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Functoriality for higher rho invariants of elliptic operators[☆]

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ABSTRACT

Let N be a closed spin manifold with positive scalar curvature and D_N the Dirac operator on N . Let M_1 and M_2 be two Galois covers of N such that M_2 is a quotient of M_1 . Then the quotient map from M_1 to M_2 naturally induces maps between the geometric C^* -algebras associated to the two manifolds. We prove, by a finite-propagation argument, that the *maximal* higher rho invariants of the lifts of D_N to M_1 and M_2 behave functorially with respect to the above quotient map. This can be applied to the computation of higher rho invariants, along with other related invariants.

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1. Introduction

An elliptic differential operator on a closed manifold has a Fredholm index. When such an operator is lifted to a covering space, one can, by taking into account the group of symmetries, define a far-reaching generalization of the Fredholm index, called the *higher index* [1,2,6,13,26]. The higher index serves as an obstruction to the existence of invariant metrics of positive scalar curvature. In the case that such a metric exists, so that the higher index of the lifted operator vanishes, a *secondary invariant* called the *higher rho invariant* [19,12] can be defined. The higher rho invariant is an obstruction to the inverse of the operator being local [4]. For some recent applications of the higher index and higher rho invariant to problems in geometry and topology, we refer the reader to [5,22,23,25,27–29].

The main purpose of this paper is to prove that the higher rho invariant behaves functorially under appropriate maps between covering spaces.

More precisely, suppose M_1 and M_2 are two Galois covers of a closed spin manifold N with deck transformation groups Γ_1 and $\Gamma_2 \cong \Gamma_1/H$, where H is a normal subgroup of Γ_1 . The natural projection $\pi: M_1 \rightarrow M_2$ induces a family of maps between various *geometric C^* -algebras* associated to M_1 and M_2 , called *folding maps*, which will be reviewed in section 2. In particular, there is a folding map

$$\Psi: C_{\max}^*(M_1)^{\Gamma_1} \rightarrow C_{\max}^*(M_2)^{\Gamma_2}$$

between maximal equivariant Roe algebras on M_1 and M_2 with the property that it preserves small propagation of operators. Consequently, Ψ induces a map

$$\Psi_{L,0}: C_{L,0,\max}^*(M_1)^{\Gamma_1} \rightarrow C_{L,0,\max}^*(M_2)^{\Gamma_2}$$

at the level of obstruction algebras. The main result of this paper is that the K -theoretic map induced by $\Psi_{L,0}$ relates the maximal higher rho invariants of Dirac operators on M_1 and M_2 :

Theorem 1.1. *Let N be a closed, spin Riemannian manifold with positive scalar curvature. Let D_N be the Dirac operator on N . Let M_1 and M_2 be Galois covers of M with deck transformation groups Γ_1 and $\Gamma_2 \cong \Gamma_1/H$ respectively, for a normal subgroup H of Γ_1 . Let D_1 and D_2 be the lifts of the Dirac operator on N to M_1 and M_2 respectively. Then*

$$(\Psi_{L,0})_*(\rho_{\max}(D_1)) = \rho_{\max}(D_2) \in K_*(C_{L,0,\max}^*(M_2)^{\Gamma_2}),$$

where $(\Psi_{L,0})_*$ is the map on K -theory induced by $\Psi_{L,0}$ and $\rho_{\max}(D_j)$ is the maximal higher rho invariant of D_j .

We also generalize this result to the non-cocompact setting using the same method of proof – see section 7 of this paper.

Theorem 1.1 is useful for computing higher rho and related invariants. It has been applied in the recent work of Wang, Xie, and Yu [24] to compute the delocalized eta invariant on a covering space of a closed manifold by showing that, under suitable geometric conditions, it can be approximated by delocalized eta invariants on finite-sheeted covers, which are more computable. We mention two of their results.

Let M be a closed spin manifold equipped with a positive scalar curvature metric. First, the authors proved that given a finitely generated discrete group Γ and a sequence $\{\Gamma_i\}_{i \in \mathbb{N}}$ of finite-index normal subgroups of Γ that *distinguishes*, in a suitable sense ([25, Definition 2.3]), a given non-trivial conjugacy class in Γ , the sequence of associated delocalized eta invariants on the Γ_i -Galois covers of M stabilizes, under the assumption that the maximal Baum-Connes assembly map for Γ is a rational isomorphism. We refer to [25, Theorem 1.1] for more details.

Second, the authors proved that if \tilde{D} is the Dirac operator associated to the Γ -Galois cover of M , then if the spectral gap of \tilde{D} at zero is sufficiently large, the delocalized eta invariant of \tilde{D} is equal to the limit of those of Dirac operators associated to the Γ_i -Galois covers of M [25, Theorem 1.4]. In particular, this is true if the group Γ has subexponential growth [25, Corollary 1.5].

Finally, we mention that in a new preprint [16] by Liu, Tang, Xie, Yao, and Yu, the authors apply Theorem 1.1 to study rigidity of *relative eta invariants*. These invariants are obtained by pairing the higher rho invariant with certain traces, similar to the way in which the delocalized eta invariant is related to the higher rho invariant through a trace [28].

Overview.

The paper is organized as follows. We begin in section 2 by recalling the geometric and operator-algebraic setup we work with. In section 3 we define the folding maps and

establish some of their basic properties. In section 4 we develop the analytical properties of the wave operator in the maximal setting. These tools are then put to use in section 5, where we give a new proof of the functoriality property of the maximal equivariant higher index. This serves as an intermediate step towards Theorem 1.1, which is proved in section 6. In section 7 we provide generalizations of our results to the non-cocompact setting.

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2. Preliminaries

In this section, we fix some notation before introducing the necessary operator-algebraic background and geometric setup for our results.

2.1. Notation

For X a Riemannian manifold, we write $B(X)$, $C_b(X)$, $C_0(X)$, and $C_c(X)$ to denote the C^* -algebras of complex-valued functions on X that are, respectively: bounded Borel, bounded continuous, continuous and vanishing at infinity, and continuous with compact support. A superscript ‘ ∞ ’ may be added where appropriate to indicate the additional requirement of smoothness.

We write d_X for the Riemannian distance function on X and $\mathbb{1}_S$ for the characteristic function of a subset $S \subseteq X$.

For any C^* -algebra A , we denote its unitization by A^+ , its multiplier algebra by $\mathcal{M}(A)$, and view A as an ideal of $\mathcal{M}(A)$.

The action of a group G on X naturally induces a G -action on spaces of functions on X as follows: given a function f on X and $g \in G$, define $g \cdot f$ by $g \cdot f(x) = f(g^{-1}x)$. More generally, for a section s of a Γ -vector bundle over X , the section $g \cdot s$ is defined by $g \cdot s(x) = s(g^{-1}x)$. We say that an operator on sections of a bundle is G -equivariant if it commutes with the G -action.

2.2. Geometric C^* -algebras

We now recall the notions of geometric modules and their associated C^* -algebras. Throughout this subsection, X is a Riemannian manifold equipped with a proper isometric action by a discrete group G .

Definition 2.1. An X - G -module is a separable Hilbert space \mathcal{H} equipped with a non-degenerate $*$ -representation $\rho: C_0(X) \rightarrow \mathcal{B}(\mathcal{H})$ and a unitary representation $U: G \rightarrow \mathcal{U}(\mathcal{H})$ such that for all $f \in C_0(X)$ and $g \in G$, we have $U_g \rho(f) U_g^* = \rho(g \cdot f)$.

For brevity, we will omit ρ from the notation when it is clear from context.

Definition 2.2. Let \mathcal{H} be an X - G -module and $T \in \mathcal{B}(\mathcal{H})$.

- The *support* of T , denoted $\text{supp}(T)$, is the complement of all $(x, y) \in X \times X$ for which there exist $f_1, f_2 \in C_0(X)$ such that $f_1(x) \neq 0$, $f_2(y) \neq 0$, and

$$f_1 T f_2 = 0;$$

- The *propagation* of T is the extended real number

$$\text{prop}(T) = \sup\{d_X(x, y) \mid (x, y) \in \text{supp}(T)\};$$

- T is *locally compact* if fT and $Tf \in \mathcal{K}(\mathcal{H})$ for all $f \in C_0(X)$;
- T is *G -equivariant* if $U_g T U_g^* = T$ for all $g \in G$;

The *equivariant algebraic Roe algebra* for \mathcal{H} , denoted $\mathbb{C}[X; \mathcal{H}]^G$, is the $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ consisting of G -equivariant, locally compact operators with finite propagation.

We will work with the maximal completion of the equivariant algebraic Roe algebra. To ensure that this completion is well-defined, we require that the module \mathcal{H} satisfy an additional admissibility condition. To define what this means, we need the following fact: if H is a Hilbert space and $\rho: C_0(X) \rightarrow \mathcal{B}(H)$ is a non-degenerate $*$ -representation, then ρ extends uniquely to a $*$ -representation $\tilde{\rho}: B(X) \rightarrow \mathcal{B}(H)$ subject to the property that, for a uniformly bounded sequence in $B(X)$ converging pointwise, the corresponding sequence in $\mathcal{B}(H)$ converges in the strong topology.

Definition 2.3 ([30]). Let \mathcal{H} be an X - G -module as in Definition 2.1. We say that \mathcal{H} is *admissible* if:

- For any non-zero $f \in C_0(X)$ we have $\rho(f) \notin \mathcal{K}(\mathcal{H})$;
- For any finite subgroup F of G and any F -invariant Borel subset $E \subseteq X$, there is a Hilbert space H' equipped with the trivial F -representation such that $\tilde{\rho}(\mathbb{1}_E)H' \cong l^2(F) \otimes H'$ as F -representations, where $\tilde{\rho}$ is defined by extending ρ as above.

If an X - G -module \mathcal{H} is admissible, we will write $\mathbb{C}[X]^G$ in place of $\mathbb{C}[X; \mathcal{H}]^G$, for the reason that $\mathbb{C}[X; \mathcal{H}]^G$ is independent of the choice of admissible module – see [26, Chapter 5].

Remark 2.4. When G acts freely and properly on X , the Hilbert space $L^2(X)$ is an admissible X - G -module. In the case that the action is not free, $L^2(X)$ can always be embedded into a larger admissible module.

Definition 2.5. The *maximal norm* of an operator $T \in \mathbb{C}[X]^G$ is

$$\|T\|_{\max} := \sup_{\phi, H'} \{ \|\phi(T)\|_{\mathcal{B}(H')} \mid \phi: \mathbb{C}[X]^G \rightarrow \mathcal{B}(H') \text{ is a } *\text{-representation} \}.$$

The *maximal equivariant Roe algebra* of M_J , denoted $C_{\max}^*(X)^G$, is the completion of $\mathbb{C}[X]^G$ in the norm $\|\cdot\|_{\max}$.

Remark 2.6. To make sense of Definition 2.5 for general X and G , one first needs to establish finiteness of the quantity $\|\cdot\|_{\max}$. It was shown in [8] that if G acts on X freely and properly with compact quotient, the norm $\|\cdot\|_{\max}$ is finite. This was generalized in [11] to the case when X has bounded geometry and the G -action satisfies a suitable geometric assumption.

Remark 2.7. Equivalently, one can obtain $C_{\max}^*(X)^G$ by taking the analogous maximal completion of the subalgebra \mathcal{S}^G of $\mathbb{C}[X]^G$ consisting of those operators given by smooth Schwartz kernels.

Definition 2.8. Consider the $*$ -algebra L of functions $f: [0, \infty) \rightarrow C_{\max}^*(X)^G$ that are uniformly bounded, uniformly continuous, and such that

$$\text{prop}(f(t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

- (i) The *maximal equivariant localization algebra*, denoted by $C_{L, \max}^*(X)^G$, is the C^* -algebra obtained by completing L with respect to the norm

$$\|f\| := \sup_t \|f(t)\|_{\max};$$

- (ii) The map $L \rightarrow \mathbb{C}[X]^G$ given by $f \mapsto f(0)$ extends to the *evaluation map*

$$\text{ev}: C_{L, \max}^*(X)^G \rightarrow C_{\max}^*(X)^G;$$

- (iii) The *maximal equivariant obstruction algebra* is $C_{L, 0, \max}^*(X)^G := \ker(\text{ev})$.

2.3. Geometric setup

We will work with the following geometric setup.

Let (N, g_N) be a closed Riemannian manifold. Let D_N be a first-order essentially self-adjoint elliptic differential operator on a bundle $E_N \rightarrow N$. We will assume throughout that if N is odd-dimensional then D_N is an ungraded operator, while if N is even-dimensional then D_N is odd-graded with respect to a \mathbb{Z}_2 -grading on E_N .

Let $p_1: M_1 \rightarrow N$ and $p_2: M_2 \rightarrow N$ be two Galois covers of N with deck transformation groups Γ_1 and Γ_2 respectively. We will assume throughout this paper that $\Gamma_2 \cong \Gamma_1/H$

for some normal subgroup H of Γ_1 , so that $M_2 \cong M_1/H$. Let $\pi: M_1 \rightarrow M_2$ be the projection map. Note that for $j = 1, 2$, the group Γ_j acts freely and properly on M_j .

For $j = 1$ or 2 , let g_j be the lift of the Riemannian metric g_N to M_j . Let E_j be the pullback of E along the covering map $M_j \rightarrow M$, equipped with the natural Γ_j -action. Since D acts locally, it lifts to a Γ_j -equivariant operator D_j on $C^\infty(E_j)$.

We will apply the notions in subsection 2.2 with $X = M_j$ and $G = \Gamma_j$. The Hilbert space $L^2(E_j)$, equipped with the natural Γ_j -action on sections and the $C_0(M_j)$ -representation defined by pointwise multiplication, is an M_j - Γ_j -module in the sense of Definition 2.2.

Moreover, the fact that the Γ_j -action on M_j is free and proper implies that $L^2(E_j)$ is admissible in the sense of Definition 2.3. To see this, choose a compact, Borel fundamental domain \mathcal{D}_j for the Γ_j -action on M_j such that for each $\gamma \in \Gamma_j$, the restriction

$$p_j|_{\gamma \cdot \mathcal{D}_j}: \gamma \cdot \mathcal{D}_j \rightarrow N$$

is a Borel isomorphism, where the projection maps p_j are as in subsection 2.3. For each $j = 1, 2$ we have a map

$$\begin{aligned} \Phi_j: L^2(E_j) &\rightarrow l^2(\Gamma_j) \otimes L^2(E_j|_{\mathcal{D}_j}), \\ s &\mapsto \sum_{\gamma \in \Gamma_j} \gamma \otimes \gamma^{-1} \chi_{\gamma \mathcal{D}_j} s. \end{aligned} \quad (2.1)$$

This is a Γ_j -equivariant unitary isomorphism with respect to the tensor product of the left-regular representation on $l^2(\Gamma_j)$ and the trivial representation on $L^2(E_j|_{\mathcal{D}_j})$. Conjugation by Φ_j induces a $*$ -isomorphism

$$\mathbb{C}[M_j]^{\Gamma_j} \cong \mathbb{C}\Gamma_j \otimes \mathcal{K}(L^2(E_j|_{\mathcal{D}_j})). \quad (2.2)$$

Remark 2.9. It follows from (2.2) that for any $r > 0$, there exists a constant C_r such that for all $T \in \mathbb{C}[M_j]^{\Gamma_j}$ with $\text{prop}(T) \leq r$, we have

$$\|T\|_{\max} \leq C_r \|T\|_{\mathcal{B}(L^2(E_j))}.$$

Thus the maximal equivariant Roe algebra $C_{\max}^*(M_j)^{\Gamma_j}$ from Definition 2.5 is well-defined in our setting. Moreover,

$$C_{\max}^*(M_j)^{\Gamma_j} \cong C_{\max}^*(\Gamma_j) \otimes \mathcal{K},$$

so that the K -theories of both sides are isomorphic.

3. Folding maps

In this section, we define certain natural $*$ -homomorphisms, called *folding maps*, between geometric C^* -algebras of covering spaces, and discuss the role they play in functoriality for higher invariants. These maps were introduced in [3, Lemma 2.12].

To begin, let us provide some motivation at the level of groups. Observe that the quotient homomorphism $\Gamma_1 \rightarrow \Gamma_2 \cong \Gamma_1/H$ induces a natural surjective $*$ -homomorphism between group algebras,

$$\alpha: \mathbb{C}\Gamma_1 \rightarrow \mathbb{C}\Gamma_2, \quad \sum_{i=1}^k c_{\gamma_i} \gamma_i \mapsto \sum_{i=1}^k c_{\gamma_i} [\gamma_i],$$

where $[\gamma]$ is the class of an element $\gamma \in \Gamma_1$ in Γ_1/H . If one views elements of $\mathbb{C}\Gamma_j$ as kernel operators on the Hilbert space $l^2(\Gamma_j)$, then the map α takes a kernel $k: \Gamma_1 \times \Gamma_1 \rightarrow \mathbb{C}$ to the kernel

$$\alpha(k): \Gamma_2 \times \Gamma_2 \rightarrow \mathbb{C}, \quad ([\gamma], [\gamma']) \mapsto \sum_{h \in H} k(h\gamma, \gamma').$$

There is an analogous map between kernels at the level of the Galois covers M_1 and $M_2 \cong M_1/H$. Given a smooth, Γ_1 -equivariant Schwartz kernel $k(x, y)$ with finite propagation on M_1 , one can define a smooth, Γ_2 -equivariant Schwartz kernel $\psi(k)$ with finite propagation on M_2 by the formula

$$\psi(k)([x], [y]) = \sum_{h \in H} k(hx, y). \quad (3.1)$$

Note that this sum is finite by properness of the H -action on M_1 . The formula (3.1) defines a map

$$\psi: \mathcal{S}^{\Gamma_1} \rightarrow \mathcal{S}^{\Gamma_2},$$

where the kernel algebras \mathcal{S}^{Γ_1} and \mathcal{S}^{Γ_2} are as in Remark 2.7. This map is a $*$ -homomorphism by [9, Lemma 5.2].

3.1. Definition of the folding map

We now define a more general version of the map (3.1), called the *folding map* Ψ , at the level of finite-propagation operators.

Let $\mathcal{B}_{\text{fp}}(L^2(E_j))^{\Gamma_j}$ denote the $*$ -algebra of bounded, Γ_j -equivariant operators on $L^2(E_j)$ with finite propagation. The folding map Ψ is a $*$ -homomorphism $\mathcal{B}_{\text{fp}}(L^2(E_1))^{\Gamma_1} \rightarrow \mathcal{B}_{\text{fp}}(L^2(E_2))^{\Gamma_2}$ with the following properties:

- For any $T \in \mathcal{B}_{\text{fp}}(L^2(E_1))^{\Gamma_1}$, we have $\text{prop}(\Psi(T)) \leq \text{prop}(T)$;

- Ψ is surjective;
- Ψ restricts to a surjective $*$ -homomorphism $\mathbb{C}[M_1]^{\Gamma_1} \rightarrow \mathbb{C}[M_2]^{\Gamma_2}$.

These properties are proved in Propositions 3.3, 3.4, and 3.5 below.

To define Ψ , it will be convenient to use partitions of unity on M_1 and M_2 that are compatible with the Γ_1 and Γ_2 -actions, as follows. Since N is compact, there exists $\epsilon > 0$ such that for each $x \in N$, the ball $B_\epsilon(x)$ is evenly covered with respect to both $p_1: M_1 \rightarrow N$ and $p_2: M_2 \rightarrow N$. Let

$$\mathcal{U}_N := \{U_i\}_{i \in I} \quad (3.2)$$

be a finite open cover of N such that each U_i has diameter at most ϵ . Let $\{\phi_i\}_{i \in I}$ be a partition of unity subordinate to \mathcal{U}_N .

For each $i \in I$, let U_i^e be a lift of U_i to M_1 via the covering map $p_1: M_1 \rightarrow N$. Similarly, let ϕ_i^e be the lift of ϕ_i to U_i^e . For each $g \in \Gamma_1$, let U_i^g and ϕ_i^g be the g -translates of U_i^e and ϕ_i^e in M_1 respectively. Then

$$\mathcal{U}_{M_1} := \{U_i^g\}_{i \in I, g \in \Gamma_1}, \quad \{\phi_i^g\}_{i \in I, g \in \Gamma_1} \quad (3.3)$$

define a locally finite, Γ_1 -invariant open cover of M_1 and a subordinate partition of unity. This partition of unity is Γ_1 -equivariant in the sense that for each $i \in I$, $g \in \Gamma$, and $x \in \text{supp}(\phi_i^e)$, we have $\phi_i^g(gx) = \phi_i^e(x)$.

After taking a quotient by the H -action, we obtain a Γ_2 -invariant open cover of M_2 , together with a subordinate Γ_2 -equivariant partition of unity:

$$\mathcal{U}_{M_2} := \{U_i^{[g]}\}_{i \in I, [g] \in \Gamma_2}, \quad \{\phi_i^{[g]}\}_{i \in I, [g] \in \Gamma_2}. \quad (3.4)$$

For each i and g , we have the following commutative diagram of isomorphisms of local structures involving the covering map π :

$$\begin{array}{ccc} E_1|_{U_i^g} & \longrightarrow & U_i^g \\ \downarrow & & \downarrow \pi \\ E_2|_{U_i^{[g]}} & \longrightarrow & U_i^{[g]}. \end{array}$$

We will write π_* and $\pi|_{U_i^g}^*$ for the maps relating a local section of E_1 to the corresponding local section of E_2 . That is, if u is a section of E_1 over U_i^g , and v is the corresponding section of E_2 over $U_i^{[g]}$, then we have

$$u \begin{array}{c} \xrightarrow{\pi_*} \\ \xleftarrow{\pi|_{U_i^g}^*} \end{array} v$$

If w is any vector in the bundle E_1 , we will also write $\pi_*(w)$ for its image under the projection $E_1 \rightarrow E_2$.

We are now ready for the definition of the folding map Ψ . Choose a set $S \subseteq \Gamma_1$ of coset representatives for $\Gamma_2 \cong H \backslash \Gamma_1$. Given an operator $T \in \mathcal{B}_{\text{fp}}(L^2(E_1))^{\Gamma_1}$, we define $\Psi(T)$ to be the operator on $L^2(E_2)$ that takes a section $u \in L^2(E_2)$ to the section

$$\Psi(T)u := \sum_{\substack{g \in \Gamma_1, \\ s \in S}} \sum_{i,j \in I} \pi_*(\phi_i^g \cdot T \circ \pi|_{U_j^s}^* \circ \phi_j^{[s]}(u)) \in L^2(E_2). \quad (3.5)$$

To clarify the presentation of this equation and others like it, we adopt the following notational convention for moving between equivalent local sections of E_1 and E_2 :

Convention 3.1. For $v \in L^2(E_1)$ and $u \in L^2(E_2)$, we will use the short-hand

- $\phi_i^{[g]}v$ to denote $\pi_*(\phi_i^g v)$;
- $\phi_i^g u$ to denote $\pi|_{U_i^g}^*(\phi_i^{[g]}u)$.

(Note that the meaning of $\phi_i^{[g]}v$ depends on the representative g .)

Using this convention, the formula (3.5) reads

$$\Psi(T)u := \sum_{\substack{g \in \Gamma_1, \\ s \in S}} \sum_{i,j \in I} \phi_i^{[g]} T(\phi_j^s u) \in L^2(E_2). \quad (3.6)$$

We call Ψ the *folding map*. The fact that the above definition is independent of the choices of the set S and compatible partitions of unity is proved in Proposition 3.3.

Before proceeding further, let us record a few straightforward identities that will help us navigate through notational clutter:

- (i) If v is a local section of E_1 supported in some neighborhood U_j^g , then for any $g' \in \Gamma_1$ we have

$$g'(\phi_i^g v) = \phi_i^{g'g} g'v, \quad (3.7)$$

$$\pi_*(g'v) = [g'] \cdot \pi_*(v). \quad (3.8)$$

- (ii) If u is a local section of E_2 supported in some neighborhood $U_j^{[g]}$, then for any $g, g' \in \Gamma_1$ and $j \in I$ we have

$$g'(\pi|_{U_j^g}^* u) = \pi|_{U_j^{g'g}}^* ([g'] \cdot u), \quad (3.9)$$

$$u = \sum_{\substack{i \in I, \\ \gamma \in \Gamma_1}} \phi_i^{[\gamma]} (\pi|_{U_j^g}^* u). \quad (3.10)$$

The following lemma provides a convenient pointwise formula for $\Psi(T)u$:

Lemma 3.2. For any $u \in L^2(E_2)$ and $x \in M_2$, we have

$$(\Psi(T)u)(x) = \pi_* \left(\sum_{\substack{j \in I, \\ g \in \Gamma_1}} T(\phi_j^g u)(y_0) \right), \quad (3.11)$$

where y_0 is any point in $\pi^{-1}(x)$, and the sum on the right-hand side is finite.

Proof. Observe that for any $j \in I$ and $g \in \Gamma_1$, the summand $T(\phi_j^g u)(y_0)$ may only be non-zero if $U_j^g \cap B_{\text{prop } T}(y_0) \neq \emptyset$. Since T has finite propagation and the Γ_1 -action is proper, the set

$$\{(g, j) \mid U_j^g \cap B_{\text{prop } T}(y_0) \neq \emptyset\}$$

is finite. Hence the sum in question is finite for every $x \in M_2$ and $y_0 \in \pi^{-1}(x)$.

To prove (3.11), let us first rewrite the right-hand side as

$$\pi_* \left(\sum_{j \in I} \sum_{\substack{h \in H, \\ s \in S}} T(\phi_j^{hs} u)(y_0) \right). \quad (3.12)$$

Next, using (3.9), (3.10), and Γ_1 -equivariance of T , each summand in (3.12) equals

$$\begin{aligned} T(\phi_j^{hs} u)(y_0) &= T(h(\phi_j^s([h] \cdot u)))(y_0) \\ &= T(h(\phi_j^s u))(y_0) \\ &= h(T(\phi_j^s u))(y_0) \\ &= \left(h \cdot \sum_{\substack{i \in I, \\ g \in \Gamma_1}} \phi_i^g T \phi_j^s u \right)(y_0), \end{aligned} \quad (3.13)$$

where we have used Convention 3.1 when writing $\phi_j^s u$. Now observe that if v is a section of E_1 supported in some open set U_i^g , then for any $x \in M_2$, we have

$$(\pi_* v)(x) = \pi_* \left(\sum_{h \in H} (hv)(y_0) \right), \quad (3.14)$$

where y_0 is an arbitrary point in the inverse image $\pi^{-1}(x)$. Keeping this in mind while summing (3.13) over $j \in I$, $s \in S$, and $h \in H$, we see that (3.12) equals

$$\pi_* \left(\sum_{h, g, s, i, j} (h(\phi_i^g T \phi_j^s u))(y_0) \right) = \left(\pi_* \sum_{g, s, i, j} \phi_i^g \cdot T(\phi_j^s u) \right)(x),$$

which is equal to $(\Psi(T)u)(x)$ by (3.6) and noting Convention 3.1. \square

Proposition 3.3. *The folding map Ψ defined in (3.6) is a $*$ -homomorphism*

$$\Psi: \mathcal{B}_{\text{fp}}(L^2(E_1))^{\Gamma_1} \rightarrow \mathcal{B}_{\text{fp}}(L^2(E_2))^{\Gamma_2}$$

and is independent of the choices of the set S of coset representatives and compatible partitions of unity used to define it.

Proof. Boundedness of $\Psi(T)$ follows from boundedness and finite propagation of T , via the formula (3.11). To see that $\Psi(T)$ is Γ_2 -equivariant, note that by (3.11), we have for all $[\gamma] \in \Gamma_2$ and $u \in L^2(E_2)$ that

$$\begin{aligned} ([\gamma]\Psi(T)[\gamma^{-1}]u)(x) &= [\gamma] \cdot \pi_* \sum_{\substack{j \in I, \\ g \in \Gamma_1}} T(\phi_j^g[\gamma^{-1}]u)(\gamma^{-1}y_0) \\ &= [\gamma] \cdot \pi_* \sum_{j,g} T\gamma^{-1}(\phi_j^{\gamma g}u)(\gamma^{-1}y_0). \end{aligned}$$

Using (3.8) and Γ_1 -equivariance of T , this is equal to

$$\pi_* \sum_{j,g} \gamma T\gamma^{-1}(\phi_j^{\gamma g}u)(y_0) = \pi_* \sum_{j,g} T(\phi_j^{\gamma g}u)(y_0) = (\Psi(T)u)(x),$$

where we used a change-of-variable for the last equality.

To see that Ψ is a $*$ -homomorphism, let Q be another operator in $\mathcal{B}_{\text{fp}}(L^2(E_1))^{\Gamma_1}$. Let $x \in M_2$ and $y_0 \in \pi^{-1}(x)$. By (3.11), and the fact that both T' and T have finite propagation, one sees that there exists a finite subset $F \subseteq \Gamma_1$ such that

$$\begin{aligned} (\Psi(T'T)u)(x) &= \pi_* \sum_{\substack{j \in I, \\ g \in F}} T'T(\phi_j^g u)(y_0) \\ &= \pi_* \left(T' \sum_{j,g} T(\phi_j^g u) \right) (y_0) \\ &= \pi_* \left(T' \sum_{\substack{i \in I, \\ \gamma \in F}} \phi_i^\gamma \sum_{j,g} T(\phi_j^g u) \right) (y_0) \\ &= \pi_* \left(\sum_{i,\gamma} T'(\phi_i^\gamma \sum_{j,g} T(\phi_j^g u)) \right) (y_0) \\ &= (\Psi(T') \circ \Psi(T)u)(x). \end{aligned}$$

One checks directly that Ψ respects $*$ -operations.

That (3.6) is independent of the choice of coset representatives $S \subseteq \Gamma_1$ is implied by the pointwise formula (3.11). To see that (3.6) is independent of partitions of unity, let $\{\varphi_i\}$, $\{\varphi_i^{[\gamma]}\}$, and $\{\varphi_i^\gamma\}$ be another set of compatible partitions of unity for N , M_2 , and M_1 with the same properties as $\{\phi_i\}$, $\{\phi_i^{[\gamma]}\}$, and $\{\phi_i^\gamma\}$. Let us write

$$\Psi^{\{\phi\}}(T) \quad \text{and} \quad \Psi^{\{\varphi\}}(T)$$

to distinguish the operators defined by (3.6) with respect to these two sets of partitions of unity. Since T has finite propagation, there exists a finite subset $F' \subseteq \Gamma_1$ such that if $g \notin F$, then $\text{supp}(\phi_j^g) \cap B_{\text{prop } T}(y_0) = \emptyset$ and $\text{supp}(\varphi_j^g) \cap B_{\text{prop } T}(y_0) = \emptyset$ for any $j \in I$. By (3.11), for each $x \in M_2$ we have

$$\begin{aligned} (\Psi^{\{\phi\}}(T)u)(x) &= \pi_* \sum_{j \in I, g \in \Gamma_1} T(\phi_j^g u)(y_0) \\ &= \pi_* \sum_{j \in I, g \in F} T(\mathbb{1}_{B_{\text{prop } T}(y_0)} \phi_j^g u)(y_0) \\ &= \pi_* \left(\left(T \sum_{j, g} \mathbb{1}_{B_{\text{prop } T}(y_0)} \phi_j^g u \right)(y_0) \right), \end{aligned} \quad (3.15)$$

where $\mathbb{1}_{B_{\text{prop } T}(y_0)}$ is the characteristic function of the set $B_{\text{prop } T}(y_0)$, and we have used that T commutes with finite sums. Likewise, we have

$$(\Psi^{\{\varphi\}}(T)u)(x) = \pi_* \left(\left(T \sum_{j, g} \mathbb{1}_{B_{\text{prop } T}(y_0)} \varphi_j^g u \right)(y_0) \right). \quad (3.16)$$

Now observe that

$$\sum_{j, g} \mathbb{1}_{B_{\text{prop } T}(y_0)} \phi_j^g u \quad \text{and} \quad \sum_{j, g} \mathbb{1}_{B_{\text{prop } T}(y_0)} \varphi_j^g u$$

are both equal to the lift of the section u to M_1 restricted to the compact subset $B_{\text{prop } T}(y_0)$. Thus the right-hand sides of (3.15) and (3.16) are equal, whence

$$(\Psi^{\{\phi\}}(T)(u))(x) = (\Psi^{\{\varphi\}}(T)(u))(x). \quad \square$$

The next proposition shows that the folding map Ψ preserves locality of operators. In particular, this means that Ψ induces a map at the level of localization algebras (see Definition 3.8), which is a crucial ingredient for our main result, Theorem 1.1.

Proposition 3.4. *For any operator $T \in \mathcal{B}_{\text{fp}}(L^2(E_1))^{\Gamma_1}$, we have*

$$\text{prop}(\Psi(T)) \leq \text{prop}(T).$$

Proof. Take a section $u \in C_c(E_2)$, and write it as a finite sum $\sum_{j, [g]} \phi_j^{[g]} u$. By (3.11),

$$(\Psi(T)u)(x) = \pi_* \sum_{j \in I, g \in \Gamma_1} T(\phi_j^g u)(y_0),$$

where y_0 is a lift of x to M_1 . Now if $d_{M_2}(x, \text{supp } u) > \text{prop}(T)$, then

$$d_{M_1}(y_0, \text{supp}(\phi_j^g u)) > \text{prop}(T)$$

for any $j \in I$ and $g \in \Gamma_1$, since distances between points do not increase under the projection π . Thus if $d_{M_2}(x, \text{supp } u) > \text{prop}(T)$, then $\Psi(T)u(x) = 0$. Since this holds for any section $u \in C_c(E_2)$, we have $\text{prop}(\Psi(T)) \leq \text{prop}(T)$. \square

Proposition 3.5. *The map Ψ is surjective and preserves local compactness of operators. Moreover, it restricts to a surjective $*$ -homomorphism*

$$\Psi: \mathbb{C}[M_1]^{\Gamma_1} \rightarrow \mathbb{C}[M_2]^{\Gamma_2}.$$

Proof. Let $S \subseteq \Gamma_1$ be the set of coset representatives used in the definition of Ψ . For any $T_2 \in \mathcal{B}_{\text{fp}}(L^2(E_2))^{\Gamma_2}$, define an operator T_1 on $L^2(E_1)$ by the formula

$$T_1(v) := \sum_{\substack{g \in \Gamma_1 \\ i \in I}} \sum_{\substack{s \in S \\ j \in I}} \phi_j^{gs} T_2(\phi_i^{[g]} v), \quad (3.17)$$

for $v \in L^2(E_1)$. We claim that $T_1 \in \mathcal{B}_{\text{fp}}(L^2(E_1))^{\Gamma_1}$ and that $\Psi(T_1) = T_2$. Indeed, for each fixed i and j , the sum of operators

$$\sum_{g \in \Gamma_1} \sum_{s \in S} \phi_j^{gs} T_2 \phi_i^{[g]}$$

converges strongly in $\mathcal{B}(L^2(E_1))$. Since I is a finite indexing set, it follows that T_1 is bounded. To see that T_1 is Γ_1 -equivariant, note that for any $\gamma \in \Gamma_1$, we have, by the identities (3.7) and (3.8), that

$$\begin{aligned} \gamma^{-1} T_1 \gamma(v) &= \sum_{g,i} \sum_{s,j} \gamma^{-1} (\phi_j^{gs} T_2(\phi_i^{[g]} \gamma v)) \\ &= \sum_{g,i} \sum_{s,j} \gamma^{-1} (\phi_j^{gs} T_2(\pi_*(\gamma \cdot (\phi_i^{\gamma^{-1}g} v)))) \\ &= \sum_{g,i} \sum_{s,j} \gamma^{-1} (\phi_j^{gs} T_2([\gamma] \cdot (\phi_i^{[\gamma^{-1}g]} v))), \end{aligned}$$

where we observe Convention 3.1 as usual. Using Γ_2 -equivariance of T_2 and (3.9), one finds this to be equal to

$$\begin{aligned} &\sum_{g,i} \sum_{s,j} \gamma^{-1} (\phi_j^{[gs]} ([\gamma] T_2(\phi_i^{[\gamma^{-1}g]} v))) \\ &= \sum_{g,i} \sum_{s,j} \phi_j^{\gamma^{-1}gs} T_2(\phi_i^{\gamma^{-1}g} v) \\ &= \sum_{\gamma^{-1}g,i} \sum_{s,j} \phi_j^{\gamma^{-1}gs} T_2(\phi_i^{[\gamma^{-1}g]} v) \end{aligned}$$

$$= T_1(v),$$

using the definition of T_1 and a change of variable for the last equality.

Next, finite propagation of T_1 follows from finite propagation of T_2 . Indeed, for any $v \in C_c(E_2)$, $g \in \Gamma_1$, and $i \in I$, the set

$$S_{T_1} := \left\{ s \in S \mid \phi_j^{[gs]} T_2(\phi_i^{[g]} v) \neq 0 \text{ for some } j \in I \right\}$$

is finite, thus the sum over S in equation (3.17) reduces to a finite sum over S_{T_1} . By the same equation, and the fact that the Γ_1 -action is isometric, we have

$$\text{prop}(T_1) \leq \sup_{\substack{s \in S_{T_1} \\ i, j \in I}} \{d_{M_1}(U_i^g, U_j^{gs})\} = \sup_{\substack{s \in S_{T_1} \\ i, j \in I}} \{d_{M_1}(U_i^e, U_j^s)\} < \infty.$$

We now show that $\Psi(T_1) = T_2$. By (3.17) and (3.5) we have, for any $u \in C_c(E_2)$,

$$\begin{aligned} \Psi(T_1)u &= \sum_{\substack{\gamma \in \Gamma_1 \\ t \in S}} \sum_{i, j \in I} \phi_i^{[\gamma]} T_1 \phi_j^t(u) \\ &= \sum_{\substack{\gamma \in \Gamma_1 \\ t \in S}} \sum_{i, j \in I} \phi_i^{[\gamma]} \left(\sum_{\substack{g \in \Gamma_1 \\ k \in I}} \sum_{\substack{s \in S \\ l \in I}} \phi_l^{gs} T_2(\phi_k^{[g]}(\phi_j^t u)) \right). \end{aligned}$$

Finite propagation of T_2 and compact support of u imply that all of these sums are finite, so by (3.10) this is equal to

$$\sum_{g, s, k, l} \sum_{\gamma, t, i, j} \phi_i^{[\gamma]} (\phi_l^{gs} T_2(\phi_k^{[g]}(\phi_j^t u))) = \sum_{g, s, k, l} \sum_{t, j} \phi_l^{[gs]} T_2(\phi_k^{[g]}(\phi_j^t u)).$$

Since $\{gs \mid s \in S\}$ is a set of coset representatives for $H \backslash \Gamma_1$ for any $g \in \Gamma_1$, the identity (3.10) implies that the above is equal to

$$\sum_{t, j} T_2 \left(\sum_{g, k} \phi_k^{[g]}(\phi_j^t u) \right) = \sum_{t, j} T_2(\phi_j^{[t]} u) = T_2 u.$$

Thus $\Psi(T_1) = T_2$ as bounded operators on $L^2(E_2)$.

Finally, that Ψ preserves local compactness of operators follows directly from (3.11), so that Ψ restricts to a map $\mathbb{C}[M_1]^{\Gamma_1} \rightarrow \mathbb{C}[M_2]^{\Gamma_2}$. To see that this map is surjective, suppose T_2 is locally compact. Then for any $f \in C_c(M_1)$, we have

$$fT_1 = f \sum_{\substack{g \in \Gamma_1 \\ s \in S}} \sum_{i, j \in I} \phi_j^{gs} T_2 \phi_i^{[g]}.$$

The sum over Γ_1 reduces to a sum over the finite set

$$F_f := \{g \in \Gamma_1 \mid U_j^{gs} \cap \text{supp}(f) \neq \emptyset \text{ for some } j \in I\},$$

hence fT_1 is equal to

$$\sum_{\substack{g \in F_f \\ s \in S}} \sum_{i, j \in I} f \phi_j^{gs} \mathbb{1}_{\pi_*(\text{supp } f)} T_2 \phi_i^{[g]}.$$

Since the subset $\pi_*(\text{supp } f) \subseteq M_2$ is compact, local compactness of T_2 means that this is a finite sum of compact operators and hence compact. A similar argument shows that the operator $T_1 f$ is compact. Thus the operator T_1 is locally compact, so Ψ restricts to a surjective $*$ -homomorphism $\mathbb{C}[M_1]^{\Gamma_1} \rightarrow \mathbb{C}[M_2]^{\Gamma_2}$. \square

3.2. Description of the folding map on invariant sections

The folding map Ψ admits an equivalent description using N -invariant sections of E_1 . Such sections, while not in general square-integrable over M_1 , form a space that is naturally isomorphic to $L^2(E_2)^{\Gamma_2}$, as we now describe. This allows computations involving Ψ to be carried out entirely on M_1 . Thus it may be a useful perspective for some applications. The discussion below slightly generalizes the averaging map from [10, subsection 5.2] applied in the context of discrete groups.

Let $c: M_1 \rightarrow [0, 1]$ be a function whose support has compact intersections with every H -orbit, and such that for all $x \in M_1$,

$$\sum_{h \in H} c(hx)^2 = 1.$$

Note that this sum is finite by properness of the Γ_1 -action.

Let $C_{\text{tc}}(E_1)^H$ denote the space of H -transversally compactly supported sections of E_1 , defined as the space of continuous, H -invariant sections of E_1 whose supports have compact images in $M_2 \cong M_1/H$ under the quotient map. Let $L_T^2(E_1)^H$ denote the Hilbert space of H -invariant, H -transversally L^2 -sections of E_1 , defined as the completion of $C_{\text{tc}}(E_1)^H$ with respect to the inner product

$$(s_1, s_2)_{L_T^2(E_1)^N} := (cs_1, cs_2)_{L^2(E_1)}.$$

Then one checks that:

Lemma 3.6. *The space $L_T^2(E_1)^H$ is naturally unitarily isomorphic to $L^2(E_2)$ and is independent of the choice of c .*

For any $T \in \mathcal{B}_{\text{fp}}(L^2(E_1))^{\Gamma_1}$, the operator $\Psi(T)$ on $L^2(E_2)$ from (3.6) can be described equivalently by its action on the isomorphic space $L_T^2(E_1)^H$ as follows. Take a partition

of unity $\{\phi_i^g\}$ of M_1 as in (3.3). Let $s \in L_T^2(E_1)^H$ be an H -transversally L^2 -section of E_1 . For any $y \in M_1$, set

$$(\Psi(T)s)(y) := \sum_{i \in I, g \in \Gamma_1} (T(\phi_i^g s)(y)). \quad (3.18)$$

The fact that T has finite propagation means that this pointwise sum is finite, and that $c(\Psi(T)s) \in L^2(E_1)$. Further, $\Psi(T)s$ is an element of $L_T^2(E_1)^H$, since for any $h \in H$ and $y \in M_1$ we have

$$(h(\Psi(T)s))(y) = \sum_{i \in I, g \in \Gamma_1} (T \circ h(\phi_i^g s)(y)) = \sum_{i \in I, g \in \Gamma_1} T(\phi_i^{hg} s)(y) = (\Psi(T)s)(y),$$

where we have used the equivariance properties of T and s . Finally, a direct comparison shows that this definition is equivalent with that given by equation (3.11).

3.3. Induced maps on geometric C^* -algebras and K -theory

By Proposition 3.5, the folding map Ψ restricts to a surjective $*$ -homomorphism

$$\Psi: \mathbb{C}[M_1]^{\Gamma_1} \rightarrow \mathbb{C}[M_2]^{\Gamma_2}.$$

By the defining property of maximal completions of $*$ -algebras, Ψ extends to a surjective $*$ -homomorphism between maximal equivariant Roe algebras,

$$\Psi: C_{\max}^*(M_1)^{\Gamma_1} \rightarrow C_{\max}^*(M_2)^{\Gamma_2}.$$

In this paper we will make use of a number natural extensions of this map to other geometric C^* -algebras, as well as the induced maps on K -theory. We refer to these maps collectively as *folding maps*.

To begin, we have the following elementary lemma:

Lemma 3.7. *Let A and B be C^* -algebras and let $\phi: A \rightarrow B$ be a surjective $*$ -homomorphism. Then ϕ extends to a $*$ -homomorphism $\tilde{\phi}: \mathcal{M}(A) \rightarrow \mathcal{M}(B)$.*

Proof. Since ϕ is surjective, we may define $\tilde{\phi}$ by requiring

$$\tilde{\phi}(m)\phi(a) = \phi(ma), \quad \phi(a)\tilde{\phi}(m) = \phi(am),$$

for $m \in \mathcal{A}$. Concretely, by way of a faithful non-degenerate representation $\sigma: B \rightarrow \mathcal{B}(H)$ on some Hilbert space H , we may identify B with a subalgebra of $\mathcal{B}(H)$. Since ϕ is surjective, the composition $\sigma \circ \phi: A \rightarrow \mathcal{B}(H)$ is a non-degenerate representation of A . Thus the above formulas define a representation of $\mathcal{M}(A)$ with values in the idealizer of $\sigma(B)$, which one identifies with $\mathcal{M}(B)$. (See also [7] Lemma I.9.14.) \square

By Lemma 3.7 and Proposition 3.5, the map Ψ extends to a $*$ -homomorphism

$$\mathcal{M}(C_{\max}^*(M_1)^{\Gamma_1}) \rightarrow \mathcal{M}(C_{\max}^*(M_2)^{\Gamma_2}).$$

Viewing $C_{\max}^*(M_j)^{\Gamma_j}$ as an ideal in \mathcal{M} , we will still denote this extended map by Ψ . This map will be essential when we apply the functional calculus for the maximal Roe algebra in sections 5 and 6.

Next, suppose we have a path $r: [0, \infty) \rightarrow C_{\max}^*(M_1)^{\Gamma_1}$ that satisfies

$$\text{prop}(r(t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Then by Proposition 3.4, the same is true of the path $\Psi \circ r: [0, \infty) \rightarrow C_{\max}^*(M_1)^{\Gamma_2}$. This allows the folding map to pass to the level of localization algebras:

Definition 3.8. Define the map

$$\begin{aligned} \Psi_L: C_{L,\max}^*(M_1)^{\Gamma_1} &\rightarrow C_{L,\max}^*(M_2)^{\Gamma_2} \\ r &\mapsto \Psi \circ r. \end{aligned}$$

Then Ψ_L restricts to a $*$ -homomorphism between obstruction algebras:

$$\Psi_{L,0}: C_{L,0,\max}^*(M_1)^{\Gamma_1} \rightarrow C_{L,0,\max}^*(M_2)^{\Gamma_2}.$$

Each of the above maps Ψ , Ψ_L , and $\Psi_{L,0}$ induces a map at the level of K -theory. In this paper, we will make use of two of them:

$$\begin{aligned} \Psi_*: K_*(C_{\max}^*(M_1)^{\Gamma_1}) &\rightarrow K_*(C_{\max}^*(M_2)^{\Gamma_2}), \\ (\Psi_{L,0})_*: K_*(C_{L,0,\max}^*(M_1)^{\Gamma_1}) &\rightarrow K_*(C_{L,0,\max}^*(M_2)^{\Gamma_2}). \end{aligned}$$

Remark 3.9. In the recent paper [21], Schick and Seyedhosseini have constructed a slightly different completion of the equivariant algebraic Roe algebra, called the *quotient completion*, which still behaves functorially under maps between covering spaces. For details of the construction, see [21, section 3].

4. Functional calculus and the wave operator

We now develop the analytical properties of the wave operator on the maximal Roe algebra that will form the basis of our work in subsequent sections.

Let us begin by recalling the functional calculus for the maximal Roe algebra, which was developed in a more general setting in [11]. The discussion here is specialized to the cocompact setting.

Throughout this section we will work in the geometric situation in subsection 2.3. To simplify notation, in this subsection and the next we will write M, Γ for either M_1, Γ_1 or

M_2, Γ_2 , and D, E for either D_1, E_1 or D_2, E_2 . In other words, Γ acts freely and properly on M with compact quotient, and D is a Γ -equivariant operator on the bundle $E \rightarrow M$.

4.1. Functional calculus on the maximal Roe algebra

We shall view the C^* -algebra $C_{\max}^*(M)^\Gamma$ as a right Hilbert module over itself. The inner product and right action on $C_{\max}^*(M)^\Gamma$ are defined naturally through multiplication: for $a, b \in C_{\max}^*(M)^\Gamma$,

$$\langle a, b \rangle = a^*b, \quad a \cdot b = ab. \quad (4.1)$$

The algebra of compact operators on this Hilbert module can be identified with $C_{\max}^*(M)^\Gamma$ via left multiplication. Similarly, the algebra of bounded adjointable operators can be identified with the multiplier algebra \mathcal{M} of $C_{\max}^*(M)^\Gamma$.

The operator D defines an unbounded operator on this Hilbert module in the following way. First note that D acts on smooth sections of the external tensor product $E \boxtimes E^* \rightarrow M \times M$ by taking a section s to the section Ds defined by

$$(Ds)(x, y) = D_x s(x, y), \quad (4.2)$$

where D_x denotes D acting on the x -variable.

Let \mathcal{S}^Γ be the $*$ -subalgebra of $\mathbb{C}[M]^\Gamma$ defined in Remark 2.7. That is, an element of \mathcal{S}^Γ is an operator T_κ given by a smooth kernel $\kappa \in C_b^\infty(E \boxtimes E^*)$ that:

- (i) is Γ -equivariant with respect to the diagonal Γ -action;
- (ii) has finite propagation, meaning that there exists a constant $c_\kappa \geq 0$ such that if $d(x, y) > c_\kappa$ then $\kappa(x, y) = 0$.

The operator D acts on elements of \mathcal{S}^Γ by acting on the corresponding smooth kernels as in (4.2). In this way, D becomes a densely defined operator on the Hilbert module $C_{\max}^*(M)^\Gamma$ that one verifies is symmetric with respect to the inner product in (4.1).

Theorem 4.1 ([11] Theorem 3.1). *There exists a real number $\mu \neq 0$ such that the operators*

$$D \pm \mu i: C_{\max}^*(M)^\Gamma \rightarrow C_{\max}^*(M)^\Gamma$$

have dense range.

Consequently, the operator D on the Hilbert module $C_{\max}^*(M)^\Gamma$ is regular and essentially self-adjoint, and so admits a continuous functional calculus (see [15, Theorem 10.9] and [14, Proposition 16]):

Theorem 4.2. *For $j = 1$ or 2 , there is a $*$ -preserving linear map*

$$\begin{aligned}\pi: C(\mathbb{R}) &\rightarrow \mathcal{R}(C_{\max}^*(M)^\Gamma), \\ f &\mapsto f(D) := \pi(f),\end{aligned}$$

where $C(\mathbb{R})$ denotes the continuous functions $\mathbb{R} \rightarrow \mathbb{C}$, and $\mathcal{R}(C_{\max}^*(M)^\Gamma)$ denotes the regular operators on $C_{\max}^*(M)^\Gamma$, such that:

- (i) π restricts to a $*$ -homomorphism $C_b(\mathbb{R}) \rightarrow \mathcal{M}$;
- (ii) If $|f(t)| \leq |g(t)|$ for all $t \in \mathbb{R}$, then $\text{dom}(g(D)) \subseteq \text{dom}(f(D))$;
- (iii) If $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C(\mathbb{R})$ for which there exists $f' \in C(\mathbb{R})$ such that $|f_n(t)| \leq |f'(t)|$ for all $t \in \mathbb{R}$, and if f_n converge to a limit function $f \in C(\mathbb{R})$ uniformly on compact subsets of \mathbb{R} , then $f_n(D)x \rightarrow f(D)x$ for each $x \in \text{dom}(f(D))$;
- (iv) $\text{Id}(D) = D$.

4.2. The wave operator

We now discuss the relationship between the wave operator formed using the functional calculus from Theorem 4.2 and the classical wave operator on L^2 . Both of these operators can be viewed as bounded multipliers of the maximal Roe algebra $C_{\max}^*(M)^\Gamma$, and we will see that:

Proposition 4.3. *For each $t \in \mathbb{R}$, we have*

$$e_{L^2}^{itD} = e^{itD} \in \mathcal{M}(C_{\max}^*(M)^\Gamma).$$

Let us begin by explaining both sides of the equation, starting with the right-hand side. For each $t \in \mathbb{R}$, the functional calculus from Theorem 4.2 allows one to form a bounded adjointable operator e^{itD} on the Hilbert module $C_{\max}^*(M)^\Gamma$. The resulting group of operators $\{e^{itD}\}_{t \in \mathbb{R}}$ is strongly continuous in the sense of Theorem 4.2 (iii) and uniquely solves the wave equation on $C_{\max}^*(M)^\Gamma$:

Lemma 4.4. *For any $\kappa \in \mathcal{S}^\Gamma$, $u(t) = e^{itD}\kappa$ is the unique solution of the problem*

$$\frac{du}{dt} = iDu, \quad u(0) = \kappa, \tag{4.3}$$

with $u: \mathbb{R} \rightarrow C_{\max}^*(M)^\Gamma$ a differentiable map taking values in $\text{dom}(D)$.

Proof. For each $t \in \mathbb{R}$, the function $s \mapsto e^{its}$ is a unitary in $C_b(\mathbb{R})$. Hence e^{itD} is bounded adjointable and unitary. Let h_n be a sequence of positive real numbers converging to 0 as $n \rightarrow \infty$. Then for each $t \in \mathbb{R}$, the sequence of functions

$$f_n(s) := \frac{e^{i(t+h_n)s} - e^{its}}{h_n}$$

converges to $f(s) := ise^{its}$ uniformly on compact subsets of \mathbb{R} in the limit $n \rightarrow \infty$. Also, each f_n is bounded above by $|1 + s|$. By Theorem 4.2 (iii), this implies (4.3).

For the uniqueness claim, let v be another solution of (4.3) with $v(0) = \kappa$. For any fixed $s \in \mathbb{R}$ and $0 \leq t \leq s$, set $w(t) = e^{itD}v(s-t)$. Then we have

$$\frac{dw}{dt} = iDe^{itD}v(s-t) - ie^{itD}Dv(s-t) = 0.$$

It follows that $w(t)$ is constant for all t , hence

$$v(s) = w(0) = w(s) = e^{isD}\kappa = u(s). \quad \square$$

On the other hand, we may apply the functional calculus on $L^2(E)$ to the essentially self-adjoint operator

$$D: L^2(E) \rightarrow L^2(E) \tag{4.4}$$

to form the *classical* wave operator $e_{L^2}^{itD}$, not to be confused with the bounded adjointable operator e^{itD} . The resulting operator group $\{e_{L^2}^{itD}\}_{t \in \mathbb{R}}$ is strongly continuous in $\mathcal{B}(L^2(E))$ and uniquely solves the wave equation on $L^2(E)$: for every $v \in C_c^\infty(E)$, we have

$$\frac{d}{dt}(e_{L^2}^{itD}(v)) = iDe_{L^2}^{itD}(v).$$

Via composition, $e_{L^2}^{itD}$ defines a map

$$e_{L^2}^{itD}: \mathcal{S}^\Gamma \rightarrow \mathcal{S}^\Gamma. \tag{4.5}$$

A standard argument involving the Sobolev embedding theorem now shows that for any $T_\kappa \in \mathcal{S}^\Gamma$, the kernel of $e_{L^2}^{itD} \circ T_\kappa$ is smooth, in addition to having finite propagation and being Γ -equivariant. Indeed, for any $m \in \mathbb{N}$, the operator $D^m \circ T_\kappa$ is bounded on $L^2(E)$, as M/Γ is compact. Together with the fact that

$$D^m \circ e_{L^2}^{itD} \circ T_\kappa = e_{L^2}^{itD} \circ D^m \circ T_\kappa,$$

Sobolev theory implies that the image of $e_{L^2}^{itD} \circ T_\kappa$ lies in the smooth sections, so that this operator also has smooth kernel. By Remark 2.9, (4.5) extends uniquely to a bounded multiplier of $C_{\max}^*(M)^\Gamma$,

$$e_{L^2}^{itD}: C_{\max}^*(M)^\Gamma \rightarrow C_{\max}^*(M)^\Gamma. \tag{4.6}$$

This is the operator on the left-hand side of Proposition 4.3.

Observe that if k is a smooth, *compactly supported* Schwartz kernel on $M \times M$, then we have the pointwise equality

$$\left(\frac{d}{dt}e_{L^2}^{itD}k\right)(x,y) = (iDe_{L^2}^{itD}k)(x,y), \quad (4.7)$$

where $e_{L^2}^{itD}k$ denotes the smooth Schwartz kernel of the composition $e_{L^2}^{itD} \circ T_k$. The proof of Proposition 4.3, involves a more general form of this observation for kernels in \mathcal{S}^Γ :

Lemma 4.5. *Let $\kappa \in \mathcal{S}^\Gamma$, and let $e_{L^2}^{itD}\kappa$ denote the smooth Schwartz kernel of $e_{L^2}^{itD} \circ T_\kappa$. Then for every $t \in \mathbb{R}$ and $x, y \in M$, we have the pointwise equality*

$$\left(\frac{d}{dt}e_{L^2}^{itD}\kappa\right)(x,y) = (iDe_{L^2}^{itD}\kappa)(x,y). \quad (4.8)$$

Moreover, the path $t \mapsto e_{L^2}^{itD}\kappa$ is continuous with respect to the operator norm, and

$$\lim_{h \rightarrow 0} \left\| \frac{e_{L^2}^{i(t+h)D} \circ T_\kappa - e_{L^2}^{itD} \circ T_\kappa}{h} - iDe_{L^2}^{itD} \circ T_\kappa \right\|_{\mathcal{B}(L^2(E))} = 0. \quad (4.9)$$

Proof. Fix $t \in \mathbb{R}$, $\epsilon > 0$, and $x, y \in M$. Take $\phi \in C_c^\infty(M)$ such that $\phi(z) = 1$ if $z \in B_{2t}(x)$, and $\phi(z) = 0$ if $z \in M \setminus B_{3t}(x)$. Since κ has finite propagation, the smooth kernel $\phi\kappa$ is compactly supported in $M \times M$, so by (4.7) we have

$$\left(\frac{d}{dt}e_{L^2}^{itD}\phi\kappa\right)(x,y) = (iDe_{L^2}^{itD}\phi\kappa)(x,y). \quad (4.10)$$

The fact that $e_{L^2}^{itD}$ has propagation at most t , together with the fact that

$$(1 - \phi)\kappa(z, y) = 0$$

for all $z \in B_{2t}(x)$, implies that for all $w \in B_t(x)$ we have $(e_{L^2}^{itD}(1 - \phi)\kappa)(w, y) = 0$. It follows that

$$(e_{L^2}^{itD}\kappa)(w, y) = (e_{L^2}^{itD}\phi\kappa)(w, y). \quad (4.11)$$

Taking $w = x$ and combining with (4.10) gives (4.8).

To establish norm continuity, it suffices to show that $t \mapsto e_{L^2}^{itD}\kappa$ is norm-continuous at $t = 0$. For each x, y , and t in an interval $[-t_0, t_0]$, we have

$$\left|e_{L^2}^{itD}\kappa(x, y) - \kappa(x, y)\right| \leq |t| \cdot \sup_{s \in [-t_0, t_0]} \left|iDe_{L^2}^{isD}\kappa(x, y)\right| \quad (4.12)$$

by the mean value theorem applied to (4.8). Since the operator $iDe_{L^2}^{isD}\kappa$ is Γ -equivariant with finite propagation, and M is cocompact, there exists a compact subset $K \subseteq M$ such that

$$\sup_{\substack{x,y \in M, \\ s \in [-t_0, t_0]}} \left| iDe_{L^2}^{isD} \kappa(x, y) \right| = \sup_{\substack{x,y \in K, \\ s \in [-t_0, t_0]}} \left| iDe_{L^2}^{isD} \kappa(x, y) \right| \leq C_{t_0} \quad (4.13)$$

for some constant C_{t_0} . Now since $e_{L^2}^{itD} \kappa - \kappa$ has finite propagation, its operator norm can be estimated using and (4.12) and (4.13):

$$\|e_{L^2}^{itD} \kappa - \kappa\|_{\mathcal{B}(L^2(E))} \leq C \cdot \sup_{x,y \in M} \left| e_{L^2}^{itD} \kappa(x, y) - \kappa(x, y) \right| \leq CC_{t_0}|t|,$$

for some constant C depending on the propagation of κ . We obtain norm continuity by taking a limit $t \rightarrow 0$. The proof of (4.9) is a straightforward adaptation of this argument. \square

With these preparations, we now prove Proposition 4.3.

Proof of Proposition 4.3. Fix $t \in \mathbb{R}$. Let $T_\kappa \in \mathcal{S}^\Gamma$ with smooth Schwartz kernel κ . We claim that the kernel $e_{L^2}^{itD} \kappa$ satisfies the wave equation in $C_{\max}^*(M)^\Gamma$. To see this, first note that by equation (4.9) we have

$$\lim_{h \rightarrow 0} \left\| \frac{e_{L^2}^{i(t+h)D} \circ T_\kappa - e_{L^2}^{itD} \circ T_\kappa}{h} - iDe_{L^2}^{itD} \circ T_\kappa \right\|_{\mathcal{B}(L^2(E))} = 0.$$

Now since the kernels $e_{L^2}^{i(t+h)D} \kappa$ and $De_{L^2}^{itD} \kappa$ each have propagation at most

$$r := \text{prop}(\kappa) + |t + h|,$$

by Remark 2.9 there exists a constant C_r such that the norm of

$$\frac{e_{L^2}^{i(t+h)D} \kappa - e_{L^2}^{itD} \kappa}{h} - iDe_{L^2}^{itD}(\kappa)$$

in $C_{\max}^*(M)^\Gamma$ is bounded above by C_r times its norm in $\mathcal{B}(L^2(E))$. Thus

$$\lim_{h \rightarrow 0} \left\| \frac{e_{L^2}^{i(t+h)D} \kappa - e_{L^2}^{itD} \kappa}{h} - iDe_{L^2}^{itD}(\kappa) \right\|_{\max} = 0,$$

so the operator group $\{e_{L^2}^{itD}\}_{t \in \mathbb{R}}$ solves the wave equation (4.3) in $C_{\max}^*(M)^\Gamma$. It follows from the uniqueness property in Lemma 4.4 that $e_{L^2}^{itD}$ and e^{itD} coincide on \mathcal{S}^Γ , and hence are equal as elements of \mathcal{M} . \square

5. Functoriality for the higher index

In this section, we discuss functoriality in the case of the maximal higher index from the point of view of folding maps and wave operators. This will prepare us for the proof of our main result, Theorem 1.1, in the next section.

Suppose M_1 and M_2 are two Galois covers of N with deck transformation groups Γ_1 and $\Gamma_2 \cong \Gamma_1/H$, for some normal subgroup H of Γ_1 . Then the quotient homomorphism $\Gamma_1 \rightarrow \Gamma_2$ induces a natural surjective $*$ -homomorphism between group algebras,

$$\alpha: \mathbb{C}\Gamma_1 \rightarrow \mathbb{C}\Gamma_2, \quad \sum_{i=1}^k c_{\gamma_i} \gamma_i \mapsto \sum_{i=1}^k c_{\gamma_i} [\gamma_i],$$

where $[\gamma]$ is the class of $\gamma \in \Gamma_1$ in Γ_1/H . In this setting, we have the following functoriality result for the maximal higher index, which follows from a theorem of Valette [17, Theorem 1.1]:

Theorem 5.1. *Let N be a closed Riemannian manifold. Let D_N be a first-order, self-adjoint elliptic differential operator acting on a bundle $E_N \rightarrow N$. Let M_1 and M_2 be Galois covers of N with deck transformation groups Γ_1 and $\Gamma_2 \cong \Gamma_1/H$ respectively, for a normal subgroup H of Γ_1 . Let D_1 and D_2 be the lifts of D_N to M_1 and M_2 . Then the map on K -theory induced by α relates the maximal higher indices of D_1 and D_2 :*

$$\alpha_*(\text{Ind}_{\Gamma_1, \max} D_1) = \text{Ind}_{\Gamma_2, \max} D_2 \in K_*(C_{\max}^*(\Gamma_2)).$$

This result was proved originally using KK -theory. To prepare for the proof of Theorem 1.1, we now give a proof of Theorem 5.1 using local properties of the wave operator and the folding map Ψ from the previous sections.

5.1. Higher index

We begin by recalling the definition of the maximal higher index of an equivariant elliptic operator. Let Γ , M , and D be as in section 4.

Let $\mathcal{Q} := \mathcal{M}/C_{\max}^*(M)^\Gamma$. Consider the short exact sequence of C^* -algebras

$$0 \rightarrow C_{\max}^*(M)^\Gamma \rightarrow \mathcal{M} \rightarrow \mathcal{Q} \rightarrow 0,$$

where \mathcal{M} is shorthand for the multiplier algebra $\mathcal{M}(C_{\max}^*(M)^\Gamma)$. This induces the following six-term exact sequence in K -theory:

$$\begin{array}{ccccc} K_0(C_{\max}^*(M)^\Gamma) & \longrightarrow & K_0(\mathcal{M}) & \longrightarrow & K_0(\mathcal{Q}) \\ \partial_0 \uparrow & & & & \downarrow \partial_1 \\ K_1(\mathcal{Q}) & \longleftarrow & K_1(\mathcal{M}) & \longleftarrow & K_1(C_{\max}^*(M)^\Gamma), \end{array}$$

where the connecting maps ∂_0 and ∂_1 , known as *index maps*, are defined as follows.

Definition 5.2.

- (i) ∂_0 : let u be an invertible matrix over \mathcal{Q} representing a class in $K_1(\mathcal{Q})$. Let v be the inverse of u . Let U and V be lifts of u and v to a matrix algebra over \mathcal{M} . Then the matrix

$$W = \begin{pmatrix} 1 & 0 \\ U & 1 \end{pmatrix} \begin{pmatrix} 1 & -V \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ U & 1 \end{pmatrix}$$

is invertible, and $P = W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^{-1}$ is an idempotent. We define

$$\partial_0[u] := [P] - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in K_0(C_{\max}^*(M)^\Gamma). \quad (5.1)$$

- (ii) ∂_1 : let q be an idempotent matrix over \mathcal{Q} representing a class in $K_0(\mathcal{Q})$. Let Q be a lift of q to a matrix algebra over \mathcal{M} . Then $e^{2\pi i Q}$ is a unitary in the unitized algebra, and we define

$$\partial_1[q] := [e^{2\pi i Q}] \in K_1(C_{\max}^*(M)^\Gamma). \quad (5.2)$$

This construction is applied to the operator D via the functional calculus from Theorem 4.2, as follows. Let $\chi: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, odd function such that

$$\lim_{x \rightarrow +\infty} \chi(x) = 1,$$

known as a *normalizing function*. Using Theorem 4.2, we obtain an element $\chi(D)$ in \mathcal{M} . We now have:

Lemma 5.3. *The class of $\chi(D)$ in $\mathcal{M}/C_{\max}^*(M)^\Gamma$ is invertible and independent of the choice of normalizing function χ .*

Proof. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of functions $\mathbb{R} \rightarrow \mathbb{C}$. Then for every $f \in \mathcal{S}(\mathbb{R})$ with compactly supported Fourier transform \widehat{f} , the operator $f(D)$ is given by a smooth kernel [18, Proposition 2.10]. Since every $f \in C_0(\mathbb{R})$ function is a uniform limit of such functions, the first part of Theorem 4.2 implies that for every such f we have $f(D) \in C_{\max}^*(M)^\Gamma$.

Now if χ is a normalizing function, then $\chi^2 - 1 \in C_0(\mathbb{R})$. Hence the class of $\chi(D)$ in $\mathcal{M}/C_{\max}^*(M)^\Gamma$ is invertible. Since any two normalizing functions differ by an element of $C_0(\mathbb{R})$, this class is independent of the choice of χ . \square

Using this lemma, one computes that

$$\frac{\chi(D) + 1}{2}$$

is an idempotent modulo $C_{\max}^*(M)^\Gamma$ and so defines element of $K_0(\mathcal{M}/C_{\max}^*(M)^\Gamma)$. This leads us to the definition of the maximal higher index of D :

Definition 5.4. For $i = 1, 2$, let ∂_i be the connecting maps from Definition 5.2. The *maximal higher index* of D is the element

$$\text{Ind}_{\Gamma, \max} D := \begin{cases} \partial_1 [\chi(D)] \in K_0(C_{\max}^*(M)^\Gamma), & \text{if } \dim M \text{ is even,} \\ \partial_0 \left[\frac{\chi(D)+1}{2} \right] \in K_1(C_{\max}^*(M)^\Gamma), & \text{if } \dim M \text{ is odd.} \end{cases}$$

5.2. Functoriality

In this subsection, we return to the geometric setup described in subsection 2.3 and give a new proof of Theorem 5.1.

A key idea is to use the local nature of the wave operator to prove:

Proposition 5.5. *For all $t \in \mathbb{R}$, we have $\Psi(e^{itD_1}) = e^{itD_2}$.*

Proof. For $j = 1, 2$, let $e_{L^2}^{itD_j}$ be the wave operator on $L^2(E_j)$. By (4.6), this operator extends uniquely to a bounded multiplier of $C_{\max}^*(M_j)^{\Gamma_j}$. By Proposition 4.3, we have $e_{L^2}^{itD_j} = e^{itD_j} \in \mathcal{M}_j$. Thus to prove this proposition, it suffices to show that $\Psi(e_{L^2}^{itD_1}) = e_{L^2}^{itD_2}$, as elements of $B_{\text{fp}}(L^2(E_j))^{\Gamma_j}$.

As a notational convenience, we will write e^{itD_j} for $e_{L^2}^{itD_j} \in B_{\text{fp}}(L^2(E_j))^{\Gamma_j}$. Let the open covers \mathcal{U}_{M_1} and \mathcal{U}_{M_2} be as in (3.3) and (3.4), so that by definition, there exists some $\epsilon > 0$ such that each ball of diameter ϵ in M_2 is evenly covered with respect to $\pi: M_1 \rightarrow M_2$.

Let $\{V_k\}$ be another open cover of M_2 such that each V_k has diameter at most $\frac{\epsilon}{2}$ and such that any compact subset of M_2 intersects only finitely many of the V_k . Let $\{\rho_k\}$ be a partition of unity subordinate to $\{V_k\}$.

Choose a positive integer n such that $\frac{t}{n} < \frac{\epsilon}{8}$. Now since

$$e^{itD_1} = \left(e^{i\frac{t}{n}D_1}\right)^n,$$

and Ψ is a $*$ -homomorphism, it suffices to show that $\Psi(e^{i\frac{t}{n}D_1}) = e^{i\frac{t}{n}D_2}$. Noting that any section $u \in C_c(E_2)$ can be written as a finite sum $u = \sum_k \rho_k u$, we have

$$\Psi(e^{i\frac{t}{n}D_1})u = \sum_k \Psi(e^{i\frac{t}{n}D_1})(\rho_k u).$$

We first claim that for each k we have

$$\Psi(e^{i\frac{t}{n}D_1})(\rho_k u) = e^{i\frac{t}{n}D_2}(\rho_k u). \quad (5.3)$$

To see this, note that the ball of radius $\frac{\epsilon}{8}$ around $\text{supp}(\rho_k u)$ has diameter at most ϵ and so is evenly covered with respect to π . Since the definition of Ψ is independent of the choice of compatible partitions of unity by Proposition 3.3, we may work with a partition of unity $\{\phi_j^{[g]}\}$ for M_2 subordinate to a cover \mathcal{U}_{M_2} , such that $B_{\frac{\epsilon}{4}}(\text{supp}(\rho_k u)) \subseteq U_{j_0}$ for some open set $U_{j_0}^{[s_0]} \in \mathcal{U}_{M_2}$ and $\phi_{j_0}^{[s_0]} \equiv 1$ on $B_{\frac{\epsilon}{8}}(\text{supp}(\rho_k u))$, for some $s_0 \in S$ and $j_0 \in I$. By the definition of the folding map (3.6) applied to this choice of open cover, we have

$$\begin{aligned} \Psi(e^{i\frac{t}{n}D_1})(\rho_k u) &= \sum_{\substack{g \in \Gamma_1, \\ s \in S}} \sum_{i, j \in I} \phi_i^{[g]}(e^{i\frac{t}{n}D_1} \phi_j^s(\rho_k u)) \\ &= \phi_{j_0}^{[s_0]}(e^{i\frac{t}{n}D_1}(\pi|_{U_{j_0}^{s_0}}^*(\rho_k u))) \\ &= \pi_*(e^{i\frac{t}{n}D_1}(\pi|_{U_{j_0}^{s_0}}^*(\rho_k u))). \end{aligned}$$

Now the wave equation on M_1 reads

$$\frac{\partial}{\partial t}(e^{i\frac{t}{n}D_1}(\pi|_{U_{j_0}^{s_0}}^*(\rho_k u))) = iD_1(e^{i\frac{t}{n}D_1}(\pi|_{U_{j_0}^{s_0}}^*(\rho_k u))).$$

Applying π_* to both sides of this equation and using that D_1 is the lift of D_2 , we obtain

$$\frac{\partial}{\partial t}(\Psi(e^{i\frac{t}{n}D_1})(\rho_k u)) = iD_2(\Psi(e^{i\frac{t}{n}D_1})(\rho_k u)),$$

whence (5.3) follows from uniqueness of the solution to the wave equation on M_2 .

Taking a sum over k now yields

$$\Psi(e^{i\frac{t}{n}D_1})u = \sum_k \Psi(e^{i\frac{t}{n}D_1})(\rho_k u) = \sum_k e^{i\frac{t}{n}D_2}(\rho_k u) = e^{i\frac{t}{n}D_2}u.$$

Thus $\Psi(e^{i\frac{t}{n}D_1}) = e^{i\frac{t}{n}D_2}$, as bounded operators on $L^2(E_2)$. By our previous remarks, this means that $\Psi(e^{itD_1}) = e^{itD_2}$. \square

Applying Fourier inversion together with Proposition 5.5 leads to:

Proposition 5.6. *For any $f \in C_0(\mathbb{R})$ we have*

$$\Psi(f(D_1)) = f(D_2) \in \mathcal{M}_2.$$

Proof. Suppose first that $f \in \mathcal{S}(\mathbb{R})$ with compactly supported Fourier transform. By the Fourier inversion formula, we have

$$f(D_j) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(t) e^{itD_j} dt, \quad (5.4)$$

where the integral converges strongly in \mathcal{M}_j . Now for any $\kappa \in C_{\max}^*(M_2)^{\Gamma_2}$, Proposition 3.5 implies that there exists $\kappa_0 \in C_{\max}^*(M_1)^{\Gamma_1}$ such that $\Psi(\kappa_0) = \kappa$. Since Ψ is a $*$ -homomorphism,

$$\Psi(f(D_1))(\kappa) = \Psi(f(D_1)\kappa_0) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(t) \Psi(e^{itD_1}\kappa_0) dt.$$

By Proposition 5.5, this equals

$$\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(t) e^{itD_2} \kappa dt = f(D_2)\kappa.$$

This proves the claim for $f \in \mathcal{S}(\mathbb{R})$. The general claim now follows from density of $\mathcal{S}(\mathbb{R})$ in $C_0(\mathbb{R})$. \square

Proof of Theorem 5.1. The expressions (5.1) and (5.2) show that, for $j = 1, 2$, the higher index of D_j is represented by a matrix $A_j(\chi)$ whose entries are operators formed using functional calculus of D_j . More precisely, if the initial operator D_N on N is ungraded, as is typically the case for M odd-dimensional, then

$$A_j(\chi) = e^{\pi i(\chi+1)}(D_j). \quad (5.5)$$

When D_j is odd-graded with respect to a \mathbb{Z}_2 -grading on the bundle $E_j = E_j^+ \oplus E_j^-$, as typically occurs when $\dim N$ is even, we have a direct sum decomposition $\chi(D_j) = \chi(D_j)^+ \oplus \chi(D_j)^-$. In this case, the index element is represented explicitly by the matrix

$$A_j(\chi) = \begin{pmatrix} (1-\chi(D_j)^- \chi(D_j)^+)^2 & \chi(D_j)^- (1-\chi(D_j)^+ \chi(D_j)^-) \\ \chi(D_j)^+ (2-\chi(D_j)^- \chi(D_j)^+) (1-\chi(D_j)^- \chi(D_j)^+) & \chi(D_j)^+ \chi(D_j)^- (2-\chi(D_j)^+ \chi(D_j)^-) - 1 \end{pmatrix} \quad (5.6)$$

Observe that in either case, each entry of A_j is an operator of the form $f(D_j)$, for some $f \in C_0(\mathbb{R})$ (modulo grading and the identity operator). By Proposition 5.6, Ψ maps each entry of A_1 to the corresponding entry of A_2 . Hence $\Psi_*[A_1] = [A_2]$, which proves the claim. \square

Remark 5.7. When $\Gamma_1 = \pi_1 N$ and Γ_2 is the trivial group, Theorem 5.1 reduces to the maximal version of Atiyah's L^2 -index theorem mentioned in section 1.

Atiyah's original L^2 -index theorem, which uses the von Neumann trace τ instead of the folding map, can be proved by an argument along lines similar to the proof of Theorem 5.1.

6. Functoriality for the higher rho invariant

The higher index is a primary obstruction to the existence of positive scalar curvature metrics on a manifold. When the manifold is spin with positive scalar curvature, so that

the higher index of the Dirac operator vanishes, one can define a secondary invariant called the *higher rho invariant*, introduced in [19,12]. This is an obstruction to the inverse of the Dirac operator being local [4]. In this section we show that the higher rho invariant behaves functorially under the map $\Psi_{L,0}$ from Definition 3.8.

6.1. Higher rho invariant

Consider the geometric situation in subsection 2.3, with the additional condition that the Riemannian manifold N is spin with positive scalar curvature. The operator D_N is then the Dirac operator acting on the spinor bundle E_N .

As the definition of the higher rho invariant is the same for either M_1 or M_2 , we will simply write M , Γ to mean either M_1 , Γ_1 or M_2 , Γ_2 . Similarly, D will refer to either of the lifted operators D_1 or D_2 acting on the equivariant spinor bundles E_1 or E_2 lifted from E_N .

Let κ be the scalar curvature function of the lifted metric on M , which is uniformly positive. Let $\nabla: C^\infty(E) \rightarrow C^\infty(T^*M \otimes E)$ be the connection on E induced by the Levi-Civita connection on M . Recall that by the Lichnerowicz formula,

$$D^2 = \nabla^* \nabla + \frac{\kappa}{4}.$$

Since κ is uniformly positive, D^2 is strictly positive as an unbounded operator on the Hilbert module $C_{\max}^*(M)^\Gamma$. Thus we may use the functional calculus from Theorem 4.2 to form the operator

$$F_0(D) := \frac{D}{|D|},$$

an element of the multiplier algebra $\mathcal{M} = \mathcal{M}(C_{\max}^*(M)^\Gamma)$. Observe that $\frac{F_0(D)+1}{2}$ is a projection in \mathcal{M} .

Since D is invertible, there exists $\epsilon > 0$ such that the spectrum of D is contained in $\mathbb{R} \setminus (-\epsilon, \epsilon)$. Let $\{F_t\}_{t \in \mathbb{R}^+}$ be a set of normalizing functions satisfying the following conditions:

- F_t has compactly supported distributional Fourier transform for each t ;
- $\text{diam}(\text{supp } \widehat{F_t}) \rightarrow 0$ as $t \rightarrow \infty$;
- $F_t \rightarrow \frac{x}{|x|}$ uniformly on $\mathbb{R} \setminus (-\epsilon, \epsilon)$ in the limit $t \rightarrow 0$.

In the limit $t \rightarrow \infty$, the propagations of $F_t(D)$ tend to 0. By Theorem 4.2 (i), as $t \rightarrow 0$, the operators $F_t(D)$ converge to $F_0(D)$ in the norm of \mathcal{M} .

Define a path $\mathbb{R}^{\geq 0} \rightarrow (C_{\max}^*(M)^\Gamma)^+$ given by

$$R_D: t \mapsto A(F_t), \tag{6.1}$$

where the matrix $A(F_t)$ is defined by (5.5) or (5.6) depending upon the dimension of N (the subscript j is omitted). Noting that

$$R_D(0) = A(F_0) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \dim N \text{ is even,} \\ 1 & \text{if } \dim N \text{ is odd,} \end{cases}$$

one sees that R_D is a matrix with entries in $(C_{L,0,\max}^*(M)^\Gamma)^+$.

Definition 6.1. The *higher rho invariant* of D on the Riemannian manifold M is

$$\rho_{\max}(D) = [R_D] \in K_i(C_{L,0,\max}^*(M)^\Gamma),$$

where $i = \dim M \pmod{2}$.

6.2. Functoriality

We are now ready to complete the proof of our main result, Theorem 1.1, using the tools we developed in sections 3, 4, and 5.

Recall from Definition 3.8 that we have a folding map at level of obstructions algebras,

$$\Psi_{L,0}: C_{L,0,\max}^*(M_1)^{\Gamma_1} \rightarrow C_{L,0,\max}^*(M_2)^{\Gamma_2}.$$

This map is well-defined because the folding map Ψ at the level of maximal equivariant Roe algebras preserves small propagation of operators, by Proposition 3.4. The induced map on K -theory,

$$(\Psi_{L,0})_*: K_*(C_{L,0,\max}^*(M_1)^{\Gamma_1}) \rightarrow K_*(C_{L,0,\max}^*(M_2)^{\Gamma_2}),$$

implements functoriality of the maximal higher rho invariant.

Proof of Theorem 1.1. For $j = 1, 2$, let the higher rho invariants of D_j be denoted by $\rho_{\max}(D_j)$, as in Definition 6.1. By (6.1), this class is represented the path

$$R_{D_j}: t \mapsto A_j(F_t),$$

where the matrix A_j is as in (5.5) and (5.6). By Definition 3.8, the map $(\Psi_L)_*$ takes the class

$$[R_{D_1}] \in K_*(C_{L,0,\max}^*(M_1)^{\Gamma_1})$$

to the class of the composed path

$$\Psi \circ R_{D_1}: t \mapsto \Psi(A_1(F_t))$$

in $K_*(C_{L,0,\max}^*(M_2)^{\Gamma_2})$. Since each entry of A_j is an operator of the form $f(D_j)$, for some $f \in C_0(\mathbb{R})$ (up to grading and the identity operator), Proposition 5.6 implies that for each $t \geq 0$, we have

$$\Psi \circ R_{D_1}(t) = \Psi(A_1(F_t)) = A_2(F_t) = R_{D_2}(t).$$

It follows that $(\Psi_{L,0})_*(\rho_{\max}(D_1)) = \rho_{\max}(D_2) \in K_*(C_{L,0,\max}^*(M_2)^{\Gamma_2})$. \square

7. Generalizations to the non-cocompact setting

The methods in this paper can be used to establish analogous results in more general geometric settings. In this final section, we give two such generalizations, both involving non-cocompact actions.

We will work with the non-cocompact analogue of the geometric setup in subsection 2.3, so that the manifold N is no longer assumed to be compact. We will assume throughout this section that the operator D_N has unit propagation speed. In place of the finite partition of unity (3.2) used to define the folding map Ψ , we take a locally finite partition of unity \mathcal{U}_N whose elements are evenly covered with respect to the projections $p_1: M_1 \rightarrow N$ and $p_2: M_2 \rightarrow N$, and with the property that any compact subset of N intersects only finitely many elements of \mathcal{U}_N . The equivariant partitions of unity \mathcal{U}_{M_1} and \mathcal{U}_{M_2} of M_1 and M_2 are defined in the same way according to (3.3) and (3.4).

The local nature of the wave operator e^{itD} means that Proposition 5.5 generalizes naturally to this setting:

Proposition 7.1. *Let N , M_1 , and M_2 be as in this section, with N not necessarily compact. Then for all $t \in \mathbb{R}$, we have $\Psi(e^{itD_1}) = e^{itD_2}$.*

Proof. We adapt the proof of Proposition 5.5, indicating only what needs to be changed. Let the open covers \mathcal{U}_{M_1} and \mathcal{U}_{M_2} be as above. The difference now is that since N may be non-compact, we cannot assume the existence of a uniformly positive covering diameter ϵ as in the proof of Proposition 5.5.

Instead, the key point is to observe that for any fixed $t \in \mathbb{R}$ and $u \in C_c(E_2)$, the section $e^{itD_2}u$ is supported within the compact subset $B_t(\text{supp } u)$. Thus we can find $\epsilon > 0$ such that for all $x \in B_t(\text{supp } u)$, the ball $B_\epsilon(x)$ is evenly covered with respect to the projection $\pi: M_1 \rightarrow M_2$. From here, we proceed precisely as in the proof of Proposition 5.5, with M_2 replaced by $B_t(\text{supp } u)$, to show that $\Psi(e^{itD_1})u = e^{itD_2}u$. Since t and u are arbitrary, we conclude. \square

7.1. Operators invertible at infinity

In this subsection, suppose that the operator D_N is *invertible at infinity*, meaning that there exists some compact subset $Z_N \subseteq N$ on whose complement we have $D_N^2 \geq a$

for some $a > 0$. An important special case is when N is spin, D_N is the Dirac operator, and the metric g_N has uniformly positive scalar curvature outside of Z_N .

For $j = 1, 2$, the lifted operator D_j then satisfies the analogous relation $D_j^2 \geq a$ on the complement of a *cocompact*, Γ_j -invariant subset $Z_j \subseteq M_j$. We can define a version of the higher index of D_j , localized around Z_j , as follows (see also [20] and [10, section 3]).

For each $R > 0$, let $C_{\max}^*(B_R(Z_j))^{\Gamma_j}$ be the maximal equivariant Roe algebra of the R -neighborhood of Z_j . Since $B_R(Z_j)$ is cocompact, this algebra is isomorphic to $C_{\max}^*(\Gamma_j) \otimes \mathcal{K}$ by Remark 2.9. One can then show that for any $f \in C_c(-a, a)$, we have

$$f(D_j) \in \lim_{R \rightarrow \infty} C_{\max}^*(B_R(Z_j))^{\Gamma_j},$$

where $\lim_{R \rightarrow \infty} C_{\max}^*(B_R(Z_j))^{\Gamma_j}$ is the direct limit of these C^* -algebras. Indeed, this limit algebra is isomorphic to $C_{\max}^*(\Gamma) \otimes \mathcal{K}$, where \mathcal{K} denotes the compact operators on a (not necessarily compact) fundamental domain of the Γ -action. The construction from subsection 5.1 then gives an index element

$$\text{Ind}_{\Gamma_j, \max} D_j \in K_*(C_{\max}^*(\Gamma_j)).$$

Similar to the cocompact case, we have the following version of Theorem 5.1 for operators that are invertible at infinity:

Theorem 7.2. *Let N be a Riemannian manifold and D_N a first-order, self-adjoint elliptic differential operator acting on a bundle $E_N \rightarrow N$. Assume that D_N has unit propagation speed. Let M_1 and M_2 be Galois covers of N with deck transformation groups Γ_1 and $\Gamma_2 \cong \Gamma_1/H$ respectively, for a normal subgroup H of Γ_1 . Let D_1 and D_2 be the lifts of D_N to M_1 and M_2 respectively. Then the map on K -theory induced by the folding map Ψ relates the maximal higher indices of D_1 and D_2 :*

$$\Psi_*(\text{Ind}_{\Gamma_1, \max} D_1) = \text{Ind}_{\Gamma_2, \max} D_2 \in K_*(C_{\max}^*(\Gamma_2)).$$

Proof. Analogous to the proof of Theorem 5.1, with Proposition 5.5 replaced by Proposition 7.1. \square

7.2. Manifolds with equivariantly bounded geometry

A second generalization involves non-cocompact coverings that satisfy certain additional geometric conditions. Under these conditions we obtain generalizations of both Theorems 5.1 and 1.1. In contrast to Theorem 7.2, here we do not require invertibility of the operator at infinity, and the Roe algebra we work with does not need to be defined in terms of cocompact sets as in subsection 7.1.

Again, suppose that we are in the setup of subsection 2.3, but with N not necessarily compact. We impose two conditions on the geometry of N and the Γ_i -action on M_i (see also [11, subsection 2.1]):

- A. The Riemannian manifold N has positive injectivity radius, and its curvature tensor is uniformly bounded across N along with all of its derivatives.
- B. For $j = 1, 2$, there exists a fundamental domain \mathcal{D}_j for the action of Γ_j on M_j such that

$$l(\gamma) \rightarrow \infty \implies d(\mathcal{D}_j, \gamma \mathcal{D}_j) \rightarrow \infty,$$

where $l: \Gamma_j \rightarrow \mathbb{N}$ is a fixed length function and d is the Riemannian distance on M_j .

It follows from these two assumptions and [11, Proposition 2.14] that the maximal equivariant Roe algebra $C_{\max}^*(M_j)^{\Gamma_j}$ is well-defined, along with a subalgebra $C_{\max,u}^*(M_j)^{\Gamma_j}$ called the maximal equivariant *uniform* Roe algebra. Using the latter, we constructed in [11] a version of the functional calculus suitable for the maximal setting.

We briefly recall the definition of $C_{\max,u}^*(M_j)^{\Gamma_j}$. Let $\mathcal{S}_u^{\Gamma_j}$ be the $*$ -subalgebra of \mathcal{S}^{Γ_j} (see section 4) whose kernels have uniformly bounded derivatives of all orders. In other words, an element of $\mathcal{S}_u^{\Gamma_j}$ is a bounded operator on $L^2(E_j)$ given by a Schwartz kernel $\kappa \in C_b^\infty(E_j \boxtimes E_j^*)$ such that

- (i) κ has finite propagation;
- (ii) $\kappa(x, y) = \kappa(\gamma x, \gamma y)$ for all $\gamma \in \Gamma_j$;
- (iii) Each covariant derivative of κ is uniformly bounded over M_j .

Then $C_{\max,u}^*(M_j)^{\Gamma_j}$ is defined to be the closure of $\mathcal{S}_u^{\Gamma_j}$ in $C_{\max}^*(M_j)^{\Gamma_j}$.

Similar to section 4, $C_{\max,u}^*(M_j)^{\Gamma_j}$ can be viewed as right Hilbert module over itself, with inner product and multiplication being defined by the same formula (4.1). The operator D_j can be viewed as a densely defined symmetric operator on $C_{\max,u}^*(M_j)^{\Gamma_j}$. By [11, Theorem 3.1], this operator is regular and so admits a functional calculus. The construction from subsection 5.1 then allows one to define the *maximal equivariant uniform index* of D_j ,

$$\text{index}_{\Gamma_j, \max, u} D_j \in K_*(C_{\max, u}^*(M_j)^{\Gamma_j}).$$

The corresponding versions of the higher rho invariant can be defined as in subsection 6.1, and we denote this by

$$\rho_{\max, u}(D_j) \in K_*(C_{L, 0, \max, u}^*(M_j)^{\Gamma_j}),$$

where $C_{L,0,\max,u}^*(M_j)^{\Gamma_j}$ is defined analogously to $C_{L,0,\max}^*(M_j)^{\Gamma_j}$, with $C_{\max,u}^*(M_j)^{\Gamma_j}$ in place of $C_{\max}^*(M_j)^{\Gamma_j}$. Constructions analogous to those in subsection 3.3 give way to folding maps at K -theory level, which we denote by

$$\begin{aligned} (\Psi_u)_* &: K_*(C_{\max,u}^*(M_1)^{\Gamma_1}) \rightarrow K_*(C_{\max,u}^*(M_2)^{\Gamma_2}), \\ (\Psi_{L,0,u})_* &: K_*(C_{L,0,\max,u}^*(M_1)^{\Gamma_1}) \rightarrow K_*(C_{L,0,\max,u}^*(M_2)^{\Gamma_2}). \end{aligned}$$

Further, Proposition 4.3 generalizes naturally to manifolds satisfying conditions A and B above:

Proposition 7.3. *For each $t \in \mathbb{R}$, we have*

$$e_{L^2}^{itD_j} = e^{itD_j} \in \mathcal{M}_j(C_{\max,u}^*(M_j)^{\Gamma_j}),$$

with notation as in subsection 4.2.

Proof. The proof proceeds exactly as for Proposition 4.3, but with Lemma 4.5 replaced by Lemma 7.4 below. \square

Lemma 7.4. *Let N , M_1 , and M_2 be as in this section, with N not necessarily compact. Let $\kappa \in \mathcal{S}_u^{\Gamma_j}$, and let $e_{L^2}^{itD_j} \kappa$ denote the smooth Schwartz kernel of $e_{L^2}^{itD_j} \circ T_\kappa$. Then the path $t \mapsto e_{L^2}^{itD_j} \kappa$ is continuous with respect to the operator norm, and we have*

$$\lim_{h \rightarrow 0} \left\| \frac{e_{L^2}^{i(t+h)D_j} \circ T_\kappa - e_{L^2}^{itD_j} \circ T_\kappa}{h} - iD_j e_{L^2}^{itD_j} \circ T_\kappa \right\|_{\mathcal{B}(L^2(E))} = 0.$$

Proof. Similar to equation (4.12), we have, for each $x, y \in M_j$ and $t \in [-t_0, t_0]$,

$$\left| e_{L^2}^{itD_j} \kappa(x, y) - \kappa(x, y) \right| \leq |t| \cdot \sup_{s \in [-t_0, t_0]} \left| iD_j e_{L^2}^{isD_j} \kappa(x, y) \right|.$$

Since M_j has bounded Riemannian geometry, it follows from Sobolev theory that the operator $iD_j e_{L^2}^{isD_j} \kappa(x, y)$ has uniformly bounded smooth kernel, along with all derivatives. Indeed, since $e_{L^2}^{isD_j}$ is unitary, $|iD_j e_{L^2}^{isD_j} \kappa(x, y)|$ is bounded above by a constant C_1 independent of s , x , and y . It follows that we can estimate the L^2 -norm of the finite-propagation operator $e_{L^2}^{itD_j} \kappa - \kappa$ by

$$\|e_{L^2}^{itD_j} \kappa - \kappa\|_{\mathcal{B}(L^2(E))} \leq C_2 \cdot \sup_{x, y \in M} \left| e_{L^2}^{itD_j} \kappa(x, y) - \kappa(x, y) \right| \leq C_1 C_2,$$

for some other constant C_2 . We can then finish the proof as in Lemma 4.5. \square

With this in hand, we arrive (via an obvious analogue of Proposition 5.6) at the following generalizations of Theorems 5.1 and 1.1.

Theorem 7.5. Let N be a Riemannian manifold and M_1 and M_2 be Galois covers of N with deck transformation groups Γ_1 and $\Gamma_2 \cong \Gamma_1/H$ respectively, for a normal subgroup H of Γ_1 . Suppose that the conditions A and B in this subsection are satisfied. Let D_N be a first-order, self-adjoint elliptic differential operator acting on a bundle $E_N \rightarrow N$, and assume that D_N has unit propagation speed. Let D_1 and D_2 be the lifts of D_N to M_1 and M_2 respectively. We then have:

$$\begin{aligned}(\Psi_{u,*})(\text{index}_{\Gamma_1,\max,u} D_1) &= \text{index}_{\Gamma_2,\max,u} D_2 \in K_*(C_{\max,u}^*(M_2)^{\Gamma_2}), \\ (\Psi_{u,*})(\text{index}_{\Gamma_1,\max} D_1) &= \text{index}_{\Gamma_2,\max} D_2 \in K_*(C_{\max}^*(M_2)^{\Gamma_2}).\end{aligned}$$

Theorem 7.6. Suppose (N, g_N) is a spin Riemannian manifold with uniformly positive scalar curvature. Let M_1 and M_2 be Galois covers of M with deck transformation groups Γ_1 and $\Gamma_2 \cong \Gamma_1/H$ respectively, for a normal subgroup H of Γ_1 . Suppose that the conditions A and B in this subsection are satisfied. Let D_N be the Dirac operator on N (with unit propagation speed). Let D_1 and D_2 be the lifts of D_N to M_1 and M_2 respectively. We then have:

$$\begin{aligned}(\Psi_{L,0,u})_*(\rho_{\max,u}(D_1)) &= \rho_{\max,u}(D_2) \in K_*(C_{L,0,\max,u}^*(M_2)^{\Gamma_2}), \\ (\Psi_{L,0,u})_*(\rho_{\max}(D_1)) &= \rho_{\max}(D_2) \in K_*(C_{L,0,\max}^*(M_2)^{\Gamma_2}).\end{aligned}$$

Remark 7.7. The second equality in Theorem 7.5 follows from the observation that the natural inclusion $C_{\max,u}^*(M_j)^{\Gamma_j} \hookrightarrow C_{\max}^*(M_j)^{\Gamma_j}$ relates $\text{index}_{\Gamma_j,\max,u} D_j$ to $\text{index}_{\Gamma_j,\max} D_j$. The second equality in Theorem 7.6 follows from a similar relationship at the level of obstruction algebras.

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