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ERRATUM: SPECTRAL BAND DEGENERACIES OF $\frac{\pi}{2}$-ROTATIONALLY INVARIANT PERIODIC SCHRÖDINGER OPERATORS*

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Abstract. We correct the statement and proof of Corollary 4.2 in Keller et al. [*Multiscale Model. Simul.*, 16 (2018), pp. 1684–1731] corresponding to the case of admissible potentials, which are also reflection invariant (ρ-invariant). We also include a short addendum on implications, in this case, for the effective Hamiltonian corresponding to states which are spectrally localized near the $\mathbf{M}$ point.

Key words. Floquet–Bloch, degenerate eigenvalues, dispersion surfaces

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Corrections.

1. page 1687: part (D) of summary of results: Equation (1.5) should read as follows:

$$\mu_\pm(\mathbf{M}+\kappa) - \mu_S = (1-\alpha)|\kappa|^2 + \mathcal{Q}_6(\kappa) \pm \sqrt{\tilde{\gamma}^2(\kappa_1^2-\kappa_2^2)^2 + 4\beta^2\kappa_1^2\kappa_2^2 \;+\; \mathcal{Q}_8(\kappa)}. \tag{A}$$

2. page 1699: The statement of Corollary 4.2 should read as follows.

Corollary 4.2. *Assume the hypotheses of Theorem* 4.1*. Assume further that with respect to the origin of coordinates,* $\mathbf{x}_c = 0$*, we have, in addition, that* V *is reflection invariant in the sense of Definition* 2.4*, i.e.,* $V(x_1,x_2) = V(x_2,x_1)$*. Then the coefficients* β *and* γ *in* (4.1) *are constrained to satisfy* $\beta\in\mathbb{R}$ *and* $\gamma = -i\tilde{\gamma}\in i\mathbb{R}$ *and we have*

$$\mu_\pm(\mathbf{M}+\kappa) - \mu_S = (1-\alpha)|\kappa|^2 + \mathcal{Q}_6(\kappa) \pm \sqrt{\tilde{\gamma}^2(\kappa_1^2-\kappa_2^2)^2 + 4\beta^2\kappa_1^2\kappa_2^2 \;+\; \mathcal{Q}_8(\kappa)}\;. \tag{B}$$

Here $\mathcal{Q}_6(\kappa)$ *and* $\mathcal{Q}_8(\kappa)$ *are now also invariant under the reflection* $(\kappa_1,\kappa_2)\mapsto(\kappa_2,\kappa_1)$.

3. pages 1710–1711: Correction to the statement of Claim 4.21 and its proof:

The correct form of Claim 4.21 is $\Re(\gamma) \;=\; 4\Re(a_{1,1}^{1,2}) \;=\; 0$.

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Correction: In section 4.4, replace all discussion from Claim 4.21 through the end of section 4.4 by the following.

Now consider the setting of Theorem 4.1; μ_S is an eigenvalue of $H = -\Delta + V$ acting in $L^2_{\mathbf{M}}$ of multiplicity 2. In particular, μ_S is a simple $L^2_{\mathbf{M},i}$ eigenvalue with corresponding eigenfunction Φ_1, and μ_S is a simple $L^2_{\mathbf{M},-i}$ eigenvalue with corresponding eigenfunction Φ_2, with $\Phi_2(\mathbf{x}) = \overline{\Phi_1(-\mathbf{x})}$. By Claim 4.20, $\rho\Phi_1 \in L^2_{\mathbf{M},-i}$, and since ρ commutes with H, we have that $\rho\Phi_1$ is an $L^2_{\mathbf{M},-i}$ eigenfunction. Thus, $\rho\Phi_1 = e^{i\nu}\Phi_2$ for some $\nu \in \mathbb{R}$ or equivalently $\rho e^{-i\frac{\nu}{2}}\Phi_1 = e^{i\frac{\nu}{2}}\Phi_2$. Hence, for the case $\nu \neq 0$ replace Φ_1 by $e^{-i\frac{\nu}{2}}\Phi_1$ and Φ_2 by $e^{i\frac{\nu}{2}}\Phi_2$ to obtain the relation

$$\rho\Phi_1 = \Phi_2 \tag{C}$$

in all cases. Recall from (4.27) of [2] that

$$\begin{aligned} \alpha &= 4a^{1,1}_{1,1} = 4\,\langle \partial_{x_1}\Phi_1, \mathcal{R}(\mu_S)\partial_{x_1}\Phi_1\rangle, \\ \beta &= 4a^{1,2}_{1,2} = 4\,\langle \partial_{x_1}\Phi_1, \mathcal{R}(\mu_S)\partial_{x_2}\Phi_2\rangle, \\ \gamma &= 4a^{1,2}_{1,1} = 4\,\langle \partial_{x_1}\Phi_1, \mathcal{R}(\mu_S)\partial_{x_1}\Phi_2\rangle. \end{aligned} \tag{D}$$

We have shown that $\alpha \in \mathbb{R}$. Using (C), we have the following constraints on β and γ.

Claim. *Assume* $[\rho, H] = 0$. *Then*

$$\beta \in \mathbb{R}, \tag{E}$$

$$\gamma = -i\tilde{\gamma}, \qquad \tilde{\gamma} \in \mathbb{R}. \tag{F}$$

Proof. We first prove (E). Since $[\rho, H] = 0$ and $\rho\partial_{x_1} = \partial_{x_2}\rho$, we have

$$\begin{aligned} \beta &= 4\,\langle \partial_{x_1}\Phi_1, \mathcal{R}(\mu_S)\partial_{x_2}\Phi_2\rangle \\ &= 4\,\langle \rho\,\partial_{x_1}\Phi_1, \rho\mathcal{R}(\mu_S)\partial_{x_2}\Phi_2\rangle \\ &= 4\,\langle \partial_{x_2}\rho\,\Phi_1, \mathcal{R}(\mu_S)\partial_{x_1}\rho\,\Phi_2\rangle \\ &= 4\,\langle \partial_{x_2}\Phi_2, \mathcal{R}(\mu_S)\partial_{x_1}\Phi_1\rangle \\ &= 4\,\langle \mathcal{R}(\mu_S)\partial_{x_2}\Phi_2, \partial_{x_1}\Phi_1\rangle \\ &= 4\,\overline{\langle \partial_{x_1}\Phi_1, \mathcal{R}(\mu_S)\partial_{x_2}\Phi_2\rangle} = \overline{\beta}. \end{aligned}$$

To prove (F), let $\kappa \in \mathbb{R}^2$ be arbitrary. Using that $\mathcal{R}[\Phi_1] = i\Phi_1$ and $\mathcal{R}[\Phi_2] = -i\Phi_2$, we have for $j_1, j_2 \in \{1, 2\}$ that

$$\begin{aligned} \kappa^T A^{j_1,j_2}\kappa &= \langle \partial_{y_l}\Phi_{j_1}, \mathcal{R}(\mu_\star)\partial_{y_m}\Phi_{j_2}\rangle_{L^2(\Omega_{\mathbf{y}})}\kappa_l\kappa_m \\ &= \langle \mathcal{R}(\rho[\partial_{y_l}\Phi_{j_1}]),\ \mathcal{R}(\mu_\star)\ \mathcal{R}(\rho[\partial_{y_m}\Phi_{j_2}])\rangle_{L^2(\Omega_{\mathbf{x}})}\ \kappa_l\kappa_m \\ &= \langle R_{ns}\rho_{ln}\partial_{x_s}\mathcal{R}[\rho\Phi_{j_1}], \mathcal{R}(\mu_\star)R_{qt}\rho_{mq}\partial_{x_t}\mathcal{R}[\rho\Phi_{j_2}]\rangle_{L^2(\Omega_{\mathbf{x}})}\ \kappa_l\kappa_m \\ &= \langle \partial_{x_s} i^{2j_2-1}\Phi_{j_2}, \mathcal{R}(\mu_\star)\partial_{x_q} i^{2j_1-1}\Phi_{j_1}]\rangle_{L^2(\Omega_{\mathbf{x}})}\ R_{ns}(\rho_{ln}\kappa_l)\ R_{qt}(\rho_{mq}\kappa_m) \\ &= i^{2(j_1-j_2)}\,\langle \partial_{x_s}\Phi_{j_2}, \mathcal{R}(\mu_\star)\partial_{x_t}\Phi_{j_1}\rangle_{L^2(\Omega_{\mathbf{x}})}\ \ R_{ns}(\rho\kappa)_n R_{qt}(\rho\kappa)_q \\ &= i^{2(j_1-j_2)}(R\rho\kappa)^T A^{j_2,j_1}(R\rho\kappa) \end{aligned}$$

for any choice of pairs (j_1, j_2) with $j_1, j_2 \in \{1,2\}$. Since κ is arbitrary,

$$A^{j_1,j_2} = i^{2(j_1-j_2)}\rho\ R^T A^{j_2,j_1} R\ \rho.$$

For any pair $j_1, j_2 \in \{1,2\}$, let

$$A^{j_1,j_2} = A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\text{(G)} \qquad R^T AR = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Consider $j_1 = 1$ and $j_2 = 2$. From the above analysis,

$$A = A^{1,2} = -\rho\ R\ A^{2,1}\ R^T\ \rho = -\rho\ R\ (A^{1,2})^\dagger\ R^T\ \rho = -\rho\ R\ A^\dagger\ R^T\ \rho$$

so that

$$\begin{aligned} -\rho\ R\ A^\dagger\ R^T\ \rho &= -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= -\begin{pmatrix} -\bar{c} & \bar{a} \\ \bar{d} & -\bar{b} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= -\begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} -\bar{a} & \bar{c} \\ \bar{b} & -\bar{d} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

In particular, $a = -\bar{a}$ and $d = -\bar{d}$. That is, $a^{1,2}_{1,1} = -\bar{a}^{1,2}_{1,1}$, and $a^{1,2}_{2,2} = -\bar{a}^{1,2}_{2,2}$, so that

$$\Re\gamma = 4\Re(a^{1,2}_{1,1}) = 4\Re(a^{1,2}_{2,2}) = 0. \qquad \square$$

Corollary 4.2 is now an immediate consequence of part 2 of Proposition 4.17.

4. page 1715: Include $V^2_{11} \neq V^2_{01}$ in the hypotheses of Corollary 5.4.
5. page 1725: At the start of Appendix C, recall the hypothesis $V^2_{11} \neq V^2_{01}$.

Addendum on effective Hamiltonians. The local behavior of wavepackets which are spectral concentrated near the quasi-momentum **M** is governed by an effective Hamiltonian, H_{eff}, which may be read off the leading order matrix Fourier symbol in (4.26) of [2]; see also Appendix B. In the reflection invariant case, we have $\gamma \equiv -i\tilde{\gamma}$ and $\tilde{\gamma} \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$; see Remark 4.16 of [2] and (E), (F) above. Thus,

$$H_{\text{eff}} = \alpha|\kappa|^2\sigma_0 + \tilde{\gamma}(\kappa_1^2 - \kappa_2^2)\sigma_2 + 2\beta\kappa_1\kappa_2\sigma_1.$$

We note that in the case of systems with the symmetries $\mathcal{P}$, $\mathcal{C}$, $\pi/2$-rotational invariance, and ρ, we have that H_{eff} is unitarily equivalent to the Hamiltonian

$$\tilde{H}_{\text{eff}} = \alpha|\kappa|^2\sigma_0 + \tilde{\gamma}(\kappa_1^2 - \kappa_2^2)\sigma_1 + 2\beta\kappa_1\kappa_2\sigma_3,$$

obtained in [1]. Indeed, $V^* H_{\text{eff}} V = \tilde{H}_{\text{eff}}$, where V is the unitary matrix

$$\text{(H)} \qquad V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

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