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Simultaneous Input and State Interval Observers for Nonlinear Systems with Full-Rank Direct Feedthrough

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***Abstract*— A simultaneous input and state interval observer is presented for Lipschitz continuous nonlinear systems with unknown inputs and bounded noise signals for the case when the direct feedthrough matrix has full column rank. The observer leverages the existence of bounding decomposition functions for mixed monotone mappings to recursively compute the maximal and minimal elements of the estimate intervals that are compatible with output/measurement signals, and are proven to contain the true state and unknown input. Furthermore, we derive a Lipschitz-like property for decomposition functions, which provides several sufficient conditions for stability of the designed observer and uniform boundedness of the sequence of estimate interval widths. Finally, the effectiveness of our approach is demonstrated using an illustrative example.**

I. INTRODUCTION

Motivation. State and unknown input estimation has recently emerged as an important and indispensable component in many engineering applications such as fault detection, urban transportation, aircraft tracking and attack (unknown input) detection and mitigation in cyber-physical systems [1]–[3]. Particularly, in bounded-error settings, interval/set-membership approaches have been proposed to provide hard accuracy bounds, which is especially useful for safety-critical systems [4]. Moreover, since the unknown inputs may be strategic in adversarial settings, the ability to simultaneously estimate states and inputs without imposing any assumption on the unknown inputs is desirable and often crucial.

Literature review. Several approaches have been proposed in the literature to design interval observers [5]–[16]. However, most of these approaches often hinge upon relatively strong assumptions about the existence of certain system properties, such as monotone dynamics, [7], [8], Metzler and/or Hurwitz partial linearization of nonlinearities [9], [10], cooperativeness [11], linear time-invariant (LTI) dynamics [12] and linear parameter-varying (LPV) dynamics that admits a diagonal Lyapunov function [13]. Moreover, the work in [14] addresses the design of interval observers for a class of continuous time nonlinear systems without unknown inputs using bounding functions by imposing somewhat restrictive assumptions on the nonlinear dynamics to conclude stability, without discussing necessary and/or sufficient conditions for the existence of bounding functions or how to compute them. The authors in [15] study the problem of interval state estimation for a class of uncertain nonlinear systems, by extracting a known nominal observable subsystem from the plant equations and designing the observer for the transformed system. However, the derived conditions for the existence and stability of the observer is not *constructive*. Moreover, there is no guarantee that the derived functional bounds have finite values, i.e., be bounded sequences. More importantly, the aforementioned works [5]–[16] do not consider unknown inputs (i.e., input, disturbance, attack or noise signals with unknown bounds/intervals) nor the reconstruction/estimation of the uncertain inputs.

The problem of designing an unknown input interval observer that satisfies L_2/L_∞ optimality criteria is investigated in [17], where the required Metzler property is formulated as a part of a semi-definite program. However, unfortunately, their approach is limited to continuous-time LPV systems. Moreover, in their setting, the (potentially unbounded) unknown inputs do not affect the output (measurement) equation. On the other hand, for systems with linear output equations and where both the state and output equations are affected by unknown inputs, the problem of simultaneously designing state and unknown input set-valued observers has been studied in our previous works for LTI [3], LPV [18] and switched linear [19] systems with bounded-norm noise.

Contributions. We consider the design of an observer that *simultaneously* returns interval-valued estimates of states and unknown inputs for a broad range of nonlinear systems. Our approach is novel in multiple ways.

First, to the best of our knowledge, all existing interval observers for nonlinear systems in the literature only return either state [5]–[16] or input [17] estimates, whereas our observer simultaneously returns both. Second, we consider arbitrary unknown input signals, i.e., no restrictive assumptions, such as *a priori* known bounds/intervals, or being bounded or stochastic with zero mean (as is often assumed for noise), are imposed on the unknown inputs. Third, leveraging decomposition functions as *nonlinear bounding mappings* of mixed monotone vector fields [20], [21], which include almost every realistic nonlinear function [22], we show that our interval estimates are guaranteed to contain the true states and unknown inputs. Fourth, we provide several sufficient conditions in the form of Linear Matrix Inequalities (LMI) for the stability of our designed observer. Finally, we provide upper bounds for the interval widths at each time step, as well as their steady-state values.

II. PRELIMINARIES

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}_{++}$ positive real numbers. For vectors $v, w \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{p \times q}$, $\|v\| \triangleq \sqrt{v^\top v}$ and $\|M\|$ denote their (induced) L_2-norm, and $v \leq w$ is an element-wise inequal-

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ity. Moreover, the transpose, Moore-Penrose pseudoinverse, (i,j)-th element and rank of M are given by $M^\top$, $M^\dagger$, $M_{i,j}$ and $\mathrm{rk}(M)$. M is called a non-negative matrix, i.e., $M \geq 0$, if $M_{i,j} \geq 0, \forall i \in \{1 \dots p\}, \forall j \in \{1 \dots q\}$. We also define $M^+, M^{++} \in \mathbb{R}^{p \times q}$ as $M^+_{i,j} = M_{i,j}$ if $M_{i,j} \geq 0$, $M^+_{i,j} = 0$ if $M_{i,j} < 0$, $M^{++} = M^+ - M$ and $|M| \triangleq M^+ + M^{++}$. For a symmetric matrix S, $S \succ 0$ and $S \prec 0$ ($S \succeq 0$ and $S \preceq 0$) are positive and negative (semi-)definite, respectively.

Next, we introduce some definitions and related results that will be useful throughout the paper.

Definition 1 (Interval, Maximal and Minimal Elements, Interval Width)**.** *A (multi-dimensional) interval $\mathcal{I} \subset \mathbb{R}^n$ is the set of all real vectors $x \in \mathbb{R}^n$ that satisfies $\underline{s} \leq x \leq \overline{s}$, where $\underline{s}$, $\overline{s}$ and $\|\overline{s} - \underline{s}\|$ are called minimal vector, maximal vector and width of $\mathcal{I}$, respectively.*

Proposition 1. *[14, Lemma 1] Let $A \in \mathbb{R}^{m \times n}$ and $\underline{b} \leq b \leq \overline{b} \in \mathbb{R}^n$. Then, $A^+\underline{b} - A^{++}\overline{b} \leq Ab \leq A^+\overline{b} - A^{++}\underline{b}$. As a corollary, if A is non-negative, then $A\underline{b} \leq Ab \leq A\overline{b}$.*

Definition 2 (Lipschitz Continuity)**.** *Vector field $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is globally L_f-Lipschitz continuous, if $\forall x_1, x_2 \in \mathbb{R}^n$, $\exists L_f \in \mathbb{R}_{++}$, such that $\|f(x_1) - f(x_2)\| \leq L_f \|x_1 - x_2\|$.*

Definition 3 (Mixed-Monotone Mappings and Decomposition Functions)**.** *[20, Definition 4] A mapping $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m$ is mixed monotone if there exists a decomposition function $f_d : \mathcal{X} \times \mathcal{X} \to \mathcal{T}$ such that:*

1) *f is embedded on the diagonal of f_d, i.e., $f_d(x,x) = f(x)$,*
2) *f_d is monotone increasing in its first argument, i.e., $x_1 \geq x_2 \Rightarrow f_d(x_1, y) \geq f_d(x_2, y)$, and*
3) *f_d is monotone decreasing in its second argument, i.e., $y_1 \geq y_2 \Rightarrow f_d(x, y_1) \leq f_d(x, y_2)$.*

Proposition 2. *[21, Theorem 1] Let $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m$ be a mixed monotone mapping with decomposition function $f_d : \mathcal{X} \times \mathcal{X} \to \mathcal{T}$ and $\underline{x} \leq x \leq \overline{x}$, where $\underline{x}, x, \overline{x} \in \mathcal{X}$. Then $f_d(\underline{x}, \overline{x}) \leq f(x) \leq f_d(\overline{x}, \underline{x})$.*

III. PROBLEM FORMULATION

System Assumptions. Consider the nonlinear discrete-time system with unknown inputs and bounded noise

$$\begin{aligned} x_{k+1} &= f(x_k) + Bu_k + Gd_k + w_k, \\ y_k &= g(x_k) + Du_k + Hd_k + v_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector at time $k \in \mathbb{N}$, $u_k \in \mathbb{R}^m$ is a known input vector, $d_k \in \mathbb{R}^p$ is an unknown input vector, and $y_k \in \mathbb{R}^l$ is the measurement vector. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^l$ are assumed to be bounded, with $\underline{w} \leq w_k \leq \overline{w}$ and $\underline{v} \leq v_k \leq \overline{v}$, where $\underline{w}$, $\overline{w}$ and $\underline{v}$, $\overline{v}$ are the known lower and upper bounds of the process and measurement noise signals, respectively. We also assume that lower and upper bounds, $\underline{x}_0$ and $\overline{x}_0$, for the initial state x_0 are available, i.e., $\underline{x}_0 \leq x_0 \leq \overline{x}_0$. The vector fields $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^l$ and matrices B, D, G and H are known and of appropriate dimensions, where G and H encode the *locations* through which the unknown input (or attack) signal can affect the system dynamics and measurements. Moreover, we assume the following:

Assumption 1. *The matrix H has full column rank.*

Assumption 2. *Vector fields $f(\cdot)$ and $g(\cdot)$ are mixed-monotone with decomposition functions $f_d(\cdot,\cdot) : \mathbb{R}^{n \times n} \to \mathbb{R}^n$ and $g_d(\cdot,\cdot) : \mathbb{R}^{n \times n} \to \mathbb{R}^l$, respectively.*

Assumption 3. *Vector fields $f(\cdot)$ and $g(\cdot)$ are globally L_f-Lipschitz and L_g-Lipschitz continuous, respectively.*

Note that Assumption 1 is a common assumption in the unknown input observer design literature, e.g., [23], while Assumption 2 is satisfied for a broad range of nonlinear functions [22]. Moreover, the decomposition function of a vector field is not unique. For instance, [16, Lemma 6] and [22] propose decomposition functions based on nonlinear optimization programs. However, due to difficulties that may arise from solving nonlinear optimization problems at run-time, we consider a tractable closed-form procedure for construction of decomposition functions, given in [20, Theorem 2]: If a vector field $h = \begin{bmatrix} h_1^\top & \dots & h_n^\top \end{bmatrix}^\top : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and its partial derivatives are bounded with known bounds, i.e., $\frac{\partial h_i}{\partial x_j} \in (a^h_{i,j}, b^h_{i,j}), \forall x \in X \in \mathbb{R}^n$, where $a^h_{i,j}, b^h_{i,j}$ are finite values, then $h(\cdot)$ is mixed monotone with a decomposition function $h_d = \begin{bmatrix} h_{d1}^\top & \dots & h_{di}^\top & \dots h_{dn}^\top \end{bmatrix}^\top$, where $h_{di}(x,y) = h_i(z) + (\alpha^h_i - \beta^h_i)^\top (x-y), \forall i \in \{1, \dots, n\}$, and $z, \alpha^h_i, \beta^h_i \in \mathbb{R}^n$ can be computed in terms of $x, y, a^h_{i,j}, b^h_{i,j}$ as given in [20, (10)–(13)]. Consequently, for $x = [x_1 \dots x_j \dots x_n]^\top$, $y = [y_1 \dots y_j \dots y_n]^\top$, we have

$$h_d(x,y) = h(z) + C_h(x-y), \tag{2}$$

with $C_h \triangleq \begin{bmatrix} [\alpha^h_1 - \beta^h_1] \dots [\alpha^h_i - \beta^h_i] & \dots [\alpha^h_m - \beta^h_m] \end{bmatrix}^\top \in \mathbb{R}^{m \times n}$, α^h_i, β^h_i as given in [20, (10)–(13)], $z = [z_1 \dots z_j \dots z_m]^\top$ and $z_j = x_j$ or y_j (dependent on the case, cf. [20, Theorem 1 and (10)–(13)] for details). On the other hand, when the precise lower and upper bounds, $a^h_{i,j}, b^h_{i,j}$, of the partial derivatives are not known or are hard to compute, we can obtain upper and lower approximations of the bounds by using *affine abstraction* algorithms, e.g., [24, Theorem 1], with the slopes set to zero, or by leveraging natural inclusions [5]. Moreover, the choice of a tighter yet tractable decomposition function (than [20]) can potentially improve the observer design.

Unknown Input (or Attack) Signal Assumptions. The unknown inputs d_k are not constrained to be a signal of any type (random or strategic) nor to follow any model, thus no prior 'useful' knowledge of the dynamics of d_k is available (independent of $\{d_\ell\}$ $\forall k \neq \ell$, $\{w_\ell\}$ and $\{v_\ell\}$ $\forall \ell$). We also do not assume that d_k is bounded or has known bounds/intervals and thus, d_k is suitable for representing adversarial attack signals.

The observer design problem can be stated as follows:

Problem 1. *Given a nonlinear discrete-time system with unknown inputs and bounded noise* (1)*, design a stable observer that simultaneously finds bounded intervals of compatible states and unknown inputs.*

IV. SIMULTANEOUS INPUT AND STATE INTERVAL OBSERVERS (SISIO)

A. Interval Observer Design

We consider a recursive two-step interval-valued observer design, composed of a *state estimation* step and an *unknown input estimation* step with the following form:

$$\textit{State Estimation: } \mathcal{I}_k^x = \mathcal{F}_x(\mathcal{I}_{k-1}^x, \mathcal{I}_{k-1}^d, u_{k-1}),$$
$$\textit{Unknown Input Estimation: } \mathcal{I}_k^d = \mathcal{F}_d(\mathcal{I}_k^x, y_k, u_k),$$

where $\mathcal{F}_x$ and $\mathcal{F}_d$ are the to-be-designed interval mappings, while $\mathcal{I}_k^x$ and $\mathcal{I}_k^d$ are the intervals of compatible states and unknown inputs at time k of the form:

$$\mathcal{I}_k^d = \{d \in \mathbb{R}^p : \underline{d}_k \le d \le \overline{d}_k\},$$
$$\mathcal{I}_k^x = \{x \in \mathbb{R}^n : \underline{x}_k \le x \le \overline{x}_k\},$$

i.e., we restrict the estimation errors to closed intervals in the Euclidean space. In this case, the observer design problem boils down to finding the minimal and maximal elements $\underline{d}_k$, $\overline{d}_k$, $\underline{x}_k$ and $\overline{x}_k$ of the intervals $\mathcal{I}_k^d$ and $\mathcal{I}_k^x$. Our interval observer can be defined at each time step $k \ge 1$ as follows (with known $\underline{x}_0$ and $\overline{x}_0$ such that $\underline{x}_0 \le x_0 \le \overline{x}_0$):

State Estimation:

$$\overline{x}_k = f_d(\overline{x}_{k-1}, \underline{x}_{k-1}) + Bu_{k-1} + G^+\overline{d}_{k-1} - G^{++}\underline{d}_{k-1} + \overline{w}, \quad (3)$$
$$\underline{x}_k = f_d(\underline{x}_{k-1}, \overline{x}_{k-1}) + Bu_{k-1} + G^+\underline{d}_{k-1} - G^{++}\overline{d}_{k-1} + \underline{w}. \quad (4)$$

Unknown Input Estimation:

$$\overline{d}_k = \min(\overline{d}_k^1, \overline{d}_k^2), \quad \underline{d}_k = \max(\underline{d}_k^1, \underline{d}_k^2), \quad (5)$$

where

$$\overline{d}_k^1 = J^+\overline{r}_k - J^{++}\underline{r}_k \quad , \quad \underline{d}_k^1 = J^+\underline{r}_k - J^{++}\overline{r}_k, \quad (6)$$
$$\overline{d}_k^2 = \begin{bmatrix}\overline{d}_{1,k}^{2\top} & \dots & \overline{d}_{p,k}^{2\top}\end{bmatrix}^\top, \quad \underline{d}_k^2 = \begin{bmatrix}\underline{d}_{1,k}^{2\top} & \dots & \underline{d}_{p,k}^{2\top}\end{bmatrix}^\top, \quad (7)$$
$$\overline{d}_{i,k}^2 = \max_{d_k \in \mathcal{D}_k} e_i d_k, \quad \underline{d}_{i,k}^2 = \min_{d_k \in \mathcal{D}_k} e_i d_k, \forall i \in \{1, \dots p\}, \quad (8)$$
$$\overline{r}_k = y_k - g_d(\underline{x}_k, \overline{x}_k) - Du_k - \underline{v}, \quad (9)$$
$$\underline{r}_k = y_k - g_d(\overline{x}_k, \underline{x}_k) - Du_k - \overline{v}, \quad (10)$$

with $J = H^\dagger$, $\tilde{H} \triangleq \begin{bmatrix}H^\top & -H^\top\end{bmatrix}^\top$, $\tilde{r}_k \triangleq \begin{bmatrix}\overline{r}_k^\top & -\underline{r}_k^\top\end{bmatrix}^\top$, $\mathcal{D}_k \triangleq \{d_k | \tilde{H} d_k \le \tilde{r}_k\}$, and $e_i \in \mathbb{R}^{1\times p}, e_i(1,i) = 1$, $e_i(1,j) = 0, \forall j \ne i$. In what follows, we will show that by choosing $J = H^\dagger$ and f_d, g_d as decomposition functions of f, g, our designed observer satisfies several desirable properties. Algorithm 1 summarizes the SISIO observer.

B. Correctness (Framer Property) of Interval Estimates

In the following, we show that the SISIO observer returns correct interval estimates in the sense that at each time step, the true states and unknown inputs are guaranteed to be within the estimated intervals given by (3)–(5). This is also known as the *framer property*, e.g., in [10]. To increase readability, all proofs will be provided in the appendix.

Theorem 1 (Correctness of the Interval Estimates)**.** *Let $\underline{x}_0 \le x_0 \le \overline{x}_0$, where $\underline{x}_0$ and $\overline{x}_0$ are known. For the system* (1)*, if Assumptions 1 and 2 hold, then the SISIO estimate intervals* (3)–(5) *with $J = H^\dagger$ and $f_d(\cdot,\cdot), g_d(\cdot,\cdot)$ as decomposition functions of $f(\cdot), g(\cdot)$ at each step k are* correct*, i.e., the true states and unknown inputs are guaranteed to satisfy $\underline{d}_k \le d_k \le \overline{d}_k$ and $\underline{x}_k \le x_k \le \overline{x}_k$.*

Algorithm 1 Simultaneous Input and State Interval Observer

Initialize: $J = H^\dagger$; maximal$(\mathcal{I}_0^x) = \overline{x}_0$; minimal$(\mathcal{I}_0^x) = \underline{x}_0$;
  Compute $J^+, J^{++}, |J|, G^+, G^{++}, |G|$; Compute L_{f_d}, L_{g_d} via Lemma 1; Compute $g_d(\overline{x}_0, \underline{x}_0), g_d(\underline{x}_0, \overline{x}_0)$ via (2);
  $\mathcal{K} \triangleq |G||J|$; $\Delta w = \overline{w} - \underline{w}$; $\Delta v = \overline{v} - \underline{v}$;
  $\overline{r}_0 = y_0 - g_d(\underline{x}_0, \overline{x}_0) - Du_0 - \underline{v}$;
  $\underline{r}_0 = y_0 - g_d(\overline{x}_0, \underline{x}_0) - Du_0 - \overline{v}$;
  $\overline{d}_0^1 = J^+\overline{r}_0 - J^{++}\underline{r}_0$; $\underline{d}_0^1 = J^+\underline{r}_0 - J^{++}\overline{r}_0$;
  $\Delta z = \Delta w + \mathcal{K}\Delta v$; $\tilde{H} \triangleq \begin{bmatrix}H^\top & -H^\top\end{bmatrix}^\top$;
  $\tilde{r}_0 \triangleq \begin{bmatrix}\overline{r}_0^\top & -\underline{r}_0^\top\end{bmatrix}^\top$; $\mathcal{D}_0 \triangleq \{d_0 | \tilde{H} d_0 \le \tilde{r}_0\}$;
  $e_i \in \mathbb{R}^{1\times p}, e_i(1,i) = 1, e_i(1,j) = 0, \forall i, j \in \{1, \dots, p\}, j \ne i$;
  $\forall i \in \{1, \dots p\}, \overline{d}_{i,0}^2 = \max_{d_0 \in \mathcal{D}_0} e_i d_0$; $\underline{d}_{i,0}^2 = \min_{d_0 \in \mathcal{D}_0} e_i d_0$;
  $\delta_0^x = \|\overline{x}_0 - \underline{x}_0\|$; $\overline{d}_0 = \min(\overline{d}_0^1, \overline{d}_0^2)$; $\underline{d}_0 = \max(\underline{d}_0^1, \underline{d}_0^2)$;
  maximal$(\mathcal{I}_0^d) = \overline{d}_0$; minimal$(\mathcal{I}_0^d) = \underline{d}_0$; $\delta_0^d = \|\overline{d}_0 - \underline{d}_0\|$;
for $k = 1$ to K **do**
  ▷ Estimation of x_k
  Compute $f_d(\overline{x}_{k-1}, \underline{x}_{k-1}), f_d(\underline{x}_{k-1}, \overline{x}_{k-1})$ via (2);
  Compute $\overline{x}_k, \underline{x}_k$ via (3) and (4);
  $\delta_k^x = \mathcal{L}^k \delta_0^x + \|\Delta z\| \left(\frac{1-\mathcal{L}^k}{1-\mathcal{L}}\right)$; $\mathcal{I}_k^x = \{x \in \mathbb{R}^n : \underline{x}_k \le x \le \overline{x}_k\}$;
  ▷ Estimation of d_k
  Compute $g_d(\overline{x}_k, \underline{x}_k), g_d(\underline{x}_k, \overline{x}_k)$ via (2);
  Compute $\overline{r}_k, \underline{r}_k$ via (9) and (10); $\tilde{r}_k \triangleq \begin{bmatrix}\overline{r}_k^\top & -\underline{r}_k^\top\end{bmatrix}^\top$;
  Compute $\overline{d}_k^1, \underline{d}_k^1$ via (6); $\mathcal{D}_k \triangleq \{d_k | \tilde{H} d_k \le \tilde{r}_k\}$;
  Compute $\overline{d}_{i,k}^2, \underline{d}_{i,k}^2$ and $\overline{d}_k, \underline{d}_k$ via (8) and (5);
  $\delta_k^d = \||J|\| L_{g_d} \delta_k^x + \||J|\Delta v\|$; $\mathcal{I}_k^d = \{d \in \mathbb{R}^p : \underline{d}_k \le d \le \overline{d}_k\}$;
end for

C. Uniform Boundedness of Estimates (Observer Stability)

In this section we study the stability of SISIO, assuming that the decomposition functions can be obtained using (2). We first derive a Lipschitz-like property for decomposition functions in Lemma 1. Then, we derive several sufficient conditions for the stability of SISIO in Theorem 2.

Lemma 1. *Let $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ be a globally L_h-Lipschitz continuous and mixed monotone vector field and $h_d(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ be the decomposition function for $h(\cdot)$, constructed using* (2)*. Consider $\underline{x} \le \overline{x}$, both in $\mathbb{R}^n$. Then $\|h_d(\overline{x}, \underline{x}) - h_d(\underline{x}, \overline{x})\| \le L_{h_d} \|\overline{x} - \underline{x}\|$, where $L_{h_d} \triangleq L_h + 2\|C_h\|$, with C_h given in* (2)*.*

Theorem 2 (Observer Stability)**.** *Consider the system* (1) *and the SISIO observer* (3)–(5)*. Suppose that Assumption 3 and all the assumptions in Theorem 1 hold and the decomposition functions f_d, g_d are constructed using* (2)*. Then, the observer is stable, in the sense that interval width sequences $\{\|\Delta_k^d\| \triangleq \|\overline{d}_k - \underline{d}_k\|, \|\Delta_k^x\| \triangleq \|\overline{x}_k - \underline{x}_k\|\}_{k=0}^\infty$ are uniformly bounded, and consequently, interval input and state estimation errors $\{\|\tilde{d}_k\| \triangleq \max(\|d_k - \underline{d}_k\|, \|\overline{d}_k - d_k\|), \|\tilde{x}_k\| \triangleq \max(\|x_k - \underline{x}_k\|, \|\overline{x}_k - x_k\|)\}_{k=0}^\infty$ are also uniformly bounded, if <u>either one</u> of the following holds*

(i) $\mathcal{L} \triangleq (L_{f_d} + L_{g_d}\|\mathcal{K}\|) \le 1$;

(ii) $$\Theta \triangleq \begin{bmatrix} F & 0 & 0 & 0 & 0 \\ * & \mathcal{K}^\top\mathcal{K} & \mathcal{K}^\top & \mathcal{K}^\top & \mathcal{K}^\top\mathcal{K} \\ * & * & I & I & \mathcal{K} \\ * & * & * & 0 & \mathcal{K} \\ * & * & * & * & 0 \end{bmatrix} \preceq 0;$$

(iii) There exist a positive definite matrix $P \succ 0$ and a positive semidefinite matrix $\Gamma \succeq 0$ in $\mathbb{R}^{n\times n}$ such that the following LMI condition is satisfied:

$$\Pi \triangleq \begin{bmatrix} P+\Gamma-I & 0 & P \\ 0 & \mathcal{L}I-P & 0 \\ P & 0 & P \end{bmatrix} \preceq 0, \qquad (11)$$

with L_{f_d} and L_{g_d} given in Lemma 1, $\mathcal{K} \triangleq |G||J|$, $F \triangleq (L_{f_d}^2 + L_{g_d}^2 \lambda_{\max}(\mathcal{K}^\top\mathcal{K}) - 1)I$ and $\lambda_{\max}(\mathcal{K}^\top\mathcal{K})$ is the maximum eigenvalue of $\mathcal{K}^\top\mathcal{K}$.

It is notable that examples of dynamic systems with only slight differences in their G and H matrices can be found, which only satisfy a subset of the three aforementioned conditions and do not satisfy the others. For instance, for the example system in Section V, we found that it satisfies Conditions (i) and (ii) but does not satisfy Condition (iii). However, when we change the G and H matrices to $G = \begin{bmatrix} 0 & 0.1 \\ 0.2 & -0.2 \end{bmatrix}$ and $H = \begin{bmatrix} 0.1 & 0.3 \\ 0.5 & -0.7 \end{bmatrix}^\top$, we observe that $\mathcal{L} = 1.284 > 1$, so Condition (i) does not hold. In addition, Condition (ii) also does not hold, but Condition (iii) holds with $P = \begin{bmatrix} 4.6714 & 0 \\ 0 & 4.6714 \end{bmatrix}$ and $\Gamma = \begin{bmatrix} 0.3802 & 0 \\ 0 & 0.3802 \end{bmatrix}$. A search for a more general condition that encompasses all of these conditions is a subject of future work.

Finally, we will provide upper bounds for the interval widths and compute their steady-state values, if they exist.

Lemma 2 (Upper Bounds of the Interval Widths and their Convergence)**.** *Consider the system* (1) *and the SISIO observer* (3)–(5)*, and suppose that Assumptions 1–3 hold. Then, there exist uniformly bounded upper bound sequences $\{\delta_k^x, \delta_k^d\}_{k=0}^\infty$, for interval width sequences $\{\|\Delta_k^x\|, \|\Delta_k^d\|\}_{k=0}^\infty$, which can be computed as follows:*

$$\|\Delta_k^x\| \le \delta_k^x = \mathcal{L}^k \delta_0^x + \|\Delta z\| \left(\frac{1-\mathcal{L}^k}{1-\mathcal{L}}\right),$$
$$\|\Delta_k^d\| \le \delta_k^d = \mathcal{G}(\delta_k^x),$$

with L_{g_d} and $\mathcal{L}$ given in Lemma 1 and Theorem 2, respectively, $\Delta z \triangleq \Delta w + \mathcal{K}\Delta v$, $\Delta w \triangleq \overline{w} - \underline{w}$, $\Delta v \triangleq \overline{v} - \underline{v}$, $\mathcal{K} \triangleq |G||J|$ and $\mathcal{G}(x) \triangleq \||J|\|L_{g_d}x + \||J|\Delta v\|$. Furthermore, if Condition (i) *in Theorem 2 holds with strict inequality, then the upper bound sequences of the interval widths converge to steady-state values as:*

$$\overline{\delta}^x \triangleq \lim_{k\to\infty} \delta_k^x = \|\Delta z\| \frac{\mathcal{L}}{1-\mathcal{L}}, \quad \overline{\delta}^d \triangleq \lim_{k\to\infty} \delta_k^d = \mathcal{G}(\overline{\delta}^x).$$

On the other hand, if Conditions (ii) *or* (iii) *in Theorem 2 hold, then the interval widths $\|\Delta_k^x\|$ and $\|\Delta_k^d\|$ are uniformly bounded by $\min\{\|\Delta_0^x\|, \Delta_0^P\}$ and $\min\{\mathcal{G}(\|\Delta_0^x\|), \mathcal{G}((\Delta_0^P)\}$, respectively, with $\Delta_0^P \triangleq \sqrt{\frac{(\Delta_0^x)^\top P \Delta_0^x}{\lambda_{\min}(P)}}$ and P being the solution for the LMI condition in Condition* (iii)*.*

Remark 1. *Note that our interval observer and its correctness in Theorem 1 are based on interval analysis (corresponding to the L_∞-norm), while the stability and convergence results in Theorem 2 and Lemma 2 are based on the L_2-norm, in order to leverage Lyapunov stability analysis techniques, similar to the interval observers in [10], [13]–[17]. Nonetheless, the sufficient conditions derived in Theorem 2 are also sufficient for uniform boundedness under any norm by the equivalence of norms, including the L_1- or L_∞-norms that may in general be more consistent with interval analysis, and similarly, the convergence property in Lemma 2 also holds for any norm with minor modifications.*

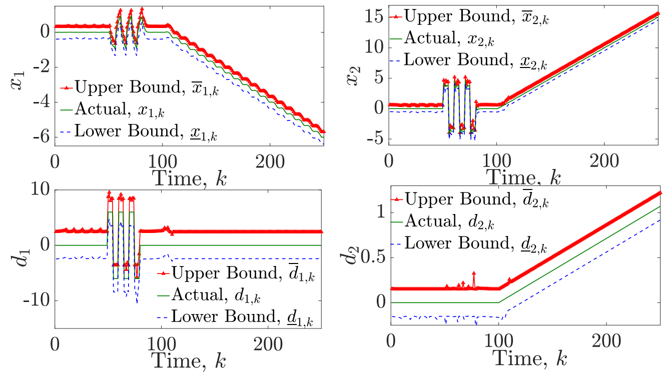

Fig. 1: Actual states and inputs, $x_{1,k}$, $x_{2,k}$, $d_{1,k}$, $d_{2,k}$, as well as their estimated maximal and minimal values, $\overline{x}_{1,k}$, $\underline{x}_{1,k}$, $\overline{x}_{2,k}$, $\underline{x}_{1,k}$, $\overline{d}_{1,k}$, $\underline{d}_{1,k}$, $\overline{d}_{2,k}$, $\underline{d}_{2,k}$.

V. Illustrative Example

We consider a nonlinear dynamical system in [25] with slight modifications to remove the uncertain parts of the matrices and to include unknown inputs, with the following parameters (cf. (1)): $n = l = p = 2$, $m = 1$, $f(x_k) = \begin{bmatrix} f_1(x_k) & f_2(x_k) \end{bmatrix}^\top$, $g(x_k) = \begin{bmatrix} g_1(x_k) & g_2(x_k) \end{bmatrix}^\top$, $B = D = 0_{2\times 1}$, $G = \begin{bmatrix} 0 & -0.1 \\ 0.2 & -0.2 \end{bmatrix}$, $H = \begin{bmatrix} -0.1 & 0.3 \\ 0.5 & -0.7 \end{bmatrix}$, $\overline{v} = -\underline{v} = \overline{w} = -\underline{w} = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}^\top$, $\overline{x}_0 = \begin{bmatrix} 2 & 1.1 \end{bmatrix}^\top$, $\underline{x}_0 = \begin{bmatrix} -1.1 & -2 \end{bmatrix}^\top$, where

$$\begin{aligned} f_1(x_k) &= 0.6x_{1,k} - 0.12x_{2,k} + 1.1\sin(0.3x_{2,k} - 0.2x_{1,k}), \\ f_2(x_k) &= -0.2x_{1,k} - 0.14x_{2,k}, \\ g_1(x_k) &= 0.2x_{1,k} + 0.65x_{2,k} + 0.8\sin(0.3x_{1,k} + 0.2x_{2,k}), \\ g_2(x_k) &= \sin(x_{1,k}), \end{aligned}$$

while the unknown input signals are depicted in Figure 1. Note that $\text{rk}(H) = 2$, thus Assumption 1 holds. Moreover, applying [24, Theorem 1], we can compute finite-valued upper and lower bounds for the partial derivatives of $f(\cdot)$ and $g(\cdot)$ as: $\begin{bmatrix} a_{11}^f & a_{12}^f \\ a_{21}^f & a_{22}^f \end{bmatrix} = \begin{bmatrix} 0.38 & -0.52 \\ -0.2-\epsilon & -0.14-\epsilon \end{bmatrix}$, $\begin{bmatrix} b_{11}^f & b_{12}^f \\ b_{21}^f & b_{22}^f \end{bmatrix} = \begin{bmatrix} 0.82 & 0.21 \\ -0.2+\epsilon & -0.14+\epsilon \end{bmatrix}$, $\begin{bmatrix} a_{11}^g & a_{12}^g \\ a_{21}^g & a_{22}^g \end{bmatrix} = \begin{bmatrix} -0.04 & 0.49 \\ -1 & -\epsilon \end{bmatrix}$, $\begin{bmatrix} b_{11}^g & b_{12}^g \\ b_{21}^g & b_{22}^g \end{bmatrix} = \begin{bmatrix} 0.44 & 0.81 \\ 1 & \epsilon \end{bmatrix}$, where ϵ is a very small positive value, ensuring that the partial derivatives are in open intervals (cf. [20, Theorem 1]). Hence, Assumption 2 is also satisfied by [20, Theorem 1]). Therefore, we expect that the interval estimates are correct by Theorem 1 (i.e., the true states and unknown inputs are within the estimate intervals), which can be verified from Figure 1 that depicts interval estimates and the true states and unknown inputs.

Furthermore, $f(\cdot)$ and $g(\cdot)$ satisfy Assumption 3 with $L_f = 0.87$ and $L_g = 1.32$. In addition, from [20, (10)–(13)]),

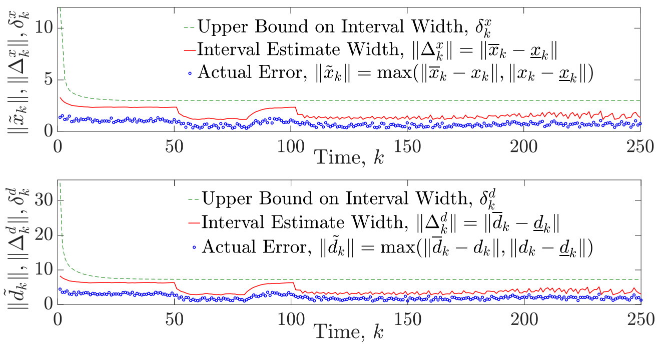

Fig. 2: Estimation errors, estimate interval widths and their upper bounds for the interval-valued estimates of states, $\|\tilde{x}_{k|k}\|$, $\|\Delta_k^x\|$, δ_k^x, and unknown inputs, $\|\tilde{d}_k\|$, $\|\Delta_k^d\|$, δ_k^d.

we obtain $C_f = \begin{bmatrix} 0.374 & .02 \\ 0.0135 & 0.407 \end{bmatrix}$, $C_g = \begin{bmatrix} -0.02 & 0.138 \\ 0.063 & -0.035 \end{bmatrix}$ using (2), which implies that $L_{f_d} = 0.885$ and $L_{g_d} = 1.32$ by Lemma 1. Consequently, $\mathcal{L} = 0.873$. Since all the required assumptions hold, including Condition (i) in Theorem 2 with strict inequality, we expect to obtain uniformly bounded and convergent interval estimate errors when applying our observer design procedure. This can be seen in Figure 2, where at each time step, the actual error sequence is upper bounded by the interval width, which in turn is bounded by the predicted upper bound sequence for the interval width. Moreover, as expected, the upper bounds converge to some steady-state values. Note that, despite our best efforts, we were unable to find interval-valued observers for nonlinear systems in the literature that simultaneously return both state and unknown input estimates for comparison with our results.

VI. Conclusion

In this paper, we proposed a simultaneous input and state interval observer for mixed monotone Lipschitz nonlinear systems with unknown inputs and bounded noise. We proved that the proposed observer recursively outputs the state and unknown input interval-valued estimates that are guaranteed to include the true states and unknown inputs. Moreover, several sufficient conditions for the stability of the observer and the uniform boundedness of the interval widths were derived. Finally, we demonstrated the effectiveness of the proposed approach with an example. For future work, we seek to relax the full-rank assumption for the direct feedthrough matrix and to find necessary conditions for observer stability.

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Appendix: Proofs

A. Proof of Theorem 1

First, note that for $r_k \triangleq Hd_k = y_k - g(x_k) - Du_k - v_k$, we can obtain $\underline{r}_k \leq r_k = Hd_k \leq \overline{r}_k$ by Assumption 2 and the fact that decomposition functions are monotone increasing in their first argument and decreasing in their second (cf. Definition 3). By left multiplying the above inequalities by $J = H^\dagger$ and from Assumption 1 and Proposition 1, we can conclude that $\underline{d}_k^1 \leq d_k \leq \overline{d}_k^1$. Moreover, since $\underline{r}_k \leq r_k = Hd_k \leq \overline{r}_k$ can be rearranged as $d_k \in \mathcal{D}_k \triangleq \{d \in \mathbb{R}^p \,|\, \tilde{H}d \leq \tilde{r}_k\}$, the linear programs (8) yield the tightest maximal and minimal elements of $\mathcal{I}_k^d$ that enclose $\mathcal{D}_k$, i.e., $\underline{d}_k^2 \leq d_k \leq \overline{d}_k^2$. Combining this and $\underline{d}_k^1 \leq d_k \leq \overline{d}_k^1$, we obtain $\underline{d}_k \leq d_k \leq \overline{d}_k$.

Similarly, using Assumption 2, the fact that decomposition functions are monotone increasing in the first argument and decreasing in the second, and applying Proposition 1 to (1), we obtain $\underline{x}_k \leq x_k \leq \overline{x}_k$ with $\underline{x}_k$ and $\overline{x}_k$ given in (3)–(4). ■

B. Proof of Lemma 1

We begin with (2) for the function f;

$$f_d(\overline{x}, \underline{x}) = f(x_1) + C_f(\overline{x} - \underline{x}), \tag{12}$$

$$f_d(\underline{x}, \overline{x}) = f(x_2) + C_f(\underline{x} - \overline{x}), \tag{13}$$

where the i-th elements of x_1 and x_2 (i.e., $x_{1,i}$ and $x_{2,i}$, $\forall i \in \{1 \ldots n\}$) are either $\overline{x}_i$, or $\underline{x_i}$, depending on the case (cf. [20, Theorem 1; (10)–(13)]). Moreover, $\underline{x} \leq \overline{x}$ and $\underline{x} \leq x_1, x_2 \leq \overline{x}$. This implies that

$$-(\overline{x} - \underline{x}) \leq x_1 - x_2 \leq \overline{x} - \underline{x} \Longrightarrow \|x_1 - x_2\| \leq \|\overline{x} - \underline{x}\|. \tag{14}$$

On the other hand, from (12) and (13),

$$f_d(\overline{x}, \underline{x}) - f_d(\underline{x}, \overline{x}) = f(x_1) - f(x_2) + 2C_f(\overline{x} - \underline{x}).$$

Then, applying triangle inequality and by the Lipschitz continuity of f, we obtain

$$\|f_d(\overline{x}, \underline{x}) - f_d(\underline{x}, \overline{x})\| \leq L_f\|x_1 - x_2\| + 2\|C_f\|\|(\overline{x} - \underline{x})\|. \tag{15}$$

Combining (14) and (15) yields the result. ■

C. Proof of Theorem 2

From (3) and (4), we obtain

$$\Delta_{k+1}^x = \Delta f_k^x + |G|\Delta_k^d + \Delta w, \tag{16}$$

where $\Delta_k^x \triangleq \overline{x}_k - \underline{x}_k$, $\Delta_k^d \triangleq \overline{d}_k - \underline{d}_k$, $\Delta f_k^x \triangleq f_d(\overline{x}_k, \underline{x}_k) - f_d(\underline{x}_k, \overline{x}_k)$ and $\Delta w \triangleq \overline{w} - \underline{w}$. Moreover, from (5) and (6),

$$\Delta_k^d \leq \Delta_k^{d,1} = |J|\Delta_k^r, \tag{17}$$

where $\Delta_k^{d,1} \triangleq \overline{d}_k^1 - \underline{d}_k^1$ and $\Delta_k^r \triangleq \overline{r}_k - \underline{r}_k$, while from the definitions of $\overline{r}_k$ and $\underline{r}_k$ in (9)–(10), we have

$$\Delta_k^r = \Delta g_k^x + \Delta v, \tag{18}$$

where $\Delta g_k^x \triangleq g_d(\overline{x}_k, \underline{x}_k) - g_d(\underline{x}_k, \overline{x}_k)$ and $\Delta v \triangleq \overline{v} - \underline{v}$. Combining (16)–(18) and using the fact that $|G|, |J| \geq 0$ and Proposition 1, we obtain

$$\Delta_{k+1}^x \leq \Delta h_k^x + \Delta z, \tag{19}$$

where $\mathcal{K} \triangleq |G||J|$, $\Delta h_k^x \triangleq \Delta f_k^x + \mathcal{K}\Delta g_k^x$ and $\Delta z \triangleq \Delta w + \mathcal{K}\Delta v$. Now, consider the following dynamical system

$$\Delta_{k+1}^s = \Delta h_k^s + \Delta z, \tag{20}$$

where $\Delta_k^s \in \mathbb{R}^n$ and $\Delta_0^s = \Delta_0^x$. By non-negativity of Δ_k^x and *Comparison Lemma* [26, Lemma 3.4], $0 \leq \Delta_k^x \leq \Delta_k^s, \forall k \geq 0$. So, uniform boundedness of $\{\Delta_k^s\}_{k=0}^\infty$ (shown below) implies uniform boundedness of $\{\Delta_k^x\}_{k=0}^\infty$.

Condition (i): Since Assumption 3 holds, the application of triangle inequality to (19) yields

$$\|\Delta_{k+1}^x\| \leq \mathcal{L}\|\Delta_k^x\| + \|\Delta z\|, \tag{21}$$

where $\mathcal{L} = L_{f_d} + L_{g_d}\|\mathcal{K}\|$, with L_{f_d} and L_{g_d} given in Lemma 1. Since $\mathcal{L} \leq 1$ (by Condition (i)), by *Comparison Lemma* [26, Lemma 3.4], $\{\|\Delta_k^x\|\}_{k=0}^\infty$ is uniformly bounded. Therefore, the interval width dynamics is stable.

Condition (ii): To show that Condition (ii) implies stability, consider a candidate Lyapunov function $V_k = \Delta_k^{s\top}\Delta_k^s$ for (20) that can be shown to satisfy $\Delta V_k \triangleq V_{k+1} - V_k \leq 0$ under Condition (ii) as follows:

$$\begin{aligned}
\Delta V_k &\triangleq V_{k+1} - V_k \\
&= \Delta f_k^{s\top}\Delta f_k^s + \Delta g_k^{s\top}\mathcal{K}^\top\mathcal{K}\Delta g_k^s + \Delta v^\top\mathcal{K}^\top\mathcal{K}\Delta v + \Delta w^\top\Delta w \\
&\quad - \Delta_k^{s\top}\Delta_k^s + 2(\Delta f_k^{s\top}\mathcal{K}\Delta g_k^s + \Delta f^{s\top}\mathcal{K}\Delta v + \Delta f^{s\top}\Delta w \\
&\quad + \Delta g^{s\top}\mathcal{K}^\top\mathcal{K}\Delta v + \Delta g^{s\top}\mathcal{K}^\top\Delta w + \Delta v^\top\mathcal{K}^\top\Delta w) \\
&\leq (L_{f_d}^2 + \lambda_{\max}(\mathcal{K}^\top\mathcal{K})L_{g_d}^2 - 1)\Delta_k^{s\top}\Delta_k^s + \Delta v^\top\mathcal{K}^\top\mathcal{K}\Delta v \\
&\quad + \Delta w^\top\Delta w + 2(\Delta f_k^{s\top}\mathcal{K}\Delta g_k^s + \Delta f^{s\top}\mathcal{K}\Delta v + \Delta f^{s\top}\Delta w \\
&\quad + \Delta g^{s\top}\mathcal{K}^\top\mathcal{K}\Delta v + \Delta g^{s\top}\mathcal{K}^\top\Delta w + \Delta v^\top\mathcal{K}^\top\Delta w) = \Delta_k^{\theta\top}\Theta\Delta_k^\theta,
\end{aligned}$$

with $\Delta_k^\theta \triangleq \left[\Delta_k^{s\top}\ \Delta v^\top\ \Delta w^\top\ \Delta f_k^{s\top}\ \Delta g_k^{s\top}\right]^\top$ and Θ given in (ii), where the first inequality holds since $\Delta f_k^{s\top}\Delta f_k^s = \|\Delta f_k^s\|^2 \leq L_{f_d}^2\|\Delta_k^s\|^2$ by Lemma 1 and $\Delta g_k^{s\top}\mathcal{K}^\top\mathcal{K}\Delta g_k^s \leq \lambda_{\max}(\mathcal{K}^\top\mathcal{K})\Delta g_k^{s\top}\Delta g_k^s = \lambda_{\max}(\mathcal{K}^\top\mathcal{K})\|\Delta g_k^s\|^2 \leq L_{g_d}^2\lambda_{\max}(\mathcal{K}^\top\mathcal{K})\|\Delta_k^s\|^2$ by using the *Rayleigh Quotient* and Lemma 1, and the last inequality holds by Condition (ii). Hence, $\{\Delta_k^s\}_{k=0}^\infty$ is uniformly bounded and so is $\{\Delta_k^x\}_{k=0}^\infty$ by the Comparison Lemma (i.e., the dynamics of Δ_k^x is stable).

Condition (iii): Similarly, we consider a candidate Lyapunov function $V_k = \Delta_k^{s\top}P\Delta_k^s$, where $P \succ 0$, which can be shown to satisfy $\Delta V_k \triangleq V_{k+1} - V_k \leq 0$ under Condition (iii). To show this, note that $\Delta h_k^{s\top}\Lambda\Delta h_k^s \leq \Delta h_k^{s\top}\Delta h_k^s \leq \mathcal{L}^2\Delta_k^{s\top}\Delta_k^s$, where the inequalities hold by choosing Γ such that $\Gamma \triangleq I - \Lambda \succeq 0$ and Lemma 1, respectively. Consequently, $\mathcal{L}^2\Delta_k^{s\top}\Delta_k^s - \Delta h_k^{s\top}\Lambda\Delta h_k^s \geq 0$. Then, inspired by a simplifying trick used in [27, Proof of Theorem 1] to satisfy $\Delta V_k \leq 0$, it suffices to guarantee that $\tilde{V}_k \triangleq \Delta V_k + \mathcal{L}^2\Delta_k^{s\top}\Delta_k^s - \Delta h_k^{s\top}\Lambda\Delta h_k^s = \Delta V_k + \mathcal{L}^2\Delta_k^{s\top}\Delta_k^s - \Delta h_k^{s\top}(I - \Gamma)\Delta h_k^s \leq 0$, where $\tilde{V}_k$ is also given by

$$\begin{aligned}
\tilde{V}_k &= \Delta h_k^{s\top}P\Delta h_k^s + \Delta z^\top P\Delta z + 2\Delta z^\top P\Delta h_k^s - \Delta_k^{s\top}P\Delta_k^s \\
&\quad + \mathcal{L}^2\Delta_k^{s\top}\Delta_k^s - \Delta h_k^{s\top}(I - \Gamma)\Delta h_k^s \\
&= \Delta h_k^{s\top}(P + \Gamma - I)\Delta h_k^s + \Delta_k^{s\top}(\mathcal{L}^2 I - P)\Delta_k^s \\
&\quad + \Delta z^\top P\Delta z + 2\Delta z^\top P\Delta h_k^s = (\Delta_k^\zeta)^\top\Pi\Delta_k^\zeta \leq 0,
\end{aligned}$$

with Π defined in (11) and $\Delta_k^\zeta \triangleq \left[\Delta h_k^{s\top}\ \Delta_k^{s\top}\ \Delta z^\top\right]^\top$. This, along with $\Gamma \succeq 0$ is equivalent to Condition (iii). Finally, since $\Delta V_k \leq 0$, $\{\Delta_k^s\}_{k=0}^\infty$ is uniformly bounded and so is the interval width sequence $\{\Delta_k^x\}_{k=0}^\infty$ by the Comparison Lemma (i.e., the dynamics of Δ_k^x is stable). ■

D. Proof of Lemma 2

Applying (21) repeatedly, we have

$$\|\Delta_k^x\| \leq \mathcal{L}^k\|\Delta_0^x\| + \sum_{i=0}^{k-1}\mathcal{L}^{k-i}\|\Delta z\| = \mathcal{L}^k\delta_0^x + \|\Delta z\|\tfrac{1-\mathcal{L}^k}{1-\mathcal{L}}.$$

Similarly, by applying Lemma 1 and triangle inequality to (17) and (18), we obtain the upper bound $\|\Delta_k^d\|$. If $\mathcal{L} < 1$, then taking the limit of k to ∞, returns $\overline{\delta}^x$ and $\overline{\delta}^d$. The rest of the results follow from the non-increasing Lyapunov functions defined in the proof of Theorem 2, as well as the fact that $\lambda_{\min}(A)\|x\|^2 \leq x^\top Ax, \forall x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$. ■