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A symplectic perspective on constrained eigenvalue problems

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Abstract

The Maslov index is a powerful tool for computing spectra of selfadjoint, elliptic boundary value problems. This is done by counting intersections of a fixed Lagrangian subspace, which designates the boundary conditions, with the set of Cauchy data for the differential operator. We apply this methodology to constrained eigenvalue problems, in which the operator is restricted to a (not necessarily invariant) subspace. The Maslov index is defined and used to compute the Morse index of the constrained operator. We then prove a constrained Morse index theorem, which says that the Morse index of the constrained problem equals the number of constrained conjugate points, counted with multiplicity, and give an application to the nonlinear Schrödinger equation.

1. Introduction

Consider the nonlinear Schrödinger equation

$$-i\frac{\partial \psi}{\partial t} = \Delta\psi + f(|\psi|^2)\psi \tag{1}$$

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on a bounded domain $\Omega \subset \mathbb{R}^n$. This admits a standing wave of the form $\psi(x,t) = e^{-i\omega t}\phi(x)$ precisely when ϕ solves the nonlinear elliptic equation

$$\Delta\phi + f(\phi^2)\phi + \omega\phi = 0. \tag{2}$$

The existence of nontrivial solutions to such equations on bounded domains can be seen as far back as the work of Pohozaev [29]. See for instance [5] for a recent generalization to compact manifolds with boundary and a fairly complete history of the problem (though note that the results therein specify power-law nonlinearities: $f(s^2) = s^p$ for $1 < p < \frac{4}{d-2}$).

Assuming the existence of a solution ϕ to (2), we can then study perturbative solutions to (1) of the form $u(x,t) = e^{-i\omega t}\big(\phi(x) + e^{\lambda t}w(x)\big)$. Plugging this ansatz into (1) and dropping higher-order terms in w yields the system of eigenvalue equations

$$L_+u = -\lambda v, \quad L_-v = \lambda u, \tag{3}$$

where we have written $w = u + iv$ and $L_\pm$ are the operators

$$L_- = -\Delta - f(\phi^2) - \omega \tag{4}$$

$$L_+ = -\Delta - f(\phi^2) - 2f'(\phi^2)\phi^2 - \omega. \tag{5}$$

The eigenvalue problem (3) is not selfadjoint, even though L_+ and L_- are. If L_- is invertible, this system is equivalent to $L_+u = -\lambda^2(L_-)^{-1}u$. However, L_- typically has a one-dimensional kernel generated by the bound state one is studying, since the standing wave equation (2) is just $L_-\phi = 0$. This lack of invertibility can be overcome by restricting the problem to the subspace $(\ker L_-)^\perp \subset L^2(\Omega)$, and so one needs to describe the spectrum of the corresponding constrained L_+ operator. (The precise functional analytic definition of the constrained operator is given in Section 3 below.) It can be shown, for instance, that unstable eigenvalues (namely those with positive real part) exist if the number of negative eigenvalues of L_+ constrained to $(\ker L_-)^\perp$ differs from the number of negative eigenvalues of L_-. See the early work of Jones [19] and Grillakis [14,15] for an analysis of this phenomenon. For a modern treatment see [21], in particular Theorem 3.2. A thorough overview of the constrained eigenvalue problem and its role in stability theory can be found in [22, §5.2] and also in [28, §4.2].

In certain cases, for instance if ϕ is the positive ground state of a constrained minimization problem, the linear stability or instability can be ascertained from a constrained Morse index calculation. In other settings, for instance those involving excited states, linear stability criteria are harder to establish and generally are computed numerically. However, the nature of the such calculations can often be related to the Krein signature, which can also be framed in terms of a constrained eigenvalue problem; see [20,24].

Motivated by the above considerations, we are thus interested in describing the spectrum, and in particular the number of negative eigenvalues, of a Schrödinger operator $L = -\Delta + V$ on a bounded domain Ω, constrained to act on a closed subspace of $L^2(\Omega)$. In this paper we give a symplectic formulation of this problem, and use it to prove a constrained version of the celebrated Morse–Smale index theorem. We begin by reviewing the symplectic formulation of the unconstrained spectral problem, which first appeared in [10], and was elaborated on in [7,8]. Throughout we will assume the following.

Hypothesis 1. $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, and $V \in L^\infty(\Omega)$.

By a slight abuse of notation we let $u\big|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ denote the Dirichlet trace of $u \in H^1(\Omega)$, and let $\partial u/\partial \nu\big|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ denote the weak Neumann trace, which is defined if $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$; see [25]. We thus define the space of Cauchy data for L

$$\mu(\lambda) = \left\{ \left(u, \frac{\partial u}{\partial \nu} \right)\bigg|_{\partial\Omega} : Lu = \lambda u \right\}, \tag{6}$$

where the equation $Lu = \lambda u$ is meant in a distributional sense. That is, $D(u, v) = \lambda \langle u, v\rangle$ for all $v \in H^1_0(\Omega)$, where $\langle \cdot, \cdot \rangle$ is the L^2 inner product and D is the bilinear form

$$D(u, v) = \int_\Omega [\nabla u \cdot \nabla v + V uv] . \tag{7}$$

It is known that $\mu(\lambda)$ defines a smooth curve of Lagrangian subspaces in the symplectic Hilbert space $H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$. A proof of this fact using standard PDE methods can be found in [8, Proposition 3.1]; a more abstract, functional analytic proof is given in [7, Proposition 4.10].

Boundary conditions are imposed by specifying the domain of the bilinear form D. The relationship between the form domain X and the induced boundary conditions is described in [8, Appendix A], following [12].

Hypothesis 2. The form domain X is a closed subspace of $H^1(\Omega)$ that contains $H^1_0(\Omega)$.

We then let β be a Lagrangian subspace of $H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$ that encodes the boundary conditions; this depends on the choice of X. For instance,

$$\beta_{\mathrm{D}} = \left\{ (0, \phi) : \phi \in H^{-1/2}(\partial\Omega) \right\} \tag{8}$$

if $X = H^1_0(\Omega)$, and

$$\beta_{\mathrm{N}} = \left\{ (x, 0) : x \in H^{1/2}(\partial\Omega) \right\} \tag{9}$$

if $X = H^1(\Omega)$. Note that $\mu(\lambda)$ intersects β_{D} nontrivially whenever there is a solution to $Lu = \lambda u$ satisfying Dirichlet boundary conditions. Similarly, the subspace β_{N} encodes Neumann boundary conditions.

Let $\mathcal{L}$ denote the selfadjoint operator corresponding to the bilinear form D in (7), with form domain X satisfying Hypothesis 2. It was shown in [8] that the subspaces $\mu(\lambda)$ and β comprise a Fredholm pair for each value of λ, so the Maslov index (a topological invariant assigned to a continuous path of Lagrangian subspaces) of μ with respect to β is well defined, and there exists a number $\lambda_\infty < 0$ such that

$$\operatorname{Mas}\left(\mu\big|_{[\lambda_\infty, 0]}; \beta \right) = -n(\mathcal{L}), \tag{10}$$

where $n(\mathcal{L})$ denotes the number of strictly negative eigenvalues (i.e. the Morse index) of $\mathcal{L}$. See [8] for a more complete discussion of boundary conditions as well as the history of the problem.

We now turn to the constrained problem. We first require an assumption on the constrained space $L^2_c(\Omega) \subset L^2(\Omega)$ where the problem will be formulated.

Hypothesis 3. $L^2_c(\Omega) = \{\phi_1, \dots, \phi_m\}^\perp$ for some functions $\phi_1, \dots, \phi_m \in H^1(\Omega)$.

In particular, this implies $L^2_c(\Omega)$ is closed and has finite codimension. This is the typical setting in which constrained index theorems are studied. While our construction of the Maslov index requires the full strength of Hypothesis 3, many of the intermediate results on constrained boundary value problems do not. We clarify this by rewriting the hypothesis as follows, where $\gamma \colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$ denotes the Dirichlet trace map.

Hypothesis 4. $L^2_c(\Omega) \subset L^2(\Omega)$ is a closed subspace such that

(i) $\gamma\left(H^1(\Omega) \cap L^2_c(\Omega)\right) = H^{1/2}(\partial\Omega)$;
(ii) $X \cap L^2_c(\Omega)$ is dense in $L^2_c(\Omega)$;
(iii) $L^2_c(\Omega)^\perp$ is continuously embedded in $H^1(\Omega)$.

Note that (ii) implies $H^1(\Omega) \cap L^2_c(\Omega)$ is dense in $L^2_c(\Omega)$, since $X \subset H^1(\Omega)$. In Section 3.4 we show that Hypotheses 3 and 4 are equivalent. Parts (i) and (ii) of the hypothesis prevent $L^2_c(\Omega)$ from being too small. In particular, the trace condition (i) guarantees that the space of Cauchy data is rich enough to fully describe the constrained spectral problem, and the density condition (ii) ensures that the constrained operator (described below) is well defined. These conditions are trivially satisfied when $L^2_c(\Omega) = L^2(\Omega)$, since $\gamma\left(H^1(\Omega)\right) = H^{1/2}(\partial\Omega)$; see, for instance, [25, Theorem 3.37]. The embedding condition (iii) means that $L^2_c(\Omega)^\perp \subset H^1(\Omega)$, and there is a constant $C > 0$ so that

$$\|\phi\|_{H^1(\Omega)} \le C\|\phi\|_{L^2(\Omega)} \tag{11}$$

for all $\phi \in L^2_c(\Omega)^\perp$. This condition implies that a weak solution u to the constrained eigenvalue problem satisfies $Lu \in L^2(\Omega)$ (by Lemma 2), hence $u \in H^2_{\mathrm{loc}}(\Omega)$ by elliptic regularity (see, for instance, [25, Theorem 4.16]).

Now consider the bilinear form (7) restricted to $X \cap L^2_c(\Omega)$. It follows from Hypothesis 4(ii) that this restriction of D is densely defined, semibounded and closed, and hence defines a unbounded, selfadjoint operator $\mathcal{L}_c$, with dense domain $D(\mathcal{L}_c) \subset L^2_c(\Omega)$; see, for instance, [30, Theorem VIII.15]. We call this the *constrained operator*. The goal of this paper is to understand its spectrum, $\sigma(\mathcal{L}_c)$; we refer to this as the constrained eigenvalue problem. In Proposition 1 we show that the constrained operator is in fact given by $\mathcal{L}_c = P\mathcal{L}\big|_{D(\mathcal{L}_c)}$, where P is the L^2-orthogonal projection onto $L^2_c(\Omega)$.

We define the space of Cauchy data for the constrained problem by

$$\mu_c(\lambda) = \left\{ \left(u, \frac{\partial u}{\partial \nu}\right)\bigg|_{\partial\Omega} : u \in H^1(\Omega) \cap L^2_c(\Omega) \text{ and } D(u,v) = \lambda\,\langle u, v\rangle \right.$$
$$\left. \text{for all } v \in H^1_0(\Omega) \cap L^2_c(\Omega) \right\}.$$

In Lemma 1 we show that the weak Neumann trace $\partial u/\partial \nu\big|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ is well defined for any weak solution u to the constrained problem. This is not a consequence of [25], which requires $Lu \in L^2(\Omega)$. Instead, we construct the trace directly, and then in Lemma 2 use its existence to prove that $Lu \in L^2(\Omega)$.

Remark 1. The existence of $\partial u/\partial \nu\big|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ only requires Hypothesis 4(i). Condition (iii) is only used in the proof of Lemma 2, which is needed to obtain $u \in H^2_{\text{loc}}(\Omega)$ for the unique continuation argument in Section 3.3. It may be possible to eliminate (iii) with a suitable unique continuation principle for the constrained equation, but do not pursue this in the present paper.

We now state our first result, relating the constrained Morse and Maslov indices.

Theorem 1. *If Hypotheses 1, 2 and 3 are satisfied, then μ_c has a well-defined Maslov index with respect to β, and there exists $\lambda_\infty < 0$ such that*

$$n(\mathcal{L}_c) = -\operatorname{Mas}\left(\mu_c\big|_{[\lambda_\infty,0]}; \beta\right).$$

In other words, the Maslov index determines the Morse index of the constrained operator $\mathcal{L}_c$.

The classical approach to the constrained eigenvalue problem (see [22,28] and references therein) is to relate $n(\mathcal{L})$ and $n(\mathcal{L}_c)$ through the index of a finite-dimensional "constraint matrix." The most general statement we are aware of is [28, Theorem 4.1], although the method of proof appeared earlier in [9,27].

Theorem 2 *([28]). Suppose $L^2_c(\Omega)$ has finite codimension, with $L^2_c(\Omega)^\perp = \operatorname{span}\{\phi_1, \dots, \phi_m\}$. The constrained and unconstrained Morse indices are related by*

$$n(\mathcal{L}) - n(\mathcal{L}_c) = \lim_{\mu \to 0^-} n(M(\mu)),$$

where $M(\mu)$ is the $m \times m$ matrix with entries $M_{ij}(\mu) = \left\langle (\mathcal{L} - \mu)^{-1}\phi_i, \phi_j \right\rangle$.

Remark 2. In [28] it is shown that the eigenvalues of $M(\mu)$ are continuous and strictly increasing as long as $\mu \notin \sigma(\mathcal{L})$. Therefore, if $\mathcal{L}$ is invertible, the matrix $M(0)$ is defined and the above result simplifies to

$$n(\mathcal{L}) - n(\mathcal{L}_c) = n(M(0)) + \dim\ker M(0).$$

A similar result appears in [21], with the added assumption that $\ker\mathcal{L} \subset L^2_c(\Omega)$. This implies $\phi_i \in (\ker\mathcal{L})^\perp = \operatorname{ran}\mathcal{L}$, so the equation $\mathcal{L}u = \phi_i$ has a unique solution $u \in L^2_c(\Omega)^\perp$, which we denote by $\mathcal{L}^{-1}\phi_i$, and hence the matrix $M(0) = \left\langle \mathcal{L}^{-1}\phi_i, \phi_j \right\rangle$ is well defined.

This result allows one to compute $n(\mathcal{L}_c)$ from the unconstrained Morse index $n(\mathcal{L})$ and the constraint matrix M. Here we take a different approach, combining Theorem 1 with a homotopy argument to compute the constrained Morse index directly, without having to first know the unconstrained index.

To do this we describe what happens when the domain Ω is deformed through a smooth one-parameter family $\{\Omega_t\}$. The result is a constrained analog of Smale's Morse index theorem [32], relating the Morse index of an operator to its conjugate points. Smale's result, which only applies to the Dirichlet problem, was originally proved by variational methods (see also [33]). A proof using the Maslov index was given in [7] for star-shaped domains, and in [8] for the general case.

We prove a general result to this effect in Section 4; for now we just state the simplest case, when Dirichlet boundary conditions are imposed and there is only one constraint function, i.e. $L^2_c(\Omega) = \{\phi\}^\perp$. We say that t is a *constrained conjugate point* for the Dirichlet problem if there exists a nonzero function $u \in H^2(\Omega_t) \cap H^1_0(\Omega_t)$ such that

$$\int_{\Omega_t} u\phi = 0, \qquad Lu = a\phi \text{ on } \Omega_t$$

for some constant a. In other words, 0 is an eigenvalue for the constrained Dirichlet problem on Ω_t. Let $d(t)$ denote its multiplicity, so that $d(t) > 0$ whenever t is a conjugate time.

Theorem 3. *Let $\{\Omega_t : 0 < t \leq 1\}$ be a smooth, increasing family of domains in $\mathbb{R}^n$, with $\Omega_1 = \Omega$. Suppose $L^2_c(\Omega) = \{\phi\}^\perp$ for some $\phi \in H^1(\Omega)$ with $\int_{\Omega_t} \phi^2 > 0$ for all $t > 0$. If $|\Omega_t| \to 0$ as $t \to 0$, then*

$$n(\mathcal{L}_c) = \sum_{t<1} d(t).$$

That is, the Morse index of the constrained operator equals the number of constrained conjugate points in $(0, 1)$, counting multiplicity. The sum on the right-hand side is well defined because $d(t)$ is only nonzero for a finite set of times. The assumption that ϕ is not identically zero on any Ω_t ensures the constraint space does not change dimension as t varies; this is a crucial ingredient in establishing the continuity properties needed to have a well defined Maslov index.

We conclude in Section 5 by giving a formal application of Theorem 3 to the ground state solution ϕ of the one-dimensional NLS. We find that there is a constrained conjugate point (hence a negative eigenvalue of L_+) if and only if the quantity

$$\frac{\partial}{\partial\omega} \int_{-\infty}^{\infty} \phi^2$$

is positive. This is the well-known Vakhitov–Kolokolov condition [34]; see also [16].

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2. A finite-dimensional example

We now give a simple illustration of Theorem 3, by computing the constrained Morse index of $L = -\Delta - C$ on $(-1, 1)$, where C is a positive constant. We do this in three different ways: first by direct computation, and then using Theorems 2 and 3.

Let $\mathcal{L}$ denote the differential operator on $(-1, 1)$ with Dirichlet boundary conditions, and $\mathcal{L}_c$ the constrained operator on the space of zero mean functions

$$L_c^2(-1,1) = \left\{ u \in L^2(-1,1) : \int_{-1}^{1} u(x)\,dx = 0 \right\}.$$

The constrained eigenvalue equation $\mathcal{L}_c u = \lambda u$ is equivalent to the conditions

$$u_{xx} + Cu + \lambda u = \text{constant}, \quad \int_{-1}^{1} u(x)\,dx = 0, \quad u(-1) = u(1) = 0.$$

From the differential equation and the zero mean condition we obtain the general solution

$$u(x) = A(\cos\gamma x - \gamma^{-1}\sin\gamma) + B\sin\gamma x$$

where $\gamma = \sqrt{C + \lambda}$. Imposing the Dirichlet boundary conditions at $x = \pm 1$, we have

$$A(\cos\gamma - \gamma^{-1}\sin\gamma) \pm B\sin\gamma = 0,$$

which implies either $\cos\gamma = \gamma^{-1}\sin\gamma$ or $\sin\gamma = 0$. Finally, observing that $\lambda < 0$ iff $\gamma < \sqrt{C}$, we find that the number of negative eigenvalues is

$$n(\mathcal{L}_c) = \#\left\{ \gamma \in (0, \sqrt{C}) : \sin\gamma = 0 \text{ or } \tan\gamma = \gamma \right\}. \tag{12}$$

We next compute the Morse index using Theorem 3, counting the number of conjugate points $t \in (0, 1)$ for the family of domains $\Omega_t = (-t, t)$. The constrained equation on Ω_t is

$$u_{xx} + Cu = \text{constant}, \quad \int_{-t}^{t} u(x)\,dx = 0.$$

Setting $\gamma = \sqrt{C}$, we can write the general solution as

$$u(x) = A(\cos\gamma x - \gamma^{-1}\sin\gamma) + B\sin\gamma x.$$

Therefore, $t \in (0, 1)$ is a conjugate point precisely when

$$A\left(\cos\gamma t - (\gamma t)^{-1}\sin\gamma t\right) \pm B\sin\gamma t = 0.$$

It follows that either $\cos\gamma t = (\gamma t)^{-1}\sin\gamma t$ or $\sin\gamma t = 0$. Recalling that $\gamma = \sqrt{C}$, we obtain

$$\#\text{conjugate points} = \#\left\{t \in (0,1) : \sin\sqrt{C}t = 0 \text{ or } \tan\sqrt{C}t = \sqrt{C}t\right\}, \tag{13}$$

which agrees with the Morse index $n(\mathcal{L}_c)$ computed in (12).

A similar computation shows that the unconstrained Morse index is

$$n(\mathcal{L}) = \#\left\{\gamma \in (0,\sqrt{C}) : \sin\gamma = 0 \text{ or } \cos\gamma = 0\right\}.$$

Comparing solutions of $\cos\gamma = 0$ and $\tan\gamma = \gamma$, we see that the constrained and unconstrained indices are related by

$$n(\mathcal{L}) = \begin{cases} n(\mathcal{L}_c) + 1 & \text{if } \tan\sqrt{C} \le \sqrt{C} \\ n(\mathcal{L}_c) & \text{if } \tan\sqrt{C} > \sqrt{C} \\ n(\mathcal{L}_c) & \text{if } \cos\sqrt{C} = 0 \end{cases} \tag{14}$$

Finally, we verify that this is consistent with the prediction of Theorem 2 by computing the constraint matrix $M(\mu)$ for small negative values of μ. Since $L^2_c(-1,1)^\perp$ is spanned by the constant function 1, $M(\mu)$ is simply the number $\left\langle(\mathcal{L}-\mu)^{-1}1, 1\right\rangle$. To compute $(\mathcal{L}-\mu)^{-1}1$ we must solve the boundary value problem

$$u_{xx} + (C+\mu)u + 1 = 0, \quad u(-1) = u(1) = 0.$$

Setting $\gamma = \sqrt{C+\mu}$, we find

$$u(x) = \frac{1}{\gamma^2}\left(\frac{\cos\gamma x}{\cos\gamma} - 1\right)$$

and so

$$\left\langle(\mathcal{L}-\mu)^{-1}1, 1\right\rangle = \int_{-1}^{1} u(x)\,dx = \frac{2}{\gamma^2}\left(\frac{\tan\gamma}{\gamma} - 1\right).$$

This is a strictly increasing function of γ (and hence of μ), so we obtain

$$\lim_{\mu\to 0^-} M(\mu) = \begin{cases} \le 0 & \text{if } \tan\sqrt{C} \le \sqrt{C} \\ > 0 & \text{if } \tan\sqrt{C} > \sqrt{C} \\ +\infty & \text{if } \cos\sqrt{C} = 0 \end{cases}$$

and hence

$$\lim_{\mu\to 0^-} n(M(\mu)) = \begin{cases} 1 & \text{if } \tan\sqrt{C} \le \sqrt{C} \\ 0 & \text{otherwise} \end{cases}$$

as expected from comparing the result in (14) with Theorem 2.

3. The constrained Maslov index

In this section we define the Maslov index for constrained eigenvalue problems in multiple dimensions. After reviewing the Fredholm–Lagrangian Grassmannian and the Maslov index, as well as some necessary details of constrained operators and boundary value problems, we define the constrained Maslov index, and prove that it equals (minus) the constrained Morse index, thus proving Theorem 1. As is common for such problems, most of the work goes into establishing the existence and regularity of the relevant paths of Lagrangian subspaces. Once this is known, the main result follows from a straightforward crossing form calculation.

Throughout the section we assume Hypothesis 1, invoking the individual parts of Hypothesis 4 only as needed.

3.1. The Maslov index in infinite dimensions

Before describing the constrained eigenvalue problem, we will review the infinite-dimensional Maslov index, following [13].

Suppose $\mathcal{H}$ is a symplectic Hilbert space: that is, a real Hilbert space equipped with a non-degenerate, skew-symmetric bilinear form ω. A subspace $\mu \subset \mathcal{H}$ is said to be isotropic if $\omega(v, w) = 0$ for all $v, w \in \mu$, and is said to be Lagrangian if it is isotropic and maximal, in the sense that it is not properly contained in any other isotropic subspace. The set of all Lagrangian subspaces is called the Lagrangian Grassmannian and is denoted $\Lambda(\mathcal{H})$. This is a smooth, contractible Banach manifold, whose differentiable structure comes from associating to each Lagrangian subspace its orthogonal projection operator. Thus a family of Lagrangian subspaces $\mu(t)$ is of class C^k if and only if the corresponding family of projections $P_{\mu(t)}$ is C^k.

We assume that the symplectic form can be written as $\omega(v, w) = \langle Jv, w\rangle$, where $J : \mathcal{H} \to \mathcal{H}$ is a skew-symmetric operator satisfying $J^2 = -I$. (Given a symplectic form ω, one can always find a complete inner product for which this is true; see [13, Proposition D.1].) If μ is a given Lagrangian subspace, and $A : \mu \to \mu$ is a bounded, selfadjoint operator, then the graph

$$\mathrm{Gr}_\mu(A) = \{v + JAv : v \in \mu\}$$

will also be Lagrangian. Moreover, the orthogonal projection onto this graph can be computed algebraically from A; see [13, Equation (2.16)]. Therefore, if $A(t)$ is a C^k family of bounded, selfadjoint operators on μ, the corresponding family $\mathrm{Gr}_\mu(A(t))$ of Lagrangian subspaces will also be of class C^k. This simple observation is our main technical tool for establishing regularity properties of paths of Lagrangian subspaces.

Since $\Lambda(\mathcal{H})$ is contractible, there is no nontrivial notion of winding for general curves of Lagrangian subspaces, and so we must restrict our attention to a smaller space in order to have a useful index theory. For a fixed Lagrangian subspace $\beta \subset \mathcal{H}$, we define the Fredholm–Lagrangian Grassmannian,

$$\mathcal{F}\Lambda_\beta(\mathcal{H}) = \{\mu \in \Lambda(\mathcal{H}) : \mu \text{ and } \beta \text{ are a Fredholm pair}\},$$

recalling that μ and β are said to be a Fredholm pair when $\mu \cap \beta$ is finite dimensional and $\mu + \beta$ is closed and has finite codimension. The Fredholm–Lagrangian Grassmannian is a smooth Banach manifold with fundamental group $\pi_1(\mathcal{F}\Lambda_\beta(\mathcal{H})) = \mathbb{Z}$. As a result, there is an integer, the Maslov index, associated to any continuous path of Lagrangian subspaces that is Fredholm with respect to β. The Maslov index is invariant with respect to fixed-endpoint homotopies, and can be thought of as a generalized winding number in the space $\mathcal{F}\Lambda_\beta(\mathcal{H})$. The utility of this index in eigenvalue problems stems from the fact that it is simply a count (with sign and multiplicity) of the nontrivial intersections between $\mu(t)$ and β.

To compute the Maslov index in practice, we use crossing forms. Suppose $\mu\colon [a,b] \to \mathcal{F}\Lambda_\beta(\mathcal{H})$ is a continuously differentiable path of Lagrangian subspaces, and $\mu(t_*) \cap \beta \neq \{0\}$ for some $t_* \in [a,b]$. Let $v(\cdot)$ be a continuously differentiable path in $\mathcal{H}$, with $v(t) \in \mu(t)$ for t close to t_* and $v(t_*) \in \mu(t_*) \cap \beta$. The crossing form is a quadratic form defined on the finite-dimensional vector space $\mu(t_*) \cap \beta$ by

$$Q(v(t_*)) = \omega\left(v, \frac{dv}{dt}\right)\bigg|_{t=t_*}.$$

It can be shown that this depends only on the vector $v(t_*)$, and not on the path $v(t)$. If Q is nondegenerate, then the crossing time t_* is isolated. Suppose that t_* is the only crossing in $[a,b]$ and let (n_+, n_-) be the signature of Q. The Maslov is then given by

$$\operatorname{Mas}\left(\mu_{[a,b]}; \beta\right) = \begin{cases} -n_- & \text{if } t_* = a, \\ n_+ - n_- & \text{if } t_* \in (a,b), \\ n_+ & \text{if } t_* = b. \end{cases}$$

The Maslov index is additive, in the sense that

$$\operatorname{Mas}\left(\mu_{[a,b]}; \beta\right) = \operatorname{Mas}\left(\mu_{[a,c]}; \beta\right) + \operatorname{Mas}\left(\mu_{[c,b]}; \beta\right)$$

for any $c \in (a,b)$, so we can use the crossing form to compute the Maslov index of any piecewise continuously differentiable curve, provided all of its crossings are nondegenerate.

If H is a real Hilbert space, with dual space H^*, then $\mathcal{H} = H \oplus H^*$ is a symplectic Hilbert space. The symplectic form is given by

$$\omega\big((x,\phi),(y,\psi)\big) = \psi(x) - \phi(y),$$

and the corresponding complex structure $J\colon \mathcal{H} \to \mathcal{H}$ is

$$J(x,\phi) = \big(R^{-1}\phi, -Rx\big),$$

where $R\colon H \to H^*$ is the isomorphism from the Riesz representation theorem.

To study selfadjoint boundary value problems we will take $H = H^{1/2}(\partial\Omega)$, hence $H^* = H^{-1/2}(\partial\Omega)$. Elements of $\mathcal{H} = H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$ will arise as the boundary values (or "traces") of weak solutions to the eigenvalue equation $Lu = \lambda u$, or its constrained analogue, via the trace map

$$\operatorname{tr} u := \left(u, \frac{\partial u}{\partial \nu}\right)\bigg|_{\partial\Omega}. \tag{15}$$

We will use integral notation to denote the dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$, so Green's second identity yields

$$\omega(\operatorname{tr} u, \operatorname{tr} v) = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) = \int_{\Omega} (u\Delta v - v\Delta u) \tag{16}$$

for any $u, v \in H^1(\Omega)$ with $\Delta u, \Delta v \in L^2(\Omega)$. This identity hints at a connection between the Lagrangian subspaces of $H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$ and selfadjoint, second-order differential operators on $L^2(\Omega)$. While the current paper utilizes a particular version of this correspondence, it is in fact part of a deeper phenomenon, which has been investigated systematically in [23].

3.2. Preliminaries on constrained boundary value problems

We define $L = -\Delta + V$ distributionally on $H^1(\Omega)$, via the bilinear form D in (7), so $Lu = F \in H^{-1}(\Omega)$ means $D(u, v) = \langle F, v \rangle$ for all $v \in H_0^1(\Omega)$. Throughout the section we assume $L_c^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, only invoking the other parts of Hypothesis 4 when needed.

To define the trace of a weak solution, as in (15), we need to know that its normal derivative is well defined. The statement and proof of the next result, a constrained version of Green's first identity, closely follow [25, Lemma 4.3].

Lemma 1. *Assume Hypothesis 4(i). Let $u \in H^1(\Omega) \cap L_c^2(\Omega)$, and suppose there exists $f \in L_c^2(\Omega)$ such that $D(u, v) = \langle f, v \rangle$ for all $v \in H_0^1(\Omega) \cap L_c^2(\Omega)$. Then there is a unique function $g \in H^{-1/2}(\partial\Omega)$ such that*

$$D(u, v) = \langle f, v \rangle + \int_{\partial\Omega} g(\gamma v) \tag{17}$$

for all $v \in H^1(\Omega) \cap L_c^2(\Omega)$. Moreover, g satisfies the estimate

$$\|g\|_{H^{-1/2}(\partial\Omega)} \le C \left(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

Proof. By Hypothesis 4(i) the constrained Dirichlet trace map

$$\gamma_c := \gamma\big|_{H^1(\Omega)\cap L_c^2(\Omega)} \colon H^1(\Omega) \cap L_c^2(\Omega) \to H^{1/2}(\partial\Omega)$$

is surjective, and hence has a bounded right inverse, $E \colon H^{1/2}(\partial\Omega) \to H^1(\Omega) \cap L_c^2(\Omega)$. Now $g \in H^{-1/2}(\partial\Omega) = H^{1/2}(\partial\Omega)^*$ can be defined by its action on $h \in H^{1/2}(\partial\Omega)$:

$$g(h) = D(u, Eh) - \langle f, Eh \rangle .$$

From the boundedness of D and E we obtain

$$|g(h)| \le C \left(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} \right) \|h\|_{H^{1/2}(\partial\Omega)}$$

for all $h \in H^{1/2}(\partial\Omega)$, so $g \in H^{-1/2}(\partial\Omega)$ and the desired estimate follows.

To see that (17) holds, let $v \in H^1(\Omega) \cap L_c^2(\Omega)$, and define $h = \gamma v$. It follows that $\gamma Eh = \gamma v$, and so $Eh - v \in H_0^1(\Omega) \cap L_c^2(\Omega)$, hence $D(u, Eh - v) = \langle f, Eh - v \rangle$. We thus obtain

$$\int_{\partial\Omega} g(\gamma v) = g(h) = D(u, Eh) - \langle f, Eh\rangle = D(u, v) - \langle f, v\rangle\,,$$

which is the desired result.

To complete the proof, we establish the uniqueness of g. If $g_1, g_2 \in H^{-1/2}(\partial\Omega)$ both satisfy (17), then

$$\int_{\partial\Omega} (g_1 - g_2)\gamma v = 0$$

for all $v \in H^1(\Omega) \cap L^2_c(\Omega)$. Since γ_c is surjective, this implies $g_1 - g_2 = 0$. $\square$

When u and v are sufficiently smooth, it follows from the classical version of Green's first identity that

$$\int_{\partial\Omega} gv = D(u, v) - \langle Lu, v\rangle = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v.$$

That is, g is just the normal derivative of u. Thus in general we will refer to the function $g \in H^{-1/2}(\partial\Omega)$ defined by Lemma 1 as the normal derivative of u.

Note that this lemma does not immediately follow from the aforementioned result in [25] because we do not know a priori that $Lu \in L^2(\Omega)$. However, using Lemma 1, we can prove a posteriori that this is the case.

Lemma 2. *Assume Hypothesis 4. If u satisfies the conditions of Lemma 1, then $Lu \in L^2(\Omega)$ and $PLu = f$.*

Proof. To prove the result we will construct a function $F \in L^2(\Omega)$ that satisfies

$$D(u, v) = \langle F, v\rangle + \int_{\partial\Omega} g(\gamma v) \tag{18}$$

for all $v \in H^1(\Omega)$. Subtracting (17) and (18), we see that if such an F exists, it must satisfy $\langle F, v\rangle = \langle f, v\rangle$ for all $v \in H^1(\Omega) \cap L^2_c(\Omega)$, and hence for all $v \in L^2_c(\Omega)$, by Hypothesis 4(ii). This implies $F = f + \phi$ for some $\phi \in L^2_c(\Omega)^\perp$.

We first claim that

$$H^1(\Omega) = \big(H^1(\Omega) \cap L^2_c(\Omega)\big) \oplus L^2_c(\Omega)^\perp.$$

This follows from writing $v = Pv + (I - P)v$. Hypothesis 4(iii) implies $(I - P)v \in L^2_c(\Omega)^\perp \subset H^1(\Omega)$, so we also have $Pv = v - (I - P)v \in H^1(\Omega)$ as required.

Now decompose $v \in H^1(\Omega)$ accordingly as $v_1 + v_2$. Using Lemma 1 we obtain

$$D(u, v) = D(u, v_1) + D(u, v_2) = \langle f, v_1\rangle + \int_{\partial\Omega} gv_1 + D(u, v_2).$$

Comparing this to the right-hand side of (18),

$$\langle F, v\rangle + \int\limits_{\partial\Omega} gv = \langle f, v_1\rangle + \langle \phi, v_2\rangle + \int\limits_{\partial\Omega} gv_1 + \int\limits_{\partial\Omega} gv_2,$$

we see that ϕ must satisfy

$$\langle \phi, v_2\rangle = D(u, v_2) - \int\limits_{\partial\Omega} gv_2 \tag{19}$$

for all $v_2 \in L^2_c(\Omega)^\perp$. The inequality (11) from Hypothesis 4(iii) implies the right-hand side of (19) is a bounded linear functional on $L^2_c(\Omega)^\perp$, so the existence of ϕ follows from the Riesz representation theorem. Setting $F = f + \phi$ completes the proof of (18). Then for any $v \in H^1_0(\Omega)$ we obtain

$$D(u, v) = \langle F, v\rangle\,,$$

hence $Lu = F \in L^2(\Omega)$ and $PLu = PF = f$ as was claimed. □

We next give a result on the solvability of a Robin-type boundary value problem that will be needed in the proof of Lemma 4. Suppose $u \in H^1(\Omega) \cap L^2_c(\Omega)$ satisfies

$$D(u, v) = \lambda\,\langle u, v\rangle + \zeta \int\limits_{\partial\Omega} (Ru)v \tag{20}$$

for every $v \in H^1(\Omega) \cap L^2_c(\Omega)$, where $R\colon\ H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is the Riesz duality map and $\zeta \in \mathbb{R}$. It follows from Lemmas 1 and 2 that $PLu = \lambda u$ and

$$\frac{\partial u}{\partial \nu} - \zeta Ru = 0,$$

and so we refer to this as a constrained Robin-type problem. Note that this is not a traditional Robin boundary value problem, even in the absence of constraints, on account of the Riesz operator R that appears in the boundary conditions.

Lemma 3. *For any fixed $\lambda_0 \in \mathbb{R}$, there exists $\zeta_0 \in \mathbb{R}$ such that the constrained Robin-type boundary value problem* (20) *is invertible for all λ sufficiently close to λ_0.*

In particular, this means the homogeneous problem only admits the zero solution, whereas the inhomogeneous problem

$$PLu = \lambda u, \qquad \frac{\partial u}{\partial \nu} - \zeta_0 Ru = h \tag{21}$$

has a unique solution for each $h \in H^{-1/2}(\partial\Omega)$, provided $|\lambda - \lambda_0| \ll 1$. This construction is the key ingredient in the proof of Lemma 4, where it will be used to write the constrained Cauchy data space $\mu_c(\lambda)$ as the graph of a selfadjoint operator on a fixed Lagrangian subspace.

Proof. We will in fact prove that ζ_0 can be chosen arbitrarily close to 0. Consider the bilinear form

$$D_\zeta(u,v) = D(u,v) - \lambda_0 \langle u, v\rangle - \zeta \int_{\partial\Omega} (R\gamma u)\gamma v$$

on $H^1(\Omega) \cap L^2_c(\Omega)$, which is dense in $L^2_c(\Omega)$ by Hypothesis 4(ii) as $X \subset H^1(\Omega)$. The boundary term satisfies $\int_{\partial\Omega}(R\gamma u)\gamma u = \|\gamma u\|^2_{H^{1/2}(\partial\Omega)} \le C\|u\|^2_{H^1(\Omega)}$, so the form D_ζ is semibounded for any $\zeta \le 0$, and for sufficiently small $\zeta > 0$. It is also closed, since $H^1(\Omega) \cap L^2_c(\Omega)$ is a closed subspace of $H^1(\Omega)$, and hence generates an unbounded, selfadjoint operator $\mathcal{L}_\zeta$; see [30, Theorem VIII.15] for the general result, and [8, Appendix A] for a discussion of this construction in the context of elliptic boundary value problems. By construction, $u \in \ker \mathcal{L}_\zeta$ if and only if u solves the homogeneous problem

$$P(L - \lambda_0)u = 0, \qquad \frac{\partial u}{\partial \nu} - \zeta R u = 0.$$

It follows immediately from the proof of Theorem 3.2 in [31] that the ordered eigenvalues of $\mathcal{L}_\zeta$ are strictly monotone with respect to ζ. Therefore, if $\mathcal{L}_0$ is not invertible, $\mathcal{L}_\zeta$ will be for any $0 < |\zeta| \ll 1$. Since $\mathcal{L}_\zeta$ has discrete spectrum, $\mathcal{L}_\zeta + (\lambda_0 - \lambda)I$ is also invertible for $|\lambda - \lambda_0| \ll 1$. □

Finally, we discuss the relationship of the selfadjoint operator $\mathcal{L}_c$ defined using the bilinear form (7) restricted to $X \cap L^2_c(\Omega)$ (which is dense in $L^2_c(\Omega)$ by Hypothesis 4(ii)) to the operator $P\mathcal{L}\big|_{L^2_c(\Omega)}$ that typically arises in the stability literature. To simplify the discussion here we will only consider $X = H^1_0(\Omega)$ (Dirichlet) or $X = H^1(\Omega)$ (Neumann). Recall that $\mathcal{L}$ is the operator corresponding to the bilinear form D with form domain $X \subset H^1(\Omega)$, whereas $\mathcal{L}_c$ corresponds to the form D restricted to $X \cap L^2_c(\Omega)$. By definition, these operators have domains

$$D(\mathcal{L}) = \left\{u \in X : \exists\, F \in L^2(\Omega) \text{ with } D(u,v) = \langle F, v\rangle \text{ for all } v \in X\right\}$$

and

$$D(\mathcal{L}_c) = \left\{u \in X \cap L^2_c(\Omega) : \exists\, f \in L^2_c(\Omega) \text{ with } D(u,v) = \langle f, v\rangle \text{ for all } v \in X \cap L^2_c(\Omega)\right\}$$

which are dense in $L^2(\Omega)$ and $L^2_c(\Omega)$, respectively.

Proposition 1. *If Hypothesis 4 is satisfied and X is either $H^1_0(\Omega)$ or $H^1(\Omega)$, then $\mathcal{L}_c = P\mathcal{L}\big|_{L^2_c(\Omega)}$.*

Proof. Let $u \in D(\mathcal{L}) \cap L^2_c(\Omega)$, with $\mathcal{L}u = F \in L^2(\Omega)$. From the definition of $\mathcal{L}$, this means $D(u,v) = \langle F, v\rangle$ for all $v \in X$. In particular, for any $v \in X \cap L^2_c(\Omega)$ we have

$$D(u,v) = \langle F, v\rangle = \langle PF, v\rangle\,,$$

hence $u \in D(\mathcal{L}_c)$ and $\mathcal{L}_c u = PF = P\mathcal{L}u$. It follows that $P\mathcal{L}\big|_{L^2_c(\Omega)} \subset \mathcal{L}_c$.

To prove the other direction, let $u \in D(\mathcal{L}_c)$, with $\mathcal{L}_c u = f$. This means $D(u, v) = \langle f, v\rangle$ for all $v \in X \cap L^2_c(\Omega)$, and hence for all $v \in H^1_0(\Omega) \cap L^2_c(\Omega)$. By Lemmas 1 and 2 there exist functions $g \in H^{-1/2}(\partial\Omega)$ and $F \in L^2(\Omega)$ such that

$$D(u, v) = \langle F, v\rangle + \int_{\partial\Omega} g\gamma v \tag{22}$$

for all $v \in H^1(\Omega)$, and hence for all $v \in X$.

If $X = H^1_0(\Omega)$, then $\gamma v = 0$ for all $v \in X$. It follows from (22) that $D(u, v) = \langle F, v\rangle$ for all $v \in X$, hence $u \in D(\mathcal{L})$ and $\mathcal{L}u = F$. On the other hand, if $X = H^1(\Omega)$, then $u \in D(\mathcal{L}_c)$ implies $D(u, v) = \langle f, v\rangle$ for all $v \in H^1(\Omega) \cap L^2_c(\Omega)$. Comparing with (22), we see that

$$\int_{\partial\Omega} g\gamma v = 0$$

for all $v \in H^1(\Omega) \cap L^2_c(\Omega)$. Using Hypothesis 4(i), we conclude that $g = 0$. It follows that $D(u, v) = \langle F, v\rangle$ for all $v \in H^1(\Omega) = X$, which means $u \in D(\mathcal{L})$ and $\mathcal{L}u = F$. Thus for either choice of X we have $u \in D(\mathcal{L})$ and $\mathcal{L}u = F$, hence $P\mathcal{L}u = PF = f = \mathcal{L}_c u$. This implies $\mathcal{L}_c \subset P\mathcal{L}\big|_{L^2_c(\Omega)}$ and thus completes the proof. □

3.3. Construction of the Maslov index

We now have all of the ingredients in place to define the constrained Maslov index, and prove that it equals (minus) the Morse index of the constrained operator $\mathcal{L}_c$. For the remainder of the section we assume Hypotheses 1, 2 and 3.

The space of weak solutions for the constrained problem, in the absence of boundary conditions, is

$$K_c(\lambda) = \left\{u \in H^1(\Omega) \cap L^2_c(\Omega) : D(u, v) = \lambda \langle u, v\rangle \text{ for all } v \in H^1_0(\Omega) \cap L^2_c(\Omega)\right\}, \tag{23}$$

where the bilinear form D is defined in (7). Any $u \in K_c(\lambda)$ satisfies the hypotheses of Lemma 1, with $f = \lambda u$, and so the boundary trace (or Cauchy data)

$$\operatorname{tr} u := \left(u, \frac{\partial u}{\partial \nu}\right)\bigg|_{\partial\Omega}$$

is a well-defined element of $H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$, and

$$\mu_c(\lambda) = \{\operatorname{tr} u : u \in K_c(\lambda)\} \tag{24}$$

defines a subspace of $H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$. In fact, from Lemma 2 we have $u \in H^2_{\text{loc}}(\Omega)$, and so it follows from a unique continuation argument (as in [2]) that

$$\operatorname{tr}\colon K_c(\lambda) \to H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$$

is injective.

Lemma 4. *The mapping $\lambda \mapsto \mu_c(\lambda)$ defines a smooth family of Lagrangian subspaces in $\mathcal{H}$.*

Proof. We first prove that $\mu_c(\lambda)$ is isotropic. Let $u, v \in K_c(\lambda)$. Then Lemma 1 implies

$$\begin{aligned}\omega(\operatorname{tr} u, \operatorname{tr} v) &= \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \\ &= D(v,u) - \langle \lambda v, u \rangle - D(u,v) + \langle \lambda u, v \rangle \\ &= 0\end{aligned}$$

because D is symmetric.

We now use the strategy of [8, Proposition 3.5] to prove that $\mu_c(\lambda)$ is Lagrangian and is smooth with respect to λ. The idea, as described in Section 3.1, is to realize each subspace $\mu_c(\lambda)$ as the graph of a bounded, selfadjoint operator $A(\lambda)$ on a fixed Lagrangian subspace. This will imply each subspace is in fact Lagrangian, and the family $\{\mu_c(\lambda)\}$ is as smooth with respect to λ as the family $\{A(\lambda)\}$ is. The operator $A(\lambda)$ will be a constrained Robin-to-Robin map for $L - \lambda$. (The Neumann-to-Dirichlet map suffices whenever it is defined, i.e. when the constrained operator with Neumann boundary conditions is invertible.) The main modification to the argument in [8] stems from using Lemma 3 to find a Robin-type boundary condition for which the constrained operator is invertible.

Since smoothness is a local property, it will suffice to construct $A(\lambda)$ in a neighborhood of a fixed λ_0. By Lemma 3 there exists $\zeta_0 \in \mathbb{R}$ so that the constrained boundary value problem (21) is invertible for $|\lambda - \lambda_0| \ll 1$. Using this fixed value of ζ_0 we define the subspace

$$\rho = \{(f,g) \in \mathcal{H} : f + \zeta_0 R^{-1} g = 0\}.$$

By construction, for any $(f,g) \in \rho$ there is a unique weak solution $u = u(\lambda) \in H^1(\Omega) \cap L^2_c(\Omega)$ to

$$PLu = \lambda u, \quad \frac{\partial u}{\partial \nu} - \zeta_0 R u = h \tag{25}$$

with $h = g - \zeta_0 R f \in H^{-1/2}(\partial\Omega)$. From this solution u we define

$$A(\lambda)(f,g) = J^{-1}\Big(\gamma u - f, \zeta_0 R(\gamma u - f)\Big).$$

Since $\big(\gamma u - f, \zeta_0 R(\gamma u - f)\big)$ is contained in the subspace $J\rho = \{(f,g) : g = \zeta_0 R f\}$, we have $A(\lambda) \colon \rho \to \rho$ as desired. The proof that $A(\lambda)$ is selfadjoint follows directly from Green's identity, as in [8], Proposition 3.5. Moreover, $A(\lambda)$ must be bounded, as it is defined on all of ρ. Therefore, the graph $\operatorname{Gr}_\rho(A(\lambda))$ is Lagrangian for each λ.

For each $(f,g) \in \rho$ we have

$$(f,g) + JA(\lambda)(f,g) = \big(\gamma u, g + \zeta_0 R(\gamma u - f)\big) = \operatorname{tr} u,$$

where u is a weak solution to $PLu = \lambda u$ and as such is contained in $K_c(\lambda)$. This implies $\operatorname{Gr}_\rho(A(\lambda)) \subset \mu_c(\lambda)$. Using the fact that $\mu_c(\lambda)$ is isotropic and $\operatorname{Gr}_\rho(A(\lambda))$ is Lagrangian (hence maximal), we conclude that $\mathrm{Gr}_\rho(A(\lambda)) = \mu_c(\lambda)$. This completes the proof that $\mu_c(\lambda)$ is Lagrangian. □

We next consider the boundary conditions. The following result was established in [8, Lemma 3.6].

Lemma 5. *The boundary space $\beta \subset \mathcal{H}$ is Lagrangian.*

Next, we study the intersection properties of $\mu_c(\lambda)$ and β.

Lemma 6. *For each $\lambda \in \mathbb{R}$, $\mu_c(\lambda)$ and β comprise a Fredholm pair, with* $\dim \mu_c(\lambda) \cap \beta = \dim \ker(\mathcal{L}_c - \lambda)$.

Proof. We follow the proof of Lemma 3.4 in [6]. Let P_β denote the orthogonal projection onto the boundary subspace $\beta \subset \mathcal{H}$, and $P_\beta^\perp = I - P_\beta$ the complementary projection. Suppose $u \in K_c(\lambda)$. Letting $v = u$ in Lemma 1, we obtain

$$\int_\Omega \left[|\nabla u|^2 + (V - \lambda) u^2 \right] = \int_{\partial\Omega} u \frac{\partial u}{\partial \nu}$$

and so

$$\|u\|^2_{H^1(\Omega)} \leq C \left(\|u\|^2_{L^2(\Omega)} + \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \right)$$

for some constant $C = C(\lambda)$. Next, from [8, Lemma 3.7] we have

$$\int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \leq C \left(\epsilon \|u\|^2_{H^1(\Omega)} + \epsilon^{-1} \left\| P_\beta^\perp (\mathrm{tr}\, u) \right\|^2_{\mathcal{H}} \right)$$

for any $\epsilon > 0$, from which we obtain

$$\|u\|_{H^1(\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \left\| P_\beta^\perp (\mathrm{tr}\, u) \right\|_{\mathcal{H}} \right). \tag{26}$$

It now follows from Peetre's Lemma [26, Lemma 3] (cf. the proof of [6, Lemma 3.4]) that $P_\beta^\perp \big|_{\mu_c(\lambda)} \colon \mu_c(\lambda) \to \mathcal{H}$ is Fredholm, hence $\mu_c(\lambda)$ and β are a Fredholm pair, by [13, Proposition 2.27].

Finally, the fact that $\dim \mu_c(\lambda) \cap \beta = \dim \ker(\mathcal{L}_c - \lambda)$ follows from the definitions of both spaces and the fact that the trace map is injective. □

Remark 3. In general, the statement of Lemma 3 of [26] is that an estimate of the form (26) only implies that $\mu_c(\lambda) + \beta$ is closed and $\mu_c(\lambda) \cap \beta$ is finite dimensional. However, since $\mu_c(\lambda)$ and β are already known to be Lagrangian (by Lemmas 4 and 5), we have

$$(\mu_c(\lambda) + \beta)^\perp = \mu_c(\lambda)^\perp \cap \beta^\perp = J(\mu_c(\lambda)) \cap J\beta = J(\mu_c(\lambda) \cap \beta).$$

Since J is an isomorphism and $\mu_c(\lambda) + \beta$ is closed, this implies $\operatorname{codim}(\mu_c(\lambda) + \beta) = \dim(\mu_c(\lambda) \cap \beta) < \infty$, so $\mu_c(\lambda)$ and β are indeed a Fredholm pair.

Combining Lemmas 4, 5 and 6, we see that $\mu_c(\lambda)$ is a smooth path in $\mathcal{F}\Lambda_\beta(\mathcal{H})$, so its Maslov index is well defined. In the final lemma of this section we relate this Maslov index to the Morse index of the constrained operator $\mathcal{L}_c$, thus completing the proof of Theorem 1.

Lemma 7. *There exists $\lambda_\infty < 0$ such that $\mu_c(\lambda) \cap \beta = \{0\}$ for all $\lambda \le \lambda_\infty$, and*

$$n(\mathcal{L}_c) = -\operatorname{Mas}\left(\mu_c\big|_{[\lambda_\infty,0]}; \beta\right).$$

Proof. We first prove the existence of λ_∞. Suppose $\mu_c(\lambda) \cap \beta \neq \{0\}$, so the constrained eigenvalue problem has a nontrivial solution. That is, there exists $u \in H^1(\Omega) \cap L^2_c(\Omega)$ satisfying $P(L - \lambda)u = 0$, with the given boundary conditions. It follows that

$$\lambda \int_\Omega u^2 = \int_\Omega \left[|\nabla u|^2 + V u^2 \right],$$

so $\lambda \ge \inf V$. Therefore any $\lambda_\infty < \inf V$ will suffice.

We next claim that the path $\mu_c(\lambda)$ is negative definite, in the sense that is always passes through β in the same direction. This means the Maslov index is equal to (minus) the number of intersections of $\mu_c(\lambda)$ with β, hence

$$\operatorname{Mas}\left(\mu_c\big|_{[\lambda_\infty,0]}; \beta\right) = -\sum_{\lambda_\infty \le \lambda < 0} \dim(\mu_c(\lambda) \cap \beta) = -\sum_{\lambda < 0} \dim(\mu_c(\lambda) \cap \beta) = -n(\mathcal{L}_c).$$

The second equality follows from the fact that there are no intersections for $\lambda < \lambda_\infty$, and the third equality is just the definition of the Morse index.

It only remains to prove the claimed monotonicity of $\mu_c(\lambda)$. We do this using crossing forms, as described in Section 3.1.

Suppose $u(\lambda)$ is a smooth curve in $K_c(\lambda)$, so $D(u, v) = \lambda \langle u, v \rangle$ for all $v \in H^1_0(\Omega) \cap L^2_c(\Omega)$, hence $D(u', v) = \langle \lambda u' + u, v \rangle$, where $'$ denotes differentiation with respect to λ. It follows from Lemma 1 that

$$\begin{aligned}
\omega\left(\operatorname{tr} u, \operatorname{tr} u'\right) &= \int_{\partial\Omega} \left(u \frac{\partial u'}{\partial \nu} - u' \frac{\partial u}{\partial \nu} \right) \\
&= \left(D(u', u) - \langle \lambda u' + u, u \rangle \right) - \left(D(u, u') - \lambda \langle u, u' \rangle \right) \\
&= -\int_\Omega u^2
\end{aligned}$$

and so the path is negative definite as claimed. $\square$

3.4. Equivalence of Hypotheses 3 and 4

Before proving the constrained Morse index theorem, we verify the claim made in the Introduction about Hypotheses 3 and 4. As usual, we assume Hypothesis 1, which implies $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.

First suppose Hypothesis 4 is satisfied. Part (iii) implies $L^2_c(\Omega)^\perp$ is compactly embedded in $L^2(\Omega)$, and hence is finite dimensional. Therefore it can be written as $L^2_c(\Omega)^\perp = \operatorname{span}\{\phi_1, \ldots, \phi_m\}$, with each $\phi_i \in H^1(\Omega)$. Since $L^2_c(\Omega)$ is closed, Hypothesis 3 follows.

Conversely, suppose Hypothesis 3 is satisfied. It follows immediately that $L^2_c(\Omega)$ is closed and has finite codimension. The remainder of Hypothesis 4 is then a consequence of the following lemma.

Lemma 8. *If $L^2_c(\Omega)^\perp = \operatorname{span}\{\phi_1, \ldots, \phi_m\}$ for functions $\phi_i \in H^1(\Omega)$, then*

$$\gamma\left(H^1(\Omega) \cap L^2_c(\Omega)\right) = H^{1/2}(\partial\Omega),$$

and

$$\overline{H^1_0(\Omega) \cap L^2_c(\Omega)} = L^2_c(\Omega).$$

Proof. Let χ_ϵ be a smooth cutoff function on Ω that vanishes on the boundary and satisfies $\chi_\epsilon(x) = 1$ whenever $\operatorname{dist}(x, \partial\Omega) > \epsilon$. We assume without loss of generality that the $\{\phi_i\}$ are orthonormal.

Suppose $f \in H^{1/2}(\partial\Omega)$ is given. Since $\gamma\colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is surjective (see, for instance, [25, Lemma 3.37]), there exists $u \in H^1(\Omega)$ with $\gamma u = f$. Now define

$$u_c = u + \chi_\epsilon \sum_{i=1}^m \alpha_i \phi_i$$

with coefficients $\alpha_1, \ldots, \alpha_m$ to be determined. Since χ_ϵ vanishes on the boundary, u_c satisfies $\gamma u_c = \gamma u = f$. Moreover, $u_c \in L^2_c(\Omega)$ if and only if

$$\sum_{i=1}^m \alpha_i \int_\Omega \chi_\epsilon \phi_i \phi_j = -\int_\Omega u\phi_j$$

for each j. This is a linear equation for the coefficients $\{\alpha_i\}$, and will have a solution if the matrix

$$M_{ij}(\epsilon) = \int_\Omega \chi_\epsilon \phi_i \phi_j$$

is invertible. The dominated convergence theorem implies

$$\lim_{\epsilon \to 0} \int_\Omega \chi_\epsilon \phi_i \phi_j = \int_\Omega \phi_i \phi_j = \delta_{ij},$$

hence $M_{ij}(\epsilon)$ is invertible for sufficiently small ϵ.

The second claim follows from a similar construction. Suppose $u \in L^2_c(\Omega)$, so there exists a sequence (u_k) in $H^1_0(\Omega)$ with $u_k \to u$ in $L^2(\Omega)$. To obtain an approximating sequence in $H^1_0(\Omega) \cap L^2_c(\Omega)$, we replace each u_k by

$$\tilde{u}_k = u_k + \chi_\epsilon \sum_{i=1}^m \alpha_i^k \phi_i,$$

where (α_i^k) solve the linear equation

$$\sum_{i=1}^m M_{ij}(\epsilon)\alpha_i^k = -\int_\Omega u_k \phi_j$$

and ϵ is chosen small enough to ensure $M_{ij}(\epsilon)$ is invertible. The fact that $u_k \to u$ implies

$$\int_\Omega u_k \phi_j \longrightarrow \int_\Omega u \phi_j = 0,$$

for each j, hence $\alpha_i^k \to 0$ as $k \to \infty$. It follows that $\tilde{u}_k - u_k \to 0$, and so $\tilde{u}_k \to u$. □

4. The constrained Morse index theorem

Now consider a one-parameter family of domains $\{\Omega_t\}_{a \le t \le b}$ in $\mathbb{R}^n$. For simplicity we will assume that each Ω_t has smooth boundary, and that the domains are increasing and varying smoothly in time. More precisely, we suppose that there is a fixed domain $\Omega = \Omega_b \subset \mathbb{R}^n$, with smooth boundary, and a one-parameter family of diffeomorphisms $\varphi_t \colon \Omega \to \Omega_t$ such that the outward normal component of the velocity

$$\nu_{\varphi_t(x)} \cdot \frac{d}{dt}\varphi_t(x)$$

is strictly positive for each $x \in \partial\Omega$ and $t \in [a, b]$. (Here $\nu_{\varphi_t(x)}$ denotes the outward unit normal to $\partial\Omega_t$ at the point $\varphi_t(x)$.) See [8, §2.2] for a description of the nonsmooth case for the unconstrained problem.

The idea is to define a Maslov index with respect to the t parameter, then use a homotopy argument to relate this to the Maslov index defined in Section 3, and hence to the Morse index of the constrained operator. There is some freedom in how one chooses the constraints on Ω_t in relation to the original constraints on Ω. Our choice is motivated by the requirement that the resulting path be monotone in t, which is necessary for the proof of Theorem 3.

4.1. The general index theorem

Let $E_t \colon L^2(\Omega_t) \to L^2(\Omega)$ denote the operator of extension by zero, and define

$$L^2_c(\Omega_t) = \left\{ u \in L^2(\Omega_t) : E_t u \in L^2_c(\Omega) \right\}. \tag{27}$$

In other words, $L^2_c(\Omega_t)$ consists of functions whose extension by zero satisfies the constraints on the larger domain Ω. To motivate this choice, suppose $L^2_c(\Omega) = \{\phi\}^\perp$ for some function ϕ. Then for any function $u \in L^2(\Omega_t)$ we have

$$u \in L^2_c(\Omega_t) \iff E_t u \in L^2_c(\Omega) \iff \int_\Omega (E_t u)\phi = 0 \iff \int_{\Omega_t} u\phi = 0,$$

and so $L^2_c(\Omega_t) = \{\phi|_{\Omega_t}\}^\perp$. We then define $\mathcal{L}^t_c$ to be the selfadjoint operator corresponding to the bilinear form (7) with form domain $H^1(\Omega_t) \cap L^2_c(\Omega_t)$ (for the Neumann problem) or $H^1_0(\Omega_t) \cap L^2_c(\Omega_t)$ (for the Dirichlet problem). Our index theorem computes the spectral flow of the family $\{\mathcal{L}^t_c\}$, i.e. the difference in Morse indices, $n(\mathcal{L}^b_c) - n(\mathcal{L}^a_c)$. To describe this, it is convenient to reformulate the problem in terms of a t-dependent family of bilinear forms on a fixed domain.

To that end, we define the bilinear form

$$D_t(u, v) = \int_{\Omega_t} \left[\nabla(u \circ \varphi_t^{-1}) \cdot \nabla(v \circ \varphi_t^{-1}) + V(u \circ \varphi_t^{-1})(v \circ \varphi_t^{-1})\right] \tag{28}$$

for $u, v \in H^1(\Omega)$, and define the subspace

$$L^2_{c,t}(\Omega) = \{u \circ \varphi_t : u \in L^2_c(\Omega_t)\} \subset L^2(\Omega).$$

Suppose $\phi \in L^2_c(\Omega)^\perp$. Then for any $u \in L^2_{c,t}(\Omega)$ we have $u \circ \varphi_t^{-1} \in L^2_c(\Omega_t)$, hence

$$0 = \int_{\Omega_t} (u \circ \varphi_t^{-1})\phi = \int_\Omega u(\phi \circ \varphi_t) \det(D\varphi_t).$$

In other words, the rescaled constraint space in $L^2(\Omega)$ is

$$L^2_{c,t}(\Omega)^\perp = \left\{\det(D\varphi_t)(\phi \circ \varphi_t) : \phi \in L^2_c(\Omega)^\perp\right\}. \tag{29}$$

This explicit description of the rescaled constraint functions will be used below in the crossing form calculation for the Dirichlet problem. However, in order to define the Maslov index, it is convenient to transform the problem into one with constraints independent of t. This motivates the following.

Hypothesis 5. For each $t_0 \in [a, b]$ there exists a smooth family of operators $T_t : H^1(\Omega) \cap L^2_{c,t_0}(\Omega) \to H^1(\Omega)$, defined in a neighborhood of t_0, with $\operatorname{ran}(T_t) = H^1(\Omega) \cap L^2_{c,t}(\Omega)$ and $\|T_t u\|_{H^1(\Omega)} \geq c\|u\|_{H^1(\Omega)}$ for some $c > 0$.

It is easy to see that this hypothesis is satisfied for a single constraint function, as in Theorem 3.

Lemma 9. *Suppose $L^2_c(\Omega) = \{\phi\}^\perp$ for some $\phi \in H^1(\Omega)$. If $\int_{\Omega_t} \phi^2 > 0$ for all t, then $L^2_{c,t}(\Omega)$ satisfies Hypothesis 5.*

Proof. Since $\phi \in H^1(\Omega)$, the t-dependent constraint functions $\phi_t = \det(D\varphi_t)(\phi \circ \varphi_t)$ found in (29) form a smooth curve in $H^1(\Omega)$, so the $L^2(\Omega)$ normalized constraints $\hat{\phi}_t$ do as well. Now define

$$T_t u = u - P_t u,$$

where $P_t u$ denotes the L^2 orthogonal projection onto $\hat{\phi}_t$. It follows from the definition that $T_t \colon H^1(\Omega) \to H^1(\Omega)$ is bounded, and in fact is smooth with respect to t.

We now fix $t_0 \in [a, b]$ and consider the restriction to $H^1(\Omega) \cap L^2_{c,t_0}$, which we again denote T_t. Let $u \in H^1(\Omega) \cap L^2_{c,t_0}$, so $\langle u, \hat{\phi}_{t_0} \rangle = 0$. We thus compute

$$\begin{aligned} \|P_t u\|_{H^1(\Omega)} &= \left|\langle u, \hat{\phi}_t - \hat{\phi}_{t_0} \rangle\right| \|\hat{\phi}_t\|_{H^1(\Omega)} \\ &\leq \|\hat{\phi}_t\|_{H^1(\Omega)} \|\hat{\phi}_t - \hat{\phi}_{t_0}\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \end{aligned}$$

so $\|P_t u\|_{H^1(\Omega)} \leq \frac{1}{2}\|u\|_{H^1(\Omega)}$ for t sufficiently close to t_0. It follows from the definition of T_t that

$$\|u\|_{H^1(\Omega)} \leq \|T_t u\|_{H^1(\Omega)} + \|P_t u\|_{H^1(\Omega)} \leq \|T_t u\|_{H^1(\Omega)} + \frac{1}{2}\|u\|_{H^1(\Omega)}$$

and so $\|T_t u\|_{H^1(\Omega)} \geq \frac{1}{2}\|u\|_{H^1(\Omega)}$ for all $u \in L^2_{c,t_0}(\Omega)$. □

In order to state the main result in this section, we observe that there is a formal differential operator L_t, and a boundary operator B_t, so that a version of Green's first identity

$$D_t(u, v) = \langle L_t u, v \rangle + \int_{\partial\Omega} (B_t u) v$$

holds if $u \in H^1(\Omega)$ and $L_t u \in L^2(\Omega)$. We thus define the space of weak solutions to the (rescaled) constrained problem

$$K_c(\lambda, t) = \left\{ u \in H^1(\Omega) \cap L^2_{c,t}(\Omega) : D_t(u, v) = \lambda \langle u, v \rangle \text{ for all } v \in H^1_0(\Omega) \cap L^2_{c,t}(\Omega) \right\},$$

and the space of Cauchy data

$$\mu_c(\lambda, t) = \{ \mathrm{tr}_t\, u : u \in K_c(\lambda, t) \},$$

using the rescaled trace map

$$\mathrm{tr}_t\, u := (u, B_t u)\Big|_{\partial\Omega}.$$

Theorem 4. *Let $\{\Omega_t\}$ be a smooth increasing family of domains, defined for $a \leq t \leq b$. If $L^2_c(\Omega)$ satisfies Hypotheses 3 and 5, then*

$$n(\mathcal{L}^a_c) - n(\mathcal{L}^b_c) = \mathrm{Mas}\left(\mu_c(0, \cdot)\big|_{[a,b]}; \beta \right). \tag{30}$$

In other words, the Maslov index computes the spectral flow of the constrained family $\{\mathcal{L}_c^t\}$.

Proof. The proof is a standard application of the homotopy invariance of the Maslov index. The space $K_c(\lambda, t)$ of weak solutions is defined in terms of the form D_t on $H^1(\Omega) \cap L^2_{c,t}(\Omega)$. By Hypothesis 5, in a neighborhood of each t_0 this is equivalent to a coercive form $D_t \circ T_t$ on the fixed (t-independent) domain $H^1(\Omega) \cap L^2_{c,t_0}(\Omega)$. Since $D_t \circ T_t$ is a smooth family of coercive forms, we can use the theory developed in [8] (which is reviewed in the proof of Lemma 4), to see that

$$\mu_c \colon [\lambda_\infty, 0] \times [a, b] \longrightarrow \mathcal{F}\Lambda_\beta(\mathcal{H})$$

is a smooth two-parameter family of Lagrangian subspaces.

This means the image under μ_c of the boundary of $[\lambda_\infty, 0] \times [a, b]$ is null-homotopic, hence its Maslov vanishes. Summing the four sides of the boundary with the appropriate orientation, we obtain

$$\operatorname{Mas}\left(\mu(\cdot, a)\big|_{[\lambda_\infty,0]}\right) + \operatorname{Mas}\left(\mu(0, \cdot)\big|_{[a,b]}\right) = \operatorname{Mas}\left(\mu(\lambda_\infty, \cdot)\big|_{[a,b]}\right) + \operatorname{Mas}\left(\mu(\cdot, b)\big|_{[\lambda_\infty,0]}\right).$$

The monotonicity computation in Lemma 7 shows that

$$\operatorname{Mas}\left(\mu(\cdot, t)\big|_{[\lambda_\infty,0]}\right) = -n(\mathcal{L}_c^t)$$

for any $t \in [a, b]$, and the choice of λ_∞ implies that

$$\operatorname{Mas}\left(\mu(\lambda_\infty, \cdot)\big|_{[a,b]}\right) = 0.$$

This completes the proof. □

4.2. The Dirichlet crossing form

We now complete the proof of Theorem 3 by computing the right-hand side of (30) when β is the Dirichlet subspace. This closely follows the crossing form computation in [8, §5]. In particular, it suffices to prove that the crossing form is negative definite at any crossing.

Let $\{u_t\}$ be a smooth family of solutions to the constrained problem with $\lambda = 0$, i.e. $\operatorname{tr} u_t \in K_c(0, t)$. This means $\int_\Omega u_t \phi_t = 0$ and $L_t u_t \propto \phi_t$, where

$$\phi_t = \det(D\varphi_t)(\phi \circ \varphi_t) \tag{31}$$

is the rescaled constraint function. More concretely, we can write $L_t u_t = a_t \phi_t$, where a_t depends only on t, so that

$$D_t(u_t, v) = a_t \langle \phi_t, v\rangle + \int_{\partial\Omega} (B_t u_t) v$$

for all $v \in H^1(\Omega)$.

Letting $v = u_t'$, we obtain

$$D_t(u_t, u_t') = a_t \left\langle \phi_t, u_t' \right\rangle + \int\limits_{\partial\Omega} (B_t u_t) u_t'.$$

On the other hand, differentiating with respect to t and then plugging in $v = u_t$, we obtain

$$D_t'(u_t, u_t) + D_t(u_t', u_t) = a_t' \left\langle \phi_t, u_t \right\rangle + a_t \left\langle \phi_t', u_t \right\rangle + \int\limits_{\partial\Omega} (B_t u_t)' u_t.$$

Therefore, the crossing form is

$$Q(\mathrm{tr}_t\, u_t) = \omega((\mathrm{tr}_t\, u_t), (\mathrm{tr}_t\, u_t)') = \int\limits_{\partial\Omega} (B_t u_t)' u_t - (B_t u_t) u_t' = D_t'(u_t, u_t) - 2a_t \left\langle \phi_t', u_t \right\rangle, \quad (32)$$

where we have used the fact that $\langle u_t, \phi_t \rangle = 0$, hence $\left\langle \phi_t, u_t' \right\rangle = -\left\langle \phi_t', u_t \right\rangle$.

To complete the computation we must find D_t'. Differentiating $D_t(u, u)$ for a fixed $u \in H^1(\Omega)$, we have

$$D_t'(u, u) = -2D_t\left(u, \nabla_X (u \circ \varphi_t^{-1}) \circ \varphi_t\right) + \int\limits_{\partial\Omega_t} \left[\left|\nabla (u \circ \varphi_t^{-1})\right|^2 + V \left|u \circ \varphi_t^{-1}\right|^2 \right] (X \cdot \nu_t). \quad (33)$$

Here X denotes the velocity of the flow $\{\varphi_t\}$, i.e. $\varphi_t'(x) = X(\varphi_t(x))$, and we have used the fact that

$$\frac{d}{dt}(u \circ \varphi_t^{-1}) = -\nabla_X (u \circ \varphi_t^{-1}),$$

which is obtained by writing

$$0 = \frac{d}{dt}(u \circ \varphi_t^{-1} \circ \varphi_t) = \frac{d}{dt}(u \circ \varphi_t^{-1}) \circ \varphi_t + \nabla_X (u \circ \varphi_t^{-1}) \circ \varphi_t.$$

We now assume that t is a crossing time, so $\mathrm{tr}_t\, u_t \in \beta$. Evaluating the first term on the right-hand side of (33) at $u = u_t \in H_0^1(\Omega)$, and defining $\widehat{u} = u_t \circ \varphi_t^{-1}$, we obtain

$$\begin{aligned} D_t\left(u_t, \nabla_X (u_t \circ \varphi_t^{-1}) \circ \varphi_t\right) &= D_t(u_t, (\nabla_X \widehat{u}) \circ \varphi_t) \\ &= \langle L_t u_t, (\nabla_X \widehat{u}) \circ \varphi_t \rangle + \int\limits_{\partial\Omega} (B_t u_t)(\nabla_X \widehat{u}) \circ \varphi_t \\ &= a_t \left\langle \phi_t, (\nabla_X \widehat{u}) \circ \varphi_t \right\rangle + \int\limits_{\partial\Omega_t} \left(\frac{\partial \widehat{u}}{\partial \nu_t} \right)^2 (X \cdot \nu_t). \end{aligned}$$

Since $\widehat{u}$ vanishes on $\partial\Omega_t$, the boundary term in (33) simplifies to

$$\int_{\partial\Omega_t} \left[|\nabla\widehat{u}|^2 + V|\widehat{u}|^2\right](X\cdot\nu_t) = \int_{\partial\Omega_t} \left(\frac{\partial\widehat{u}}{\partial\nu_t}\right)^2 (X\cdot\nu_t)$$

and so

$$D_t'(u_t,u_t) = -2a_t\,\langle\phi_t,(\nabla_X\widehat{u})\circ\varphi_t\rangle - \int_{\partial\Omega_t} \left(\frac{\partial u}{\partial\nu_t}\right)^2 (X\cdot\nu_t)\,d\mu_t. \tag{34}$$

Combining this with (32), we find that

$$Q(\mathrm{tr}_t\,u_t) = -2a_t\,\langle\phi_t,(\nabla_X\widehat{u})\circ\varphi_t\rangle - 2a_t\left\langle\phi_t',u_t\right\rangle - \int_{\partial\Omega_t} \left(\frac{\partial\widehat{u}}{\partial\nu_t}\right)^2 (X\cdot\nu_t). \tag{35}$$

This expression for the crossing form is generally valid, in the sense that it holds for any smooth family of constraint functions $\{\phi_t\}$. We now show that our choice of ϕ_t is such that the first two terms on the right-hand side cancel, resulting in a form that is sign definite.

Differentiating (31), we obtain

$$\phi_t' = \det(D\varphi_t)\,\mathrm{div}(X)(\phi\circ\varphi_t) + \det(D\varphi_t)(\nabla_X\phi)\circ\varphi_t = \det(D\varphi_t)\,\mathrm{div}(\phi X)\circ\varphi_t.$$

On the other hand, we can use the divergence theorem, together with the fact that $\widehat{u}$ vanishes on $\partial\Omega_t$, to write

$$\begin{aligned}
\langle\phi_t,(\nabla_X\widehat{u})\circ\varphi_t\rangle &= \int_\Omega \det(D\varphi_t)(\phi\circ\varphi_t)(\nabla_X\widehat{u})\circ\varphi_t \\
&= \int_{\Omega_t} \phi\nabla_X\widehat{u} \\
&= -\int_{\Omega_t} \widehat{u}\,\mathrm{div}(\phi X) \\
&= -\int_\Omega u_t\det(D\varphi_t)\,\mathrm{div}(\phi X)\circ\varphi_t \\
&= -\left\langle u_t,\phi_t'\right\rangle.
\end{aligned}$$

Thus the first two terms on the right-hand side of (35) cancel, and the crossing form simplifies to

$$Q(\mathrm{tr}\,u_t) = -\int_{\partial\Omega_t} \left(\frac{\partial\widehat{u}}{\partial\nu_t}\right)^2 (X\cdot\nu_t). \tag{36}$$

Remark 4. The above computation readily generalizes to any finite number of constraints. The constrained equation becomes $L_t u_t = \sum a_t^i \phi_t^i$, where $\{\phi_t^i\}$ are the rescaled constraint functions, and the right-hand side of (35) simply becomes

$$-2\sum a_t^i \left[\left\langle \phi_t^i, (\nabla_X \widehat{u}) \circ \varphi_t \right\rangle + \left\langle (\phi_t^i)', u_t \right\rangle\right] - \int\limits_{\partial\Omega_t} \left(\frac{\partial \widehat{u}}{\partial \nu_t}\right)^2 (X \cdot \nu_t) = - \int\limits_{\partial\Omega_t} \left(\frac{\partial \widehat{u}}{\partial \nu_t}\right)^2 (X \cdot \nu_t),$$

where each of the terms in brackets vanishes, as in the case of a single constraint.

5. An application to the nonlinear Schrödinger equation

As a final application of Theorem 3, we study the stability of the ground state solution of the nonlinear Schrödinger equation (1). Under a mild condition on f and ω (see, for instance, [3, Appendix]), there exists an even, positive solution ϕ to

$$\phi_{xx} + f(\phi^2)\phi + \omega\phi = 0, \qquad \lim_{|x|\to\infty} \phi(x) = 0 \tag{37}$$

that is decreasing for $x > 0$. It follows that $L_-\phi = 0$, hence $n(L_-) = 0$ since $\phi > 0$ is then the ground state eigenfunction of L_-. We are interested in computing the Morse index of L_+, constrained to $(\ker L_-)^\perp = \{\phi\}^\perp$. We do this by applying Theorem 3 to the semi-infinite domain $\Omega_t = (-\infty, t)$.

Remark 5. The following computation is not, strictly speaking, an application of Theorem 3, which was only proved for bounded domains. A Morse–Maslov index theorem for semi-infinite domains recently appeared in [1]; see also [4,17,18]. Instead of focusing on technical details, we simply compute the conjugate points, and observe that a formal application of Theorem 3 yields the well-known stability criterion of Vakhitov and Kolokolov.

Differentiating (37) with respect to x, we obtain $L_+(\phi_x) = 0$. Moreover, assuming the map $\omega \mapsto \phi$ is C^1, we also have $L_+(\phi_\omega) = \phi$. As proved in [35, Appendix A], the homogeneous equation $L_+ u = 0$ has one linearly independent solution that is in L^2 as $x \to -\infty$, namely ϕ_x, and so as 0 is an isolated eigenvalue of L_+, any solution to the inhomogeneous equation $L_+ u = \phi$ that decays at $-\infty$ is of the form

$$u = A\phi_x + \phi_\omega$$

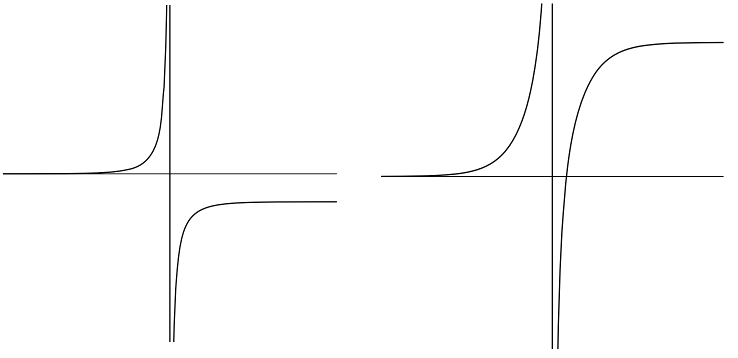
for some $A \in \mathbb{R}$. The constraint on $(-\infty, t)$ is

$$0 = \int_{-\infty}^{t} u\phi = \int_{-\infty}^{t} (A\phi\phi_x + \phi\phi_\omega) = \frac{1}{2}\left(A\phi^2(t) + \frac{\partial}{\partial\omega}\int_{-\infty}^{t} \phi^2(x)\,dx\right) \tag{38}$$

and the Dirichlet boundary condition at t is

$$A\phi_x(t) + \phi_\omega(t) = 0. \tag{39}$$

By definition, $t \in \mathbb{R}$ is a conjugate point when both (38) and (39) are satisfied.



(a) $p = 1$ (b) $p = 3$

Fig. 1. The function $c(t)$ for the power-law nonlinearity $f(\phi^2) = (p+1)\phi^{2p}$.

Now define

$$g(u) = u^{-1} \int_0^u f(v)\,dv, \quad u > 0,$$

so that $ug'(u) + g(u) = f(u)$. It follows that $\phi_x^2 + [\omega + g(\phi^2)]\phi^2$ is constant for any solution to (37). Since $\phi \in H^1(\mathbb{R})$, we have $\phi_x^2 + [\omega + g(\phi^2)]\phi^2 = 0$. Differentiating this equation with respect to ω, we find that

$$\phi_x \phi_{x\omega} + g'(\phi^2)\phi^3\phi_\omega + \frac{1}{2}\phi^2 + [\omega + g(\phi^2)]\phi\phi_\omega = 0. \tag{40}$$

Since ϕ is positive, this implies ϕ_x and ϕ_ω do not simultaneously vanish. Together with (39), this implies $\phi_x(t) \neq 0$ if t is a conjugate point, hence $A = -\phi_\omega(t)/\phi_x(t)$. Substituting this into (38), we find that t is a conjugate point precisely when it is a root of the function

$$c(t) = -\frac{\phi^2(t)\phi_\omega(t)}{\phi_x(t)} + \frac{\partial}{\partial\omega} \int_{-\infty}^t \phi^2(x)\,dx. \tag{41}$$

This function is plotted in Fig. 1 for the power law $f(\phi^2) = \phi^{2p}$. In this case it is easily verified that there is a conjugate point if and only if $p > 2$.

It is not difficult to verify that the function c has the following properties:

(i) $\lim_{t \to -\infty} c(t) = 0$;
(ii) $\lim_{t \to 0-} c(t) = \infty$;
(iii) $\lim_{t \to 0+} c(t) = -\infty$;
(iv) $\lim_{t \to \infty} c(t) = \frac{\partial}{\partial\omega} \int_{-\infty}^{\infty} \phi^2(x)\,dx$;
(v) $c'(t) > 0$ for $t \neq 0$.

In particular, for (v) we differentiate to obtain

$$c'(t) = \phi^2 \left(\frac{\phi_\omega \phi_{xx} - \phi_{\omega x}\phi_x}{\phi_x^2} \right) \Big|_{x=t}$$

for $t \neq 0$. Using (37) and (40) to compute ϕ_{xx} and $\phi_{x\omega}$ we find that

$$\phi_\omega \phi_{xx} - \phi_{\omega x}\phi_x = \frac{1}{2}\phi^2.$$

Since $\phi > 0$, this implies $c'(t) > 0$.

It follows immediately that there are no conjugate points in $(-\infty, 0)$, and there is a single conjugate point in $(0, \infty)$ if and only if the "slope" $\partial_\omega \int \phi^2$ is positive. Hence, by a formal application of Theorem 3, the slope condition determines whether or not the number of negative eigenvalues of L_+ constrained to $(\ker L_-)^\perp$ differs from the number of negative eigenvalues of L_-. As a result, the (in)stability of the resulting non-selfadjoint operator for the ground state follows from the results of Jones [19] and Grillakis [14,15], as recalled the discussion in the Introduction. Note that this also suggests that for a domain of sufficiently small volume, an unstable problem can be stabilized. As seen in the recent paper [11], this phenomenon can be seen in excited states as well and is a natural further direction for analysis involving the constrained Maslov index tools developed here.

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