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C-DOC: Co-State Desensitized Optimal Control

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Abstract— **In this paper, co-states are used to develop a framework that desensitizes the optimal cost. A general formulation for an optimal control problem with fixed final time is considered. The proposed scheme involves elevating the parameters of interest into states, and further augmenting the co-state equations of the optimal control problem to the dynamical model. A running cost that penalizes the co-states of the targeted parameters is then added to the original cost function. The solution obtained by minimizing the augmented cost yields a control which reduces the dispersion of the original cost with respect to parametric variations. The relationship between co-states and the cost-to-go function, for any given control law, is established substantiating the approach.**

I. Introduction

Obtaining robust solutions against parametric variations in optimal control problems is a requirement in various applications, particularly in the fields of aerospace and robotics. For many problems, it is essential that for a given performance criterion, one can also ensure minimal dispersion in the total cost, in spite of variations in the problem parameters. Previous attempts have primarily aimed at stability and performance criteria defined over an infinite horizon [1], [2]. Questions regarding the sensitivity of the trajectory (or the cost) explicitly and its implications on the performance have been largely overlooked. In [3], the authors have addressed the implications of parametric uncertainties over a finite horizon using linear matrix inequalities (LMIs), however, the problem does not address the sensitivity of the performance with respect to the parameters. Traditionally, robust optimal control [4]–[6] and feedback control synthesis [7] have been used to address the issue of parametric uncertainty, with an inherent trade-off between cost and robustness to be decided. Indeed, the increased cost is incurred due to additional control effort, in magnitude or over time. The main goal of *desensitized optimal control* (DOC) is to alleviate the additional effort induced onto the control feedback loop by, instead, picking a trajectory which is less sensitive to variations under parametric uncertainty.

Early work on trajectory sensitivity design include those of Winsor and Roy [8], who developed a technique to design controllers that provide assurance for system performance under mathematical modeling inaccuracy. The feasibility of the technique was established with appropriate simulation results. However, their work has been restricted to linear systems. Following that work, several approaches including sensitivity-reduction for linear regulators, using increased-order augmented system [9], modification of weighting matrix [10], feedback [11], [12], and an augmented cost function [13], [14], were all thoroughly analyzed. The approach of using an augmented cost function was further tested on the linear quadratic regulator (LQR) problem, which was later applied for active suspension control in passenger cars [14]. The work by Seywald et al. on desensitized optimal control makes use of sensitivity matrices to obtain an optimal open-loop trajectory that is insensitive to first-order parametric variations [15], [16]. However, the sensitivity matrix based approaches [15], [16] requires propagating the original states, the targeted parameters, and the elements in the sensitivity matrix, resulting to a total of $(n{+}\ell)^2{+}n{+}\ell$ number of states. An alternative approach was presented in [17] where the dimensionality of the state-space for the augmented problem is reduced to $n + n\ell$, using traditional sensitivity functions.

Desensitization of a solution can include addressing the problem of minimizing variations: a) in the optimal trajectory; b) in the final state; or c) in the optimal cost under variation in model parameters, for a given optimal control problem. The former two cases were dealt with in our previous work [17]. Three different formulations that employ sensitivity functions were put forth which desensitize either entire trajectories or the state at a particular time (e.g., final time).

An approach to desensitize the cost for an optimal control problem with fixed final time is presented in this paper. To this end, we recall that the co-states in an optimal control problem are a measure of the sensitivity of the value function with respect to the states along the optimal trajectory [18], [19]. In this paper, we first prove that the co-states indeed capture sensitivity of the *cost-to-go* function with respect to perturbations in the state given *any* prescribed control law $u(t)$, not just the optimal one. Using this fact, a new approach to solve the DOC problem is presented.

II. Problem Formulation

Consider a standard optimal control problem of the form

$$\inf_{u} \quad \mathcal{J}(u,p) \triangleq \phi(x(t_f),t_f) + \int_{t_0}^{t_f} L(x(t),u(t),t)\,\mathrm{d}t, \quad (1)$$

subject to

$$\dot{x} = f(x,p,u,t), \quad x(t_0) = x_0,, \quad (2a)$$

$$\psi(x(t_f),t_f) = 0, \quad (2b)$$

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where $t \in [t_0, t_f]$ denotes time, with t_0 being the initial time and t_f being the final time (both assumed to be fixed), $p \in \mathcal{P} \subset \mathbb{R}^\ell$ are ℓ unknown, possibly time-varying, model parameters, $x(t) \in \mathbb{R}^n$ denotes the state, with x_0 being the fixed state at t_0. The control $u \in \mathcal{U} = \{\text{Piecewise Continuous (PWC)}, u(t) \in U, \ \forall t \in [t_0, t_f]\}$, with $U \subseteq \mathbb{R}^m$ being the set of allowable values of $u(t)$, $\phi : \mathbb{R}^n \times [t_0, t_f] \to \mathbb{R}$, the terminal cost function, and $L : \mathbb{R}^n \times \mathbb{R}^m \times [t_0, t_f] \to \mathbb{R}$, the running cost. Finally, $\psi : \mathbb{R}^n \times [t_0, t_f] \to \mathbb{R}^k$ is a function representing k-number of constraint equations at the final time. The above problem is to be solved by finding the optimal control $u^* \in \mathcal{U}$ that minimizes the cost function in (1). The solution involves the optimal trajectory $x^*(t)$, $t \in [t_0, t_f]$, determined from $\dot{x}^*(t) = f(x^*(t), p, u^*(t), t)$ subject to $x^*(t_0) = x_0$, and the optimal cost $\mathcal{J}^* = \phi(x^*(t_f), t_f) + \int_{t_0}^{t_f} L(x^*(t), u^*(t), t)\,\mathrm{d}t$.

The system dynamics represented by the function $f(x, p, u, t)$ contains the model parameters p which are assumed to be constant. It is understood that the optimal solution (constituting the cost, and the trajectory) is model-sensitive and, if changes in the parameters p occur at any time $t \in [t_0, t_f]$, then the optimality of the obtained solution is not guaranteed. Consequently, the optimal control problem has to be resolved for each new value of the parameter vector. If the optimal solution u^* is used despite the parameter variations, one can expect a dispersion in the optimal trajectory (and/or cost $\mathcal{J}^*$). With a motivation to minimize the dispersion of the final state $x^*(t_f)$ of the optimal solution, under parametric uncertainties, Seywald and Kumar constructed an augmented cost function using sensitivity matrices [15]. The approach goes as follows.

First, the parameters of interest and the corresponding entries in the sensitivity matrix are elevated to states, and the augmented state $[\tilde{x}^\top \ (\operatorname{vec} S)^\top]^\top$, where $\tilde{x} = [x^\top \ p^\top]^\top$, along with the corresponding dynamics and initial conditions are derived. The sensitivity of the vector $\tilde{x}(t)$[1] at time t with respect to perturbations in the initial state vector $\tilde{x}(t_0) = \tilde{x}_0$ is denoted as $S(t|t_0, \tilde{x}_0)$ That is,

$$S(t|t_0, \tilde{x}_0) = \frac{\partial \tilde{x}(t|t_0, \tilde{x}_0)}{\partial \tilde{x}_0}. \tag{3}$$

The dynamics of the state $\tilde{x}$ can be written as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}, u, t) = [f^\top(x, p, u, t) \quad 0_{1\times\ell}]^\top, \\ \tilde{x}(t_0) &= \tilde{x}_0 = [x_0^\top \quad p_0^\top]^\top, \end{aligned} \tag{4}$$

and

$$\dot{S}(t|t_0, \tilde{x}_0) = \frac{\partial \tilde{f}}{\partial \tilde{x}} S(t|t_0, \tilde{x}_0), \quad S(t_0|t_0, \tilde{x}_0) = I_{(n+\ell)}, \tag{5}$$

where p_0 is the nominal value of the parameter vector, and where $S(t|t_0, \tilde{x}_0)$ represents the sensitivity of the vector $\tilde{x}(t)$ at time t with respect to perturbations in the initial state vector $\tilde{x}(t_0)$.

The augmented cost function, given in (6) below, is then minimized to obtain an optimal solution with the final state being "desensitized" with respect to the parameter variations

$$\begin{aligned} \mathcal{J}_s(u, p) =& \mathcal{J}(u, p) \\ &+ \int_{t_0}^{t_f} \| \operatorname{vec}\left(S(t_f|t_0, \tilde{x}_0) S(t|t_0, \tilde{x}_0)^{-1}\right) \|^2_{Q(t)} \Big) \mathrm{d}t, \end{aligned} \tag{6}$$

with $Q(t) \geq 0$, for all $t \geq t_0$. Note that the sensitivity matrix of Seywald in (3) has the form of a state transition matrix and its properties are exploited to construct the sensitivity of the final state with respect to the variations in the state at time $t \in [0, t_f]$, which is then plugged into the running cost (6). This is elaborated upon in Ref. [15]. The approach achieves the desensitization of the final state.

A. *Problem Formulation*

In this work, we are interested in desensitizing the cost itself. By denoting

$$\mathcal{J}(u, p) = \int_{t_0}^{t} L(x(s), u(s), s)\mathrm{d}t + C(\tilde{x}(t), u, t), \tag{7}$$

$$C(\tilde{x}(t), u, t) = \int_{t}^{t_f} L(x(s), u(s), s)\mathrm{d}t + \phi(x(t_f), t_f), \tag{8}$$

we immediately notice that the parametric variation at time t, affects the total cost $\mathcal{J}(u, p)$ only through the cost-to-go $C(\tilde{x}(t), u, t)$. Thus, the sensitivity of the total cost for a parametric variation at time t from its nominal value p_0 can be captured through the term

$$S_C(x(t), p_0, u, t) = \frac{\partial C}{\partial p}(\tilde{x}(t), u, t)\Big|_{p=p_0}. \tag{9}$$

There are several ways to capture the effect of the parametric variations on the cost, one of which is to consider the following sensitivity cost

$$\mathcal{J}_c(u, p_0) = \int_{t_0}^{t_f} \|S_C(x(t), p_0, u, t)\|^2_{Q(t)}\mathrm{d}t, \tag{10}$$

for some $Q(t) \geq 0$, for all $t \geq t_0$.

There are three major formulations relevant to the problem of cost-based desensitization, which are as follows.

Problem 2.1: Solve

$$\inf_{u\in\mathcal{U}} \quad \mathcal{J}(u, p_0), \tag{11a}$$

$$\text{subject to} \quad \mathcal{J}_c(u, p_0) \leq D. \tag{11b}$$

Let us denote the solution of Problem 2.1 to be the "cost-desensitization" function $\mathsf{J}(D)$ which represents the optimal cost given a bound on the sensitivity metric. A similar problem is to consider minimizing the sensitivity of the cost for a given bound on the performance index, as presented below.

Problem 2.2: Solve

$$\inf_{u\in\mathcal{U}} \quad \mathcal{J}_c(u, p_0), \tag{12a}$$

$$\text{subject to} \quad \mathcal{J}(u, p_0) \leq J. \tag{12b}$$

Let us denote the solution of Problem 2.2 to be the "desensitization-cost" function $\mathsf{D}(J)$. Finding analytical or numerical solutions to $\mathsf{J}(D)$ or $\mathsf{D}(J)$ are challenging. How-

[1]From time to time we will denote $\tilde{x}(t)$ as $\tilde{x}(t|t_0, \tilde{x}_0)$ to explicitly represent the dependency on the initial conditions $\tilde{x}_0 = [x(t_0)^\top \ p(t_0)^\top]$

ever, $\mathsf{J}(D)$ or $\mathsf{D}(J)$ can be constructed by solving the following family of optimization problems for all $\alpha \in [0, \infty)$.

Problem 2.3: Solve

$$\inf_{u \in \mathcal{U}} \mathcal{J}(u, p_0) + \alpha \mathcal{J}_c(u, p_0) \tag{13}$$

By observing that the scalar α can be absorbed into the matrix $Q(t)$, we will rewrite the objetive function in Problem 2.3 as

$$\mathcal{J}_s(u) = \mathcal{J}(u, p_0) + \mathcal{J}_c(u, p_0).$$

When the sensitivity cost has zero weight ($Q(t) \equiv 0$), we solve problem (1) and retrieve $\limsup_{D \to \infty} \mathsf{J}(D)$, and as we increase the weight on the sensitivity cost (through $Q(t)$), we arrive at an optimal control whose performance is more insensitive to the variations in the parameters. In the limit when $Q(t) \to \infty$ for all t, we retrieve $\limsup_{J \to \infty} \mathsf{D}(J)$. In this work, we will focus on minimizing $\mathcal{J}_s(u)$. Detailed analysis of $\mathsf{J}(D)$ and $\mathsf{D}(J)$ will appear elsewhere.

The new optimization problem we are interested in solving is

$$\inf_{u} \quad \mathcal{J}_s(u), \tag{14a}$$
$$\text{subject to } \dot{x} = f(x, p_0, u, t), \quad x(t_0) = x_0, \tag{14b}$$
$$\psi(x(t_f), t_f) = 0. \tag{14c}$$

The following section presents a formal theorem for the fact that the co-states capture the sensitivity of the cost-to-go function for any given control input $\bar{u}(t)$, that satisfies the terminal constraint (14c) with nominal value of the parameter p_0. The result would allow us to penalize a weighted norm of the co-states, with their dynamics obtained from the adjoint equations, that desensitizes the cost function with respect to the variations in the targeted parameters.

III. Co-states and Desensitized Optimal Control

In this section we characterize the cost-sensitivity $S_C(x(t), p_0, u, t)$ in terms of the co-state process associated with the optimal control problem given by (1)-(2b). The following theorem shows that the sensitivity of the cost-to-go function with respect to the state at time t can be represented by a co-state process λ with certain boundary conditions at the final time t_f.

Theorem 3.1: Consider the dynamical system $\dot{x} = f(x, u, t)$, evolving under a given control law $\bar{u} \in \bar{\mathcal{U}} \subseteq \mathcal{U}$, where

$$\bar{\mathcal{U}} = \Big\{ \bar{u} : [t_0, t_f] \to \mathbb{R}^m \text{ is PWC}, \bar{u}(t) \in U, \ \psi(x(t_f), t_f) = 0, \ x(t_f) = x_0 + \int_{t_0}^{t_f} f(x(t), \bar{u}(t), t)\, \mathrm{d}t \Big\}.$$

Then, for a cost-to-go function (associated with the cost functional (1)) with $x = x(t)$

$$C(x, \bar{u}, t) = \phi(x(t_f), t_f) + \int_t^{t_f} L(x(\tau), \bar{u}(\tau), \tau)\, \mathrm{d}\tau, \tag{15}$$

under the control $\bar{u} \in \bar{\mathcal{U}}$, the sensitivity of the cost-to-go function with respect to the state x at time t is,

$$\lambda^\top(t) = \frac{\partial C}{\partial x}(x(t), \bar{u}, t), \tag{16}$$

which obeys the dynamics

$$\dot{\lambda}^\top(t) = -\frac{\partial H}{\partial x}(x(t), \bar{u}, \lambda(t), t), \tag{17}$$

where

$$H(x, u, \lambda, t) = L(x, u, t) + \lambda^\top f(x, u, t). \tag{18}$$

Furthermore, the terminal condition for (17) is given by

$$\lambda(t_f) = \frac{\partial \phi}{\partial x}(x(t_f), t_f). \tag{19}$$

Proof: The proof is presented in Appendix A. ■

It is interesting to note that the theorem holds not only for the optimal control (a result that follows directly from the maximum principle [20]), but for any control law that is piecewise-continuous and ensures that the terminal constraint is met. The C-DOC problem can now be fully formulated using this result.

For the C-DOC problem the augmented state is $\tilde{x} = [x^\top, p^\top]$ with dynamics given in (4). The Hamiltonian, defined in Theorem 3.1, for this system, can be written as

$$\begin{aligned} H(\tilde{x}, u, \lambda, \mu, t) &= L(x, u, t) + \lambda^\top \dot{x} + \mu^\top \dot{p} \\ &= L(x, u, t) + \lambda^\top f(x, p, u, t), \end{aligned} \tag{20}$$

where λ and μ are the co-states corresponding to state x and vector of parameters defined by p, respectively. The corresponding adjoint equations are given by

$$\begin{aligned} \dot{\lambda}^\top &= -\frac{\partial H}{\partial x}(\tilde{x}, u, \lambda, \mu, t) \\ &= -\lambda^\top \frac{\partial f}{\partial x}(x, p, u, t) - \frac{\partial L}{\partial x}(x, u, t), \end{aligned} \tag{21}$$

$$\dot{\mu}^\top = -\frac{\partial H}{\partial p}(\tilde{x}, u, \lambda, \mu, t) = -\lambda^\top \frac{\partial f}{\partial p}(x, p, u, t). \tag{22}$$

Since the co-states represent the sensitivity of the cost-to-go function for a given control input $u(t)$ (Theorem 3.1), they can be expressed as

$$\lambda(t)^\top = \frac{\partial C}{\partial x}(\tilde{x}(t), u, t), \tag{23}$$

$$\mu(t)^\top = \frac{\partial C}{\partial p}(\tilde{x}(t), u, t), \tag{24}$$

for a given control $u \in \bar{\mathcal{U}}$, this results in the trajectory $x(t)$ for $t_0 \leq t \leq t_f$, where

$$C(\tilde{x}, u, t) = \phi(x(t_f), t_f) + \int_t^{t_f} L(x(\tau), u(\tau), \tau)\, \mathrm{d}\tau.$$

Note that p is an augmented state in the given problem and affects the cost $\mathcal{J}$ through the state x, whose dynamics is a function of p. Since we have used $\dot{p} = 0$ and $p(t_0) = p_0$, we have ensured that $p(t) = p_0$. Thus, by comparing equations (9) and (24), we obtain $\mu(t) = S_C(x(t), p_0, u, t)$. Therefore, weighting the co-state in the existing cost function will ensure that the solution of the augmented problem minimizes the sensitivity of the cost $\mathcal{J}$ with respect to parametric variations. This results in an updated optimal control problem with an augmented cost, accounting for the sensitivity component, given by

$$\mathcal{J}_s(u) = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} \left[L(x(t), u(t), t) + \mu^\top(t) Q(t) \mu(t) \right] \mathrm{d}t. \quad (25)$$

Minimizing the cost (25) subject to the dynamics (4), terminal constraint (2b), and the transversality conditions (19) with

$$\mu(t_f) = 0, \quad (26)$$

yields a desensitized optimal control problem for the original problem. Here, $Q(t) \in \mathbb{R}^{\ell \times \ell}$ is a user-defined positive semi-definite weighting function and is generally of the form

$$Q(t) \equiv \mathrm{diag}(\alpha_1(t), \ldots, \alpha_\ell(t)). \quad (27)$$

This co-state based approach requires formulating $2(n + \ell)$ number of states, as compared to the higher $2(n+\ell)^2 + n + \ell$ states in [15], employing sensitivity matrices for an optimal control problem. The resulting problem (25) is typically solved by the off-the-shelf existing solvers.

IV. Numerical Examples

The following section presents some numerical examples that will aid in understanding the implementation of this technique and will elucidate its subtleties. The simulations are obtained using GPOPS-II [21].

Consider an optimal control problem of minimizing a quadratic cost

$$\mathcal{J}(u) = \int_0^{t_f} \frac{1}{2} (x^\top R_1 x + u^\top R_2 u) \, \mathrm{d}t, \quad (28)$$

given the n-dimensional linear dynamics with parameter vector p

$$\dot{x} = A(p)x + B(p)u, \quad (29)$$
$$\dot{p} = 0, \quad (30)$$

with initial conditions $x(0) = x_0, \quad p(0) = p_0$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $p \in \mathbb{R}^\ell$, $A : \mathbb{R}^\ell \to \mathbb{R}^{n \times n}$, $B : \mathbb{R}^\ell \to \mathbb{R}^{n \times m}$, $R_1 \geq 0$, $R_2 > 0$, and t_f is fixed. The goal is to desensitize the cost with respect to the parameter p. Following the steps to construct the cost term for desensitization, the Hamiltonian is given by

$$H = \frac{1}{2}(x^\top R_1 x + u^\top R_2 u) + \lambda^\top \dot{x} + \mu^\top \dot{p}, = \frac{1}{2}(x^\top R_1 x + u^\top R_2 u) + \lambda^\top (A(p)x + B(p)u). \quad (31)$$

The adjoint equations are

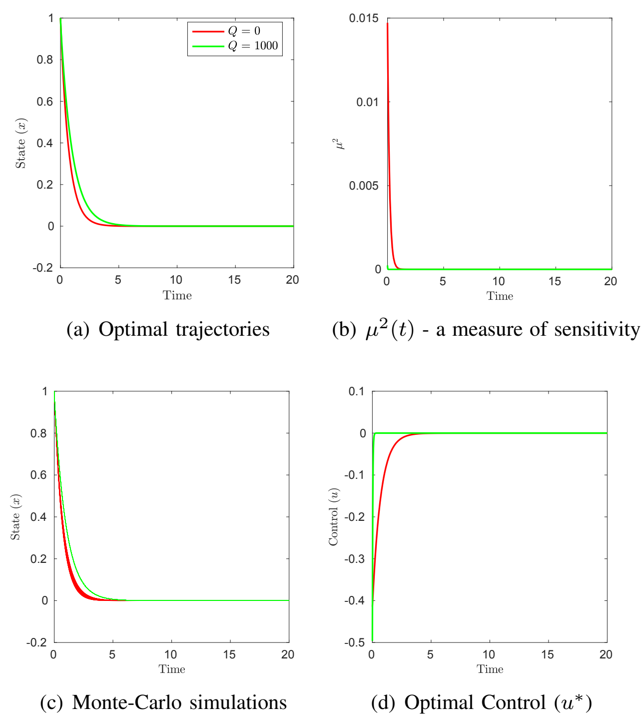
$$\dot{\lambda}^\top = -\frac{\partial H}{\partial x} = -x^\top R_1 - \lambda^\top A(p), \quad (32)$$

$$\dot{\mu}^\top = -\frac{\partial H}{\partial p} = -(x^\top \otimes \lambda^\top) \frac{\partial}{\partial p} \mathrm{vec}(A(p)) - (u^\top \otimes \lambda^\top) \frac{\partial}{\partial p} \mathrm{vec}(B(p)). \quad (33)$$

where λ and μ are the co-states of x and p, respectively. Since the cost has to be desensitized with respect to p, the augmented cost that has to be minimized for the C-DOC problem is given by

$$\mathcal{J}_s(u) = \int_0^{t_f} \frac{1}{2} (x^\top R_1 x + u^\top R_2 u + \mu^\top Q \mu) \, \mathrm{d}t. \quad (34)$$

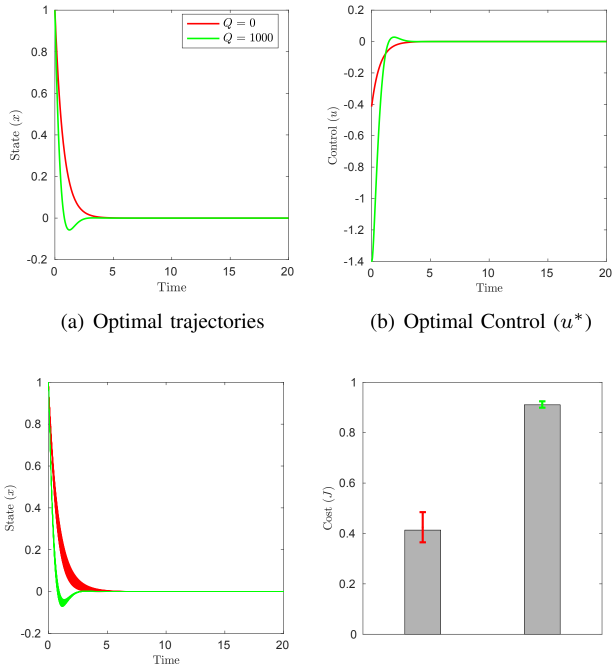
To demonstrate the results, we consider a one-dimensional linear system with the dynamics $\dot{x} = ax + bu$ with initial condition $x(0) = 1$, and let $R_1 = R_2 = 2$, $t_f = 20$. We first analyze the case where b is the uncertain parameter with its nominal value as $b_0 = 1$, and $a = -1$. The solutions obtained for $Q = 0$ and $1,000$ are shown in Fig. 1. Note that the sensitivity measure ($\mu^2(t)$) in Fig. 1(b) is lower for the desensitized solution. Since b is the source of uncertainty that perturbs the trajectory (and eventually the cost), by introducing desensitization ($Q = 1,000$), it can be observed from Fig. 1(d) that the control goes to zero earlier compared to the non-desensitized solution. By making the control zero, the source of uncertainty is removed from the system. The results obtained from the Monte-Carlo simulations with $b \in [0.8b_0, 1.2b_0]$ are shown in Fig. 1(c), which suggests that the variation in the cost for the desensitized solution is significantly lower.



(a) Optimal trajectories
(b) $\mu^2(t)$ - a measure of sensitivity
(c) Monte-Carlo simulations
(d) Optimal Control (u^*)

Fig. 1. Results obtained for an LQR problem with an uncertain B matrix.

The results for the case where a is the uncertain parameter with its nominal value as $a_0 = -1$ (stable), and $b = 1$ are shown in Fig. 2. Since a is the source of uncertainty, by switching on the desensitization ($Q = 1,000$), it can be observed from Fig. 2(a) that the state approaches zero faster compared to the non-desensitized solution. Consequently, from the Monte-Carlo simulations ($a \in [0.8a_0, 1.2a_0]$), it can be observed that the variations in the optimal trajectory (Fig. 2(c)), and the cost (Fig. 2(d)) are significantly lower for the desensitized solution, though the cost for the same is higher which is a trade-off. The error bars in Fig. 2(d) represent the minimum and the maximum costs obtained form the Monte-Carlo results where the corresponding grey bars represent the nominal costs with $a = a_0$.



(a) Optimal trajectories (b) Optimal Control (u^*)

(c) Monte-Carlo simulations (d) Cost (J)

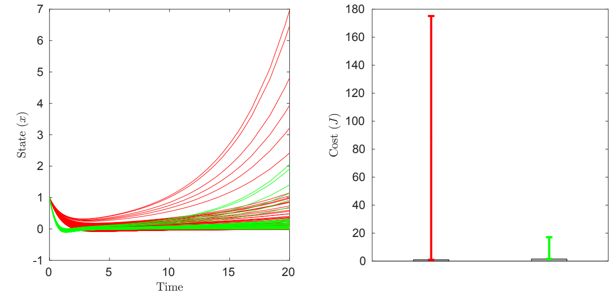
Fig. 2. Results obtained for an LQR problem with a stable and uncertain A matrix.

A more interesting case is a marginally stable system with $a_0 = 0$, and $a \in [-0.2, 0.2]$. The corresponding results can be found in Fig. 3. In the previous cases, although a parametric variation in a is studied, such variations did not change the stability of the system, i.e., if the nominal system is stable, then the system with parametric variation is stable as well. Since a can be both stable and unstable, the optimal control obtained for the nominal system without desensitization will be less effective combating the instabilities compared to the desensitized solution, as can be seen from the dispersion in trajectories (and costs) in Fig. 3.

V. Discussion: Relation between the Sensitivity Matrix and Co-States

In this section we address the relationship between the sensitivity matrix defined in (3) and the co-states λ. Let us note that, $\lambda^\top(t) = \frac{\partial C}{\partial x}(x(t), u, t)$, which can be expressed as

$$\lambda^\top(t) = \frac{\partial C(x(t), u, t)}{\partial x_0} \left[\frac{\partial x(t)}{\partial x_0}\right]^{-1}, \\ = \frac{\partial C(x(t), u, t)}{\partial x_0} S(t|t_0, x_0)^{-1}, \quad (35)$$



(a) Monte-Carlo simulations (b) Cost (J)

Fig. 3. Results obtained for the LQR problem with $a = 0$ (red: $Q = 0$, green: $Q = 1,000$).

where $S(t|t_0, x_0) = \frac{\partial x(t)}{\partial x_0}$ is the sensitivity of the state at time t along the trajectory with respect to variation in its initial condition x_0 [15]. Note that the dependency of $x(t)$, $t \geq t_0$ on t_0 and x_0 (initial conditions) is implicit. The relationship between the co-state and the sensitivity matrix in (35) can be generalized to obtain the sensitivity of the solution with regards to the state at any other time t' as

$$\lambda^\top(t) = \frac{\partial C(x(t), u, t)}{\partial x(t')} \frac{\partial x(t')}{\partial x(t)}, \quad (36)$$

$$= \frac{\partial C(x(t), u, t)}{\partial x(t')} S(t|t', x(t'))^{-1} \ \forall \ t, t' \in [t_0, t_f]. \quad (37)$$

Therefore,

$$\lambda(t) = S(t|t', x(t'))^{-\top} \left[\frac{\partial C(x(t), u, t)}{\partial x(t')}\right]^\top, \quad (38)$$

$$\left[\frac{\partial C(x(t), u, t)}{\partial x(t')}\right]^\top = S(t|t', x(t'))^\top \lambda(t). \quad (39)$$

From the above expressions, we observe that the sensitivity matrix $S(t|t', x(t'))^\top$ is essentially the transition matrix between the co-states $\lambda(t)$ and the partial of the cost-to-go function at time t with respect to the state at time t', i.e., the sensitivity of the cost-to-go function at time t with respect to the state at time $t' < t$.

VI. Conclusion

We attempt to exploit the co-states to obtain trajectories that are *less sensitive* to parametric variations. It is established that the co-states, defined by the Hamiltonian and the adjoint equations, capture the sensitivity of a cost-to-go function for *any* arbitrary control law. This has led to the idea of inserting the co-states into the cost function and then studying its implications in the context of DOC. In particular, this approach has been used to solve the problem of cost desensitization with respect to variations in the system parameters. The results suggest that variations to parametric uncertainty and optimality can be balanced using this approach through the choice of an appropriate weighting parameter. The numerical simulations show promising results that validate the theory. The proposed approach can be used to look at some other interesting problems, especially related to robust optimal control.

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APPENDIX

A. Proof of Theorem 3.1

For a fix control $\bar{u}$, the cost-to-go from any state x at time t is

$$C(x,\bar{u},t) = \phi(x(t_f),t_f) + \int_t^{t_f} L(x(s),\bar{u}(s),s)\mathrm{d}s$$

where $x(t) = x$.

Let the perturbed state at time t be represented by $x(t,\alpha) = x(t) + \alpha\delta x(t)$ where $\alpha \in [0,\alpha_0)$ for some $\alpha_0 > 0$, and $\delta x(t) \in \mathbb{R}^n$. With this perturbation the new cost-to-go is

$$C(x+\alpha\delta x,\bar{u},t) = \phi(x(t_f,\alpha),t_f) + \int_t^{t_f} L(x(s,\alpha),\bar{u}(s),s)\mathrm{d}s$$

where $x(s,\alpha)$ denotes the perturbed state at time $s \geq t$. By denoting $x(s,\alpha) = x(s,0) + \alpha\delta x(s)$, for all $s \geq t$, we obtain

$$\delta\dot{x}(s) = f_x(x(s,0),\bar{u}(s),s)\delta x(s) + O(\alpha),$$

where $O(\alpha)$ is such that $\lim_{\alpha\to 0} O(\alpha) = 0$. Consequently, $\delta x(s) = \Gamma(s,t)\delta x(t) + O(\alpha)$ where $\Gamma(s,t)$ is the state transition matrix corresponding to the matrix $f_x(x(s,0),u(s),s)$.

Therefore,

$$C(x+\alpha\delta x,\bar{u},t) - C(x,\bar{u},t) = \alpha\phi_x(x(t_f,0),t_f)\Gamma(t_f,t)\delta x + \alpha\left[\int_t^{t_f} L_x(x(s,0),u(s),s)\Gamma(s,t)\mathrm{d}s\right]\delta x + O(\alpha^2)$$

and thus,

$$\lim_{\alpha\to 0^+} \frac{C(x+\alpha\delta x,\bar{u},t) - C(x,\bar{u},t)}{\alpha} = \Big[\phi_x(x(t_f,0),t_f)\Gamma(t_f,t) + \int_t^{t_f} L_x(x(s,0),u(s),s)\Gamma(s,t)\mathrm{d}s\Big]\delta x,$$

and

$$C_x(x,\bar{u},t) = \phi_x(x(t_f,0),t_f)\Gamma(t_f,t) + \int_t^{t_f} L_x(x(s,0),\bar{u}(s),s)\Gamma(s,t)\mathrm{d}s.$$

At this point, if we denote

$$\lambda^\top \triangleq C_x(x,\bar{u},t).$$

We then have

$$\dot{\lambda}^\top = -\lambda^\top f_x - L_x,$$

since $\dot{\Gamma}(s,t) = -\Gamma(s,t)f_x(x(t,0),u(t),t)$. Furthermore, $\lambda(s)$ satisfies the terminal condition $\lambda(t_f) = \phi_x(x(t_f),t_f)$. Thus, if we define the Hamiltonian as $H = L + \lambda^\mathsf{T} f$, it follows that

$$\dot{\lambda}^\top = -\frac{\partial H}{\partial x}$$

and this λ represents the first order variation in cost-to-go with boundary condition

$$\lambda(t_f) = \phi_x(x(t_f),t_f).$$

The result follows. ■

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