} : A - \mathrm{alg} \to \mathrm{Sets}$$

be the functor

$$S \mapsto R \otimes_A S.$$

(1) A polynomial law $D : R \to A$ is a natural transformation $\underline{R} \to \underline{A}$. If B is a commutative A-algebra, we let $D_B : R \otimes_A B \to B$ be the induced map.
(2) A polynomial law D is homogeneous of degreee n if $D_B(xb) = b^n D(x)$ for all $b \in B, x \in R \otimes_A B$.
(3) A polynomial law D is *multiplicative* if $D_B(1) = 1$ and $D_B(xy) = D_B(x)D_B(y)$ for all B and all $x, y \in R \otimes_A B$.
(4) A d-dimensional pseudorepresentation of R is a multiplicative, homogeneous polynomial law $D : R \to A$ of degree d.
(5) If G is a group, then a d-dimensional A-pseudorepresentation of G is a homogeneous pseudorepresentation $D : A[G] \to A$ of degree d.
(6) If D is a d-dimensional A-pseudorepresentation of G, we define its characteristic polynomial $\chi(g, t)$ by
$$\chi(g, t) := D_{A[t]}(t - g) = \sum_{i=0}^{d} (-1)^i \Lambda_i^D(g) t^{d-i}$$
for $g \in G$. We define the trace of D to be Λ_1^D.
(7) If A is a topological commutative ring and G is a topological group, we say that a d-dimensional pseudorepresentation D of G is *continuous* if $\Lambda_i^D : G \to A$, defined as above, is continuous for all i. (See [Che14, Section 2.30].)

Of course if

$$\rho : G \to GL_n(R)$$

is a continuous representation, then $\det \circ \rho$ is a continuous n-dimensional pseudorepresentation of G.

We will use throughout that if R is an algebraically closed field of characteristic zero, then a pseudorepresentation is determined uniquely by its trace [Che14, Proposition 1.29], and that

$$\rho \mapsto \det \circ \rho$$

gives a bijection between isomorphism classes of (continuous) semisimple representations into $GL_n(R)$ and (continuous) n-dimensional R-pseudorepresentations [Che14, Theorem 2.12].

Given a d-dimensional pseudorepresentation $\bar{r} : G \to \mathbb{F}_{\ell^r}$, we let

$$D^{\mathrm{ps}}_{\bar{r}} : \mathscr{C}_\Lambda \to \mathrm{Sets}$$

be the functor which assigns to an Artin Λ-algebra A with residue field $\mathbb{F}_{\ell^r}$ the set of d-dimensional pseudorepresentations $G \to A$ lifting $\bar{r}$. Chenevier shows [Che14, Proposition 3.3] that $D^{\mathrm{ps}}_{\bar{r}}$ is pro-representable by a local Noetherian Λ-algebra $A(\bar{r})$. Given an n-dimensional residual representation $\bar{\rho}$, there is a natural map $D_{\bar{\rho}} \to D^{\mathrm{ps}}_{\det \circ \bar{\rho}}$, given by sending a deformation of $\bar{\rho}$ to its determinant.

We will also at several places in this text use the rigid-analytic moduli space of pseudorepresentations. Briefly, if An is the category of rigid-analytic spaces over $\mathbb{Q}_\ell$, and

$$E^{\mathrm{an}} : \mathrm{An} \to \mathrm{Sets}$$

is the functor which associates to X the set of d-dimensional pseudorepresentations $G \to \mathscr{O}(X)$, Chenevier shows that E^{an} is represented by a quasi-Stein rigid analytic space, which we will denote E_d [Che14, Theorem D]. Chenevier shows that E_d is the disjoint union of the rigid generic fibers of the $A(\bar{r})$, where $\bar{r}$ ranges over all residual d-dimensional pseudorepresentations [Che14, Theorem F].

Finally, if $\bar{\rho} : G \to GL_n(\mathbb{F}_{\ell^r})$ is an n-dimensional representation of G, there is for each $c \in \mathbb{R}$ with $1 \geq c > 0$ an affinoid subdomain of E_n given by the set of pseudorepresentations $D : G \to \overline{\mathbb{Q}_\ell}$ with $\chi(g,t) \equiv \mathrm{ch}(\bar{\rho}) \bmod \ell^c$. We denote this subdomain by $E_{\bar{\rho},c}$. Let

$$E_{\bar{\rho}} = \bigcup_{1 \geq c > 0} E_{\bar{\rho},c}.$$

2.2. **Galois actions on deformation rings.** Let X be a normal, geometrically connected variety over a finitely generated field k of characteristic different from ℓ; let $x \in X(k)$ be a rational point. Choose an algebraic closure $\bar{k}$ of k, and let $\bar{x} \in X(\bar{k})$ be the geometric point of X associated to x. The fact that x is a rational point means that the pair $(X_{\bar{k}}, \bar{x})$ has an action by $G_k := \mathrm{Gal}(\bar{k}/k)$; hence G_k acts on $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$.

Let

$$\bar{\rho} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r})$$

be a continuous representation. We will now apply the discussion of the previous section to the case $G = \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$.

As $\ell \neq \mathrm{char}(k)$, $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ satisfies Mazur's finiteness condition (Φ_ℓ) (by the finite-generation of $H^1_{\text{ét}}(X'_{\bar{k}}, \mathbb{F}_\ell)$, where X' is any finite étale cover of X), and hence $D^{\square}_{\bar{\rho}}$ is pro-representable by a local Noetherian Λ-algebra $R^{\square}_{\bar{\rho}}$.

Now G_k acts on $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$, and hence on the space of pseudorepresentations of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$, denoted E_n as above. Moreover, as $GL_n(\mathbb{F}_{\ell^r})$ is finite, there exists a finite index subgroup $H_1 \subset G_k$ so that for each $h \in H_1, \bar{\rho}^h = \bar{\rho}$. That is, for each $\gamma \in \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}), g \in H_1$, we have

$$\bar{\rho}(\gamma^g) = \bar{\rho}(\gamma).$$

Let $k_1 = \bar{k}^{H_1}$, so $H_1 = G_{k_1}$. Then for any A-deformation

$$\tilde{\rho} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(A)$$

of $\bar{\rho}$, and any $g \in G_{k_1}$, the representation

$$\tilde{\rho}^\gamma := (g \mapsto \tilde{\rho}(\gamma^g))$$

is another A-deformation of $\bar{\rho}$. Hence G_{k_1} acts on $D_{\bar{\rho}}$ via its action on $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$, and hence on $R_{\bar{\rho}}$ if it exists. An identical argument shows that G_{k_1} acts on $E_{\bar{\rho},c}$ for each c with $1 \geq c > 0$, and on $A(\det \circ \bar{\rho})$.

3. Properties of arithmetic representations

3.1. **Basic properties.** We now establish some basic properties of arithmetic representations.

Proposition 3.1.1. *Let $K = \overline{\mathbb{Q}_\ell}$ or $K = \mathbb{C}_\ell := \widehat{\overline{\mathbb{Q}_\ell}}$. Let*

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(K)$$

be a continuous, semisimple representation. Then the following are equivalent:

(1) *There exists a finite extension k' of k and a continuous representation*

$$\tilde{\rho} : \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \to GL_n(K)$$

such that $\tilde{\rho}|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})} \simeq \rho$.

(2) *ρ is arithmetic.*

(3) *There exists an open subgroup $H \subset G_k$ such that for all $h \in H, \rho^h \simeq \rho$.*

(4) *There exists an open subgroup $H \subset G_k$ such that for all $h \in H$, $\mathrm{Tr}(\rho^h(g)) = \mathrm{Tr}(\rho(g))$.*

Proof. Clearly (1) $\implies$ (2) and (3) $\implies$ (4).

We now show that (2) $\implies$ (3). By definition, there exists a finite extension k' of k and a continuous representation

$$\tilde{\rho} : \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \to GL_n(K)$$

such that ρ is a subquotient of $\tilde{\rho}|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$. Let $\tilde{S}$ be the (finite) set of isomorphism classes of irreducible representations of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ appearing as subquotients of $\tilde{\rho}|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$, and let $S \subset \tilde{S}$ be the set of irreducible representations appearing as subquotients of ρ. $G_{k'}$ permutes $\tilde{S}$ (acting via its outer action on $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$), and thus we may set H to be the finite-index subgroup of $G_{k'}$ fixing each element of S. As ρ is semisimple, this implies that (3) holds.

The fact that (4) $\implies$ (3) is immediate from [Tay91, Theorem 1].

Finally, we show (3) $\implies$ (1). We may immediately reduce to the case that ρ is irreducible. Now let $k' = \bar{k}^H$ be the fixed field of $H \subset G_k$, so that $H = G_{k'}$. Without loss of generality, we may assume that the exact sequence

$$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \to G_{k'} \to 1$$

splits (if not, replace k' with a finite extension so that X has a rational point, giving such a splitting), so that

$$\pi_1^{\text{ét}}(X_{k'}, \bar{x}) \simeq \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \rtimes G_{k'}.$$

For each $h \in H$, choose an isomorphism

$$\gamma_h : \rho^h \xrightarrow{\sim} \rho$$

— as ρ is irreducible, γ_h is well-defined up to scaling, by Schur's lemma. Thus the assignment

$$\gamma : H \to PGL_n(K)$$
$$h \mapsto [\gamma_h]$$

is a well-defined (continuous) homomorphism. We claim that γ lifts to an honest representation $H \to GL_n(K)$. Indeed, the obstruction to lifting γ from a projective representation to an honest representation is a class $\alpha \in H^2(H, \mathbb{Q}/\mathbb{Z})$. Choose H' such that $\mathrm{Res}^H_{H'}(\alpha) = 0$. Let $\tilde{\gamma} : H' \to GL_n(K)$ be a choice of lift.

Now we may set

$$\tilde{\rho} = \rho \rtimes \tilde{\gamma}.$$

$\square$

Remark 3.1.2. A similar argument appears in [EG20, Proposition 4.6] and [EG18, Proposition 3.1].

Corollary 3.1.3. *Let $K = \overline{\mathbb{Q}_\ell}$ or $\mathbb{C}_\ell$. Let $\rho : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(K)$ be arithmetic. Then its semisimplification ρ^{ss} is arithmetic as well.*

Proof. Let S be the set of irreducible subquotients of ρ. The arithmeticity of ρ implies that there exists an open subgroup of G_k which stabilizes S; hence ρ^{ss} is arithmetic by Proposition 3.1.1(4). $\square$

Motivated by Proposition 3.1.1, we make the following definitions.

Definition 3.1.4 (Field of moduli and field of definition)**.** Let $\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$ be a semisimple arithmetic representation.

(1) Let $H \subset G_k$ be the stabilizer of ρ; by Proposition 3.1.1(3), H is open. We say that the fixed field of H, $\bar{k}^H$, is the *field of moduli* of ρ.
(2) If k' is such that there exists $\tilde{\rho} : \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$ such that $\tilde{\rho}|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})} \simeq \rho$, we say that k is a *field of definition* of ρ.

If k is a number field, and ρ is irreducible, the field of moduli equals the field of definition:

Theorem 3.1.5. *Suppose k is a number field, X a geometrically connected k-variety, and $x \in X(k)$ is a rational point. Let $\bar{k}$ be an algebraic closure of k and $\bar{x}$ the geometric point of X associated to x. Let*

$$\rho : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

be a continuous irreducible representation. Then the following are equivalent:

(1) For each $g \in G_k := \mathrm{Gal}(\bar{k}/k)$, $\rho^g \simeq \rho$.
(2) There exists a representation $\tilde{\rho} : \pi_1^{\acute{e}t}(X, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$ such that

$$\rho \simeq \tilde{\rho}|_{\pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x})}.$$

Proof. Clearly (2) $\implies$ (1). We now show that (1) $\implies$ (2). As in the proof of Proposition 3.1.1, we obtain a projective representation

$$\gamma : G_k \to PGL_n(\overline{\mathbb{Q}_\ell});$$

as the sequence

$$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}) \to G_k \to 1$$

splits (because x is a rational point), it is enough to lift γ to an honest representation

$$\tilde{\gamma} : G_k \to GL_n(\overline{\mathbb{Q}_\ell}).$$

But such a lift exists by a result of Tate (see [Ser77, Theorem 4]). $\square$

Definition 3.1.6. Let L be an algebraic extension of $\mathbb{Q}_\ell$. A continuous representation

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(L)$$

(assumed to be tame when restricted to every curve in $X_{\bar{k}}$ if $\text{char}(k) > 0$) *arises from geometry* if there exists an algebraically closed field F with $k \subset F$, and a smooth proper map $f : Y \to U$, (where Y is an F-variety and $\iota : U \hookrightarrow X_F$ is a Zariski-open subset), such that $\rho \otimes \overline{\mathbb{Q}_\ell}$ is conjugate to the monodromy representation associated to a lisse subquotient of $R^i(\iota \circ f)_* \overline{\mathbb{Q}_\ell}$.

Remark 3.1.7. See [KS10] for a discussion of various notions of tameness for covers of higher-dimensional varieties. These subtleties will be more or less irrelevant for us, as our main results are all about the case of curves.

Remark 3.1.8. The tameness assumption in Definition 3.1.6 is necessary to allow arbitrary algebraically extensions of the base field; in general if $\overline{k} \subset F$ is an extension of algebraically closed fields, the map

$$\pi_1^{\text{ét}}(X_F, \bar{x}) \to \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$$

is not an isomorphism, if $\text{char}(k) > 0$.

Proposition 3.1.9. *Let L be an algebraic extension of $\mathbb{Q}_\ell$, and let*

$$\rho : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(L)$$

be a representation which arises from geometry. Then ρ is arithmetic.

Proof. This is a standard spreading-out and specialization argument; we include a sketch for the reader's convenience.

Let F be an algebraically closed extension of k such that there exists an F-variety Y and a smooth proper map $f : Y \to U$ (where $\iota : U \hookrightarrow X_F$ is a Zariski-open subset) with $\rho \otimes \overline{\mathbb{Q}_\ell}$ conjugate to the monodromy representation associated to a lisse subquotient of $R^i(\iota \circ f)_* \overline{\mathbb{Q}_\ell}$. There exists a finitely-generated k-algebra $R \subset F$, R-schemes $\mathscr{Y}$ and $\mathscr{U}$, isomorphisms $s : Y \xrightarrow{\sim} \mathscr{Y}_F, t : U \xrightarrow{\sim} \mathscr{U}_F$, a smooth proper morphism $g : \mathscr{Y} \to \mathscr{U}$, and an open embedding $\tilde{\iota} : \mathscr{U} \to X_R$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "s", "\sim"'] \arrow[d, "f"] & \mathscr{Y}_F \arrow[d, "g_F"] \\ U \arrow[r, "t", "\sim"'] & \mathscr{U}_F \end{tikzcd}$$

commutes. Now specializing to a closed point z of $\text{Spec}(R)$, we find that $(R^i(\tilde{\iota} \circ g)_{z*} \overline{\mathbb{Q}_\ell})$ has a lisse subquotient with the same geometric monodromy representation as the lisse subquotient of $R^i(\iota\circ f)_*\overline{\mathbb{Q}_\ell}$ associated to ρ. (That is, under the canonical surjection

$$\pi_1^{\text{tame}}(X_{\bar{k}},\bar{x}) \twoheadrightarrow \pi_1^{\text{tame}}(X_{\overline{\kappa(z)}},\bar{x})$$

the two representations are identified.) Hence the representation in question is arithmetic as desired. □

3.2. **Rigidity.** We now prove a rigidity statement for arithmetic representations; this, along with some results in ℓ-adic dynamics proved in Section 4, will be the main ingredient in the proof of Theorem 1.1.3.

Lemma 3.2.1 (Compare to [BHK⁺19, Corollary 5.12])**.** *Let $k=\mathbb{F}_q$ be a finite field with algebraic closure $\bar{k}$, C/k a smooth affine curve, and ℓ a prime not dividing* $\mathrm{char}(k)$*. Let $x\in C(k)$ be a rational point, and $\bar{x}$ the associated geometric point of C. Let*

$$\rho:\pi_1^{\text{ét}}(C_{\bar{k}},\bar{x})\to GL_n(\overline{\mathbb{Q}_\ell})$$

be a continuous irreducible arithmetic representation. Let A be a local Artinian $\overline{\mathbb{Q}_\ell}$-algebra with residue field $\overline{\mathbb{Q}_\ell}$ and let

$$\tilde{\rho}:\pi_1^{\text{ét}}(C_{\bar{k}},\bar{x})\to GL_n(A)$$

be a deformation of ρ such that $\mathrm{Tr}(\tilde{\rho}^{\phi_x})=\mathrm{Tr}(\tilde{\rho})$, where ϕ_x is the Frobenius at x. Then $\tilde{\rho}\simeq\rho\otimes_{\overline{\mathbb{Q}_\ell}}A$.

Proof. Let $W(k):=\mathbb{Z}\cdot\mathrm{Frob}\subset\mathrm{Gal}(\bar{k}/k)$ be the subgroup of the Galois group of k generated by Frobenius. Let the Weil group of C, denoted $W(C)$, be the fiber product

$$\begin{tikzcd} W(C):=W(k)\times_{\mathrm{Gal}(\bar{k}/k)}\pi_1^{\text{ét}}(C,\bar{x}) \arrow[r] \arrow[d] & W(k) \arrow[d, hook] \\ \pi_1^{\text{ét}}(C,\bar{x}) \arrow[r] & \mathrm{Gal}(\bar{k}/k). \end{tikzcd}$$

Observe that as ρ is arithmetic, we may after replacing k with a finite extension assume that there exists an isomorphism $\gamma:\rho\xrightarrow{\sim}\rho^{\phi_x}$; as ρ is irreducible, γ is well-defined up to scaling. Our choice of γ extends ρ to a $W(C)$ representation, well-defined up to a character of $W(k)$. After twisting by a character of $W(k)$ we may assume that this $W(C)$-representation has finite determinant, by [Del80, Proposition 1.3.4], and thus that corresponds to a lisse ℓ-adic sheaf of weight zero on C, by work of Lafforgue [Laf02, Corollaire VII.8]. Thus $\rho\otimes\rho^\vee$ (which doesn't change when one twists ρ by a character and is hence well-defined as a representation of $W(C)$) has weight zero as well.

Let $\mathfrak{m}_A$ be the maximal ideal of A, and let $I\subset A$ be a non-zero ideal with $\mathfrak{m}_A\cdot I=0$, so that

$$0\to I\to A\to B\to 0$$

is a small extension of local Artinian $\overline{\mathbb{Q}_\ell}$-algebras with residue field $\overline{\mathbb{Q}_\ell}$. Let

$$\tilde{\rho}:\pi_1^{\text{ét}}(C_{\bar{k}},\bar{x})\to GL_n(A)$$

be a deformation of ρ satisfying the hypotheses of the theorem, i.e. $\tilde{\rho}\otimes_B k\simeq\rho$ and $\mathrm{Tr}(\tilde{\rho}^{\phi_x})=\mathrm{Tr}(\tilde{\rho})$. By induction on the length of A we may assume that $\tilde{\rho}\otimes_A B\simeq\rho\otimes_{\overline{\mathbb{Q}_\ell}}B$.

Now $\tilde{\rho}$ and $\rho \otimes_{\overline{\mathbb{Q}_\ell}} A$ are deformations of $\rho \otimes_{\overline{\mathbb{Q}_\ell}} B$; the space of deformations of $\rho \otimes_{\overline{\mathbb{Q}_\ell}} B$ is a torsor for

$$H^1(\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}), \rho \otimes \rho^\vee) \otimes I = H^1(C_{\bar{k}}, \rho \otimes \rho^\vee) \otimes I$$

(where the equality follows as affine curves are $K(\pi, 1)$'s). Thus $[\tilde{\rho}] - [\rho \otimes_{\overline{\mathbb{Q}_\ell}} A]$ gives a well-defined class in $H^1(\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}), \rho \otimes \rho^\vee) \otimes I$.

By [Car94, Théorème 1], $\tilde{\rho}^{\phi_x} \simeq \tilde{\rho}$; clearly $\rho \otimes_{\overline{\mathbb{Q}_\ell}} A$ is also ϕ_x-invariant. Hence

$$[\tilde{\rho}] - [\rho \otimes_{\overline{\mathbb{Q}_\ell}} A] \in (H^1(\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}), \rho \otimes \rho^\vee) \otimes I)^{G_k}.$$

But recall that $\rho \otimes \rho^\vee$ has weight zero; thus by Weil II [Del80], $H^1(\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}), \rho \otimes \rho^\vee)$ has weights in $\{1, 2\}$. In particular $(H^1(\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}), \rho \otimes \rho^\vee) \otimes I)^{G_k} = 0$. So $\tilde{\rho} \simeq \rho \otimes_{\overline{\mathbb{Q}_\ell}} A$ as desired. ∎

Remark 3.2.2. The proof of Lemma 3.2.1 is the only place in this paper where the work of Lafforgue is used, where we need above that $\rho \otimes \rho^\vee$ has weight zero.

We now deduce that all semisimple arithmetic representations into $GL_n(\mathbb{C}_\ell)$ are defined over $\overline{\mathbb{Q}}_\ell$. Note that by compactness any continuous representation ρ of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ into $GL_n(\mathbb{C}_\ell)$ is conjugate to a representation into $GL_n(\mathscr{O}_{\mathbb{C}_\ell})$; hence $\operatorname{ch}(\rho)$ is valued in $\mathscr{O}_{\mathbb{C}_\ell}$. We will use this fact below without comment.

Theorem 3.2.3. *Let X be a normal, geometrically connected curve over a finite field k, and let ℓ be a prime different from the characteristic of k. Suppose that*

$$\rho : \pi_1^{ét}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{C}_\ell)$$

is semisimple arithmetic, and that there exists r such that the characteristic polynomial of ρ satisfies

$$\operatorname{ch}(\rho)(g) \bmod \mathfrak{m}_{\mathscr{O}_{\mathbb{C}_\ell}} \in \mathbb{F}_{\ell^r}[t]$$

for all $g \in \pi_1^{ét}(X_{\bar{k}}, \bar{x})$. Then ρ is in fact defined over $\overline{\mathbb{Q}_\ell}$, i.e. there exists a representation

$$\tilde{\rho} : \pi_1^{ét}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

such that the representation $\tilde{\rho} \otimes_{\overline{\mathbb{Q}_\ell}} \mathbb{C}_\ell$ is conjugate to ρ.

Before the proof, we need a few simple lemmas. Recall below that $\Lambda = W(\mathbb{F}_{\ell^r})$.

Lemma 3.2.4. *Let m be an integer and $A \subset \mathscr{O}_{\mathbb{C}_\ell}/\ell^m$ a finitely-generated $W_m(\mathbb{F}_{\ell^r})$-algebra with $A/(A \cap \mathfrak{m}_{\mathscr{O}_{\mathbb{C}_\ell}/\ell^m}) = \mathbb{F}_{\ell^r}$. Then A is in $\mathscr{C}_\Lambda$, i.e. it is a local Artinian $W(\mathbb{F}_{\ell^r})$-algebra with residue field $\mathbb{F}_{\ell^r}$. In particular, A is finite.*

Proof. A is finitely generated over $W(\mathbb{F}_{\ell^r})$, hence Noetherian.

We claim A is local with maximal ideal $\mathfrak{m}_A = A \cap \mathfrak{m}_{\mathscr{O}_{\mathbb{C}_\ell}/\ell^m}$. Indeed, $A/\mathfrak{m}_A = \mathbb{F}_{\ell^r}$ by assumption, so $\mathfrak{m}_A$ is maximal; we claim that every element of $\mathfrak{m}_A$ is nilpotent. But indeed, this is true for every element of $\mathfrak{m}_{\mathscr{O}_{\mathbb{C}_\ell}/\ell^m}$.

All that remains is to prove that A is Artinian, i.e. that $\mathfrak{m}_A$ itself is nilpotent. But A is a Noetherian ring, so this follows from the fact that every element of $\mathfrak{m}_A$ is nilpotent. ∎

Lemma 3.2.5. *Let G be a group and $H \subset G$ a finite-index subgroup. Let $k \subset k'$ be an extension of algebraically closed fields of characteristic zero. Suppose*

$$\rho : G \to GL_n(k')$$

is a representation, such that for each $h \in H$, $\mathrm{Tr}(\rho(h)) \in k$. Then for all $g \in G$, $\mathrm{Tr}(\rho(g)) \in k$.

Proof. It suffices to show that if A is a matrix with entries in k', and

$$\mathrm{Tr}(A^{rs}) \in k$$

for some positive integer r and all integers s, then $\mathrm{Tr}(A) \in k$. But the statement implies immediately (using the assumption that k has characteristic zero) that the eigenvalues of A^r lie in k; hence by the fact that k is algebraically closed, the same is true for the eigenvalues of A. □

Lemma 3.2.6. *Let*

$$\bar{\rho} : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{F}_\ell})$$

be a representation whose associated pseudorepresentation $\mathrm{ch}(\bar{\rho}) : G \to \overline{\mathbb{F}_\ell}[t]$ factors through $\mathbb{F}_{\ell^r}[t]$ for some integer r, as in the statement of Theorem 3.2.3. Then ρ has finite image.

Proof. The result follows by an argument identical to the proof of [dJ01, Proposition 2.8-Lemma 2.10]. □

Finally, we record the following fact for which we were unable to find a reference:

Lemma 3.2.7. *Let*

$$\rho : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{C}_\ell)$$

be a continuous representation. Then ρ is conjugate to a representation into $GL_n(\mathscr{O}_{\mathbb{C}_\ell})$, i.e. there exists a $\pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x})$-stable $\mathscr{O}_{\mathbb{C}_\ell}$-lattice.

Proof. The proof is essentially the same as the usual proof for representations into $GL_n(K)$, with K finite over $\mathbb{Q}_\ell$, with an additional complication arising from the fact that $\mathscr{O}_{\mathbb{C}_\ell}$ is not Noetherian. Let $\Lambda \subset \mathbb{C}_\ell^n$ be any $\mathscr{O}_{\mathbb{C}_\ell}$-lattice. Then Λ is open in $\mathbb{C}_\ell^n$ and hence its stabilizer Γ in $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ is open. Thus Γ is of finite index in $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$. Now let Λ' be the sum of $g\Lambda, g \in \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$. As Γ is finite index in $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$, Λ' is finitely-generated, and it is evidently $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$-stable.

It remains to show that Λ' is isomorphic to $\mathscr{O}_{\mathbb{C}_\ell}^n$ as an $\mathscr{O}_{\mathbb{C}_\ell}$-module. But Λ' is finitely-generated and torsion-free, hence flat (as $\mathscr{O}_{\mathbb{C}_\ell}$ is a Bézout domain), hence free, as desired. □

Proof of Theorem 3.2.3. The proof is analogous to the proof of the fact that a rigid complex representation of a finitely generated (discrete) group is defined over $\overline{\mathbb{Q}}$, where rigidity comes from Lemma 3.2.1; we must overcome technical difficulties coming from the fact that we are working with profinite, as opposed to discrete, groups. We first reduce to the case that ρ is tame; then we reduce to the case that it is both tame and irreducible; then we show (using the theory of deformation rings recalled in Section 2) that ρ is conjugate to a representation defined over a $\mathbb{Z}_\ell$-algebra which is topologically of finite type. The rigidity statement of Lemma 3.2.1 then implies that we may take this algebra to be finite over $\mathbb{Z}_\ell$.

By [Tay91, Theorem 1], it suffices to show that for each $g \in \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$, $\mathrm{Tr}(\rho(g)) \in \overline{\mathbb{Q}_\ell}$.

Step 1. We first reduce to the case that ρ is tame. By Lemma 3.2.7 we may let

$$\rho_{\text{int}} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathscr{O}_{\mathbb{C}_\ell})$$

be an integral model of ρ (i.e. $\rho_{\text{int}} \otimes \mathbb{C}_\ell$ is conjugate to ρ), and let

$$\bar{\rho} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{F}_\ell})$$

be the residual representation associated to ρ_{int}. As the statement about traces is insensitive to conjugation, we may replace ρ with $\rho_{\text{int}} \otimes \mathbb{C}_\ell$. By the assumption on characteristic polynomials and Lemma 3.2.6, $\bar{\rho}$ factors through $GL_n(\mathbb{F}_{\ell^{r'}})$ for some r'; we rename r' as r and abuse notation to refer to the representation

$$\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r})$$

as $\bar{\rho}$ as well. Let X' be a finite étale cover of X so that $\bar{\rho}|_{\pi_1^{\text{ét}}(X'_{\bar{k}}, \bar{x})}$ is trivial; then $\rho|_{\pi_1^{\text{ét}}(X'_{\bar{k}}, \bar{x})}$ has pro-ℓ image and is hence tame. Note that this restriction operation preserves both semisimplicity and arithmeticity. We claim it suffices to prove that the tame representation $\rho|_{\pi_1^{\text{ét}}(X'_{\bar{k}}, \bar{x})}$ is defined over $\overline{\mathbb{Q}_\ell}$.

Indeed, $\pi_1^{\text{ét}}(X'_{\bar{k}}, \bar{x})$ is finite-index in $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$, so the claim follows from Lemma 3.2.5. So for the rest of the proof, we assume ρ is tame and semisimple arithmetic.

Step 2. We now reduce to the case where ρ is both tame and irreducible. Indeed, our (tame, semisimple, arithmetic) ρ is a direct sum of irreducible constituents; if each is defined over $\overline{\mathbb{Q}_\ell}$, then their direct sum is as well. Each irreducible constituent remains tame and arithmetic.

So for the rest of the proof, we assume ρ is tame, irreducible, and arithmetic. After replacing ρ with a conjugate representation (which remains, tame, irreducible, and arithmetic by definition), we may assume that there exists

$$\rho_{\text{int}} : \pi_1(X_{\bar{k}}, \bar{x}) \to GL_n(\mathscr{O}_{\mathbb{C}_\ell})$$

with $\rho = \rho_{\text{int}} \otimes \mathbb{C}_\ell$, with residual representation $\bar{\rho}$. Again it suffices to show that the traces of such a ρ lie in $\overline{\mathbb{Q}_\ell}$. We fix such a ρ, with integral model ρ_{int} and residual representation $\bar{\rho}$ for the rest of the proof.

Step 3. We now show that such a (tame, irreducible, arithmetic) ρ is in fact defined over a $\mathbb{Z}_\ell$-algebra topologically of finite type.

Let $R_{\bar{\rho}}^{\square}$ be the framed deformation ring of $\bar{\rho}$, as in Section 2. For each positive integer m, let ρ_m be the representation defined by the composition

$$\rho_m : \pi_1^{\text{tame}}(X_{\bar{k}}, \bar{x}) \xrightarrow{\rho_{\text{int}}} GL_n(\mathscr{O}_{\mathbb{C}_\ell}) \to GL_n(\mathscr{O}_{\mathbb{C}_\ell}/\ell^m).$$

The group $\pi_1^{\text{tame}}(X_{\bar{k}}, \bar{x})$ is topologically finitely generated; choose a finite set of topological generators γ_i. Let A_m be the subring of $\mathscr{O}_{\mathbb{C}_\ell}/\ell^m$ generated over $W_m(\mathbb{F}_{\ell^r})$ by the matrix entries of $\rho_m(\gamma_i)$. A_m is in $\mathscr{C}_\Lambda$ by Lemma 3.2.4. Hence A_m is its own profinite completion (indeed, it is finite), so the representation ρ_m factors continuously through $GL_n(A_m)$.

Hence, ρ_m is classified by a map

$$R_{\bar{\rho}}^{\square} \to A_m \to \mathscr{O}_{\mathbb{C}_\ell}/\ell^m$$

for all m. There is a natural surjection $A_m \to A_{m-1}$ induced by reducing the matrix entries of $\rho_m(\gamma_i)$ mod ℓ^{m-1}, turning the (A_m) into a pro-system of objects in $\mathscr{C}_\Lambda$. Hence for all m we have a commutative diagram

$$\begin{tikzcd} R_{\bar{\rho}}^{\square} \arrow[r] \arrow[dr] & A_m \arrow[r] \arrow[d] & \mathscr{O}_{\mathbb{C}_\ell}/\ell^m \arrow[d] \\ & A_{m-1} \arrow[r] & \mathscr{O}_{\mathbb{C}_\ell}/\ell^{m-1}. \end{tikzcd}$$

Thus the representation ρ_{int} is classified by a map

$$\tilde{f}_\rho : R^{\square}_{\bar{\rho}} \to \mathscr{O}_{\mathbb{C}_\ell};$$

let S be the image of $\tilde{f}_\rho$ inside of $\mathscr{O}_{\mathbb{C}_\ell}$ (note that we have here shown that $R^{\square}_{\bar{\rho}}$ satisfies a somewhat stronger property than its defining universal property, namely that it represents the framed deformation functor on *all* inverse limits of finite local Λ-algebras). The surjection $R^{\square}_{\bar{\rho}} \to S$ gives a representation

$$\rho_S : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(S)$$

such that $\rho_S \otimes_S \mathbb{C}_\ell \simeq \rho$, by construction. That is, ρ_S is a model of ρ over a Noetherian local $W(\mathbb{F}_{\ell^r})$-algebra. Indeed, S is topologically of finite type over $\mathbb{Z}_\ell$ — it is a continuous quotient of $R^{\square}_{\bar{\rho}}$, and $R^{\square}_{\bar{\rho}}$ is topologically of finite type over $\mathbb{Z}_\ell$.

Step 4. We now complete the proof. There exists a finite-index subgroup $H \subset \text{Gal}(\bar{k}/k)$ such that for each $h \in H$, $\text{Tr}(\rho_S^h) = \text{Tr}(\rho_S)$, as the same is true for ρ, by the definition of arithmeticity. Now by e.g. [Che14, Proposition G], there exists a Zariski-open subset $U^{\text{irr}} \subset \text{Spec}(S[1/\ell])$ such that for each point $z : \text{Spec}(\overline{\mathbb{Q}_\ell}) \to U^{\text{irr}}$ of U^{irr}, the representation $\rho_z := \rho_S \otimes_S \overline{\mathbb{Q}_\ell}$ is irreducible. Choose such a point, and let $\widehat{S}$ be the completion of S at z; as S is an integral domain, S injects into $\widehat{S}$. But by Lemma 3.2.1, $\text{Tr}(\rho_S \otimes_S \widehat{S})(g) \subset \overline{\mathbb{Q}_\ell}$ for each $g \in \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$. Hence the same is true for ρ, and thus ρ is defined over $\overline{\mathbb{Q}_\ell}$ by [Tay91, Theorem 1], as desired. $\square$

4. Dynamics of deformation rings

4.1. **The local dynamical Mordell-Lang conjecture.** We now prove some general facts about continuous pro-finite group actions on $\Lambda[[x_1, \cdots, x_N]]$, which will be used in the proof of Theorem 1.1.3 and related results. The main technical tool is a uniform local version of the dynamical Mordell-Lang conjecture (Lemma 4.1.3); it is surely well-known to experts, but we include a proof as we were unable to find a version with the required uniformities in the literature.

As above, let $\Lambda = W(\mathbb{F}_{\ell^r})$; endow Λ with the usual absolute value $|\cdot|_\ell$, so that $|\ell|_\ell = 1/\ell$. Let $R = \Lambda[[x_1, \cdots, x_N]]$. Let $\Lambda\langle x_1, \cdots, x_N\rangle$ be the Tate algebra on N variables, i.e. $\Lambda\langle x_1, \cdots, x_N\rangle \subset R$ is the set of power series

$$\sum_{I \in \mathbb{Z}^N_{\geq 0}} a_I x^I$$

with $|a_I|_\ell \to 0$ as $|I| = \sum_{j=1}^N i_j \to \infty$.

We now recall the "ℓ-adic analytic arc lemma," in a form due to Poonen [Poo14], building on results of Bell, Ghioca, and Tucker [BGT10]:

Lemma 4.1.1 (ℓ-adic analytic arc lemma, [Poo14, Theorem 1])**.** *Let $f \in \Lambda\langle x_1, \cdots, x_N\rangle^N$ satisfy $f(\mathbf{x}) = \mathbf{x} \bmod \ell^c$ for some $c > \frac{1}{\ell-1}$. Then there exists $g \in \Lambda\langle x_1, \cdots, x_N, n\rangle^N$ with $g(\mathbf{x}, m) = f^m(\mathbf{x})$ for each $m \in \mathbb{Z}_{\geq 0}$.*

In other words, iterates of analytic self-maps of the closed unit ball which are sufficiently close to the identity may be ℓ-adically interpolated.

Using Lemma 4.1.1, we will prove a uniform local version of the dynamical Mordell-Lang conjecture. Let $R = \Lambda[[x_1, \cdots, x_n]]$, and let $U(R)$ be the rigid generic fiber of R; this is the open unit ball. If L is a ℓ-adic field, an L-point of $U(R)$ is a continuous Λ-algebra homomorphism $R \to L$. For $1 \geq c > 0$ a positive real number, let $U_c(R) \subset U(R)$ be the closed ball of radius ℓ^{-c} around any $\Lambda[1/\ell]$-point of $U(R)$.

Note that $U_c(R)$ is independent of the choice of $\Lambda[1/\ell]$-point, by the ultrametric inequality. In particular, if

$$\varphi : \Lambda[[x_1, \cdots, x_N]] \xrightarrow{\sim} \Lambda[[x_1, \cdots, x_N]]$$

is a continuous automorphism, then φ restricts to an automorphism of $U_c(R)$ for all $c \leq 1$. An L-point $z : R \to L$ of $U(R)$ lands in $U_c(R)$ if $|z(x_i)|_\ell \leq \ell^{-c}$ for all i. Each $U_c(R)$ is an affinoid subdomain of $U(R)$.

If $\mathscr{I} \subset \mathscr{O}_{\mathbb{C}_\ell}[[x_1, \cdots, x_N]]$ is an ideal, we let $V(\mathscr{I}) \subset U(R)$ be the set

$$V(\mathscr{I}) := \{z \mid z(\mathscr{I}) = \{0\}\}.$$

We call a set which arises this way a *closed analytic subset* of $U(R)$. The dynamical Mordell-Lang conjecture asks for a characterization of the set of $m \in \mathbb{Z}_{\geq 0}$ such that $\varphi^m(z) \subset V(\mathscr{I})$, where φ is an analytic automorphism of $U(R)$ and $z \in U(R)$ — it asserts that this set is semilinear.

Definition 4.1.2 (Semilinear sets)**.** A set $A \subset \mathbb{Z}_{\geq 0}$ is *semilinear with period* M if it is the union of a finite set with finitely many residue classes modulo M. That is, it is a union

$$S \cup \{a_i + jM \mid a_i \subset T, j \in \mathbb{Z}_{\geq 0}\}$$

where $S \subset \mathbb{Z}_{\geq 0}$ is finite and $T \subset \{0, 1, \cdots, M-1\}$.

Lemma 4.1.3 (Uniform local dynamical Mordell-Lang conjecture)**.** *Let $c \in \mathbb{R}$ be a real number with $1 > c > 0$. Then there exists $M = M(c, \ell^r, N)$ with the following property: if*

$$\varphi : R = \Lambda[[x_1, \cdots, x_N]] \xrightarrow{\sim} \Lambda[[x_1, \cdots, x_N]]$$

is a continuous automorphism of Λ-algebras, $z \in U_c(R)$, and $\mathscr{I} \subset \mathscr{O}_{\mathbb{C}_\ell}[[x_1, \cdots, x_N]]$ is a proper ideal, then the set

$$\{m \in \mathbb{Z}_{\geq 0} \mid \varphi^m(z) \in V(\mathscr{I})\}$$

is semilinear with period M.

Proof. Let c' be a rational number with $c > c' > 0$, so that $U_{c'}(R)$ contains $U_c(R)$; it suffices to prove the theorem with c replaced by c'. Choose a (possibly ramified) finite extension Λ' of Λ so that there exists $\varpi \in \Lambda'$ with $|\varpi|_\ell = \ell^{-c'}$. There exists M_1 depending only on c', N, ℓ^r such that $\varphi^{M_1}(\mathbf{x}) = \mathbf{x} \bmod \varpi$. Let

$$\tilde{\varphi}(\mathbf{x}) = \frac{1}{\varpi}\varphi^{M_1}(\varpi \cdot \mathbf{x}).$$

Note that $\tilde{\varphi}$ lies in $\Lambda'\langle x_1, \cdots, x_N\rangle^N$. Then there exists $M_2 > 0$ depending only on c', ℓ^r, N such that $\tilde{\varphi}^{M_2}$ satisfies the hypotheses of Lemma 4.1.1; let $\vartheta \in \Lambda'\langle x_1, \cdots, x_N, n\rangle^N$ be such that $\vartheta(\mathbf{x}, m) = \tilde{\varphi}^{M_2 m}(\mathbf{x})$ for each $m \in \mathbb{Z}_{\geq 0}$, and let $M = M_1 M_2$. Note that ϑ is an N-tuple of convergent power series in the x_i and the *variable* n, which we view as a variable in an auxiliary Tate algebra.

Let $f \in \mathscr{I}$. Without loss of generality $|f(\mathbf{0})|_\ell \leq \ell^{-c'}$, as otherwise $V(\mathscr{I}) \cap U_{c'}(R) = \emptyset$; in particular, $\vartheta(\frac{1}{\varpi} f, m)$ converges for any $m \in \Lambda'$. We have

$$\begin{aligned}\{m \in \mathbb{Z}_{\geq 0} \mid \varphi^m(z) \in V(f)\} &= \{m \in \mathbb{Z}_{\geq 0} \mid z \circ \varphi^m(f) = 0\} \\ &= \bigcup_{j=0}^{M-1} \{j + Mm \in \mathbb{Z}_{\geq 0} \mid z \circ \varphi^{j+Mm}(f) = 0, m \in \mathbb{Z}_{\geq 0}\} \\ &= \{j + Mm \in \mathbb{Z}_{\geq 0} \mid z \circ \varphi^j \circ (\varpi \cdot \vartheta(\frac{1}{\varpi} \cdot f, m)) = 0, m \in \mathbb{Z}_{\geq 0}\}.\end{aligned}$$

But the functions $m \mapsto z \circ \varphi^j \circ (\varpi \cdot \vartheta(\frac{1}{\varpi} \cdot f, m)), j = 0, \cdots M-1$ are ℓ-adic analytic in m; hence they either have finitely many zeroes in $\mathbb{Z}_{\geq 0}$ or are identically zero.

Thus for each $f \in \mathscr{I}$, the set of $m \in \mathbb{Z}_{\geq 0}$ such that $\varphi^m(z) \in V(f)$ is semilinear with period M. We have

$$\{m \in \mathbb{Z}_{\geq 0} \mid \varphi^m(z) \in V(\mathscr{I})\} = \bigcap_{f \in \mathscr{I}} \{m \in \mathbb{Z}_{\geq 0} \mid \varphi^m(z) \in V(f)\}.$$

But an arbitrary intersection of sets which are semilinear with period M is semilinear with period M, from which we may conclude the result. □

Remark 4.1.4. Note that the constant M of Lemma 4.1.3 is independent of φ.

Corollary 4.1.5. *Let R be a local Noetherian Λ-algebra. Then for any affinoid subdomain U of the rigid generic fiber of R, there exists an integer $M = M(U)$ such that: If $\varphi : R \xrightarrow{\sim} R$ is an automorphism such that U is φ-stable, any φ-periodic point z of U satisfies $\varphi^M(z) = z$.*

Proof. Let $\mathfrak{m}_R$ be the maximal ideal of R. Write $R = S/\mathscr{J}$, where $S = \Lambda[[x_1, \cdots, x_N]]$ with $N = \dim \mathfrak{m}_R/\mathfrak{m}_R^2$ and $\mathscr{J} \subset \mathfrak{m}_S^2$ is an ideal, so the rigid generic fiber of R is a closed analytic subset of the open unit ball. We may lift φ to an automorphism $\tilde{\varphi}$ of $\Lambda[[x_1, \cdots, x_N]]$ (indeed any lift of φ to an endomorphism of $\Lambda[[x_1, \cdots, x_n]]$ is an isomorphism, as the induced map on $\mathfrak{m}_S/\mathfrak{m}_S^2$ is invertible by our choice of S). By quasicompactness of affinoids, U is contained in $U_c(S)$ for some c with $1 \geq c > 0$. Now let $z : R \to L$ be a φ-periodic point of U; it is a $\tilde{\varphi}$-periodic point of $U_c(S)$. Let $\mathscr{I} \subset S$ be the maximal ideal cutting out z, i.e. $\mathscr{I} = \ker(S \to R \xrightarrow{z} L)$. Now the result follows from Lemma 4.1.3, applied to $z, \mathscr{I}$; note that the integer M coming from Lemma 4.1.3 only depends on U, and not on φ, z, etc. □

We now apply this result to the case where R is a deformation ring. Recall from Section 2 that if $\bar{\rho}$ is a residual representation of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ and $\det \circ \bar{\rho}$ is the associated pseudorepresentation, we denoted the deformation ring of $\det \circ \bar{\rho}$ by $A(\det \circ \bar{\rho})$. For c with $1 \geq c > 0$, we let $E_{\bar{\rho},c}$ be the affinoid of the rigid generic fiber of $A(\det \circ \bar{\rho})$ consisting of pseudorepresentations with trace equal to the trace of $\bar{\rho}$ mod ℓ^c.

Corollary 4.1.6. *Let X be a smooth, geometrically connected curve over a finite field k of characteristic p, and let ℓ be a prime different from p; let* Frob $\in G_k$ *be the Frobenius element. Let*

$$\bar{\rho} : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r})$$

be a continuous representation such that $\bar{\rho}^{\mathrm{Frob}} \simeq \bar{\rho}$, so Frob *acts on $E_{\bar{\rho}}$. Then for any c with $1 \geq c > 0$, there exists $M \in \mathbb{Z}_{>0}$ such that any* Frob*-periodic point of $E_{\bar{\rho},c}$ is fixed by* Frob^M.

Proof. By the assumption that $\bar{\rho}^{\mathrm{Frob}} \simeq \bar{\rho}$, Frob acts on $A(\det \circ \bar{\rho})$; now we are in precisely the situation of Corollary 4.1.5, setting $R = A(\det \circ \bar{\rho}), \varphi = \mathrm{Frob}$, and $U = E_{\bar{\rho},c}$. □

Remark 4.1.7. Unwinding the proof, we used Lemma 4.1.1 to interpolate the action of the powers of Frob on $A(\det \circ \bar{\rho})$. We knew a priori that there was a continuous interpolation (coming from the action of G_k on $A(\det \circ \bar{\rho})$) — the input of Lemma 4.1.1 is required to see that this action is locally analytic.

4.2. **Finiteness.** We now prove Theorem 1.1.3 and Corollary 1.1.5.

Proof of Theorem 1.1.3. The proof proceeds by reduction to the case where k is a finite field.

Step 1. We first prove the theorem under the assumption that k is finite. Let

$$\bar{\rho}: \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r})$$

be a residual representation as in the statement of the theorem; let $A(\det\circ\bar{\rho})$ be the deformation ring of the pseudorepresentation corresponding to $\bar{\rho}$, defined in Section 2. Let $E_{\bar{\rho},c}$ be the affinoid of the rigid generic fiber of $A(\det\circ\bar{\rho})$, also defined in Section 2. After extending k, we have that $\bar{\rho}$ is G_k-fixed (as it has finite image) and hence that G_k acts on $A(\det\circ\bar{\rho})$ and $E_{\bar{\rho},c}$; arithmetic representations correspond to G_k-periodic $\mathbb{C}_\ell$-points of $E_{\bar{\rho},c}$ (after Theorem 3.2.3, we may assume they are $\overline{\mathbb{Q}_\ell}$-points). It suffices to show that there are finitely many such points of $E_{\bar{\rho},c}$.

But we are in the situation of Corollary 4.1.6 — there exists M such that any G_k-periodic point of $E_{\bar{\rho},c}$ is fixed by Frob^M. Consider the set of all Frob^M-fixed points; this is an analytic subset of $E_{\bar{\rho},c}$. By the Weierstrass preparation theorem, if it is infinite, it is in fact a *positive-dimensional* rigid space. Hence it contains a $\mathbb{C}_\ell$-point not defined over $\overline{\mathbb{Q}_\ell}$. But such a point would be the pseudorepresentation associated to a semisimple arithmetic representation (by [Tay91, Theorem 1]) over $\mathbb{C}_\ell$, not defined over $\overline{\mathbb{Q}_\ell}$, contradicting Theorem 3.2.3.

Step 2. We now reduce to the case of finite fields via a spreading-out and specialization argument. Let X' be the finite étale cover of $X_{\bar{k}}$ defined by $\ker(\bar{\rho}) \subset \pi_1(X_{\bar{k}}, \bar{x})$. After replacing k with a finite extension, we may assume that X' and the finite étale map $X' \to X_{\bar{k}}$ are in fact defined over k. Let $\bar{x}'$ be a geometric point of X' lying over $\bar{x}$. It suffices to prove the theorem with X replaced by X' and $\bar{\rho}$ replaced with the trivial representation. Indeed, any semisimple representation ρ of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ is a subquotient of $\text{Ind}_{\pi_1^{\text{ét}}(X'_{\bar{k}}, \bar{x}')}^{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}(\rho|_{\pi_1^{\text{ét}}(X'_{\bar{k}}, \bar{x}')})$. There are finitely many such subquotients. So we rename X' as X and assume $\bar{\rho}$ is trivial.

Now let $\overline{X}$ be the unique smooth proper geometrically connected curve containing X, and let $D = \overline{X} \setminus X$; after extending k we may assume $D = \{x_1, \cdots, x_i\}$ where the x_i are rational points of $\overline{X}$; after a further extension, we may assume the geometric point $\bar{x}$ of X comes from a k-rational point x of X. There exists an algebra $R \subset k$, finitely generated over $\mathbb{Z}$, and a smooth curve $\overline{\mathscr{X}}/R$ with disjoint R-points $\xi, \xi_1, \cdots, \xi_i$ so that $(\overline{\mathscr{X}}, \xi, \xi_1, \cdots, \xi_i)_k \simeq (\overline{X}, x, x_1, \cdots, x_i)$. Let $z \in \text{Spec}(R)$ be a closed point with residue field $\kappa(z)$ such that $\text{char}(\kappa(z)) \neq \ell$, and choose an algebraic closure $\overline{\kappa(z)}$ of $\kappa(z)$. The specialization map

$$\text{sp}: \pi_1^\ell(X_{\bar{k}}, \bar{x}) \to \pi_1^\ell((\overline{\mathscr{X}} \setminus \{\xi_1, \cdots, \xi_i\})_{\overline{\kappa(z)}}, \xi_{\overline{\kappa(z)}}) \tag{4.2.1}$$

is an isomorphism, where π_1^ℓ denotes the pro-ℓ completion of $\pi_1^{\text{ét}}$ (see [LO10, Corollary A.12], for example). Let $\bar{z} \in \text{Spec}(R)(\overline{\kappa(z)})$ be the geometric point of $\text{Spec}(R)$ associated to z and our choice of algebraic closure of $\kappa(z)$, and let $F \in G_{\kappa(z)}$ be the Frobenius.

Let $\widehat{R}$ be the completion of R at z and let $\widehat{K}$ be the fraction field of $\widehat{R}$. Choose an algebraically closed field $\overline{\widehat{K}}$ containing $\widehat{K}$ and an embedding $\bar{k} \hookrightarrow \overline{\widehat{K}}$. These choices give a natural map $G_{\widehat{K}} \to G_k$ (and hence a natural action of $G_{\widehat{K}}$ on $\pi_1^\ell(X_{\bar{k}}, \bar{x})$. There is also a natural surjective map $\pi: G_{\widehat{K}} \to G_{\kappa(z)}$, again given by specialization.

The key property of the map 4.2.1 we will use is that it is equivariant for the Galois action, in the sense that for $g \in G_{\widehat{K}}, \gamma \in \pi_1^{\ell}(X_{\bar{k}}, \bar{x})$,

$$\mathrm{sp}(g \cdot \gamma) = \pi(g) \cdot \mathrm{sp}(\gamma).$$

This follows from e.g. the proof of [LO10, Corollary A.12].

Now we claim that any arithmetic representation ρ of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ which is residually trivial (and hence factors through $\pi_1^{\ell}(X_{\bar{k}}, \bar{x})$) gives an arithmetic representation of $\tilde{\rho} : \pi_1^{\text{ét}}((\overline{\mathscr{X}} \setminus \{\xi_1, \cdots, \xi_i\})_{\overline{\kappa(z)}}, \xi_{\overline{\kappa(z)}})$, via the isomorphism 4.2.1. Indeed, it suffices to show that there exists M such that $\tilde{\rho}^{F^M} \simeq \tilde{\rho}$. By arithmeticity, for any element of $g \in G_k$ in the image of $G_{\widehat{K}}$, some power of it fixes ρ; hence the same is true for the action of $\pi(g)$ on $\tilde{\rho}$, by the Galois-equivariance of the map 4.2.1. As π is surjective, we may choose g with $\pi(g) = F$, giving the claim.

Thus we may replace k with $\kappa(z)$ and X with $\overline{\mathscr{X}} \setminus \{\xi_1, \cdots, \xi_i\}$; as $\kappa(z)$ is finite, we have reduced to the case of finite fields, which was proven in Step 1. □

We now deduce Corollary 1.1.5.

Proof of Corollary 1.1.5. We first prove the statements about semisimple arithmetic representations.

Let L be a finite extension of $\mathbb{Q}_\ell$, as in the statement, with valuation ring $\mathscr{O}_L$, maximal ideal $\mathfrak{m}_L$, and residue field $\mathbb{F}_{\ell^r}$. There are finitely many continuous representations

$$\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r}),$$

(resp. $\pi_1^{\text{tame}}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{F}_{\ell^r})$ in characteristic $p > 0$) as $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ (resp. $\pi_1^{\text{tame}}(X_{\bar{k}}, \bar{x})$) is topologically finitely-generated. Given this, (2) and (3) follow from (1). So we fix $\bar{\rho}$ as in the statement; we wish to show that there are finitely many semisimple arithmetic $GL_n(L)$-valued representations ρ with $\mathrm{ch}(\rho) \equiv \mathrm{ch}(\bar{\rho}) \bmod \mathfrak{m}_L$.

Let $c \in \mathbb{R}$ be such that $0 < c < v_\ell(x)$ for any $x \in \mathfrak{m}_L$; such a c exists because L is discretely valued. Now any arithmetic representation ρ admits a $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$-stable $\mathscr{O}_L$-lattice M; let $\bar{\rho}_M = M/\mathfrak{m}_L M$. By our choice of c, we have

$$\mathrm{ch}(\rho) \equiv \mathrm{ch}(\bar{\rho}) \bmod \ell^c.$$

Hence there are only finitely many possibilities for ρ, by Theorem 1.1.3.

We now deduce the required statements for representations which arise from geometry. But such representations are arithmetic by Proposition 3.1.9, and semisimple by [Del80, Corollaire 3.4.13]. □

5. An analogue of the Frey-Mazur conjecture

We now begin preparations for the proof of Theorem 1.1.11.

5.1. **Weight filtrations on deformation rings.** Let X be a smooth, affine, geometrically connected curve over a finitely generated field k of characteristic 0, and let c be a rational point of X; choosing an algebraic closure of k, c gives rise to a geometric point $\bar{c}$. Let $\overline{X}$ be the unique smooth, proper, geometrically connected k-curve containing X as an open subscheme. Let

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{c}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

be an arithmetic representation of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{c})$. The Leray spectral sequence for the inclusion

$$j : X \hookrightarrow \overline{X}$$

gives an exact sequence

$$0 \to H^1(\overline{X}_{\bar{k}}, j_*\rho) \to H^1(X_{\bar{k},\text{ét}}, \rho) \to H^0(\overline{X}_{\bar{k}}, R^1 j_*\rho) \to H^2(\overline{X}_{\bar{k}}, j_*\rho) \tag{5.1.1}$$

where we here identify ρ with the associated lisse ℓ-adic sheaf.

Definition 5.1.2. The *weight filtration* on $H^1(X_{\bar{k},\text{ét}}, \rho)$ is defined by

$$W^i H^1(X_{\bar{k},\text{ét}}, \rho) = 0 \text{ for } i \leq 0$$

$$W^1 H^1(X_{\bar{k},\text{ét}}, \rho) = H^1(\overline{X}_{\bar{k}}, j_*\rho)$$

$$W^i H^1(X_{\bar{k},\text{ét}}, \rho) = H^1(X_{\bar{k},\text{ét}}, \rho) \text{ for } i \geq 2.$$

By [Del80, Théorème 2], this agrees with the usual, geometrically-defined weight filtration on $H^1(X_{\bar{k},\text{ét}}, \rho)$ if ρ arises from geometry and is pure of weight zero.

Now suppose ρ is irreducible, and let S_ρ be its deformation ring (pro-representing the functor of continuous deformations of ρ on the category of local Artinian $\overline{\mathbb{Q}_\ell}$-algebras with residue field $\overline{\mathbb{Q}_\ell}$). See e.g. [Che14, Section 4] for the relation between this ring and Chenevier's moduli of pseudorepresentations. Note that for two given basepoints $\bar{c}, \bar{c}'$ of X, the associated deformation rings are canonically isomorphic, so we do not include $\bar{c}$ in the notation. Let

$$\rho_{\text{univ}} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{c}) \to GL_n(S_\rho)$$

be the universal deformation of ρ.

Because X is an affine curve,

$$S_\rho \simeq \overline{\mathbb{Q}_\ell}[[x_1, \cdots, x_N]]$$

non-canonically, where $N = \dim H^1(X_{\bar{k}}, \rho \otimes \rho^\vee)$. Let $\mathfrak{m}_\rho$ be the maximal ideal of S_ρ. We have

$$\mathfrak{m}_\rho/\mathfrak{m}_\rho^2 \simeq H^1(X_{\bar{k}}, \rho \otimes \rho^\vee)^\vee$$

canonically; let $W^i(\mathfrak{m}_\rho/\mathfrak{m}_\rho^2)$ be the dual filtration to the filtration given in Definition 5.1.2.

Definition 5.1.3 (Weight filtration on S_ρ)**.** Let

$$W^i S_\rho = S_\rho \text{ for } i \geq 0,$$

$$W^{-1} S_\rho = \mathfrak{m}_\rho$$

$$W^{-2} S_\rho = \mathfrak{m}_\rho^2 + W^{-2}(\mathfrak{m}_\rho/\mathfrak{m}_\rho^2)$$

and

$$W^{-m} S_\rho = \sum_{i+j=m} (W^{-i} S_\rho) \cdot (W^{-j} S_\rho) \text{ for } m > 2.$$

Remark 5.1.4. If X is proper, $W^{-i} = \mathfrak{m}_\rho^i$ for $i \geq 0$. In general, it is immediate from the definition that

$$W^{-2i} \subset \mathfrak{m}_\rho^i \subset W^{-i}$$

for $i \geq 0$.

Replace k with a finite extension so that G_k acts on S_ρ. The goal of this section is to construct, for $\alpha \in \mathbb{Z}_\ell^\times$ sufficiently close to 1, an element $\sigma_\alpha \in G_k$ such that σ_α acts on $\mathrm{gr}_W^{-i} S_\rho$ via the scalar α^i. We first prove the analogous statement for $H^1(X_{\bar{k},\text{ét}}, \rho \otimes \rho^\vee)$, where ρ is irreducible and arises from geometry.

So we assume ρ is irreducible and arises from geometry. Note that, as ρ is irreducible, there exists by definition (after replacing k with a finite extension) a representation

$$\tilde{\rho} : \pi_1^{\text{ét}}(X_k, \bar{c}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

such that $\tilde{\rho}|_{\pi_1^{\text{ét}}(X_{\bar{k}},\bar{c})} \simeq \rho$, arising from the ℓ-adic cohomology of some X-scheme. Given this choice of $\tilde{\rho}$, G_k acts canonically on $H^1(X_{\bar{k}}, \rho \otimes \rho^\vee)$. As ρ is irreducible $\rho \otimes \rho^\vee$ is pure of weight zero, and hence Weil II [Del80] implies $H^1(X_{\bar{k}}, \rho \otimes \rho^\vee)$ is mixed with weights in $\{1, 2\}$, with the weight filtration given as in Definition 5.1.2.

Lemma 5.1.5. *Let*

$$\rho : \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{c}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

be an irreducible representation which arises from geometry. Then for $\alpha \in \mathbb{Z}_\ell^\times$ sufficiently close to 1, there exists $\sigma_\alpha \in G_k$ such that σ_α acts on $\mathrm{gr}_W^1 H^1(X_{\bar{k}}, \rho \otimes \rho^\vee)$ via $\alpha \cdot \mathrm{Id}$, and on

$$(\mathrm{gr}_W^2 H^1(X_{\bar{k}}, \rho \otimes \rho^\vee)) \oplus H^0(\overline{X}_{\bar{k}}, R^1 j_*(\rho \otimes \rho^\vee))$$

via $\alpha^2 \cdot \mathrm{Id}$.

Proof. This is similar to [Lit18, Lemma 2.10], [Hin88, Lemme 12], or [Bog80a, Theorem 3], with some mild additional complications arising from the fact that we do not assume ρ arises from the monodromy action on the Tate module of an Abelian C-scheme. For simplicity of notation, we set

$$V = H^1(X_{\bar{k}}, \rho \otimes \rho^\vee) \oplus H^0(\overline{X}_{\bar{k}}, R^1 j_*(\rho \otimes \rho^\vee)),$$

and we let

$$\gamma : G_k \to GL(V)$$

be the Galois representation we are studying. We assume k is a number field; the general case follows by the argument of [Ser13, Letter to Ribet of 1/1/1981, §1]. The weight filtration on V is inherited from that of $H^1(X_{\bar{k}}, \rho \otimes \rho^\vee)$; $H^0(\overline{X}_{\bar{k}}, R^1 j_*(\rho \otimes \rho^\vee))$ is pure of weight 2.

Step 1. Let $\tilde{\rho} : \pi_1^{\text{ét}}(C_k, \bar{c}) \to GL_n(\overline{\mathbb{Q}_\ell})$ be an extension of ρ to the arithmetic fundamental group, as in the discussion before the statement of the lemma; then $\tilde{\rho}$ is defined over L for some $L/\mathbb{Q}_\ell$ finite. We abuse notation and write

$$\tilde{\rho} : \pi_1^{\text{ét}}(C_k, \bar{c}) \to GL_n(L)$$

for some choice of descent of our original representation to a model over L. Let V_L be the descent of V to L obtained from $\tilde{\rho}$, and let

$$\tilde{\gamma} : G_k \to GL(V_L)$$

be the associated Galois representation.

Viewing V_L as a $\mathbb{Q}_\ell$-vector space (of dimension $[L : \mathbb{Q}_\ell] \cdot \dim_L V_L$), we let

$$\gamma' : G_k \to GL_{n[L:\mathbb{Q}_\ell]}(\mathbb{Q}_\ell)$$

be the associated $\mathbb{Q}_\ell$-adic represenation. We also fix a G_k-stable $\mathbb{Z}_\ell$-lattice in this representation; when we refer to $\mathbb{Z}_\ell$-points in the following paragraph, we are giving $GL_{n[L:\mathbb{Q}_\ell],\mathbb{Q}_\ell}$ the $\mathbb{Z}_\ell$-structure coming from this lattice.

We first show that the image of γ' is open in the $\mathbb{Z}_\ell$-points of its Zariski-closure; in Step 2, we will study this Zariski closure. By [Bog80b, Théorème 1], it is enough to check that γ' is Hodge-Tate at primes above ℓ. But for any prime v above ℓ, the lisse sheaf associated to $\tilde{\rho}\otimes\tilde{\rho}^\vee$ is a de Rham local system on C_{k_v} in the sense of [LZ17], and hence its cohomology group

$$H^1(C_{\bar{k}}, \tilde{\rho}\otimes\tilde{\rho}^\vee)$$

is de Rham by [DLLZ18, Theorem 1.1], for example.

Moreover, we claim that $H^0(\overline{X}_{\bar{k}}, R^1j_*(\rho\otimes\rho^\vee))$ is de Rham. This is a local computation at each point of $\overline{X}\setminus X$; after replacing the local ring at each point by a ramified extension, we may assume by resolution of singularities that ρ comes from the cohomology of a semistable $\overline{X}$-scheme and conclude by e.g. the weight spectral sequence for this semistable scheme. (Alternately, this can be deduced from [DLLZ18, Corollary 4.3.4].) Hence it is Hodge-Tate, which completes the proof of openness.

Step 2. We now show that for any $\alpha\in\mathbb{Q}_\ell^\times$, the Zariski-closure of $\mathrm{im}(\gamma')$ contains elements σ_α acting as required. By Step 1, this suffices. Let $F\in G_k$ be a Frobenius element acting on $\mathrm{gr}^i_W(V)$ with weight i; let $Z\subset\overline{\mathrm{im}(\gamma')}$ be the identity component of the Zariski-closure of $\{F^n\}_{n\in\mathbb{Z}}$. Z is a commutative, connected, algebraic group over a field of characteristic zero, and hence $Z\simeq T\times U$ canonically, where T is a torus and U is unipotent. After replacing F with a power, we may assume it lies in Z. In particular F admits a unique decomposition $F=F_sF_u$, where $F_s\in T$ is semisimple and $F_u\in U$; F_s and F have the same eigenvalues. After possibly replacing $\mathbb{Q}_\ell$ with a finite extension, choose a basis of eigenvectors $\{e_i\}_{i=1,\cdots,\dim_{\mathbb{Q}_\ell}(V_L)}$ for F_s, with eigenvalues $\{\lambda_i\}$, and let $D\subset GL_n$ be the diagonal torus for this basis. Let $I_1\subset\{1,\cdots,\dim(V)\}$ be the set of indices i such that λ_i has weight 1, and I_2 the set of indices i such that λ_i has weight 2. The inclusion $T\hookrightarrow D$ induces a surjection on cocharacter lattices $X(D)\twoheadrightarrow X(T)$ with kernel K; T is precisely the subtorus of D cut out by the characters in K. If we identify D with $\mathbb{Z}^{\dim V}$ via the choice of basis $\{e_i\}$, K consists of the vectors $\underline{a}=(a_1,\cdots,a_{\dim V})$ such that

$$\prod_i \lambda_i^{a_i}=1.$$

But if $\underline{a}\in K$,

$$\prod_i |\lambda_i|^{a_i}=1,$$

where $|\cdot|$ denotes any Archimedean absolute value on $\overline{\mathbb{Q}}$, by the multiplicativity of absolute value. In other words,

$$\prod_{i\in I_1} q^{a_i/2}\cdot\prod_{j\in I_2} q^{a_j}=1,$$

where q is the size of the residue field of the prime corresponding to F, and hence

$$\sum_{i\in I_1} a_i+2\sum_{j\in I_2} a_j=0.$$

Hence if $V=\mathrm{gr}^1_W(V)\oplus\mathrm{gr}^2_W(V)$ is the unique F_s-equivariant splitting, we have that

$$\alpha\cdot\mathrm{Id}_{\mathrm{gr}^1_W(V)}\bigoplus\alpha^2\cdot\mathrm{Id}_{\mathrm{gr}^2_W(V)}\in T,$$

as desired. $\square$

Remark 5.1.6. This lemma is the only place in the proof of Theorem 1.1.11 in which the geometricity of ρ is used. In fact, the weaker condition that ρ extends to a pure local system on C which is *geometric in the sense of Fontaine-Mazur* suffices — see e.g. the conjecture on page 2 of [LZ17] for a discussion of this notion. In this case one may deduce the desired p-adic Hodge-theoretic properties for $H^1(C_{\bar{k}}, \rho\otimes\rho^\vee), H^0(\overline{C}_{\bar{k}}, R^1j_*(\rho\otimes\rho^\vee))$ for such ρ from [DLLZ18, Theorem 1.1 and Corollary 4.3.4], for example.

The set of α for which the desired σ_α exist is an important invariant of the representation ρ, which we record in the following definition.

Definition 5.1.7 (Index of homothety)**.** Let ρ be as in Lemma 5.1.5. Let $Z\subset\mathbb{Z}_\ell^\times$ be the set of α for which there exists σ_α satisfying the conclusions of Lemma 5.1.5. Then the *index of homothety* of ρ, denoted $c(\rho)$, is the index of Z in $\mathbb{Z}_\ell^\times$. By Lemma 5.1.5, Z is open in $\mathbb{Z}_\ell^\times$, so $c(\rho)$ is finite.

We now give the analogous statement for S_ρ.

Theorem 5.1.8. *Let*
$$\rho:\pi_1^{\acute{e}t}(X_{\bar{k}},\bar{c})\to GL_n(\overline{\mathbb{Q}_\ell})$$
be an irreducible representation which arises from geometry. Then for $\alpha\in\mathbb{Z}_\ell^\times$ sufficiently close to 1, there exists $\sigma_\alpha\in G_k$ such that σ_α acts on $\mathrm{gr}^i_W S_\rho$ via $\alpha^i\cdot\mathrm{Id}$.

In fact we will be able to choose the element σ_α in Theorem 5.1.8 to be inverse to the element σ_α constructed in Lemma 5.1.5.

Before giving the proof, we need to analyze the contribution of the inertia subgroups of $\pi_1^{\text{ét}}(X_{\bar{k}},\bar{c})$ to the geometry of S_ρ.

Recall that $\overline{X}$ is the smooth curve compactifying X. Let $D=\overline{X}\setminus X$, and pass to a finite extension of k so that $D=\{x_1,\cdots,x_n\}$ for rational points $x_i\in\overline{X}(k)$. For $i=1,\cdots,n$, let R_i be the complete local ring of $\overline{X}$ at x_i; by the Cohen structure theorem, each $R_i\simeq k[[t]]$. Let $\overline{R_i}$ be the completion of $R_i\otimes\bar{k}$ at its maximal ideal, and let $z_i=\mathrm{Spec}(\mathrm{Frac}(\overline{R_i}))$. Let
$$\eta_i:z_i\to X_{\bar{k}}$$
be the natural inclusion. A local computation shows that
$$H^0(\overline{X}_{\bar{k}},R^1j_*(\rho\otimes\rho^\vee))\simeq\bigoplus_i H^1(z_i,\eta_i^*(\rho\otimes\rho^\vee)),$$
and that under this identification, the map
$$H^1(X_{\bar{k}},\rho\otimes\rho^\vee)\to H^0(\overline{X}_{\bar{k}},R^1j_*(\rho\otimes\rho^\vee))\simeq\bigoplus_i H^1(z_i,\eta_i^*(\rho\otimes\rho^\vee))\qquad(5.1.9)$$
is given by $\oplus_i\eta_i^*$.

Let $\overline{K_i}$ be an algebraic closure of $K_i:=\mathrm{Frac}(\overline{R_i})$, and let $\bar{z}_i$ be the geometric point of X associated to z_i by this choice. The inclusion $\eta_i:z_i\to X_{\bar{k}}$ induces a Galois-equivariant inclusion $\gamma_i:\widehat{\mathbb{Z}}(1)\simeq\mathrm{Gal}(\overline{K_i}/K_i)\hookrightarrow\pi_1^{\text{ét}}(X_{\bar{k}},\bar{z}_i)$. Let $\rho_i=\rho\circ\gamma_i$ be the restriction of ρ to $\mathrm{Gal}(\overline{K_i}/K_i)$. (Here we view ρ as a representation of $\pi_1^{\text{ét}}(X_{\bar{k}},\bar{z}_i)$ rather than $\pi_1^{\text{ét}}(X_{\bar{k}},\bar{c})$ via a change-of-basepoint isomorphism, well-defined up to conjugacy. Recall from before that for any two choices of basepoint, the resulting deformation rings S_ρ are canonically isomorphic, so this indeterminacy will not affect our later constructions.)

Let S_i be a hull for the deformation functor D_{ρ_i} for ρ_i (that is, the functor which associates to an Artinian $\overline{\mathbb{Q}_\ell}$-algebra A the set of isomorphism classes of representations $\mathrm{Gal}(\overline{K_i}/K_i) \to GL_n(A)$ lifting ρ_i). As K_i has cohomological dimension 1, S_i is a smooth complete local Noetherian $\overline{\mathbb{Q}_\ell}$-algebra, with

$$\mathfrak{m}_i/\mathfrak{m}_i^2 \simeq H^1(z_i, \eta_i^*(\rho \otimes \rho^\vee))^\vee$$

canonically, where $\mathfrak{m}_i$ is the maximal ideal of S_i.

The map γ_i gives a S_ρ-point of D_{ρ_i}; we may choose a lift to S_i, giving a map $\gamma_i^* : S_i \to S_\rho$ for each i. We must deal with some subtle issues coming from the non-canonicity of this choice.

Lemma 5.1.10. *Suppose $|D| \geq 2$. Then*

(1) the natural transformation $D_\rho \to D_{\rho_i}$ is an epimorphism,
(2) any lift of this map to a map $\gamma_i^* : S_i \to S_\rho$ is injective,
(3) its image is stable under the action of G_k, and
(4) the induced G_k-action on S_i lifts its action on D_{ρ_i}.

Proof. We first prove (1). It is enough to show that for each Artin $\overline{\mathbb{Q}_\ell}$-algebra A, the map $D_\rho \to D_{\rho_i}$ is surjective. Let

$$\tilde{\rho}_i : \mathrm{Gal}(\overline{K_i}/K_i) \to GL_n(A)$$

be an A-point of D_{ρ_i}. We wish to lift it to an A-point of D_ρ.

Recall that if X has genus g, then

$$\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{z}_i) \simeq \widehat{\langle a_1, b_1, \cdots, a_g, b_g, \lambda_1, \cdots, \lambda_{|D|} | \prod_{i=1}^{g}[a_i, b_i] \cdot \prod_{j=1}^{|D|} \lambda_j \rangle},$$

where without loss of generality, $\mathrm{Gal}(\overline{K_i}/K_i) \subset \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{z}_i)$ is topologically generated by λ_1. We wish to find a representation $\tilde{\rho}$ of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{z}_i)$ into $GL_n(A)$, lifting ρ, where the value of λ_1 is specified, say $\tilde{\rho}(\lambda_1) = M$. But we may choose arbitrary lifts of $\rho(a_1), \rho(b_1), \cdots, \rho(a_g), \rho(b_g)$, set $\tilde{\rho}(\lambda_1) = M$, choose $(\tilde{\rho}(\lambda_i))$ to be arbitrary lifts of $\rho(\lambda_i)$ for $i = 2, \cdots, |D| - 1$, and set

$$\tilde{\rho}(\lambda_{|D|}) = \left(\prod_{i=1}^{g} [\tilde{\rho}(a_i), \tilde{\rho}(b_i)] \cdot \prod_{j=1}^{|D|-1} \tilde{\rho}(\lambda_j) \right)^{-1}.$$

We now prove (2). S_i and S_ρ are both power series rings over $\overline{\mathbb{Q}_\ell}$, since $\mathrm{Gal}(\overline{K_i}/K_i)$ and $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{z}_i)$ have cohomological dimension 1. So it is enough to show that the induced map $\mathfrak{m}_i/\mathfrak{m}_i^2 \to \mathfrak{m}_\rho/\mathfrak{m}_\rho^2$ is injective. But this follows by applying (1) to the map

$$D_\rho(\overline{\mathbb{Q}_\ell}[\epsilon]/\epsilon^2) \to D_{\rho_i}(\overline{\mathbb{Q}_\ell}[\epsilon]/\epsilon^2).$$

We now prove (3). The image of γ_i^* is generated by the matrix entries of $\rho_{\text{univ}}(\mathrm{im}(\gamma_i))$. But G_k preserves $\mathrm{im}(\gamma_i)$ by our choice of basepoint. Hence the image is stable under the G_k-action, as desired. Hence we have a G_k-action on S_i so that γ_i^* is G_k-equivariant, by the injectivity of γ_i^*.

Finally, we prove (4). For $\sigma \in G_k$, we wish to show that the diagram

$$\begin{tikzcd} S_i \arrow[r, "\sigma"] \arrow[d] & S_i \arrow[d] \\ D_{\rho_i} \arrow[r, "\sigma"] & D_{\rho_i} \end{tikzcd}$$

commutes, where σ acts as described in the previous paragraph. But this follows from the fact that the diagram

$$\begin{tikzcd} D_\rho \arrow[r, "\sigma"] \arrow[d] & D_\rho \arrow[d] \\ D_{\rho_i} \arrow[r, "\sigma"] & D_{\rho_i} \end{tikzcd}$$

commutes, and the fact that

$$D_\rho \xrightarrow{\gamma_i^*} \mathrm{Hom}(S_i, -)$$

is an epimorphism, by (2). $\square$

Proof of Theorem 5.1.8. It suffices to prove the theorem after deleting several closed points of X, so we may assume that $|D_{\bar{k}}| \geq 2$, where $D = \overline{X} \setminus X$.

From Lemma 5.1.5, we know that for $\alpha \in \mathbb{Z}_\ell^\times$ sufficiently close to 1, there exists $\sigma_\alpha \in G_k$ acting on

$$\mathrm{gr}^i_W \mathfrak{m}_\rho/\mathfrak{m}_\rho^2 = \mathrm{gr}^i_W H^1(X_{\bar{k}}, \rho \otimes \rho^\vee)^\vee$$

via $\alpha^{-i} \cdot \mathrm{Id}$. We claim that for such a σ_α, σ_α^{-1} acts on $\mathrm{gr}^i_W S_\rho$ in the desired manner.

Step 1. We first claim that it suffices to show that the exact sequence

$$0 \to \mathfrak{m}_\rho^2/\mathfrak{m}_\rho^3 \to \mathfrak{m}_\rho/\mathfrak{m}_\rho^3 \to \mathfrak{m}_\rho/\mathfrak{m}_\rho^2 \to 0 \tag{5.1.11}$$

splits σ_α^{-1}-equivariantly. Indeed, multiplication gives an isomorphism

$$\mathrm{Sym}^j(\mathfrak{m}_\rho/\mathfrak{m}_\rho^2) \to \mathfrak{m}_\rho^j/\mathfrak{m}_\rho^{j+1},$$

so the eigenvalues of the σ_α^{-1}-action on $\mathfrak{m}_\rho^j/\mathfrak{m}_\rho^{j+1}$ are contained in $\{\alpha^j, \cdots, \alpha^{2j}\}$. Thus (by the completeness of S_ρ), $\mathfrak{m}_\rho \to \mathfrak{m}_\rho/\mathfrak{m}_\rho^2$ splits σ_α^{-1}-equivariantly if and only if sequence (5.1.11) does, because neither α nor α^2 appears as a generalized eigenvalue for the σ_α^{-1}-action on $\mathfrak{m}_\rho^3$. Such a splitting induces a σ_α^{-1}-equivariant isomorphism

$$\mathrm{Sym}^*(\mathfrak{m}_\rho/\mathfrak{m}_\rho^2) \xrightarrow{\sim} S_\rho,$$

which respects the weight filtrations, where the weight filtration on $\mathrm{Sym}^*(\mathfrak{m}_\rho/\mathfrak{m}_\rho^2)$ is induced from the filtration on $\mathfrak{m}_\rho/\mathfrak{m}_\rho^2$, by the multiplicativity of the weight filtration. The element $\sigma_\alpha^{-1} \in G_k$ clearly acts on $\mathrm{Sym}^*(\mathfrak{m}_\rho/\mathfrak{m}_\rho^2)$ as desired (again by multiplicativity), so we are done.

Step 2. We now show the sequence (5.1.11) does indeed split σ_α^{-1}-equivariantly. Write

$$\mathfrak{m}_\rho/\mathfrak{m}_\rho^2 = V_1 \oplus V_2,$$

where V_i is the α^i-eigenspace of σ_α^{-1}. The σ_α^{-1}-action on $\mathfrak{m}_\rho^2/\mathfrak{m}_\rho^3$ has eigenvalues in $\{\alpha^2, \alpha^3, \alpha^4\}$, so the projection

$$\mathfrak{m}_\rho/\mathfrak{m}_\rho^3 \to \mathfrak{m}_\rho/\mathfrak{m}_\rho^2 \to V_1$$

splits σ_α^{-1}-equivariantly. Thus it suffices to show that the projection

$$\mathfrak{m}_\rho/\mathfrak{m}_\rho^3 \to \mathfrak{m}_\rho/\mathfrak{m}_\rho^2 \to V_2$$

splits σ_α^{-1}-equivariantly.

We first give an explicit description of the subspace $V_2 \subset \mathfrak{m}_\rho/\mathfrak{m}_\rho^2$. Recall that $\mathfrak{m}_\rho/\mathfrak{m}_\rho^2 \simeq H^1(X_{\bar{k}}, \rho\otimes\rho^\vee)^\vee$ canonically. As before, let $\overline{X}$ be the unique smooth, geometrically connected, proper curve containing X, and $j : X \hookrightarrow \overline{X}$ the natural inclusion. By sequence (5.1.1), V_2 is the image of the dual of the natural map

$$H^1(X_{\bar{k}}, \rho\otimes\rho^\vee) \to H^0(\overline{X}_{\bar{k}}, R^1 j_*(\rho\otimes\rho^\vee))$$

(the second map in sequence (5.1.1)). Recall that this map is described in line (5.1.9), i.e. it is given by $\oplus_i \eta_i^*$, where we identify

$$H^0(\overline{X}_{\bar{k}}, R^1 j_*(\rho\otimes\rho^\vee) \simeq \bigoplus_i H^1(z_i, \eta_i^*(\rho\otimes\rho^\vee))$$

and

$$\eta_i^* : H^1(X_{\bar{k}}, \rho\otimes\rho^\vee) \to H^1(z_i, \eta_i^*(\rho\otimes\rho^\vee))$$

is induced by the inclusion

$$\eta_i : \overline{z_i} \to X_{\bar{k}}.$$

Now let $S_i, \mathfrak{m}_i$ be as in Lemma 5.1.10. The map $\mathfrak{m}_i/\mathfrak{m}_i^2 \to \mathfrak{m}_\rho/\mathfrak{m}_\rho^2$ induced by γ_i is dual to η_i^*, so

$$\bigoplus_i \eta_i^{*\vee} : \bigoplus_i \mathfrak{m}_i/\mathfrak{m_i}^2 \to V_2$$

is surjective. As the map above evidently splits σ_α^{-1}-equivariantly (since σ_α^{-1} acts semisimply on its source), it suffices to lift this map to a σ_α^{-1}-equivariant map

$$\bigoplus_i \mathfrak{m}_i/\mathfrak{m_i}^2 \to S_\rho.$$

By Lemma 5.1.10, we have a Galois-equivariant map

$$\gamma_i^* : S_i \to S_\rho,$$

equivariantly lifting the map $D_\rho \to D_{\rho_i}$. The element $\sigma_\alpha^{-1} \in G_k$ acts on $\mathfrak{m}_i^r/\mathfrak{m}_i^{r+1}$ via $\alpha^{2r}\cdot \mathrm{Id}$, so the decomposition of S_i into σ_α^{-1}-eigenspaces gives an isomorphism

$$S_i \simeq \prod_{r\geq 0} \mathfrak{m}_i^r/\mathfrak{m}_i^{r+1}.$$

Let $p_i : \mathfrak{m}_i/\mathfrak{m}_i^2 \to S_\rho$ be the map induced by this decomposition.

Then the map

$$\bigoplus_i p_i : \bigoplus_i \mathfrak{m}_i/\mathfrak{m}_i^2 \to \mathfrak{m}_\rho \to S_\rho$$

is the desired lift of $\bigoplus_i \eta_i^{*\vee}$, completing the proof. $\square$

5.2. **The integral analysis.** In the previous section, we constructed natural elements $\sigma_\alpha \in G_k$ whose action on the deformation ring of an irreducible arithmetic $\overline{\mathbb{Q}_\ell}$-representation we understand well. We now consider an absolutely irreducible representation

$$\bar{\rho} : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{c}) \to GL_n(\mathbb{F}_{\ell^r}),$$

and we use the results of the previous section to analyze the action of σ_α on $R_{\bar{\rho}}$, assuming $\bar{\rho}$ admits a lift which arises from geometry.

Lemma 5.2.1. *Let V be a finite free $\overline{\mathbb{Z}_\ell}$-module, and $T : V \to V$ an endomorphism. Let $W \subset V$ be a T-stable submodule with $W, V/W$ free $\overline{\mathbb{Z}_\ell}$-modules. Suppose that $T|_W = \alpha \cdot \mathrm{Id}$, and that T acts on V/W via $\beta \cdot \mathrm{Id}$, where $\alpha, \beta \in \overline{\mathbb{Z}_\ell}, \alpha \neq \beta$. Let $v \in V \otimes \overline{\mathbb{Q}_\ell}$ be such that*

(1) $Tv = \beta v$, and
(2) $v \in V + (W \otimes \overline{\mathbb{Q}_\ell})$.

Then $(\alpha - \beta) \cdot v \in V$.

Proof. Choose a $\overline{\mathbb{Z}_\ell}$-basis $\{e_i\}$ of V such that $e_1, \cdots, e_{\dim W}$ are a $\overline{\mathbb{Z}_\ell}$-basis of W. Then $v = \sum a_i e_i$, where the $a_i \in \overline{\mathbb{Q}_\ell}$, and $a_i \in \overline{\mathbb{Z}_\ell}$ for $i > \dim W$. We have

$$Te_i = \begin{cases} \alpha e_i & i \leq \dim W \\ \beta e_i + \sum_{j=1}^{\dim W} b_{ij} e_j & i > \dim W \end{cases}$$

where the $b_{ij} \in \overline{\mathbb{Z}_\ell}$. Now

$$\begin{aligned} \beta \cdot v &= Tv \\ &= \sum_{i=1}^{\dim W} \alpha a_i e_i + \sum_{i=\dim W+1}^{\dim V} a_i \left(\beta e_i + \sum_{j=1}^{\dim W} b_{ij} e_j \right) \end{aligned}$$

Equating coefficients for e_i, we have for each $i \leq \dim W$,

$$\beta a_i = \alpha a_i + \sum_{j=\dim W+1}^{\dim V} a_j b_{ji}.$$

Hence

$$a_i = \frac{\sum_{j=\dim W+1}^{\dim V} a_j b_{ji}}{\beta - \alpha},$$

whence the result follows. □

Now if

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{c}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

is a lift of $\bar{\rho}$ (hence irreducible), the natural induced map $R_{\bar{\rho}} \to S_\rho$ exhibits S_ρ as the completion of $R_{\bar{\rho}} \otimes \overline{\mathbb{Q}_\ell}$ at the maximal ideal corresponding to ρ. The map $R_{\bar{\rho}} \to S_\rho$ gives S_ρ a natural integral structure (namely the image of the induced map $R_{\bar{\rho}} \otimes \overline{\mathbb{Z}_\ell} \to S_\rho$). We now apply the computation in Lemma 5.2.1 to estimate the denominators required to write down eigenvectors for the σ_α-action on S_ρ, relative to this integral structure.

Lemma 5.2.2. *Let*

$$\rho : \pi_1(X_{\bar{k}}, \bar{c}) \to GL_n(\overline{\mathbb{Z}_\ell})$$

be a continuous representation lifting $\bar{\rho} \otimes \overline{\mathbb{F}_\ell}$, such that $\rho \otimes \overline{\mathbb{Q}_\ell}$ arises from geometry. Let $S_\rho^{int} \subset S_\rho$ be the image of the induced map $R_{\bar{\rho}} \otimes \overline{\mathbb{Z}_\ell} \to S_\rho$.

Let σ_α be as in Theorem 5.1.8. Let $x \in S_\rho$ be such that

(1) $\sigma_\alpha \cdot x = \alpha^i x$, for $i = 1$ or $i = 2$, and
(2) $x \in S_\rho^{int} + W^{-i-1}$.

Then for any $r \geq i$,

$$\left(\prod_{j=i+1}^{r} (\alpha^i - \alpha^j) \right) \cdot x \in S_\rho^{int} + W^{-r-1}.$$

(Here if $r = i$ we take the product above to be the empty product, i.e. 1.)

Proof. This follows by induction on r from Lemma 5.2.1. The case $r = i$ follows from hypothesis (2) of the lemma. Now let $r > i$. Suppose the result holds for $r - 1$; we now prove it for r. Let $\bar{x} \in S_\rho^{\text{int}}$ be any element such that

$$\bar{x} = \left(\prod_{j=i+1}^{r-1} (\alpha^i - \alpha^j) \right) \cdot x \bmod W^{-r}.$$

Set

$$V = (S_\rho^{\text{int}} \cap W^{-r} + \overline{\mathbb{Z}_\ell} \cdot \bar{x})/(S_\rho^{\text{int}} \cap W^{-r-1}),$$

$$W = (S_\rho^{\text{int}} \cap W^{-r})/(S_\rho^{\text{int}} \cap W^{-r-1}),$$

and

$$v = \left(\prod_{j=i+1}^{r-1} (\alpha^i - \alpha^j) \right) \cdot x \bmod W^{-r-1}.$$

Then the hypotheses of Lemma 5.2.1 are satisfied by the induction hypothesis, giving the proof. □

Lemma 5.2.3. *Let $\alpha \in \mathbb{Z}_\ell^\times$ be an ℓ-adic unit which is not a root of unity. Let s be the least positive integer such that $\alpha^s = 1 \bmod \ell$; let $\epsilon = 1$ if $\ell = 2$ and 0 otherwise. Then for any $n \geq 1$,*

$$\sum_{i=1}^{n} v_\ell(1 - \alpha^i) \leq C(\alpha) \cdot n,$$

where

$$C(\alpha) = \frac{1}{s} \left(v_\ell(\alpha^s - 1) + \frac{1}{\ell - 1} \right) + \epsilon.$$

Proof. This is Lemma 3.10 of [Lit18]. □

5.3. **The proof.** We now give the proof of Theorem 1.1.11.

Proof of theorem 1.1.11. Let $E_{\bar\rho}$ be the rigid generic fiber of $R_{\bar\rho}$ (this agrees with the notation of Section 2 by [Che14, Theorems B and F]); as X is an affine curve, $E_{\bar\rho}$ is (non-canonically) analytically isometric to the open unit ball. Then ρ gives a $\overline{\mathbb{Q}_\ell}$-point z_ρ of $E_{\bar\rho}$; we may view z_ρ as a map

$$z_\rho : R_{\bar\rho}[1/\ell] \to \overline{\mathbb{Q}_\ell}.$$

Let $\mathfrak{m} \subset R_{\bar\rho}[1/\ell]$ be the kernel of this map, and let S'_ρ be the completion of $R_{\bar\rho}[1/\ell]$ at $\mathfrak{m}$, so that $S_\rho \simeq S'_\rho \widehat{\otimes} \overline{\mathbb{Q}_\ell}$. Let $\mathfrak{m}'_\rho$ be the maximal ideal of S'_ρ. Let L be the residue field of S'_ρ, so that S'_ρ is non-canonically isomorphic to $L[[x_1, \cdots, x_m]]$; let $S^{\mathrm{int}'}_\rho = R_{\bar\rho} \otimes_{W(\mathbb{F}_{\ell^r})} \mathscr{O}_L \subset S'_\rho$ be the extension of scalars of $R_{\bar\rho}$ to $\mathscr{O}_L$, so that $S^{\mathrm{int}'}_\rho$ is non-canonically isomorphic to $\mathscr{O}_L[[x_1, \cdots, x_m]]$, where $\{x_1, \cdots, x_m\}$ generate $\mathfrak{m}'_\rho \cap S^{\mathrm{int}'}_\rho$. Choose such an isomorphism. The rigid generic fiber of $S^{\mathrm{int}'}_\rho$ is the open unit ball over L.

Choose $\alpha \in \mathbb{Z}_\ell^\times$ not a root of unity such that

1. there exists $\sigma_\alpha \in G_k$ as in Theorem 5.1.8, and
2. $C(\alpha)$ is minimal among all α satisfying (1).

(Here $C(\alpha)$ is defined as in Lemma 5.2.3.)

We claim that we may take $N = N(c(\rho), \ell)$ to be any rational number greater than $3C(\alpha)$; note that this may be bounded from above purely in terms of ℓ and $c(\rho)$, as the notation suggests.

Let $\sigma_\alpha \in G_k$ be as in Theorem 5.1.8. For $1 \geq r > 0$, let $U_r \subset E_{\bar\rho}$ be the closed ball of radius r around ρ — set-theoretically, this is the set

$$\{\tilde\rho \in E_{\bar\rho} \big| |\operatorname{ch}(\tilde\rho) - \operatorname{ch}(\rho)|_\ell \leq r\}.$$

Note that $U_{\ell^{-s}}$ is stable under the σ_α-action on $E_{\bar\rho}$. There is a unique $\pi_1^{\text{ét}}(X_{\bar k}, \bar c)$-lattice $V_\rho \subset \rho$, up to homothety. By [Car94, Théorème 1], for $s \in \mathbb{Q}$, $U_{\ell^{-s}}$ is the same as the set of $\tilde\rho$ admitting an $\pi_1^{\text{ét}}(X_{\bar k}, \bar c)$-stable $\overline{\mathbb{Z}_\ell}$-lattice $V_{\tilde\rho}$ such that $V_{\tilde\rho}/\ell^s V_{\tilde\rho} \simeq V_\rho/\ell^s V_\rho$.

Let $\mathscr{O}_{U_{\ell^{-s}}}$ be the set of functions on $U_{\ell^{-s}}$. Explicitly, under our chosen isomorphism $S^{\mathrm{int}'}_\rho \simeq \mathscr{O}_L[[x_1, \cdots, x_m]]$ and the induced isomorphism $S'_\rho \simeq L[[x_1, \cdots, x_m]]$, $\mathscr{O}_{U_{\ell^{-s}}} \subset S'_\rho$ is the subring

$$\left\{\sum a_I x^I \in S'_\rho \mid v_\ell(a_I) + |I| \cdot s \to \infty\right\},$$

topologized via the Gauss norm (which makes it into a Tate algebra). (Here $I = (i_1, \cdots, i_m)$ is a multi-index with $i_j \geq 0$, $x^I = x_1^{i_1} \cdots x_m^{i_m}$, and $|I| = \sum_j i_j$.)

Choose a basis of integral σ_α-eigenvectors of $\mathfrak{m}'_\rho/{\mathfrak{m}'_\rho}^2$, and lift it to a set of σ_α-eigenvectors $\{e_1, \cdots, e_m\}$ of S'_ρ (we may do this by the proof of Theorem 5.1.8). After a linear change of coordinates, we may assume $e_i \equiv x_i \bmod (\mathfrak{m}'_\rho)^2$. We claim that for $s > 2C(\alpha)$, we have $e_1, \cdots, e_m \subset \mathscr{O}_{U_{\ell^{-s}}}$. Indeed, by Lemma 5.2.2, we have

$$\left(\prod_{i=1}^{2r-1} (1 - \alpha^i)\right) \cdot e_j \in S^{\mathrm{int}'}_\rho + W^{-2r} \subset S^{\mathrm{int}'}_\rho + \mathfrak{m}'^r_\rho,$$

and hence if

$$e_i = \sum a_{I,i} x^I,$$

we have

$$v_\ell(a_{I,i}) \geq -\sum_{i=1}^{2|I|-1} v_\ell(1-\alpha^i) \geq -(2|I|-1)C(\alpha) \tag{5.3.1}$$

by Lemma 5.2.3, whence the claim follows.

Now choose any $s > 3C(\alpha)$, and suppose that $\tilde{\rho} \in U_{\ell^{-s}}$ is arithmetic. We wish to show that $\tilde{\rho} \simeq \rho$. We may view $\tilde{\rho}$ as a map

$$z_{\tilde{\rho}} : \mathscr{O}_{U_{\ell^{-s}}} \to \overline{\mathbb{Q}_\ell},$$

where $v_\ell(z_{\tilde{\rho}}(x_i)) \geq s$. By the arithmeticity of $\tilde{\rho}$, there exists an integer M such that $z_{\tilde{\rho}}$ is equivariant for the natural σ_α^M-action on $\mathscr{O}_{U_{\ell^{-s}}}$, and the trivial action on $\overline{\mathbb{Q}_\ell}$. In particular, for each i, we have $z_{\tilde{\rho}}(e_i) = 0$, as e_i is a σ_α-eigenvector.

In other words, viewing the e_i as power series in the x_j, we have that the

$$e_i = x_i + \text{higher order terms},$$

and $(z_{\tilde{\rho}}(x_i))_i$ is a zero of the e_j satisfying $v_\ell(z_{\tilde{\rho}}(x_i)) \geq s$. We would like to show that this implies $z_{\tilde{\rho}}(x_i) = 0$ from our estimate 5.3.1 on the coefficients of the e_i. But indeed, we have

$$\begin{aligned} z_{\tilde{\rho}}(e_i) &= z_{\tilde{\rho}}(x_i) + \sum_{|I|\geq 2} a_{I,i} z_{\tilde{\rho}}(x^I) \\ &= 0. \end{aligned}$$

Now choose i such that $v_\ell(z_{\tilde{\rho}}(x_i))$ is minimized. Then estimate 5.3.1 and the ultrametric inequality immediately imply that $z_{\tilde{\rho}}(x_i) = 0$, as by our choice of s,

$$v_\ell(a_{I,i} z_{\tilde{\rho}}(x^I)) > v_\ell(z_{\tilde{\rho}}(x_i))$$

for any $|I| \geq 2$. As we chose i minimizing $v_\ell(z_{\tilde{\rho}}(x_i))$, this implies that all $z_{\tilde{\rho}}(x_j) = 0$, as desired. □

Remark 5.3.2. From the proof, we see that we may take N to be any rational number greater than $3C(\alpha)$, where $\alpha \in \mathbb{Z}_\ell^\times$ is such that there exists σ_α as in Theorem 5.1.8 (or Lemma 5.1.5).

5.4. **The case of representations with finite image.** Finally, we prove Theorem 1.1.13. The proof is essentially identical to that of the main theorem of [Lit18], with additional input from a result of Serre, of which we were unaware at the time of writing [Lit18]. We indicate how to use this additional input below.

Proof of Theorem 1.1.13. By passing to the cover of X defined by $\ker(\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to G$, we may assume ρ is trivial. Now the proof is essentially identical to that of Theorem 1.2 of [Lit18] (which is stated for $\mathbb{Q}_\ell$-representations, but works equally well for $\overline{\mathbb{Q}_\ell}$-representations). While Remark 4.3 of [Lit18] indicates that the constant N of that theorem depends on the index of the image of $G_k \to GL(H^1(X_{\bar{k}}, \mathbb{Z}_\ell)$ in the $\mathbb{Z}_\ell$-points of its Zariski closure, in fact it depends only on the set Z of $\alpha \in \mathbb{Z}_\ell^\times$ such that there exists $\sigma_\alpha \in G_k$ such that σ_α acts on $\mathrm{gr}^i_W H^1(X_{\bar{k}}, \mathbb{Z}_\ell)$ via $\alpha^i \cdot \mathrm{Id}$.

By Remark 3.11 of [Lit18], it suffices to show that the index of Z in $\mathbb{Z}_\ell^\times$ is bounded independent of ℓ; indeed, the proof of Theorem 1.2 of [Lit18] shows that any arithmetic representation trivial modulo ℓ^s, with s greater than the constant r_α defined in [Lit18, Remark 3.11], is trivial. But direct computation shows that this constant tends to zero as $\ell \to \infty$ if the index of Z in $\mathbb{Z}_\ell^\times$ is bounded.

The case where X has genus zero is covered in [Lit18], so we assume the genus of X is at least one.

Let $\overline{X}$ be the unique smooth proper geometrically connected k-curve containing X, and let $D = \overline{X} \setminus X$. Replace k with a finite extension, so that $D = \{x_1, \cdots, x_m\}$ for $x_i \in \overline{X}(k)$.

Now we claim that Z contains the set of $\alpha \in \mathbb{Z}_\ell^\times$ so that there exists $\sigma'_\alpha \in G_k$ such that σ'_α acts on $\mathrm{gr}^1_W H^1(X_{\bar{k}}, \mathbb{Z}_\ell)$ via $\alpha \cdot \mathrm{Id}$. Indeed, $\mathrm{gr}^2_W H^1(X_{\bar{k}}, \mathbb{Z}_\ell)$ is a direct sum of copies of the cyclotomic character, so by Poincaré duality, any such σ'_α in fact acts on $\mathrm{gr}^2_W H^1(X_{\bar{k}}, \mathbb{Z}_\ell)$ as desired.

But now, as desired, the index of Z in $\mathbb{Z}_\ell^\times$ is bounded independent of ℓ by a result of Serre [Ser13, Lettre à Ken Ribet, p. 60], using that $\mathrm{gr}^1_W H^1(X_{\bar{k}}, \mathbb{Z}_\ell)$ is dual to the ℓ-adic Tate module of the Jacobian of $\overline{X}$. $\square$

Remark 5.4.1. It is natural to conjecture that if ρ_ℓ is a compatible system of irreducible lisse sheaves on a curve, $c(\rho_\ell)$ is bounded independently of ℓ. More precisely, fix $p, q \in \mathbb{Z}_{\geq 0}$. If k is a number field and $f : X \to Y$ is a smooth proper morphism over k, let $H_\ell \subset \mathbb{Z}_\ell^\times$ be the set of $\alpha \in \mathbb{Z}_\ell^\times$ such that there exists $\sigma_\alpha \in G_k$ with σ_α acting on $\mathrm{gr}^i_W H^p(Y_{\bar{k}}, R^q f_* \mathbb{Q}_\ell)$ via $\alpha^i \cdot \mathrm{id}$. We conjecture that the index of $H_\ell \subset \mathbb{Z}_\ell^\times$ is bounded independent of ℓ, and indeed that this index is equal to 1 for almost all ℓ.

Assuming the Tate conjecture, one may show this index is uniformly bounded via the argument of [Win02, §2.3]. This gives, by the proof of Theorem 1.1.11, a much stronger version of Theorem 1.1.11. It implies (on the Tate conjecture) that if ρ_ℓ is a compatible system of ℓ-adic representations arising from geometry (i.e. the monodromy representation underlying $R^q f_* \mathbb{Q}_\ell$ as above), then the constants $N(c(\rho_\ell), \ell)$ of Theorem 1.1.11 tend to zero as $\ell \to \infty$.

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