2020 59th IEEE Conference on Decision and Control (CDC)
Jeju Island, Republic of Korea, December 14-18, 2020

Covariance Steering for Discrete-Time Linear-Quadratic Stochastic Dynamic Games*

Venkata Ramana Makkapati[1] Tanmay Rajpurohit[2] Kazuhide Okamoto[3] Panagiotis Tsiotras[4]

Abstract— **This paper addresses the problem of steering a discrete-time linear dynamical system from an initial Gaussian distribution to a final distribution in a game-theoretic setting. One of the two players strives to minimize a quadratic payoff, while at the same time tries to meet a given mean and covariance constraints at the final time-step. The other player maximizes the same payoff, but it is assumed to be *indifferent* to the terminal constraint. At first, the unconstrained version of the game is examined, and the necessary conditions for the existence of a saddle point are obtained. We show that obtaining a solution for the *one-sided* constrained dynamic game is not guaranteed, and subsequently the players' best responses are analyzed. Finally, we propose to numerically solve the problem of steering the distribution under adversarial scenarios using the Jacobi iteration method.**

I. Introduction

Stochastic games, introduced by Shapley in 1953, deal with instances where a stochastic process is jointly controlled by two players, a *controller* and a *stopper*, along with an underlying payoff function that is common to both players [1]. The stopper tries to maximize the payoff function, while the controller strives to minimize it. The current work addresses a class of linear-quadratic (LQ) stochastic dynamic games in discrete-time with finite-time horizon. It is assumed that the players have perfect measurements of the state at each time instant and that the initial state is sampled from a given Gaussian distribution. First, the problem of steering the covariance in an LQ game setting without any constraints is analyzed, and the associated saddle point equilibrium is identified. Subsequently, the problem of steering the initial distribution to a specified terminal distribution (which is also Gaussian) under adversarial situations, which can be categorized as a *general constrained game* (GCG) [2], is considered.

Owing to the fact that a Gaussian distribution can be fully defined using its first two moments, the problem discussed in this paper can be decomposed into mean and covariance steering problems [3]. The mean steering problem is essentially a deterministic dynamic game. The necessary and sufficient conditions for the existence of a solution to the discrete-time LQ dynamic game was provided by Pachter and Pham, along with a closed-form solution [4].

The idea of covariance steering has its genesis in the 1980s. First introduced by Hotz and Skelton [5], the problem of infinite-horizon covariance assignment for continuous and discrete-time systems has been analyzed by various researchers [6], [7]. The finite-horizon equivalent of the problem in continuous-time was investigated only recently by Chen et al. [8], [9], where it was shown that the related solutions have theoretical connections to the Schrödinger bridges and the optimal mass transport problems. The solution to the problem of covariance steering in finite time is also of great importance to entry, descent, and landing problems [10].

The contributions of this work are as follows. i) A novel LQ formulation for driving a Gaussian to a given terminal distribution under an adversarial setting is introduced. The adversary is assumed to be indifferent to the controller's terminal constraint which is unique to the literature on covariance steering. ii) It is shown that the proposed game theoretic formulation can be decomposed into two independent games, mean steering and covariance steering games, which makes the problem tractable. iii) The existence of equilibrium solutions is discussed for both unconstrained and constrained versions of the games. iv) A condition in terms of *relative controllability* is identified in the mean steering game with controller constraints for discrete systems. v) A simple Jacobi procedure for finding saddle points is introduced to solve the constrained covariance steering game, assuming a linear feedback control structure.

At this point, it is worth mentioning that the attitude of a player towards its opponent's constraints influences the outcome of the GCG [2]. In the case where a player's main goal is to prevent the opponent from meeting its constraints, his attitude is to be understood as being aggressive. Analyzing the scenario where the stopper has an aggressive attitude towards the controller's constraints is beyond the scope of this work. The proofs for some lemmas in this paper are omitted for brevity, and can be found in Ref. [11].

II. Problem Formulation

Consider the following discrete-time linear stochastic system

$$x_{k+1} = A_k x_k + B_k u_k + C_k v_k + D_k w_k, \tag{1}$$

where $k = 0,\ 1, \ldots, N-1$ is the time-step. At the k^{th} time-step, $x_k \in \mathbb{R}^n$ denotes the state, $u_k \in \mathbb{R}^m$ is the controller input, $v_k \in \mathbb{R}^\ell$ is the stopper input, and $w_k \in \mathbb{R}^r$ is a zero-mean white Gaussian noise with unit covariance. It is assumed that $\mathbb{E}[x_{k_1} w_{k_2}^\top] = 0, \quad 0 \le k_1 \le k_2 \le N$. The

*This work has been supported by NSF award CMMI-1662542.

[1]PhD Candidate, School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA. Email: mvramana@gatech.edu

[2]Assistant Vice President, Cora AI, Genpact Innovation Center, Palo Alto, CA 94304, USA. Email: tanmay.rajpurohit@genpact.digital

[3]Software Engineer (Planning and Control), Zoox, Foster City, CA 94404, USA.

[4] Professor and David and Andrew Lewis Chair, Institute for Robotics and Intelligent Machines, School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA. Email: tsiotras@gatech.edu

initial state x_0 is a sample from the Gaussian distribution $\mathcal{N}(\mu_0, \Sigma_0)$, where $\mu_0 \in \mathbb{R}^n$ is the initial state mean, and $\Sigma_0 \in \mathbb{R}^{n\times n}$ is the initial state covariance, with $\Sigma_0 \succeq 0$. The payoff function is

$$J(u_0, \dots, u_{N-1}, v_0, \dots, v_{N-1}) = \mathbb{E}\left[\sum_{k=0}^{N-1} \left(x_k^\top Q_k x_k + u_k^\top R_k u_k - v_k^\top S_k v_k\right)\right]. \quad (2)$$

It is assumed that $Q_k \succeq 0$, $R_k, S_k \succ 0$ for all $k = 0, \dots, N-1$. The set of control inputs $\{u_0, \dots, u_{N-1}\}$ is chosen by the controller to minimize the payoff function (2), and the control inputs $\{v_0, \dots, v_{N-1}\}$ are chosen by the stopper to maximize (2).

Using the notation introduced in [12], the system dynamics in (1) can be alternatively expressed as

$$X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{C}V + \mathcal{D}W, \quad (3)$$

where $X = [x_1^\top, \ x_2^\top, \dots, x_N^\top]^\top$, $U = [u_0^\top, \ u_1^\top, \dots, u_{N-1}^\top]^\top$, $V = [v_0^\top, \ v_1^\top, \dots, v_{N-1}^\top]^\top$, and $W = [w_0^\top, \ w_1^\top, \dots, w_{N-1}^\top]^\top$ for some appropriately constructed matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. Note that $\mathbb{E}[x_0 x_0^\top] = \Sigma_0 + \mu_0\mu_0^\top$, $\mathbb{E}[x_0 W^\top] = 0$, $\mathbb{E}[WW^\top] = I$. Consequently, the payoff function (2) can be expressed as

$$J(U, V) = \mathbb{E}[X^\top \bar{Q} X + U^\top \bar{R} U - V^\top \bar{S} V], \quad (4)$$

where $\bar{Q} = \text{blkdiag}(Q_0, \dots, Q_{N-1}, 0) \in \mathbb{R}^{(N+1)n\times(N+1)n}$, $\bar{R} = \text{blkdiag}(R_0, R_1, \dots, R_{N-1}) \in \mathbb{R}^{Nm\times Nm}$, and $\bar{S} = \text{blkdiag}(S_0, S_1, \dots, S_{N-1}) \in \mathbb{R}^{N\ell\times N\ell}$. Also, since $Q_k \succeq 0$ and $R_k, S_k \succ 0$ for all $k = 0, \dots, N-1$, it follows that $\bar{Q} \succeq 0$ and $\bar{R}, \bar{S} \succ 0$.

The mean and the covariance of the initial state x_0 can be written in terms of X as

$$\mu_0 = E_0\mathbb{E}[X], \ \Sigma_0 = E_0(\mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top)E_0^\top, \quad (5)$$

where $E_0 \triangleq [I_n, 0, \dots, 0] \in \mathbb{R}^{n\times(N+1)n}$.

Definition II.1. *The* upper game *is a scheme in which the stopper chooses V based on the information it has on the control U, and the* upper value *is defined by*

$$\mathcal{V}^+ = \inf_{U\in\mathbb{R}^{Nm}} \sup_{V\in\mathbb{R}^{N\ell}} J(U, V). \quad (6)$$

Similarly, the lower game *is a scheme in which the controller chooses U based on the information it has on the control V, and the* lower value *is defined by*

$$\mathcal{V}^- = \sup_{V\in\mathbb{R}^{N\ell}} \inf_{U\in\mathbb{R}^{Nm}} J(U, V). \quad (7)$$

It is well known that, in general $\mathcal{V}^- \leq \mathcal{V}^+$. If the *Isaacs minimax condition* holds, then $\mathcal{V}^- = \mathcal{V}^+$, and the corresponding set of control actions (U^*, V^*) is called the equilibrium solution or saddle point [13]. The unconstrained Gaussian steering problem to be addressed in this paper can now be stated as follows.

Problem 1. Find the saddle point (U^*, V^*) for the *unconstrained dynamic game* (UDG), described by the payoff function (4), the system (3), and the initial conditions (5).

In this paper, as mentioned earlier, we propose to analyze the one-sided constrained dynamic game. To this end, let

$$E_N X = x_N \sim \mathcal{N}(\mu_N, \Sigma_N), \quad (8)$$

where $E_N \triangleq [0, \dots, 0, I_n] \in \mathbb{R}^{n\times(N+1)n}$, be terminal state that the controller strives to achieve at the final time-step. Note that it is only the controller who is concerned about meeting the terminal condition (8), and hence (8) is a one-sided constraint. It is assumed that the stopper is aware of the controller's terminal constraint. However, it is indifferent to this constraint, and it is solely interested in maximizing the payoff (4). Furthermore, since the terminal constraint (8) is dependent on the control inputs of both players, and it is a one-sided constraint, the problem of interest can be categorized as a GCG [2]. The terminal condition (8) can be used to enforce probabilistic capture in the case of a two-player pursuit-evasion game with $\mu_N = 0$, when (1) represents the relative motion between the pursuer and the evader.

We will now formally define the saddle point in the one-sided constrained dynamic game using the corresponding upper and lower values. For a given stopper action V, let $\mathcal{U}(V)$ denote the set of controls $U \in \mathbb{R}^{Nm}$ that drive the system to the terminal Gaussian distribution in (8), and let $\mathcal{R} \triangleq \bigcup_{V\in\mathbb{R}^{N\ell}} \mathcal{U}(V) \subseteq \mathbb{R}^{Nm}$.

Definition II.2. *The* constrained upper value *is defined by*

$$\mathcal{V}_c^+ = \inf_{U\in\mathbb{R}^{Nm}} \sup_{V\in\mathbb{R}^{N\ell}} J(U, V), \quad (9)$$

and the constrained lower value *is defined by*

$$\mathcal{V}_c^- = \sup_{V\in\mathbb{R}^{N\ell}} \inf_{U\in\mathcal{R}} J(U, V). \quad (10)$$

Finally, a saddle point in the constrained game can be defined as (U_c^*, V_c^*) for which the $\mathcal{V}_c^+$ and $\mathcal{V}_c^-$ exist, and are equal.

Problem 2. Find necessary conditions such that the controller can drive the system to the final state given by (8), while the stopper tries to maximize the payoff function (4), given the system dynamics (3) and the initial conditions (5). Furthermore, find the optimal control inputs for both players. Hereafter, this problem will be referred to as the *constrained dynamic game* (CDG).

III. Separation of Mean and Covariance Steering Problems

It can be easily shown that that

$$\bar{X} \triangleq \mathbb{E}[X] = \mathcal{A}\mu_0 + \mathcal{B}\bar{U} + \mathcal{C}\bar{V}, \quad (11)$$

and

$$\tilde{X} \triangleq X - \mathbb{E}[X] = \mathcal{A}\tilde{x}_0 + \mathcal{B}\tilde{U} + \mathcal{C}\tilde{V} + \mathcal{D}W. \quad (12)$$

The objective function (4) can be further rewritten as

$$J(U, V) = J_\mu(\bar{U}, \bar{V}) + J_\Sigma(\tilde{U}, \tilde{V}), \quad (13)$$

where

$$J_\mu(\bar{U}, \bar{V}) = \bar{X}^\top \bar{Q}\bar{X} + \bar{U}^\top \bar{R}\bar{U} - \bar{V}^\top \bar{S}\bar{V}, \quad (14)$$

and

$$J_\Sigma(\tilde{U},\tilde{V}) = \mathrm{tr}(\bar{Q}\mathbb{E}[\tilde{X}\tilde{X}^\top]) + \mathrm{tr}(\bar{R}\mathbb{E}[\tilde{U}\tilde{U}^\top]) - \mathrm{tr}(\bar{S}\mathbb{E}[\tilde{V}\tilde{V}^\top]). \tag{15}$$

Lemma III.1. *For the UDG, the saddle point controls (U^*, V^*) that solve the problem (if they exist) are given by $U^* = \bar{U}^* + \tilde{U}^*$ and $V^* = \bar{V}^* + \tilde{V}^*$, where $(\bar{U}^*, \bar{V}^*)$ solves the* unconstrained mean steering game *(UMSG), defined using the payoff function in (14) along with (11), and $(\tilde{U}^*, \tilde{V}^*)$ solves the* unconstrained covariance steering game *defined using the payoff function in (15) along with (12)*

Lemma III.2. *For the CDG, the solution (U_c^*, V_c^*) can be characterized as $U_c^* = \bar{U}_c^* + \tilde{U}_c^*$, $V_c^* = \bar{V}_c^* + \tilde{V}_c^*$, where $(\bar{U}_c^*, \bar{V}_c^*)$ solves the* constrained mean steering game *(CMSG), defined using payoff function in (14) along with (11) and controller constraint*

$$\mu_N = E_N\bar{X} = \bar{A}_N\mu_0 + \bar{B}_N\bar{U} + \bar{C}_N\bar{V}, \tag{16}$$

and $(\tilde{U}_c^*, \tilde{V}_c^*)$ solves the constrained covariance steering game *(CCSG), defined using payoff function in (15) along with (12) and controller constraint*

$$\Sigma_N = E_N\left(\mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top\right)E_N^\top, \tag{17}$$

where the constraints (16) and (17), as stated earlier, are of concern only for the controller.

Note that non-existence of saddle point in either CMSG or CCSG or both, implies non-existence of saddle point in CDG. For the analysis of mean steering game in the following section, we introduce the set $\bar{\mathcal{R}}$. For a given stopper action $\bar{V}$ in CMSG, let $\bar{\mathcal{U}}(\bar{V})$ denote the set of mean controllers $\bar{U} \in \mathbb{R}^{Nm}$ that satisfies the constraint in (16), and let $\bar{\mathcal{R}} \triangleq \bigcup_{\bar{V}\in\mathbb{R}^{N\ell}} \bar{\mathcal{U}}(\bar{V}) \subseteq \mathbb{R}^{Nm}$.

IV. Mean Steering Game

The solution to the UMSG is given in the following proposition.

Proposition IV.1. *Assume that*

$$\bar{S} - \mathcal{C}^\top\bar{Q}\mathcal{C} \succ 0, \tag{18}$$

then the saddle point $(\bar{U}^*, \bar{V}^*)$ that solves the UMSG is given by

$$\begin{bmatrix} \bar{U}^* \\ \bar{V}^* \end{bmatrix} = -\begin{bmatrix} \mathcal{B}^\top\bar{Q}\mathcal{B} + \bar{R} & \mathcal{B}^\top\bar{Q}\mathcal{C} \\ \mathcal{C}^\top\bar{Q}\mathcal{B} & \mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{B}^\top\bar{Q}\mathcal{A} \\ \mathcal{C}^\top\bar{Q}\mathcal{A} \end{bmatrix}\mu_0 \tag{19}$$

and this solution is unique.

Proof. The payoff function of the UMSG can be expressed as $J_\mu(\bar{U},\bar{V}) = (\mathcal{A}\mu_0 + \mathcal{B}\bar{U} + \mathcal{C}\bar{V})^\top\bar{Q}(\mathcal{A}\mu_0 + \mathcal{B}\bar{U} + \mathcal{C}\bar{V}) + \bar{U}^\top\bar{R}\bar{U} - \bar{V}^\top\bar{S}\bar{V}$. The first-order necessary conditions [14] for a saddle point yield

$$\nabla_{\bar{U}}J_\mu = (\mathcal{B}^\top\bar{Q}\mathcal{B} + \bar{R})\bar{U} + \mathcal{B}^\top\bar{Q}\mathcal{C}\bar{V} + \mathcal{B}^\top\bar{Q}\mathcal{A}\mu_0 = 0, \tag{20a}$$

$$\nabla_{\bar{V}}J_\mu = (\mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S})\bar{V} + \mathcal{C}^\top\bar{Q}\mathcal{B}\bar{U} + \mathcal{C}^\top\bar{Q}\mathcal{A}\mu_0 = 0. \tag{20b}$$

The above two equations can be expressed as

$$\begin{bmatrix} \mathcal{B}^\top\bar{Q}\mathcal{B} + \bar{R} & \mathcal{B}^\top\bar{Q}\mathcal{C} \\ \mathcal{C}^\top\bar{Q}\mathcal{B} & \mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S} \end{bmatrix} \begin{bmatrix} \bar{U}^* \\ \bar{V}^* \end{bmatrix} = -\begin{bmatrix} \mathcal{B}^\top\bar{Q}\mathcal{A} \\ \mathcal{C}^\top\bar{Q}\mathcal{A} \end{bmatrix}\mu_0, \tag{21}$$

Let

$$\mathcal{T}_m = \begin{bmatrix} \mathcal{B}^\top\bar{Q}\mathcal{B} + \bar{R} & \mathcal{B}^\top\bar{Q}\mathcal{C} \\ \mathcal{C}^\top\bar{Q}\mathcal{B} & \mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S} \end{bmatrix}, \tag{22}$$

and from (18), $\mathcal{B}^\top\bar{Q}\mathcal{C}(\mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top\bar{Q}\mathcal{B} \prec 0$. As a result, $\mathcal{B}^\top\bar{Q}\mathcal{B} + \bar{R} - \mathcal{B}^\top\bar{Q}\mathcal{C}(\mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top\bar{Q}\mathcal{B} \succ 0$. Therefore, $\det(\mathcal{T}_m) = \det(\mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S})\det(\mathcal{B}^\top\bar{Q}\mathcal{B} + \bar{R} - \mathcal{B}^\top\bar{Q}\mathcal{C}(\mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top\bar{Q}\mathcal{B}) \neq 0$, and $\mathcal{T}_m$ is invertible. Equation (19) then follows immediately from (21). From (18), the second order derivatives yield $\nabla_{\bar{U}\bar{U}}J_\mu = \mathcal{B}^\top\bar{Q}\mathcal{B} + \bar{R} \succ 0$, $\nabla_{\bar{V}\bar{V}}J_\mu = \mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S} \prec 0$. Therefore, the payoff function is convex in $\bar{U}$, and concave in $\bar{V}$. Hence $(\bar{U}^*, \bar{V}^*)$ is the only saddle point that solves the given dynamic game [14]. □

Next, we analyze the CMSG. As this is a constrained zero-sum game, we obtain the following inequality. A similar result can be found in Ref. [2] (Theorem III.1).

Lemma IV.2. *Assuming that the UMSG has a saddle point equilibrium (Proposition IV.1), the CMSG satisfies*

$$\inf_{\bar{U}\in\mathbb{R}^{Nm}} \sup_{\bar{V}\in\mathbb{R}^{N\ell}} J_\mu(\bar{U},\bar{V}) \leq \sup_{\bar{V}\in\mathbb{R}^{N\ell}} \inf_{\bar{U}\in\bar{\mathcal{R}}} J_\mu(\bar{U},\bar{V}). \tag{23}$$

Proof. Given that the UMSG has a saddle point equilibrium, it follows that

$$\inf_{\bar{U}\in\mathbb{R}^{Nm}} \sup_{\bar{V}\in\mathbb{R}^{N\ell}} J_\mu(\bar{U},\bar{V}) = \sup_{\bar{V}\in\mathbb{R}^{N\ell}} \inf_{\bar{U}\in\mathbb{R}^{Nm}} J_\mu(\bar{U},\bar{V}). \tag{24}$$

Since $\bar{\mathcal{R}} \subseteq \mathbb{R}^{Nm}$,

$$\inf_{\bar{U}\in\mathbb{R}^{Nm}} J_\mu(\bar{U},\bar{V}) \leq \inf_{\bar{U}\in\bar{\mathcal{R}}} J_\mu(\bar{U},\bar{V}). \tag{25}$$

Hence,

$$\sup_{\bar{V}\in\mathbb{R}^{N\ell}} \inf_{\bar{U}\in\mathbb{R}^{Nm}} J_\mu(\bar{U},\bar{V}) \leq \sup_{\bar{V}\in\mathbb{R}^{N\ell}} \inf_{\bar{U}\in\bar{\mathcal{R}}} J_\mu(\bar{U},\bar{V}), \tag{26}$$

and from (24), the result follows. □

As a result, a pure-strategy equilibrium might not exist for the CMSG, and only players' best responses can be obtained [2]. To this end, the constrained upper and lower games for the CMSG problem can be examined. As stated in Definition II.2, in the constrained lower game, the stopper has to choose its input first, while the controller has the advantage of obtaining the stopper input, and then choosing his best response accordingly.

Lemma IV.3. *Assuming that the discrete-time linear dynamical system (1) is controllable for $C_k = 0$ and $D_k = 0$ (i.e., rank$[\bar{B}_N] = n$), the controller's feasible set (the set of controllers for which the constraint (16) is met given the stopper input) is non-empty for any $\bar{V} \in \mathbb{R}^{N\ell}$.*

From the above lemma, it is obvious that the controller can meet the constraint (16), if the condition $\mathrm{rank}[\bar{B}_N] = n$ is satisfied. In the upper game, the controller input is obtained first and the stopper best responds accordingly. The terminal condition (16) depends on the stopper input. Note that it is assumed that the stopper is indifferent to this constraint, and in this regard, the sufficient condition for which the controller's terminal constraint is met is derived in Lemma IV.4 below.

From the first-order necessary conditions in (20a) and (20b), the players' best responses as a function of their opponent's response can be obtained as

$$\bar{U} = -(\mathcal{B}^\top \bar{Q}\mathcal{B} + \bar{R})^{-1}(\mathcal{B}^\top \bar{Q}\mathcal{C}\bar{V} + \mathcal{B}^\top \bar{Q}\mathcal{A}\mu_0), \tag{27a}$$

$$\bar{V} = -(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}(\mathcal{C}^\top \bar{Q}\mathcal{B}\bar{U} + \mathcal{C}^\top \bar{Q}\mathcal{A}\mu_0). \tag{27b}$$

In the upper game, where the controller plays first, the stopper input as a function of $\bar{U}$ is given by (27b). Given the stopper input (as per (27b)), from the constraint (16), it follows that $\mu_N = \big(\bar{A}_N - \bar{C}_N(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top \bar{Q}\mathcal{A}\big)\mu_0 + \big(\bar{B}_N - \bar{C}_N(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top \bar{Q}\mathcal{B}\big)\bar{U}$. For the sake of brevity, let $\mathcal{G} = \bar{B}_N - \bar{C}_N(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top \bar{Q}\mathcal{B}$.

Lemma IV.4. *For the case of CMSG, in the associated upper game, the constraint (16) is satisfied if and only if*

$$\textit{rank}\left[\mathcal{G} \quad \mu_N - \big(\bar{A}_N - \bar{C}_N(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top \bar{Q}\mathcal{A}\big)\,\mu_0\right] = \textit{rank}\left[\mathcal{G}\right]. \tag{28}$$

Note that the matrix $\mathcal{G}$ can be treated as a *relative controllability matrix*, similar to the one introduced in Ref. [15] for continuous systems. The optimal control sequences $\bar{U}_*$ and $\bar{V}_*$ that solve the upper game can be found as follows. From (27b), the upper game can be expressed in terms of the following minimization problem.

$$\begin{cases} \min\limits_{\bar{U}\in\mathbb{R}^{Nm}} \bar{X}^\top \bar{Q}\bar{X} + \bar{U}^\top \bar{R}\bar{U} - \bar{V}^\top \bar{S}\bar{V}, \\ \text{subject to } \mu_N = \bar{A}_N\mu_0 + \bar{B}_N\bar{U} + \bar{C}_N\bar{V}, \end{cases} \tag{29}$$

where $\bar{X} = \mathcal{A}\mu_0 + \mathcal{B}\bar{U} + \mathcal{C}\bar{V}$, and $\bar{V} = -(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}(\mathcal{C}^\top \bar{Q}\mathcal{B}\bar{U} + \mathcal{C}^\top \bar{Q}\mathcal{A}\mu_0)$.

Proposition IV.5. *Under the assumption*

$$\textit{rank}\ \mathcal{G} = n, \tag{30}$$

the optimal control sequence $\bar{U}_*$ that solves the minimization problem in (29) is given by

$$\bar{U}_* = \mathcal{R}^{-1}\left(\mathcal{M} + \mathcal{G}^\top\lambda/2\right), \tag{31}$$

where $\mathcal{R} = \bar{R} + \mathcal{B}^\top \bar{Q}\mathcal{B} - \mathcal{B}^\top \bar{Q}\mathcal{C}(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top \bar{Q}\mathcal{B}$ $\mathcal{M} = \big(\mathcal{B}^\top \bar{Q}\mathcal{C}(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top - \mathcal{B}^\top\big)\bar{Q}\mathcal{A}\mu_0$, $\lambda = 2\big(\mathcal{G}\mathcal{R}^{-1}\mathcal{G}^\top\big)^{-1}\big(\mu_N - \bar{A}_N\mu_0 + \bar{C}_N(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top \bar{Q}\mathcal{A}\mu_0 - \mathcal{G}\mathcal{R}^{-1}\mathcal{M}\big)$.

Proof. The Lagrangian for the constrained minimization problem (29) can be written as

$$\begin{aligned} \mathcal{L}(\bar{U},\lambda) = {} & (\mathcal{A}\mu_0 + \mathcal{B}\bar{U} + \mathcal{C}\bar{V})^\top \bar{Q}(\mathcal{A}\mu_0 + \mathcal{B}\bar{U} + \mathcal{C}\bar{V}) \\ & + \bar{U}^\top \bar{R}\bar{U} - \bar{V}^\top \bar{S}\bar{V} \\ & + \lambda^\top(\mu_N - \bar{A}_N\mu_0 - \bar{B}_N\bar{U} - \bar{C}_N\bar{V}), \end{aligned} \tag{32}$$

where $\lambda \in \mathbb{R}^n$. The first-order optimality condition yields

$$\begin{aligned} \nabla_{\bar{U}}\mathcal{L} = {} & 2(\mathcal{A}\mu_0 + \mathcal{B}\bar{U} + \mathcal{C}\bar{V})^\top \bar{Q}\left(\mathcal{B} + \mathcal{C}\frac{\partial\bar{V}}{\partial\bar{U}}\right) + 2\bar{U}^\top\bar{R} \\ & - 2\bar{V}^\top\bar{S}\frac{\partial\bar{V}}{\partial\bar{U}} + \lambda^\top\left(-\bar{B}_N - \bar{C}_N\frac{\partial\bar{V}}{\partial\bar{U}}\right) = 0, \end{aligned} \tag{33}$$

and (31) follows from the fact that $\dfrac{\partial\bar{V}}{\partial\bar{U}} = -(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top \bar{Q}\mathcal{B}$ (obtained using (27b)), and from the second-order optimality condition

$$\begin{aligned} \frac{\nabla_{\bar{U}\bar{U}}\mathcal{L}}{2} &= \left(\mathcal{B} + \mathcal{C}\frac{\partial\bar{V}}{\partial\bar{U}}\right)^\top \bar{Q}\left(\mathcal{B} + \mathcal{C}\frac{\partial\bar{V}}{\partial\bar{U}}\right) + \bar{R} - \frac{\partial\bar{V}}{\partial\bar{U}}^\top\bar{S}\frac{\partial\bar{V}}{\partial\bar{U}} \\ &= \big(\bar{R} + \mathcal{B}^\top\bar{Q}\mathcal{B} - \mathcal{B}^\top\bar{Q}\mathcal{C}(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top\bar{Q}\mathcal{B}\big) = \mathcal{R} \succ 0 \end{aligned} \tag{34}$$

The Lagrange multiplier λ can be found by substituting (31) into the terminal constraint, obtaining $\big(\mathcal{G}\mathcal{R}^{-1}\mathcal{G}^\top\big)\lambda = 2\big(\mu_N - \bar{A}_N\mu_0 + \bar{C}_N(\mathcal{C}^\top \bar{Q}\mathcal{C} - \bar{S})^{-1}\mathcal{C}^\top \bar{Q}\mathcal{A}\mu_0 - \mathcal{G}\mathcal{R}^{-1}\mathcal{M}\big)$. Note that since $\mathcal{R}$ is invertible and $\mathcal{G}$ has full row rank, $\mathcal{G}\mathcal{R}^{-1}\mathcal{G}^\top$ is invertible. □

V. Covariance Steering Game

The methodology to solve the UCSG and the CCSG is presented in this section. Assuming a linear feedback control structure for steering the covariance, we express $\tilde{U}$ and $\tilde{V}$ as

$$\tilde{u}_k = K_k y_k, \quad \tilde{v}_k = L_k y_k, \tag{35}$$

where $K_k \in \mathbb{R}^{m\times n}$, $L_k \in \mathbb{R}^{\ell\times n}$,

$$y_{k+1} = A_k y_k + D_k w_k, \quad y_0 = x_0 - \mu_0, \tag{36a}$$

and $y_k \in \mathbb{R}^n$. Note that $\mathbb{E}[y_0] = 0$ and $\mathbb{E}[y_0 y_0^\top] = \Sigma_0$. Further, it can be obtained that

$$Y = \mathcal{A}y_0 + \mathcal{D}W, \tag{37}$$

where $Y = [y_0^\top, \ldots, y_N^\top]^\top \in \mathbb{R}^{(N+1)n}$, using the matrices introduced in Section II. Therefore, $\tilde{X}$ in (12) can be rewritten as

$$\tilde{X} = (I + \mathcal{B}K + \mathcal{C}L)(\mathcal{A}y_0 + \mathcal{D}W). \tag{38}$$

where,

$$K = \begin{bmatrix} K_0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & K_{N-1} & 0 \end{bmatrix}, \tag{39}$$

$$L = \begin{bmatrix} L_0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & L_{N-1} & 0 \end{bmatrix}, \tag{40}$$

are the controller and the stopper gain matrices, respectively. Here $K \in \mathbb{K}$ and $L \in \mathbb{L}$, where $\mathbb{K}$ is the set of $Nm \times (N+1)n$ matrices that have the structure shown in (39), and similarly, $\mathbb{L}$ is the set of $N\ell \times (N+1)n$ matrices that have the structure shown in (40). From (35), (37), and (38), we have $\mathbb{E}[\tilde{X}\tilde{X}^\top] = (I + \mathcal{B}K + \mathcal{C}L)\Sigma_s(I + \mathcal{B}K + \mathcal{C}L)^\top$, $\mathbb{E}[\tilde{U}\tilde{U}^\top] = K\Sigma_s K^\top$, $\mathbb{E}[\tilde{V}\tilde{V}^\top] = L\Sigma_s L^\top$, where $\Sigma_s =$

$\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top$. Therefore, the covariance cost function $J_\Sigma(\tilde{U},\tilde{V})$ can be converted to the following quadratic form in terms of K and L:

$$J_\Sigma(K,L) = \mathrm{tr}(((I+\mathcal{B}K+\mathcal{C}L)^\top\bar{Q}(I+\mathcal{B}K+\mathcal{C}L)+ K^\top\bar{R}K - L^\top\bar{S}L)\Sigma_s), \quad (41)$$

and the terminal constraint (17) can be rewritten as

$$\Sigma_N = E_N(I+\mathcal{B}K+\mathcal{C}L)\Sigma_s(I+\mathcal{B}K+\mathcal{C}L)^\top E_N^\top. \quad (42)$$

For the sake of analysis, we introduce the set $\tilde{\mathcal{R}}$. Given stopper gain L in CCSG, let $\mathcal{K}(L)$ denote the set of gains $K \in \mathbb{K}$ for which the controller satisfies the constraint in (42), and let $\tilde{\mathcal{R}} \triangleq \bigcup_{L\in\mathbb{L}} \mathcal{K}(L) \subseteq \mathbb{K}$.

We first analyze the UCSG. Since the gain matrices K and L have constraints on their structure with zeros, as shown in (39) and (40), with a slight abuse of notation, the Lagrangian can be written as

$$\begin{aligned}\mathcal{L}(K,L,\Theta,\Xi) &= \mathrm{tr}(((I+\mathcal{B}K+\mathcal{C}L)^\top\bar{Q}(I+\mathcal{B}K+\mathcal{C}L) \\ &\quad + K^\top\bar{R}K - L^\top\bar{S}L)\Sigma_s)/2 \\ &\quad + \sum_{i=1}^{Nm}\sum_{j\in\mathscr{J}_k(i)} \theta_{ij}e_i^\top K e_j + \sum_{i=1}^{N\ell}\sum_{j\in\mathscr{J}_l(i)} \xi_{ij}e_i^\top L e_j, \quad (43)\end{aligned}$$

where the functions $\mathscr{J}_k(.)$ and $\mathscr{J}_l(.)$ map each row number to the set of columns in which the gains K and L, respectively, have zero elements. The matrices $\Theta \in \mathbb{R}^{Nm\times(N+1)n}$ and $\Xi \in \mathbb{R}^{N\ell\times(N+1)n}$ are Lagrange multipliers of sizes equal to K and L, respectively. Note that the blocks in Θ and Ξ (corresponding to K_k and L_k) are zeros, and θ_{ij} and ξ_{ij} are the non-zero elements of these matrices. The first-order necessary conditions for the existence of a saddle point can be obtained by taking derivatives of the Lagrangian in (43) with respect to K and L as

$$\nabla_K\mathcal{L} = \left[\mathcal{B}^\top\bar{Q} + \bar{R}K + \mathcal{B}^\top\bar{Q}\mathcal{B}K + \mathcal{B}^\top\bar{Q}\mathcal{C}L\right]\Sigma_s + \Theta = 0, \quad (44a)$$

$$\nabla_L\mathcal{L} = \left[\mathcal{C}^\top\bar{Q} - \bar{S}L + \mathcal{C}^\top\bar{Q}\mathcal{B}K + \mathcal{C}^\top\bar{Q}\mathcal{C}L\right]\Sigma_s + \Xi = 0. \quad (44b)$$

The candidate solutions for the UCSG can be obtained by solving the linear system of equations given in (44). Since the gradients are linear, the second-order sufficient conditions, using the bordered Hessians, can be invoked to find the saddle points among the candidate solutions numerically [16]. Next, we analyze the CCSG. A result similar to the one proposed for the CMSG (Lemma IV.2) follows for the CCSG and is given below.

Lemma V.1. *Assuming that the UCSG with payoff function (41) has a saddle point equilibrium, then the CCSG (17), with the terminal constraint (42) imposed only for the controller, satisfies*

$$\inf_{K\in\mathbb{K}}\sup_{L\in\mathbb{L}} J_\Sigma(K,L) \le \sup_{L\in\mathbb{L}}\inf_{K\in\tilde{\mathcal{R}}} J_\Sigma(K,L). \quad (45)$$

Similarly, in the CCSG, a pure-strategy equilibrium need not exist. To this end, consider a simple Jacobi procedure given in Algorithm 1 to arrive at an equilibrium solution, assuming one exists. For Algorithm 1 to converge to an equilibrium solution for any K_0, L_0, the solution has to be a stable one [17]. The conditions for the existence of a stable equilibrium for the case where the cost is convex in K and concave in L can be found in Ref. [17].

Algorithm 1 Jacobi procedure to obtain saddle points

procedure JACOBI(K_0,L_0)
  for i = 0,1,2,... **do**
    $L_{i+1} := \arg\max_{L\in\mathbb{L}} J_\Sigma(K_i, L)$
    $K_{i+1} := \arg\min_{K\in\mathcal{K}(L_i)} J_\Sigma(K, L_i)$
  return K_{i+1}, L_{i+1}

Subsequently, under the assumptions that $\Sigma_s \otimes (\mathcal{B}^\top\bar{Q}\mathcal{B} + \bar{R}) \succ 0$ (convex in K) and $\Sigma_s \otimes (\mathcal{C}^\top\bar{Q}\mathcal{C} - \bar{S}) \prec 0$ (concave in L), we can formulate the successive minimization and maximization problems as convex programming problems by relaxing the equality constraint in (42) to an inequality constraint,

$$\Sigma_N \succeq E_N(I+\mathcal{B}K+\mathcal{C}L)\Sigma_s(I+\mathcal{B}K+\mathcal{C}L)^\top E_N^\top. \quad (46)$$

Lemma V.2. *Assuming $\Sigma_N \succ 0$, the inequality constraint (46) can be expressed as*

$$\|\Sigma_N^{-1/2}E_N(I+\mathcal{B}K+\mathcal{C}L)\Sigma_s^{1/2}\|_2 - 1 \le 0. \quad (47)$$

Proof. Since $\Sigma_N \succ 0$, (46) can be rewritten as $I - \Sigma_N^{-1/2}E_N(I+\mathcal{B}K+\mathcal{C}L)\Sigma_s(I+\mathcal{B}K+\mathcal{C}L)^\top E_N^\top\Sigma_N^{-1/2} \succeq 0$. As it is symmetric, the matrix $\Sigma_N^{-1/2}E_N(I+\mathcal{B}K+\mathcal{C}L)\Sigma_s(I+\mathcal{B}K+\mathcal{C}L)^\top E_N^\top\Sigma_N^{-1/2}$ is diagonalizable via an orthogonal matrix $T \in \mathbb{R}^{n\times n}$ as $T(I_n - \mathrm{diag}(\lambda_1,\ldots,\lambda_n))T^\top \succeq 0$, where $\lambda_1,\ldots,\lambda_n$ are its eigenvalues. Consequently, we have

$$1 - \lambda_{max}(\Sigma_N^{-1/2}E_N(I+\mathcal{B}K+\mathcal{C}L)\Sigma_s\times (I+\mathcal{B}K+\mathcal{C}L)^\top E_N^\top\Sigma_N^{-1/2}) \ge 0. \quad (48)$$

$$\implies 1 - \|\Sigma_N^{-1/2}E_N(I+\mathcal{B}K+\mathcal{C}L)\Sigma_s^{1/2}\|_2 \ge 0. \quad (49)$$

Hence proved. □

VI. Numerical Simulations

As mentioned earlier, in the lower game of the mean steering case, the controller has an advantage to drive the distribution to a given terminal Gaussian, assuming the system is controllable. A more challenging case is that of the upper game, where the controller has to ensure that the terminal constraint (16) is met while choosing its input first. In this section, we first present test examples for the upper game of the CMSG with linear time-invariant systems. For the covariance steering part, YALMIP [18] in conjunction with MOSEK [19] was used to solve the successive convex optimization problems in the Jacobi procedure. The convergence criterion for the iterative method is $\epsilon_k, \epsilon_\ell \le \epsilon$, where $\epsilon_k = \|K_{i+1} - K_i\|$ and $\epsilon_\ell = \|L_{i+1} - L_i\|$.

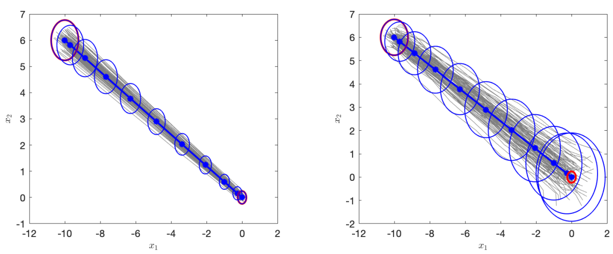
Consider the linear system

$$z_{k+1} = Az_k + Bu_k + Cv_k + Dw_k \quad (50)$$

where $z_k = [x_1, x_2, x_3, x_4]^\top \in \mathbb{R}^4$, $u_k, v_k \in \mathbb{R}^2$, $w_k \in \mathbb{R}^4$,

$$A = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \Delta t^2/2 & 0 \\ 0 & \Delta t^2/2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}, \tag{51}$$

$C = -B$, and $D = 0.01I_4$. Note that x_1, x_2 can be understood as relative coordinates, and x_3, x_4 are the relative velocities along the x_1 and x_2 axes, respectively, with $\Delta t = 0.2$ being the time-step size. Finally, u_k and v_k are the accelerations of the pursuer (controller) and the evader (stopper), respectively.



(a) Case where the covariance condition is met by the controller.

(b) Case where the terminal covariance constraint is not met.

Fig. 1: Numerical example of a constrained dynamic game

The initial condition is chosen to be $\mu_0 = [-10, 6, 0, 0]$, $\Sigma_0 = \mathrm{diag}(0.05, 0.05, 0.01, 0.01)$, and the terminal constraint is $\mu_N = [0, 0, 0, 0]$, $\Sigma_N = \mathrm{diag}(0.005, 0.005, 0.001, 0.001)$. The time horizon is fixed at $N = 10$, and the cost matrices are $Q_k = I_4$, and $R_k = I_2$ $S_k = 100I_2$, for all $k \geq 1$. The CCSG is solved using the Jacobi procedure illustrated in Algorithm 1. For the CDG, the relative controllability matrix is found to have full row rank, and therefore the mean can be driven to the specified terminal value. Also, the covariance steering problem is feasible with $\epsilon = 10^{-5}$, and the result is illustrated in Fig. 1(a). The red ellipses in Fig. 1(a) denote the 3σ error of the initial and the desired terminal state distributions of x_1 and x_2 coordinates. The blue solid line illustrates the mean trajectory, and the blue ellipses illustrate the covariance evolution over the time horizon. The gray lines are the trajectories simulated for 100 different initial conditions that are sampled from $\mathcal{N}(\mu_0, \Sigma_0)$. From Fig. 1(a), it can be observed that the covariance constraint is satisfied.

Fig. 1(b) illustrates the case where $D = 0.1I_4$, while the rest of the values are kept unchanged. Since changing the matrix D does not change the behavior of the mean, in this case, the mean converges to the specified terminal value. However, the covariance constraint cannot be achieved in this case and from Fig. 1(b), it can be observed that the covariance ellipse grows with time. The result in Fig. 1(b) is for the set of optimal gains (K^*, L^*), obtained by minimizing the cost (41) subject to the constraint (44b), since the constraint (46) cannot be met.

VII. Conclusion

This work addressed the problem of steering a Gaussian in adversarial scenarios using the theory of general constrained games. The problem is posed from a perspective of the player that desires to drive the distribution to a given terminal Gaussian while minimizing a quadratic cost. The player that tries to maximize the cost is assumed to be indifferent to the terminal constraint. It is shown that the game need not have a saddle point equilibrium. Subsequently, we obtained necessary conditions for the controller to drive the mean to the specified value in the upper game. The covariance steering problem is solved numerically using the well-known Jacobi procedure. The approach is illustrated via numerical examples.

References

[1] L. S. Shapley, "Stochastic games," *Proceedings of the National Academy of Sciences*, vol. 39, no. 10, pp. 1095–1100, 1953.

[2] E. Altman and E. Solan, "Constrained games: The impact of the attitude to adversary's constraints," *IEEE Transactions on Automatic Control*, vol. 54, no. 10, pp. 2435–2440, 2009.

[3] K. Okamoto, M. Goldshtein, and P. Tsiotras, "Optimal covariance control for stochastic systems under chance constraints," *IEEE Control Systems Letters*, vol. 2, no. 2, pp. 266–271, 2018.

[4] M. Pachter and K. D. Pham, "Discrete-time linear-quadratic dynamic games," *Journal of Optimization Theory and Applications*, vol. 146, no. 1, pp. 151–179, 2010.

[5] A. F. Hotz and R. E. Skelton, "A covariance control theory," in *IEEE Conference on Decision and Control*, vol. 24, Fort Lauderdale, FL, Dec. 1985, pp. 552–557.

[6] K. M. Grigoriadis and R. E. Skelton, "Minimum-energy covariance controllers," *Automatica*, vol. 33, no. 4, pp. 569–578, 1997.

[7] E. Collins and R. Skelton, "A theory of state covariance assignment for discrete systems," *IEEE Transactions on Automatic Control*, vol. 32, no. 1, pp. 35–41, 1987.

[8] Y. Chen, T. T. Georgiou, and M. Pavon, "Optimal steering of a linear stochastic system to a final probability distribution, Part I," *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1158–1169, 2016.

[9] ——, "Optimal steering of a linear stochastic system to a final probability distribution, Part II," *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1170–1180, 2016.

[10] J. Ridderhof and P. Tsiotras, "Uncertainty quantication and control during Mars powered descent and landing using covariance steering," in *AIAA Guidance, Navigation, and Control Conference*, Kissimmee, FL, Jan. 2018.

[11] V. R. Makkapati, T. Rajpurohit, K. Okamoto, and P. Tsiotras, "Covariance steering for discrete-time linear-quadratic stochastic dynamic games," *arXiv preprint arXiv:2003.03045*, 2020.

[12] M. Goldshtein and P. Tsiotras, "Finite-horizon covariance control of linear time-varying systems," in *IEEE Conference on Decision and Control*, Melbourne, Australia, Dec. 12 –15, 2017, pp. 3606–3611.

[13] W. H. Fleming and P. E. Souganidis, "On the existence of value functions of two-player, zero-sum stochastic differential games," *Indiana University Mathematics Journal*, vol. 38, no. 2, pp. 293–314, 1989.

[14] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, Second ed. SIAM, 1999, ch. 4.

[15] R. Behn and Y.-C. Ho, "On a class of linear stochastic differential games," *IEEE Transactions on Automatic Control*, vol. 13, no. 3, pp. 227–240, 1968.

[16] J. R. Magnus and H. Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, Third ed. Hoboken, Nj, USA: Wiley, 1999.

[17] S. Li and T. Başar, "Distributed algorithms for the computation of noncooperative equilibria," *Automatica*, vol. 23, no. 4, pp. 523–533, 1987.

[18] J. Löfberg, "YALMIP : A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[19] MOSEK, ApS. (2017) The MOSEK optimization toolbox for MATLAB manual. version 8.1. [Online]. Available: http://docs.mosek.com