

# A discontinuous Galerkin method and its error estimate for nonlinear fourth-order wave equations

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**Abstract:** In this paper, an ultra-weak local discontinuous Galerkin (UWLDG) method for a class of nonlinear fourth-order wave equations is designed and analyzed. The UWLDG method is a new DG method designed for solving partial differential equations (PDEs) with high order spatial derivatives. We prove the energy conserving property of our scheme and its optimal error estimates in the  $L^2$ -norm for the solution itself as well as for the auxiliary variables approximating the derivatives of the solution. Compatible high order energy conserving time integrators are also proposed. The theoretical results are confirmed by numerical experiments.

**Keywords:** nonlinear fourth-order wave equation, discontinuous Galerkin method, energy conserving, error estimates.

## 1 Introduction

In recent years, many numerical methods have been defined and analyzed for the wave equations [2, 4, 8, 12, 13, 23, 38, 41, 42]. The nonlinear fourth-order wave equations arise commonly from the studies of vibration of beams and thin plates [25]. In this paper, we are interested in the numerical methods for a class of nonlinear fourth-order wave equations [7, 24, 27–32, 40],

$$u_{tt} + \Delta^2 u + u + f(u) = 0, \quad \mathbf{x} \in \Omega, \quad t \in [0, T], \quad (1.1)$$

with the initial conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = v_0(\mathbf{x}). \quad (1.2)$$

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For the sake of simplicity, we only consider the periodic boundary condition. The solution  $u = u(\mathbf{x}, t)$ ,  $\mathbf{x} \in \Omega \in \mathbb{R}^d$ ,  $d = 1, 2$ ,  $t \in [0, T]$  is a real-valued function, and the initial conditions  $u_0$  and  $v_0$  are assumed to be as smooth as necessary. Levandosky [30] proved that our problem (1.1) admits a unique local solution for nonlinearity  $f(u)$  which satisfies

$$f(0) = 0; \tag{1.3}$$

$$f \in C^1(\mathbb{R}) \text{ and } |f'(u)| \leq c|u|^{p-1}, \text{ for } 1 < p \leq 2^{**} - 1, \tag{1.4}$$

where  $2^{**} = \infty$  for  $1 \leq d \leq 4$  and  $2^{**} = \frac{2d}{d-4}$  for  $d \geq 5$ , denotes the critical exponent for the embedding of  $H^2(\mathbb{R}^d)$  into  $L^q(\mathbb{R}^d)$ ,  $2 \leq q \leq 2^{**}$ .

There are many numerical methods proposed in the literature for solving the fourth-order equations [1, 3, 5, 6, 22, 26, 33]. In [1], Achouri designed a second-order conservative finite difference scheme for the two-dimensional fourth-order nonlinear wave equation. The mixed finite elements for the fourth-order wave equations also have been studied by He et al. in [26]. They considered mixed finite element method with explicit and implicit discretization in time and derived the optimal error estimate in the  $L^2$  norm. In [5, 6], Baccouch applied the local discontinuous Galerkin (LDG) method for the fourth-order Euler-Bernoulli partial differential equation (PDE) in one dimension, including superconvergence analysis and a posterior error estimate.

We consider an ultraweak-local discontinuous Galerkin (UWLDG) method introduced in [36] for (1.1). The DG method is a class of finite element methods using completely discontinuous basis functions. The first DG method was introduced in 1973 by Reed and Hill [34] in the framework of neutron transport. It was later developed for time-dependent nonlinear hyperbolic conservation laws, coupled with the Runge-Kutta time discretization, by Cockburn et al. [14–16]. Since then, the DG method has been intensively studied and successfully applied to various problems in a wide range of applications due to its flexibility with meshing, its compactness and its high parallel efficiency. The UWLDG method is a discontinuous Galerkin method designed for PDEs with high order spatial derivatives, which combines the advantage of the LDG method and the ultra-weak DG (UWDG) method. The idea of the LDG method [17, 18, 38, 39] is to rewrite the equations with higher order spatial derivatives into a first order system, then apply the DG method to this system and design suitable numerical fluxes to ensure stability. The UWDG method [9] is based on repeated integration by parts to move all spatial derivatives to the test function in the weak formulation, and to ensure stability by carefully choosing numerical fluxes. In our method, at first, we rewrite the equation (1.1) as a second-order system. Then we repeat the application of integration by parts, and choose suitable numerical fluxes to ensure stability. Compared to the LDG method, we introduce fewer auxiliary variables, thereby reducing memory and computational costs. Compared to the UWDG method, we do not need any internal penalty terms to ensure stability.

We define the energy

$$E_u = \int_{\Omega} \left( \frac{1}{2}(u_t)^2 + \frac{1}{2}(\Delta u)^2 + \frac{1}{2}u^2 + F(u) \right) dx,$$

where  $F'(s) = f(s)$  and  $F(0) = 0$ . For the equation (1.1)  $E_u$  is a constant. Therefore, we would like to design a numerical method that conserves the energy  $E_u$ . Energy conserving DG methods for wave equations have been developed in [11, 19–21, 37]. Recently, Chou et al. [11, 37] developed an optimal energy-conserving local DG method for multi-dimensional second-order wave equation in heterogeneous media. Later, Fu and Shu [20] proposed an optimal energy conserving DG method for linear symmetric hyperbolic systems on general unstructured meshes. They proved a priori optimal error estimates for the semi-discrete scheme in one dimension, and also in multi-dimensions for Cartesian meshes when using tensor-product polynomials. They also proposed an energy-conserving ultra-weak DG method for the generalized Korteweg-de Vries (KdV) equations in one dimension [21], and proved its optimal error estimate. In this work, we design an optimally convergent energy-conserving method for the nonlinear fourth-order equations. We choose the alternating fluxes, and prove that the energy is conserved both in one-dimensional and two-dimensional cases. We also prove the optimal error estimates in the  $L^2$ -norm for the solution itself as well as for the auxiliary variables.

The organization of the paper is as follows. In Section 2, we introduce some notations and the UWLDG method. In Section 3, the energy conserving property of our scheme will be discussed. In Section 4, we will introduce some projections and give the optimal error estimates in the  $L^2$ -norm for one-dimensional and two-dimensional cases. Time discretization will be shown in the Section 5. The theoretical results are confirmed numerically in Section 6. In Section 7, we give some concluding remarks.

## 2 The UWLDG scheme

### 2.1 Notations

Let us introduce some notations. Throughout this paper, we adopt standard notations for the Sobolev spaces such as  $W^{m,q}(D)$  on the subdomain  $D \subset \Omega$  equipped with the norm  $\|\cdot\|_{W^{m,q}(D)}$ . If  $D = \Omega$ , we omit the index  $D$ ; and if  $q = 2$ , we set  $W^{m,q}(D) = H^m(D)$ ,  $\|\cdot\|_{W^{m,q}(D)} = \|\cdot\|_{H^m(D)}$ ; and we use  $\|\cdot\|_D$  to denote the  $L^2$  norm in  $D$ .

Let  $\Omega_h$  denote a tessellation of  $\Omega$  with shape-regular elements  $K$ , and the union of the boundary faces of elements  $K \in \Omega_h$ , denoted as  $\partial\Omega = \bigcup_{K \in \Omega_h} \partial K$ . We denote the diameter of  $K$  by  $h_K$ , and set  $h = \max_K h_K$ . For example, in the one-dimensional case,  $K$  is a subinterval; in the two-dimensional case,  $K$  is a rectangle for Cartesian meshes. The finite element space with the mesh  $\Omega_h$  is of the form

$$W_h = \{\eta \in L^2(\Omega) : \eta|_K \in \mathcal{Q}^k(K), \forall K \in \Omega_h\},$$

where  $\mathcal{Q}^k(K)$  is the space of tensor product of polynomials of degree at most  $k \geq 0$  in each variable defined on  $K$ . In the one-dimensional case,  $\mathcal{Q}^k(K) = \mathcal{P}^k(K)$  which is the space of polynomial of degree at most  $k \geq 0$  on  $K$ .

For any  $v \in W_h$ , in the one-dimensional case,  $\Omega_h = \bigcup_{j=1}^N [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$  and

$$K = I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}), \quad j \in \mathcal{Z}_N = \{1, 2, \dots, N\},$$

denote the cells and cell centers, respectively. We use  $v_{j+\frac{1}{2}}^+$  and  $v_{j+\frac{1}{2}}^-$  to denote the right and left limit values of  $v$  at  $x_{j+\frac{1}{2}}$ , respectively. As usual, the average and the jump of the function  $v$  at  $x_{j+\frac{1}{2}}$  are denoted as

$$\{\!\!\{v\}\!\!\}_{j+\frac{1}{2}} = \frac{1}{2}(v_{j+\frac{1}{2}}^+ + v_{j+\frac{1}{2}}^-), \quad \llbracket v \rrbracket_{j+\frac{1}{2}} = v_{j+\frac{1}{2}}^+ - v_{j+\frac{1}{2}}^-,$$

respectively. In the two-dimensional case, we associate to this partition  $\Omega_h$  the set of all faces  $\Gamma_h$ . Let  $e \in \Gamma_h$  be an edge shared by two elements  $K_L$  and  $K_R$ , (we refer to [39] for a proper definition of “left” and “right” in our context, for rectangular meshes these are the usual left and bottom directions denoted as “left” and right and top directions denoted as “right”). The normal vectors  $\nu_L$  and  $\nu_R$  on the edge  $e$  point exterior to  $K_L$  and  $K_R$  respectively. Assuming  $\psi$  is a function defined on  $K_L$  and  $K_R$ , let  $\psi^-$  denote  $(\psi|_{K_L})|_e$  and  $\psi^+$  denote  $(\psi|_{K_R})|_e$ , the left and right traces, respectively. We denote the jump and the average of  $\varphi$  on the edge  $e$  by

$$\llbracket \varphi \rrbracket = \varphi^+ - \varphi^-, \quad \{\!\!\{\varphi\}\!\!\} = \frac{1}{2}(\varphi^+ + \varphi^-).$$

## 2.2 The UWLDG method

In this subsection, we will define the semi-discrete DG method for the nonlinear wave equation (1.1). First of all, we rewrite the equation as a second-order system:

$$u_{tt} + \Delta w + u + f(u) = 0, \tag{2.1}$$

$$w - \Delta u = 0. \tag{2.2}$$

Then the discontinuous Galerkin method is defined as follows: find  $u_h, w_h \in W_h$ , such that for all  $\varphi, \psi \in W_h$  we have

$$((u_h)_{tt}, \varphi)_K + (w_h, \Delta \varphi)_K + \langle \widetilde{\nabla} w_h \cdot \mathbf{n}, \varphi \rangle_{\partial K} - \langle \widetilde{w}_h, \nabla \varphi \cdot \mathbf{n} \rangle_{\partial K} + (u_h, \varphi)_K + (f(u_h), \varphi)_K = 0, \tag{2.3}$$

$$(w_h, \psi)_K - (u_h, \Delta \psi)_K - \langle \widehat{\nabla} u_h \cdot \mathbf{n}, \psi \rangle_{\partial K} + \langle \widehat{u}_h, \nabla \psi \cdot \mathbf{n} \rangle_{\partial K} = 0. \tag{2.4}$$

Here  $\mathbf{n}$  denotes the outward unit vector to  $\partial K$ , and

$$(\varphi, \psi)_K := \int_K \varphi(\mathbf{x})\psi(\mathbf{x})d\mathbf{x}, \quad \langle \varphi, \nabla\psi \cdot \mathbf{n} \rangle := \int_{\partial K} \varphi(\mathbf{x})(\nabla\psi(\mathbf{x}) \cdot \mathbf{n})d\gamma,$$

for any  $\varphi, \psi \in H^1(\Omega)$ . The tilde terms  $\widetilde{\nabla}w_h$  and  $\widetilde{w}_h$ , and the hat terms  $\widehat{\nabla}u_h$ ,  $\widehat{u}_h$  are cell boundary terms obtained from integration by parts, and they are the so-called numerical fluxes. To complete the definition of the DG scheme we need to define the numerical fluxes  $\widehat{u}_h$ ,  $\widehat{\nabla}u_h$ ,  $\widetilde{w}_h$ ,  $\widetilde{\nabla}w_h$ . Here, we choose the alternating fluxes [36]:

$$\widehat{u}_h = u_h^+, \quad \widehat{\nabla}u_h = (\nabla u_h)^+, \quad \widetilde{w}_h = w_h^-, \quad \widetilde{\nabla}w_h = (\nabla w_h)^-; \quad (2.5)$$

or

$$\widehat{u}_h = u_h^-, \quad \widehat{\nabla}u_h = (\nabla u_h)^-, \quad \widetilde{w}_h = w_h^+, \quad \widetilde{\nabla}w_h = (\nabla w_h)^+; \quad (2.6)$$

or

$$\widehat{u}_h = u_h^-, \quad \widehat{\nabla}u_h = (\nabla u_h)^+, \quad \widetilde{w}_h = w_h^-, \quad \widetilde{\nabla}w_h = (\nabla w_h)^+; \quad (2.7)$$

or

$$\widehat{u}_h = u_h^+, \quad \widehat{\nabla}u_h = (\nabla u_h)^-, \quad \widetilde{w}_h = w_h^+, \quad \widetilde{\nabla}w_h = (\nabla w_h)^-. \quad (2.8)$$

It is crucial that  $\widehat{w}_h$  and  $\widehat{\nabla}u_h$  come from the opposite sides, and  $\widetilde{\nabla}w_h$  and  $\widetilde{u}_h$  come from the opposite sides (alternating fluxes).

**Remark 2.1.** For the numerical fluxes, we can also take the general case,

$$\begin{aligned} \widehat{u}_h &= \{u_h\} + \alpha_1[[u_h]] + \beta_1[[\nabla u_h]], \quad \alpha_1, \beta_1 \in \mathbb{R}, \\ \widehat{\nabla}u_h &= \{[\nabla u_h]\} + \alpha_2[[\nabla u_h]] + \beta_2[[u_h]], \quad \alpha_2, \beta_2 \in \mathbb{R}, \\ \widetilde{w}_h &= \{w_h\} - \alpha_2[[w_h]] + \beta_1[[\nabla w_h]], \\ \widetilde{\nabla}w_h &= \{[\nabla w_h]\} - \alpha_1[[\nabla w_h]] + \beta_2[[w_h]]. \end{aligned}$$

For simplicity, in this paper we will only consider the alternating fluxes (2.5).

### 3 Energy conservation

In this section, we will demonstrate that the UWLDG scheme (2.3)-(2.4) conserves the discrete energy. Experience shows that the scheme conserving the discrete energy can often behave better, especially in long time simulation.

**Theorem 3.1.** *The energy*

$$E_h(t) = \int_{\Omega} \left( \frac{1}{2}(u_h)_t^2 + \frac{1}{2}w_h^2 + \frac{1}{2}u_h^2 + F(u_h) \right) d\mathbf{x}, \quad (3.1)$$

is conserved by the semi-discrete UWLDG method (2.3)-(2.4), with numerical fluxes (2.5)-(2.8) for all time.

*Proof.* Without loss of generality, we choose the flux (2.5). In equation (2.3), we take the test function to be  $\varphi = (u_h)_t$ :

$$\begin{aligned} & ((u_h)_{tt}, (u_h)_t)_K + (w_h, \Delta(u_h)_t)_K + \langle (\nabla w_h)^- \cdot \mathbf{n}, (u_h)_t \rangle_{\partial K} - \langle w_h^-, \nabla(u_h)_t \cdot \mathbf{n} \rangle_{\partial K} \\ & + (u_h, (u_h)_t)_K + (f(u_h), (u_h)_t)_K = 0. \end{aligned} \quad (3.2)$$

By taking the time derivative of equation (2.4), and choosing the test function  $\psi = w_h$ , we can obtain

$$((w_h)_t, w_h)_K - ((u_h)_t, \Delta w_h)_K - \langle (\nabla u_h)_t^+ \cdot \mathbf{n}, w_h \rangle_{\partial K} + \langle (u_h)_t^+, \nabla w_h \cdot \mathbf{n} \rangle_{\partial K} = 0. \quad (3.3)$$

Addition of equations (3.2) and (3.3) becomes

$$\begin{aligned} & ((u_h)_{tt}, (u_h)_t)_K + ((w_h)_t, w_h)_K + (w_h, \Delta(u_h)_t)_K - \langle w_h^-, \nabla(u_h)_t \cdot \mathbf{n} \rangle_{\partial K} + \langle (\nabla w_h)^- \cdot \mathbf{n}, (u_h)_t \rangle_{\partial K} \\ & - ((u_h)_t, \Delta w_h)_K + \langle (u_h)_t^+, \nabla w_h \cdot \mathbf{n} \rangle_{\partial K} - \langle (\nabla u_h)_t^+ \cdot \mathbf{n}, w_h \rangle_{\partial K} + (u_h, (u_h)_t)_K + (f(u_h), (u_h)_t)_K = 0. \end{aligned}$$

We define

$$B_K^1(w, \varphi) = (w, \Delta \varphi)_K - \langle w_h^-, (\nabla \varphi \cdot \mathbf{n}) \rangle_{\partial K} + \langle (\nabla w_h)^- \cdot \mathbf{n}, \varphi \rangle_{\partial K}, \quad (3.4)$$

$$B_K^2(u, \psi) = (u, \Delta \psi)_K - \langle u_h^+, (\nabla \psi \cdot \mathbf{n}) \rangle_{\partial K} + \langle (\nabla u_h)^+ \cdot \mathbf{n}, \psi \rangle_{\partial K}. \quad (3.5)$$

Then we integrate by parts, and sum over  $K$  to obtain

$$\sum_K (B_K^1(w_h, (u_h)_t) - B_K^2((u_h)_t, w_h)) = 0.$$

Therefore,

$$((u_h)_{tt}, (u_h)_t)_{\Omega_h} + ((w_h)_t, w_h)_{\Omega_h} + (u_h, (u_h)_t)_{\Omega_h} + (f(u_h), (u_h)_t)_{\Omega_h} = 0,$$

and

$$\frac{d}{dt} \int_{\Omega_h} \left( \frac{1}{2}(u_h)_t^2 + \frac{1}{2}w_h^2 + \frac{1}{2}u_h^2 + F(u_h) \right) d\mathbf{x} = 0.$$

□

## 4 Error estimates

We study the optimal error estimates for the UWLDG method defined in (2.3)-(2.4) for the equation (1.1). In subsection 4.1, we introduce some projections and inequalities that will be used in our proof. In subsection 4.2, we give the error estimate in the  $L^2$  norm.

### 4.1 Projections

In this subsection, we will introduce some projections that will be used in our analysis on different types of meshes. First, let us define the commonly used  $L^2$  projection  $P_h$ : For  $w \in L^2(\Omega_h)$  and  $\forall K \in \Omega_h$ ,  $P_h w \in W_h$

$$(P_h w - w, \xi)_K = 0, \quad \forall \xi \in \mathcal{Q}^k(K). \quad (4.1)$$

#### 4.1.1 Projection in the one-dimensional case

First, we introduce some projections in the one dimensional case. For  $k \geq 1$  we can define the projections  $P_{1h}^\pm$  into  $W_h$  which satisfy:  $\forall j \in Z_N$ , where  $Z_r = \{1, 2, \dots, r\}$ ,

$$\int_{I_j} uv_h dx = \int_{I_j} P_{1h}^\pm uv_h dx, \quad (4.2)$$

for any  $v_h \in \mathcal{P}^{k-2}(I_j)$  and

$$P_{1h}^\pm u \left( x_{j \mp \frac{1}{2}}^\pm \right) = u \left( x_{j \mp \frac{1}{2}} \right), \quad (P_{1h}^\pm u)_x \left( x_{j \mp \frac{1}{2}}^\pm \right) = u_x \left( x_{j \mp \frac{1}{2}} \right). \quad (4.3)$$

#### 4.1.2 Projection for the Cartesian mesh in the two-dimensional case

For the Cartesian mesh in the two-dimensional case, we use the tensor products of the projections in the one-dimensional case [36]. On a rectangle  $K_{i,j} = I_i \times J_j$ , for  $u \in W^{1,\infty}(\bar{K})$ , we define

$$\Pi^\pm u := (P_{1hx}^\pm \otimes P_{1hy}^\pm)u, \quad (4.4)$$

with the subscripts indicating the application of the one-dimensional operators  $P_{1h}^\pm$  with respect to the corresponding variable. **To be more specific, we shall list explicitly the formulations for  $\Pi^- u$  on a rectangular element  $K = I_i \times J_j := (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ .** We have

$$\int_K \Pi^- u(x, y) v_h(x, y) dx dy = \int_K u(x, y) v_h(x, y) dx dy, \quad (4.5a)$$

$$\int_{I_i} \Pi^- u(x, y_{j+\frac{1}{2}}^-) v_h(x, y_{j+\frac{1}{2}}^-) dx = \int_{I_i} u(x, y_{j+\frac{1}{2}}^-) v_h(x, y_{j+\frac{1}{2}}^-) dx, \quad (4.5b)$$

$$\int_{I_i} (\Pi^- u)_y(x, y_{j+\frac{1}{2}}^-) v_h(x, y_{j+\frac{1}{2}}^-) dx = \int_{I_i} u_y(x, y_{j+\frac{1}{2}}^-) v_h(x, y_{j+\frac{1}{2}}^-) dx, \quad (4.5c)$$

$$\int_{J_j} \Pi^- u(x_{i+\frac{1}{2}}^-, y) v_h(x_{i+\frac{1}{2}}^-, y) dy = \int_{J_j} u(x_{i+\frac{1}{2}}^-, y) v_h(x_{i+\frac{1}{2}}^-, y) dy, \quad (4.5d)$$

$$\int_{J_j} (\Pi^- u)_x(x_{i+\frac{1}{2}}^-, y) v_h(x_{i+\frac{1}{2}}^-, y) dy = \int_{J_j} u_x(x_{i+\frac{1}{2}}^-, y) v_h(x_{i+\frac{1}{2}}^-, y) dy, \quad (4.5e)$$

$$\Pi^- u(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-) = u(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-), \quad (4.5f)$$

$$(\Pi^- u)_x(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-) = u_x(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-), \quad (4.5g)$$

$$(\Pi^- u)_y(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-) = u_y(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-), \quad (4.5h)$$

$$(\Pi^- u)_{xy}(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-) = u_{xy}(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^-), \quad (4.5i)$$

for all  $v_h \in \mathcal{Q}^{k-2}(K)$  and  $K \in \Omega_h$ . Similarly, we can define the projection  $\Pi^+$ .

### 4.1.3 Approximation property of projections and inequalities

The projections defined above have the following approximation properties [10], for any  $u \in H^{k+1}$ :

$$\|u^e\| + h\|u^e\|_\infty + h^{\frac{1}{2}}\|u^e\|_{\Gamma_h} \leq Ch^{k+1}\|u\|_{H^{k+1}}, \quad (4.6)$$

where  $u^e = \pi_h u - u$ ,  $\pi_h = P_h$ ,  $P_{1h}^\pm$ ,  $\Pi^\pm$ , and  $C$  is a positive constant dependent on  $k$  but not on  $h$ .

## 4.2 A priori error estimate

Let us state the a priori error estimate for the nonlinear fourth-order equations (1.1). In the following analysis, without loss of generality, we choose the fluxes (2.5). Let

$$e_u = u - u_h, \quad e_w = w - w_h,$$

be the errors between the numerical and exact solutions. Since  $u$  and  $w$  clearly satisfy the scheme (2.3)-(2.4) as well, we can obtain the cell error equations: for all  $\varphi, \psi \in W_h$

$$((e_u)_{tt}, \varphi)_K + (e_w, \Delta \varphi)_K + \langle (\nabla e_w)^- \cdot \mathbf{n}, \varphi \rangle_{\partial K} - \langle (e_w)^-, \nabla \varphi \cdot \mathbf{n} \rangle_{\partial K} + (e_u, \varphi)_K + (f(u) - f(u_h), \varphi)_K = 0, \quad (4.7)$$

$$(e_w, \psi)_K - (e_u, \Delta \psi)_K - \langle (\nabla e_u)^+ \cdot \mathbf{n}, \psi \rangle_{\partial K} + \langle (e_u)^+, \nabla \psi \cdot \mathbf{n} \rangle_{\partial K} = 0. \quad (4.8)$$

We denote

$$\eta_u = u - \mathbb{P}_1 u, \quad \xi_u = u_h - \mathbb{P}_1 u, \quad \eta_w = w - \mathbb{P}_2 w, \quad \xi_w = w_h - \mathbb{P}_2 w,$$



where  $\mathbb{P}_1$  and  $\mathbb{P}_2$  are some projections onto the finite element space. For the one-dimensional case, we can choose  $(\mathbb{P}_1, \mathbb{P}_2) = (P_{1h}^+, P_{1h}^-)$ , and  $(\mathbb{P}_1, \mathbb{P}_2) = (\Pi^+, \Pi^-)$  for the two-dimensional Cartesian mesh. Taylor expansion on  $f(u)$  provides the identity

$$f(u_h) = f(u) - f'(u + \theta(u_h - u))e_u, \quad (4.9)$$

where  $\theta$  is a constant in  $[0, 1]$ . To obtain an optimal error estimates, we need some superconvergence results of  $B_K^1$  and  $B_K^2$  in two dimensions [36].

**Lemma 4.1.** *Let  $B_K^1(\eta_w, \varphi)$  and  $B_K^2(\eta_u, \psi)$  be defined by (3.4) and (3.5). Then we have for  $k \geq 1$ ,*

$$B_K^1(\eta_w, \varphi) = 0, \quad B_K^2(\eta_u, \psi) = 0, \quad \forall u, w \in \mathcal{P}^{k+2}(K), \quad \varphi, \psi \in \mathcal{Q}^k(K). \quad (4.10)$$

**Lemma 4.2.** *Let  $B_K^1(\eta_w, p)$  and  $B_K^2(\eta_u, q)$  defined by (3.4) and (3.5). Then we have*

$$|B_K^1(\eta_w, \varphi)| \leq Ch^{k+2} \|w\|_{W^{2k+4, \infty}(\Omega_h)} \|\varphi\|_{L^2(K)}, \quad (4.11)$$

$$|B_K^2(\eta_u, \psi)| \leq Ch^{k+2} \|u\|_{W^{2k+4, \infty}(\Omega_h)} \|\psi\|_{L^2(K)}, \quad (4.12)$$

where  $\varphi, \psi \in \mathcal{Q}^k(K)$  and the constant  $C$  is independent of  $h$ .

#### 4.2.1 Error estimates for the initial conditions

Since initial conditions play an important role in the proof of optimal error estimate of the UWLDG scheme (2.3)-(2.4), we need to choose suitable projections for the initial conditions. Here we have two initial conditions  $u(\mathbf{x}, 0)$  and  $u_t(\mathbf{x}, 0)$ . We take the initial condition  $u_h(x, 0) = P_{1h}^+(u(x, 0))$  and  $u_h(x, y, 0) = \Pi^+ u(x, y, 0)$  in one dimension and two dimensions respectively. For the other initial condition  $(u_h)_t(0)$  we take the standard  $L^2$  projections  $(u_h)_t(0) = P_h(u_t(0))$ , and we have following lemma.

**Lemma 4.3.** *Suppose the initial conditions of the UWLDG scheme (2.3)-(2.4) are given by*

- *One-dimensional case:*

$$u_h(x, 0) = P_{1h}^+(u(x, 0)), \quad (u_h)_t(x, 0) = P_h(u_t(x, 0)); \quad (4.13)$$

- *Two-dimensional case:*

$$u_h(x, y, 0) = \Pi^+(u(x, y, 0)), \quad (u_h)_t(x, y, 0) = P_h(u_t(x, y, 0)). \quad (4.14)$$

There holds the following estimate

$$\|\xi_u(0)\| = 0, \quad \|(\xi_u)_t(0)\| \leq Ch^{k+1}, \quad \|\xi_w(0)\| \leq Ch^{k+1}, \quad (4.15)$$

where  $C$  is a constant independent of  $h$ .

*Proof.* We choose  $t = 0$  in equation (4.8), in one dimension, due to the choice of  $u_h(x, 0)$  we have

$$(e_w(0), \psi)_K = 0,$$

we choose  $\psi = w_h(x, 0) - P_h(w(x, 0))$ , and obtain  $\|e_w(0)\| \leq Ch^{k+1}$ , and  $\|\xi_w(0)\| \leq Ch^{k+1}$ .

In two dimensions, we have

$$(e_w(0), \psi)_K = B_K^2(e_u(0), \psi),$$

by Lemma 4.2,

$$|B_K^2(e_u(0), \psi)| \leq Ch^{k+2} \|u\|_{W^{2k+4, \infty}(\Omega_h)} \|\psi\|_{L^2(K)},$$

we choose  $\psi = w_h(0) - P_h(w(0))$ , then we have

$$|(e_w(0), w_h(0) - P_h(w(0)))_{\Omega_h}| \leq Ch^{k+1} \|w_h(0) - P_h(w(0))\| \leq Ch^{k+1} \|e_w(0)\| + Ch^{2k+2},$$

therefore

$$\begin{aligned} \|e_w(0)\|^2 &= (e_w(0), w(0) - P_h(w(0))) + (e_w(0), P_h(w(0)) - w_h(0)) \\ &\leq Ch^{k+1} \|e_w(0)\| + Ch^{2k+2}, \end{aligned}$$

we obtain  $\|e_w(0)\| \leq Ch^{k+1}$ . □

#### 4.2.2 Error estimates for $t > 0$

Here we assume the nonlinear term  $f(u)$  satisfies  $|f'(u)| \leq C_f |u|^{p-1}$ ,  $p > 1$ . To estimate the error between  $u$  and  $u_h$ , we need to first estimate  $(e_u)_t = u_t - (u_h)_t$ , and we have the following lemma.

**Lemma 4.4.** *For the one-dimensional case and two-dimensional Cartesian meshes, if  $u$  and  $w$  are the smooth solutions of the equation (2.1) and (2.2), and  $u_h, w_h$  are numerical solutions of the scheme (2.3)-(2.4) with the smooth initial condition and periodic boundary condition,  $W_h$  is the space of piecewise polynomials with degree  $k \geq 1$ , then we have the following error estimate:*

$$\|(e_u)_t\| + \|e_w\| \leq Ch^{k+1} + C \int_0^t A ds, \quad (4.16)$$

where  $A = \left(1 + h^{-(p-1)d/2} \|\xi_u\|^{p-1} + h^{k(p-1)}\right) \left(\|\xi_u\| + h^{k+1}\right)$ , and  $C$  is a constant independent of  $h$ .

*Proof.* Along the same line in the proof of Theorem 3.1, we take the time derivative of the equation (4.8), and take  $\varphi = (\xi_u)_t$  and  $\psi = \xi_w$ , to obtain

$$\begin{aligned} & ((e_u)_{tt}, (\xi_u)_t)_K + (e_w, \Delta(\xi_u)_t)_K + \langle (\nabla e_w)^- \cdot \mathbf{n}, (\xi_u)_t \rangle_{\partial K} - \langle (e_w)^-, \nabla(\xi_u)_t \cdot \mathbf{n} \rangle_{\partial K} \\ & + (e_u, (\xi_u)_t)_K + (f(u) - f(u_h), (\xi_u)_t)_K = 0, \end{aligned} \quad (4.17)$$

$$((e_w)_t, \xi_w)_K - ((e_u)_t, \Delta \xi_w)_K - \langle (\nabla e_u)_t^+ \cdot \mathbf{n}, \xi_w \rangle_{\partial K} + \langle (e_u)_t^+, \nabla \xi_w \cdot \mathbf{n} \rangle_{\partial K} = 0. \quad (4.18)$$

First, we estimate the nonlinear part, for any  $v \in W_h$  we have

$$\begin{aligned} (f(u) - f(u_h), v)_{\Omega_h} &= \int_{\Omega_h} f'(u + \theta(u_h - u))(u - u_h)v d\mathbf{x} \\ &\leq C_f \int_{\Omega_h} |u + \theta(u_h - u)|^{p-1} (u - u_h)v d\mathbf{x} \\ &\leq 2^{p-1} C_f \int_{\Omega_h} (|u|^{p-1} + |e_u|^{p-1}) |u - u_h| |v| d\mathbf{x} \\ &\leq 2^{p-1} C_f \left( \|u\|_{L^\infty}^{p-1} + (\|\xi_u\|_{L^\infty} + \|\eta_u\|_{L^\infty})^{p-1} \right) (\|\eta_u\| + \|\xi_u\|) \|v\|, \end{aligned}$$

where for the first equality we have used the Taylor expansion (4.9) of  $f$ . By the Sobolev embedding results we have  $\|u\|_{L^\infty} \leq C\|u\|_{k+1}$ , for  $k > d/2 - 1$ . We also have  $\|\xi_u\|_{L^\infty} \leq h^{-d/2}\|\xi_u\|$ ,  $\|\eta_u\|_{L^\infty} \leq h^k$ . Therefore,

$$\begin{aligned} (f(u) - f(u_h), v)_{\Omega_h} &\leq C \left( 1 + h^{-(p-1)d/2} \|\xi_u\|^{p-1} + h^{k(p-1)} \right) (\|\xi_u\| + h^{k+1}) \|v\| \\ &= CA \|v\|. \end{aligned} \quad (4.19)$$

- **One-dimensional case.**

For the one-dimensional case, since  $k \geq 1$ , we can choose  $(\mathbb{P}_1, \mathbb{P}_2) = (P_{1h}^+, P_{1h}^-)$ . By the stability results and properties of the projections  $P_{1h}^\pm$  we have

$$((\xi_u)_{tt}, (\xi_u)_t)_{\Omega_h} + ((\xi_w)_t, \xi_w)_{\Omega_h} = ((\eta_u)_{tt}, (\xi_u)_t)_{\Omega_h} + ((\eta_w)_t, \xi_w)_{\Omega_h} + (e_u, (\xi_u)_t)_{\Omega_h} + (f(u) - f(u_h), (\xi_u)_t)_{\Omega_h}.$$

By (4.19), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|(\xi_u)_t\|^2 + \|\xi_w\|^2) &\leq Ch^{k+1} (\|(\xi_u)_t\| + \|\xi_w\|) + C(h^{k+1} + \|\xi_u\|) \|(\xi_u)_t\| + CA \|(\xi_u)_t\| \\ &\leq Ch^{k+1} (\|(\xi_u)_t\| + \|\xi_w\|) + C(\|\xi_u\| + A) (\|(\xi_u)_t\| + \|\xi_w\|). \end{aligned}$$

Since  $\|\xi_u\| \leq (1 + \|\xi_u\|^{p-1})\|\xi_u\| \leq A$ ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|(\xi_u)_t\|^2 + \|\xi_w\|^2) &\leq Ch^{k+1} (\|(\xi_u)_t\| + \|\xi_w\|) + CA (\|(\xi_u)_t\| + \|\xi_w\|) \\ &\leq C(h^{k+1} + A) (\|(\xi_u)_t\|^2 + \|\xi_w\|^2)^{\frac{1}{2}}. \end{aligned}$$

Combining with the initial condition (4.13), we obtain

$$\left(\|(\xi_u)_t\|^2 + \|\xi_w\|^2\right)^{\frac{1}{2}} \leq C \left( h^{k+1} + \int_0^t Ads \right),$$

where  $C$  is dependent on  $\|u_{tt}\|_{H^{k+1}}$ ,  $\|u_t\|_{H^{k+3}}$ ,  $\|u\|_{H^{k+1}}$  and independent on the mesh size  $h$ .

- **Two-dimensional case.**

For the two-dimensional Cartesian mesh, we choose  $(\mathbb{P}_1, \mathbb{P}_2) = (\Pi^+, \Pi^-)$ , and have

$$\begin{aligned} ((\xi_u)_{tt}, (\xi_u)_t)_K + B_K^1(\xi_w, (\xi_u)_t) &= ((\eta_u)_{tt}, (\xi_u)_t)_K + B_K^1(\eta_w, (\xi_u)_t) \\ &\quad + (e_u, (\xi_u)_t)_K + (f(u) - f(u_h), (\xi_u)_t)_K, \end{aligned} \quad (4.20)$$

$$((\xi_w)_t, \xi_w)_K - B_K^2((\xi_u)_t, \xi_w) = ((\eta_w)_t, \xi_w)_K - B_K^2((\eta_u)_t, \xi_w). \quad (4.21)$$

Summing over  $K$  in (4.20) and (4.21), and adding the two equations we have

$$\begin{aligned} ((\xi_u)_{tt}, (\xi_u)_t)_{\Omega_h} + ((\xi_w)_t, \xi_w)_{\Omega_h} &= ((\eta_u)_{tt}, (\xi_u)_t)_{\Omega_h} + ((\eta_w)_t, \xi_w)_{\Omega_h} + (f(u) - f(u_h), (\xi_u)_t)_{\Omega_h} \\ &\quad + (e_u, (\xi_u)_t)_{\Omega_h} + \sum_K (B_K^1(\eta_w, (\xi_u)_t) - B_K^2((\eta_u)_t, \xi_w)). \end{aligned}$$

By Lemma 4.2,

$$|B_K^1(\eta_w, \varphi)| \leq Ch^{k+2} \|w\|_{W^{2k+4, \infty}(\Omega_h)} \|\varphi\|_{L^2(K)}, \quad (4.22)$$

$$|B_K^2((\eta_u)_t, \psi)| \leq Ch^{k+2} \|u_t\|_{W^{2k+4, \infty}(\Omega_h)} \|\psi\|_{L^2(K)}, \quad (4.23)$$

where  $\varphi, \psi \in \mathcal{Q}^k(K)$  and the constant  $C$  is independent of  $h$ . Then by the Cauchy-Schwartz inequality and (4.19), we have

$$\frac{1}{2} \frac{d}{dt} (\|(\xi_u)_t\|^2 + \|\xi_w\|^2) \leq Ch^{k+1} (\|(\xi_u)_t\| + \|\xi_w\|) + C(h^{k+1} + \|\xi_u\|) \|(\xi_u)_t\| + CA \|(\xi_u)_t\|.$$

Next, by the Gronwall's inequality and choosing  $u_h(0) = \Pi^+ u(0)$ , similar to the one-dimensional case we have

$$\left(\|(\xi_u)_t\|^2 + \|\xi_w\|^2\right)^{\frac{1}{2}} \leq C \left( h^{k+1} + \int_0^t Ads \right),$$

where  $C$  is dependent on  $\|u_{tt}\|_{H^{k+1}}$ ,  $\|u_t\|_{H^{k+3}}$ ,  $\|u\|_{H^{k+1}}$ ,  $\|u_t\|_{W^{2k+4, \infty}}$ ,  $\|u\|_{W^{2k+6, \infty}}$  and independent on the mesh size  $h$ . Together with the properties of projections, we get the error estimate (4.16).  $\square$

**Theorem 4.1.** *For the one-dimensional case and two-dimensional Cartesian meshes, if  $u$  and  $w$  are the smooth solutions of the equation (2.1) and (2.2), and  $u_h, w_h$  are numerical solutions of the scheme (2.3)-(2.4) with the smooth initial conditions (4.13), (4.14) and periodic boundary condition,  $W_h$  is the space of piecewise polynomials with degree  $k \geq 1$ , then we have the following*

error estimate:

$$\|e_u\| \leq Ch^{k+1}, \quad (4.24)$$

where  $C$  is a constant independent of  $h$ , and dependent on the solution  $u$  and  $T$ .

*Proof.* Here we just give the proof for the two-dimensional Cartesian mesh, as the proof for the one-dimensional case is similar and simpler. First, by using product rule in the time derivative, we obtain

$$-((\xi_u)_t, \varphi_t)_K - B_K^1(e_w, \varphi) = ((\eta_u)_{tt}, \varphi)_K - \frac{d}{dt}((\xi_u)_t, \varphi)_K + (e_u, \varphi)_K + (f(u) - f(u_h), \varphi)_K, \quad (4.25)$$

for any fixed time  $\tau \leq T$ . We denote the time integral of the errors by

$$\begin{aligned} E_u &= \int_t^\tau e_u(s) ds, & E_u^\eta &= \int_t^\tau \eta_u(s) ds, & E_u^\xi &= \int_t^\tau \xi_u(s) ds, \\ E_w &= \int_t^\tau e_w(s) ds, & E_w^\eta &= \int_t^\tau \eta_w(s) ds, & E_w^\xi &= \int_t^\tau \xi_w(s) ds. \end{aligned}$$

Take  $\varphi = E_u^\xi$  in (4.25), then  $\varphi_t = -\xi_u$ ,

$$\begin{aligned} ((\xi_u)_t, \xi_u)_K + B_K^1(\xi_w, E_u^\xi) &= ((\eta_u)_{tt}, E_u^\xi)_K + B_K^1(\eta_w, E_u^\xi) - \frac{d}{dt}((\xi_u)_t, E_u^\xi)_K \\ &\quad + (e_u, E_u^\xi)_K + (f(u) - f(u_h), E_u^\xi)_K. \end{aligned} \quad (4.26)$$

Integrating the equation (4.8) in time from time  $t$  to  $\tau$  and taking  $\psi = \xi_w$ , we get

$$(E_w^\xi, \xi_w)_K - B_K^2(E_u^\xi, \xi_w) = (E_w^\eta, \xi_w)_K - B_K^2(E_u^\eta, \xi_w). \quad (4.27)$$

Next, we add (4.26) and (4.27), since  $\xi_w = -(E_w^\xi)_t$ , and sum over  $K$

$$\sum_K (B_K^1(\xi_w, E_u^\xi) - B_K^2(E_u^\xi, \xi_w)) = 0,$$

we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\xi_u\|^2 - \|E_w^\xi\|^2) &= ((\eta_u)_{tt}, E_u^\xi)_{\Omega_h} + \sum_K B_K^1(\eta_w, E_u^\xi) - \frac{d}{dt} ((\xi_u)_t, E_u^\xi)_{\Omega_h} \\ &\quad + (e_u, E_u^\xi)_{\Omega_h} + (f(u) - f(u_h), E_u^\xi)_{\Omega_h} + (E_w^\eta, \xi_w)_{\Omega_h} - \sum_K B_K^2(E_u^\eta, \xi_w). \end{aligned}$$

We estimate these terms one by one.

- **Estimates of  $((\eta_u)_{tt}, E_u^\xi)_{\Omega_h}$ .**

$$|((\eta_u)_{tt}, E_u^\xi)_{\Omega_h}| \leq \|(\eta_u)_{tt}\| \|E_u^\xi\| \leq Ch^{k+1} \|E_u^\xi\|,$$

where

$$\|E_u^\xi\| = \left( \int_{\Omega_h} \left( \int_t^\tau \xi_u ds \right)^2 dx \right)^{\frac{1}{2}} \leq (\tau - t)^{\frac{1}{2}} \left( \int_{\Omega_h} \int_t^\tau \xi_u^2 ds dx \right)^{\frac{1}{2}} \leq (\tau - t)^{\frac{1}{2}} \left( \int_t^\tau \|\xi_u\|^2 ds \right)^{\frac{1}{2}},$$

we obtain

$$|((\eta_u)_{tt}, E_u^\xi)_{\Omega_h}| \leq Ch^{k+1} \left( \int_t^\tau \|\xi_u\|^2 ds \right)^{\frac{1}{2}}.$$

- **Estimates of  $(E_w^\eta, \xi_w)_{\Omega_h}$ .**

By Lemma 4.4, we get

$$|(E_w^\eta, \xi_w)_{\Omega_h}| \leq \|E_w^\eta\| \|\xi_w\| \leq Ch^{k+1} \|\xi_w\| \leq Ch^{2k+2} + C \left( \int_0^\tau Ads \right)^2.$$

- **Estimates of  $\sum_K |B_K^1(\eta_w, E_u^\xi)| + \sum_K |B_K^2(E_u^\eta, \xi_w)|$ .**

One has

$$|B_K^1(\eta_w, E_u^\xi)| \leq Ch^{k+2} \|E_u^\xi\|_{L^2(K)},$$

similarly, we have

$$|B_K^2(E_u^\eta, \xi_w)| \leq Ch^{k+2} \|\xi_w\|_{L^2(K)}.$$

Then,

$$\sum_K |B_K^1(\eta_w, E_u^\xi)| + \sum_K |B_K^2(E_u^\eta, \xi_w)| \leq Ch^{2k+2} + C \int_0^\tau \|\xi_u\|^2 ds + C \left( \int_0^\tau Ads \right)^2.$$

- **Estimates of  $(e_u, E_u^\xi)_{\Omega_h}$  and  $(f(u) - f(u_h), E_u^\xi)_{\Omega_h}$ .**

By applying (4.19) we obtain

$$\begin{aligned} |(e_u, E_u^\xi)_{\Omega_h}| + |(f(u) - f(u_h), E_u^\xi)_{\Omega_h}| &\leq C \|\eta_u\| \|E_u^\xi\| + C \|\xi_u\| \|E_u^\xi\| + CA \|E_u^\xi\| \\ &\leq Ch^{2k+2} + C \int_0^\tau \|\xi_u\|^2 ds + C \|\xi_u\|^2 + CA \left( \int_0^\tau \|\xi_u\|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Combining these inequalities together, we have

$$\frac{1}{2} \frac{d}{dt} (\|\xi_u\|^2 - \|E_w^\xi\|^2) \leq Ch^{2k+2} + C \int_0^\tau \|\xi_u\|^2 ds + C \|\xi_u\|^2$$

$$+ C \left( \int_0^\tau Ads \right)^2 + CA \left( \int_0^\tau \|\xi_u\|^2 ds \right)^{\frac{1}{2}} - \frac{d}{dt} \left( (\xi_u)_t, E_u^\xi \right)_{\Omega_h}.$$

Integrating with respect to time in the above equation from 0 to  $\tau$ ,

$$\frac{1}{2} \|\xi_u(\tau)\|^2 - \frac{1}{2} \|\xi_u(0)\|^2 + \frac{1}{2} \|E_w^\xi(0)\|^2 \leq Ch^{2k+2} + C \left( \int_0^\tau Ads \right)^2 + C \int_0^\tau \|\xi_u\|^2 ds + ((\xi_u)_t(0), E_u^\xi(0))_{\Omega_h}.$$

Hence, by choosing initial conditions (4.14) and Lemma 4.3, we have

$$\frac{1}{2} \|\xi_u(\tau)\|^2 \leq Ch^{2k+2} + C \int_0^\tau \|\xi_u\|^2 ds + C \left( \int_0^\tau Ads \right)^2,$$

and

$$\frac{1}{2} \left( \frac{\|\xi_u\|}{h^{k+1}} \right)^2 \leq C + C \int_0^\tau \left( \frac{\|\xi_u\|}{h^{k+1}} \right)^2 ds + C \int_0^\tau \left( \frac{A}{h^{k+1}} \right)^2 ds, \quad (4.28)$$

where

$$\left( \frac{A}{h^{k+1}} \right)^2 = \left( 1 + h^{(k+1-d/2)(p-1)} \left( \frac{\|\xi_u\|}{h^{k+1}} \right)^{p-1} + h^{k(p-1)} \right)^2 \left( \frac{\|\xi_u\|}{h^{k+1}} + 1 \right)^2.$$

We choose  $h < 1$ , then  $\varepsilon = h^{(k+1-d/2)(p-1)} < 1$  and

$$\begin{aligned} \left( \frac{A}{h^{k+1}} \right)^2 &\leq C \left( 1 + \varepsilon \left( \frac{\|\xi_u\|}{h^{k+1}} \right)^{p-1} \right)^2 \left( \frac{\|\xi_u\|}{h^{k+1}} + 1 \right)^2 \\ &\leq C \left( 1 + \varepsilon^2 \left( \frac{\|\xi_u\|}{h^{k+1}} \right)^{2p-2} \right) \left( \left( \frac{\|\xi_u\|}{h^{k+1}} \right)^2 + 1 \right). \end{aligned} \quad (4.29)$$

Next, we denote

$$D = \left( \frac{\|\xi_u\|}{h^{k+1}} \right)^2, \quad (4.30)$$

and  $D$  satisfies

$$D \leq C \int_0^\tau (D + (1 + \varepsilon^2 D^{p-1})(D + 1)) ds.$$

We can prove there exist a constant  $C$  such that satisfies  $D \leq C$  (we provide the proof in Appendix A). That is

$$\|\xi_u\| \leq Ch^{k+1},$$

we can conclude

$$\|e_u\| \leq Ch^{k+1},$$

where the constant  $C$  is independent of  $h$ .  $\square$

**Corollary 4.1.** *For the one-dimensional case and two-dimensional Cartesian meshes, if  $u$  and  $w$  are the smooth solutions of the equation (2.1) and (2.2), and  $u_h, w_h$  are numerical solutions of the scheme (2.3)-(2.4) with smooth initial condition and periodic boundary condition,  $W_h$  is the space of piecewise polynomials with degree  $k \geq 1$ , then we have the following error estimate:*

$$\|(e_u)_t\| + \|e_w\| \leq Ch^{k+1}, \quad (4.31)$$

where  $C$  is independent on  $h$ .

*Proof.* Combining Theorem 4.1 and Lemma 4.4, we can easily get (4.31).  $\square$

## 5 Time discretization

In this section, we consider the fully discrete method of scheme (2.3)-(2.4). We use the UWLDG method for the spacial discretization, it can be of high order accuracy. Therefore, we also would like to introduce an explicit, energy conserving, high order time stepping method. As in LDG method, the auxiliary variable  $w_h$  in our method could be solved in terms of  $u_h$  in an element-by-element fashion. After eliminating  $w_h$ , we can get a linear second-order ordinary differential system as follows:

$$M\ddot{u}_h(t) = Au_h(t).$$

Next, we consider a fourth-order time discretization. Here  $0 \leq t_0 < t_1 < t_2 < \dots < t_N = T$  is a partition for the time travel  $[0, T]$  with the uniform time step  $\Delta t = t_n - t_{n-1}$ . Then, a fourth-order accuracy fully discrete approximation  $u_h^n$  to  $u(\cdot, t_n)$  is constructed as follows [11, 35]: for  $n = 1, \dots, N-1$ ,  $u_h^{n+1}$  is given by

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} = M^{-1}Au_h^n + \frac{\Delta t^2}{12}(M^{-1}A)^2u_h^n, \quad (5.1)$$

more precisely, we can rewrite it in the form of a second-order predictor step, for any  $\varphi, \psi \in W_h$

$$\left(\frac{u_h^* - 2u_h^n + u_h^{n-1}}{\Delta t^2}, \varphi\right)_K + B_K^1(w_h^n, \varphi) + (u_h^n, \varphi)_K + (f(u_h^n), \varphi)_K = 0, \quad (5.2)$$

$$(w_h^n, \psi)_K - B_K^2(u_h^n, \psi) = 0. \quad (5.3)$$

and the corrector step, for any  $\varphi, \psi \in W_h$

$$v_h^n = \frac{u_h^* - 2u_h^n + u_h^{n-1}}{\Delta t^2}, \quad (5.4)$$

$$(s_h^n, \psi)_K - B_K^2(v_h^n, \psi) = 0, \quad (5.5)$$



$$(u_h^{n+1}, \varphi)_K = (u_h^*, \varphi)_K - \frac{\Delta t^4}{12} B_K^1(s_h^n, \varphi) - \frac{\Delta t^4}{12} (v_h^n, \varphi)_K - \frac{\Delta t^4}{12} (f(v_h^n), \varphi)_K. \quad (5.6)$$

Since we need initial conditions for two time steps, we take Taylor expansion of  $u$  at  $t = 0$ .

$$u(\Delta t) = u(0) + \Delta t u_t(0) + \frac{\Delta t^2}{2} u_{tt}(0) + \frac{\Delta t^3}{6} u_{ttt}(0) + \mathcal{O}(\Delta t^4).$$

We could get the terms  $u_{tt}(0)$ ,  $u_{ttt}(0)$  by the equation

$$\begin{aligned} u_{tt} + \Delta^2 u + u + f(u) &= 0, \\ u_{tt}(0) &= -(\Delta^2 u(0) + u(0) + f(u(0))), \\ u_{ttt}(0) &= -(\Delta^2 u_t(0) + u_t(0) + f'(u(0))u_t(0)). \end{aligned}$$

**Remark 5.1.** For the linear case  $f(u) = 0$ , we can easily prove the fully discrete UWLDG method (5.1), conserves the energy

$$\begin{aligned} E^{n+1} &= \left\| \frac{u_h^{n+1} - u_h^n}{\Delta t} \right\|^2 + \left( \left\| \frac{u_h^{n+1} + u_h^n}{2} \right\|^2 + \left\| \frac{w_h^{n+1} + w_h^n}{2} \right\|^2 - \frac{\Delta t^2}{12} \left\| \frac{v_h^{n+1} + v_h^n}{2} \right\|^2 \right) \\ &\quad - \frac{\Delta t^2}{4} \left( \left\| \frac{u_h^{n+1} - u_h^n}{2} \right\|^2 + \left\| \frac{w_h^{n+1} - w_h^n}{2} \right\|^2 - \frac{\Delta t^2}{12} \left\| \frac{v_h^{n+1} - v_h^n}{2} \right\|^2 \right), \end{aligned}$$

for all  $n$ .

## 6 Numerical examples

In this section, we present numerical examples to verify our theoretical convergence properties of the UWLDG method.

### Example 6.1.

First example, we consider the linear fourth-order equations in one-dimension with the periodic boundary condition.

$$u_{tt} + u_{xxxx} = 0, \quad (x, t) \in [0, 2\pi] \times (0, 10],$$

and the initial conditions

$$u(x, 0) = \cos(x), \quad u_t(x, 0) = -\sin(x).$$

Table 6.1: Errors and the corresponding convergence rates for Example 6.1 when using  $\mathcal{P}^k$  polynomials on a uniform mesh of  $N$  cells. Final time  $t = 10$ .

	$N$	Error of $u$				Error of $w$			
		$L^2$ error	order	$L^\infty$ error	order	$L^2$ error	order	$L^\infty$ error	order
$\mathcal{P}^1$	10	4.60E-01	–	7.51E-01	–	4.37E-01	–	6.93E-01	–
	20	1.13E-01	2.02	1.74E-01	2.11	1.12E-01	1.97	1.73E-01	2.00
	40	2.82E-02	2.01	4.25E-02	2.03	2.77E-02	2.01	4.43E-02	1.96
	80	7.04E-03	2.00	1.05E-02	2.02	7.00E-03	1.98	1.23E-02	1.84
	160	1.76E-03	2.00	2.61E-03	2.01	1.78E-03	1.97	3.03E-03	2.03
	320	4.40E-04	2.00	6.51E-04	2.00	4.39E-04	2.02	7.45E-04	2.02
$\mathcal{P}^2$	10	3.07E-03	–	7.26E-03	–	3.50E-03	–	1.03E-02	–
	20	3.03E-04	3.34	1.10E-03	2.72	4.90E-04	2.84	1.77E-03	2.53
	40	3.48E-05	3.12	1.49E-04	2.88	5.60E-05	3.13	2.06E-04	3.10
	80	4.26E-06	3.03	1.94E-05	2.94	9.04E-06	2.63	3.61E-05	2.52
	160	5.30E-07	3.01	2.48E-06	2.97	1.20E-06	2.91	3.91E-06	3.21
	320	6.61E-08	3.00	3.15E-07	2.98	1.38E-07	3.12	5.49E-07	2.83
$\mathcal{P}^3$	10	7.44E-05	–	4.15E-04	–	9.66E-05	–	2.82E-04	–
	20	4.68E-06	3.99	2.70E-05	3.94	1.68E-05	2.52	8.45E-05	1.74
	40	2.92E-07	4.00	1.69E-06	4.00	7.92E-07	4.41	3.76E-06	4.49
	80	1.82E-08	4.00	1.06E-07	4.00	6.28E-08	3.66	3.04E-07	3.63
	160	1.14E-09	4.00	6.61E-09	4.00	4.87E-09	3.69	2.25E-08	3.75
	320	7.12E-11	4.00	4.13E-10	4.00	2.76E-10	4.14	1.19E-09	4.24

The exact solution of the problem is

$$u(x, t) = \cos(x + t).$$

We implement the DG method (2.3)-(2.4) with the alternating fluxes (2.5) and use the time discrete method (5.1). The errors of  $u$  and  $w$ , and numerical orders of accuracy for  $\mathcal{P}^k$  elements with  $1 \leq k \leq 3$  are listed in Table 6.1. We observe that our scheme gives the optimal  $(k + 1)$ -th order of the accuracy.

The errors between the numerical solution and the exact solution at time  $t = 500$  and  $t = 1000$  are shown in Figure 6.1. From these figures we observe that our numerical scheme works well in long time simulation. We also display the energy error  $E_h^n - E_h^0$  at various time  $t^n$ , where  $|E_h^n|$  is defined in Remark 5.1 in Figure 6.2. We can observe that the magnitude of energy error is smaller than  $10^{-6}$ , indicating that our scheme indeed conserves the discrete energy.

### Example 6.2.

In this example, we present the two-dimensional linear fourth-order equations with the periodic boundary condition on Cartesian meshes

$$u_{tt} + \Delta^2 u = 0, \quad (x, y) \in [0, 2\pi] \times [0, 2\pi], \quad t \in (0, 1],$$

Figure 6.1: The error of numerical solution and exact solution at  $t = 500$  (left) and  $t = 1000$  (right) for Example 6.1 when using  $\mathcal{P}^2$  polynomial on a uniform mesh of  $N = 20, 40, 80$ .

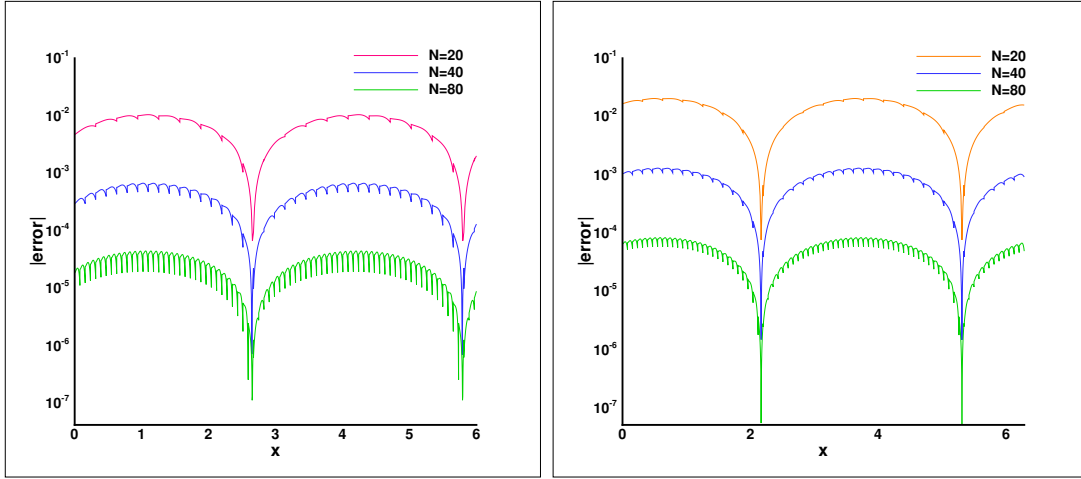
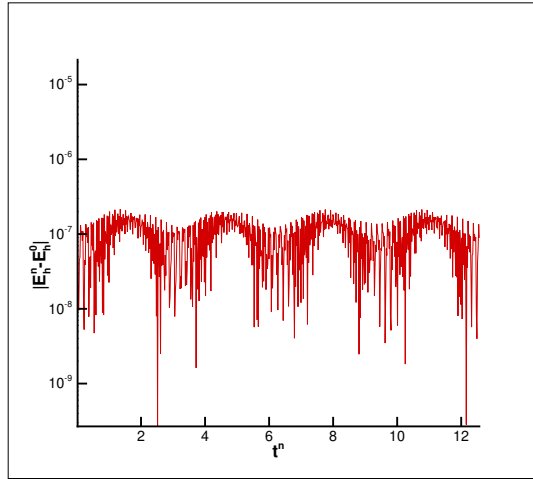


Figure 6.2: We compute the  $|E_h^n - E_h^0|$  for Example 6.1 when using  $\mathcal{P}^2$  polynomial on a uniform mesh of  $N = 80$ .



the initial conditions are

$$u(x, y, 0) = \cos(x + y), \quad u_t(x, y, 0) = -2 \sin(x + y),$$

and the exact solution is

$$u(x, y, t) = \cos(x + y + 2t).$$

We test the space  $\mathcal{Q}^k$ ,  $1 \leq k \leq 3$ , and list the errors of  $u$  and  $w$ , and corresponding orders in Table 6.2. **Since our scheme is energy conserving rather than energy dissipative, the initial error cannot be quickly damped and numerical orders of convergence tend to fluctuate. This is common for energy conserving methods, for example see [37].**

### Example 6.3.

Table 6.2: Errors and the corresponding convergence rates for Example 6.2 when using  $\mathcal{Q}^k$  polynomials on a uniform mesh of  $N \times N$  cells. Final time  $t = 1$ .

	$N \times N$	Error of $u$				Error of $w$			
		$L^2$ error	order	$L^\infty$ error	order	$L^2$ error	order	$L^\infty$ error	order
$\mathcal{Q}^1$	$8 \times 8$	1.86E+00	–	8.36E-01	–	2.44E+00	–	9.28E-01	–
	$16 \times 16$	4.59E-01	2.02	1.94E-01	2.11	7.13E-01	1.78	2.48E-01	1.90
	$32 \times 32$	1.13E-01	2.02	4.63E-02	2.07	2.04E-01	1.80	1.05E-01	1.25
	$64 \times 64$	2.80E-02	2.01	1.12E-02	2.04	4.94E-02	2.05	2.27E-02	2.20
	$128 \times 128$	6.98E-03	2.00	2.78E-03	2.01	1.47E-02	1.75	8.06E-03	1.49
	$256 \times 256$	1.74E-03	2.00	6.92E-04	2.01	2.92E-03	2.34	1.16E-03	2.79
$\mathcal{Q}^2$	$4 \times 4$	4.40E-01	–	3.43E-01	–	1.43E+00	–	1.10E+00	–
	$8 \times 8$	3.96E-02	3.47	3.19E-02	3.43	3.87E-02	3.45	3.51E-02	3.45
	$16 \times 16$	4.82E-03	3.04	4.04E-03	2.98	1.93E-02	2.55	1.15E-02	2.91
	$32 \times 32$	5.82E-04	3.05	5.00E-04	3.01	1.88E-03	3.36	1.20E-03	3.27
	$64 \times 64$	7.26E-05	3.00	6.32E-05	2.98	1.77E-04	3.41	1.39E-04	3.10
	$128 \times 128$	9.07E-06	3.00	7.93E-06	2.99	3.27E-05	2.43	2.52E-05	2.47
$\mathcal{Q}^3$	$4 \times 4$	2.59E-02	–	2.03E-02	–	7.94E-02	–	3.78E-02	–
	$8 \times 8$	1.59E-03	4.03	1.34E-03	3.93	4.95E-03	4.01	3.53E-03	3.42
	$16 \times 16$	1.00E-04	3.99	8.33E-05	4.01	4.14E-04	3.58	3.04E-04	3.54
	$32 \times 32$	6.20E-06	4.01	5.22E-06	4.00	1.93E-05	4.43	1.67E-05	4.19
	$64 \times 64$	3.88E-07	4.00	3.27E-07	4.00	1.85E-06	3.38	1.13E-06	3.88
	$128 \times 128$	2.42E-08	4.00	2.05E-08	4.00	1.04E-07	4.15	7.21E-08	3.98

We consider the nonlinear fourth-order wave equation (1.1), with a nonlinear term  $f(u) = u^3$ .

$$u_{tt} + \Delta^2 u + u + f(u) = g(x, y, t), \quad (x, y, t) \in \Omega \times (0, 1],$$

the initial conditions are

$$u(x, y, 0) = \cos(x + y), \quad u_t(x, y, 0) = -2 \sin(x + y),$$

where  $\Omega = [0, 2\pi] \times [0, 2\pi]$ , and  $u(x, y, t) = \cos(x + y + 2t)$ .

We test the space  $\mathcal{Q}^k$ ,  $1 \leq k \leq 3$ , and list the errors of  $u$  and  $w$ , and the corresponding orders in Table 6.3.

## 7 Concluding remarks

In this paper, we have developed an UWLDG method for a class of nonlinear fourth-order wave equations. The UWLDG methods combine the LDG and UWDG methods for solving time-dependent PDEs with high order spatial derivatives. The numerical fluxes have been carefully chosen to make our scheme energy conserving. We have proved the optimal error estimate in the  $L^2$ -norm for the solution itself as well as for the auxiliary variables

Table 6.3: Errors and the corresponding convergence rates for Example 6.3 when using  $Q^k$  polynomials on a uniform mesh of  $N \times N$  cells. Final time  $t = 1$ .

	$N \times N$	Error of $u$				Error of $w$			
		$L^2$ error	order	$L^\infty$ error	order	$L^2$ error	order	$L^\infty$ error	order
$Q^1$	$8 \times 8$	9.86E-01	–	6.05E-01	–	1.50E+00	2.47	8.95E-01	–
	$16 \times 16$	2.31E-01	2.10	1.35E-01	2.17	2.51E-01	2.58	1.39E-01	2.69
	$32 \times 32$	5.54E-02	2.06	3.27E-02	2.04	1.10E-01	1.19	6.52E-02	1.09
	$64 \times 64$	1.35E-02	2.04	8.05E-03	2.02	2.47E-02	2.15	1.39E-02	2.23
$Q^2$	$4 \times 4$	3.75E-01	–	3.21E-01	–	1.21E+00	–	9.39E-01	–
	$8 \times 8$	3.83E-02	3.29	3.29E-02	3.29	1.04E-01	3.54	7.69E-02	3.61
	$16 \times 16$	4.78E-03	3.00	4.12E-03	3.00	1.87E-02	2.47	1.10E-02	2.81
	$32 \times 32$	5.80E-04	3.04	5.06E-04	3.02	1.87E-03	3.32	1.19E-03	3.21
	$64 \times 64$	7.25E-05	3.00	6.36E-05	2.99	1.68E-04	3.48	1.28E-04	3.22
$Q^3$	$4 \times 4$	2.54E-02	–	2.04E-02	–	6.76E-02	–	2.42E-02	–
	$8 \times 8$	1.57E-03	4.01	1.34E-03	3.93	3.64E-03	4.22	2.37E-03	3.35
	$16 \times 16$	1.00E-04	3.97	8.35E-05	4.00	3.48E-04	3.38	2.62E-04	3.18
	$32 \times 32$	6.20E-06	4.02	5.22E-06	4.00	1.80E-05	4.28	1.03E-05	4.66
	$64 \times 64$	3.88E-07	4.00	3.27E-07	4.00	1.01E-06	4.15	4.89E-07	4.40

approximating its derivatives in the semi-discrete method, and have also shown that our scheme preserves energy in the semi-discrete sense. Compatible high order energy conserving time integrators are also proposed. The theoretical findings are confirmed by numerical experiments. **The algorithm can be easily designed on two-dimensional unstructured triangular meshes and the stability can be easily proven, however optimal error estimates would be more challenging and we would like to leave it for our future work.**

## Appendix A Estimate for $D = \left( \frac{\|\xi_u\|}{h^{k+1}} \right)^2$ in Theorem 4.1.

*Proof.* By (4.28) and (4.29), we have

$$D \leq C \int_0^\tau (D + (1 + \varepsilon^2 D^{p-1})(D + 1)) ds. \quad (\text{A.1})$$

Here we denote  $H(D) = D + (1 + \varepsilon^2 D^{p-1})(D + 1)$ , and  $M(\tau) = \int_0^\tau H(D) ds$ , then

$$\frac{d}{d\tau} M = H(D).$$

By (A.1) we obtain,

$$D \leq CM.$$

Therefore, the proof of D could be bounded by a constant independent of  $h$ , equal to prove that there exist a constant  $C^*$  independent on  $h$ , such that satisfy

$$M \leq C^*. \quad (\text{A.2})$$

Since  $H(s)$  is increasing for  $s > 0$ , and  $D \leq CM$  we have

$$\frac{d}{d\tau}M = H(D) \leq H(CM) \leq C_1H(M).$$

Firstly, we have

$$\frac{d}{d\tau}M \leq C_1H(M) = C_1(M + (1 + \varepsilon^2M^{p-1})(M + 1)),$$

and

$$\frac{dL(M)}{d\tau} = L'(M)\frac{d}{d\tau}M = \frac{1}{H(M)}\frac{d}{d\tau}M \leq C_1, \quad (\text{A.3})$$

where

$$L(s) := \int_1^s \frac{dz}{H(z)} = \int_1^s \frac{dz}{z + (1 + \varepsilon^2z^{p-1})(z + 1)}.$$

Integrate with respect to time in (A.3), we have

$$L(M(\tau)) \leq L(M(0)) + C_1T \leq C_1T, \quad \tau \in (0, T].$$

If  $M(\tau) \leq 1$ , the proof is done. If  $M(\tau) > 1$ , we have

$$\begin{aligned} L(M) &= \int_1^M \frac{dz}{z + (1 + \varepsilon^2z^{p-1})(z + 1)} \\ &\geq \int_1^M \frac{dz}{(z + 1) + (1 + \varepsilon^2z^{p-1})(z + 1)} \\ &\geq \frac{1}{2} \int_1^M \frac{dz}{(2 + \varepsilon^2z^{p-1})z} \\ &= \frac{1}{2} \int_{\varepsilon^{\frac{2}{p-1}}}^{M\varepsilon^{\frac{2}{p-1}}} \frac{dy}{(2 + y^{p-1})y} \quad (\varepsilon^2z^{p-1} = y^{p-1}) \\ &= \frac{1}{4} \int_{(\frac{1}{2}\varepsilon^2)^{\frac{1}{p-1}}}^{M(\frac{1}{2}\varepsilon^2)^{\frac{1}{p-1}}} \frac{dx}{(1 + x^{p-1})x} \quad (2x^{p-1} = y^{p-1}) \\ &= \frac{1}{4(1-p)} \log \left( 1 + \frac{2M^{1-p}}{\varepsilon^2} \right) - \frac{1}{4(1-p)} \log \left( 1 + \frac{2}{\varepsilon^2} \right). \end{aligned}$$

Therefore,

$$\frac{1}{4(1-p)} \log \left( 1 + \frac{2M^{1-p}}{\varepsilon^2} \right) - \frac{1}{4(1-p)} \log \left( 1 + \frac{2}{\varepsilon^2} \right) \leq C_1 T,$$

then we have

$$M \leq \left( \frac{2e^a}{2 - \varepsilon^2(e^a - 1)} \right)^{\frac{1}{p-1}},$$

where  $a = 4(p-1)C_1 T$ , we choose  $h$  sufficient small, so that  $\varepsilon^2 \leq \frac{1}{e^a - 1}$ , we obtain

$$M \leq (2e^a)^{\frac{1}{p-1}} \leq C^*.$$

Hence,  $M \leq \max\{1, C^*\}$ . □

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