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Markov chain approximation and measure change for time-inhomogeneous stochastic processes

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ABSTRACT

In this paper, we propose a general time-inhomogeneous continuous-time Markov chain (CTMC) framework for the approximation of the general one-dimensional and two-dimensional time-inhomogeneous diffusion processes. For the approximating CTMC, we can perform a change of measure, choose the minimal relative entropy measure to determine the measure uniquely, and finally establish the convergence. Therefore, the proposed methodology covers the stochastic processes that are hard to perform a change of measure, and is applicable to valuation problems driven by models not only under the risk-neutral probability measure but also under the physical probability measure.

1. Introduction

1.1. Continuous time Markov chain approximation technique

Asset prices and interest rates are widely assumed to follow continuous-time stochastic processes with continuous state space, while there are only a limited number of models for a limited number of financial derivatives having analytical formulas. Mijatović and Pistorius [1] introduced such a continuous-time Markov chain (CTMC) approximation based valuation framework to solve option pricing problems. They successfully applied the CTMC approximation technique with one-dimensional stochastic processes in the pricing of European and continuously monitored barrier options, and further established rigorous convergence properties for the approximation. The accuracy of the CTMC approximation method over other numerical methods has been fully tested and verified in different settings, and proved in terms of error analysis and convergence rate in [2] and [3].

The CTMC technique was used to price other options with one-dimensional stochastic processes including, but are not limited to, the following excellent works: Eriksson and Pistorius [4] analyzed the solution of the optimal stopping problem

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associated with the valuation of American options driven by CTMC; The discretely and continuously monitored arithmetic Asian options through numerical inversion of a double transform are studied in [5]; Song et al. [6] extended the valuation framework to a one-dimensional Markov process with regime-switching coefficients; Chatterjee et al. [7] developed a CTMC based approximation method for the pricing and hedging of short-maturity arithmetic Asian options; Meier et al. [8] developed the CTMC approximation of one-dimensional diffusions with a lower sticky boundary and priced bond in a sticky short rate model for low-interest environment; Siu [9] priced bond under a Markovian regime-switching jump-augmented Vasicek model, provided that the market parameters switch over time according to a CTMC.

Approximating a correlated two-dimensional diffusion process is much more challenging to all existing algorithms, for example the trees approach needs a transformation to give constant volatility or decorrelate the Brownian motions ([10] and [11]) while that transformation is not intuitive. Cui et al. [12] and [13] developed a two-layer CTMC method to approximate the stochastic (local) volatility model; Cui et al. [14] and [15] provided an alternative double-layer CTMC approximation for time-changed Markov processes and skew stochastic (local) volatility models, respectively; In contrast to the existing double-layer approach, Xi et al. [16] proposed a simultaneous two-dimensional CTMC approximation method, to approximate the general two-dimensional fully coupled Markov diffusion processes; Cai et al. [17] invented a unified analytical CTMC approximation framework for practically useful quantities including first passage times, running extrema, and time integrals; Elliott et al. [18] modeled unobservable states of the economy by a hidden CTMC and adopted a regime switching random Esscher transform to determine an equivalent martingale pricing measure; Kirkby et al. [19] proposed a general CTMC approximation for multi-asset option pricing with systems of correlated diffusions; Kirkby et al. [20] considered Bermudan and barrier options using CTMC approximation of stochastic volatility; Kirkby and Nguyen [21] efficiently priced Asian option under regime-switching jump diffusions and stochastic volatility models by combining CTMC approximation with Fourier pricing techniques.

1.2. Motivation

Firstly, in practical applications, model parameters such as the short rate or the volatility function are often taken to be time-dependent, such as for short rate models that perfectly fit the initial yield curve (see, e.g., [22,23]) or for other calibration purposes. Various efforts have been made to explicitly model the dependence of parameters on time, such as [24–26]. In recent years, time-dependent parameter modeling with financial time series have been researched with new techniques (see, e.g., [27–29]). Stochastic processes that use time-dependent drift and/or diffusion coefficients, are time-inhomogeneous. When it comes to the valuation of financial derivatives whose underlying asset prices modeled by a time-inhomogeneous stochastic process, only very few models for certain types of options have analytical formulas available. In this paper, we are going to develop an efficient and accurate pricing framework that is generally applicable for the time-inhomogeneous one-dimensional and two-dimensional models.

Secondly, in mathematical finance, the fundamental theorem of asset pricing reveals that for a stochastic process the existence of an equivalent martingale measure plays a very important role, because it is essentially equivalent to the market being arbitrage-free (see [30]). Starting from the economically meaningful assumption that a stochastic process, which does not allow arbitrage profits, enables the real probability measure to be replaced by an equivalent probability measure such that the process becomes a martingale, and then it is possible to use the rich machinery of martingale theory.

The Girsanov theorem (see [31]) is especially important in the theory of financial mathematics as it tells how to convert from the physical measure to the risk-neutral measure, which is a very useful tool for pricing derivatives on the underlying instrument. The Novikov's condition (see [32]) is a widely used sufficient condition for a stochastic process that takes the form of the Radon-Nikodym derivative in the Girsanov's theorem to be a martingale. The Kazamaki's condition (see [33]), as a more general condition than the Novikov's condition, gives a sufficient criterion ensuring that the Doléans-Dade exponential of a local martingale is a true martingale. However, there are many cases for which the Girsanov theorem is not applicable either because both Novikov's condition and Kazamaki's condition fail, or these two conditions are challenging to meet (see [34]), which generates difficulties in performing a change of measure.

1.3. Our contributions and structure of the paper

So far all the stochastic processes being approximated by the CTMC technique for option pricing purposes are in the risk-neutral measure. In this paper, starting with the general time-inhomogeneous stochastic processes in the physical probability measure, we provide a systematic approach, from time-inhomogeneous CTMC approximation, to settle the "closest" martingale measure relative to the physical measure, and to price options based on the time-inhomogeneous approximating CTMC after measure change. Our methodology is applicable to general stochastic processes that are hard to perform a change of measure.

In Section 2, we construct the time-inhomogeneous approximating CTMC. We firstly approximate the general one-dimensional time-inhomogeneous diffusion process given in (2.1) by a time-inhomogeneous CTMC in Section 2.1, using the technique proposed in [35]. The two-dimensional time-inhomogeneous diffusion process given in (2.7) is approximated by a double-layer time-inhomogeneous CTMC in Section 2.2, using the auxiliary process technique proposed in [13] but in the time-inhomogeneous setting.

In Section 3, we perform a change of measure on the approximating time-inhomogeneous CTMC. For the reason that Markov chains have random jumps, there does not exist a unique equivalent martingale measure. To tackle this problem, we determine a “closest” martingale measure to the physical measure by choosing the one having the minimal relative entropy with respect to the physical measure. In the risk-neutral world, there exists possible destruction of the martingale property after approximating, while the proposed approach is able to preserve the martingale property after approximation. The convergence of the “chosen” equivalent martingale measure is established in Section 4, by transforming the problem to determining whether the sequence of the Radon–Nikodym derivatives is convergent, as the cardinality of the discrete state space of the approximating CTMC goes to infinity.

In Section 5, we apply the approximating framework and obtain approximating formulae for pricing options in European and barrier types. In Section 6, we provide CTMC approximated option valuation and demonstrate the accuracy of the proposed methodology. In Section 7, we conclude and remark.

2. The CTMC approximation

2.1. One-dimensional diffusion

We consider the following general diffusion process $\{S_t\}_{t\in[0,T]}$ with state space $\mathbb{D}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration and $\mathbb{P}$ is the physical measure. Here, S evolves according to

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dB_t \tag{2.1}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$. Suitable conditions are imposed on the functions μ and σ so that (2.1) has a unique solution (see, e.g., Definition 1.1. and Theorem 1.2. on page 4 of [36], for sufficient conditions for a strong solution and the uniqueness, respectively). Let $\mathbb{F}^S = \mathbb{F}^B$ be the natural filtration generated by B such that $\mathbb{F}^B \subseteq \mathbb{F}$. That is, we consider the general diffusion case that $\mu(t, x)$ and $\sigma(t, x)$ can be time-dependent functions. We further assume that S_t is a Feller process whose Feller property guarantees that there exists a version of the process with cádlág paths satisfying the strong Markov property.

Next, we illustrate the methodology introduced in [1] on the approximation of CTMC to S. For the general time-inhomogeneous Markov process described in (2.1), on the same probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ its approximating process is a time-inhomogeneous CTMC $X^{(N)}$ on the time interval $[0, T]$, whose finite state space is given by

$$\mathbb{G}^{(N)} = \{x_1, \dots, x_N\} \subset \mathbb{D}$$

where x_1 represents the smallest element and x_N represents the largest element. Define the boundary of $\mathbb{G}^{(N)}$ as $\partial\mathbb{G}^{(N)} := \{x_1, x_N\}$ and denote the interior of $\mathbb{G}^{(N)}$ simply as $\mathbb{G}^{(N)}\backslash\partial\mathbb{G}^{(N)}$. We further partition the time interval $[0, T]$ in the way that $T_l^{(m)} = \frac{l}{m}T$ for $l \in \{0, 1, \dots, m\}$.

Denote the time-dependent generator of $X^{(N)}$ as $\Lambda_t^{(N)} = \left(\lambda_{i,j}^{(N)}(t)\right)_{i,j\in\{1,\dots,N\}}$. In line with [37], $\Lambda_t^{(N)}$ is generated in each time interval $[T_{l-1}^{(m)}, T_l^{(m)})$ for $l \in \{1, \dots, m\}$. Denote $\lambda_{i,j}^{(N)}(t) := \sum_{l=1}^{m} \lambda_{i,j}^{(l,N)} \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t)$, then $\lambda_{i,j}^{(l,N)}$ is constructed as follows:

- For $i = 2, \dots, N-1$, set

$$\begin{cases} \lambda_{i,j}^{(l,N)} \geq 0, & \forall j \neq i. \\ \sum_{j\neq i} \lambda_{i,j}^{(l,N)}(x_j - x_i) = \mu(T_{l-1}^{(m)}, x_i), & j = i-1, i+1. \\ \sum_{j\neq i} \lambda_{i,j}^{(l,N)}(x_j - x_i)^2 = \sigma^2(T_{l-1}^{(m)}, x_i), & j = i-1, i+1. \\ \lambda_{i,i}^{(l,N)} = -\lambda_{i,i-1}^{(l,N)} - \lambda_{i,i+1}^{(l,N)}, & j = i. \\ \lambda_{i,j}^{(l,N)}, & j \neq i-1, i, i+1. \end{cases} \tag{2.2}$$

- For $i = 1, N$, set

$$\lambda_{i,j}^{(l,N)} = 0, \qquad \forall j \in \{1, \dots, N\}.$$

Remark 1. The state space $\{x_i\}_{i\in\{1,\dots,N\}}$ are formed by a nonuniform grid. For fixed x_1 and x_N, we define $\{x_2, \dots, x_{N-1}\}$ proposed in [38]: Assume that $X_0 = x_0 > 0$, and for $\alpha \in (0, 1)$,

$$x_j := x_0 + \alpha \cdot (x_N - x_1) \cdot \sinh\left(k_2 \frac{j}{N} + k_1\left(1 - \frac{j}{N}\right)\right), \quad j = 2, \dots, N-1, \tag{2.3}$$

where

$$k_1 := \operatorname{arcsinh}\left(\frac{x_1 - x_0}{\alpha(x_N - x_j)}\right), \quad k_2 := \operatorname{arcsinh}\left(\frac{x_N - x_0}{\alpha(x_N - x_j)}\right). \tag{2.4}$$

The sequence of time-inhomogeneous generator matrices $\Lambda_t^{(N)}$ defined on the time-space grids $[0,T]\times\mathbb{G}^{(N)}$, needs to be close to the time-dependent generator of the time-inhomogeneous Markov process $\{S_t\}_{t\in[0,T]}$. Consider the time-space Markov process (Z, Y). Here, $Z=(Z_t)_{t\geq 0}$ and $Y=(Y_t)_{t\geq 0}$ are defined as

$$Z_t := (Z_0+t)\wedge T, \qquad Y_t := S_{Z_t}, \tag{2.5}$$

for $t>0$ and $Z_0\in[0,T]$, where the state space of Y is $\mathbb{D}$. For any $s, t>0$ and bounded Borel function $f:[0,T]\times\mathbb{D}\mapsto\mathbb{R}$, the time-space Markov process (Z, Y) with state space $[0,T]\times\mathbb{D}$, satisfies

$$(\mathbf{P}_t f)(s,x) := \mathbb{E}[f(Z_t,Y_t)|Z_0=s, Y_0=x] = \mathbb{E}[f((s+t)\wedge T, S_{(s+t)\wedge T})|S_s=x].$$

Then $\mathbf{P}_t$ generates a semigroup as defined in [35], and its corresponding infinitesimal generator $\mathcal{L}'$ is defined as

$$\mathcal{L}'f(s,x) = \lim_{t\downarrow 0} t^{-1}[\mathbf{P}_t f(s,x) - f(s,x)], \tag{2.6}$$

for all f in the domain of $\mathcal{L}'$, where suitable regularity condition are imposed on (Z, Y) to ensure the existence of the limit. The convergence result is given in Corollary 2 of [37].

2.2. Two-dimensional diffusion

In this subsection, we consider that the underlying security follows a two-dimensional diffusion process,

$$\begin{cases} dS_t = \mu(t,S_t,\mathcal{W}_t)dt + \nu(S_t)\sigma(\mathcal{W}_t)dB_t^{(1)}, \\ d\mathcal{W}_t = \alpha(\mathcal{W}_t)dt + \gamma(\mathcal{W}_t)dB_t^{(2)}, \end{cases} \tag{2.7}$$

where $\mathbb{E}^{\mathbb{P}}\left[dB_t^{(1)}dB_t^{(2)}\right]=\rho dt$ with $-1\leq\rho\leq 1$. We consider the general case that the coefficients $\mu(t, x, y)$ is a time-dependent function. Suitable conditions are imposed on the functions μ, σ, α, γ, and $\nu>0$, so that model (2.7) has a unique-in-law weak solution that exits in its state space. Assume that $\frac{1}{\nu}$ is locally integrable on the state space $\mathbb{S}$ of S_t, and $\frac{\sigma}{\gamma}$ is locally integrable on the state space $\mathbb{W}$ of $\mathcal{W}_t$. We consider the case that $(S_t,\mathcal{W}_t)$ is a Feller process whose Feller property guarantees that there exists a version of the process with cádlág paths satisfying the strong Markov property.

The rest of this section is to approximate this two-dimensional model (2.7) by a double-layer CTMC, using the auxiliary process technique proposed in [13].

2.2.1. The auxiliary process

Define the functions g and h as

$$g(x) := \int_{\cdot}^{x}\frac{1}{\nu(u)}du \quad\text{and}\quad h(x) := \int_{\cdot}^{x}\frac{\sigma(u)}{\gamma(u)}du, \tag{2.8}$$

and denote $B_t^* = \frac{B_t^{(1)}-\rho B_t^{(2)}}{\sqrt{1-\rho^2}}$ is a standard Brownian motion independent of $B_t^{(2)}$. We create an auxiliary process with state space $\mathbb{X}$ as $X_t := g(S_t)-\rho h(\mathcal{W}_t)$, then we can rewrite model (2.7) as

$$\begin{cases} dX_t = \widetilde{\theta}(t,X_t,\mathcal{W}_t)dt + \sqrt{(1-\rho^2)}\sigma(\mathcal{W}_t)dB_t^*, \\ d\mathcal{W}_t = \alpha(\mathcal{W}_t)dt + \gamma(\mathcal{W}_t)dB_t^{(2)}, \end{cases} \tag{2.9}$$

where

$$\widetilde{\theta}(t,X_t,\mathcal{W}_t) := \theta(t, g^{-1}(X_t+\rho h(\mathcal{W}_t)),\mathcal{W}_t),$$

and

$$\theta(t,x,y) = \left(\frac{\mu(t,x,y)}{\nu(x)} - \frac{\nu'(x)}{2}\sigma^2(y)\right) - \rho\left(\alpha(y)\frac{\sigma(y)}{\gamma(y)} + \frac{1}{2}\big(\gamma(y)\sigma'(y)-\gamma'(y)\sigma(y)\big)\right). \tag{2.10}$$

Now, S is driven by $\mathcal{W}$'s approximating CTMC $\omega^{(n_1)}$ instead, and denote the resulting S process as S^{ω}. We further denote

$$\widetilde{X}_t := g(S_t^{\omega}) - \rho h(\omega_t^{(n_1)}), \tag{2.11}$$

by Proposition 1 in [13], one can show that

$$d\widetilde{X}_t = \widetilde{\theta}(t,\widetilde{X}_t,\omega_t^{(n_1)})dt + \sqrt{(1-\rho^2)}\sigma(\omega_t^{(n_1)})dB_t^*. \tag{2.12}$$

Note that $\widetilde{X}$ can be regarded as a regime-switching diffusion.

2.2.2. CTMC approximation of $(X_t, \mathcal{W}_t)$

Since $\widetilde{X} = (\widetilde{X}_t)_{t\geq 0}$ is a time-dependent process, we can approximate it in a similar way as that of in Section 2.1:

- **First layer:** We firstly approximate the diffusive variance process $\mathcal{W} = (\mathcal{W}_t)_{t\geq 0}$ by a CTMC $\omega^{(n_1)} = (\omega_t^{(n_1)})_{t\geq 0}$ with state space $\mathbb{W}^{(n_1)} = \{w_1, \ldots, w_{n_1}\}$.
  Denote the generator of $\omega_t^{(n_1)}$ as $Q = \left(q_{i,j}\right)_{i,j\in\{1,\ldots,n_1\}}$, which is generated as a tri-diagonal matrix with zero row sums and nonnegative elements as follows:
  For $i = 2, \ldots, n_1 - 1$,
$$\begin{cases} \sum_{j\neq i} q_{i,j}(w_j - w_i) = \alpha(w_i), & j = i-1, i+1 \\ \sum_{j\neq i} q_{i,j}(w_j - w_i)^2 = \gamma^2(w_i), & j = i-1, i+1, \\ q_{i,i} = -q_{i,i-1} - q_{i,i+1}, & j = i, \\ 0, & j \neq i-1, i, i+1, \end{cases} \tag{2.13}$$
  and $q_{1,i} = q_{n_1,i} = 0$ for $j \in \{1, \ldots, n_1\}$.
  Similarly, the states of $\omega^{(n_1)}$ are nonuniform grids, which satisfy (2.3) and (2.4).
- **Second layer:** Then we approximate the process $\widetilde{X} = (\widetilde{X}_t)_{t\geq 0}$ by a CTMC $\widetilde{X}^{(n_2)} = (\widetilde{X}_t^{(n_2)})_{t\geq 0}$ with state space $\mathbb{X}^{(n_2)} = \{\widetilde{x}_1, \ldots, \widetilde{x}_{n_2}\}$.
  For $\omega^{(n_1)}$ taking value $w_k \in \{w_1, \ldots, w_{n_1}\}$ with $k \in \{1, \ldots, n_1\}$, the time-dependent generator matrix of $\widetilde{X}^{(n_2)}$ is defined accordingly as $\Lambda_k(t) := \left(\lambda_{i,j}^k(t)\right)_{i,j\in\{1,\ldots,n_2\}}$, where for $l \in \{1, \ldots, m\}$
$$\lambda_{i,j}^k(t) = \sum_{l=1}^{m} \lambda_{i,j}^{k(l)} \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t). \tag{2.14}$$
  Then $\lambda_{ij}^{(l,N)}$ should be constructed in the time interval $[T_{l-1}^{(m)}, T_l^{(m)})$ as
$$\begin{cases} \sum_{j\neq i} \lambda_{i,j}^{k(l)}(x_j - x_i) = \widetilde{\theta}(T_{l-1}^{(m)}, x_i, w_k), & j = i-1, i+1, \\ \sum_{j\neq i} \lambda_{i,j}^{k(l)}(x_j - x_i)^2 = (1-\rho^2)\sigma^2(w_k), & j = i-1, i+1, \\ \lambda_{i,i}^{k(l)} = -\lambda_{i,i-1}^{k(l)} - \lambda_{i,i+1}^{k(l)}, & j = i, \\ 0, & j \neq i-1, i, i+1, \end{cases} \tag{2.15}$$
  for $i = 2, \ldots, n_2 - 1$,
  and $\lambda_{1,i}^{k(l)} = \lambda_{n_1,i}^{k(l)} = 0$ for $i \in \{1, \ldots, n_2\}$.

Moreover, for notational convenience, we define the following sets $\mathbb{X}^k$, which correspond to $\Lambda_k(t)$ for each $\omega_t^{(n_1)} = w_k,\ k \in \{1, \ldots, n_1\}$:

$$\mathbb{X}^k := \{\widetilde{x}_1, \ldots, \widetilde{x}_{n_2}\}.$$

One can see that the elements of $\mathbb{X}^k$ are the same as in $\mathbb{X}^{(n_2)}$. An intuitive perception is that once $\omega_t^{(n_2)} = w_k$ is fixed, $\mathbb{X}^k$ is the state space of the approximating CTMC with generator matrix $\Lambda_k(t)$. It is not hard to see that

$$\mathbb{X}^1 = \cdots = \mathbb{X}^{n_1}.$$

2.2.3. Regime-switching CTMC approximation to the second layer

By Theorem 1 in [6], we can embed the CTMC $\widetilde{X}^{(n_2)}$ (the second layer covered in Session 2.2.2) in a Markov chain to transfer the two-dimensional process $(X_t, \mathcal{W}_t)$ to an equivalent one-dimensional process $\overline{\overline{X}}^{(N)}$ with an enlarged state space

$$\overline{\overline{\mathbb{X}}}^{(N)} = \mathbb{X}^1 \times \mathbb{X}^2 \times \cdots \times \mathbb{X}^{n_1} := \{\overline{\overline{x}}_1, \overline{\overline{x}}_2, \ldots, \overline{\overline{x}}_N\}, \quad N = n_1 \cdot n_2.$$

As a natural extension, we have a result similar to [6] in time-inhomogenuous case:

Proposition 2. *Consider $\widetilde{X}^{(n_2)}$ as the discrete state time-inhomogeneous regime-switching CTMC given in Section 2.2.2, and consider another one-dimensional time-inhomogeneous CTMC $\overline{\overline{X}}^{(N)}$ with state space $\overline{\overline{\mathbb{X}}}^{(N)}$ and $N \times N$ transition rate matrix*

$$\mathbf{\Lambda}^{(N)}(t) = \begin{pmatrix} q_{1,1}\mathbf{I}_{n_2} + \Lambda_1(t) & q_{1,2}\mathbf{I}_{n_2} & \cdots & q_{1,n_1}\mathbf{I}_{n_2} \\ q_{2,1}\mathbf{I}_{n_2} & q_{2,2}\mathbf{I}_{n_2} + \Lambda_2(t) & \cdots & q_{2,n_1}\mathbf{I}_{n_2} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n_1,1}\mathbf{I}_{n_2} & q_{n_1,2}\mathbf{I}_{n_2} & \cdots & q_{n_1,n_1}\mathbf{I}_{n_2} + \Lambda_{n_1}(t) \end{pmatrix}, \tag{2.16}$$

where $\mathbf{I}_{n_2}$ *is the* $n_2 \times n_2$ *identity matrix,* $Q = (q_{i,j})_{n_1\times n_1}$ *is the time-independent generator matrix of* $\omega^{(n_1)}$*, and* $\Lambda_k(t)$ *is the time-dependent generator matrix of* $\widetilde{X}^{(n_2)}$ *when* $\omega^{(n_1)}$ *takes value* w_k*. Define the mapping* $\phi : \mathbb{X}^{(n_2)} \times \{1, \ldots, n_1\} \to \overline{\overline{\mathbb{X}}}^{(N)}$ *by*

$$\phi(x_i, k) = \overline{\overline{x}}_{(k-1)n_2+i},$$

and its inverse $\phi^{-1} : \overline{\overline{\mathbb{X}}}^{(N)} \to \mathbb{X}^{(n_2)} \times \{1, \ldots, n_1\}$ *by*

$$\phi^{-1}(\overline{\overline{x}}_n) = (x_i, k), \qquad \text{s.t. } n = (k-1)n_2 + i.$$

Further define

$$\widetilde{X}^{(n_2)} := \left(\widetilde{X}^{(n_2)}_t\right)_{0\le t\le T}, \quad \omega^{(n_1)} := \left(\omega^{(n_1)}_t\right)_{0\le t\le T}, \quad \overline{\overline{X}}^{(N)} := \left(\overline{\overline{X}}^{(N)}_t\right)_{0\le t\le T},$$

and then we have

$$\left(\widetilde{X}^{(n_2)}, \omega^{(n_1)}\right)\Big|\omega^{(n_1)}_0 = w_k, \widetilde{X}^{(n_2)}_0 = x_i\Big]$$
$$= \mathbb{E}\left[\varphi\circ\phi^{-1}\left(\overline{\overline{X}}^{(N)}\right)\Big|\overline{\overline{X}}^{(N)}_0 = \overline{\overline{x}}_{(k-1)n_2+i}\right]$$

for any path-dependent payoff function φ *such that the expectation on the left-hand side is finite.*

2.2.4. Weak convergence

Denote the state space of the Markov process $(t, X_t, \mathcal{W}_t)$ as $[0, T] \times \mathbb{X} \times \mathbb{W}$. For any *s*, $t > 0$ and bounded Borel function $f : [0, T] \times \mathbb{X} \times \mathbb{W} \to \mathbb{R}$, set f_t: $(x, w) \mapsto f(t, x, w)$. For function $f_t \in \mathcal{C}^{2,2}_c(\mathbb{X} \times \mathbb{W})$, by Eq. (2.9), the corresponding infinitesimal generator $\mathcal{L}_t$ acting on f_t is given by

$$\mathcal{L}_t f_t(x, w) = \widetilde{\theta}(t, x, w)\frac{\partial f_t}{\partial x} + \frac{1}{2}(1-\rho^2)\sigma^2(w)\frac{\partial^2 f_t}{\partial x^2} + \alpha(w)\frac{\partial f_t}{\partial w} + \frac{1}{2}\gamma^2(w)\frac{\partial^2 f_t}{\partial w^2}.$$

Denote

$$\mathbb{G}^{(n_1,n_2)} := \{\widetilde{x}_1, \ldots, \widetilde{x}_{n_2}\} \times \{w_1, \ldots, w_{n_1}\},$$

the approximation error over $\mathbb{G}^{(n_1,n_2)}$ is measured by

$$\epsilon^{(n_1,n_2)}(f) := \max_{t\in[0,T],\,(\widetilde{x},w)\in\mathbb{G}^{(n_1,n_2)}} |\mathbf{\Lambda}^{(N)} f_{t,n_1,n_2}(\widetilde{x}, w) - \mathcal{L}_t f_t(\widetilde{x}, w)|, \tag{2.17}$$

where function $f_{t,n_1,n_2} = f_t|_{\mathbb{G}^{(n_1,n_2)}}$ denotes the restriction of f_t to $\mathbb{G}^{(n_1,n_2)}$, and $\mathbf{\Lambda}^{(N)} = \mathbf{\Lambda}^{(N)}(t)$ is given in (2.16).

In the following proposition, we provide the weak convergence result of approximating two-dimensional time-inhomogeneous diffusions, which represents an extension of Proposition 4 of [13] to the time-inhomogenous case. In its proof, we employ the standard convergence results for Markov process in [35] and the techniques used in proving the weak convergence result regarding one-dimensional time-inhomogeneous diffusions in the classical work [37].

Proposition 3. *Let* $\mathcal{D}^*(\mathcal{L}_t) \subset \mathcal{D}(\mathcal{L}_t)$ *denote a core of* $\mathcal{L}_t$*. For any function* $f_t \in \mathcal{D}^*(\mathcal{L}_t)$*, if* $\epsilon^{(n_1,n_2)}(f_t) \mapsto 0$ *as* $n_1, n_2 \to \infty$*, then*

(a) $(\widetilde{X}^{(n_2)}_t, \omega^{(n_1)}_t) \Longrightarrow (X_t, \mathcal{W}_t)$ *as* $n_1, n_2 \to \infty$*,*

(b) $\widetilde{S}^{(N)}_t \Longrightarrow S_t$ *as* $n_1, n_2 \to \infty$*, for* $\widetilde{S}^{(N)}_t := g^{-1}\left(\widetilde{X}^{(n_2)}_t + \rho f(\omega^{(n_1)}_t)\right)$*,*

where "$\Longrightarrow$*" denotes weak convergence.*

Proof. See Appendix A. □

Remark 4. We note that the proposed CTMC approximation methodology is applicable on general multidimensional diffusion processes if the corresponding Brownian motions are perfectly correlated or totally uncorrelated. That is, considering $\mathbf{X} := \{X^{(1)}, \ldots, X^{(n)}\}$ for $n > 2$, where

$$X^{(i)}_t = \mu_i(t, X^{(i)}_t)dt + \sigma_i(t, X^{(i)}_t)dW^{(i)}_t$$

and $\mathbb{E}[dW^{(i)}_t dW^{(j)}_t] = \rho_{ij}dt$ with $|\rho_{ij}| \le 1$ for $i, j = 1, \ldots, n$ and $i \ne j$, the proposed CTMC approximation methodology can construct the approximating CTMCs if $\rho_{ij} = -1, 0, 1$. However, when $0 < \rho_{ij} < 1$, there are multidimensional dependencies in $\{X_i\}_i$ which should be tackled to proceed.

3. The equivalent martingale measure

In this section, we perform a change of equivalent martingale measure for the time-inhomogeneous CTMC, by means of the techniques in Section 11.3 of [39] and the references therein. Because Markov chains have random jumps, there does not exist a unique equivalent martingale measure. To determine the "closest" martingale measure to the physical measure, we choose the one having the minimal relative entropy with respect to the physical measure.

3.1. Change of measure for time-inhomogeneous CTMC

Recall that $X^{(N)}$ is the time-inhomogeneous CTMC with finite state-space $\mathbb{G}^{(N)} = \{x_1, \dots, x_N\}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let $\mathbb{F}^{X^{(N)}}$ be the filtration generated by $X^{(N)}$ such that $\mathbb{F}^{X^{(N)}} \subseteq \mathbb{F}$.

Lemma 5. *[Section 11.3, [39]] On the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, for $i, j \in \{1, \dots, N\}$, $i \neq j$, and $0 \le t \le T$, let*

$$H_t^{i(N)} := \mathbf{1}_{\{X_t^{(N)} = x_i\}}, \qquad H_t^{ij(N)} := \sum_{0 \le u \le t} H_{u-}^{i(N)} H_u^{j(N)},$$

and then we have that the process $(M_t^{ij(N)})_{0 \le t \le T}$ defined as

$$M_t^{ij(N)} = H_t^{ij(N)} - \int_0^t \lambda_{i,j}^{(N)}(u) H_u^{i(N)} du$$

is an $\mathbb{F}$-martingale under $\mathbb{P}$.

Consider a family $\{\kappa_{ij}^{(N)}(t)\}_{i,j \in \{1,\dots,N\}}$ of bounded real-valued processes which is $\mathbb{F}$-predictable, such that $\kappa_{ij}^{(N)}(t) > -1$ and $\kappa_{ii}^{(N)}(t) = 0$. Define $(\eta_t^{(N)})_{0 \le t \le T}$ as

$$\eta_t^{(N)} = 1 + \int_0^t \sum_{i,j=1}^N \eta_{u-}^{(N)} \kappa_{ij}^{(N)}(u) dM_u^{ij} \tag{3.1}$$

and we have the following proposition.

Proposition 6. *The solution to Eq. (3.1) can be expressed as*

$$\eta_t^{(N)} = e^{-M_t^c} \prod_{0 < u \le t} \left(1 + \sum_{i,j=1}^N \kappa_{ij}^{(N)}(u)(H_u^{ij(N)} - H_{u-}^{ij(N)})\right), \tag{3.2}$$

where $M_t^c = \int_0^t \sum_{i,j=1}^N \kappa_{ij}^{(N)}(u) \lambda_{i,j}^{(N)}(u) H_u^{i(N)} du$. Furthermore, $\eta_t^{(N)}$ is a strictly positive $\mathbb{F}$-martingale under $\mathbb{P}$ such that $\mathbb{E}[\eta_T^{(N)}] = 1$.

Proof. By Lemma 5, we can rewrite Eq. (3.1) as

$$\eta_t^{(N)} = 1 + \int_0^t \sum_{i,j=1}^N \eta_{u-}^{(N)} \kappa_{ij}^{(N)}(u) dH_u^{ij(N)} - \int_0^t \sum_{i,j=1}^N \eta_{u-}^{(N)} \kappa_{ij}^{(N)}(u) \lambda_{i,j}^{(N)}(u) H_u^{i(N)} du.$$

Let $M_t^c = \int_0^t \sum_{i,j=1}^N \kappa_{ij}^{(N)}(u) \lambda_{i,j}^{(N)}(u) H_u^{i(N)} du$ and then we have

$$\begin{aligned}\eta_t^{(N)} &= e^{-M_t^c} \prod_{0 < u \le t} \left(1 + \sum_{i,j=1}^N \kappa_{ij}^{(N)}(u)(M_u^{ij} - M_{u-}^{ij})\right) \\ &= e^{-M_t^c} \prod_{0 < u \le t} \left(1 + \sum_{i,j=1}^N \kappa_{ij}^{(N)}(u)(H_u^{ij(N)} - H_{u-}^{ij(N)})\right).\end{aligned}$$

From the definition of $H_u^{ij(N)}$, we know that at most one $H_u^{ij(N)} - H_{u-}^{ij(N)}$ is equal to 1 while the rest is equal to 0, and then $1 + \sum_{i,j=1}^N \kappa_{ij}^{(N)}(u)(H_u^{ij(N)} - H_{u-}^{ij(N)})$ is either equal to $1 + \kappa_{ij}^{(N)}(u)$ for some $i \neq j$, or equal to 1. Hence $\eta_t^{(N)}$ is strictly positive. By Lemma 5, we know that $M_t^{ij(N)}$ is a martingale, and then by Eq. (3.1), we have that $\eta_t^{(N)}$ is a martingale such that $\mathbb{E}[\eta_T^{(N)}] = 1$. □

Now, an equivalent probability measure $\mathbb{Q}^{(N)}$ on $(\Omega, \mathcal{F}_t)$ with any $t \in [0, T]$ can be defined by

$$\left.\frac{d\mathbb{Q}^{(N)}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \eta_t^{(N)}, \qquad \mathbb{P} - a.s. \tag{3.3}$$

The following result demonstrates that the approximating CTMC $X^{(N)}$ defined in Section 2.1 for the one-dimensional diffusion case, is also a Markov chain with generator $\widetilde{\Lambda}_t^{(N)}$ under $\mathbb{Q}^{(N)}$. For the approximating CTMC $\overline{\overline{X}}^{(N)}$ defined in Proposition 2, its generator matrix $\widetilde{\mathbf{\Lambda}}^{(N)}(t)$ under equivalent martingale measure $\mathbb{Q}^{(N)}$ can be generated analogously.

Proposition 7. *[Section 11.3, [39]] For the probability measure $\mathbb{Q}^{(N)}$ defined in Eq. (3.3), if the process $\{X_t^{(N)}\}_{0 \le t \le T}$ is a Markov chain under $\mathbb{P}$ with infinitesimal generator $\Lambda_t^{(N)} = \left(\lambda_{i,j}^{(N)}(t)\right)_{i,j \in \{1,\dots,N\}}$, then $\{X_t^{(N)}\}_{0 \le t \le T}$ is also a Markov chain under $\mathbb{Q}^{(N)}$ whose corresponding infinitesimal generator $\widetilde{\Lambda}_t^{(N)} = \left(\widetilde{\lambda}_{ij}^{(N)}(t)\right)_{i,j \in \{1,\dots,N\}}$ satisfies*

$$\begin{aligned}&\widetilde{\lambda}_{ii}^{(N)}(t) = -\textstyle\sum_{j \neq i} \widetilde{\lambda}_{ij}^{(N)}(t), \\ &\widetilde{\lambda}_{ij}^{(N)}(t) = (1 + \kappa_{ij}^{(N)}(t)) \lambda_{ij}^{(N)}(t), \quad \text{for } i \neq j.\end{aligned} \tag{3.4}$$

3.2. Martingale conditions

3.2.1. One-dimensional models

Consider $\mathbb{Q}^{(N)}$ as a risk-neutral measure such that $\{e^{-rt}X_t^{(N)}\}$ is a martingale under $\mathbb{Q}^{(N)}$ with risk-free rate r. If we set

$$\mathbf{P}_t^{(N)}f(x):=\mathbb{E}_x^{\mathbb{Q}^{(N)}}[f(X_t^{(N)})]$$

with any Borel functions f, it is know that $\mathbf{P}_t^{(N)}$ forms a semigroup such that

$$\mathbf{P}_{t+s}^{(N)}f=\mathbf{P}_t^{(N)}(\mathbf{P}_s^{(N)}f),$$

for all s, $t\geq 0$ and $\mathbf{P}_0^{(N)}f=f$. Taking $f(x)=x$ and by the Markov property, we have

$$\mathbb{E}^{\mathbb{Q}^{(N)}}[e^{-r(t+s)}X_{t+s}^{(N)}|\mathbb{F}_s]=E^{\mathbb{Q}^{(N)}}[e^{-r(t+s)}X_{t+s}^{(N)}|X_s^{(N)}]=e^{-r(t+s)}\mathbf{P}_t^{(N)}(X_s^{(N)}). \tag{3.5}$$

Assume that the time-inhomogeneous generator satisfies commutativity, i.e., $\mathbf{P}_t^{(N)}\mathbf{P}_s^{(N)}=\mathbf{P}_s^{(N)}\mathbf{P}_t^{(N)}$, for s, $t\geq 0$. Then, by the fact that $\mathbf{P}_t^{(N)}=e^{t\widetilde{\Lambda}_t^{(N)}}$ where $\widetilde{\Lambda}_t^{(N)}=(\widetilde{\lambda}_{i,j}^{(N)}(t))_{i,j\in\{1,\ldots,N\}}$ is the generator of $X^{(N)}$ under $\mathbb{Q}^{(N)}$, and by the Eq. (3.5) and with the fact that $\{e^{-rt}X_t^{(N)}\}$ is a martingale, under risk-neutral measure $\mathbb{Q}^{(N)}$, we have

$$\begin{aligned} rx_i &= \textstyle\sum_j \widetilde{\lambda}_{i,j}^{(N)}(t)\cdot x_j \\ &= \textstyle\sum_{j\neq i}(1+\kappa_{ij}^{(N)}(t))\lambda_{i,j}^{(N)}(t)\cdot x_j - x_i\sum_{j\neq i}(1+\kappa_{ij}^{(N)}(t))\lambda_{i,j}^{(N)}(t), \end{aligned} \tag{3.6}$$

for every $i\in\{1,\ldots,N\}$. Setting $\kappa_{ij}^{(N)}(t)=\sum_{l=1}^{m}\kappa_{ij}^{(l,N)}(t)\mathbf{1}_{[T_{l-1}^{(m)},T_l^{(m)})}(t)$ and $\kappa_{ii}^{(l)}(t)=0$ for fixed $t\in[0,T]$, Eq. (3.6) can be written as

$$rx_i=\sum_{j\neq i}(1+\kappa_{ij}^{(l,N)}(t))\lambda_{i,j}^{(l,N)}(x_j-x_i), \tag{3.7}$$

which implies that $\kappa_{ij}^{(l,N)}(t)\equiv\kappa_{ij}^{(l,N)}$ constant for fixed i, j.

3.2.2. Two-dimensional models

In the case of two-dimensional models, by the analysis of Section 2.2, we can obtain analogous results as those in Section 3.1 of [13] that for any measurable function ψ, it holds that $\mathbb{E}_{s,w}^{\mathbb{Q}}[e^{-rT}\psi(S_T)]$ can be approximated by

$$\mathbb{E}_{s_i,w_k}^{\mathbb{Q}^{(N)}}[e^{-rT}\psi(\widetilde{S}_T^{(N)})]=\mathbb{E}_{\bar{\bar{x}}_n}^{\mathbb{Q}^{(N)}}[e^{-rT}\widetilde{\psi}(\overline{\overline{X}}_T^{(N)})], \tag{3.8}$$

where $n=(k-1)n_2+i$, $\bar{\bar{x}}_n=g(s_i)-\rho h(w_k)$, $\widetilde{\psi}(x)=\psi\circ g^{-1}(x+\rho h(w_k))$,

$$\mathbb{E}_{s,w}^{\mathbb{Q}}[\cdot]=\mathbb{E}^{\mathbb{Q}}[\cdot|S_0=s,\mathcal{W}_0=w],$$

$$\mathbb{E}_{(s_i,w_k)}^{\mathbb{Q}^{(N)}}[\cdot]=\mathbb{E}^{\mathbb{Q}^{(N)}}[\cdot|\widetilde{S}_0^{(N)}=s_i,\omega_0^{(n_1)}=w_k].$$

Hence, we only need to focus on identifying the risk-neutral measure with the underlying $\overline{\overline{X}}^{(N)}$.

Consider $\mathbb{Q}^{(N)}$ as a risk-neutral measure such that the discounted value of the regime-switching CTMC $\left\{e^{-rt}\overline{\overline{X}}_t^{(N)}\right\}$ is a martingale under $\mathbb{Q}^{(N)}$ with risk-free rate r. Denote $\widetilde{\mathbf{\Lambda}}^{(N)}(t)$ as the generator matrix of $\overline{\overline{X}}_t^{(N)}$ under measure $\mathbb{Q}^{(N)}$ whose corresponding parameters $\boldsymbol{\kappa}^{(N)}(t)$ satisfying

$$\boldsymbol{\kappa}^{(N)}(t)=\begin{pmatrix} \widetilde{\boldsymbol{\kappa}}_{11}^{(N)}(t) & \boldsymbol{\kappa}_{1,2}^{(n_2)} & \cdots & \boldsymbol{\kappa}_{1,n_1}^{(n_2)} \\ \boldsymbol{\kappa}_{2,1}^{(n_2)} & \widetilde{\boldsymbol{\kappa}}_{22}^{(N)}(t) & \cdots & \boldsymbol{\kappa}_{2,n_1}^{(n_2)} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\kappa}_{n_1,1}^{(n_2)} & \boldsymbol{\kappa}_{n_1,2}^{(n_2)} & \cdots & \widetilde{\boldsymbol{\kappa}}_{n_1n_1}^{(N)}(t) \end{pmatrix}, \tag{3.9}$$

where

$$\widetilde{\boldsymbol{\kappa}}_{ii}^{(N)}(t)=\sum_{l=1}^{m}\boldsymbol{\kappa}_{i,i}^{(l,n_2)}\mathbf{1}_{[T_{l-1}^{(m)},T_l^{(m)})}(t),\qquad i\in\{1,\ldots,n_1\}.$$

Here, $\boldsymbol{\kappa}_{i,i}^{(l,n_2)}$ and $\boldsymbol{\kappa}_{i,j}^{(n_2)}$ are $n_2\times n_2$-dimensional matrices for $i,j\in\{1,\ldots,n_1\}$. Thus, similar to Eq. (3.6), we can obtain that

$$\widetilde{\mathbf{\Lambda}}^{(N)}(t)\cdot(\overline{\overline{\mathbb{X}}}^{(N)})^T=r\cdot(\overline{\overline{\mathbb{X}}}^{(N)})^T, \tag{3.10}$$

where $\overline{\overline{\mathbb{X}}}^{(N)}$ is the state space of $\overline{\overline{X}}_t^{(N)}$, and the superscript T denotes transpose. By an analogous result to Proposition 7 in the one-dimensional case, (3.10) is equivalent to

$$\overline{\overline{x}}_i r_i = \sum_{j\neq i}(1+\kappa_{ij}^{(N)})\lambda_{i,j}^{(N)}(\overline{\overline{x}}_j - \overline{\overline{x}}_i), \tag{3.11}$$

for fixed i and $i, j \in \{1, 2, \dots, N\}$, where $\kappa_{ij}^{(N)}$ is the entry of $\boldsymbol{\kappa}^{(N)}(t)$ given in (3.9) and $\lambda_{i,j}^{(N)}$ is the entry of $\boldsymbol{\Lambda}^{(N)}(t)$ given in (2.16). Observing the expression of (3.11), one can set the parameters $\boldsymbol{\kappa}^{(N)}(t)$ as

$$\boldsymbol{\kappa}^{(N)}(t) = \begin{pmatrix} \widetilde{\boldsymbol{\kappa}}_{11}^{(N)}(t) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \widetilde{\boldsymbol{\kappa}}_{22}^{(N)}(t) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \widetilde{\boldsymbol{\kappa}}_{n_1 n_1}^{(N)}(t) \end{pmatrix}. \tag{3.12}$$

Remark 8. From the form of (3.12) we can see that the martingale condition is independent of the first layer CTMC (i.e. different regimes). Similar conclusions in the regime-switching diffusion case were obtained in [40].

3.3. Minimal relative entropy martingale measure

The relative entropy (also called Kullback–Leibler divergence) is a measure of how one probability distribution is different from another probability distribution. Similar to [41], in this paper, we consider the minimal entropy martingale measure for CTMC. Specifically, the relative entropy $H(\mathbb{Q}^{(N)} \parallel \mathbb{P})$ of measure $\mathbb{Q}^{(N)}$ with respect to $\mathbb{P}$ is defined as

$$H(\mathbb{Q}^{(N)} \parallel \mathbb{P}) = \int \log\left(\frac{d\mathbb{Q}^{(N)}}{d\mathbb{P}}\right) d\mathbb{Q}^{(N)} = \int \frac{d\mathbb{Q}^{(N)}}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}^{(N)}}{d\mathbb{P}}\right) d\mathbb{P}.$$

When there does not exist a unique equivalent martingale measure, a natural strategy is to find the martingale measure with the minimal relative entropy. Considering that $X^{(N)}$ is bounded, we have the uniqueness result below.

Theorem 9 (Theorem 2.1, [42]). *If $H(\mathbb{Q}^{(N)} \parallel \mathbb{P}) < \infty$, then there exists a unique minimal entropy martingale measure.*

We firstly consider the one-dimensional case. For an equivalent martingale measure $\mathbb{Q}^{(N)}$ defined by (3.2), the relative entropy in terms of the $\mathbb{Q}^{(N)}$-martingale is given by

$$\begin{aligned} &H(\mathbb{Q}^{(N)} \parallel \mathbb{P}) \\ &= \mathbb{E}^{\mathbb{Q}^{(N)}}\left[\log\left(\frac{d\mathbb{Q}^{(N)}}{d\mathbb{P}}\right)\Big|_{\mathcal{F}_T}\right] \\ &= \mathbb{E}^{\mathbb{Q}^{(N)}}\Big[\sum_{i,j=1}^{N}\Big(-\int_0^T \kappa_{ij}^{(N)}(t)\lambda_{i,j}^{(m,N)}(t)H_t^{i(N)}dt \\ &\qquad + \sum_{0<u\le T}\log\left(1+\kappa_{ij}^{(N)}(u)(H_u^{ij(N)} - H_{u-}^{ij(N)})\right)\Big)\Big] \\ &= \mathbb{E}^{\mathbb{Q}^{(N)}}\Big[\sum_{i,j=1}^{N}\sum_{l=1}^{m}\Big[-\kappa_{ij}^{(l,N)}\lambda_{i,j}^{(l,N)}\int_0^T \mathbf{1}_{[T_{l-1}^{(m)},T_l^{(m)})}(t)H_t^{i(N)}dt \\ &\qquad + \sum_{0<u\le T}\log(1+\kappa_{ij}^{(l,N)}(H_u^{ij(N)} - H_{u-}^{ij(N)}))\mathbf{1}_{[T_{l-1}^{(m)},T_l^{(m)})}(u)\Big]\Big], \end{aligned}$$

where in the second equality we can move $\sum_{i,j=1}^{N}$ to the outside because at most one $H_u^{ij(N)} - H_{u-}^{ij(N)}$ is equal to 1 while the rest is equal to 0. The minimization problem of finding the equivalent martingale measure with the minimum relative entropy, is equivalent to minimizing

$$\sum_j\left[-\kappa_{ij}^{(l,N)}\lambda_{i,j}^{(l,N)} + \log(1+\kappa_{ij}^{(l,N)})\right],$$

for fixed t and i, subject to (3.7).

We apply the method of Lagrange multipliers. Let γ be the Lagrange multiplier associated with the constraint (3.7) and L be the associated Lagrangian

$$L\Big(\gamma,\kappa_{ij}^{(l,N)}\Big)=\sum_{j}\Big[-\kappa_{ij}^{(l,N)}\lambda_{i,j}^{(l,N)}+\log\Big(1+\kappa_{ij}^{(l,N)}\Big)\Big]+\gamma\Bigg(\sum_{j\neq i}\Big(1+\kappa_{ij}^{(l,N)}\Big)\lambda_{i,j}^{(l,N)}\big(x_j-x_i\big)-rx_i\Bigg)$$
$$=\sum_{j\neq i}\Big[-\kappa_{ij}^{(l,N)}\lambda_{i,j}^{(l,N)}+\log\Big(1+\kappa_{ij}^{(l,N)}\Big)\Big]+\gamma\Bigg(\sum_{j\neq i}\Big(1+\kappa_{ij}^{(l,N)}\Big)\lambda_{i,j}^{(l,N)}\big(x_j-x_i\big)-rx_i\Bigg),$$

where the last equality follows from $\kappa_{ii}^{(l)}(t)=0$. Note that $L(\gamma,\kappa_{ij}^{(l,N)})$ is a convex function in $\kappa_{ij}^{(l,N)}\in(-1,\infty)$, for fixed $i,j=1,\dots,N$ and $j\neq i$ we let

$$\begin{cases}\frac{\partial}{\partial\kappa_{ij}^{(l,N)}}L(\gamma,\kappa_{ij}^{(l,N)})=-\lambda_{i,j}^{(l,N)}+\frac{1}{1+\kappa_{ij}^{(l,N)}}+\gamma\lambda_{i,j}^{(l,N)}(x_j-x_i)=0,\\ \frac{\partial}{\partial\gamma}L(\gamma,\kappa_{ij}^{(l,N)})=\sum_{j\neq i}(1+\kappa_{ij}^{(l,N)})\lambda_{i,j}^{(l,N)}(x_j-x_i)-rx_i=0,\end{cases}\tag{3.13}$$

by which the equivalent martingale measure with the minimum relative entropy can be determined.

An analogous result for the two-dimensional case can be obtained similarly, for fixed $i,j=1,\dots,N$ and $j\neq i$,

$$\begin{cases}\frac{\partial}{\partial\kappa_{ij}^{(N)}}L(\gamma,\kappa_{ij}^{(N)})=-\lambda_{i,j}^{(N)}+\frac{1}{1+\kappa_{ij}^{(N)}}+\gamma\lambda_{i,j}^{(N)}(\bar{\bar{x}}_j-\bar{\bar{x}}_i)=0,\\ \frac{\partial}{\partial\gamma}L(\gamma,\kappa_{ij}^{(N)})=\sum_{j\neq i}(1+\kappa_{ij}^{(N)})\lambda_{i,j}^{(N)}(\bar{\bar{x}}_j-\bar{\bar{x}}_i)-r\bar{\bar{x}}_i=0,\end{cases}\tag{3.14}$$

where $\kappa_{ij}^{(N)}$ is the entry of $\boldsymbol{\kappa}^{(N)}(t)$ given in (3.12) and $\lambda_{i,j}^{(N)}$ is the entry of $\boldsymbol{\Lambda}^{(N)}(t)$ given in (2.16).

4. Convergence of equivalent martingale measures

In this section, we investigate the limiting behavior of the equivalent martingale measure $\mathbb{Q}^{(N)}$ as N goes to infinity. Note that by Proposition 6 we know that $\eta_t^{(N)}$ is a strictly positive $\mathbb{F}$-martingale under $\mathbb{P}$ such that $\mathbb{E}[\eta_T^{(N)}]=1$, and then by Theorem 11.4 (Martingale Convergence Theorem) of [43] we know that $\eta_t^{(N)}$ converges almost surely under $\mathbb{P}$. Furthermore, by Eqs. (3.9) and (3.10) on page 29 of [44], and from the construction of $\mathbb{Q}^{(N)}$ in Eq. (3.3), we know that the convergence of $\eta_t^{(N)}$ implies the convergence of $\mathbb{Q}^{(N)}$, as N goes to infinity. For a fixed $T\in(0,\infty)$, we consider $(\eta_t^{(N)})_{0\le t\le T}$ as a stochastic process on a Skorohod topology space $D_{\mathcal{S}}[0,T]$ with complete and separable metric space $(\mathcal{S},d)$. In the following, we investigate the limiting behavior of $\eta_t^{(N)}$ as N goes to infinity.

4.1. Preliminary analysis

We firstly show that $\left\{\eta_t^{(N)}\right\}_{N\in\mathbb{N}}$ is relatively compact by means of the technique in Chapter 3 of [44] and Chapter 3 of [35]. The following two propositions conclude that the sequence of $\eta_t^{(N)}$ is relatively compact.

Proposition 10. *For every $T>0$, it holds that*

$$\lim_{R\to\infty}\sup_{N\in\mathbb{N}}\mathbb{P}\Big(\sup_{t\in[0,T]}|\eta_t^{(N)}|\ge R\Big)=0.$$

Proof. By submartingale inequality, and by the fact that $\eta_t^{(N)}$ is strictly positive and $\mathbb{E}[\eta_T^{(N)}]=1$, we have that for every $N\in\mathbb{N}$,

$$\mathbb{P}\Big(\sup_{t\in[0,T]}|\eta_t^{(N)}|\ge R\Big)\le\frac{\mathbb{E}\big[\eta_T^{(N)}\big]}{R}=\frac{1}{R}\to0,\quad\text{as }R\to\infty.$$

□

Proposition 11. *For $q(x,y):=|x-y|\wedge1$, there exist constants $C>0$ and $\theta>1$, such that for all N, one has*

$$\mathbb{E}^{\mathbb{P}}[q^2(\eta_{t+u}^{(N)},\eta_t^{(N)})|\mathcal{F}_t^{\eta_t^{(N)}}]\le Cu^{\theta},\qquad0\le t\le T,\quad0\le u\le T-t.$$

Proof. See Appendix B. □

4.2. Convergence for one-dimensional models

Next, we investigate the limiting behavior of the sequence of R-N derivatives $\{\eta_t^{(N)}\}$ as N goes to infinity. We adopt the techniques in [45,46] and [47] by firstly defining an operator $\mathcal{A}$ acting on $f \in \mathcal{S}$ as

$$\mathcal{A}_t f(x) = \frac{1}{2} f''(x) x^2 \int_0^\infty \frac{(\mu(t,y) - ry)^2}{\sigma^2(t,y)} \mathbf{1}_{\{S_t=y\}} dy, \quad \text{for } x \in \mathbb{R}. \tag{4.1}$$

Proposition 12. *For $\eta_t^{(N)}$ given in Eq. (3.1), it holds that*

$$\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}} \left[\left(f(\eta_{t_{k+1}}^{(N)}) - f(\eta_{t_k}^{(N)}) - \int_{t_k}^{t_{k+1}} \mathcal{A}_t f(\eta_s^{(N)}) ds \right) \prod_{l=1}^{k} h_l(\eta_{t_l}^{(N)}) \right] = 0, \tag{4.2}$$

where $0 \le t_1 < \cdots < t_{k+1} < \infty$ for $k \in \mathbb{N}$ and h_l is a bounded measurable functions on $\mathcal{S}$ for $l = 1, 2, \ldots, k$.

Proof. See Appendix C. □

4.3. Convergence for two-dimensional model

In this subsection, we investigate the weak limit of R-N densities for the two-dimensional model.

By the construction of approximating regime-switching Markov chain $\widetilde{X}^{(n_2)}$ in Eqs. (2.14) and (2.15), the martingale condition in Eq. (3.11), and the elements of $\overline{\overline{\mathbb{X}}}^{(N)}$ are n_1 times repetition of the elements of $\mathbb{X}^{(n_2)}$, we have that for every $i \in \{1, 2, \ldots, n_2\}$ and $\omega_t^{(n_1)} = w_k$,

$$\begin{cases} \sum_{j\neq i} \lambda_{ij}^k(t)(\widetilde{x}_j - \widetilde{x}_i) = \sum_{l=1}^{m} \widetilde{\theta}(T_{l-1}^{(m)}, \widetilde{x}_i, w_k) \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t), \\ \sum_{j\neq i} \lambda_{ij}^k(t)(\widetilde{x}_j - \widetilde{x}_i)^2 = (1-\rho^2)\sigma^2(w_k), \\ \sum_{j\neq i} (1+\kappa_{ij}^k(t)) \lambda_{ij}^{(N)}(t)(\widetilde{x}_j - \widetilde{x}_i) = r\widetilde{x}_i, \end{cases} \tag{4.3}$$

where $\kappa_{ij}^k(t)$ is the element of matrix $\widetilde{\boldsymbol{\kappa}}_{kk}^{(N)}$ defined in (3.12), i.e., $\kappa_{ij}^k(t) = (\widetilde{\boldsymbol{\kappa}}_{kk}^{(N)})_{ij}$.

Denote

$$\tilde{\boldsymbol{\theta}}(\cdot,\cdot) := \left(\tilde{\theta}(\cdot,\cdot,w_1), \tilde{\theta}(\cdot,\cdot,w_2), \ldots, \tilde{\theta}(\cdot,\cdot,w_{n_1})\right)$$

and

$$\boldsymbol{\sigma} := (\sigma(w_1), \sigma(w_2), \ldots, \sigma(w_{n_1})).$$

Hence one has the following equation in terms of $\overline{\overline{\mathbb{X}}}^{(N)}$:

$$\frac{\left(\sum_{j\neq i} \lambda_{i,j}^{(N)}(t) \kappa_{ij}^{(N)}(t)(\bar{\bar{x}}_j - \bar{\bar{x}}_i)\right)^2}{\sum_{j\neq i} \lambda_{i,j}^{(N)}(t)(\bar{\bar{x}}_j - \bar{\bar{x}}_i)^2} = \frac{\left(\sum_{l=1}^{m} \tilde{\boldsymbol{\theta}}(T_{l-1}^{(m)}, \bar{\bar{x}}_i) \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t) - r\bar{\bar{x}}_i\right)^2}{(1-\rho^2)\boldsymbol{\sigma}^2}. \tag{4.4}$$

We adopt the techniques in [40] by firstly defining an operator $\mathcal{B}$ acting on $f \in \mathcal{S}$ as

$$\mathcal{B}_t f(x) = \frac{1}{2} f''(x) x^2 \int_0^\infty \frac{\left(\tilde{\boldsymbol{\theta}}(t,y) - ry\right)^2}{(1-\rho^2)\boldsymbol{\sigma}^2} \mathbf{1}_{\{\widetilde{X}_t = y\}} dy, \quad \text{for } x \in \mathbb{R}, \tag{4.5}$$

where $\widetilde{X}$ is the regime-switching diffusion evolving according to (2.12).

Proposition 13. *For $\eta_t^{(N)}$ given in Eq. (3.1), it holds that*

$$\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}} \left[\left(f(\eta_{t_{k+1}}^{(N)}) - f(\eta_{t_k}^{(N)}) - \int_{t_k}^{t_{k+1}} \mathcal{B}_s f(\eta_s^{(N)}) ds \right) \prod_{l=1}^{k} h_l(\eta_{t_l}^{(N)}) \right] = 0, \tag{4.6}$$

where $0 \le t_1 < \cdots < t_{k+1} < \infty$ for $k \in \mathbb{N}$ and h_l is a bounded measurable functions on $\mathcal{S}$ for $l = 1, 2, \ldots, k$.

Proof. See Appendix D. □

5. Applications to option pricing

Recall that our approximation approach is under physical measure $\mathbb{P}$. For one-dimensional diffusion case, under $\mathbb{P}$ we have constructed the approximating CTMC $X^{(N)}$ in Section 2.1 with the time-dependent generator matrix $\Lambda_t^{(N)} = \left(\lambda_{i,j}^{(N)}(t)\right)_{i,j\in\{1,\dots,N\}}$, where

$$\lambda_{i,j}^{(N)}(t) = \sum_{l=1}^{m} \lambda_{i,j}^{(l,N)} \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t).$$

Under the equivalent martingale measure $\mathbb{Q}^{(N)}$, by Proposition 7, $X_t^{(N)}$ is a CTMC with generator matrix

$$\tilde{\Lambda}_t^{(N)} = \sum_{l=1}^{m} \tilde{\Lambda}^{(l,N)} \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t),$$

where $\tilde{\Lambda}^{(l,N)} = \left(\tilde{\lambda}_{ij}^{(l,N)}\right)_{i,j\in\{1,\dots,N\}}$ satisfies

$$\tilde{\lambda}_{ii}^{(l,N)} = -\sum_{j\neq i} \tilde{\lambda}_{ij}^{(l,N)}, \qquad \tilde{\lambda}_{ij}^{(l,N)} = (1+\kappa_{ij}^{(l,N)})\lambda_{ij}^{(l,N)}, \quad i \neq j.$$

For the approximating CTMC $\overline{\overline{X}}^{(N)}$ defined in Proposition 2 for the two-dimensional diffusion case, whose generator matrix is $\boldsymbol{\Lambda}^{(N)}(t)$ under $\mathbb{P}$, its generator matrix $\widetilde{\boldsymbol{\Lambda}}^{(N)}(t)$ under equivalent martingale measure $\mathbb{Q}^{(N)}$ can be generated analogously as the one-dimensional case

$$\widetilde{\boldsymbol{\Lambda}}^{(N)}(t) =: \sum_{l=1}^{m} \widetilde{\boldsymbol{\Lambda}}^{(l,N)} \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t).$$

In this section, we apply the approximating framework and obtain approximating formulae for pricing options in European and barrier types.

5.1. European options

- **One-dimensional models:** For any payoff function $f: E \to \mathbb{R}$, the value of the European option with maturity T for the underlying S can be approximated as

$$\mathbb{E}_x\left[e^{-rT} f\left(X_T^{(N)}\right)\right] = \exp(-rT)\left(\exp\left(\Delta T_1 \widetilde{\Lambda}^{(1,N)}\right)\cdots\exp\left(\Delta T_m \widetilde{\Lambda}^{(m,N)}\right) f\right)(x).$$

- **Two-dimensional models:** Assume $\omega_0^{(n_1)} = w_k$. By Eq. (3.8), for any payoff function $f: E \to \mathbb{R}$, the value of the European option with maturity T shall approximately be

$$\mathbb{E}_{\bar{\bar{x}}_n}\left[e^{-rT} \tilde{f}\left(\overline{\overline{X}}_T^{(N)}\right)\right] = \exp(-rT)\left(\exp\left(\Delta T_1 \widetilde{\boldsymbol{\Lambda}}^{(1,N)}\right)\cdots\exp\left(\Delta T_m \widetilde{\boldsymbol{\Lambda}}^{(m,N)}\right)\tilde{f}\right)\left(\bar{\bar{x}}_n\right),$$

where $\widetilde{f}(x) = f \circ g^{-1}(x + \rho h(w_k))$.

5.2. Barrier options

Assume a knock-out barrier option with maturity T have the continuation region $\hat{E}$ and the knock-out region $\widehat{E}^c = E \backslash \widehat{E}$, where E is the state space of the approximating CTMC (i.e., $E = \mathbb{G}^{(N)}$ for the one-dimensional case and $E = \overline{\overline{\mathbb{X}}}^{(N)}$ for the two-dimensional case). Denote τ as the first passage time that the approximating CTMC belongs to the knock-out region.

- **One-dimensional models:** Define

$$\tilde{\Lambda}_r^{(l)}(i,j) = \begin{cases} \tilde{\Lambda}^{(l,N)}(x_i, x_j) - r, & x_i \in \widehat{E},\ i = j, \\ \tilde{\Lambda}^{(l,N)}(x_i, x_j), & x_i \in \widehat{E},\ x_j \in E,\ i \neq j, \\ 0, & x_i \in \hat{E}^c,\ x_j \in E, \end{cases}$$

where $l = 1, \dots, m$ and r is the risk-free rate. Then the knock-out barrier option prices can be approximated as:

Proposition 14. *([37]) For any payoff function $\phi: E \to \mathbb{R}$, it holds that*

$$\mathbb{E}_x\left[e^{-r(T\wedge\tau)}\phi(X_{T\wedge\tau}^{(N)})\right] = \left(\exp\left(\Delta T_1 \tilde{\Lambda}_r^{(1)}\right)\cdots\exp\left(\Delta T_m \tilde{\Lambda}_r^{(m)}\right)\phi\right)(x).$$

Table 1
Numerical results of option pricing on diffusion processes with time-dependent diffusion function. The results are obtained by implementing the number of time partition $m = 50$ and states $N = 500, 1000$. Common parameters are $\mu = 0.1, r = 0.035, \alpha = -30, T = 0.25, S_0 = 100$ and $\beta_t = 0.2$ for each test.

		CTMC		Closed
T	K	N=500	N=1000	
0.25	95	7.9667	7.9533	7.9629
	100	5.1224	5.1102	5.1070
	105	3.0487	3.0445	3.0433
0.5	95	9.8973	9.8889	9.8856
	100	7.1184	7.1167	7.1125
	105	4.9282	4.9278	4.9266
1	95	12.5508	12.5474	12.5455
	100	9.7598	9.7569	9.7563
	105	7.4206	7.4165	7.4153

- **Two-dimensional models:** Assume $\omega_0^{(n_1)} = w_k$. Similarly, define

$$\widetilde{\mathbf{\Lambda}}_r^{(l)}(i,j) = \begin{cases} \widetilde{\mathbf{\Lambda}}^{(l,N)}(\bar{\bar{x}}_i, \bar{\bar{x}}_j) - r, & \bar{\bar{x}}_i \in \widehat{E},\ i = j, \\ \widetilde{\mathbf{\Lambda}}^{(l,N)}(\bar{\bar{x}}_i, \bar{\bar{x}}_j), & \bar{\bar{x}}_i \in \widehat{E},\ \bar{\bar{x}}_j \in E,\ i \neq j, \\ 0, & \bar{\bar{x}}_i \in \hat{E}^c,\ \bar{\bar{x}}_j \in E, \end{cases}$$

where $l = 1, \ldots, m$ and r is the risk-free rate. By Eq. (3.8), for any payoff function $\phi : E \to \mathbb{R}$, the knock-out barrier option prices can be approximated by

$$\mathbb{E}_{\bar{\bar{x}}_n}\left[e^{-r(T\wedge\tau)}\widetilde{\phi}(\bar{\bar{X}}_{T\wedge\tau})\right] = \left(\exp\left(\Delta T_1 \widetilde{\mathbf{\Lambda}}_r^{(1)}\right)\cdots\exp\left(\Delta T_m \widetilde{\mathbf{\Lambda}}_r^{(m)}\right)\widetilde{\phi}\right)(\bar{\bar{x}}_n),$$

where $\widetilde{\phi}(x) = \phi \circ g^{-1}(x + \rho h(w_k))$.

6. Numerical examples

6.1. Diffusion process with time-dependent diffusion function

Brigo and Mercurio [48] and [49] proposed a shifted geometric Brownian motion process with deterministic coefficients that allows for skews in the implied volatility, and then this model can be used to fit market volatility structures that are skewed. Here, the term "skew" describes the shape where low-strikes implied volatilities are higher than high-strikes implied volatilities. [48] assumed that the asset process S has a constant drift coefficient and a time-dependent diffusion coefficient $\nu(t, S_t)$. To determine ν, we start from a process X, which is a simple extension of the Black–Scholes model, evolving according to

$$dX_t = \mu X_t dt + \beta_t X_t dB_t$$

where μ is a constant and β_t is a time-dependent (deterministic) volatility function. Consider that the asset process S is given by an affine transformation of X, i.e., $S_t = X_t + \alpha e^{\mu t}$, where α is a constant. Then S is driven by the following time-dependent model:

$$dS_t = \mu S_t dt + \beta_t (S_t - \alpha e^{\mu t}) dB_t. \tag{6.1}$$

The explicit form of European option prices is given by

$$C_t = \left(S_t - \alpha e^{rt}\right)e^{(r-r)(T-t)}\Phi\left(\frac{\ln\frac{S_t-\alpha e^{rt}}{K-\alpha e^{rT}} + \int_t^T\left(r+\frac{1}{2}\beta_u^2\right)du}{\sqrt{\int_t^T \beta_u^2 du}}\right) - \left(K - \alpha e^{rT}\right)e^{-r(T-t)}\Phi\left(\frac{ln\frac{S_t-\alpha e^{rt}}{K-\alpha e^{rT}} + \int_t^T\left(r-\frac{1}{2}\beta_u^2\right)du}{\sqrt{\int_t^T \beta_u^2 du}}\right), \tag{6.2}$$

where $0 \le t < T$, r denotes the risk-free rate, and Φ is the standard normal distribution function.

To illustrate the accuracy of our method, Table 1 reports the results of European call option prices under the model (6.1) based on our CTMC method and the closed-form (6.2), from which we can see that our results are very close to the closed-form in all cases.

6.2. Diffusion process with time-dependent drift function

Ball and Torous [50] first put forward the dynamics of a default-free pure discounted bond price following a Brownian bridge process, and investigated the equilibrium valuation of a call option. They realized that the Brownian bridge process

Table 2
Numerical results of option pricing on diffusion processes with time-dependent drift function. European call options were priced with strike prices 100, 105, and 110, and the double barrier knock-out call options were priced with strike prices 95, 100, and 105, under the geometric Brownian bridge model. Other parameters we used are maturity $T = 0.8$, $b = 5$, $S_0 = 100$, and the number of time partition $m = 50$. The lower and upper barriers are $L = 90$ and $U = 130$, respectively. N stands for the number of the states of the approximating CTMC.

	European			Barrier		
N	K=100	K=105	K=110	K=95	K=100	K=105
500	9.7145	5.0979	0.5256	14.3288	9.7132	5.0977
1000	9.7128	5.0955	0.5253	14.3253	9.7097	5.0941
1500	9.7098	5.0948	0.5250	14.3246	9.7090	5.0934
2000	9.7090	5.0932	0.5249	14.3244	9.7088	5.0932
2500	9.7086	5.0930	0.5248	14.3243	9.7087	5.0931

deserves many further financial applications. Next, we provide an approximate valuation for the Brownian bridge process. Consider a security S whose logarithm $\log S$ is a Brownian bridge model with diffusion coefficient σ under physical measure $\mathbb{P}$, i.e., S is a geometric Brownian bridge process and satisfies

$$\begin{cases} dS_t = (\frac{b-\log S_t}{T-t} + \frac{1}{2}\sigma^2)S_t dt + \sigma S_t dW_t, \qquad 0 \le t < T, \\ S_0 = e^a. \end{cases} \tag{6.3}$$

Table 2 reports the valuation results of European call options and double barrier knock-out call options under model (6.3), using the proposed CTMC approximation methodology. We can see fast convergences to stable limiting prices under scenarios of three strike prices.

7. Conclusion

In this paper, we have proposed a time-inhomogeneous CTMC approximation, equivalent martingale measure changing, and valuation methodology, for one-dimensional and two-dimensional time-inhomogeneous diffusion processes in the physical probability measure. Considering that there does not exist a unique equivalent martingale measure, the “chosen” equivalent martingale measure is the one having the minimal relative entropy with respect to the physical measure, whose convergence as the cardinality of the discrete state space of the approximating CTMC goes to infinity, is established by transforming the problem to determining whether the sequence of the Radon–Nikodym derivatives is convergent. Numerical examples have confirmed the accuracy of the proposed methodology. While this paper shed the light on option pricing with time-inhomogeneous two-dimensional correlated stochastic processes in the physical probability measure, by means of the time-inhomogeneous CTMC approximation technique, studying the effects of ambiguous volatility [51] and ambiguous correlation [52] could be promising topics for further research.

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Appendix A. Proof of Proposition 3

For a fixed time T, define two processes $Z = (Z_t)_{t\ge 0}$ and $Y = (Y_t)_{t\ge 0}$ by

$$Z_t := (Z_0 + t) \wedge T, \qquad Y_t := X_{Z_t}, \tag{A.1}$$

for $t > 0$ and $Z_0 \in [0, T]$. For any $s, t > 0$ and bounded Borel function $f : [0,T] \times \mathbb{X} \times \mathbb{W} \to \mathbb{R}$, it holds that

$$\begin{aligned}(\mathbf{P}_t f)(s,x,w) :=& \mathbb{E}[f(Z_t, Y_t, \mathcal{W}_t)|Z_0 = s, Y_0 = x, \mathcal{W}_0 = w] \\ =& \mathbb{E}[f\big((s+t)\wedge T, X_{(s+t)\wedge T}, \mathcal{W}_t\big)|X_s = x, \mathcal{W}_0 = w].\end{aligned}$$

Then $\mathbf{P}_t$ defines a strongly continuous contractive semigroup on $\mathcal{C}_0(E)$ (see [35] for details), and the corresponding infinitesimal generator $\mathcal{L}'$ is defined by

$$\mathcal{L}' f(s,x,w) = \lim_{t\downarrow 0} \frac{\mathbf{P}_t f(s,x,w) - f(s,x,w)}{t}, \tag{A.2}$$

We prove (a) by firstly approximating the time-space Feller Markov process $(Z, Y, \mathcal{W})$ by a time-homogeneous Markov chain $(\widetilde{Z}, \widetilde{X}^{(n_2)}, \omega^{(n_1)})$ on $\mathbb{T}^{(N)} \times \mathbb{G}^{(n_1,n_2)}$ with $\mathbb{T}^{(N)} = \{0\} \cup \delta_m \mathbb{N}$ and $\delta_m = T/(mN)$ for $m \in \mathbb{N}$. Assume that $\mathbf{\Lambda}^{(i,N)}$, $i = 1, \dots, N$ are generator matrices approximating the generator $\mathcal{L}_{iT/N}$. Then the generator of Markov chain $(\widetilde{Z}, \widetilde{X}^{(n_2)}, \omega^{(n_1)})$ is given by

$$\mathbf{\Lambda}^{(N,m)} f(t, x, \omega) = \frac{f(t+\delta_m, x, w) - f(t, x, w)}{\delta_m} + \sum_{i=1}^{N} (\mathbf{\Lambda}^{(i,N)} f_t)(x, w) \mathbf{1}_{\{(i-1)m\delta_m \le t < im\delta_m\}},$$

for $(t, x, w) \in \mathbb{T}^{(N)} \times \mathbb{G}^{(n_1,n_2)}$. From the analysis in [37], we can describe $\widetilde{Z}$ as δ_m times a Poisson process with rate $\lambda = 1/\delta_m$. That is, for any $t > 0$, $\widetilde{Z}_t$ follows a Poisson distribution with mean t and variance $\delta_m t = tT/(mN)$. Then, $\widetilde{Z}_t$ tends to t for every fixed $t > 0$. When $\widetilde{Z}_t$ taking value in $[(i-1)\delta_m, i\delta_m)$ and ω_t is independent to $\widetilde{Z}_t$, $(\widetilde{X}^{(n_2)}, \omega)$ is a Markov chain with state space $\mathbb{G}^{(n_1,n_2)}$ and generator matrix $\Lambda^{(i,N)}$.

For any natural number sequence $(m(N))_{N\in\mathbb{N}}$, consider the sequence of Markov chains $(\widetilde{Z}, \widetilde{X}^{(n_2)}, \omega^{(n_1)})$ with state space $\mathbb{T}^{(N)} \times \mathbb{G}^{(n_1,n_2)}$ and generator matrices $\mathbf{\Lambda}^{(N,m)}$. Given the condition that $\epsilon^{(n_1,n_2)}(f_t) \mapsto 0$ as n_1, $n_2 \to \infty$ for any function $f_t \in \mathcal{D}^*(\mathcal{L}_t)$, the construction of $\mathbf{\Lambda}^{(N,m)}$ implies that, for all functions f in a core of $\mathcal{L}'$ which is the generator of $(Z, Y, \mathcal{W})$ given in Eq. (A.2), as m, n_1, $n_2 \to \infty$ one has

$$\epsilon^{(n_1,n_2,m)}(f) = \sup_{t\in\mathbb{T}^{(N)}, (x,w)\in\mathbb{G}^{(n_1,n_2)}} |\mathbf{\Lambda}^{(N,m)} f_{n_1,n_2}(t, x, w) - \mathcal{L}' f(t, x, w)| \to 0, \tag{A.3}$$

where $f_{n_1,n_2} = f|_{[0,T]\times\mathbb{G}^{(n_1,n_2)}}$.

Recall that

$$(\mathbf{P}_t f)(s, x, w) = \mathbb{E}[f(Z_t, Y_t, \mathcal{W}_t)|Z_0 = s, Y_0 = x, \mathcal{W}_0 = w]$$

and define

$$(\mathbf{P}_t^{(m,n_1,n_2)} f)(s, x, w) := \mathbb{E}[f(\widetilde{Z}_t, \widetilde{X}_t^{(n_2)}, \omega_t)|\widetilde{Z}_0 = s, \widetilde{X}_0^{(n_2)} = x, \omega_0^{(n_1)} = w]$$

for $t \in [0, T]$. By Theorem 1.6.1 in [35], Eq. (A.3) implies that

$$(\mathbf{P}_t^{(m,n_1,n_2)} f)(s, x, w) \to (\mathbf{P}_t f)(s, x, w).$$

Hence, for any initial value (s, x, w) and applying Theorem 4.2.11 in [35], we have that

$$(\widetilde{X}_t^{(n_2)}, \omega_t^{(n_1)}) \Longrightarrow (X_t, \mathcal{W}_t).$$

Note that h and g^{-1} defined in Eq. (2.8) are continuous functions, and by continuous mapping theorem, (b) can be concluded from (a).

Appendix B. Proof of Proposition 11

Let

$$\Delta M_t^{ij(N)} := M_t^{ij(N)} - M_{t-}^{ij(N)},$$

$$\Delta H_t^{ij(N)} := H_t^{ij(N)} - H_{t-}^{ij(N)},$$

and then by Eq. (5) we know that $\Delta M_t^{ij(N)} = \Delta H_t^{ij(N)}$. By Itô's formula for semimartingales (Proposition 8.19, [53]) we obtain

$$\begin{aligned}
(M_t^{ij(N)})^2
&= 2\int_0^t M_{s-}^{ij(N)} dM_s^{ij(N)} + \langle M^{ij(N)}, M^{ij(N)} \rangle_t^c \\
&\quad + \sum_{0<s\le t} [(M_{s-}^{ij(N)} + \Delta M_s^{ij(N)})^2 - (M_{s-}^{ij(N)})^2 - 2M_{s-}^{ij(N)} \cdot \Delta M_s^{ij(N)}] \\
&= 2\int_0^t M_{s-}^{ij(N)} dM_s^{ij(N)} + \sum_{0<s\le t} (\Delta M_s^{ij(N)})^2 \\
&= 2\int_0^t M_{s-}^{ij(N)} dM_s^{ij(N)} + \sum_{0<s\le t} (\Delta H_s^{ij(N)})^2 \\
&= 2\int_0^t M_{s-}^{ij(N)} dM_s^{ij(N)} + \sum_{0<s\le t} \Delta H_s^{ij(N)} \\
&= 2\int_0^t M_{s-}^{ij(N)} dM_s^{ij(N)} + \int_0^t \left(dH_s^{ij(N)} - \lambda_{i,j}^{(N)}(s) H_s^{i(N)} ds\right) \\
&\quad + \int_0^t \lambda_{i,j}^{(N)}(s) H_s^{i(N)} ds \\
&= \widetilde{M}_t + \int_0^t \lambda_{i,j}^{(N)}(s) H_s^{i(N)} ds
\end{aligned}$$

where

$$\widetilde{M}_t := 2\int_0^t M_{s-}^{ij(N)} dM_s^{ij(N)} + M_t^{ij(N)}$$

is a martingale. Then

$$\langle M^{ij(N)}, M^{ij(N)}\rangle_t = \int_0^t \lambda_{i,j}^{(N)}(s) H_s^{i(N)} ds.$$

By Eq. (3.1) and the independence of the family of martingales $\{(M_t^{ij(N)})_{t\geq 0}\}$ for different combinations of $i,j \in \{1,\dots,N\}$ and $i \neq j$ (see Lemma 5),

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}[\eta_{t+u}^{(N)}]^2 &= \mathbb{E}_t^{\mathbb{P}}\left[1 + \int_t^{t+u} \eta_{s-}^{(N)} \sum_{i,j} \kappa_{ij}^{(N)}(s) dM_s^{ij(N)}\right]^2 \\
&= 1 + 2\mathbb{E}_t^{\mathbb{P}}\left[\int_t^{t+u} \eta_{s-}^{(N)} \sum_{i,j} \kappa_{ij}^{(N)}(s) dM_s^{ij(N)}\right] + \sum_{i,j}\int_t^{t+u} \mathbb{E}_t^{\mathbb{P}}\left[(\eta_{s-}^{(N)})^2 (\kappa_{ij}^{(N)}(s))^2 \lambda_{i,j}^{(N)}(s) H_s^{i(N)}\right] ds \\
&= 1 + \sum_{i,j}\int_t^{t+u} \mathbb{E}_t^{\mathbb{P}}\left[(\eta_{s-}^{(N)})^2 (\kappa_{ij}^{(N)}(s))^2 \lambda_{i,j}^{(N)}(s) H_s^{i(N)}\right] ds.
\end{aligned}$$

For $\widehat{\kappa}_{ij}^{(N)} = \sup_{t\in[0,T]} \kappa_{ij}^{(N)}(t)$ and $\widehat{\lambda}_{i,j}^{(N)} = \sup_{t\in[0,T]} \lambda_{i,j}^{(N)}(t)$, considering that $H_t^{i(N)}$ indicates $X_t^{(N)}$ taking a specific value x_i, we have

$$\mathbb{E}_t^{\mathbb{P}}[\eta_{t+u}^{(N)}]^2 \leq 1 + \sup_i \sum_j (\widehat{\kappa}_{ij}^{(N)})^2 \widehat{\lambda}_{i,j}^{(N)} \int_t^{t+u} \mathbb{E}_t^{\mathbb{P}}[(\eta_s^{(N)})^2] ds.$$

Set $M = \sup_N \sup_i \sum_j (\widehat{\kappa}_{ij}^{(N)})^2 \widehat{\lambda}_{i,j}^{(N)}$ and by Gronwall's Lemma,

$$\mathbb{E}_t^{\mathbb{P}}[\eta_{t+u}^{(N)}]^2 \leq 1 - e^{Mt} + e^{M(t+u)}. \tag{B.1}$$

Thus,

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}\left[\eta_{t+u}^{(N)} - \eta_t^{(N)}\right]^2 &= \mathbb{E}_t^{\mathbb{P}}\left[\int_t^{t+u} \eta_{s-}^{(N)} \sum_{i,j} \kappa_{ij}^{(N)} dM_s^{ij(N)}\right]^2 \\
&= \sum_{i,j}\int_t^{t+u} \mathbb{E}_t^{\mathbb{P}}\left[(\eta_{s-}^{(N)})^2 (\kappa_{ij}^{(N)}(s))^2 \lambda_{i,j}^{(N)}(s) H_s^{i(N)}\right] ds \\
&\leq M\int_t^{t+u} \mathbb{E}_t^{\mathbb{P}}[\eta_s^{(N)}]^2 ds \\
&\leq M\int_t^{t+u} (1 - e^{Mt} + e^{Ms}) ds \\
&\leq Mu(1 - e^{Mt}) + e^{M(t+u)} - e^{Mt} \\
&\leq e^{MT}(e^{Mu} - 1).
\end{aligned}$$

It is not hard to obtain that there exist $C > 0$ and $\theta > 1$, such that

$$e^{MT}(e^{Mu} - 1) \leq Cu^{\theta},$$

which completes the proof.

Appendix C. Proof of Proposition 12

By Itô's formula, we have

$$\begin{aligned}
f(\eta_t^{(N)}) =& f(1) + \int_0^t f'(\eta_{s-}^{(N)}) d\eta_s^{(N)} + \frac{1}{2}\int_0^t f''(\eta_{s-}^{(N)}) d\langle \eta^{(N)}, \eta^{(N)}\rangle_s^c \\
&+ \sum_{0<s\leq t} \{f(\eta_{s-}^{(N)} + \Delta\eta_s^{(N)}) - f(\eta_{s-}^{(N)}) - f'(\eta_{s-}^{(N)}) \cdot \Delta\eta_s^{(N)}\} \\
=& f(1) + \int_0^t f'(\eta_{s-}^{(N)}) d\eta_s^{(N)} + \sum_{0<s\leq t} \Big\{ f\Big(\eta_{s-}^{(N)}(1 + \sum_{i,j} \kappa_{ij}^{(N)}(s) \Delta M_s^{ij(N)})\Big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - f(\eta_{s-}^{(N)}) - f'(\eta_{s-}^{(N)}) \cdot \eta_{s-}^{(N)} \sum_{i,j} \kappa_{ij}^{(N)}(s) \Delta M_s^{ij(N)} \Big\} \\
&= f(1) + \int_0^t f'(\eta_{s-}^{(N)}) d\eta_s^{(N)} + \sum_{0<s\le t} \Big\{ \sum_{i,j} \big[f(\eta_{s-}^{(N)}(1+\kappa_{ij}^{(N)}(s))) \\
&\qquad - f(\eta_{s-}^{(N)}) - f'(\eta_{s-}^{(N)}) \cdot \eta_{s-}^{(N)} \kappa_{ij}^{(N)}(s) \big] \Delta H_s^{ij(N)} \Big\} \\
&= f(1) + M_t + \sum_{i,j} \int_0^t \Big[f(\eta_{s-}^{(N)}(1+\kappa_{ij}^{(N)}(s)) - f(\eta_{s-}^{(N)}) \\
&\qquad - f'(\eta_{s-}^{(N)}) \cdot \eta_{s-}^{(N)} \kappa_{ij}^{(N)}(s) \Big] \times \lambda_{i,j}^{(N)}(s) H_s^{i(N)} ds,
\end{aligned} \tag{C.1}
$$

where

$$
\begin{aligned}
M_t := \int_0^t f'(\eta_{s-}^{(N)}) d\eta_s^{(N)} + \sum_{i,j} \int_0^t \Big(f(\eta_{s-}^{(N)}(1+\kappa_{ij}^{(N)}(s))) - f(\eta_{s-}^{(N)}) \\
- f'(\eta_{s-}^{(N)}) \cdot \eta_{s-}^{(N)} \kappa_{ij}^{(N)}(s) \Big) dM_s^{ij(N)}
\end{aligned}
$$

is a martingale under $\mathbb{P}$. Recalling the construction of approximating Markov chain in Eq. (2.2) and the martingale condition in Eq. (3.7), for every $i \in \{1, \ldots, N\}$, we have the following equations

$$
\begin{cases}
\sum_j \lambda_{ij}^{(N)}(t)(x_j - x_i) = \sum_{l=1}^{m} \mu(T_{l-1}^{(m)}, x_i) \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t), \\
\sum_j \lambda_{ij}^{(N)}(t)(x_j - x_i)^2 = \sum_{l=1}^{m} \sigma^2(T_{l-1}^{(m)}, x_i) \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t), \\
\sum_j (1+\kappa_{ij}^{(N)}(t)) \lambda_{ij}^{(N)}(t)(x_j - x_i) = r x_i,
\end{cases} \tag{C.2}
$$

then

$$
\frac{\Big(\sum_j \lambda_{i,j}^{(N)}(t) \kappa_{ij}^{(N)}(t)(x_j - x_i) \Big)^2}{\sum_j \lambda_{i,j}^{(N)}(t)(x_j - x_i)^2} = \frac{\Big(\sum_{l=1}^{m} \mu(T_{l-1}^{(m)}, x_i) \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t) - r x_i \Big)^2}{\sum_{l=1}^{m} \sigma^2(T_{l-1}^{(m)}, x_i) \mathbf{1}_{[T_{l-1}^{(m)}, T_l^{(m)})}(t)}. \tag{C.3}
$$

For the left side of Eq. (C.3), by the Cauchy–Schwarz inequality, for fixed $t \in [0, T]$, it holds that

$$
\frac{\Big(\sum_j \lambda_{i,j}^{(N)}(t) \kappa_{ij}^{(N)}(t)(x_j - x_i) \Big)^2}{\sum_j \lambda_{i,j}^{(N)}(t)(x_j - x_i)^2} \le \sum_j \lambda_{i,j}^{(N)}(t)(\kappa_{ij}^{(N)}(t))^2. \tag{C.4}
$$

Furthermore, the equality in the above equation holds by the linear relationship between $\kappa_{ij}^{(N)}(t)$ and $(x_j^{(N)} - x_i^{(N)})$, which holds for every N for the reason that $x_j^{(N)}$ and $x_i^{(N)}$ are finite grids and $\kappa_{ij}^{(N)}(t)$ is finite solved in (3.13). Hence, for fixed i, we have

$$
\frac{\Big(\sum_{j\neq i} \lambda_{i,j}^{(N)}(t) \kappa_{ij}^{(N)}(t)(x_j - x_i) \Big)^2}{\sum_{j\neq i} \lambda_{i,j}^{(N)}(t)(x_j - x_i)^2} \longrightarrow \sum_{j\neq i} \lambda_{i,j}^{(N)}(t)(\kappa_{ij}^{(N)}(t))^2, \tag{C.5}
$$

as $N \to \infty$.

Combining (4.1) and (C.1), we get that as $N \to \infty$

$$
\begin{aligned}
&\mathbb{E}\Big[f(\eta_t^{(N)}) - f(1) - \int_0^t \mathcal{A}_s f(\eta_s^{(N)}) ds \Big] \\
&= \mathbb{E}\Big[\int_0^t \Big(\sum_{i,j} \Big(f(\eta_{s-}^{(N)}(1+\kappa_{ij}^{(N)}(s))) - f(\eta_{s-}^{(N)}) - f'(\eta_{s-}^{(N)}) \cdot \eta_{s-}^{(N)} \kappa_{ij}^{(N)}(s) \Big) \\
&\qquad \times \lambda_{i,j}^{(N)}(s) H_s^{i(N)} - \frac{1}{2} f''(\eta_s^{(N)}) \big(\eta_s^{(N)}\big)^2 \int_0^\infty \Big(\frac{\mu(s,y) - ry}{\sigma(s,y)}\Big)^2 dy \Big) ds \Big]
\end{aligned}
$$

$$\simeq \mathbb{E}\Big[\int_0^t \Big(\sum_{i,j} \frac{1}{2} f''(\eta_{s-}^{(N)})(\eta_{s-}^{(N)})^2(\kappa_{ij}^{(N)}(s))^2 \lambda_{i,j}^{(N)}(s) H_s^{i(N)} - \frac{1}{2} f''(\eta_s^{(N)})\big(\eta_s^{(N)}\big)^2 \int_0^\infty \frac{(\mu(s,y)-ry)^2}{\sigma^2(t,y)} dy\Big) ds\Big]$$

$$= \mathbb{E}\Big[\frac{1}{2}\int_0^t \Big(\sum_{i=1}^N f''(\eta_s^{(N)})\big(\eta_s^{(N)}\big)^2 \sum_{l=1}^m \frac{\Big(\mu(T_{l-1}^{(m)},x_i)\mathbf{1}_{[T_{l-1}^{(m)},T_l^{(m)})}(s) - rx_i\Big)^2}{\sigma^2(T_{l-1}^{(m)},x_i)\mathbf{1}_{[T_{l-1}^{(m)},T_l^{(m)})}(s)} \times \mathbf{1}_{\{X_s=x_i\}} - \frac{1}{2} f''(\eta_s^{(N)})\big(\eta_s^{(N)}\big)^2 \int_0^\infty \frac{(\mu(s,y)-ry)^2}{\sigma^2(t,y)} \mathbf{1}_{\{S_s=y\}} dy\Big) ds\Big]$$

$$\longrightarrow 0,$$

where $a_N \simeq b_N$ means that $\lim_{N\to\infty} |a_N - b_N| = 0$. The second equality follows from Eqs. (C.3) and (C.5). The second limit holds for $X_t^{(N)} \Rightarrow S_t$, as $N \to \infty$. Considering that $\prod_{l=1}^k h_l(\eta_{t_l}^{(N)})$ is bounded and $\mathbb{F}_{t_l}$-measurable, completes the proof.

Appendix D. Proof of Proposition 13

By Eq. (C.1), we know that

$$f\big(\eta_t^{(N)}\big) = f(1) + M_t + \sum_{i,j}\int_0^t \Big[f\Big(\eta_{s-}^{(N)}\Big(1+\kappa_{ij}^{(N)}(s)\Big)\Big) - f\big(\eta_{s-}^{(N)}\big) - f'\big(\eta_{s-}^{(N)}\big)\cdot \eta_{s-}^{(N)}\kappa_{ij}^{(N)}(s)\Big] \times \lambda_{i,j}^{(N)}(s) H_s^{i(N)} ds,$$

where

$$M_t := \int_0^t f'(\eta_{s-}^{(N)}) d\eta_s^{(N)} + \sum_{i,j}\int_0^t \Big(f(\eta_{s-}^{(N)}(1+\kappa_{ij}^{(N)}(s))) - f(\eta_{s-}^{(N)}) - f'(\eta_{s-}^{(N)})\cdot \eta_{s-}^{(N)}\kappa_{ij}^{(N)}(s))\Big) dM_s^{ij(N)},$$

is a martingale under $\mathbb{P}$. Similar to the one-dimensional case,

$$\mathbb{E}\Big[f(\eta_t^{(N)}) - f(1) - \int_0^t \mathcal{B}_s f(\eta_s^{(N)}) ds\Big]$$

$$= \mathbb{E}\Big[\int_0^t \Big(\sum_{i,j}\Big(f(\eta_{s-}^{(N)}(1+\kappa_{ij}^{(N)}(s))) - f(\eta_{s-}^{(N)}) - f'(\eta_{s-}^{(N)})\cdot \eta_{s-}^{(N)}\kappa_{ij}^{(N)}(s)\Big) \times \lambda_{i,j}^{(N)}(s) H_s^{i(N)} - \frac{1}{2} f''(\eta_s^{(N)})\big(\eta_s^{(N)}\big)^2 \int_0^\infty \Big(\frac{\tilde{\boldsymbol{\theta}}(s,y)-ry}{\sqrt{(1-\rho^2)\boldsymbol{\sigma}}}\Big)^2 dy\Big) ds\Big]$$

$$\simeq \mathbb{E}\Big[\int_0^t \Big(\sum_{i,j}\frac{1}{2} f''(\eta_{s-}^{(N)})(\eta_{s-}^{(N)})^2(\kappa_{ij}^{(N)}(s))^2 \lambda_{i,j}^{(N)}(s) H_s^{i(N)} - \frac{1}{2} f''(\eta_s^{(N)})\big(\eta_s^{(N)}\big)^2 \int_0^\infty \frac{\Big(\tilde{\boldsymbol{\theta}}(s,y)-ry\Big)^2}{(1-\rho^2)\boldsymbol{\sigma}^2} dy\Big) ds\Big]$$

$$= \mathbb{E}\Big[\frac{1}{2}\int_0^t \Big(\sum_{i=1}^N f''(\eta_s^{(N)})\big(\eta_s^{(N)}\big)^2 \sum_{l=1}^m \frac{\Big(\tilde{\boldsymbol{\theta}}(T_{l-1}^{(m)},x_i)\mathbf{1}_{[T_{l-1}^{(m)},T_l^{(m)})}(s) - rx_i\Big)^2}{(1-\rho^2)\boldsymbol{\sigma}^2} \times \mathbf{1}_{\{\bar{\bar{X}}_s^{(N)}=\bar{\bar{x}}_i\}} - \frac{1}{2} f''(\eta_s^{(N)})\big(\eta_s^{(N)}\big)^2 \int_0^\infty \frac{\Big(\tilde{\boldsymbol{\theta}}(t,y)-ry\Big)^2}{(1-\rho^2)\boldsymbol{\sigma}^2} \mathbf{1}_{\{\tilde{X}_s=y\}} dy\Big) ds\Big]$$

$$\longrightarrow 0,$$

which ends this proof.

References

[1] A. Mijatović, M. Pistorius, Continuously monitored barrier options under Markov processes, Math. Financ.: Int. J. Math. Stat. Financ. Econ. 23 (1) (2013) 1–38.
[2] L. Li, G. Zhang, Error analysis of finite difference and Markov chain approximations for option pricing, Math. Financ. 28 (3) (2018) 877–919.
[3] G. Zhang, L. Li, Analysis of Markov chain approximation for option pricing and hedging: grid design and convergence behavior, Oper. Res. (2019).
[4] B. Eriksson, M. Pistorius, American option valuation under continuous-time Markov chains, Adv. Appl. Probab. 47 (2) (2015) 378–401.
[5] N. Cai, Y. Song, S. Kou, A general framework for pricing asian options under Markov processes, Oper. Res. 63 (3) (2015) 540–554.
[6] Y. Song, N. Cai, S. Kou, A unified framework for options pricing under regime switching models, Technical Report, Working paper, 2016.
[7] R. Chatterjee, Z. Cui, J. Fan, M. Liu, An efficient and stable method for short maturity asian options, J. Futures Markets 38 (12) (2018) 1470–1486.
[8] C. Meier, L. Li, G. Zhang, Markov chain approximation of one-dimensional sticky diffusions, arXiv:1910.14282(2019).
[9] T.K. Siu, Bond pricing under a Markovian regime-switching jump-augmented Vasicek model via stochastic flows, Appl. Math. Comput. 216 (11) (2010) 3184–3190.
[10] J.E. Hilliard, A. Schwartz, Binomial option pricing under stochastic volatility and correlated state variables, J. Derivat. 4 (1) (1996) 23–39.
[11] E. Akyildirim, Y. Dolinsky, H.M. Soner, Approximating stochastic volatility by recombinant trees, Ann. Appl. Probab. 24 (5) (2014) 2176–2205.
[12] Z. Cui, J.L. Kirkby, D. Nguyen, A general framework for discretely sampled realized variance derivatives in stochastic volatility models with jumps, Eur. J. Oper. Res. 262 (1) (2017) 381–400.
[13] Z. Cui, J.L. Kirkby, D. Nguyen, A general valuation framework for SABR and stochastic local volatility models, SIAM J. Financ. Math. 9 (2) (2018) 520–563.
[14] Z. Cui, J.L. Kirkby, D. Nguyen, A general framework for time-changed Markov processes and applications, Eur. J. Oper. Res. 273 (2) (2019) 785–800.
[15] K. Ding, Z. Cui, Y. Wang, A Markov chain approximation scheme for option pricing under skew diffusions, Quant. Financ. (2020) 1–20.
[16] Y. Xi, K. Ding, N. Ning, Simultaneous two-dimensional continuous-time Markov chain approximation of two-dimensional fully coupled Markov diffusion processes, Available at SSRN (2019).
[17] N. Cai, S. Kou, Y. Song, A unified framework for regime-switching models, Available at SSRN (2019).
[18] R.J. Elliott, L. Chan, T.K. Siu, Option pricing and Esscher transform under regime switching, Ann. Financ. 1 (4) (2005) 423–432.
[19] J.L. Kirkby, D.H. Nguyen, D. Nguyen, A general continuous time Markov chain approximation for multi-asset option pricing with systems of correlated diffusions, Appl. Math. Comput. 386 (2020) 125472.
[20] J.L. Kirkby, D. Nguyen, Z. Cui, A unified approach to bermudan and barrier options under stochastic volatility models with jumps, J. Econ. Dyn. Control 80 (2017) 75–100.
[21] J.L. Kirkby, D. Nguyen, Efficient asian option pricing under regime switching jump diffusions and stochastic volatility models, Ann. Financ. (2020) 1–45.
[22] Y. Maghsoodi, Solution of the extended CIR term structure and bond option valuation, Math. Financ. 6 (1) (1996) 89–109.
[23] D. Filipović, J. Teichmann, On the geometry of the term structure of interest rates, Proc. R. Soc. Lond. Series A: Math. Phys. Eng. Sci. 460 (2041) (2004) 129–167.
[24] T.S. Ho, S.-B. Lee, Term structure movements and pricing interest rate contingent claims, J. Financ. 41 (5) (1986) 1011–1029.
[25] J. Hull, A. White, Pricing interest-rate-derivative securities, Rev. Financ. Stud. 3 (4) (1990) 573–592.
[26] F. Black, E. Derman, W. Toy, A one-factor model of interest rates and its application to treasury bond options, Financ. Anal. J. 46 (1) (1990) 33–39.
[27] J. Qiu, S.R. Jammalamadaka, N. Ning, Multivariate Bayesian structural time series model, J. Mach. Learn. Res. 19 (1) (2018) 2744–2776.
[28] S.R. Jammalamadaka, J. Qiu, N. Ning, Predicting a stock portfolio with the multivariate Bayesian structural time series model: do news or emotions matter? Int. J. Artif. Intell. 17 (2) (2019) 81–104.
[29] J. Qiu, S.R. Jammalamadaka, N. Ning, Multivariate time series analysis from a Bayesian machine learning perspective, Ann. Math. Artif. Intell. (2020) 1–22.
[30] F. Delbaen, W. Schachermayer, A general version of the fundamental theorem of asset pricing, Mathematische Annalen 300 (1) (1994) 463–520.
[31] I.V. Girsanov, On transforming a certain class of stochastic processes by absolutely continuous substitution of measures, Theory Probab. Appl. 5 (3) (1960) 285–301.
[32] A.A. Novikov, On conditions for uniform integrability of continuous non-negative martingales, Theory Probab. Appl. 24 (4) (1980) 820–824.
[33] N. Kazamaki, On a problem of Girsanov, Tohoku Math. J. Second Series 29 (4) (1977) 597–600.
[34] J. Yong, Some estimates on exponentials of solutions to stochastic differential equations, J. Appl. Math. Stochast. Anal. 2004 (4) (2004) 287–316.
[35] S.N. Ethier, T.G. Kurtz, Markov Processes: Characterization and Convergence, 282, John Wiley & Sons, 2009.
[36] R. Carmona, Lectures on BSDEs, Stochastic Control, and Stochastic Differential Games with Financial Applications, SIAM, 2016.
[37] M. Pistorius, A. Mijatovic, Continuously monitored barrier options under Markov processes (unabridged version with MATLAB code), 2009. SSRN: https://ssrn.com/abstract=1462822or https://doi.org/10.2139/ssrn.1462822
[38] D. Tavella, C. Randall, Pricing Financial Instruments: The Finite Difference Method., 13, John Wiley & Sons, 2000.
[39] T.R. Bielecki, M. Rutkowski, Credit Risk: Modeling, Valuation and Hedging, Springer Science & Business Media, 2002.
[40] L. Bo, D. Tang, Y. Wang, Optimal investment of variance-swaps in jump-diffusion market with regime-switching, J. Econ. Dyn. Control 83 (2017) 175–197.
[41] Y. Lee, T. Rheinlnder, Optimal investment of variance-swaps in jump-diffusion market with regime-switching, J. Appl. Probab. 50 (2) (2013) 344–358.
[42] M. Frittelli, The minimal entropy martingale measure and the valuation problem in incomplete markets, Math. Financ. 10 (1) (2000) 39–52.
[43] A. Klenke, Probability Theory: A Comprehensive Course, Springer Science & Business Media, 2008.
[44] P. Billingsley, Convergence of Probability Measures (second ed.), John Wiley & Sons, 1999.
[45] L. Bo, A. Capponi, Bilateral credit valuation adjustment for large credit derivatives portfolios, Financ. Stochast. 18 (2) (2014) 431–482.
[46] L. Bo, A. Capponi, Systemic risk in interbanking networks, SIAM J. Financ. Math. 6 (1) (2015) 386–424.
[47] L. Bo, Y. Wang, The pricing of basket options: a weak convergence approach, Oper. Res. Lett. 45 (2) (2017) 119–125.
[48] D. Brigo, F. Mercurio, Displaced and mixture diffusions for analytically-tractable smile models, in: Mathematical Finance - Bachelier Congress 2000, 2002, pp. 151–174.
[49] D. Brigo, F. Mercurio, Lognormal-mixture dynamics and calibration to market volatility smiles, Int. J. Theore. Appl. Financ. 5 (04) (2002) 427–446.
[50] C.A. Ball, W.N. Torous, Bond price dynamics and options, J. Financ. Quant. Anal. 18 (4) (1983) 517–531.
[51] J.-P. Fouque, N. Ning, Uncertain volatility models with stochastic bounds, SIAM J. Financ. Math. 9 (4) (2018) 1175–1207.
[52] J.-P. Fouque, C.S. Pun, H.Y. Wong, Portfolio optimization with ambiguous correlation and stochastic volatilities, SIAM J. Control Optim. 54 (5) (2016) 2309–2338.
[53] R. Cont, P. Tankov, Financial Modelling with Jump Processes, Chapman and Hall/CRC, 2003.