Global Mild Solutions of the Landau and Non-Cutoff Boltzmann Equations

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Abstract

This paper proves the existence of small-amplitude global-in-time unique mild solutions to both the Landau equation including the Coulomb potential and the Boltzmann equation without angular cutoff. Since the well-known works [45] and [3, 43] on the construction of classical solutions in smooth Sobolev spaces which in particular are regular in the spatial variables, it still remains an open problem to obtain global solutions in an $L_{x,v}^\infty$ framework, similar to that in [49], for the Boltzmann equation with the cutoff assumption in general bounded domains. One main difficulty arises from the interaction between the transport operator and the velocity-diffusion-type collision operator in the non-cutoff Boltzmann and Landau equations; another major difficulty is the potential formation of singularities for solutions to the boundary value problem.

In the present work we introduce a new function space with low regularity in the spatial variable to treat the problem in cases when the spatial domain is either a torus or a finite channel with boundary. For the latter case, either the inflow boundary condition or the specular reflection boundary condition is considered. An important property of the function space is that the $L_T^\infty L_v^2$ norm, in velocity and time, of the distribution function is in the Wiener algebra $A(\Omega)$ in the spatial variables. Besides the construction of global solutions in these function spaces, we additionally study the large-time behavior of solutions for both hard and soft potentials, and we further justify the property of propagation of regularity of solutions in the spatial variables. © 2020 Wiley Periodicals, Inc.

Contents

1 Introduction

The Landau equation with Coulomb potential and the non-cutoff Boltzmann equation for the long-range interaction potentials are two fundamental mathematical models in collisional kinetic theory that describe the dynamics of a nonequilibrium rarefied gas. These two equations are connected in several ways; for example, the Landau equation can be formally derived from the Boltzmann equation by taking the grazing limit. Further, the collision operators in both equations each feature velocity diffusion that can induce a gain of spatial regularity for spatially inhomogeneous solutions as a result of the interplay with the free transport operator.

The existence theory for both equations has a long history. A well-established framework in which to study global well-posedness is to look for solutions that are close to the global Maxwellian equilibria in different function spaces. However, for these two equations it is a big open problem to characterize the optimal mathematical space of initial data with lower regularity in space and velocity variables such that unique solutions may exist globally in time. The main goal of this work is to prove the global existence of unique solutions in a new function space with mild regularity for both equations in the perturbative framework.

1.1 Equation

In this article we are concerned with the collisional kinetic equation of the form

$$(1.1) \qquad\qquad \partial_t F + v \cdot \nabla_x F = Q(F, F).$$

Here the unknown $F = F(t, x, v) \geq 0$ stands for the density function of particles with position $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t > 0$, and it is supplemented with initial data

$$F(0, x, v) = F_0(x, v).$$

1.2 Collision Term

The collision term on the right-hand side of (1.1) that we will consider in this paper is described by either the Landau operator or the non-cutoff Boltzmann operator, which are given as described below.

Landau Collision Operator

For the Landau collision operator, $Q(\cdot,\cdot)$ is given by

$$(1.2) \quad Q(G,F)(v) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \psi(v-u)[G(u)\nabla_v F(v) - F(v)\nabla_u G(u)]du \right\}.$$

Then (1.1) with (1.2) is called the Landau equation. Equivalently, the Landau collision operator can be written as

$$Q(G,F)(v)$$

$$(1.3) \qquad = \sum_{j,m=1}^{3} \partial_{v_j} \int_{\mathbb{R}^3} \psi^{jm}(v-u)[G(u)\partial_{v_m}F(v) - F(v)\partial_{u_m}G(u)]du.$$

The Landau collision kernel ψ in (1.2) or (1.3) is a nonnegative symmetric matrix-valued function defined for $0 \neq z = (z_1, z_2, z_3) \in \mathbb{R}^3$ as

$$(1.4) \qquad \psi^{jm}(z) = \left\{ \delta_{jm} - \frac{z_j z_m}{|z|^2} \right\} |z|^{\gamma+2}, \quad j,m = 1,2,3,$$

where δ_{jm} is the Kronecker delta and $-3 \leq \gamma \leq 1$ is a parameter determined by the interaction potential between particles. It is customary to use the term "hard potentials" when $0 < \gamma \leq 1$, "Maxwell molecules" when $\gamma = 0$, "moderately soft potentials" when $-2 \leq \gamma < 0$, and "very soft potentials" when $-3 \leq \gamma < -2$ (cf. [84]). The case $\gamma = -3$ corresponds to the classical Coulomb potential (cf. [45,82]). However, see the discussion below (1.26) for the terminology that we will use in the rest of this paper.

Non-Cutoff Boltzmann Collision Operator

For the Boltzmann collision operator without angular cutoff, $Q(\cdot,\cdot)$ takes the form of

$$(1.5) \quad Q(G,F)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\sigma)[G(u')F(v') - G(u)F(v)]d\sigma\,du,$$

where the velocity pairs (v,u) and (v',u') satisfy

$$\begin{cases} v' = \frac{v+u}{2} + \frac{|v-u|}{2}\sigma, \\ u' = \frac{v+u}{2} - \frac{|v-u|}{2}\sigma, \end{cases} \qquad \sigma \in \mathbb{S}^2.$$

Then (1.1) with (1.5) is called the Boltzmann equation. The Boltzmann collision kernel $B(v-u,\sigma)$ is a nonnegative function, depending only on the relative velocity $|v-u|$ and on the deviation angle θ given by

$$\cos\theta = \left\langle \sigma, \frac{v-u}{|v-u|} \right\rangle.$$

Without loss of generality we may assume that $B(v-u,\sigma)$ is supported on $\cos\theta \geq 0$, i.e., $0 \leq \theta \leq \frac{\pi}{2}$. Otherwise, we can reduce to this situation with the following standard symmetrization (cf. [38,83]):

$$B(v-u,\sigma) = [B(v-u,\sigma) + B(v-u,-\sigma)]\mathbf{1}_{\cos\theta\geq 0}.$$

Here and in the rest of the paper, $\mathbf{1}_A$ denotes the usual indicator function of the set A. Throughout this paper we further assume that $B(v - u, \sigma)$ takes the product form as follows:

$$B(v - u, \sigma) = C_B |v - u|^\gamma b(\cos \theta),$$

for a constant $C_B > 0$, where $|v - u|^\gamma$ is called the kinetic part with $\gamma > -3$, and $b(\cos \theta)$ is called the angular part satisfying that there are $C_b > 0, 0 < s < 1$, such that

$$\frac{1}{C_b \theta^{1+2s}} \leq \sin \theta b(\cos \theta) \leq \frac{C_b}{\theta^{1+2s}} \quad \forall \theta \in (0, \tfrac{\pi}{2}].$$

For the angular non-cutoff Boltzmann operator, it is also customary to use the terms "hard potentials" when $\gamma \geq 0$, "moderately soft potentials" when $0 > \gamma \geq -2s$, and "very soft potentials" when $-3 < \gamma < -2s$. However, see the discussion below (1.26) for the terminology that we will use in the rest of this paper.

One physical example is given when the collision kernel is derived from a spherical intermolecular repulsive potential of the inverse power law form $\phi(r) = r^{-(\ell-1)}$ with $2 < \ell < \infty$ corresponding to which B satisfies the assumptions above with $\gamma = (\ell - 5)/(\ell - 1)$ and $s = 1/(\ell - 1)$ (cf. [18]).

Through the paper, in the Boltzmann case we further require that

$$\gamma > \max\left\{-3, -\frac{3}{2} - 2s\right\},$$

due to the mild regularity setting of the results in this article. Indeed, the above condition is satisfied for the full range of the inverse power law model.

1.3 Spatial Domain and Boundary Condition

In this paper we focus on two kinds of specific bounded domains $\Omega \subset \mathbb{R}^3$. We consider either a torus or a finite channel with prescribed boundary conditions.

Case of the Torus

In this case, we set

$$(1.6) \qquad \Omega = \mathbb{T}^3 := [0, 2\pi]^3.$$

Correspondingly, $F(t, x, v)$ is assumed to be spatially periodic in $x \in \mathbb{T}^3$.

Case of the Finite Channel

In this case we set

$$(1.7) \quad \begin{aligned} \Omega &= I \times \mathbb{T}^2 \\ &= \left\{x = (x_1, \bar{x}), x_1 \in I := (-1, 1), \bar{x} := (x_2, x_3) \in \mathbb{T}^2 = [0, 2\pi]^2\right\}. \end{aligned}$$

Correspondingly, $F(t, x, v)$ is assumed to be spatially periodic for $\bar{x} \in \mathbb{T}^2$ and satisfy the following two kinds of boundary conditions at $x_1 = \pm 1$:

- *Inflow* boundary condition:

$$(1.8) \quad F(t, -1, \bar{x}, v)|_{v_1 > 0} = G_-(t, \bar{x}, v), \quad F(t, 1, \bar{x}, v)|_{v_1 < 0} = G_+(t, \bar{x}, v).$$

- *Specular reflection* boundary condition:

$$
\begin{aligned}
F(t, -1, \bar{x}, v_1, \bar{v})|_{v_1 > 0} &= F(t, -1, \bar{x}, -v_1, \bar{v}), \\
F(t, 1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} &= F(t, 1, \bar{x}, -v_1, \bar{v}),
\end{aligned}
$$
(1.9)

where $\bar{v} = (v_2, v_3)$.

1.4 Reformulation

Consider the following global Maxwellian equilibrium state:

$$
\mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.
$$

Note that μ is a spatially homogeneous steady solution to (1.1) for either the Landau case or the Boltzmann case. We look for a solution to (1.1) of the form

$$
F(t, x, v) = \mu + \mu^{1/2} f(t, x, v).
$$

Then, the new unknown $f = f(t, x, v)$ satisfies

$$
\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f),
$$
(1.10)

with initial data

$$
f(0, x, v) = f_0(x, v) := \mu^{-1/2}[F_0(x, v) - \mu].
$$
(1.11)

Here, the linearized collision operator L and nonlinear collision operator Γ are given by

$$
Lf = -\mu^{-1/2}\{Q(\mu, \mu^{1/2} f) + Q(\mu^{\frac{1}{2}} f, \mu)\}
$$
(1.12)

and

$$
\Gamma(f, f) = \mu^{-\frac{1}{2}} Q(\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} f),
$$
(1.13)

respectively. In the case of a finite channel, corresponding to (1.8) or (1.9), the boundary conditions along x_1 are given by

$$
f(t, -1, \bar{x}, v)|_{v_1 > 0} = g_-(t, \bar{x}, v), \quad f(t, 1, \bar{x}, v)|_{v_1 < 0} = g_+(t, \bar{x}, v),
$$
(1.14)

for the inflow boundary, and

$$
\begin{aligned}
f(t, -1, \bar{x}, v_1, \bar{v})|_{v_1 > 0} &= f(t, -1, \bar{x}, -v_1, \bar{v}), \\
f(t, 1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} &= f(t, 1, \bar{x}, -v_1, \bar{v}),
\end{aligned}
$$
(1.15)

for the specular reflection boundary.

1.5 Problem

In this paper we will study the following two problems **(PT)** and **(PC)** as described below. For each problem, both the Landau operator (1.2) and the Boltzmann operator (1.5) will be included in our study.

(PT): Consider (1.10) and (1.11) in the torus domain Ω as in (1.6). Due to the conservation laws, we may always assume that $f(t, x, v)$ satisfies

$$(1.16) \qquad \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f(t, x, v) dv\, dx = 0,$$

$$(1.17) \qquad \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v_i \mu^{\frac{1}{2}} f(t, x, v) dv\, dx = 0, \quad i = 1, 2, 3,$$

$$(1.18) \qquad \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |v|^2 \mu^{\frac{1}{2}} f(t, x, v) dv\, dx = 0,$$

for any $t \geq 0$. In other words if these are satisfied initially, then the equation shows that they will continue to be satisfied for any $t \geq 0$.

(PC): Consider (1.10), (1.11), and (1.14) or (1.15) in a finite channel domain Ω as in (1.7).

For the specular reflection boundary condition (1.15), we assume further that

$$(1.19) \qquad \int_{-1}^{1} \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f(t, x_1, \overline{x}, v) dv\, d\overline{x}\, dx_1 = 0,$$

$$(1.20) \qquad \int_{-1}^{1} \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} v_i \mu^{\frac{1}{2}} f(t, x_1, \overline{x}, v) dv\, d\overline{x}\, dx_1 = 0, \quad i = 1, 2, 3,$$

$$(1.21) \qquad \int_{-1}^{1} \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} |v|^2 \mu^{\frac{1}{2}} f(t, x_1, \overline{x}, v) dv\, d\overline{x}\, dx_1 = 0,$$

for any $t \geq 0$. Indeed, (1.19), (1.20) with $i = 2, 3$, and (1.21) are the direct consequence of the conservation of mass, momentum, and energy, respectively. To obtain (1.20) for $i = 1$, we further assume that the initial data satisfy the following symmetry assumption:

$$(1.22) \qquad F_0(x_1, \overline{x}, v_1, \overline{v}) = F_0(-x_1, \overline{x}, -v_1, \overline{v}),$$

and hence the solution will also satisfy

$$F(t, x_1, \overline{x}, v_1, \overline{v}) = F(t, -x_1, \overline{x}, -v_1, \overline{v}),$$

which directly gives (1.20) when $i = 1$. We only assume the symmetry condition (1.22) for the specular reflection boundary condition (1.15).

1.6 Function Space

To study the well-posedness of the problems **(PT)** and **(PC)**, we first introduce a function space X_T with $0 < T \leq \infty$, which is a key point in this work. We will

present the motivation for this space later below. For the problem **(PT)** in a torus $\mathbb{T}^3$, we define

$$X_T := L_k^1 L_T^\infty L_v^2$$

with norm

(1.23) $\|f\|_{X_T} := \int_{\mathbb{Z}^3} \sup_{0<t<T} \|\widehat{f}(t,k,\cdot)\|_{L_v^2} \, d\Sigma(k) < \infty,$

where the Fourier transformation of $f(t,x,v)$ with respect to $x \in \mathbb{T}^3$ is denoted

$$\widehat{f}(t,k,v) = \mathscr{F}_x f(t,k,v) = \int_{\mathbb{T}^3} e^{-ik\cdot x} f(t,x,v)dx, \quad k \in \mathbb{Z}^3,$$

In this paper, for convenience we would use $d\Sigma(k)$ to denote the discrete measure in $\mathbb{Z}^3$, namely,

$$\int_{\mathbb{Z}^3} g(k)d\Sigma(k) = \sum_{k\in\mathbb{Z}^3} g(k).$$

Remark 1.1. The main reason for using the integral form $\int_{\mathbb{Z}^3} d\Sigma(k)$ is to make it clear that when the torus domain $\mathbb{T}_x^3$ changes to the whole space domain $\mathbb{R}_x^3$, one may directly change $d\Sigma(k)$ over $k \in \mathbb{Z}^3$ into dk over $k \in \mathbb{R}_k^3$, and the similar results in this paper also hold true in $\mathbb{R}_x^3$ after taking into account the appropriate modifications to the time decay of solutions.

Now, similar to the above, for the problem **(PC)** in a finite channel, we define

$$X_T := L_k^1 L_T^\infty L_{x_1,v}^2$$

with norm

$$\|f\|_{X_T} := \int_{\mathbb{Z}_{\bar{k}}^2} \sup_{0<t<T} \|\widehat{f}(t,\bar{k},\cdot)\|_{L_{x_1,v}^2} \, d\Sigma(\bar{k}) < \infty,$$

with $\bar{k} = (\bar{k}_2, \bar{k}_3) \in \mathbb{Z}^2$ and $\bar{x} = (x_2, x_3) \in \mathbb{T}^2$ we take the Fourier transform as

$$\widehat{f}(t,x_1,\bar{k},v) = \mathscr{F}_{\bar{x}} f(t,x_1,\bar{k},v) = \int_{\mathbb{T}^2} e^{-i\bar{k}\cdot\bar{x}} f(t,x_1,\bar{x},v)d\bar{x}, \quad \bar{k} \in \mathbb{Z}^2.$$

Notice that Fourier transform is taken only with respect to $\bar{x} = (x_2, x_3)$.

We also introduce the velocity-weighted norm

$$\|f\|_{X_T^w} := \int_{\mathbb{Z}^3} \sup_{0\leq t\leq T} \|w\widehat{f}(t,k,\cdot)\|_{L_v^2} \, d\Sigma(k)$$

(1.24)   or $\int_{\mathbb{Z}^2} \sup_{0\leq t\leq T} \|w\widehat{f}(t,\bar{k},\cdot)\|_{L_{x_1,v}^2} \, d\Sigma(\bar{k}),$

for **(PT)** or **(PC)**, respectively. Here $w = w_{q,\vartheta}(v)$ is a velocity-weighted function defined as

(1.25) $w_{q,\vartheta}(v) = e^{\frac{q\langle v\rangle^\vartheta}{4}}, \quad \langle v\rangle = \sqrt{1 + |v|^2},$

with two parameters q and ϑ. Note that whenever there is no confusion, we will omit the dependence of $w_{q,\vartheta}(v)$ on q and ϑ, and write $w_{q,\vartheta}(v) = w(v)$ for brevity. In the situation when $q = 0$, we have $w_{q,\vartheta}(v) \equiv 1$ and hence the function space X_T^w with velocity weight is reduced to X_T without weight. Throughout this paper we require that (q, ϑ) satisfies the following hypothesis in terms of γ and s:

(1.26)   **(H)** $\begin{cases} \text{Landau case:} & \text{if } -2 \le \gamma \le 1, \text{ then } q = 0; \\ & \text{if } -3 \le \gamma < -2, \text{ then } q > 0 \text{ and } 0 < \vartheta \le 2 \\ & \text{with the restriction that } 0 < q < 1 \text{ if } \vartheta = 2. \\ \\ \text{Boltzmann case:} & \text{if } \gamma + 2s \ge 0, \text{ then } q = 0; \\ & \text{if } -3 < \gamma < -2s, \text{ then } q > 0 \text{ and } \vartheta = 1. \end{cases}$

For convenience of terminology in relation to (1.26), in the rest of this paper we will call them *soft* potentials if either $\gamma + 2 < 0$ in the Landau case or $\gamma + 2s < 0$ in the Boltzmann case, and we will call them the *hard* potentials otherwise.

In general, we define the $L_k^p L_T^q L_v^r$ mixed Lebesgue space norms as follows:

$$\|f\|_{L_k^p L_T^q L_v^r} := \left(\int_{\mathbb{Z}_k^3} \left(\int_0^T \left(\int_{\mathbb{R}^3} |\hat{f}(t, k, v)|^r \, dv \right)^{q/r} dt \right)^{p/q} d\Sigma(k) \right)^{1/p},$$

where $1 \le p, q, r < \infty$, and we use the standard modification of the above when any of p, q, or r is equal to ∞. Analogously for the case of a finite channel in problem **(PC)** we will define standard $L_{\bar{k}}^p L_T^q L_{x_1, v}^r$ mixed Lebesgue space norms as

$$\|f\|_{L_{\bar{k}}^p L_T^q L_{x_1, v}^r} :=$$
$$\left(\int_{\mathbb{Z}_{\bar{k}}^2} \left(\int_0^T \left(\int_{-1}^1 \int_{\mathbb{R}^3} w |\hat{f}(t, x_1, \bar{k}, v)|^r \, dv \, dx_1 \right)^{q/r} dt \right)^{p/q} d\Sigma(\bar{k}) \right)^{1/p},$$

where again $1 \le p, q, r < \infty$ and we use the same standard modification when any of p, q or r is equal to ∞. In this paper we typically use $p = 1$, $q = 2$ or $q = \infty$, and $r = 2$ in the above definitions.

We note further that for ease of the notation we will use the general norm notation $\| \cdot \|$ in different sections to denote different norms, the specific norm that we are using is identified within the section before it is used each time.

1.7 Goal

Using the above preliminary notations, for the problems **(PT)** and **(PC)** our main goal of this paper is to study the following three issues: • *Global-in-time existence of small amplitude solutions in* X_T. For the problem **(PT)** we are able to show that if

$$\|w f_0\|_{L_k^1 L_v^2} := \int_{\mathbb{Z}^3} \|w \hat{f}_0(k, \cdot)\|_{L_v^2} d\Sigma(k)$$

is suitably small and $F_0(x, v) = \mu + \mu^{1/2} f_0(x, v) \geq 0$, then **(PT)** admits a unique global-in-time mild solution $f(t, x, v)$ such that $F(t, x, v) = \mu + \mu^{\frac{1}{2}} f(t, x, v) \geq 0$, and $\|f\|_{X_T^w}$ is uniformly bounded for all $T > 0$. We prove similar results for the problem **(PC)** with physical boundaries when the solution space is replaced by

$$\|f\|_{X_T^w} + \|\nabla_x f\|_{X_T^w}.$$

We will give the precise statements for all of these results in the next section.

- *Large-time behavior of solutions.* To obtain the rate of convergence, under the hypothesis **(H)**, associated with the velocity weight function $w = w_{q,\vartheta}(v)$ and γ, we define a parameter κ in the Landau case as

$$(1.27) \qquad \kappa = \begin{cases} 1 & \text{for } q = 0, -2 \leq \gamma \leq 1, \\ \dfrac{\vartheta}{\vartheta + |\gamma + 2|} & \text{for } q > 0, -3 \leq \gamma < -2, \end{cases}$$

and in the non-cutoff Boltzmann case we define

$$(1.28) \qquad \kappa = \begin{cases} 1 & \text{for } q = 0, \gamma + 2s \geq 0, \\ \dfrac{\vartheta}{\vartheta + |\gamma + 2s|} & \text{for } q > 0, -3 < \gamma < -2s, \vartheta = 1. \end{cases}$$

Note that $\kappa \in (0, 1]$, and $0 < \vartheta \leq 2$ in (1.26). Additionally we use $\vartheta = 1$ only in the Boltzmann case as in (1.26) and (1.28) because of Lemma 4.1 from [29].
We then are able to show that the obtained solutions decay in time as follows:

$$\|f(t)\|_{L_k^1 L_v^2} \lesssim e^{-\lambda t^\kappa},$$

for the problem **(PT)**, and

$$\|f(t)\|_{L_{\bar{k}}^1 L_{x_1,v}^2} + \|\nabla_x f(t)\|_{L_{\bar{k}}^1 L_{x_1,v}^2} \lesssim e^{-\lambda t^\kappa}$$

for the problem **(PC)**, where $\lambda > 0$ is a suitably small uniform constant that is independent of time.

- *Propagation of spatial regularity.* For the problem **(PT)**, we will establish the propagation of spatial regularity of solutions. Specifically, we are able to show that if the initial data f_0 additionally satisfies that $\|\langle \nabla_x \rangle^m f_0\|_{L_k^1 L_v^2}$ is small enough with $m \geq 0$, then the solution $f(t, x, v)$ satisfies that $\|\langle \nabla_x \rangle^m f\|_{L_k^1 L_T^\infty L_v^2}$ is bounded uniformly for all $T > 0$, where $\langle \nabla_x \rangle^m f = \mathscr{F}_x^{-1}(\langle k \rangle^m \hat{f})$. For the problem **(PC)**, similar results also hold true but instead they regard the propagation of $\bar{x}$-regularity, namely, along the tangential direction of $x_1 \in (-1, 1)$, and we will give the precise statement of these results in the next section.

1.8 Motivation of the Current Work

The global-in-time existence theory in the perturbation framework for the Landau equation and the Boltzmann equation without angular cutoff has been well established for suitably small initial data in smooth Sobolev spaces. Particularly, for the Landau equation in a periodic box with $\gamma \geq -3$ including the physical Coulomb potential, Guo [45] gave the first result on the construction of global classical solutions near Maxwellian with $f(t, x, v) \in L^\infty(0, \infty; H^8_{x,v})$. As pointed out in [45, p. 393], such a high Sobolev norm is a standard choice due to the nonlinearity of the Landau operator since the L^∞ norm in x is easily controlled by the Sobolev embedding and then a high Sobolev norm of a product is bounded by a product of the same norms. For the Boltzmann equation without angular cutoff in a periodic box, Gressman-Strain [40] first constructed the global small-amplitude solution $f(t, x, v)$ in $L^\infty(0, \infty; L^2_v H^2_x)$ for the hard potential case $\gamma + 2s \geq 0$ (including $\gamma + 2s \geq -\frac{3}{2}$) and in $L^\infty(0, \infty; H^4_{x,v})$ for the general soft potential case $\gamma + 2s < 0$. A similar result was also independently obtained by AMUXY in their series of works (cf. [2,3], for instance) in the case of the whole space. A key point in those well-known works is to characterize the dissipation property in the L^2 norm in v for the linearized self-adjoint operator (cf. [1,67]) and further carry out the energy estimates by controlling the trilinear term in an appropriate way.

Regarding the solution space, a natural question is to ask whether it is possible to construct the global solutions $f(t, x, v)$ in some function spaces with much lower regularity in space and velocity. In fact, we remark that the local existence with mild regularity for the Boltzmann equation with or without cutoff was discussed in AMUXY [4] in the function space $L^\infty(0, T_*; L^2_v H^s_x)$ with $s > 3/2$ and some $T_* > 0$ for even large initial data. Note that if one uses the Sobolev space $L^2_v H^s_x$ it seems necessary to require $s > 3/2$ in order to obtain the L^∞ bound in x by the imedding. Motivated by [4] as well as [73], for the Boltzmann equation with angular cutoff in the whole space, Duan-Liu-Xu [28] found a Chemin-Lerner-type space $\widetilde{L}^\infty_T B^{3/2}_x L^2_v$ (cf. [8, 19]) such that the solution of small amplitude can be bounded in this function space uniformly in time $T > 0$, where $B^s_x = B^s_{2,1}(\mathbb{R}^3_x)$ for $s \geq 0$ denotes the usual Besov space in x, and $f(t, x, v) \in \widetilde{L}^\infty_T B^s_x L^2_v$ means

$$\left\| \left(2^{qs} \sup_{0 \leq t \leq T} \|\Delta_q f(t, \cdot, \cdot)\|_{L^2_{x,v}} \right)_{q \geq -1} \right\|_{\ell^1_q}$$

$$= \sum_{q \geq -1} 2^{qs} \sup_{0 \leq t \leq T} \|\Delta_q f(t, \cdot, \cdot)\|_{L^2_{x,v}} < \infty.$$

Above the Δ_q are the standard frequency projection operators in Fourier space. A key observation is that the L^∞ norm in x still can be controlled by the embedding $B^{3/2}_x \subset L^\infty_x$ in three dimensions. One feature of the Chemin-Lerner-type space is that the supremum over $0 < t < T$ is taken before the ℓ^1_q norm, and this feature plays a role in controlling the quadratically nonlinear term in terms of the chosen energy functional and the dissipation rate functional. Later, Morimoto-Sakamoto

[65] carried over the same idea to treat global existence for the Boltzmann equation without angular cutoff in the case $\gamma > \max\{-3, -\frac{3}{2} - 2s\}$. Duan-Sakamoto [30] obtained the time decay rates of solutions for both hard and soft potentials with angular cutoff. We remark that it seems still true that those results in [30,65] can also be obtained for the Landau equation with $\gamma \geq -3$ including the Coulomb potential.

On the other hand, we notice that the space $L^\infty_{x,v}$ with suitable velocity weight seems the simplest one with mild regularity in space and velocity variables for directly treating the nonlinearity to obtain the global existence; see an $L^2 \cap L^\infty$ approach by Ukai-Yang [80] based on spectral theory and the decomposition $L = -\nu + K$ (cf. [40,41]) for the Boltzmann equation with cutoff hard potentials in the whole space. In fact, for the Boltzmann equation with cutoff hard potentials in the torus or even in a general bounded domain, Guo [49] developed a mathematical theory of global existence of small-amplitude $L^\infty_{x,v}$ solutions by an L^2-L^∞ interplay method. Since then, there have been extensive generalizations and further developments of the Boltzmann global existence theory in the angular cutoff setting, for instance, [10,13,26,31,50,51,57,58,63,68]. Particularly, motivated by [26,49], Nishimura [68] provided an interesting result on global existence of the angular cutoff Boltzmann equation in the space $L^p_v L^\infty_{t,x}$ with suitable velocity weight for $1 < p \leq \infty$; this work extends to the case of finite $p < \infty$. The case of $p = 2$ is of its own interest because this means that it may no longer be necessary to study the interplay between L^2_v and L^p_v if $p \neq 2$.

However, so far there are not many results on the global existence theory in $L^\infty_{x,v}$ for the angular non-cutoff Boltzmann equation. The main reason is that the collision operator in the non-cutoff case exposes the fractional velocity diffusion property so that it is very difficult to apply the characteristic approach as in [49] to obtain the L^∞ bounds in x for the solutions. The same situation occurs to the Landau equation with velocity diffusion. Recently Kim-Guo-Hwang [59] developed an L^2 to L^∞ approach to the Landau equation in the torus domain, where initial data are required to be small in $L^\infty_{x,v}$ but additionally belong to $H^1_{x,v}$. Since the Sobolev embedding can no longer be used, a key point of the proof in [59] is to control the L^∞ bound by the L^2 estimates via De Giorgi's method [39], and also to control the velocity derivatives to ensure uniqueness by the Hölder estimates again via De Giorgi's method [39]. The De Giorgi's method applied to the Landau equation has been recently developed in Golse-Imbert-Mouhot-Vasseur [39]. Notice that for the Boltzmann equation without cutoff, the regularity issue has also been studied in many recent works, for instance, Silvestre [72], Imbert-Silvestre [56], Imbert-Mouhot-Silvestre [55], and Chen-Hu-Li-Zhan [20]. See also a recent survey by Mouhot [66] and references therein.

Moreover, we observe that there recently have been several research works to apply the Wiener algebra $A(\Omega)$, or in the notation of this paper L^1_k, to study the global-in-time existence and gain of analyticity for solutions to some evolution equations with diffusions, for instance, Constantin-Córdoba-Gancedo-Strain [22],

Constantin–Córdoba–Gancedo–Rodríguez-Piazza–Strain [21], Patel-Strain [70], Gancedo–García-Juárez–Patel–Strain [37], Liu-Strain [62], Granero-Belinchón–Magliocca [42], and Ambrose [7]. See also a recent work by Lei-Lin [60] for the global well-posedness of mild solutions to the three-dimensional, incompressible Navier-Stokes equations if the initial data satisfy $\nabla_x f_0 \in L_k^1$. Here a function $f(x)$ with $x \in \mathbb{T}^3$ is in L_k^1 if its Fourier transform $\widehat{f}(k)$ is summable in $k \in \mathbb{Z}^3$, namely,

$$\int_{\mathbb{Z}_k^3} |\widehat{f}(k)| \, d\Sigma(k) = \sum_{k \in \mathbb{Z}_k^3} |\widehat{f}(k)| < \infty.$$

It is obvious to see that for two functions $f(x)$ and $g(x)$ with $x \in \mathbb{T}^3$,

$$\begin{aligned}
\|fg\|_{L_k^1} &\leq \int_{\mathbb{Z}^3} d\Sigma(k) \, |\widehat{fg}(k)| \\
&\leq \int_{\mathbb{Z}_k^3} d\Sigma(k) \int_{\mathbb{Z}_l^3} d\Sigma(l) \, |\widehat{f}(k-l)\widehat{g}(l)| = \|f\|_{L_k^1} \|g\|_{L_k^1},
\end{aligned}$$

where we have used Fubini's theorem together with the change of variables $k - l \to k$ in the last identity. Thus using this Banach algebra property one may expect that the L_k^1 norm can play a similar role to the L_x^∞ norm when studying the Landau equation or the non-cutoff Boltzmann equation. Note that, as pointed out in [7], the space $L_k^1 L_T^\infty$ is also a Banach algebra, where $f = f(t,x)$ with $t \geq 0$ and $x \in \mathbb{T}^3$ is in $L_k^1 L_T^\infty$ if

$$\int_{\mathbb{Z}^3} \sup_{0 \leq t \leq T} |\widehat{f}(t,k)| d\Sigma(k) < \infty.$$

Those observations together with the aforementioned works [28, 30, 65] motivated us to introduce the function space $X_T = L_k^1 L_T^\infty L_v^2$ with the norm $\|\cdot\|_{X_T}$ as defined previously in (1.23).

1.9 Related Literature

In what follows we recall some known results on the Landau and Boltzmann equations with a focus on the topics under consideration in this paper, particularly on global existence and large-time behavior of solutions to the spatially inhomogeneous equations in the perturbation framework. For global solutions to the renormalized equation with large initial data, we mention the classical works by DiPerna-Lions [24, 25], Lions [61], Villani [82, 83], Desvillettes-Villani [23], and Alexandre-Villani [5].

When the spatial domain is either the whole space or a torus, we mention Ukai [78, 79], Caflisch [11, 12], Guo [46, 47], Liu-Yang-Yu [64], and Strain-Guo [75, 76] for the Boltzmann equation with cutoff. In the non-cutoff case, besides those works [2, 3, 43] mentioned before, we also mention the recent works by Alonso-Morimoto-Sun-Yang [6], He-Jiang [53], and Hérau-Tonon-Tristani [54] for global existence and large-time behavior of solutions with the polynomial velocity weight for the Boltzmann equation in the torus, as well as Carrapatoso-Mischler [14] in

the case of the Landau equation. Here, [6, 14, 54] are motivated by the recent work Gualdani-Mischler-Mouhot [44] in the case of the cutoff Boltzmann equation in the torus, while [53] is based on a new energy-entropy approach in terms of quasi-linear equations.

Moreover, for the large-time behavior of solutions in soft potential cases, particularly for the optimal time-decay rates, we mention Caflisch [12], Strain-Guo [75, 76], and Strain [74]; see also Sohinger-Strain [73], in different settings. We also mention the recent numerical study on the possible sharp $2/3$ rate of large time decay for the Landau equation with Coulomb interactions in Bobylev-Gamba-Zhang [9]; we obtain this decay rate in (2.9), (2.14), and (2.17) with (1.27) when $\gamma = -3$ and $\vartheta = 2$.

Since one part of this paper is concerned with the boundary value problem on the Landau and Boltzmann equations, we also make some comments on the related known results in a perturbation framework. We mainly focus on the case of the Boltzmann equation, as it seems there are few results in the case of the Landau equation with boundaries. In fact, the mathematical study for the the boundary value problems of the Boltzmann equation date back to the 1960s. Cercignani [15, 16] proved the existence and uniqueness for the inflow boundary value problem of the discrete time linearized Boltzmann equation with angular cutoff in two parallel plates, which was immediately extended to the nonlinear case by Pao [69]. With the aid of these approaches as well as the work by Caflish [12], Esposito-Lebowitz-Marra [34, 35] studied the hydrodynamic limits of the stationary Boltzmann equation with diffuse reflection boundary condition in a slab.

For the general bounded domains, Shizuta-Asano [71] in 1977 announced that the global existence and time decay rates to the equilibrium for the Boltzmann equation with the specular reflection boundary condition could be established by using the Vidav's multi-iteration method [81]. Almost ten years later, Ukai [79] in 1986 constructed the famous trace theorem for the Boltzmann equation, which was then used by Hamdache [52] to construct the normalized weak solutions for the Boltzmann equation with Maxwell boundary condition. Later on, Ukai's trace theorem was improved by Cercignani [17] so that Hamdache's result could be eventually extended to more general cases.

As mentioned before, with the foundational work by Guo [49] on the L^2-L^∞ method, a great many of achievements have been made over the past decades in the study of the initial boundary value problems for the kinetic equations, particularly for the Boltzmann equation. For instance, Esposito-Guo-Kim-Marra [32] constructed a genuine nonequilibrium stationary solution. Briant-Guo [10] partially proved the stability of the Boltzmann equation with Maxwell boundary condition. Kim-Lee [58] removed the analytical boundary condition that was required in [49] for the study of the sole specular reflection boundary value problem by iterating the Duhamel's formula three times. Liu-Yang [63] extended the results in [49] to the cutoff soft potential case. Duan-Wang [31] showed the global well-posedness for a class of large-amplitude initial data. Guo-Liu [51] constructed the global existence

for the Boltzmann equation with specular reflection boundary condition around the polynomial initial data differing from a global equilibrium Maxwellian. Esposito-Guo-Kim-Marra [33] and Duan-Liu [27] studied the hydrodynamic limits of the Boltzmann equation in bounded domains based on an improved L^2-L^∞ method.

The main difficulty in studying the initial boundary value problems for the Boltzmann equation is that there should be a singularity at the boundary [50], and actually this singularity may propagate to the interior of the domains [57]. In this sense, unlike the Cauchy problem, it is quite hard to establish the global existence in Sobolev spaces for the Boltzmann equation with physical boundary. Thus, although the well-posedness for the Cauchy problem of the Landau equation [45, 76] and the non-cutoff Boltzmann equation [3, 43] have already been established several years ago, the boundary value problem for either the Landau equation [45] or the non-cutoff Boltzmann equation is left as a big open problem. Our results in this paper seem to be the first ones concerning this issue.

1.10 Organization of the Paper

The rest of this paper is organized as follows. In Section 2, we give the precise statements of the main results of the paper regarding the global existence, large-time behavior, and propagation of spatial regularity of solutions to the problems **(PT)** and **(PC)**. In Section 3, we present our strategy of proof by showing the uniform a priori estimates without any velocity weight for both the non-cutoff Boltzmann equation and the Landau equation in the torus. Motivated by our strategy of proof, Section 4 is concerned with the trilinear estimates with velocity weights in the corresponding function spaces. Section 5 is then devoted to establishing the macroscopic estimates both in the torus case and in the finite channel case. In Sections 6 and 7, we will give the proof of the main results in the case of a torus and a finite channel, respectively. In Section 8, we explain the local-in-time existence of mild solutions for completeness. Then at the end of the paper, the appendix includes some basic lemmas that are used in the previous sections.

2 Main Results

2.1 Notations

To state the main results in this section, we will now introduce some more notation. Recall that to characterize the energy functional for the problem **(PT)** or **(PC)**, we have introduced the function space

$$X_T = L_k^1 L_T^\infty L_v^2 \ \text{ or } \ L_{\bar{k}}^1 L_T^\infty L_{x_1,v}^2,$$

respectively, as well as the velocity-weighted space X_T^w as in (1.24), where the velocity weight $w = w_{q,\vartheta}(v)$ is defined in (1.25) under the assumption **(H)** as in (1.26). In what follows, we further define the corresponding energy dissipation rate functionals. In this section, and in the rest of the paper, we will use f, g, and h as generic smooth real-valued functions in our estimates, when f is not being used as the solution to an equation such as (1.10). Then since we are taking the Fourier

transform, we will also use the standard complex conjugate as $\overline{f}$. Now, for the Landau equation, we recall the Landau kernel in (1.4). Then we define

$$\sigma^{jm} = \sigma^{jm}(v) = \int_{\mathbb{R}^3} \psi^{jm}(v-u)\mu(u)du.$$

It is convenient to define the following velocity-weighted D-norm:

$$|w_{q,\vartheta}f|_D^2 = \sum_{j,m=1}^{3} \int_{\mathbb{R}^3} w_{q,\vartheta}^2 \left\{ \sigma^{jm} \partial_{v_j} f \partial_{v_m} \overline{f} + \frac{1}{4}\sigma^{jm} v_j v_m f\overline{f} \right\} dv.$$

In the case of the finite channel, we also define

$$
\begin{aligned}
&\|w_{q,\vartheta}f\|_D^2 \\
(2.1)\quad &= \sum_{j,m=1}^{3} \int_I \int_{\mathbb{R}^3} w_{q,\vartheta}^2 \left\{ \sigma^{jm} \partial_{v_j} f \partial_{v_m} \overline{f} + \frac{1}{4}\sigma^{jm} v_j v_m f\overline{f} \right\} dv\, dx_1,
\end{aligned}
$$

by including an extra integration in $x_1 \in I = (-1,1)$. For the case of the non-cutoff Boltzmann equation, we define accordingly

$$(2.2)\quad |w_{q,\vartheta}f|_D^2 =$$

$$\int_{\mathbb{R}_v^3} \int_{\mathbb{R}_u^3} \int_{\mathbb{S}^2} B(v-u,\sigma)w_{q,\vartheta}^2(v)\mu(u)(f(v')-f(v))\overline{(f(v')-f(v))}\,d\sigma\,du\,dv$$

$$+ \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_u^3} \int_{\mathbb{S}^2} B(v-u,\sigma)w_{q,\vartheta}^2(v)f(u)\overline{f(u)}(\mu^{\frac{1}{2}}(v')-\mu^{\frac{1}{2}}(v))^2\,d\sigma\,du\,dv,$$

and in the presence of the spatial variable in the finite channel we have

$$(2.3)\quad \|w_{q,\vartheta}g\|_D^2 =$$

$$\int_I \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_u^3} \int_{\mathbb{S}^2} B(v-u,\sigma)w_{q,\vartheta}^2(v)\mu(u)\big(f(v')-f(v)\big)$$

$$\cdot \overline{(f(v')-f(v))}\,d\sigma\,du\,dv\,dx_1$$

$$+ \int_I \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_u^3} \int_{\mathbb{S}^2} B(v-u,\sigma)w_{q,\vartheta}^2(v)f(u)\overline{f(u)}$$

$$\cdot \big(\mu^{\frac{1}{2}}(v')-\mu^{\frac{1}{2}}(v)\big)^2\,d\sigma\,du\,dv\,dx_1.$$

Then, corresponding to the energy functional X_T^w, we define the weighted dissipation rate functionals:

$$\|w_{q,\vartheta}f\|_{L_k^1 L_T^2 L_{v,D}^2} = \int_{\mathbb{Z}_k^3} \left(\int_0^T |w_{q,\vartheta}\mathscr{F}_x f(t,k)|_D^2\, dt \right)^{1/2} d\Sigma(k)$$

and

$$\|w_{q,\vartheta} f\|_{L^1_{\bar{k}} L^2_T L^2_{x_1, v, D}} = \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \|w_{q,\vartheta} \mathscr{F}_{\bar{x}} f(t, \bar{k})\|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{k}),$$

for the torus and finite channel domains, respectively. In the case of a finite channel, we need to include an extra first-order derivative in x, and thus we define the total energy functional and energy dissipation rate functional, respectively, as

$$(2.4) \qquad \mathscr{E}_{T,w}(f) = \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f\|_{L^1_{\bar{k}} L^\infty_T L^2_{x_1, v}}$$

and

$$(2.5) \qquad \begin{aligned} \mathscr{D}_{T,w}(f) &= \sum_{|\alpha| \leq 1} \|\partial^\alpha [a, b, c]\|_{L^1_{\bar{k}} L^2_T L^2_{x_1}} \\ &\quad + \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \{\mathbf{I} - \mathbf{P}\} \partial^\alpha f\|_{L^1_{\bar{k}} L^2_T L^2_{x_1, v, D}}, \end{aligned}$$

where $\partial^\alpha = \partial^\alpha_x = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3}$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ a standard multi-index. Further, the macroscopic part (a, b, c) is defined by (5.1) below, and the norm $\|\cdot\|_{L^1_{\bar{k}} L^2_T L^2_{x_1}}$ only in the t- and x- variables for a function $g = g(t, x)$ is understood in the same integration order as the norm $\|\cdot\|_{L^1_{\bar{k}} L^2_T L^2_{x_1, v, D}}$:

$$\|g\|_{L^1_{\bar{k}} L^2_T L^2_{x_1}} = \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \int_{-1}^1 |\mathscr{F}_{\bar{x}} g(t, \bar{k})|^2 \, dx_1 \, dt \right)^{1/2} d\Sigma(\bar{k}).$$

When $w \equiv 1$, we will use $\mathscr{E}_T(f)$ and $\mathscr{D}_T(f)$ for brevity to denote the norms (2.4) and (2.5) without the weight (1.25).

For any given $t \geq 0$, we define the following norms in x and v:

$$\|w_{q,\vartheta} f(t)\|_{L^1_k L^2_v} = \int_{\mathbb{Z}^3_k} \|w_{q,\vartheta} \mathscr{F}_x f(t, k)\|_{L^2_v} \, d\Sigma(k)$$

and

$$\|w_{q,\vartheta} f(t)\|_{L^1_{\bar{k}} L^2_{x_1, v}} = \int_{\mathbb{Z}^2_{\bar{k}}} \|w_{q,\vartheta} \mathscr{F}_{\bar{x}} f(t, \bar{k})\|_{L^2_{x_1, v}} \, d\Sigma(\bar{k}).$$

The corresponding high-order norms are defined by

$$\|w_{q,\vartheta} f(t)\|_{L^1_{k,m} L^2_v} = \int_{\mathbb{Z}^3_k} \langle k \rangle^m \|w_{q,\vartheta} \mathscr{F}_x f(t, k)\|_{L^2_v} \, d\Sigma(k)$$

and

$$\|w_{q,\vartheta} f(t)\|_{L^1_{\bar{k},m} L^2_{x_1, v}} = \int_{\mathbb{Z}^2_{\bar{k}}} \langle \bar{k} \rangle^m \|w_{q,\vartheta} \mathscr{F}_{\bar{x}} f(t, \bar{k})\|_{L^2_{x_1, v}} \, d\Sigma(\bar{k}),$$

where m is an integer.

For the inflow boundary value problem in the finite channel case, we also define the following norms to capture the boundary effect of the given functions $g_\pm$:

$$E_{\bar{k}}(\widehat{g_\pm}) = \sum_\pm \int_0^T \int_{\pm v_1 < 0} |v_1|^{-1} |\widehat{\partial_t g_\pm}|^2 \, dv \, dt$$

$$+ \sum_\pm \int_0^T \int_{\pm v_1 < 0} |v_1|^{-1} |\bar{k} \cdot \bar{v}|^2 |\widehat{g_\pm}|^2 \, dv \, dt$$

(2.6)
$$+ \sum_\pm \int_0^T \int_{\pm v_1 < 0} |v_1|^{-1} |L\widehat{g_\pm}|^2 \, dv \, dt$$

$$+ \sum_\pm \int_0^T \int_{\pm v_1 < 0} |v_1|^{-1} |\Gamma(\widehat{g_\pm}, \widehat{g_\pm})|^2 \, dv \, dt$$

$$+ \sum_\pm \int_0^T \int_{\pm v_1 < 0} |v_1|(1 + |\bar{k}|^2) |\widehat{g_\pm}|^2 \, dv \, dt,$$

and

(2.7)
$$E(\widehat{g_\pm}) = \int_{\mathbb{Z}_{\bar{k}}^2} \sqrt{E_{\bar{k}}(\widehat{g_\pm})} \, d\Sigma(\bar{k}).$$

Other notations. Throughout this paper, C denotes a generic positive (generally large) uniform constant and λ denotes a generic positive (generally small) uniform constant, where both C and λ may take different values in different places. $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leq CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. We also typically use $\eta > 0$ to denote a constant that can be made arbitrarily small.

2.2 Case of the Torus

To state the main results, we are first concerned with the problem **(PT)** for the Landau equation or the non-cutoff Boltzmann equation in a torus.

THEOREM 2.1 (Existence and large-time behavior). *Let $\Omega = \mathbb{T}^3$. Assume that $f_0(x, v)$ satisfies* (1.16), (1.17), *and* (1.18). *Let $w_{q,\vartheta}$ be chosen under the assumption* **(H)** *in* (1.26). *There is $\epsilon_0 > 0$ such that if $F_0(x, v) = \mu + \mu^{1/2} f_0(x, v) \geq 0$ and*

$$\|w_{q,\vartheta} f_0\|_{L_k^1 L_v^2} \leq \epsilon_0,$$

then there exists a unique global mild solution $f(t, x, v)$, $t > 0$, $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$ to the problem **(PT)** (1.10) *and* (1.11) *for the Landau equation or the non-cutoff Boltzmann equation, satisfying that $F(t, x, v) = \mu + \mu^{1/2} f(t, x, v) \geq 0$ and*

(2.8)
$$\|w_{q,\vartheta} f\|_{L_k^1 L_T^\infty L_v^2} + \|w_{q,\vartheta} f\|_{L_k^1 L_T^2 L_{v,D}^2} \lesssim \|w_{q,\vartheta} f_0\|_{L_k^1 L_v^2},$$

for any $T > 0$. Moreover, let $\kappa \in (0, 1]$ be defined in (1.27) or (1.28) for the Landau case or the non-cutoff Boltzmann case, respectively; then there is a $\lambda > 0$ such that the solution also enjoys the time decay estimate

$$(2.9) \qquad \|f(t)\|_{L_k^1 L_v^2} \lesssim e^{-\lambda t^\kappa} \|w_{q,\vartheta} f_0\|_{L_k^1 L_v^2}$$

for any $t \geq 0$.

THEOREM 2.2 (Propagation of spatial regularity). *Let all the conditions in Theorem 2.1 be satisfied; then for any integer $m \geq 0$, there is an $\epsilon_0 > 0$ such that if*

$$(2.10) \qquad \|w_{q,\vartheta} f_0\|_{L_{k,m}^1 L_v^2} \leq \epsilon_0,$$

then the solution $f(t, x, v)$ to (1.10)–(1.11) established in Theorem 2.1 satisfies

$$
\begin{aligned}
(2.11) \qquad & \int_{\mathbb{Z}^3} \langle k \rangle^m \sup_{0 \leq t \leq T} \|w_{q,\vartheta} \hat{f}(t, k)\|_{L_v^2} \, d\Sigma(k) \\
& + \int_{\mathbb{Z}^3} \langle k \rangle^m \left(\int_0^T |w_{q,\vartheta} \hat{f}(t, k)|_D^2 \, dt \right)^{1/2} d\Sigma(k) \\
& \lesssim \|w_{q,\vartheta} f_0\|_{L_{k,m}^1 L_v^2},
\end{aligned}
$$

for any $T > 0$.

2.3 Case of the Finite Channel

Next, we are concerned with the problem **(PC)** for the Landau equation or the non-cutoff Boltzmann equation in the finite channel.

THEOREM 2.3 (Inflow boundary condition). *Let $\Omega = I \times \mathbb{T}^2$. Let $w_{q,\vartheta}$ be chosen under the assumption* **(H)** *in (1.26). There are $\epsilon_0 > 0$ and $C > 0$ such that if $F_0(x_1, \bar{x}, v) = \mu + \mu^{1/2} f_0(x_1, \bar{x}, v) \geq 0$, $F(t, \pm 1, \bar{x}, v) = \mu + \mu^{1/2} g_\pm(t, \bar{x}, v) \geq 0$ for $v_1 > 0$ at $x_1 = -1$ and $v_1 < 0$ at $x_1 = 1$, and*

$$(2.12) \qquad \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L_{\bar{k}}^1 L_{x_1, v}^2} + E(w_{q,\vartheta} \widehat{g_\pm}) \leq \epsilon_0,$$

then there exists a unique mild solution $f(t, x_1, \bar{x}, v)$ to the inflow boundary problem **(PC)** *(1.10), (1.11), and (1.14) for the Landau equation or the non-cutoff Boltzmann equation, satisfying that $F(t, x_1, \bar{x}, v) = \mu + \mu^{1/2} f(t, x_1, \bar{x}, v) \geq 0$, and*

$$(2.13) \quad \mathscr{E}_{T,w}(f) + \mathscr{D}_{T,w}(f) \leq C \left\{ \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L_{\bar{k}}^1 L_{x_1, v}^2} + E(w_{q,\vartheta} \widehat{g_\pm}) \right\},$$

for any $T > 0$, where $\mathscr{E}_{T,w}(f)$, $\mathscr{D}_{T,w}(f)$, and $E(w_{q,\vartheta} \widehat{g_\pm})$ are defined in (2.4), (2.5), and (2.7), respectively. Moreover, let $\kappa \in (0, 1]$ be defined in (1.27) or (1.28) for the Landau case or the non-cutoff Boltzmann case, respectively; then there is $\lambda > 0$ such that if

$$E(w_{q,\vartheta} \widehat{g_\pm}) + \sup_{s > 0} E(e^{\lambda s^\kappa} \widehat{g_\pm}) \leq \epsilon_0$$

for $\epsilon_0 > 0$ further small enough, then it holds that

$$
\sum_{|\alpha| \leq 1} \|\partial^\alpha f(t)\|_{L^1_{\frac{1}{k}} L^2_{x_1, v}} \lesssim e^{-\lambda t^\kappa} \sum_{|\alpha| \leq 1} \|w_{q, \vartheta} \partial^\alpha f_0\|_{L^1_{\frac{1}{k}} L^2_{x_1, v}}
$$

(2.14)

$$
+ e^{-\lambda t^\kappa} \left\{ E(w_{q, \vartheta} \widehat{g_\pm}) + \sup_{s > 0} E(e^{\lambda s^\kappa} \widehat{g_\pm}) \right\},
$$

for any $t \geq 0$.

THEOREM 2.4 (Specular reflection boundary condition). *Let $\Omega = I \times \mathbb{T}^2$. Let $w_{q, \vartheta}$ be chosen under the assumption* (**H**) *in* (1.26). *Assume the symmetry condition* (1.22), *and let $f_0(x, v)$ satisfy* (1.19), (1.20), *and* (1.21). *There are $\epsilon_0 > 0$ and $C > 0$ such that if $F_0(x_1, \bar{x}, v) = \mu + \mu^{1/2} f_0(x_1, \bar{x}, v) \geq 0$ and*

$$
\sum_{|\alpha| \leq 1} \|w_{q, \vartheta} \partial^\alpha f_0\|_{L^1_{\frac{1}{k}} L^2_{x_1, v}} \leq \epsilon_0,
$$
(2.15)

then there exists a unique mild solution $f(t, x_1, \bar{x}, v)$ to the specular reflection boundary problem (**PC**) (1.10), (1.11), *and* (1.15) *for the Landau equation or the non-cutoff Boltzmann equation, satisfying that*

$$
F(t, x_1, \bar{x}, v) = \mu + \mu^{1/2} f(t, x_1, \bar{x}, v) \geq 0
$$

with $f(t, -x_1, \bar{x}, -v_1, \bar{v}) = f(t, x_1, \bar{x}, v_1, \bar{v})$ and

$$
\mathscr{E}_{T, w}(f) + \mathscr{D}_{T, w}(f) \leq C \sum_{|\alpha| \leq 1} \|w_{q, \vartheta} \partial^\alpha f_0\|_{L^1_{\frac{1}{k}} L^2_{x_1, v}}
$$
(2.16)

for any $T > 0$, where $\mathscr{E}_{T, w}(f)$ and $\mathscr{D}_{T, w}(f)$ are defined in (2.4) *and* (2.5), *respectively. Moreover, let $\kappa \in (0, 1]$ be defined in* (1.27) *or* (1.28) *for the Landau case or the non-cutoff Boltzmann case, respectively; then there is $\lambda > 0$ such that*

$$
\sum_{|\alpha| \leq 1} \|\partial^\alpha f(t)\|_{L^1_{\frac{1}{k}} L^2_{x_1, v}} \lesssim e^{-\lambda t^\kappa} \sum_{|\alpha| \leq 1} \|w_{q, \vartheta} \partial^\alpha f_0\|_{L^1_{\frac{1}{k}} L^2_{x_1, v}},
$$
(2.17)

for any $t \geq 0$.

THEOREM 2.5 (Propagation of spatial regularity in $\bar{x}$). *Let all of the conditions in Theorem 2.3 and Theorem 2.4 be satisfied, then for any integer $m \geq 0$, there are $\epsilon_0 > 0$ and $C > 0$ such that if*

$$
\sum_{|\alpha| \leq 1} \|w_{q, \vartheta} \partial^\alpha f_0\|_{L^1_{\frac{1}{k}, m} L^2_{x_1, v}} + E(w_{q, \vartheta} \langle \bar{k} \rangle^m \widehat{g_\pm}) \leq \epsilon_0,
$$
(2.18)

and

$$
\sum_{|\alpha| \leq 1} \|w_{q, \vartheta} \partial^\alpha f_0\|_{L^1_{\frac{1}{k}, m} L^2_{x_1, v}} \leq \epsilon_0,
$$
(2.19)

hold in the place of (2.12) *and* (2.15), *respectively, then we obtain that*

$$
\begin{aligned}
(2.20) \quad & \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \langle \bar{k} \rangle^m \sup_{0 \leq t \leq T} \| w_{q,\vartheta} \widehat{\partial^\alpha f}(t,\bar{k}) \| d\Sigma(\bar{k}) \\
& + \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \langle \bar{k} \rangle^m \left(\int_0^T \| w_{q,\vartheta} \widehat{\partial^\alpha f} \|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{k}) \\
& \lesssim \left\{ \sum_{|\alpha|\leq 1} \| w_{q,\vartheta} \partial^\alpha f_0 \|_{L^1_{\bar{k},m} L^2_{x_1,v}} + E(w_{q,\vartheta} \langle \bar{k} \rangle^m \widehat{g_\pm}) \right\},
\end{aligned}
$$

for the inflow boundary condition, and

$$
\begin{aligned}
(2.21) \quad & \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \langle \bar{k} \rangle^m \sup_{0 \leq t \leq T} \| w_{q,\vartheta} \widehat{\partial^\alpha f}(t,\bar{k}) \| d\Sigma(\bar{k}) \\
& + \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \langle \bar{k} \rangle^m \left(\int_0^T \| w_{q,\vartheta} \widehat{\partial^\alpha f} \|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{k}) \\
& \lesssim \sum_{|\alpha|\leq 1} \| w_{q,\vartheta} \partial^\alpha f_0 \|_{L^1_{\bar{k},m} L^2_{x_1,v}},
\end{aligned}
$$

for the specular reflection boundary condition, respectively.

2.4 Comments

Here a few comments are in order on the results of this paper.

(a) As mentioned before, the global existence of classical solutions close to the global Maxwellians was established in [45] for the Landau equation and in [43] for the Boltzmann equation without angular cutoff both in the torus. To the best of our knowledge, the current results seem to be the first ones to revisit these global existence theories in a new function space with mild regularity in both space and velocity variables. The proof can be carried out in a unified way for the Landau and Boltzmann cases. The function space

$$
L^1_k L^\infty_T L^2_v \cap L^1_k L^2_T L^2_{v,D}
$$

appears to be very useful for treating the global dynamics of Landau and non-cutoff Boltzmann solutions in perturbation framework.

(b) In the case of a finite channel, to the best of our knowledge these results may be the first ones to provide an elementary understanding of the existence theories for the Landau or non-cutoff Boltzmann equations in the situation where the spatial domain has physical boundaries.

As mentioned before, the singularities can generally form for solutions to the boundary value problem. When the boundary condition is specular reflection, we have made an essential use of the symmetric condition (1.22) on the initial data, which then also remains true for the solution. The singularity is killed by such a symmetric property of solutions. It turns out

that without this symmetric assumption, the corresponding results are still unknown. For the inflow boundary value problem, the singularity is transformed to the boundary data; see the boundary energy functional as in (2.6) or (2.7), which is induced by the estimates on the normal derivatives $\partial_{x_1} f$ in terms of the equations as in (7.4) below.

Another issue is related to the case of the diffusive reflection boundary for which it seems too hard at the moment to treat estimates on the boundary terms in our settings. Of course, with the developments in this work, we are still far away from dealing with the existence of solutions, even local in time, for the Landau or non-cutoff Boltzmann equation in general bounded domains.

(c) Regarding results on the propagation of spatial regularity, it should be pointed out that it may be unnecessary to require the smallness assumptions on the mth-order derivatives of the initial data as in (2.10), (2.18), and (2.19). Indeed, since the initial data is assumed to be sufficiently small in the function space without the Fourier mode $\langle k \rangle^m$, one could try to use an induction argument together with interpolation techniques and time decay properties of solutions to obtain the similar results to those in Theorem 2.2 and Theorem 2.5 under the only assumption that the m-order derivatives of initial data have a finite norm in the space $L_k^1 L_v^2$ with an appropriate velocity weight. For brevity of the presentation we will not pursue such improved results in this paper.

(d) The question of the regularity of the obtained solutions in our situation of the global existence theory is an interesting issue to be further studied. This is because either the Landau operator or the non-cutoff Boltzmann operator both enjoy the velocity diffusion property, and the spatial regularity could also be gained through the hypoellipticity techniques of such collisional kinetic equations; see an aforementioned work [20] and references therein.

3 Strategy of the Proof

In this section we shall present the strategy of the proof. In particular, we will explain how to obtain the uniform a priori estimates on the solutions in the function space $X_T = L_k^1 L_T^\infty L_v^2$ for the problem (**PT**) in the torus case $\Omega = \mathbb{T}^3$. For the problem (**PC**) in the finite channel case $\Omega = (-1, 1) \times \mathbb{T}^2$, we can carry out the same strategy to treat the periodic variable $\bar{x} = (x_2, x_3) \in \mathbb{T}^2$ with some extra consideation for the variable $x_1 \in (-1, 1)$; this will be clarified in Section 7 later. In this section we will not include any velocity weight in order to increase the claritiy of the exposition. The local-in-time existence of solutions will be studied in Section 8.

3.1 Non-Cutoff Boltzmann Case

For convenience of the presentation we first start from the Boltzmann equation without angular cutoff. Let $f = f(t, x, v)$, $0 \le t \le T$, $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$, be a smooth solution to

$$(3.1) \qquad \partial_t f + v \cdot \nabla_x f + L f = \Gamma(f, f),$$

with prescribed initial data $f(0, x, v) = f_0(x, v)$. Taking the Fourier transform in $x \in \mathbb{T}^3$, we obtain

$$(3.2) \qquad \partial_t \widehat{f}(t, k, v) + i v \cdot k \widehat{f}(t, k, v) + L \widehat{f}(t, k, v) = \widehat{\Gamma}(\widehat{f}, \widehat{f})(t, k, v).$$

Here, for brevity, we have denoted the operator

$$(3.3) \quad \widehat{\Gamma}(\widehat{f}, \widehat{g})(k, v)$$
$$= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \sigma) \mu^{1/2}(u) \big([\widehat{f}(u') * \widehat{g}(v')](k) - [\widehat{f}(u) * \widehat{g}(v)](k) \big) d\sigma \, du,$$

where the convolutions are taken with respect to $k \in \mathbb{Z}^3$:

$$[\widehat{f}(u') * \widehat{g}(v')](k) := \int_{\mathbb{Z}_l^3} \widehat{f}(k - l, u') \widehat{g}(l, v') d\Sigma(l),$$

$$[\widehat{f}(u) * \widehat{g}(v)](k) := \int_{\mathbb{Z}_l^3} \widehat{f}(k - l, u) \widehat{g}(l, v) d\Sigma(l).$$

Then for this system we have the uniform estimate in the following proposition.

PROPOSITION 3.1. *There is a universal constant $C > 0$ such that*

$$\int_{\mathbb{Z}^3} \sup_{0 \le t \le T} \|\widehat{f}(t, k, \cdot)\|_{L_v^2} d\Sigma(k)$$

$$(3.4) \qquad + \int_{\mathbb{Z}^3} \left(\int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k, \cdot)|_D^2 \, dt \right)^{1/2} d\Sigma(k)$$

$$\lesssim \|f_0\|_{L_k^1 L_v^2}$$

$$+ \left(\eta + \frac{\|f\|_{L_k^1 L_T^\infty L_v^2}}{4\eta} \right) \int_{\mathbb{Z}^3} \left(\int_0^T |\widehat{f}(t, k, \cdot)|_D^2 \, dt \right)^{1/2} d\Sigma(k),$$

for any $T > 0$, where the constant $\eta > 0$ can be arbitrarily small.

PROOF. Taking the product of (3.2) with the complex conjugate of $\widehat{f}(t, k, v)$ and further taking the real part of the resulting equation, we have

$$\frac{1}{2} \frac{d}{dt} |\widehat{f}(t, k, v)|^2 + \mathscr{R}(L\widehat{f}, \widehat{f}) = \mathscr{R}(\widehat{\Gamma}(\widehat{f}, \widehat{f}), \widehat{f}),$$

where $(\cdot, \cdot)$ denotes the complex inner product over the complex field, i.e., $(f, g) = f \cdot \overline{g}$, and $\mathscr{R}$ denotes the real part of a complex number. Integrating the above

identity with respect to v and then t, we have

$$
\frac{1}{2}\|\widehat{f}(t,k,\cdot)\|_{L_v^2}^2 + \int_0^t \mathscr{R}(L\widehat{f},\widehat{f})_{L_v^2}\,d\tau
$$

(3.5)

$$
= \frac{1}{2}\|\widehat{f_0}(k,\cdot)\|_{L_v^2}^2 + \int_0^t \mathscr{R}(\widehat{\Gamma}(\widehat{f},\widehat{f}),\widehat{f})_{L_v^2}\,d\tau,
$$

where correspondingly $(\cdot,\cdot)_{L_v^2}$ denotes the complex inner product over L_v^2, i.e.,

$$
(f,g)_{L_v^2} = \int_{\mathbb{R}_v^3} f(v)\overline{g(v)}dv.
$$

Recall that by the coercivity estimate of L (cf. Lemma A.2), there is $\delta_0 > 0$ such that

$$
\delta_0 |\{\mathbf{I}-\mathbf{P}\}g|_D^2 \le \mathscr{R}(Lg,g)_{L_v^2}.
$$

Thus it follows from (3.5) that

$$
\frac{1}{2}\|\widehat{f}(t,k,\cdot)\|_{L_v^2}^2 + \delta_0 \int_0^t |\{\mathbf{I}-\mathbf{P}\}\widehat{f}|_D^2\,d\tau
$$

$$
\le \frac{1}{2}\|\widehat{f_0}(k,\cdot)\|_{L_v^2}^2 + \int_0^t \mathscr{R}(\widehat{\Gamma}(\widehat{f},\widehat{f}),\widehat{f})_{L_v^2}\,d\tau.
$$

Taking the square root on both sides and using the elementary inequalities

$$
\frac{1}{\sqrt{2}}(A+B) \le \sqrt{A^2+B^2} \le A+B,
$$

we further have

$$
\frac{1}{\sqrt{2}}\|\widehat{f}(t,k,\cdot)\|_{L_v^2} + \sqrt{\delta_0}\left(\int_0^t |\{\mathbf{I}-\mathbf{P}\}\widehat{f}(\tau,k,\cdot)|_D^2\,d\tau\right)^{1/2}
$$

$$
\le \|\widehat{f_0}(k,\cdot)\|_{L_v^2} + \sqrt{2}\left(\int_0^t |\mathscr{R}(\widehat{\Gamma}(\widehat{f},\widehat{f}),\widehat{f})_{L_v^2}|\,d\tau\right)^{1/2}.
$$

So, we have derived, for any $0 \le t \le T$ and $k \in \mathbb{Z}^3$, that

(3.6)

$$
\|\widehat{f}(t,k,\cdot)\|_{L_v^2} + \left(\int_0^t |\{\mathbf{I}-\mathbf{P}\}\widehat{f}(\tau,k,\cdot)|_D^2\,d\tau\right)^{1/2}
$$

$$
\le C_0\left\{\|\widehat{f_0}(k,\cdot)\|_{L_v^2} + \left(\int_0^t \left|\mathscr{R}(\widehat{\Gamma}(\widehat{f},\widehat{f}),\widehat{f})_{L_v^2}\right|\,d\tau\right)^{1/2}\right\},
$$

with

$$
C_0 = \frac{\sqrt{2}}{\min\{1/\sqrt{2},\sqrt{\delta_0}\}} > 0.
$$

Moreover, taking $\sup_{0 \le t \le T}$ on both sides of (3.6) and then integrating the resulting inequality with respect to $d\Sigma(k)$ over $\mathbb{Z}^3$, we have

$$(3.7) \quad \begin{aligned} &\int_{\mathbb{Z}^3} \sup_{0 \le t \le T} \|\widehat{f}(t, k, \cdot)\|_{L_v^2} \, d\Sigma(k) \\ &\quad + \int_{\mathbb{Z}^3} \left(\int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k, \cdot)|_D^2 \, dt \right)^{1/2} d\Sigma(k) \\ &\qquad \le C_0 \left\{ \|\widehat{f}_0\|_{L_k^1 L_v^2} + \int_{\mathbb{Z}^3} \left(\int_0^T |(\widehat{\Gamma}(\widehat{f}, \widehat{f}), \widehat{f})_{L_v^2}| dt \right)^{1/2} d\Sigma(k) \right\}. \end{aligned}$$

In what follows we will estimate the last term on the right-hand side of (3.7). Indeed, it can be bounded as

$$(3.8) \quad \begin{aligned} &\int_{\mathbb{Z}^3} \left(\int_0^T |(\widehat{\Gamma}(\widehat{f}, \widehat{f}), \widehat{f})_{L_v^2}| dt \right)^{1/2} d\Sigma(k) \\ &\quad \le \int_{\mathbb{Z}^3} \left(\int_0^T \int_{\mathbb{Z}^3} \|\widehat{f}(k-l)\|_{L_v^2} |\widehat{f}(l)|_D |\widehat{f}(k)|_D d\Sigma(l) dt \right)^{1/2} d\Sigma(k), \end{aligned}$$

in terms of the trilinear estimate in the following lemma.

LEMMA 3.2. *It holds that*

$$(3.9) \quad \left| (\widehat{\Gamma}(\widehat{f}, \widehat{g})(k), \widehat{h}(k))_{L_v^2} \right| \le C \int_{\mathbb{Z}^3} \|\widehat{f}(k-l)\|_{L_v^2} |\widehat{g}(l)|_D |\widehat{h}(k)|_D \, d\Sigma(l).$$

PROOF. By the definition (3.3) of $\widehat{\Gamma}(\cdot, \cdot)$ as well as Fubini's theorem, we obtain

$$\begin{aligned} &\left(\widehat{\Gamma}(\widehat{f}, \widehat{g})(k), \widehat{h}(k) \right)_{L_v^2} \\ &= \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_u^3} \int_{\mathbb{S}^2} B(v - u, \sigma) \mu^{1/2}(u) \\ &\qquad \times \{ [\widehat{f}(u') * \widehat{g}(v')](k) - [\widehat{f}(u) * \widehat{g}(v)](k) \} \overline{\widehat{h}}(v, k) d\sigma \, du \, dv \\ &= \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_u^3} \int_{\mathbb{S}^2} B(v - u, \sigma) \mu^{1/2}(u) \\ &\qquad \times \int_{\mathbb{Z}^3} \{ \widehat{f}(k - l, u') \widehat{g}(l, v') - \widehat{f}(k - l, u) \widehat{g}(l, v) \} \overline{\widehat{h}}(v, k) d\Sigma(l) d\sigma \, du \, dv \\ &= \int_{\mathbb{Z}^3} \int_{\mathbb{R}_v^3} \Gamma(\widehat{f}(k - l), \widehat{g}(l)) \overline{\widehat{h}}(k) dv \, d\Sigma(l). \end{aligned}$$

Therefore, it follows that

$$(3.10) \quad \left| (\widehat{\Gamma}(\widehat{f}, \widehat{g})(k), \widehat{h}(k))_{L_v^2} \right| \le \int_{\mathbb{Z}^3} \left| (\Gamma(\widehat{f}(k - l), \widehat{g}(l)), \widehat{h}(k))_{L_v^2} \right| d\Sigma(l).$$

Recall either from [4, theorem 1.2] or from [43, theorem 2.1, p. 782] that one has

$$(3.11) \quad |(\Gamma(f, g), h)_{L_v^2}| \lesssim \|f\|_{L_v^2} |g|_D |h|_D.$$

With the above inequality the desired estimate (3.9) follows from (3.10). This completes the proof of Lemma 3.2. □

Remark 3.3. We remark here, in regards to the previous proof and what follows, that the norm (2.2) is equivalent to the norm $N^{s,\gamma}$ from [43, eq. (1.8), page 774]; this can be shown directly for example by using the estimates in [43]. See for example [43, eq. (2.13), p. 784].

We continue to estimate the upper bound in (3.8). Applying the Cauchy-Schwarz inequality with respect to $\int_0^T (\cdot) dt$ and further using Young's inequality with an arbitrary small constant $\eta > 0$, we have

$$
\int_{\mathbb{Z}_k^3} \left(\int_0^T \int_{\mathbb{Z}_l^3} \|\widehat{f}(t, k-l)\|_{L_v^2} |\widehat{f}(t,l)|_D |\widehat{f}(t,k)|_D d\Sigma(l) dt \right)^{1/2} d\Sigma(k)
$$

$$
\leq \int_{\mathbb{Z}_k^3} \left(\int_0^T \left(\int_{\mathbb{Z}_l^3} \|\widehat{f}(t,k-l)\|_{L_v^2} |\widehat{f}(t,l)|_D d\Sigma(l) \right)^2 dt \right)^{1/4}
$$

(3.12)
$$
\times \left(\int_0^T |\widehat{f}(t,k)|_D^2 dt \right)^{1/4} d\Sigma(k)
$$

$$
\leq \eta \int_{\mathbb{Z}_k^3} \left(\int_0^T |\widehat{f}(t,k)|_D^2 dt \right)^{1/2} d\Sigma(k)
$$

$$
+ \frac{1}{4\eta} \int_{\mathbb{Z}_k^3} \left(\int_0^T \left(\int_{\mathbb{Z}_l^3} \|\widehat{f}(t,k-l)\|_{L_v^2} |\widehat{f}(t,l)|_D d\Sigma(l) \right)^2 dt \right)^{1/2} d\Sigma(k).
$$

To treat the second term on the right-hand side of (3.12), we first use Minkowski's inequality to obtain

(3.13)
$$
\left\| \| \cdot \|_{L_l^1} \right\|_{L_t^2} \leq \left\| \| \cdot \|_{L_t^2} \right\|_{L_l^1}.
$$

Then we have

$$
\left(\int_0^T \left(\int_{\mathbb{Z}_l^3} \|\widehat{f}(t,k-l)\|_{L_v^2} |\widehat{f}(t,l)|_D d\Sigma(l) \right)^2 dt \right)^{1/2}
$$

$$
\leq \int_{\mathbb{Z}_l^3} \left(\int_0^T \|\widehat{f}(t,k-l)\|_{L_v^2}^2 |\widehat{f}(t,l)|_D^2 dt \right)^{1/2} d\Sigma(l),
$$

and hence it follows that

$$
\int_{\mathbb{Z}_k^3} \left(\int_0^T \left(\int_{\mathbb{Z}_l^3} \|\widehat{f}(t,k-l)\|_{L_v^2} |\widehat{f}(t,l)|_D d\Sigma(l) \right)^2 dt \right)^{1/2} d\Sigma(k)
$$

$$
\leq \int_{\mathbb{Z}_k^3} \int_{\mathbb{Z}_l^3} \sup_{0 \leq t \leq T} \|\widehat{f}(t,k-l)\|_{L_v^2} \left(\int_0^T |\widehat{f}(t,l)|_D^2 dt \right)^{1/2} d\Sigma(l) d\Sigma(k).
$$

Further, by Fubini's theorem and translation invariance, the upper bound in the above inequality can be computed as

$$\int_{\mathbb{Z}_k^3} \int_{\mathbb{Z}_l^3} \sup_{0 \le t \le T} \|\widehat{f}(t, k - l)\|_{L_v^2} \left(\int_0^T |\widehat{f}(t, l)|_D^2 \, dt \right)^{1/2} d\Sigma(l) d\Sigma(k)$$

$$= \int_{\mathbb{Z}_l^3} d\Sigma(l) \left(\int_0^T |\widehat{f}(t, l)|_D^2 \, dt \right)^{1/2} \int_{\mathbb{Z}_k^3} d\Sigma(k) \sup_{0 \le t \le T} \|\widehat{f}(t, k - l)\|_{L_v^2}$$

$$= \|\widehat{f}\|_{L_k^1 L_T^\infty L_v^2} \int_{\mathbb{Z}_l^3} \left(\int_0^T |\widehat{f}(t, l)|_D^2 \, dt \right)^{1/2} d\Sigma(l).$$

Then, applying those estimates above to the second term on the right-hand side of (3.12) and further using (3.7) and (3.8) leads to the desired estimate (3.4). This completes the proof of Proposition 3.1. □

We remark that we have the identity

$$\left(\widehat{\Gamma}(\widehat{f}, \widehat{f}), \widehat{f} \right)_{L_v^2} = \left(\widehat{\Gamma}(\widehat{f}, \widehat{f}), \{\mathbf{I} - \mathbf{P}\}\widehat{f} \right)_{L_v^2}.$$

Then using this identity one can modify the proof above slightly to obtain

$$\int_{\mathbb{Z}^3} \sup_{0 \le t \le T} \|\widehat{f}(t, k, \cdot)\|_{L_v^2} d\Sigma(k) + \int_{\mathbb{Z}^3} \left(\int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k, \cdot)|_D^2 \, dt \right)^{1/2} d\Sigma(k)$$

$$\lesssim \|f_0\|_{L_k^1 L_v^2} + \|f\|_{L_k^1 L_T^\infty L_v^2} \int_{\mathbb{Z}^3} \left(\int_0^T |\widehat{f}(t, k, \cdot)|_D^2 \, dt \right)^{1/2} d\Sigma(k).$$

This indicates that as long as one can further appropriately estimate (such as in Theorem 5.1 below) the macroscopic dissipation,

$$\int_{\mathbb{Z}^3} \left(\int_0^T |\mathbf{P}\widehat{f}(t, k, \cdot)|_D^2 \, dt \right)^{1/2} d\Sigma(k)$$

$$\sim \int_{\mathbb{Z}^3} \left(\int_0^T |\widehat{(a, b, c)}(t, k)|^2 \, dt \right)^{1/2} d\Sigma(k),$$

where (a, b, c) is defined in (5.1), we can then obtain the uniform estimates under the smallness assumption on $\|f\|_{L_k^1 L_T^\infty L_v^2}$ that can be closed provided that f_0 is suitably small in $L_k^1 L_v^2$.

3.2 Landau Case

When the non-cutoff Boltzmann operator is replaced by the Landau operator as in (1.2), we are still able to carry out the same strategy. Indeed, the nonlinear

Landau operator (1.13) takes the following form (cf. [45, p. 395]):

$$\Gamma(f, g) = \sum_{i,j=1}^{3} \left[\partial_{v_i} [\{\psi^{ij} * (\mu^{\frac{1}{2}} f)\} \partial_{v_j} g] - \{\psi^{ij} * (v_i \mu^{\frac{1}{2}} f)\} \partial_{v_j} g \right.$$
$$\left. - \partial_{v_i} [\{\psi^{ij} * (\mu^{\frac{1}{2}} \partial_{v_j} f)\} g] + \{\psi^{ij} * (v_i \mu^{\frac{1}{2}} \partial_{v_j} f)\} g \right].$$

We consider the linearized Landau equation of the same form as in (3.1). Taking the Fourier transform in $x \in \mathbb{T}^3$ gives the same form (3.2) with the nonlinear part now taking the form

$$\widehat{\Gamma}(\widehat{f}, \widehat{g})$$

$$(3.14) \quad = \sum_{i,j=1}^{3} \left[\partial_{v_i} [\{\psi^{ij} *_v (\mu^{\frac{1}{2}} \widehat{f})\} *_k \partial_{v_j} \widehat{g}] - \{\psi^{ij} *_v (v_i \mu^{\frac{1}{2}} \widehat{f})\} *_k \partial_{v_j} \widehat{g} \right.$$
$$- \partial_{v_i} [\{\psi^{ij} *_v (\mu^{\frac{1}{2}} \partial_{v_j} \widehat{f})\} *_k \widehat{g}]$$
$$\left. + \{\psi^{ij} *_v (v_i \mu^{\frac{1}{2}} \partial_{v_j} \widehat{f})\} *_k \widehat{g} \right],$$

where $*_v$ denotes the convolution in $v \in \mathbb{R}^3$ and $*_k$ denotes the convolution in $k \in \mathbb{Z}^3$. We now apply the estimate [45, theorem 3, p. 406] to directly obtain

LEMMA 3.4. *The following estimate holds uniformly:*

$$|(\widehat{\Gamma}(\widehat{f}, \widehat{g})(k), \widehat{h}(k))_{L_v^2}| \lesssim (\|\widehat{f}\|_{L_v^2} *_k |\widehat{g}|_D)(k)|\widehat{h}(k)|_D,$$

More precisely, the estimate above can be found in [77, prop. 1, p. 621], which is a simplification of the original estimate from [45, theorem 3, p. 406]. We remark that the estimates in those papers involve weights and derivatives, however the above estimate in Lemma 3.4 only differs in the lack of weights and derivatives and the inclusion of the convolution $*_k$ in the terms. And Lemma 3.4 follows directly from the exact same proofs. Based on the above lemma, one can obtain the same estimate as stated in Proposition 3.1. For brevity we omit the rest of the details in the Landau case since they are the same as in the Boltzmann case.

4 Trilinear Estimates

In this section we will treat trilinear estimates by three parts. The first part is concerned with the velocity-weighted trilinear estimates as used in (3.11). Moreover, corresponding to estimating the left-hand term of (3.8), the second and third parts are devoted to considering the velocity-weighted trilinear estimates on mixed variables in the cases of the torus and the finite channel, respectively.

4.1 Trilinear Estimates in L_v^2 with Velocity Weight

We first give some basic estimates on the velocity weighted trilinear terms in the following lemma whose proof can be found in [76, lemma 10, p. 327] and [29, 36, lemma 2.3, p. 176; lemma 2.4, p. 121], respectively. Note that for *hard* potentials

corresponding to either $-2 \leq \gamma \leq 1$ in the Landau case or $\gamma + 2s \geq 0$ in the Boltzmann case, we have $q = 0$ and hence $w_{q,\vartheta} \equiv 1$, namely, it is not necessary to include any velocity weight in the *hard* potential cases. In fact, in all the cases under the hypothesis (**H**), there is no need to include any velocity weight for the purposes of establishing the existence theory. However, the velocity weight will be needed for deducing the subexponential time decay in the *soft* potential cases.

LEMMA 4.1. *Let (q, ϑ) in the velocity weight function $w_{q,\vartheta}$ be chosen in terms of the hypothesis* (**H**) *in* (1.26). *Then, for the Landau operator, it holds that*

$$
(4.1) \quad \begin{aligned}
&\left| \left(\Gamma(f, g), w_{q,\vartheta}^2 h \right)_{L_v^2} \right| \\
&\lesssim \left(\|w_{q,\vartheta} f\|_{L_v^2} |w_{q,\vartheta} g|_D + |w_{q,\vartheta} f|_D \|w_{q,\vartheta} g\|_{L_v^2} \right) |w_{q,\vartheta} h|_D.
\end{aligned}
$$

Similarly, for the non-cutoff Boltzmann operator, it holds that

$$
(4.2) \quad \begin{aligned}
&\left| \left(\Gamma(f, g), w_{q,\vartheta}^2 h \right)_{L_v^2} \right| \\
&\lesssim \left(\|w_{q,\vartheta} f\|_{L_v^2} |w_{q,\vartheta} g|_D + |w_{q,\vartheta} f|_D \|w_{q,\vartheta} g\|_{L_v^2} \right) |w_{q,\vartheta} h|_D,
\end{aligned}
$$

if $\gamma + 2s \geq 0$, and

$$
(4.3) \quad \begin{aligned}
&\left| \left(\Gamma(f, g), w_{q,\vartheta}^2 h \right)_{L_v^2} \right| \\
&\lesssim \left\{ \left\| \langle v \rangle^{\gamma/2+s} w_{q,\vartheta} f \right\|_{L_v^2} |g|_D + \left\| \langle v \rangle^{\gamma/2+s} g \right\|_{L_v^2} |w_{q,\vartheta} f|_D \right\} |w_{q,\vartheta} h|_D \\
&\quad + \min \left\{ \|w_{q,\vartheta} f\|_{L_v^2} \left\| \langle v \rangle^{\gamma/2+s} g \right\|_{L_v^2}, \|g\|_{L_v^2} \left\| \langle v \rangle^{\gamma/2+s} w_{q,\vartheta} f \right\|_{L_v^2} \right\} |w_{q,\vartheta} h|_D \\
&\quad + \|w_{q,\vartheta} g\|_{L_v^2} \left\| \langle v \rangle^{\gamma/2+s} w_{q,\vartheta} f \right\|_{L_v^2} |w_{q,\vartheta} h|_D,
\end{aligned}
$$

if $\gamma + 2s < 0$.

4.2 Trilinear Estimates on Mixed Variables in Isotropic Case

Recall that we have derived (3.4) without any velocity weights. Given Lemma 4.1, we may employ the same idea to include the velocity weight $w_{q,\vartheta}$ in our estimates.

LEMMA 4.2. *Let (q, ϑ) in the velocity weight function $w_{q,\vartheta}$ be chosen in terms of the hypothesis* (**H**) *in* (1.26). *Then, for both the Landau and Boltzmann cases it holds that*

$$
(4.4) \quad \begin{aligned}
&\int_{\mathbb{Z}_k^3} \left(\int_0^T \left| \left(\widehat{\Gamma(f, g)}, w_{q,\vartheta}^2 \hat{h} \right)_{L_v^2} \right| dt \right)^{1/2} d\Sigma(k) \\
&\leq C_\eta \left(\|w_{q,\vartheta} f\|_{L_k^1 L_T^\infty L_v^2} \|w_{q,\vartheta} g\|_{L_k^1 L_T^2 L_{v,D}^2} \right. \\
&\qquad\qquad \left. + \|w_{q,\vartheta} f\|_{L_k^1 L_T^2 L_{v,D}^2} \|w_{q,\vartheta} g\|_{L_k^1 L_T^\infty L_v^2} \right) \\
&\qquad + \eta \|w_{q,\vartheta} h\|_{L_k^1 L_T^2 L_{v,D}^2},
\end{aligned}
$$

where the Fourier transform $\widehat{\cdot}$ is taken in $x = (x_1, x_2, x_3) \in \mathbb{T}^3$, $\eta > 0$ is an arbitrary small constant, and C_η is a universal large constant depending only on η.

PROOF. In the rest of this proof $(\cdot, \cdot)$ denotes the L_v^2 complex inner product. We consider Proposition 3.1 and its proof. In particular, we recall (3.12) and the estimates below it. Then to show (4.4), it suffices to verify that

$$
\begin{aligned}
(4.5) \quad & \left| \left(\widehat{\Gamma(f, g)}, w_{q,\vartheta}^2 \widehat{h} \right) \right| \\
& \lesssim \int_{\mathbb{Z}_l^3} \left(\left\| w_{q,\vartheta} \widehat{f}(k - l) \right\|_{L_v^2} |w_{q,\vartheta} \widehat{g}(l)|_D + |w_{q,\vartheta} \widehat{f}(k - l)|_D \left\| w_{q,\vartheta} \widehat{g}(l) \right\|_{L_v^2} \right) \\
& \quad \times \left| w_{q,\vartheta} \widehat{h}(k) \right|_D \, d\Sigma(l).
\end{aligned}
$$

Indeed, by (1.13), it holds that

$$
(4.6) \quad \left(\widehat{\Gamma(f, g)}(k), w_{q,\vartheta}^2 \widehat{h}(k) \right) = \int_{\mathbb{R}_v^3} w_{q,\vartheta}^2 \mu^{-\frac{1}{2}}(v) \mathscr{F}_x Q(\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} g)(k) \overline{\widehat{h}(k)} \, dv.
$$

Then, it follows from (1.3) and (4.6) together with Fubini's theorem that

$$
\begin{aligned}
& \left| \left(\widehat{\Gamma(f, g)}(k), w_{q,\vartheta}^2 \widehat{h}(k) \right) \right| \\
& = \left| \int_{\mathbb{R}_v^3} dv \, w_{q,\vartheta}^2 \mu^{-\frac{1}{2}}(v) \int_{\mathbb{Z}_l^3} d\Sigma(l) \right. \\
& \quad \times \partial_{v_j} \left\{ \left[\psi^{jm} *_v \left(\mu^{\frac{1}{2}} \widehat{f}(k - l) \right) \right] \mu^{\frac{1}{2}}(v) \left[\partial_{v_m} \widehat{g}(l) - \frac{v_m}{2} \widehat{g}(l) \right] \right\} \overline{\widehat{h}(k)} \\
& \quad - \int_{\mathbb{R}_v^3} dv \, w_{q,\vartheta}^2 \mu^{-\frac{1}{2}}(v) \int_{\mathbb{Z}_l^3} d\Sigma(l) \\
& \quad \left. \times \partial_{v_j} \left\{ \left[\psi^{jm} *_v \left\{ \left(\partial_{v_m} \widehat{f} - \frac{v_m}{2} \widehat{f} \right)(k - l) \mu^{\frac{1}{2}} \right\} \right] \mu^{\frac{1}{2}}(v) \widehat{g}(l) \right\} \overline{\widehat{h}(k)} \right| \\
& = \left| \int_{\mathbb{Z}_l^3} d\Sigma(l) \int_{\mathbb{R}_v^3} dv \, \Gamma(\widehat{f}(k - l), \widehat{g}(l)) w_{q,\vartheta}^2 \overline{\widehat{h}(k)} \right| \\
& \leq \int_{\mathbb{Z}_l^3} \left| \left(\Gamma(\widehat{f}(k - l), \widehat{g}(l)), w_{q,\vartheta}^2 \widehat{h}(k) \right) \right| d\Sigma(l),
\end{aligned}
$$

for the Landau collision operator, and

$$
\begin{aligned}
&\left|\left(\widehat{(\Gamma(f,g))}(k), \hat{h}(k)\right)\right| \\
&= \left|\int_{\mathbb{R}_v^3} dv \int_{\mathbb{R}_u^3} du \int_{\mathbb{S}^2} d\sigma \, B\mu^{\frac{1}{2}}(u) w_{q,\vartheta}^2 \left(\widehat{f(u')g(v')}(k) - \widehat{f(u)g(v)}(k)\right)\overline{\hat{h}}(k)\right| \\
&= \left|\int_{\mathbb{R}_v^3} dv \int_{\mathbb{R}_u^3} du \int_{\mathbb{S}^2} d\sigma \, B\mu^{\frac{1}{2}}(u) w_{q,\vartheta}^2 \right. \\
&\qquad \left. \times \int_{\mathbb{Z}_l^3} d\Sigma(l)\left(\widehat{f(u')}(k-l)\widehat{g(v')}(l) - \widehat{f(u)}(k-l)\widehat{g(v)}(l)\right)\overline{\hat{h}}(k)\right| \\
&= \left|\int_{\mathbb{Z}_l^3} d\Sigma(l) \int_{\mathbb{R}_v^3} dv \, \Gamma(\hat{f}(k-l), \hat{g}(l))w_{q,\vartheta}^2 \overline{\hat{h}}(k)\right| \\
&\leq \int_{\mathbb{Z}_l^3} \left|\left(\Gamma(\hat{f}(k-l), \hat{g}(l)), w_{q,\vartheta}^2 \hat{h}(k)\right)\right| d\Sigma(l),
\end{aligned}
$$

for the Boltzmann collision operator. Hence, from the above two estimates, one can further derive (4.5) with the help of (4.1) for the Landau operator, and (4.2) and (4.3) for the Boltzmann operator. This completes the proof of Lemma 4.2. □

The following lemma will be useful for dealing with the nonlinear terms arising from the macroscopic estimates in Section 5.1. The proof of Lemma 4.3 is omitted for brevity of presentation, since it is similar to and much easier than that of Lemma 4.5 below in the case of the finite channel, where the proof will be provided.

LEMMA 4.3. *Assume that $\zeta(v)$ depends only on v and decays rapidly at infinity. Then, for $|\alpha| = 0, 1$, it holds that*

$$
\begin{aligned}
&\int_{\mathbb{Z}_k^3} \left(\int_0^T \left|\left(\mathscr{F}_{\bar{x}}\Gamma(f,g), \zeta(v)\right)_{L_v^2}\right|^2 dt\right)^{1/2} d\Sigma(k) \\
&\qquad \leq C\left(\|f\|_{L_k^1 L_T^\infty L_v^2}\|g\|_{L_k^1 L_T^2 L_{v,D}^2} + \|f\|_{L_k^1 L_T^2 L_{v,D}^2}\|g\|_{L_k^1 L_T^\infty L_v^2}\right).
\end{aligned}
$$

4.3 Trilinear Estimates on Mixed Variables in Anisotropic Case

Similar to the proof of Lemma 4.2, the proof of the following lemma is also based on Lemma 4.1 and some significant properties of the $L_k^{\frac{1}{k}}$ Wiener algebra space $A(\mathbb{T}^2)$.

LEMMA 4.4. *Let (q, ϑ) in the velocity weight function $w_{q,\vartheta}$ be chosen in terms of the hypothesis* **(H)** *in* (1.26). *Then, for both Landau and Boltzmann cases it holds*

that for $|\alpha| = 0, 1$, we have

$$\int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \left| \left(\mathscr{F}_{\bar{x}} \Gamma(\partial^\alpha f, g), w^2_{q,\vartheta} \hat{h} \right) \right| dt \right)^{1/2} d\Sigma(\bar{k})$$

$$(4.7) \qquad \leq C_\eta \left(\left\| w_{q,\vartheta} \partial^\alpha f \right\|_{L^1_{\bar{k}} L^\infty_T L^2_{x_1,v}} \left\| w_{q,\vartheta} g \right\|_{L^1_{\bar{k}} L^2_T H^1_{x_1} L^2_{v,D}} \right.$$

$$\left. + \left\| w_{q,\vartheta} \partial^\alpha f \right\|_{L^1_{\bar{k}} L^2_T L^2_{x_1} L^2_{v,D}} \left\| w_{q,\vartheta} g \right\|_{L^1_{\bar{k}} L^\infty_T H^1_{x_1} L^2_v} \right)$$

$$+ \eta \left\| w_{q,\vartheta} h \right\|_{L^1_{\bar{k}} L^2_T L^2_{x_1} L^2_{v,D}}$$

and

$$\int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \left| \left(\mathscr{F}_{\bar{x}} \Gamma(f, \partial^\alpha g), w^2_{q,\vartheta} \hat{h} \right) \right| dt \right)^{1/2} d\Sigma(\bar{k})$$

$$(4.8) \qquad \leq C_\eta \left(\left\| w_{q,\vartheta} \partial^\alpha g \right\|_{L^1_{\bar{k}} L^\infty_T L^2_{x_1,v}} \left\| w_{q,\vartheta} f \right\|_{L^1_{\bar{k}} L^2_T H^1_{x_1} L^2_{v,D}} \right.$$

$$\left. + \left\| w_{q,\vartheta} \partial^\alpha g \right\|_{L^1_{\bar{k}} L^2_T L^2_{x_1} L^2_{v,D}} \left\| w_{q,\vartheta} f \right\|_{L^1_{\bar{k}} L^\infty_T H^1_{x_1} L^2_v} \right)$$

$$+ \eta \left\| w_{q,\vartheta} h \right\|_{L^1_{\bar{k}} L^2_T L^2_{x_1} L^2_{v,D}},$$

where the Fourier transform $\hat{\cdot}$ is taken in $\bar{x} = (x_2, x_3) \in \mathbb{T}^2$, the inner product $(\cdot, \cdot)$ is taken over $L^2_{x_1,v}$ where $x_1 \in I = (-1, 1)$, $\eta > 0$ is an arbitrary constant, and C_η is a universal constant depending only on η.

PROOF. We only prove (4.7), since the proof of (4.8) is very similar. For this purpose, we first show that

$$\left| \left(\mathscr{F}_{\bar{x}} \Gamma(\partial^\alpha f, g), w^2_{q,\vartheta} \hat{h} \right)_{L^2_{x_1,v}} \right|$$

$$(4.9) \qquad \lesssim \int_{\mathbb{Z}^2_{\bar{l}}} \int_I \left(\left\| w_{q,\vartheta} \widehat{\partial^\alpha f}(\bar{k} - \bar{l}) \right\|_{L^2_v} \left| w_{q,\vartheta} \hat{g}(\bar{l}) \right|_D \right.$$

$$\left. + \left| w_{q,\vartheta} \widehat{\partial^\alpha f}(\bar{k} - \bar{l}) \right|_D \left\| w_{q,\vartheta} \hat{g}(\bar{l}) \right\|_{L^2_v} \right) \left| w_{q,\vartheta} \hat{h} \right|_D dx_1 d\Sigma(\bar{l}).$$

To verify (4.9) for the Landau operator or Boltzmann operator, similarly to the proof of Lemma 4.2, we get from (1.13) and Fubini's theorem that

$$(4.10) \qquad \left| \left(\mathscr{F}_{\bar{x}} \Gamma(\partial^\alpha f, g)(\bar{k}), w^2_{q,\vartheta} \hat{h}(\bar{k}) \right)_{L^2_{x_1,v}} \right|$$

$$= \left| \int_I dx_1 \int_{\mathbb{R}^3_v} dv \, w^2_{q,\vartheta} \mu^{-\frac{1}{2}}(v) \mathscr{F}_{\bar{x}} Q(\mu^{\frac{1}{2}} \partial^\alpha f, \mu^{\frac{1}{2}} g)(\bar{k}) \overline{\hat{h}}(\bar{k}) \right|$$

$$= \left| \int_I dx_1 \int_{\mathbb{R}^3_v} dv \int_{\mathbb{Z}^2_{\bar{l}}} d\Sigma(\bar{l}) \, \Gamma(\widehat{\partial^\alpha f}(\bar{k} - \bar{l}), \hat{g}(\bar{l})) w^2_{q,\vartheta} \overline{\hat{h}}(\bar{k}) \right| =$$

$$= \left| \int_{\mathbb{Z}^2_{\bar{l}}} d\Sigma(\bar{l}) \int_I dx_1 \int_{\mathbb{R}^3_v} dv \, \Gamma(\widehat{\partial^\alpha f}(\bar{k} - \bar{l}), \widehat{g}(\bar{l})) w^2_{q,\vartheta} \overline{\widehat{h}}(\bar{k}) \right|$$

$$\leq \int_{\mathbb{Z}^2_{\bar{l}}} \left| \left(\Gamma(\widehat{\partial^\alpha f}(\bar{k} - \bar{l}), \widehat{g}(\bar{l})), w^2_{q,\vartheta} \widehat{h}(\bar{k}) \right) \right| d\Sigma(\bar{l}).$$

Hence (4.9) follows from (4.10) with the help of Lemma 4.1. Next, we let J denote the left-hand term of (4.7):

$$J := \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \left| \left(\mathscr{F}_{\bar{x}} \Gamma(\partial^\alpha f, g), w^2_{q,\vartheta} \widehat{h} \right)_{L^2_{x_1,v}} \right| dt \right)^{1/2} d\Sigma(\bar{k}).$$

Then by applying (4.9) together with the functional Sobolev embedding inequality $\|g\|_{L^\infty(I)} \lesssim \|g\|_{H^1(I)}$, one has

$$J \lesssim \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \int_{\mathbb{Z}^2_{\bar{l}}} \sum_{\alpha' \leq 1} \|w \widehat{\partial^\alpha f}(\bar{k} - \bar{l})\| \|w \partial^{\alpha'}_{x_1} \widehat{g}(\bar{l})\|_D \, d\Sigma(\bar{l}) \|w \widehat{h}(\bar{k})\|_D \, dt \right)^{1/2} d\Sigma(\bar{k})$$

$$+ \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \int_{\mathbb{Z}^2_{\bar{l}}} \sum_{\alpha' \leq 1} \|w \widehat{\partial^\alpha f}(\bar{k} - \bar{l})\|_D \|w \partial^{\alpha'}_{x_1} \widehat{g}(\bar{l})\| \, d\Sigma(\bar{l}) \|w \widehat{h}(\bar{k})\|_D \, dt \right)^{1/2} d\Sigma(\bar{k}).$$

Now above and in the rest of this proof the norm $\|\cdot\|$ is $L^2_{x_1,v}$ and the norm $\|\cdot\|_D$ is as defined in (2.1). Also the above integrals both contain an implicit $d\Sigma(\bar{k})$, and the weight $w = w_{q,\vartheta}$ as usual.

It further follows from the Cauchy-Schwarz inequality in the time integral that

$$J \lesssim C_\eta \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \left[\int_{\mathbb{Z}^2_{\bar{l}}} \sum_{\alpha' \leq 1} \|w_{q,\vartheta} \widehat{\partial^\alpha f}(\bar{k} - \bar{l})\| \|w_{q,\vartheta} \partial^{\alpha'}_{x_1} \widehat{g}(\bar{l})\|_D \, d\Sigma(\bar{l}) \right]^2 dt \right)^{1/2} d\Sigma(\bar{k})$$

$$+ C_\eta \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \left[\int_{\mathbb{Z}^2_{\bar{l}}} \sum_{\alpha' \leq 1} \|w_{q,\vartheta} \widehat{\partial^\alpha f}(\bar{k} - \bar{l})\|_D \|w_{q,\vartheta} \partial^{\alpha'}_{x_1} \widehat{g}(\bar{l})\| \, d\Sigma(\bar{l}) \right]^2 dt \right)^{1/2} d\Sigma(\bar{k})$$

$$+ \eta \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \|w_{q,\vartheta} \widehat{h}(\bar{k})\|^2_D \, dt \right)^{1/2} d\Sigma(\bar{k}),$$

for an arbitrary small constant $\eta > 0$. Now we use Minkowski's inequality as in (3.13) to deduce from the above estimate that

$$J \lesssim C_\eta \int_{\mathbb{Z}^2_{\bar{k}}} \sup_{0 \leq t \leq T} \|w_{q,\vartheta} \widehat{\partial^\alpha f}(\bar{k})\| \, d\Sigma(\bar{k}) \int_{\mathbb{Z}^2_{\bar{l}}} \left(\int_0^T \sum_{\alpha' \leq 1} \|w_{q,\vartheta} \partial^{\alpha'}_{x_1} \widehat{g}(\bar{l})\|^2_D \, dt \right)^{1/2} d\Sigma(\bar{l})$$

$$+ C_\eta \int_{\mathbb{Z}^2_{\bar{l}}} \sup_{0 \leq t \leq T} \sum_{\alpha' \leq 1} \|w_{q,\vartheta} \partial^{\alpha'}_{x_1} \widehat{g}(\bar{l})\| \, d\Sigma(\bar{l}) \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \|w_{q,\vartheta} \widehat{\partial^\alpha f}(\bar{k})\|^2_D \, dt \right)^{1/2} d\Sigma(\bar{k})$$

$$+ \eta \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \|w_{q,\vartheta} \widehat{h}(\bar{k})\|^2_D \, dt \right)^{1/2} d\Sigma(\bar{k}),$$

that is,

$$
\begin{aligned}
J \lesssim\ & C_\eta \big\| w_{q,\vartheta} \partial^\alpha f \big\|_{L^1_{\frac{1}{k}} L^\infty_T L^2_{x_1,v}} \big\| w_{q,\vartheta} g \big\|_{L^1_{\frac{1}{k}} L^2_T H^1_{x_1} L^2_{v,D}} \\
&+ C_\eta \big\| w_{q,\vartheta} \partial^\alpha f \big\|_{L^1_{\frac{1}{k}} L^2_T L^2_{x_1} L^2_{v,D}} \big\| w_{q,\vartheta} g \big\|_{L^1_{\frac{1}{k}} L^\infty_T H^1_{x_1} L^2_v} \\
&+ \eta \big\| w_{q,\vartheta} \widehat{h} \big\|_{L^1_{\frac{1}{k}} L^2_T L^2_{x_1} L^2_{v,D}}.
\end{aligned}
$$

This then proves (4.7), and thus completes the proof of Lemma 4.4.   □

Similar to Lemma 4.3, the following lemma will be used for treating the nonlinear term in the macroscopic estimates. We will provide a brief proof for completeness.

LEMMA 4.5. *Assume that $\zeta(v)$ depends only on v and decays rapidly at infinity. Then, for $|\alpha| = 0, 1$, it holds that*

$$
\begin{aligned}
&\int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \big\| \big(\mathscr{F}_{\bar{x}} \Gamma(\partial^\alpha f, g), \zeta(v) \big)_{L^2_v} \big\|^2_{L^2_{x_1}} dt \right)^{1/2} d\Sigma(\bar{k}) \\
&\quad \leq C \big(\| \partial^\alpha f \|_{L^1_{\frac{1}{k}} L^\infty_T L^2_{x_1,v}} \| g \|_{L^1_{\frac{1}{k}} L^2_T H^1_{x_1} L^2_{v,D}} \\
&\qquad + \| \partial^\alpha f \|_{L^1_{\frac{1}{k}} L^2_T L^2_{x_1} L^2_{v,D}} \| g \|_{L^1_{\frac{1}{k}} L^\infty_T H^1_{x_1} L^2_v} \big)
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \big\| \big(\mathscr{F}_{\bar{x}} \Gamma(f, \partial^\alpha g), \zeta(v) \big)_{L^2_v} \big\|^2_{L^2_{x_1}} dt \right)^{1/2} d\Sigma(\bar{k}) \\
&\quad \leq C \big(\| \partial^\alpha g \|_{L^1_{\frac{1}{k}} L^\infty_T L^2_{x_1,v}} \| f \|_{L^1_{\frac{1}{k}} L^2_T H^1_{x_1} L^2_{v,D}} \\
&\qquad + \| \partial^\alpha g \|_{L^1_{\frac{1}{k}} L^2_T L^2_{x_1} L^2_{v,D}} \| f \|_{L^1_{\frac{1}{k}} L^\infty_T H^1_{x_1} L^2_v} \big).
\end{aligned}
\tag{4.12}
$$

PROOF. As in the proof for Lemma 4.4, we prove (4.11) only, since the proof of (4.12) is similar. Firstly, to treat the left-hand term of (4.12), we write that

$$
\begin{aligned}
&\left(\int_0^T \big\| \big(\mathscr{F}_{\bar{x}} \Gamma(\partial^\alpha f, g), \zeta(v) \big)_{L^2_v} \big\|^2_{L^2_{x_1}} dt \right)^{1/2} \\
&= \left(\int_0^T \int_I \left(\int_{\mathbb{R}^3_v} \int_{\mathbb{Z}^2_{\bar{l}}} \Gamma(\widehat{\partial^\alpha f}(\bar{k} - \bar{l}), \widehat{g}(\bar{l})) \zeta(v) \, d\Sigma(\bar{l}) \, dv \right)^2 dt \, dx_1 \right)^{1/2}.
\end{aligned}
\tag{4.13}
$$

From Lemma 4.1, the above term is bounded by

$$
\begin{aligned}
&C_\zeta \left(\int_0^T \int_I \left(\int_{\mathbb{Z}^2_{\bar{l}}} \big\| \widehat{\partial^\alpha f}(\bar{k} - \bar{l}) \big\|_{L^2_v} |\widehat{g}(\bar{l})|_D \, d\Sigma(\bar{l}) \right)^2 dt \, dx_1 \right)^{1/2} \\
&+ C_\zeta \left(\int_0^T \int_I \left(\int_{\mathbb{Z}^2_{\bar{l}}} \big\| \widehat{g}(\bar{l}) \big\|_{L^2_v} |\widehat{\partial^\alpha f}(\bar{k} - \bar{l})|_D \, d\Sigma(\bar{l}) \right)^2 dt \, dx_1 \right)^{1/2},
\end{aligned}
$$

where $C_\zeta > 0$ is a universal constant depending only on ζ. Furthermore, by Minkowski's inequality $\||\|\cdot\|_{L_{\bar{l}}^1}\|_{L_{t,x_1}^2} \leq \||\|\cdot\|_{L_{t,x_1}^2}\|_{L_{\bar{l}}^1}$, (4.13) is then bounded by

$$
C_\zeta \int_{\mathbb{Z}_{\bar{l}}^2} \left(\int_0^T \int_I (\|\widehat{\partial^\alpha f}(\bar{k}-\bar{l})\|_{L_v^2} |\widehat{g}(\bar{l})|_D)^2 \, dt \, dx_1 \right)^{1/2} d\Sigma(\bar{l})
$$

$$
+ C_\zeta \int_{\mathbb{Z}_{\bar{l}}^2} \left(\int_0^T \int_I (\|\widehat{g}(\bar{l})\|_{L_v^2} |\widehat{\partial^\alpha f}(\bar{k}-\bar{l})|_D)^2 \, dt \, dx_1 \right)^{1/2} d\Sigma(\bar{l}).
$$

According to Sobolev's inequality $\|g\|_{L^\infty(I)} \lesssim \|g\|_{H^1(I)}$, the above term can be bounded by

(4.14)
$$
C_\zeta \int_{\mathbb{Z}_{\bar{l}}^2} \sup_{0 \leq t \leq T} \|\widehat{\partial^\alpha f}(\bar{k}-\bar{l})\|_{L_{x_1,v}^2} \sum_{0 \leq \alpha_1 \leq 1} \left(\int_0^T \|\widehat{\partial_{x_1}^{\alpha_1} g}(\bar{l})\|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{l})
$$

$$
+ C_\zeta \int_{\mathbb{Z}_{\bar{l}}^2} \sup_{0 \leq t \leq T} \sum_{0 \leq \alpha_1 \leq 1} \|\widehat{\partial_{x_1}^{\alpha_1} g}(\bar{l})\|_{L_{x_1,v}^2} \left(\int_0^T \|\widehat{\partial^\alpha f}(\bar{k}-\bar{l})\|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{l}).
$$

Consequently, (4.11) follows from taking the integration of (4.14) in $\bar{k} \in \mathbb{Z}_{\bar{k}}^2$ with respect to $d\Sigma(\bar{k})$ and further applying Fubini's theorem. This completes the proof of Lemma 4.5. $\qquad\square$

5 Macroscopic Estimates

Throughout this section, we let $T > 0$ be an arbitrary fixed constant. We further emphasize that the universal constant $C > 0$ in all estimates below is independent of T. Let us start from the macro-micro decomposition of the solution f; i.e., we split f as $f = \mathbf{P}f + \{\mathbf{I} - \mathbf{P}\}f$, where

(5.1)
$$
\mathbf{P}f = \{a + b \cdot v + (|v|^2 - 3)c\}\mu^{\frac{1}{2}}.
$$

This expression above defines $[a, b, c]$, where $[\cdot, \cdot, \cdot]$ represents a vector.

5.1 Isotropic Case

In this section we will derive the uniform a priori estimates for the macroscopic part of a solution to the equation

(5.2)
$$
\partial_t f + v \cdot \nabla_x f + Lf = H, \quad t > 0, \ x \in \mathbb{T}^3, v \in \mathbb{R}^3,
$$

with initial data given as in (1.11), where generally the inhomogeneous source term $H = H(t, x, v)$ is assumed to be a functional of an arbitrary distribution $h(t, x, v)$ and $H = H(h(x, v)) \in \mathcal{N}^\perp(L)$, where $\mathcal{N}^\perp(L)$ is the perpendicular to the null space of L for any t and x. For the proof we will follow the same strategy as in [32] by the dual argument.

THEOREM 5.1. *Under the assumptions of Theorem* 2.1, *it holds that*

$$\|[a, b, c]\|_{L_k^1 L_T^2} \lesssim \|\{\mathbf{I} - \mathbf{P}\}f\|_{L_k^1 L_T^2 L_{v,D}^2} + \|f\|_{L_k^1 L_T^\infty L_v^2} + \|f_0\|_{L_k^1 L_v^2}$$

(5.3)
$$+ \int_{\mathbb{Z}_k^3} \left(\int_0^T |(\hat{H}(t, k), \mu^{\frac{1}{4}})|^2 \, dt \right)^{1/2} d\Sigma(k),$$

where the inner product in the last term is taken over L^2 in v.

PROOF. By taking the velocity moments

$$\mu^{\frac{1}{2}}, \ v_j \mu^{\frac{1}{2}}, \ \frac{1}{6}(|v|^2 - 3)\mu^{\frac{1}{2}}, \ (v_j v_m - 1)\mu^{\frac{1}{2}}, \ \frac{1}{10}(|v|^2 - 5)v_j \mu^{\frac{1}{2}}$$

with $1 \le j, m \le 3$ for the equation (1.10) with $\Gamma(f, f)$ replaced by H, one sees
that the coefficient functions $[a, b, c] = [a, b, c](t, x)$ satisfy the fluid-type system

(5.4)
$$\begin{cases} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b + \nabla_x (a + 2c) + \nabla_x \cdot \Theta(\{\mathbf{I} - \mathbf{P}\}f) = 0, \\ \partial_t c + \frac{1}{3}\nabla_x \cdot b + \frac{1}{6}\nabla_x \cdot \Lambda(\{\mathbf{I} - \mathbf{P}\}f) = 0, \\ \partial_t [\Theta_{jm}(\{\mathbf{I} - \mathbf{P}\}f) + 2c\delta_{jm}] + \partial_j b_m + \partial_m b_j = \Theta_{jm}(\mathbb{r} + \mathbb{h}), \\ \partial_t \Lambda_j(\{\mathbf{I} - \mathbf{P}\}f) + \partial_j c = \Lambda_j(\mathbb{r} + \mathbb{h}), \end{cases}$$

where the high-order moment functions

$$\Theta = (\Theta_{jm}(\cdot))_{3 \times 3} \quad \text{and} \quad \Lambda = (\Lambda_j(\cdot))_{1 \le j \le 3}$$

are, respectively, defined by

$$\Theta_{jm}(f) = ((v_j v_m - 1)\mu^{\frac{1}{2}}, f), \quad \Lambda_j(f) = \frac{1}{10}((|v|^2 - 5)v_j \mu^{\frac{1}{2}}, f),$$

with the inner product taken with respect to the velocity variable v only, and the
terms $\mathbb{r}$ and $\mathbb{h}$ on the right are given by

$$\mathbb{r} = -v \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\}f, \quad \mathbb{h} = -L\{\mathbf{I} - \mathbf{P}\}f + H.$$

Note that the conservation laws (1.16), (1.17), and (1.18) imply that

$$\widehat{[a, b, c]}(t, 0) = 0$$

for any $t \ge 0$.

In order to carry out the estimate in a unified way, we take a general function as

$$\hat{\Phi}(t, k, v) \in C^1((0, \infty) \times \mathbb{Z}^3 \times \mathbb{R}^3),$$

which will be fixed later according to the different cases that we study. Applying
the Fourier transform to (5.2), taking the inner product of it and $\hat{\Phi}$ in L_v^2, then

integrating the resultant over $[0, T]$, we have

$$(\hat{f}, \hat{\Phi})|_{t=T} - (\hat{f}, \hat{\Phi})|_{t=0} - \int_0^T (\hat{f}, \partial_t \hat{\Phi}) dt$$

$$- \int_0^T (\hat{f}, v \cdot i k \hat{\Phi}) dt + \int_0^T (L\hat{f}, \hat{\Phi}) dt = \int_0^T (\hat{H}, \hat{\Phi}) dt.$$

Note that above and in the rest of this proof, $(\cdot, \cdot) = (\cdot, \cdot)_{L_v^2}$. Here we have used the shorthand notations $(\hat{f}, \hat{\Phi})|_{t=T} = (\hat{f}, \hat{\Phi})(T)$ and $(\hat{f}, \hat{\Phi})|_{t=0} = (\hat{f}, \hat{\Phi})(0)$. Plugging in the macro-micro decomposition yields

$$- \int_0^T (\mathbf{P}\hat{f}, v \cdot i k \hat{\Phi}) dt$$

$$= (\hat{f}, \hat{\Phi})(0) - (\hat{f}, \hat{\Phi})(T) + \int_0^T (\hat{f}, \partial_t \hat{\Phi}) dt$$

$$+ \int_0^T ((\mathbf{I} - \mathbf{P})\hat{f}, v \cdot i k \hat{\Phi}) dt - \int_0^T (L\hat{f}, \hat{\Phi}) dt + \int_0^T (\hat{H}, \hat{\Phi}) dt.$$

Now we will define the following notation:

$$\begin{cases} S_1 = (\hat{f}, \hat{\Phi})(0) - (\hat{f}, \hat{\Phi})(T), \\ S_2 = \int_0^T (\hat{f}, \partial_t \hat{\Phi}) dt, \\ S_3 = \int_0^T ((\mathbf{I} - \mathbf{P})\hat{f}, v \cdot i k \hat{\Phi}) dt, \\ S_4 = - \int_0^T (L\hat{f}, \hat{\Phi}) dt + \int_0^T (\hat{H}, \hat{\Phi}) dt. \end{cases}$$

First, we consider the estimate of c. We choose

$$\hat{\Phi} = \hat{\Phi}_c = (|v|^2 - 5)\{v \cdot i k \hat{\phi}_c(t, k)\} \mu^{\frac{1}{2}},$$

where ϕ_c satisfies

(5.5) $$|k|^2 \hat{\phi}_c(t, k) = \hat{c}(t, k).$$

Note that since $\hat{c}(t, 0) \equiv 0$, we formally write $\hat{\phi}_c(t, k) = \hat{c}(t, k)/|k|^2$ for any $k \in \mathbb{Z}^3$ with the understanding that we define $\hat{\phi}_c(t, 0) = 0$. By this choice, we can calculate that

$$- \int_0^T (\mathbf{P}\hat{f}, v \cdot i k \hat{\Phi}) dt$$

$$= - \sum_j \int_0^T (\{\hat{a} + \hat{b} \cdot v + (|v|^2 - 3)\hat{c}\} \mu^{\frac{1}{2}}, v_j i k_j \hat{\Phi}_c) dt$$

$$= - \sum_{j,n} \int_0^T (\{\hat{a} + \hat{b} \cdot v + (|v|^2 - 3)\hat{c}\} \mu^{\frac{1}{2}}, v_j v_n (|v|^2 - 5) \mu^{\frac{1}{2}} (-k_j k_n) \hat{\phi}_c) dt$$

$$= \int_0^T (\hat{c}, |k|^2 \hat{\phi}_c) dt = \int_0^T |\hat{c}(t, k)|^2 dt,$$

where we have used the orthogonality of the different integrands for the third equality.

Now we estimate the S_j's. Due to $k \in \mathbb{Z}^3$ it holds that

$$|\widehat{\Phi}_c(t,k)| \leq C\mu^{\frac{1}{4}}|k||\widehat{\phi}_c(t,k)| \leq C\mu^{\frac{1}{4}}|k|^2|\widehat{\phi}_c(t,k)| = C\mu^{\frac{1}{4}}|\widehat{c}(t,k)|,$$

and then we have $\|\widehat{\Phi}_c(t,k)\|_{L_v^2} \lesssim |\widehat{c}(t,k)| \lesssim \|\widehat{f}(t,k)\|_{L_v^2}$. Thus

$$|S_1| \lesssim \|\widehat{f}(k,T)\|_{L_v^2}^2 + \|\widehat{f_0}(k)\|_{L_v^2}^2.$$

To obtain the estimate of S_2, we first notice

$$|\partial_t \widehat{c}(t,k)| \lesssim |k|(|\widehat{b}(t,k)| + |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t,k)|_D)$$

by the third equation of (5.4). Therefore, we have

$$\begin{aligned}
|S_2| &\leq \int_0^T |(\widehat{f}, \partial_t \widehat{\Phi}_c)|dt = \int_0^T |(\{\mathbf{I} - \mathbf{P}\})\widehat{f}, \partial_t \widehat{\Phi}_c)|dt \\
&\lesssim \eta \int_0^T \|\partial_t \widehat{\Phi}_c(t,k)\|_{L_v^2}^2 dt + C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t,k)|_D^2 dt \\
&\lesssim \eta \int_0^T \frac{|\partial_t \widehat{c}(t,k)|^2}{|k|^2} dt + C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t,k)|_D^2 dt \\
&\lesssim \eta \int_0^T |\widehat{b}(t,k)|^2 dt + C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t,k)|_D^2 dt.
\end{aligned}$$

Note that the equality above is due to the choice of the velocity moments in the definition of $\widehat{\Phi}_c$. By Hölder's inequality and (5.5), we have

$$|S_3| \lesssim \eta \int_0^T |\widehat{c}(t,k)|^2 dt + C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t,k)|_D^2 dt.$$

Regarding the estimate of S_4, we can also show that

$$\begin{aligned}
|S_4| &\lesssim \eta \int_0^T |\widehat{c}(t,k)|^2 dt + C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t,k)|_D^2 dt \\
&\quad + C_\eta \int_0^T |(|\widehat{H}(t,k)|, \mu^{\frac{1}{4}})|^2 dt.
\end{aligned}$$

In particular, in S_4 we use that $L\widehat{f} = L\{\mathbf{I} - \mathbf{P}\}\widehat{f}$, and then we can use the estimate [43, eq. (6.12) on p. 819], recalling also Remark 3.3; then, similarly to the estimate for S_1, the above estimate follows. Summing up the estimates of S_j's with $\eta > 0$

suitably small, we obtain

$$\int_0^T |\hat{c}(t,k)|^2 \, dt$$

(5.6)
$$\lesssim \|\hat{f}(k,T)\|_{L_v^2}^2 + \|\hat{f}_0(k)\|_{L_v^2}^2 + \eta \int_0^T |\hat{b}(t,k)|^2 \, dt$$

$$+ C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\hat{f}(t,k)|_D^2 \, dt + C_\eta \int_0^T |(|\hat{H}(t,k)|, \mu^{\frac{1}{4}})|^2 \, dt.$$

Next, we consider the estimate of b. For this purpose we choose

$$\hat{\Phi} = \hat{\Phi}_b = \sum_{m=1}^3 \hat{\Phi}_b^{J,m}, \quad J = 1, 2, 3,$$

where

$$\hat{\Phi}_b^{J,m} = \begin{cases} \{|v|^2 v_m v_J i k_m \hat{\phi}_J(t,k) - \frac{7}{2}(v_m^2 - 1) i k_J \hat{\phi}_J(t,k)\}\mu^{\frac{1}{2}}, & J \neq m, \\ \frac{7}{2}(v_J^2 - 1) i k_J \hat{\phi}_J(t,k)\mu^{\frac{1}{2}}, & J = m, \end{cases}$$

and

(5.7)
$$|k|^2 \hat{\phi}_J(t,k) = \hat{b}_J(t,k).$$

Note again that since $\hat{b}_J(t,0) \equiv 0$, it is valid to write $\hat{\phi}_J(t,k) = \hat{b}_J(t,k)/|k|^2$ for any $k \in \mathbb{Z}^3$ with $\hat{\phi}_J(t,0) = 0$. Under this choice we have

$$-\sum_{m=1}^3 \int_0^T (\mathbf{P}\hat{f}, v \cdot i k \hat{\Phi}_b^{J,m}) dt$$

$$= -\sum_{m=1}^3 \int_0^T \sum_{j=1}^3 (\{\hat{a} + \hat{b} \cdot v + (|v|^2 - 3)\hat{c}\}\mu^{\frac{1}{2}}, v \cdot i k \hat{\Phi}_b^{J,m}) dt$$

$$= -\sum_{m=1, m \neq J}^3 \int_0^T (v_m v_J \mu^{\frac{1}{2}} \hat{b}_J, |v|^2 v_m v_J \mu^{\frac{1}{2}}(-k_m^2)\hat{\phi}_J) dt$$

$$- \sum_{m=1, m \neq J}^3 \int_0^T (v_m v_J \mu^{\frac{1}{2}} \hat{b}_m, |v|^2 v_m v_J \mu^{\frac{1}{2}}(-k_J k_m)\hat{\phi}_J) dt$$

$$+ 7 \sum_{m=1, m \neq J}^3 \int_0^T (\hat{b}_m, (-k_J k_m)\hat{\phi}_J) dt - 7 \int_0^T (\hat{b}_J, (-k_J^2)\hat{\phi}_J) dt$$

$$= -7 \sum_{m=1}^3 \int_0^T (\hat{b}_J, (-k_m^2)\hat{\phi}_J) dt = 7 \int_0^T |\hat{b}_J(t,k)|^2 \, dt.$$

The cancellation of the terms above is due to the choice of $\widehat{\Phi}_b$. Here we will only establish the estimate of S_2 since the other terms can be controlled similarly using the methods in the previous case. Since the second equation of (5.4) gives

$$|\partial_t \widehat{b}(t, k)| \lesssim |k| \big(|(a + c)(t, k)| + |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k)|_D\big),$$

it holds by (5.7) that

$$|S_2| \leq \sum_m \int_0^T \big|\big(\{\mathbf{I} - \mathbf{P}\}\widehat{f}, \partial_t \widehat{\Phi}_b^{J,m}\big)\big| dt + \sum_m \int_0^T \big|\big(\mathbf{P}\widehat{f}, \partial_t \widehat{\Phi}_b^{J,m}\big)\big| dt$$

$$\lesssim \eta \sum_m \int_0^T \big\|\partial_t \widehat{\Phi}_b^{J,m}(t, k)\big\|_{L_v^2}^2 dt + C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k)|_D^2 dt$$

$$+ C_\eta \int_0^T |\widehat{c}(t, k)|^2 dt$$

$$\lesssim \eta \sum_m \int_0^T |\partial_t k \widehat{\phi}_J(t, k)|^2 dt + C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k)|_D^2 dt$$

$$+ C_\eta \int_0^T |\widehat{c}(t, k)|^2 dt$$

$$\lesssim \eta \int_0^T |\widehat{a}(t, k)|^2 dt + C_\eta \int_0^T |\widehat{c}(t, k)|^2 dt + C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k)|_D^2 dt.$$

Here in the second line we again have used the orthogonality in the v-integration for the estimate of

$$\int_0^T \big|\big(\mathbf{P}\widehat{f}, \partial_t \widehat{\Phi}_b^{J,m}\big)\big| dt.$$

One can then deduce that

$$\int_0^T |\widehat{b}(t, k)|^2 dt$$

(5.8)
$$\lesssim \|\widehat{f}(k, T)\|_{L_v^2}^2 + \|\widehat{f}_0(k)\|_{L_v^2}^2 + \eta \int_0^T |\widehat{a}(t, k)|^2 dt$$

$$+ C_\eta \int_0^T |\widehat{c}(t, k)|^2 dt + C_\eta \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k)|_D^2 dt$$

$$+ C_\eta \int_0^T |(|\widehat{H}(t, k)|, \mu^{\frac{1}{4}})|^2 dt,$$

where $\eta > 0$ can be arbitrarily small.

Lastly, we consider a. We set

$$\widehat{\Phi} = \widehat{\Phi}_a = (|v|^2 - 10)\{v \cdot i k \widehat{\phi}_a(t, k)\}\mu^{\frac{1}{2}},$$

where $\widehat{\phi}_a$ is a solution to

$$|k|^2 \widehat{\phi}_a = \widehat{a}.$$

Note again that since $\widehat{a}(t, 0) \equiv 0$, it is valid to write $\widehat{\phi}_a(t, k) = \widehat{a}(t, k)/|k|^2$ for any $k \in \mathbb{Z}^3$ with the understanding that we define $\widehat{\phi}_a(t, 0) = 0$. For this choice the inner product of the macroscopic part is recast as

$$
-\int_0^T (\mathbf{P}\widehat{f}, v \cdot i k \widehat{\Phi}_a) dt
$$
$$
= -\sum_{j,n} \int_0^T (\{\widehat{a} + \widehat{b} \cdot v + (|v|^2 - 3)\widehat{c}\} \mu^{\frac{1}{2}}, v_j v_n (|v|^2 - 10)\mu^{\frac{1}{2}}(-k_j k_n \widehat{\phi}_a)) dt
$$
$$
= 5 \int_0^T (\widehat{a}, |k|^2 \widehat{\phi}_a) dt = 5 \int_0^T |\widehat{a}(t, k)|^2 dt.
$$

By the first equation of (5.4), one has

$$
|\partial_t \widehat{a}(t, k)| \lesssim |k| |\widehat{b}(t, k)|,
$$

and then we have

$$
|S_2| \leq \int_0^T |(\{\mathbf{I} - \mathbf{P}\}\widehat{f}, \partial_t \widehat{\Phi}_a)| dt + \int_0^T |(\mathbf{P}\widehat{f}, \partial_t \widehat{\Phi}_a)| dt
$$
$$
\lesssim \int_0^T \|\partial_t \widehat{\Phi}_a(t, k)\|_{L_v^2}^2 dt + \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k)|_D^2 dt + \int_0^T |\widehat{b}(t, k)|^2 dt
$$
$$
\lesssim \int_0^T |\partial_t k \widehat{\phi}_a(t, k)|^2 dt + \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k)|_D^2 dt + \int_0^T |\widehat{b}(t, k)|^2 dt
$$
$$
\lesssim \int_0^T |\widehat{b}(t, k)|^2 dt + \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k)|_D^2 dt.
$$

Therefore similar to the previous cases we obtain

$$
\int_0^T |\widehat{a}(t, k)|^2 dt
$$
(5.9)
$$
\lesssim \|\widehat{f}(k, T)\|_{L_v^2}^2 + \|\widehat{f}_0(k)\|_{L_v^2}^2 + \int_0^T |\widehat{b}(t, k)|^2 dt
$$
$$
+ \int_0^T |\{\mathbf{I} - \mathbf{P}\}\widehat{f}(t, k)|_D^2 dt + \int_0^T |(\widehat{H}(t, k), \mu^{\frac{1}{4}})|^2 dt.
$$

Combining those estimates (5.6), (5.8), and (5.9) yields the desired estimate (5.3). This completes the proof of Lemma 5.1. □

5.2 Anisotropic Case with Boundary

In this section, we derive the important estimates for the macroscopic part of the solutions to the problem (1.10), (1.11), (1.14), and (1.15). Recall the macro-micro decomposition (5.1). Given a function $H = H(h(x, v)) \in \mathcal{N}^\perp(L)$, we

then consider the linear problem

$$\partial_t f + v_1 \partial_{x_1} f + \bar{v} \cdot \nabla_{\bar{x}} f + Lf = H, \quad t > 0,$$

(5.10)
$$(x_1, \bar{x}) \in \Omega, \quad v = (v_1, \bar{v}) \in \mathbb{R}^3,$$

with the same initial data (1.11) and the boundary condition (1.14) or (1.15). As in the torus case, the H^1-estimates of the coefficients $[a, b, c](t, x_1, \bar{x})$ will be obtained by using the dual argument as in [32]. We would like to emphasize that the computations in our situation are more complicated than those in [32], since the function space that we are using here is anisotropic. We shall give the full details of the proof for completeness. Before doing this, we first define the boundary integral functionals:

(5.11)
$$|\Upsilon_{T,w}^+(h)|^2 = \int_0^T \int_{v_1>0} |v_1| w_{q,\vartheta}^2 |h(t, 1, v)|^2 \, dv \, dt$$
$$+ \int_0^T \int_{v_1<0} |v_1| w_{q,\vartheta}^2 |h(t, -1, v)|^2 \, dv \, dt,$$

for particles with outgoing velocities on boundaries $x_1 = \pm 1$, and

(5.12)
$$|\Upsilon_{T,w}^-(h)|^2 = \int_0^T \int_{v_1<0} |v_1| w_{q,\vartheta}^2 |h(t, 1, v)|^2 \, dv \, dt$$
$$+ \int_0^T \int_{v_1>0} |v_1| w_{q,\vartheta}^2 |h(t, -1, v)|^2 \, dv \, dt,$$

for particles with incoming velocities on boundaries $x_1 = \pm 1$, where the argument $h = h(t, x_1, v)$ is a distribution function that is well-defined on the boundaries. For the shorthand notation, we set $\Upsilon_T^\pm(\cdot) := \Upsilon_{T,w}^\pm(\cdot)$ in the case when $w \equiv 1$, i.e., when there is no velocity weight.

THEOREM 5.2. *Assume all the conditions listed in Theorems 2.3 and 2.4 are valid and let $H(h(-v_1)) = H(h)(-v_1)$. For $|\alpha| = 0$ or 1, it holds that*

$$\|\partial^\alpha [a, b, c]\|_{L_{\bar{k}}^1 L_T^2 L_{x_1,v}^2}$$

$$\lesssim \sum_{|\alpha|\leq 1} \|\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f\|_{L_{\bar{k}}^1 L_T^2 L_{x_1}^2 L_{v,D}^2} + \sum_{|\alpha|\leq 1} \|\partial^\alpha f\|_{L_{\bar{k}}^1 L_T^\infty L_{x_1,v}^2}$$

(5.13)
$$+ \sum_{|\alpha|\leq 1} \|\partial^\alpha f_0\|_{L_{\bar{k}}^1 L_{x_1,v}^2} + E(\widehat{g_\pm}) + \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} |\Upsilon_T^+(\widehat{\partial^\alpha f})| d\Sigma(\bar{k})$$

$$+ \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \left(\int_0^T \|(\widehat{\partial^\alpha H}, \mu^{\frac{1}{4}})_{L_v^2}\|^2 \, dt\right)^{1/2} d\Sigma(\bar{k})$$

$$+ \int_{\mathbb{Z}^2} \left|\Upsilon_T^-\left(\frac{\widehat{H}}{|v_1|}\right)\right| d\Sigma(\bar{k}),$$

for the inflow boundary condition (1.14), *and*

$$\|\partial^\alpha [a, b, c]\|_{L_{\bar{k}}^1 L_T^2 L_{x_1,v}^2}$$

$$(5.14)$$

$$\lesssim \sum_{|\alpha| \le 1} \|\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f\|_{L_{\bar{k}}^1 L_T^2 L_{x_1}^2 L_{v,D}^2}$$

$$+ \sum_{|\alpha| \le 1} \|\partial^\alpha f\|_{L_{\bar{k}}^1 L_T^\infty L_{x_1,v}^2} + \sum_{|\alpha| \le 1} \|\partial^\alpha f_0\|_{L_{\bar{k}}^1 L_{x_1,v}^2}$$

$$+ \sum_{|\alpha| \le 1} \int_{\mathbb{Z}^2} \left(\int_0^T \left\| (\widehat{\partial^\alpha H}, \mu^{\frac{1}{4}})_{L_v^2} \right\|^2 dt \right)^{1/2} d\Sigma(\bar{k}),$$

for the specular reflection boundary condition (1.15).

PROOF. In what follows we only show that (5.13) and (5.14) are valid for $\partial^\alpha = \partial_{x_1}$ with $\alpha = (1, 0, 0)$, since the proof in other cases is quite similar.

Acting $\partial := \partial_{x_1}$ to (5.10) and taking the Fourier transform of the resulting equations with respect to $\bar{x}$, we have

$$(5.15) \qquad \partial_t \widehat{\partial f} + v_1 \partial_{x_1} \widehat{\partial f} + i\bar{k} \cdot \bar{v} \widehat{\partial f} + L \widehat{\partial f} = \widehat{\partial H},$$

with the corresponding initial data

$$(5.16) \qquad \widehat{f}(0, x_1, \bar{k}, v) = \widehat{f_0}(x_1, \bar{k}, v),$$

and the inflow boundary condition

$$(5.17) \quad \widehat{f}(t, -1, \bar{k}, v)|_{v_1 > 0} = \widehat{g_-}(t, \bar{k}, v), \quad \widehat{f}(t, 1, \bar{k}, v)|_{v_1 < 0} = \widehat{g_+}(t, \bar{k}, v),$$

or the specular reflection condition

$$(5.18) \qquad \begin{aligned} \widehat{f}(t, -1, \bar{k}, v_1, \bar{v})|_{v_1 > 0} &= \widehat{f}(t, -1, \bar{k}, -v_1, \bar{v}), \\ \widehat{f}(t, 1, \bar{k}, v_1, \bar{v})|_{v_1 < 0} &= \widehat{f}(t, 1, \bar{k}, -v_1, \bar{v}), \end{aligned}$$

We note for clarity that in the proof of Theorem 5.2 the Fourier transform is always taken with respect only to $\bar{x}$.

Remark 5.3. Since $H(h(-v_1)) = H(h)(-v_1)$, it is easy to see that if $f(t, x_1, \bar{x}, v_1, \bar{v})$ is a solution of (5.15), (5.16), and (5.18) with $f_0(x_1, v_1) = f(-x_1, -v_1)$, then

$$f(t, x_1, \bar{x}, v_1, \bar{v}) = f(t, -x_1, \bar{x}, -v_1, \bar{v}).$$

We also note that if $H = \Gamma(h, h)$, then the property proved in [48, lemma 3.1, p. 637] guarantees that $H(h(-v_1)) = H(h)(-v_1)$.

Similar to obtaining (5.4), one has the following fluid-type system for the coefficient functions $[a, b, c]$:

$$(5.19) \quad \begin{cases} \partial_t a + \partial_{x_1} b_1 + \partial_{\bar{x}} \cdot \bar{b} = 0, \ \bar{b} = (b_2, b_3), \\ \partial_t b + \nabla_x (a + 2c) + \nabla_x \cdot \Theta(\{\mathbf{I} - \mathbf{P}\} f) = 0, \\ \partial_t c + \frac{1}{3} \nabla_x \cdot b + \frac{1}{6} \nabla_x \cdot \Lambda(\{\mathbf{I} - \mathbf{P}\} f) = 0, \\ \partial_t [\Theta_{jm}(\{\mathbf{I} - \mathbf{P}\} f) + 2c \delta_{jm}] + \partial_j b_m + \partial_m b_j = \Theta_{jm}(\mathbb{r} + \mathbb{h}), \\ \partial_t \Lambda_j(\{\mathbf{I} - \mathbf{P}\} f) + \partial_j c = \Lambda_j(\mathbb{r} + \mathbb{h}), \end{cases}$$

where the terms $\mathbb{r}$ and $\mathbb{h}$ are given by

$$\mathbb{r} = -v_1 \partial_{x_1} \{\mathbf{I} - \mathbf{P}\} f - \bar{v} \cdot \nabla_{\bar{x}} \{\mathbf{I} - \mathbf{P}\} f, \quad \mathbb{h} = -L\{\mathbf{I} - \mathbf{P}\} f + H.$$

We also recall the definitions of Λ_j and Θ_{jm} below (5.4).

Let $\hat{\Phi}(t, x_1, \bar{k}, v) \in C^1((0, +\infty) \times (-1, 1) \times \mathbb{R}^3)$ with $\bar{k} = (k_2, k_3) \in \mathbb{Z}^2$ be a test function. Taking the inner product of $\hat{\Phi}(t, x_1, \bar{k}, v)$ and (5.15) with respect to (x_1, v) and integrating the resulting identity with respect to t over $[0, T]$ for any $T > 0$, we obtain

$$\widehat{(\partial f}, \hat{\Phi})(T) - \widehat{(\partial f}, \hat{\Phi})(0) - \int_0^T (\widehat{\partial f}, \partial_t \hat{\Phi}) dt - \int_0^T (\widehat{\partial f}, v \cdot \widehat{\nabla_{x_1, \bar{x}} \Phi}) dt$$

$$+ \int_0^T \langle v_1 \widehat{\partial f}(1), \hat{\Phi}(1) \rangle dt - \int_0^T \langle v_1 \widehat{\partial f}(-1), \hat{\Phi}(-1) \rangle dt + \int_0^T (L \widehat{\partial f}, \hat{\Phi}) dt$$

$$= \int_0^T (\widehat{\partial H}, \hat{\Phi}) dt.$$

In the rest of this proof the inner product $(\cdot, \cdot)$ is always with respect to $L^2_{x_1, v}$, and the inner product $\langle \cdot, \cdot \rangle$ is always with respect to L^2_v. The above identity together with $\hat{f} = \mathbf{P}\hat{f} + \{\mathbf{I} - \mathbf{P}\}\hat{f}$ implies that

$$(5.20) \qquad\qquad -\int_0^T (\widehat{\partial \mathbf{P} f}, v \cdot \widehat{\nabla_{x_1, \bar{x}} \Phi}) dt = \sum_{j=1}^5 S_j,$$

where S_j $(1 \le j \le 5)$ are defined by

$$\begin{cases} S_1 = \widehat{(\partial f}, \hat{\Phi})(0) - \widehat{(\partial f}, \hat{\Phi})(T), \\ S_2 = \int_0^T (\widehat{\partial f}, \partial_t \hat{\Phi}) dt, \\ S_3 = \int_0^T (\{\mathbf{I} - \mathbf{P}\}\widehat{\partial f}, v \cdot \widehat{\nabla_{x_1, \bar{x}} \Phi}) dt, \\ S_4 = -\int_0^T (L\widehat{\partial f}, \hat{\Phi}) dt + \int_0^T (\widehat{\partial H}, \hat{\Phi}) dt, \\ S_5 = -\int_0^T \langle v_1 \widehat{\partial f}(1), \hat{\Phi}(1) \rangle dt + \int_0^T \langle v_1 \widehat{\partial f}(-1), \hat{\Phi}(-1) \rangle dt. \end{cases}$$

Estimates on $\hat{c}(t, x_1, \bar{k})$: We choose the test function

$$\hat{\Phi} = \hat{\Phi}_c = (|v|^2 - 5)\{v \cdot \widehat{\nabla_{x_1, \bar{x}} \phi_c}(t, x_1, \bar{k})\} \mu^{\frac{1}{2}},$$

where

$$(5.21) \qquad -\partial_{x_1}^2 \widehat{\phi}_c + |\bar{k}|^2 \widehat{\phi}_c(\bar{k}) = \widehat{\partial c}(\bar{k}), \quad \widehat{\phi}_c(\pm 1, \bar{k}) = 0.$$

It is straightforward to see that

$$(5.22) \qquad \|\widehat{\phi}_c\|_{H^2_{x_1}} + |\bar{k}| \|\widehat{\phi}_c\| \lesssim \|\widehat{\partial c}\|.$$

Here and in the rest of the proof of Theorem 5.2, for brevity we shall use $\|\cdot\| = \|\cdot\|_{L^2_{x_1}}$ to denote the L^2 norm in x_1 only if the function inside the norm only depends on x_1. If the function inside the norm depends on both x_1 and v, then in the rest of this proof we shall also use $\|\cdot\| = \|\cdot\|_{L^2_{x_1,v}}$ without any risk of ambiguity. Using the definition (5.1), we have

$$\mathbf{P}\widehat{f} = \{\widehat{a} + \widehat{b} \cdot v + (|v|^2 - 3)\widehat{c}\}\mu^{\frac{1}{2}}.$$

Then one has

$$-\int_0^T (\partial \widehat{\mathbf{P}f}, v \cdot \widehat{\nabla_{x_1,\bar{x}}\Phi})dt$$

$$= -\sum_j \int_0^T \left(\{\widehat{\partial a} + \widehat{\partial b} \cdot v + (|v|^2 - 3)\widehat{\partial c}\}\mu^{\frac{1}{2}}, v_j \widehat{\partial_j \Phi_c}\right)dt$$

$$= -\sum_{j,n} \int_0^T \left(\{\widehat{\partial a} + \widehat{\partial b} \cdot v + (|v|^2 - 3)\widehat{\partial c}\}\mu^{\frac{1}{2}}, v_j v_n (|v|^2 - 5)\mu^{\frac{1}{2}} \widehat{\partial_j \partial_n \phi_c}\right)dt$$

$$= \sum_j \int_0^T (\widehat{\partial c}, -\widehat{\partial_j^2 \phi_c})dt = \int_0^T \|\widehat{\partial c}(\bar{k})\|^2 \, dt.$$

We now turn to estimating the S_j $(1 \le j \le 5)$ term by term. By Hölder's inequality and the elliptic estimate (5.22), it follows that

$$|S_1| \lesssim \|\widehat{\partial f}(T)\|^2 + \|\widehat{\partial f}_0\|^2.$$

Above we also used that $\|\widehat{\partial c}\| \lesssim \|\widehat{\partial f}\|$. For the delicate term S_2, we first get from the third equation of (5.19) that

$$(5.23) \qquad \begin{aligned} \|\partial_t \partial_{x_1} \widehat{c}(\bar{k})\|_{H^{-1}_{x_1}} &\lesssim \|\partial_{x_1} \widehat{b}_1(\bar{k})\| + |\bar{k}| \|\widehat{(b_2, b_3)}(\bar{k})\| \\ &\quad + \|\partial_{x_1} \{\mathbf{I} - \mathbf{P}\}\widehat{f}\|_D + |\bar{k}| \|\{\mathbf{I} - \mathbf{P}\}\widehat{f}\|_D, \end{aligned}$$

and for $k \ne 0$

$$(5.24) \qquad \begin{aligned} |\bar{k}|^{-1} \|\partial_t \widehat{\partial_{\bar{x}} c}(\bar{k})\| &\lesssim \|\partial_{x_1} \widehat{b}_1(\bar{k})\| + |\bar{k}| \|\widehat{(b_2, b_3)}(\bar{k})\| \\ &\quad + \|\partial_{x_1} \{\mathbf{I} - \mathbf{P}\}\widehat{f}\|_D + |\bar{k}| \|\{\mathbf{I} - \mathbf{P}\}\widehat{f}\|_D. \end{aligned}$$

On the other hand, it follows from (5.21) that

$$\|\partial_t \widehat{\phi}_c\|_{H^1_{x_1}} \lesssim \|\partial_t \widehat{\partial c}\|_{H^{-1}_{x_1}}, \quad |\bar{k}| \|\partial_t \widehat{\phi}_c\|^2 \lesssim |\bar{k}|^{-1} \|\partial_t \widehat{\partial c}\|^2, \ k \ne 0.$$

Then we compute

$$
\begin{aligned}
|S_2| &\le \int_0^T \left|(\widehat{\partial f}, \partial_t \hat{\Phi})\right| dt = \int_0^T \left|(\{\mathbf{I} - \mathbf{P}\}\widehat{\partial f}, \partial_t \hat{\Phi})\right| dt \\
&\lesssim \eta \int_0^T \|\partial_t \hat{\Phi}\|^2 \, dt + C_\eta \int_0^T \left\|\{\mathbf{I} - \mathbf{P}\}\widehat{\partial f}\right\|_D^2 \, dt \\
&\lesssim \eta \int_0^T \|\partial_t \partial_{x_1} \hat{\phi}_c(t, \bar{k})\|^2 \, dt + \eta |\bar{k}|^2 \int_0^T \|\partial_t \hat{\phi}_c(t, \bar{k})\|^2 \, dt \\
&\quad + C_\eta \int_0^T \left\|\{\mathbf{I} - \mathbf{P}\}\widehat{\nabla_x f}\right\|_D^2 \, dt \\
&\lesssim \eta \int_0^T \left\|\widehat{\nabla_x b}(\bar{k})\right\|^2 \, dt + C_\eta \int_0^T \left\|\{\mathbf{I} - \mathbf{P}\}\widehat{\nabla_x f}\right\|_D^2 \, dt.
\end{aligned}
$$

Thanks to (5.22) and the Cauchy-Schwarz inequality, S_3 is bounded as follows:

$$
|S_3| \lesssim \eta \int_0^T \left\|\widehat{\partial c}(\bar{k})\right\|^2 \, dt + C_\eta \int_0^T \left\|\{\mathbf{I} - \mathbf{P}\}\widehat{\nabla_x f}\right\|_D^2 \, dt.
$$

For the term S_4, applying the elliptic estimate (5.22) yields

$$
\begin{aligned}
|S_4| &\le \eta \int_0^T \left\|\widehat{\partial c}(\bar{k})\right\|^2 \, dt + C_\eta \int_0^T \left\|\{\mathbf{I} - \mathbf{P}\}\widehat{\partial f}\right\|_D^2 \, dt \\
&\quad + C_\eta \int_0^T \left\|(\widehat{\partial H}, \mu^{\frac{1}{4}})_{L_v^2}\right\|^2 \, dt.
\end{aligned}
$$

We further recall the explanation of the torus case of the estimate S_4 above (5.6). In particular, we also use the estimate [43, eq. (6.12) on p. 819], then the above estimate follows after the Cauchy-Schwarz inequality is applied.

For the boundary term S_5, we first consider the inflow boundary condition (5.17), noticing that

$$
\hat{\Phi}_c(\pm 1) = (|v|^2 - 5)v_1 \partial_{x_1} \hat{\phi}_c(t, \pm 1, \bar{k})\mu^{\frac{1}{2}},
$$

and by the trace theory

$$
|\partial_{x_1} \hat{\phi}_c(t, \pm 1, \bar{k})| \lesssim \|\hat{\phi}_c(t, \bar{k})\|_{H_{x_1}^2} \lesssim \|\widehat{\partial c}\|.
$$

Then one has by using the Cauchy-Schwarz inequality that

$$
\begin{aligned}
|S_5| \lesssim{} & \eta \int_0^T \|\widehat{\partial c}(\bar{k})\|^2 \, dt + C_\eta \int_0^T \int_{\mathbb{R}^3} |v_1| |\widehat{\partial f}(1)|^2 \, dv \, dt \\
& + C_\eta \int_0^T \int_{\mathbb{R}^3} |v_1| |\widehat{\partial f}(-1)|^2 \, dv \, dt \\
\lesssim{} & \eta \int_0^T \|\widehat{\partial c}(\bar{k})\|^2 \, dt + C_\eta \int_0^T \int_{v_1 > 0} |v_1| |\widehat{\partial f}(1)|^2 \, dv \, dt \\
& + C_\eta \int_0^T \int_{v_1 < 0} |v_1| |\widehat{\partial f}(1)|^2 \, dv \, dt \\
& + C_\eta \int_0^T \int_{v_1 > 0} |v_1| |\widehat{\partial f}(-1)|^2 \, dv \, dt \\
& + C_\eta \int_0^T \int_{v_1 < 0} |v_1| |\widehat{\partial f}(-1)|^2 \, dv \, dt.
\end{aligned}
\tag{5.25}
$$

Next, in view of (5.17), we get from equation (5.15) that

$$
\begin{aligned}
& \int_0^T \int_{v_1 < 0} |v_1| |\widehat{\partial_{x_1} f}(1)|^2 \, dv \, dt \\
& \lesssim \int_0^T \int_{v_1 < 0} |v_1|^{-1} |\widehat{\partial_t g_+}|^2 \, dv \, dt \\
& \quad + \int_0^T \int_{v_1 < 0} |v_1|^{-1} |\bar{k} \cdot \bar{v}|^2 |\widehat{g_+}|^2 \, dv \, dt \\
& \quad + \int_0^T \int_{v_1 < 0} |v_1|^{-1} |L \widehat{g_+}|^2 \, dv \, dt \\
& \quad + \int_0^T \int_{v_1 < 0} |v_1|^{-1} |\widehat{H(1)}|^2 \, dv \, dt \\
& \lesssim E_{\bar{k}}(\widehat{g_+}) + \int_0^T \int_{v_1 < 0} |v_1|^{-1} |\widehat{H(1)}|^2 \, dv \, dt.
\end{aligned}
\tag{5.26}
$$

Similarly, it holds that

$$
\begin{aligned}
& \int_0^T \int_{v_1 > 0} |v_1| |\widehat{\partial_{x_1} f}(-1)|^2 \, dv \, dt \\
& \lesssim E_{\bar{k}}(\widehat{g_-}) + \int_0^T \int_{v_1 > 0} |v_1|^{-1} |\widehat{H}(-1)|^2 \, dv \, dt.
\end{aligned}
\tag{5.27}
$$

It is straightforward to see that if ∂_{x_1} is replaced by $\partial_{\bar{x}}$, then it follows that

$$\int_0^T \int_{v_1<0} |v_1||\widehat{\partial_{\bar{x}} f}(1)|^2 \, dv \, dt = \int_0^T \int_{v_1<0} |v_1||\widehat{\partial_{\bar{x}} g_+}|^2 \, dv \, dt,$$

$$\int_0^T \int_{v_1>0} |v_1||\widehat{\partial_{\bar{x}} f}(-1)|^2 \, dv \, dt = \int_0^T \int_{v_1>0} |v_1||\widehat{\partial_{\bar{x}} g_-}|^2 \, dv \, dt.$$

Therefore we only consider the case $\partial = \partial_{x_1}$ in the following.

As to the specular reflection boundary condition (5.18), the symmetric property plays a key role in dealing with the intractable boundary term S_5. In fact, from $\hat{f}(-x_1, -v_1) = \hat{f}(x_1, v_1)$ for all $x_1 \in [-1, 1]$ and from (5.18) we have that

$$\hat{f}(\pm 1, v_1)|_{v_1 \neq 0} = \hat{f}(\pm 1, -v_1)$$

at the boundary. Then we can show that S_5 vanishes at this stage. Since

$$\hat{c}(t, x_1, \bar{k}) = \frac{1}{6} \int_{\mathbb{R}^3} \hat{f}(t, x_1, \bar{k}, v)(|v|^2 - 3)\mu^{\frac{1}{2}}(v) dv,$$

we then see that $\hat{c}(t, x_1, \bar{k})$ is even with respect to x_1, which further implies that $\hat{\phi}_c$ is odd w.r.t. x_1 according to (5.21). Therefore one has

$$\hat{\Phi}_c(t, -1, \bar{k}, -v_1) = -\hat{\Phi}_c(t, 1, \bar{k}, v_1)$$

by the definition of $\hat{\Phi}_c$. Note the fact that $\widehat{\partial_{\bar{x}} \phi_c}(\pm 1) = 0$ was also used here. Moreover, by $\hat{f}(-1, v_1) = \hat{f}(1, -v_1) = \hat{f}(1, v_1)$ for $v_1 \neq 0$, and using the equation (5.15) to define the derivative ∂_{x_1}, one also has

$$\widehat{\partial_{x_1} f}(-1, -v_1) = \widehat{\partial_{x_1} f}(-1, v_1) = \widehat{\partial_{x_1} f}(1, v_1) \quad \text{for } v_1 \neq 0.$$

Consequently, by a change of variable $v_1 \to -v_1$, we have

$$S_5 = \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(1), \hat{\Phi}_c(1)\rangle dt - \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(-1), \hat{\Phi}_c(-1)\rangle dt$$

$$= \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(1, v_1), \hat{\Phi}_c(1)\rangle dt + \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(-1, -v_1), \hat{\Phi}_c(-1, -v_1)\rangle dt$$

$$= \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(1, v_1), \hat{\Phi}_c(1)\rangle dt - \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(1, v_1), \hat{\Phi}_c(1, v_1)\rangle dt = 0.$$

Combining the above estimates for the S_j $(1 \leq j \leq 5)$ together, we now arrive at

$$(5.28) \qquad \int_0^T \|\widehat{\partial c}(\bar{k})\|^2 \, dt$$

$$\lesssim \|\widehat{\partial f}(T)\|^2 + \|\widehat{\partial f}(0)\|^2 + E_{\bar{k}}(g_\pm)$$

$$+ C_\eta \int_0^T \int_{v_1>0} |v_1||\widehat{\partial f}(1)|^2 \, dv \, dt +$$

$$+ C_\eta \int_0^T \int_{v_1 < 0} |v_1| |\widehat{\partial f}(-1)|^2 \, dv \, dt + \eta \int_0^T \left\| \widehat{\nabla_x b}(\bar{k}) \right\|^2 dt$$

$$+ C_\eta \int_0^T \left\| \{\mathbf{I} - \mathbf{P}\} \widehat{\nabla_x f} \right\|_D^2 dt + C_\eta \int_0^T \left\| (\widehat{\partial^\alpha H}, \mu^{\frac{1}{4}})_{L_v^2} \right\|^2 dt$$

$$+ \int_0^T \int_{v_1 < 0} |v_1|^{-1} |\widehat{H(1)}|^2 \, dv \, dt$$

$$+ \int_0^T \int_{v_1 > 0} |v_1|^{-1} |\widehat{H(-1)}|^2 \, dv \, dt,$$

for the inflow boundary condition (5.17), and

$$\int_0^T \| \widehat{\partial c}(\bar{k}) \|^2 \, dt \lesssim \| \widehat{\partial f}(T) \|^2 + \| \widehat{\partial f}_0 \|^2 + \eta \int_0^T \left\| \widehat{\nabla_x b}(\bar{k}) \right\|^2 dt$$

$$(5.29) \qquad\qquad + C_\eta \int_0^T \left\| \{\mathbf{I} - \mathbf{P}\} \widehat{\nabla_x f} \right\|_D^2 dt$$

$$+ C_\eta \int_0^T \left\| (\widehat{\partial H}, \mu^{\frac{1}{4}})_{L_v^2} \right\|^2 dt,$$

for the specular reflection boundary condition (5.18). We thus conclude the estimates on c.

Estimates on $\hat{b}(t, x_1, \bar{k})$: In this case, we choose the test function

$$\hat{\Phi} = \sum_{m=1}^3 \hat{\Phi}_b^{J,m}, \ J = 1, 2, 3,$$

with

$$\hat{\Phi}_b^{J,m} = \begin{cases} |v|^2 v_m v_J \widehat{\partial_m \phi}_J(t, x_1, \bar{k}) - \frac{7}{2}(v_m^2 - 1)\widehat{\partial_J \phi}_J(t, x_1, \bar{k})\mu^{\frac{1}{2}}, & J \neq m, \\ \frac{7}{2}(v_J^2 - 1)\widehat{\partial_J \phi}_J(t, x_1, \bar{k})\mu^{\frac{1}{2}}, & J = m, \end{cases}$$

where

$$(5.30) \qquad -\partial_{x_1}^2 \hat{\phi}_J + |\bar{k}|^2 \hat{\phi}_J(\bar{k}) = \widehat{\partial b}_J(\bar{k}) \quad \text{and} \quad \hat{\phi}_J(\pm 1, \bar{k}) = 0.$$

In this paper we use the notation $(\partial_1, \partial_2, \partial_3) \overset{\text{def}}{=} (\partial_{x_1}, \partial_{\bar{x}})$.

Standard elliptic estimates yield the estimate

$$\| \hat{\phi}_J \|_{H_{x_1}^2} + |\bar{k}| \| \hat{\phi}_J \| \lesssim C \| \widehat{\partial b}_J \|.$$

With the above choices in hand, we now turn to estimate both sides of (5.20) term by term. Notice

$$
-\sum_{m=1}^{3}\int_{0}^{T}\big(\widehat{\partial\mathbf{P}f},v\cdot\widehat{\nabla\Phi}_{b}^{J,m}\big)dt
$$

$$
=-\sum_{m=1}^{3}\int_{0}^{T}\sum_{j=1}^{3}\Bigg(\Big\{\widehat{\partial a}+\sum_{n=1}^{3}\widehat{\partial b_{n}}v_{n}+(|v|^{2}-3)\widehat{\partial c}\Big\}\mu^{\frac{1}{2}},v_{j}\widehat{\partial_{j}\Phi}_{b}^{J,m}\Bigg)dt.
$$

Then it follows that

$$
-\sum_{m=1}^{3}\int_{0}^{T}\big(\widehat{\partial\mathbf{P}f},v\cdot\widehat{\nabla\Phi}_{b}^{J,m}\big)dt
$$

$$
=-\sum_{m=1,m\neq J}^{3}\int_{0}^{T}\big(v_{m}v_{J}\mu^{\frac{1}{2}}\widehat{\partial b_{J}},|v|^{2}v_{m}v_{J}\mu^{\frac{1}{2}}\widehat{\partial_{m}^{2}\phi_{J}}(t,x_{1},\bar{k})\big)dt
$$

$$
-\sum_{m=1,m\neq J}^{3}\int_{0}^{T}\big(v_{m}v_{J}\mu^{\frac{1}{2}}\widehat{\partial b_{m}},|v|^{2}v_{m}v_{J}\mu^{\frac{1}{2}}\widehat{\partial_{J}\partial_{m}\phi_{J}}(t,x_{1},\bar{k})\big)dt
$$

$$
+7\sum_{m=1,m\neq J}^{3}\int_{0}^{T}\big(\widehat{\partial b_{m}},\widehat{\partial_{m}\partial_{J}\phi_{J}}(t,x_{1},\bar{k})\big)dt
$$

$$
-7\int_{0}^{T}\big(\widehat{\partial b_{J}},\widehat{\partial_{J}^{2}\phi_{J}}(t,x_{1},\bar{k})\big)dt.
$$

Therefore one has

$$
-\sum_{m=1}^{3}\int_{0}^{T}\big(\widehat{\partial\mathbf{P}f},v\cdot\widehat{\nabla\Psi}_{b}^{J,m}\big)dt=-7\sum_{m=1}^{3}\int_{0}^{T}\big(\widehat{\partial b_{J}},\widehat{\partial_{m}^{2}\phi_{J}}(t,x_{1},\bar{k})\big)dt
$$

$$
=7\int_{0}^{T}\big\|\widehat{\partial b_{J}}\big\|^{2}dt.
$$

In what follows, we only estimate S_2 and S_5, since the other terms are similar to obtaining (5.28). By using the second equation of (5.19), as in (5.23) and (5.24), we obtain

$$
\|\partial_{t}\partial_{x_{1}}\widehat{b}(\bar{k})\|_{H_{x_{1}}^{-1}}\lesssim\big\|\partial_{x_{1}}(\widehat{a}+\widehat{c})(\bar{k})\big\|+|\bar{k}|\big\|(\widehat{a}+\widehat{c})(\bar{k})\big\|
$$

$$
+\big\|\partial_{x_{1}}\{\mathbf{I}-\mathbf{P}\}\widehat{f}\big\|_{D}+|\bar{k}|\big\|\{\mathbf{I}-\mathbf{P}\}\widehat{f}\big\|_{D},
$$

and for $k\neq0$ we have

$$
|\bar{k}|^{-1}\|\partial_{t}\widehat{\partial_{\bar{x}}b}(\bar{k})\|\lesssim\big\|\partial_{x_{1}}(\widehat{a}+\widehat{c})(\bar{k})\big\|+|\bar{k}|\big\|(\widehat{a}+\widehat{c})(\bar{k})\big\|
$$

$$
+\big\|\partial_{x_{1}}\{\mathbf{I}-\mathbf{P}\}\widehat{f}\big\|_{D}+|\bar{k}|\big\|\{\mathbf{I}-\mathbf{P}\}\widehat{f}\big\|_{D}.
$$

As a consequence, it follows that

$$
\begin{aligned}
|S_2| &\leq \sum_m \int_0^T \left| (\widehat{\partial f}, \partial_t \widehat{\Phi}_b^{J,m}) \right| dt \\
&\leq \sum_m \int_0^T \left| (\{\mathbf{I} - \mathbf{P}\}\widehat{\partial f}, \partial_t \widehat{\Phi}_b^{J,m}) \right| dt + \sum_m \int_0^T \left| (\mathbf{P}\widehat{\partial f}, \partial_t \widehat{\Phi}_b^{J,m}) \right| dt \\
&\lesssim \eta \sum_m \int_0^T \|\partial_t \widehat{\Phi}_b^{J,m}\|^2 \, dt + C_\eta \int_0^T \left\| \{\mathbf{I} - \mathbf{P}\}\widehat{\nabla_x f} \right\|_D^2 \, dt \\
&\quad + C_\eta \int_0^T \left\| \widehat{\nabla_x c}(\bar{k}) \right\|^2 \, dt \\
&\lesssim \eta \int_0^T \|\partial_t \partial_{x_1} \widehat{\phi}_J(t, \bar{k})\|^2 \, dt + \eta \int_0^T \|\partial_t i k \widehat{\phi}_J(t, \bar{k})\|^2 \, dt \\
&\quad + C_\eta \int_0^T \left\| \{\mathbf{I} - \mathbf{P}\}\widehat{\nabla_x f} \right\|_D^2 \, dt + C_\eta \int_0^T \left\| \widehat{\nabla_x c}(\bar{k}) \right\|^2 \, dt \\
&\lesssim \eta \int_0^T \left\| \widehat{\nabla_x a}(\bar{k}) \right\|^2 \, dt + C_\eta \int_0^T \left\| \widehat{\nabla_x c}(\bar{k}) \right\|^2 \, dt \\
&\quad + C_\eta \int_0^T \left\| \{\mathbf{I} - \mathbf{P}\}\widehat{\nabla_x f} \right\|_D^2 \, dt,
\end{aligned}
$$

where from (5.30) we used the following elliptic estimates:

$$
\|\partial_t \widehat{\phi}_J\|_{H_{x_1}^1} \lesssim \|\partial_t \widehat{\partial b}\|_{H_{x_1}^{-1}}, \qquad |\bar{k}|\|\partial_t \widehat{\phi}_J\|^2 \lesssim |\bar{k}|^{-1}\|\partial_t \widehat{\partial b}\|^2, \ k \neq 0.
$$

We also then used the estimates just above in the upper bounds of these estimates. For the inflow boundary condition (5.17), S_5 enjoys similar estimates to (5.25), (5.26), and (5.27). Notice that only the case $\partial = \partial_{x_1}$ should be considered. We now *claim* that S_5 also vanishes for the specular reflection boundary condition (5.18). From

$$
\widehat{b}(t, x_1, \bar{k}) = \int_{\mathbb{R}^3} \widehat{f}(t, x_1, \bar{k}, v) v \mu^{\frac{1}{2}}(v) \, dv
$$

and $\widehat{f}(-x_1, -v_1) = \widehat{f}(x_1, v_1)$, we see that $\widehat{b}_1(t, x_1, \bar{k})$ is odd w.r.t. x_1, which further implies that $\widehat{\phi}_1$ is even according to (5.30). Hence it follows that

$$
\widehat{\Phi}_b^{1,m}(t, -1, \bar{k}, -v_1) = -\widehat{\Phi}_b^{1,m}(t, 1, \bar{k}, v_1)
$$

by the definition of $\widehat{\Phi}^{J,m}$. Eventually, from

$$
\widehat{\partial_{x_1} f}(-1, -v_1) = \widehat{\partial_{x_1} f}(1, v_1) \quad \text{for } v_1 \neq 0,
$$

and further with the change of variable $v_1 \to -v_1$, we obtain

$$S_5 = \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(1), \widehat{\Phi}_b^{1,m}(1) \rangle dt - \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(-1), \widehat{\Phi}_b^{1,m}(-1) \rangle dt$$

$$= \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(1, v_1), \widehat{\Phi}_b^{1,m}(1) \rangle dt$$

$$+ \int_0^T \langle v_1 \widehat{\partial_{x_1} f}(-1, -v_1), \widehat{\Phi}_b^{1,m}(-1, -v_1) \rangle dt$$

$$= 0.$$

For $J = 2$ and 3, both $\widehat{b}_2$ and $\widehat{b}_3$ are even w.r.t. x_1, which further yields that $\widehat{\phi}_2$ and $\widehat{\phi}_3$ are odd w.r.t. x_1. Then we still have

$$\widehat{\Phi}_b^{J,m}(t, -1, \bar{k}, -v_1) = -\widehat{\Phi}_b^{J,m}(t, 1, \bar{k}, v_1), \quad J = 2, 3.$$

Thus S_5 also vanishes in the case of $\widehat{\Phi}_b^{J,m}(t, 1, \bar{k}, v_1)$ with $J = 2$ and 3. We now have the following conclusion as in (5.28) and (5.29):

$$\int_0^T \|\widehat{\partial b}(\bar{k})\|^2 \, dt$$

$$\lesssim \|\widehat{\partial f}(T)\|^2 + \|\widehat{\partial f}_0\|^2 + E_{\bar{k}}(g_\pm)(T)$$

$$+ C_\eta \int_0^T \int_{v_1 > 0} |v_1| |\widehat{\partial f}(1)|^2 \, dv \, dt + C_\eta \int_0^T \int_{v_1 < 0} |v_1| |\widehat{\partial f}(-1)|^2 \, dv \, dt$$

$$+ \eta \int_0^T \|\widehat{\nabla_x a}(\bar{k})\|^2 \, dt + C_\eta \int_0^T \|\widehat{\nabla_x c}(\bar{k})\|^2 \, dt$$

$$+ C_\eta \int_0^T \|\{\mathbf{I} - \mathbf{P}\}\widehat{\nabla_x f}\|_D^2 \, dt + C_\eta \int_0^T \|(\widehat{\partial H}, \mu^{\frac{1}{4}})_{L_v^2}\|^2 \, dt$$

$$+ \int_0^T \int_{v_1 < 0} |v_1|^{-1} |\widehat{H(1)}|^2 \, dv \, dt + \int_0^T \int_{v_1 > 0} |v_1|^{-1} |\widehat{H(-1)}|^2 \, dv \, dt,$$

for the inflow boundary condition (5.17), and

$$\int_0^T \|\widehat{\partial b}(\bar{k})\|^2 \, dt \lesssim \|\widehat{\partial f}(T)\|^2 + \|\widehat{\partial f}(0)\|^2$$

$$+ \eta \int_0^T \|\widehat{\nabla_x a}(\bar{k})\|^2 \, dt + C_\eta \int_0^T \|\widehat{\nabla_x c}(\bar{k})\|^2 \, dt$$

$$+ C_\eta \int_0^T \|\{\mathbf{I} - \mathbf{P}\}\widehat{\nabla_x f}\|_D^2 \, dt + C_\eta \int_0^T \|(\widehat{\partial H}, \mu^{\frac{1}{4}})_{L_v^2}\|^2 \, dt,$$

for the specular reflection boundary condition (5.18). This then concludes the estimates on b.

Estimates on $\widehat{a}(t, x_1, \overline{k})$: In this case, we choose the test function

$$\widehat{\Phi} = \widehat{\Phi}_a = (|v|^2 - 10)\{v \cdot \widehat{\nabla_{x_1,\overline{x}}\phi}_a(t, x_1, \overline{k})\}\mu^{\frac{1}{2}},$$

where

(5.31) $\qquad -\partial_{x_1}^2 \widehat{\phi}_a + |\overline{k}|^2 \widehat{\phi}_a(\overline{k}) = \widehat{\partial a}(\overline{k}) \quad \text{and} \quad \widehat{\phi}_a(\pm 1, \overline{k}) = 0$

for the inflow boundary condition (5.17), and

(5.32) $\qquad -\partial_{x_1}^2 \widehat{\phi}_a + |\overline{k}|^2 \widehat{\phi}_a(\overline{k}) = \widehat{\partial a}(\overline{k}) \quad \text{and} \quad \partial_{x_1} \widehat{\phi}_a(\pm 1, \overline{k}) = 0$

for the specular reflection boundary condition (5.18).

We compute both sides of (5.20) with $\widehat{\Phi}$ replaced by $\widehat{\Phi}_a$. For the left-hand side, we have

$$-\int_0^T (\widehat{\partial \mathbf{P}f}, v \cdot \widehat{\nabla_{x_1,\overline{x}}\Phi})\,dt$$

$$= -\sum_j \int_0^T \left(\{\widehat{\partial a} + \widehat{\partial b} \cdot v + (|v|^2 - 3)\widehat{\partial c}\}\mu^{\frac{1}{2}}, v_j \widehat{\partial_j \Phi}_a\right)dt$$

$$= -\sum_{j,n} \int_0^T \left(\{\widehat{\partial a} + \widehat{\partial b} \cdot v + (|v|^2 - 3)\widehat{\partial c}\}\mu^{\frac{1}{2}}, v_j v_n (|v|^2 - 10)\mu^{\frac{1}{2}}\widehat{\partial_j \partial_n \phi}_a\right)dt$$

$$= -5\sum_j \int_0^T (\widehat{\partial a}, -\widehat{\partial_j^2 \phi}_c)\,dt = -5\int_0^T \|\widehat{\partial a}(\overline{k})\|^2\,dt.$$

As to the right-hand side, similar to the estimates for $\widehat{b}(t, x_1, \overline{k})$, we only show the estimates for S_2 and S_5 as the others follow in the same way as done previously. For this, we first have from (5.31) or (5.32) that

$$\|\partial_t \widehat{\phi}_a\|_{H_{x_1}^1} \lesssim \|\partial_t \widehat{\partial a}\|_{H_{x_1}^{-1}}, \quad |\overline{k}|\|\partial_t \widehat{\phi}_c\|^2 \lesssim |\overline{k}|^{-1}\|\partial_t \widehat{\partial a}\|^2, \ k \neq 0,$$

and

$$\|\widehat{\phi}_a\|_{H_{x_1}^2} + |\overline{k}|\|\widehat{\phi}_a\| \lesssim \|\widehat{\partial a}\|.$$

Moreover, one gets from the first equation of (5.19) that

$$\|\partial_t \partial_{x_1} \widehat{a}(\overline{k})\|_{H_{x_1}^{-1}} \lesssim \|\partial_{x_1} \widehat{b}_1(\overline{k})\| + |\overline{k}|\|\widehat{(b_2, b_3)}(\overline{k})\|,$$

and

$$|\overline{k}|^{-1}\|\partial_t \widehat{\partial_{\overline{x}} a}(\overline{k})\| \lesssim \|\partial_{x_1} \widehat{b}_1(\overline{k})\| + |\overline{k}|\|\widehat{(b_2, b_3)}(\overline{k})\|,$$

provided $k \neq 0$.

With the above estimates in hand, we now compute

$$
\begin{aligned}
|S_2| &\le \int_0^T \big|(\widehat{\partial f}, \partial_t \hat{\Phi}_a)\big|\, dt \le \int_0^T \big|(\{\mathbf{I}-\mathbf{P}\}\widehat{\partial f}, \partial_t \hat{\Phi}_a)\big|\, dt \\
&\quad + \int_0^T \big|(\mathbf{P}\widehat{\partial f}, \partial_t \hat{\Phi}_a)\big|\, dt \\
&\lesssim \int_0^T \|\partial_t \hat{\Phi}_a\|^2\, dt + \int_0^T \big\|\{\mathbf{I}-\mathbf{P}\}\widehat{\partial f}\big\|_D^2\, dt + \int_0^T \big\|\widehat{\partial b}(\bar{k})\big\|^2\, dt \\
&\lesssim \int_0^T \|\partial_t \partial_{x_1} \hat{\phi}_a(t,\bar{k})\|^2\, dt + |\bar{k}|^2 \int_0^T \|\partial_t \hat{\phi}_a(t,\bar{k})\|^2\, dt \\
&\quad + \int_0^T \big\|\{\mathbf{I}-\mathbf{P}\}\widehat{\nabla_x f}\big\|_D^2\, dt \\
&\lesssim \int_0^T \big\|\widehat{\nabla_x b}(\bar{k})\big\|^2\, dt + \int_0^T \big\|\{\mathbf{I}-\mathbf{P}\}\widehat{\nabla_x f}\big\|_D^2\, dt.
\end{aligned}
$$

We now turn to estimating S_5. For the inflow boundary condition (5.17), S_5 shares the similar estimates with (5.25), (5.26), and (5.27). Noticing that $\hat{\phi}_a$ possesses the same symmetry as $\hat{\phi}_c$ w.r.t. x_1, one can show that S_5 also vanishes for the specular reflection boundary condition (5.18) by the same arguments as were given for the estimates on $\hat{c}(t, x_1, \bar{k})$ and $\hat{b}(t, x_1, \bar{k})$. In conclusion, the following estimates for $\hat{a}(t, x_1, \bar{k})$ hold:

$$
\begin{aligned}
\int_0^T \|\widehat{\partial a}(\bar{k})\|^2\, dt &\lesssim \|\widehat{\partial f}(T)\|^2 + \|\widehat{\partial f}_0\|^2 + E_{\bar{k}}(g_\pm) \\
&\quad + \int_0^T \int_{v_1>0} |v_1| |\widehat{\partial f}(1)|^2\, dv\, dt \\
&\quad + \int_0^T \int_{v_1<0} |v_1| |\widehat{\partial f}(-1)|^2\, dv\, dt + \int_0^T \big\|\widehat{\nabla_x b}(\bar{k})\big\|^2\, dt \\
&\quad + \int_0^T \big\|\{\mathbf{I}-\mathbf{P}\}\widehat{\nabla_x f}\big\|_D^2\, dt + \int_0^T \big\|(\widehat{\partial H}, \mu^{\frac{1}{4}})_{L_v^2}\big\|^2\, dt \\
&\quad + \int_0^T \int_{v_1<0} |v_1|^{-1} |\widehat{H(1)}|^2\, dv\, dt \\
&\quad + \int_0^T \int_{v_1>0} |v_1|^{-1} |\widehat{H(-1)}|^2\, dv\, dt
\end{aligned}
$$

for the inflow boundary condition (5.17), and

$$
\begin{aligned}
\int_0^T \|\widehat{\partial a}(\bar{k})\|^2\, dt &\lesssim \|\widehat{\partial f}(T)\|^2 + \|\widehat{\partial f}_0\|^2 + \int_0^T \big\|\widehat{\nabla_x b}(\bar{k})\big\|^2\, dt \\
&\quad + \int_0^T \big\|\{\mathbf{I}-\mathbf{P}\}\widehat{\nabla_x f}\big\|_D^2\, dt + \int_0^T \big\|(\widehat{\partial H}, \mu^{\frac{1}{4}})_{L_v^2}\big\|^2\, dt,
\end{aligned}
$$

for the specular reflection boundary condition (5.18). Finally, combining all the above estimates on $[a, b, c]$ together, we see that (5.13) and (5.14) hold true for the case when $\partial = \partial_{x_1}$. The same estimate with the other derivatives also holds, and the proof is analogous but easier. This completes the proof of Theorem 5.2. $\qquad\square$

6 Proof of the Main Results in the Torus

In this section, we shall obtain the global existence and large-time behavior (stated in Theorem 2.1). We further obtain the propagation of regularity in the x-variable (stated in Theorem 2.2) for solutions to the initial value problem **(PT)** (1.10) and (1.11) in the torus (1.6). We shall prove Theorem 2.1 and Theorem 2.2.

PROOF OF THEOREM 2.1. We first consider the uniform a priori estimates without any velocity weight. As shown in Section 3, by applying the Fourier transform to (1.10), as in (3.2) with (3.3) or (3.14), and then taking the complex inner product of the result with $\hat{f}$ in L_v^2, we have

$$(\partial_t \hat{f}, \hat{f}) + i(k \cdot v \hat{f}, \hat{f}) + (L\hat{f}, \hat{f}) = (\widehat{\Gamma(f, f)}, \hat{f}).$$

Taking the real part of this identity and integrating over $[0, T]$ for fixed $T > 0$ gives

$$\int_{\mathbb{Z}^3} \sup_{0 \le t \le T} \|\hat{f}(t, k)\|_{L_v^2} \, d\Sigma(k) + \int_{\mathbb{Z}^3} \left(\int_0^T |\{\mathbf{I} - \mathbf{P}\}\hat{f}(t, k)|_D^2 \, dt \right)^{1/2} d\Sigma(k)$$

$$\lesssim \int_{\mathbb{Z}^3} \|\hat{f_0}(k)\|_{L_v^2} \, d\Sigma(k) + \int_{\mathbb{Z}^3} \left(\int_0^T \left| \mathscr{R}(\widehat{\Gamma(f, f)}, \{\mathbf{I} - \mathbf{P}\}\hat{f}) \right| dt \right)^{1/2} d\Sigma(k)$$

$$\lesssim \int_{\mathbb{Z}^3} \|\hat{f_0}(k)\|_{L_v^2} \, d\Sigma(k) + C_\eta \|f\|_{L_k^1 L_T^\infty L_v^2} \|f\|_{L_k^1 L_T^2 L_{v,D}^2}$$

$$+ \eta \|\{\mathbf{I} - \mathbf{P}\}f\|_{L_k^1 L_T^2 L_{v,D}^2},$$

for an arbitrary constant $\eta > 0$. Together with Theorem 5.1 and Lemma 4.3, the above estimate yields

(6.1)
$$\|f\|_{L_k^1 L_T^\infty L_v^2} + \|f\|_{L_k^1 L_T^2 L_{v,D}^2}$$
$$\lesssim \|f_0\|_{L_k^1 L_v^2} + \|f\|_{L_k^1 L_T^\infty L_v^2} \|f\|_{L_k^1 L_T^2 L_{v,D}^2}.$$

This is the main estimate that is needed for the *hard* potentials.

In case of *soft* potentials, in order to treat the time decay of solutions, we need to make additional velocity-weighted estimates with the weight $w_{q,\vartheta}$ given as in (1.25) under the assumption **(H)** (1.26). Indeed, with the help of Lemma A.3, one can also show that

(6.2)
$$\|w_{q,\vartheta} f\|_{L_k^1 L_T^\infty L_v^2} + \|w_{q,\vartheta} f\|_{L_k^1 L_T^2 L_{v,D}^2}$$
$$\lesssim \|w_{q,\vartheta} f_0\|_{L_k^1 L_v^2} + \|w_{q,\vartheta} f\|_{L_k^1 L_T^\infty L_v^2} \|w_{q,\vartheta} f\|_{L_k^1 L_T^2 L_{v,D}^2}.$$

Note that the negative term $-C\|f\|_{L_k^1 L_T^2 L_{v,D}^2}$ appearing in the use of Lemma A.3 can be handled in terms of (6.1), since one has $w_{q,\vartheta} \ge 1$. For brevity we

omit these details in verifying (6.2). Hence, under the smallness assumption on $\|w_{q,\vartheta} f_0\|_{L_k^1 L_v^2}$, one can then obtain the closed estimate

$$\|w_{q,\vartheta} f\|_{L_k^1 L_T^\infty L_v^2} + \|w_{q,\vartheta} f\|_{L_k^1 L_T^2 L_{v,D}^2} \lesssim \|w_{q,\vartheta} f_0\|_{L_k^1 L_v^2}.$$

This completes the proof of the uniform a priori estimate (2.8) under the smallness assumption on $\|w_{q,\vartheta} f_0\|_{L_k^1 L_v^2}$. Combining this with the local existence to be discussed in Section 8 and applying the standard continuity argument, we obtain the global existence and uniqueness of global mild solutions. The positivity of solutions is also guaranteed by the local existence result as in Theorem 8.1.

Next, we consider the rate of convergence of the obtained solutions. We only treat the case of the non-cutoff Boltzmann equation since the same method can be applied to the Landau case. Moreover, we focus on the soft potentials $\gamma + 2s < 0$, since it is similar to carry out the time-weighted estimates with the exponential weight $e^{\lambda t}$ for a suitably small constant $\lambda > 0$ in case of hard potentials $\gamma + 2s \geq 0$.

Therefore, for the *soft* potentials, let

$$\widehat{h} = e^{\lambda t^p} \widehat{f}$$

with $\lambda > 0$ and $0 < p < 1$ chosen later. As f solves (1.10), then $\widehat{h}$ satisfies

$$\partial_t \widehat{h} + i k \cdot v \widehat{h} + L\widehat{h} = e^{-\lambda t^p} \widehat{\Gamma(h,h)} + \lambda p t^{p-1} \widehat{h},$$

with initial data

$$\widehat{h}(0, k, v) = \widehat{h}_0(k, v).$$

With the aid of the arguments used to derive (6.1), we have

$$
\begin{aligned}
\int_{\mathbb{Z}^3} \sup_{0 \leq t \leq T} &\|\widehat{h}(t,k)\|_{L_v^2} \, d\Sigma(k) + \int_{\mathbb{Z}^3} \left(\int_0^T |\widehat{h}(t,k)|_D^2 \, dt \right)^{1/2} d\Sigma(k) \\
(6.3) \qquad &\lesssim \int_{\mathbb{Z}^3} \|\widehat{f}_0(k)\|_{L_v^2} \, d\Sigma(k) \\
&\quad + \sqrt{\lambda p} \int_{\mathbb{Z}^3} \left(\int_0^T t^{p-1} \|\widehat{h}(t,k)\|_{L_v^2}^2 \, dt \right)^{1/2} d\Sigma(k).
\end{aligned}
$$

For $\rho > 0$ to be small enough later on and $p' > 0$ to be chosen later depending upon p, we define a set

$$\mathbf{E} = \{\langle v \rangle \leq \rho t^{p'}\}$$

and make the decomposition $1 = \mathbf{1_E} + \mathbf{1_{E^c}}$, cf. [76]. Then the second term on the right-hand side of (6.3) can be bounded by

$$
\sqrt{\lambda p} \int_{\mathbb{Z}^3} \left(\int_0^T t^{p-1} \mathbf{1_E} |\widehat{h}|^2 \, dv \, dt \right)^{1/2} d\Sigma(k)
$$

$$
+ \sqrt{\lambda p} \int_{\mathbb{Z}^3} \left(\int_0^T t^{p-1} \mathbf{1_{E^c}} |\widehat{h}|^2 \, dv \, dt \right)^{1/2} d\Sigma(k) =: I_1 + I_2.
$$

We will define $p = (\gamma + 2s)p' + 1$ in the Boltzmann case (or $p = (\gamma + 2)p' + 1$ in the Landau case). Here, then over $\mathbf{E}$, I_1 is controlled by

$$I_1 \leq \sqrt{\lambda p} \rho^{-(p-1)/p'} \int_{\mathbb{Z}^3} \left(\int_0^T \langle v \rangle^{(p-1)/p'} e^{2\lambda t^p} |\widehat{f}|^2 \, dv \, dt \right)^{1/2} d\Sigma(k).$$

We choose p as in (1.28) in the Boltzmann case (or (1.27) in the Landau case), such that $(p-1)/p' = \gamma + 2s < 0$ (or $(p-1)/p' = \gamma + 2 < 0$ for the Landau case), we then further obtain

$$I_1 \leq \sqrt{\lambda p} \rho^{-(p-1)/p'} \int_{\mathbb{Z}^3} \left(\int_0^T |\widehat{h}(t,k)|_D^2 \, dt \right)^{1/2} d\Sigma(k).$$

As $\lambda > 0$ and $\rho > 0$ can be chosen arbitrarily small, the above term is absorbed into the left-hand side of (6.3). For I_2, since it holds that $w_{q,\vartheta}^{-2} \leq e^{-q\rho^\vartheta t^{p'\vartheta}/2}$ on $\mathbf{E}^c$, and we notice that we can take $p = p'\vartheta$ and $2\lambda < q\rho^\vartheta/2$, we have

$$I_2 \leq \sqrt{\lambda p} \int_{\mathbb{Z}^3} \left(\int_0^T \int_{\mathbf{E}^c} t^{p-1} e^{2\lambda t^p} e^{-\frac{q}{2}\rho^\vartheta t^{p'\vartheta}} w_{q,\vartheta}^2 |\widehat{f}|^2 \, dv \, dt \right)^{1/2} d\Sigma(k)$$

$$\leq \sqrt{\lambda p} \int_{\mathbb{Z}^3} \sup_{0 \leq t \leq T} \|w_{q,\vartheta} \widehat{f}(t,k)\|_{L_v^2} \left(\int_0^T t^{p-1} e^{2\lambda t^p} e^{-\frac{q}{2}\rho^\vartheta t^p} \, dt \right)^{1/2} d\Sigma(k)$$

$$\leq C \sqrt{\lambda p} \int_{\mathbb{Z}^3} \sup_{0 \leq t \leq T} \|w_{q,\vartheta} \widehat{f}(t,k)\|_{L_v^2} \, d\Sigma(k),$$

due to the finiteness of the t-integral; cf. [76]. By the existence result, it further holds that

$$I_2 \lesssim \int_{\mathbb{Z}^3} \|w_{q,\vartheta} \widehat{f}_0(k)\|_{L_v^2} \, d\Sigma(k).$$

Plugging the above estimates back into (6.3) gives

$$\int_{\mathbb{Z}^3} \sup_{0 \leq t \leq T} \|\widehat{h}(t,k)\|_{L_v^2} \, d\Sigma(k) + \int_{\mathbb{Z}^3} \left(\int_0^T |\widehat{h}(t,k)|_D^2 \, dt \right)^{1/2} d\Sigma(k)$$

$$\lesssim \int_{\mathbb{Z}^3} \|w_{q,\vartheta} \widehat{f}_0(k)\|_{L_v^2} \, d\Sigma(k).$$

We use Minkowski's inequality $\left\| \|\cdot\|_{L_k^1} \right\|_{L_T^\infty} \leq \left\| \|\cdot\|_{L_T^\infty} \right\|_{L_k^1}$ and the expression $h = e^{\lambda t^p} f$ to obtain the time decay estimate (2.9) with $\kappa = p$ in the soft potential case. This then completes the proof of Theorem 2.1. $\qquad \square$

We will now give the proof of Theorem 2.2.

PROOF OF THEOREM 2.2. Following closely the proofs of Lemma 4.2 and Theorem 5.1, one can show that

$$\int_{\mathbb{Z}^3}\left(\int_0^T |(\widehat{\Gamma(f,g)},\langle k\rangle^{2m}w_{q,\vartheta}^2\widehat{h})|dt\right)^{1/2}d\Sigma(k)$$
$$\leq C_\eta\|w_{q,\vartheta}f\|_{L_{k,m}^1L_T^\infty L_v^2}\|w_{q,\vartheta}g\|_{L_{k,m}^1L_T^2L_{v,D}^2}$$
$$+ C_\eta\|w_{q,\vartheta}f\|_{L_{k,m}^1L_T^2L_{v,D}^2}\|w_{q,\vartheta}g\|_{L_{k,m}^1L_T^\infty L_v^2}+\eta\|w_{q,\vartheta}h\|_{L_{k,m}^1L_T^2L_{v,D}^2}$$

and

$$\|[a,b,c]\|_{L_{k,m}^1L_T^2}$$

(6.4)
$$\lesssim \|\{\mathbf{I}-\mathbf{P}\}f\|_{L_{k,m}^1L_T^2L_{v,D}^2}+\|f\|_{L_{k,m}^1L_T^\infty L_v^2}+\|f_0\|_{L_{k,m}^1L_T^\infty L_v^2}$$
$$+\int_{\mathbb{Z}^3}\left(\int_0^T |(\langle k\rangle^m|\hat{H}(t,k)|,\mu^{\frac{1}{4}})_{L_v^2}|^2\,dt\right)^{1/2}d\Sigma(k).$$

Taking the L_v^2 inner product of the Fourier transform of (1.10) and $\langle k\rangle^{2m}w_{q,\vartheta}^2\hat{f}$, we have

$$\left(\partial_t\hat{f},\langle k\rangle^{2m}w_{q,\vartheta}^2\hat{f}\right)+i\left(k\cdot v\hat{f},\langle k\rangle^{2m}w_{q,\vartheta}^2\hat{f}\right)+\left(L\hat{f},\langle k\rangle^{2m}w_{q,\vartheta}^2\hat{f}\right)$$
$$= (\widehat{\Gamma(f,f)},\langle k\rangle^{2m}w_{q,\vartheta}^2\hat{f}).$$

Then by using the same argument that was used to derive (6.1) and (6.2), we have

$$\int_{\mathbb{Z}^3}\sup_{0\leq t\leq T}\|\langle k\rangle^m\hat{f}(t,k)\|_{L_v^2}d\Sigma(k)$$
$$+\int_{\mathbb{Z}^3}\left(\int_0^T \|\langle k\rangle^m\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k)\|_D^2\,dt\right)^{1/2}$$

(6.5)
$$\lesssim \int_{\mathbb{Z}^3}\|\langle k\rangle^m\hat{f}_0\|d\Sigma(k)$$
$$+\int_{\mathbb{Z}^3}\left(\int_0^T |(\widehat{\Gamma(f,f)},\langle k\rangle^{2m}\{\mathbf{I}-\mathbf{P}\}\hat{f})|dt\right)^{1/2}d\Sigma(k)$$
$$\lesssim \int_{\mathbb{Z}^3}\|\langle k\rangle^m\hat{f}_0\|_{L_v^2}d\Sigma(k)+C_\eta\|f\|_{L_{k,m}^1L_T^\infty L_v^2}\|f\|_{L_{k,m}^1L_T^2L_{v,D}^2}$$
$$+\eta\|\{\mathbf{I}-\mathbf{P}\}f\|_{L_{k,m}^1L_T^2L_{v,D}^2},$$

and we further deduce the velocity-weighted estimate

$$\int_{\mathbb{Z}^3} \sup_{0 \leq t \leq T} \|\langle k \rangle^m w_{q,\vartheta} \widehat{f}(t,k)\|_{L^2_v} d\Sigma(k)$$

$$+ \int_{\mathbb{Z}^3} \left(\int_0^T \|\langle k \rangle^m w_{q,\vartheta} \widehat{f}\|_D^2 \, dt \right)^{1/2} d\Sigma(k)$$

$$- C \int_{\mathbb{Z}^3} \left(\int_0^T \|\langle k \rangle^m \widehat{f}\|_D^2 \, dt \right)^{1/2} d\Sigma(k)$$

(6.6)
$$\lesssim \int_{\mathbb{Z}^3} \|\langle k \rangle^m \widehat{f_0}\|_{L^2_v} d\Sigma(k)$$

$$+ \int_{\mathbb{Z}^3} \left(\int_0^T |(\widehat{\Gamma(f,f)}, \langle k \rangle^{2m} w_{q,\vartheta}^2 \widehat{f})| dt \right)^{1/2} d\Sigma(k)$$

$$\lesssim \int_{\mathbb{Z}^3} \|\langle k \rangle^m w_{q,\vartheta} \widehat{f_0}\|_{L^2_v} d\Sigma(k)$$

$$+ C_\eta \|w_{q,\vartheta} f\|_{L^1_{k,m} L^\infty_T L^2_v} \|w_{q,\vartheta} f\|_{L^1_{k,m} L^2_T L^2_{v,D}}$$

$$+ \eta \|w_{q,\vartheta} f\|_{L^1_{k,m} L^2_T L^2_{v,D}}.$$

A combination of the estimates (6.4), (6.5), and (6.6) gives

$$\int_{\mathbb{Z}^3} \sup_{0 \leq t \leq T} \|\langle k \rangle^m w_{q,\vartheta} \widehat{f}(t,k)\|_{L^2_v} d\Sigma(k)$$

$$+ \int_{\mathbb{Z}^3} \left(\int_0^T \|\langle k \rangle^m w_{q,\vartheta} \widehat{f}\|_D^2 \, dt \right)^{1/2} d\Sigma(k)$$

$$\lesssim \int_{\mathbb{Z}^3} \|\langle k \rangle^m w_{q,\vartheta} \widehat{f_0}\|_{L^2_v} d\Sigma(k),$$

which implies (2.11). This completes the proof of Theorem 2.2. □

7 Proof of the Main Results in the Finite Channel

In this section, we shall obtain the global existence, large-time behavior, and propagation of regularity in the $\overline{x}$-variable for solutions to the initial boundary value problem (PC) (1.10), (1.11), (1.14), and (1.15) in the case when the spatial domain is the finite channel.

PROOF OF THEOREM 2.3 AND THEOREM 2.4. We divide the proof into three parts as follows. First of all, we start from the proof of global existence. Then we explain the time decay rates. After that we explain the positivity of a solution and uniqueness.

Global existence. In this section we give a sequence of uniform a priori energy estimates assuming the smallness assumption and using the local existence as stated

in Theorem 8.1. For the sake of brevity, in what follows we only present the proof of the a priori energy estimates. The computation is divided into two cases in terms of the two prescribed boundary conditions.

Case 1. *Inflow boundary condition.* Let $|\alpha| \leq 1$, then apply ∂^α to (1.10), and then take the Fourier transform $\widehat{\cdot} = \mathscr{F}_{\bar{x}}$ with respect to $\bar{x} = (x_2, x_3)$ to obtain

$$(7.1) \qquad \partial_t \widehat{\partial^\alpha f} + v_1 \partial_{x_1} \widehat{\partial^\alpha f} + i\bar{k} \cdot \bar{v} \widehat{\partial^\alpha f} + L \widehat{\partial^\alpha f} = \mathscr{F}_{\bar{x}} \{\partial^\alpha \Gamma(f, f)\},$$

with initial data

$$\widehat{\partial^\alpha f}(0, x_1, \bar{k}, v) = \widehat{\partial^\alpha f}_0(x_1, \bar{k}, v),$$

and the inflow boundary condition

$$(7.2) \qquad \hat{f}(t, -1, \bar{k}, v)|_{v_1 > 0} = \widehat{g_-}(t, \bar{k}, v), \qquad \hat{f}(t, 1, \bar{k}, v)|_{v_1 < 0} = \widehat{g_+}(t, \bar{k}, v).$$

Particularly, it follows from (7.2) that if $|\alpha| = 1$, then for $\partial^\alpha = \partial_{\bar{x}}$, one has

$$(7.3) \qquad \begin{aligned} \widehat{\partial_{\bar{x}} f}(t, -1, \bar{k}, v)|_{v_1 > 0} &= \widehat{\partial_{\bar{x}} g_-}(t, \bar{k}, v), \\ \widehat{\partial_{\bar{x}} f}(t, 1, \bar{k}, v)|_{v_1 < 0} &= \widehat{\partial_{\bar{x}} g_+}(t, \bar{k}, v), \end{aligned}$$

while for $\partial^\alpha = \partial_{x_1}$, one has by using (7.1) with $\alpha = 0$ and Remark 5.3 that

$$(7.4) \qquad \begin{aligned} &\widehat{\partial_{x_1} f}(t, -1, \bar{k}, v)\big|_{v_1 > 0} \\ &= -\frac{1}{v_1} \{ \widehat{\partial_t g_-}(t, \bar{k}, v) + \bar{v} \cdot \widehat{\partial_{\bar{x}} g_-} + L\widehat{g_-} - \mathscr{F}_{\bar{x}}\{\Gamma(g_-, g_-)\} \}, \\ &\widehat{\partial_{x_1} f}(t, 1, \bar{k}, v)\big|_{v_1 < 0} \\ &= -\frac{1}{v_1} \{ \widehat{\partial_t g_+}(t, \bar{k}, v) + \bar{v} \cdot \widehat{\partial_{\bar{x}} g_+} + L\widehat{g_+} - \mathscr{F}_{\bar{x}}\{\Gamma(g_+, g_+)\} \}. \end{aligned}$$

In the rest of this section the complex inner product $(\cdot, \cdot) = (\cdot, \cdot)_{L^2_{x_1, v}}$ is in the space $L^2_{x_1, v}$. Now taking the inner product of (7.1) and $\widehat{\partial^\alpha f}$ with respect to (x_1, v):

$$(7.5) \qquad \begin{aligned} &\left(\partial_t \widehat{\partial^\alpha f}, \widehat{\partial^\alpha f} \right) + \left(v_1 \partial_{x_1} \widehat{\partial^\alpha f}, \widehat{\partial^\alpha f} \right) \\ &+ \left(i\bar{k} \cdot \bar{v} \widehat{\partial^\alpha f}, \widehat{\partial^\alpha f} \right) + (L\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}) = \left(\mathscr{F}_{\bar{x}}\{\partial^\alpha \Gamma(f, f)\}, \widehat{\partial^\alpha f} \right). \end{aligned}$$

In the rest of this proof we use the following brief notation $\|\cdot\| = \|\cdot\|_{L^2_{x_1, v}}$ for the norm with $x_1 \in I$ and $v \in \mathbb{R}^3$. Then, by taking the real part of (7.5) and integrating the resultant identity with respect to t over $[0, T]$, we obtain that

$$(7.6) \qquad \begin{aligned} &\sup_{0 \leq t \leq T} \|\widehat{\partial^\alpha f}\|^2 + 2\delta_0 \int_0^T \|\{I - P\}\widehat{\partial^\alpha f}\|_D^2 \, dt + \left| \Upsilon_T^+(\widehat{\partial^\alpha f}) \right|^2 \\ &\leq 2\|\widehat{\partial^\alpha f}_0\|^2 + |\Upsilon_T^-(\widehat{\partial^\alpha f})|^2 \\ &\quad + 2 \int_0^T \left| \mathscr{R}\left(\mathscr{F}_{\bar{x}}\{\partial^\alpha \Gamma(f, f)\}, \{I - P\}\widehat{\partial^\alpha f} \right) \right| dt, \end{aligned}$$

for any $\bar{k} \in \mathbb{Z}^2$, where Υ_T^+ and Υ_T^- are defined in (5.11) and (5.12), respectively, and the dissipation norm $\|\cdot\|_D$ is defined in (2.1) or (2.3) for the Landau or non-cutoff Boltzmann equation, respectively. Furthermore, taking the square root on both sides of (7.6) and then integrating in $\bar{k} \in \mathbb{Z}^2$, we obtain

$$
(7.7) \quad
\begin{aligned}
&\int_{\mathbb{Z}^2} \sup_{0 \leq t \leq T} \|\widehat{\partial^\alpha f}\| \, d\Sigma(\bar{k}) + \int_{\mathbb{Z}^2} \left(\int_0^T \|\{\mathbf{I} - \mathbf{P}\}\widehat{\partial^\alpha f}\|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{k}) \\
&\quad + \int_{\mathbb{Z}^2} |\Upsilon_T^+(\widehat{\partial^\alpha f})| \, d\Sigma(\bar{k}) \\
&\lesssim \int_{\mathbb{Z}^2} \|\widehat{\partial^\alpha f_0}\| \, d\Sigma(\bar{k}) + \int_{\mathbb{Z}^2} |\Upsilon_T^-(\widehat{\partial^\alpha f})| \, d\Sigma(\bar{k}) \\
&\quad + \int_{\mathbb{Z}^2} \left(\int_0^T \left| \mathscr{R}\big(\mathscr{F}_{\bar{x}}\{\partial^\alpha \Gamma(f,f)\}, \{\mathbf{I} - \mathbf{P}\}\widehat{\partial^\alpha f}\big) \right| dt \right)^{1/2} d\Sigma(\bar{k}).
\end{aligned}
$$

By using (7.2), (7.3), and (7.4) together with Lemma 4.4, it follows from (7.7) that

$$
(7.8) \quad
\begin{aligned}
&\int_{\mathbb{Z}^2} \sup_{0 \leq t \leq T} \|\widehat{\partial^\alpha f}\| \, d\Sigma(\bar{k}) + \int_{\mathbb{Z}^2} \left(\int_0^T \|\{\mathbf{I} - \mathbf{P}\}\widehat{\partial^\alpha f}\|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{k}) \\
&\quad + \int_{\mathbb{Z}^2} |\Upsilon_T^+(\widehat{\partial^\alpha f})| \, d\Sigma(\bar{k}) \\
&\lesssim \|\partial^\alpha f_0\|_{L_{\bar{k}}^1 L_{x_1,v}^2} + E(\widehat{g_\pm}) \\
&\quad + C_\eta \|\partial^\alpha f\|_{L_{\bar{k}}^1 L_T^\infty L_{x_1,v}^2} \|f\|_{L_{\bar{k}}^1 L_T^2 H_{x_1}^1 L_{v,D}^2} \\
&\quad + C_\eta \|\partial^\alpha f\|_{L_{\bar{k}}^1 L_T^2 L_{x_1}^2 L_{v,D}^2} \|f\|_{L_{\bar{k}}^1 L_T^\infty H_{x_1}^1 L_v^2} \\
&\quad + \eta \|\{\mathbf{I} - \mathbf{P}\}\partial^\alpha f\|_{L_{\bar{k}}^1 L_T^2 L_{x_1}^2 L_{v,D}^2},
\end{aligned}
$$

where we have used (7.4) and the norm $E(\cdot)$ defined in (2.7) to control the boundary term on the right-hand side. Consequently, a suitable linear combination of the above estimate (7.8) and the macroscopic dissipation estimate (5.13) gives rise to

$$
(7.9) \quad \mathscr{E}_T(f) + \mathscr{D}_T(f) \lesssim \sum_{|\alpha| \leq 1} \|\partial^\alpha f_0\|_{L_{\bar{k}}^1 L_{x_1,v}^2} + \mathscr{E}_T(f)\mathscr{D}_T(f) + E(\widehat{g_\pm}),
$$

where $\mathscr{E}_T(f)$ and $\mathscr{D}_T(f)$ are defined as in (2.4) and (2.5), respectively. Performing similar calculations to those used in obtaining (7.9), recalling also the methods used to obtain (6.2), then one can also show the following velocity-weighted estimate:

$$
(7.10) \quad
\begin{aligned}
&\mathscr{E}_{T,w}(f) + \mathscr{D}_{T,w}(f) \\
&\lesssim \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L_{\bar{k}}^1 L_{x_1,v}^2} + \mathscr{E}_{T,w}(f)\mathscr{D}_{T,w}(f) + E(w_{q,\vartheta} \widehat{g_\pm}).
\end{aligned}
$$

Here we recall that the velocity weight is useful to obtain the subexponential time decay only in case of *soft* potentials. In fact, according to Lemma A.3, one has

$$\mathscr{R}\big(L\widehat{\partial^\alpha f}, w_{q,\vartheta}^2 \widehat{\partial^\alpha f}\big)_{L_{\bar{x}_1,v}^2} \geq \delta_0 \|w_{q,\vartheta}\widehat{\partial^\alpha f}\|_D^2 - C\|\widehat{\partial^\alpha f}\|_D^2.$$

Applying this inequality, we are able to obtain an estimate similar to (7.7) with its right-hand term containing an extra term:

$$\int_{\mathbb{Z}^2} \Big(\int_0^T \|\widehat{\partial^\alpha f}\|_D^2 \, dt \Big)^{1/2} d\Sigma(\bar{k}),$$

where the above term can be controlled by taking a linear combination with the estimate (7.9). Hence, similar to how we derived (7.8) from (7.9), we then obtain (7.10) analogously. This concludes the proof in the case of the inflow boundary condition.

Case 2. *Specular reflection boundary condition.* Compared with Case 1 on the inflow boundary condition (7.2), the difference in this case stems from the boundary term; i.e., we have to control

(7.11)
$$\int_{\mathbb{Z}^2} \Upsilon_{T,w}^-(\widehat{\partial^\alpha f}) d\Sigma(\bar{k})$$
$$= \int_{\mathbb{Z}^2} \Big(-\int_0^T \int_{v_1<0} v_1 w^2 |\widehat{\partial^\alpha f}(1)|^2 \, dv \, dt$$
$$+ \int_0^T \int_{v_1>0} v_1 w^2 |\widehat{\partial^\alpha f}(-1)|^2 \, dv \, dt \Big)^{1/2} d\Sigma(\bar{k}),$$

with

(7.12)
$$\widehat{f}(-1,\bar{k},v_1,\bar{v})|_{v_1>0} = \widehat{f}(-1,\bar{k},-v_1,\bar{v}),$$
$$\widehat{f}(1,\bar{k},v_1,\bar{v})|_{v_1<0} = \widehat{f}(1,\bar{k},-v_1,\bar{v}).$$

Note that above and below we use the notation $w^2 = w_{q,\vartheta}^2$ for brevity. It is straightforward to see that if $\alpha = 0$ or $\partial^\alpha = \partial_{\bar{x}}$, then by the change of variable $v_1 \to -v_1$ and (7.12), (7.11) can be rewritten as

$$\int_{\mathbb{Z}^2} \Big(\int_0^T \int_{v_1>0} v_1 w^2 |\widehat{\partial^\alpha f}(1)|^2 \, dv \, dt$$
$$- \int_0^T \int_{v_1<0} v_1 w^2 |\widehat{\partial^\alpha f}(-1)|^2 \, dv \, dt \Big)^{1/2} d\Sigma(\bar{k})$$
$$= \int_{\mathbb{Z}^2} \Upsilon_{T,w}^+(\partial^\alpha f) d\Sigma(\bar{k}),$$

which is just the corresponding boundary term on the left-hand side of (7.6) and hence they can be cancelled. If $\partial^\alpha = \partial_{x_1}$, we first have from equation (7.1) with

$\alpha = 0$ and the boundary condition (7.12) and Remark 5.3 that

$$
\begin{aligned}
v_1 \partial_{x_1} \widehat{f}(1, \bar{k}, v_1) &= -\partial_t \widehat{f}(1, \bar{k}, v_1) - i\bar{k} \cdot \bar{v} \widehat{f}(1, \bar{k}, v_1) \\
&\quad - L\widehat{f}(1, \bar{k}, v_1) + \widehat{\Gamma(f, f)}(1, \bar{k}, v_1) \\
&= -\partial_t \widehat{f}(-1, \bar{k}, -v_1) - i\bar{k} \cdot \bar{v} \widehat{f}(-1, \bar{k}, -v_1) \\
&\quad - L\widehat{f}(-1, \bar{k}, -v_1) + \widehat{\Gamma(f, f)}(-1, \bar{k}, -v_1) \\
&= -\partial_t \widehat{f}(-1, \bar{k}, v_1) - i\bar{k} \cdot \bar{v} \widehat{f}(-1, \bar{k}, v_1) \\
&\quad - L\widehat{f}(-1, \bar{k}, v_1) + \widehat{\Gamma(f, f)}(-1, \bar{k}, v_1) \\
&= v_1 \partial_{x_1} \widehat{f}(-1, \bar{k}, v_1),
\end{aligned}
$$
(7.13)

where the second identity is valid due to the fact that we looked for solutions to
(7.1) satisfying the symmetric property $\widehat{f}(t, -x_1, \bar{k}, -v_1) = \widehat{f}(t, x_1, \bar{k}, v_1)$. After
that we used the boundary condition (7.12). Note that in (7.13) we have suppressed
the time variable for brevity. Then (7.13) implies that the boundary terms on both
sides of (7.6) are equal and thus they can also cancel each other. Therefore, similar
to how we obtained (7.10), it follows that we have

$$
(7.14) \quad \mathscr{E}_{T,w}(f) + \mathscr{D}_{T,w}(f) \lesssim \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k}} L^2_{x_1,v}} + \mathscr{E}_{T,w}(f) \mathscr{D}_{T,w}(f).
$$

Once (7.10) and (7.14) are obtained, then (2.13) and (2.16) follow from the stan-
dard continuity argument; cf. [28] and [45]. This concludes the proof of the global
existence of mild solutions.

Time decay rates. The proof is quite similar to the torus case. We shall only show
(2.14) for the inflow boundary value problem, since (2.17) for the specular re-
flection boundary value problem can be obtained in the same way. For the same
reason as in the torus case, we only focus on the soft potentials. Let $\widehat{h} = e^{\lambda t^p} \widehat{f}$
with $\lambda > 0$ and $0 < p < 1$ to be determined later. Then $\widehat{h}$ satisfies

$$
\partial_t \widehat{\partial^\alpha h} + v_1 \partial_{x_1} \widehat{\partial^\alpha h} + i\bar{k} \cdot \bar{v} \widehat{\partial^\alpha h} + L\widehat{\partial^\alpha h} = e^{-\lambda t^p} \mathscr{F}_{\bar{x}} \{\partial^\alpha \Gamma(h, h)\} + \lambda p t^{p-1} \widehat{h},
$$

with initial data

$$
\widehat{\partial^\alpha h}(0, x_1, \bar{k}, v) = \widehat{\partial^\alpha f_0}(x_1, \bar{k}, v),
$$

and for the inflow boundary condition we have

$$
\widehat{h}(t, -1, \bar{k}, v)|_{v_1 > 0} = e^{\lambda t^p} \widehat{g_-}(t, \bar{k}, v), \quad \widehat{h}(t, 1, \bar{k}, v)|_{v_1 < 0} = e^{\lambda t^p} \widehat{g_+}(t, \bar{k}, v).
$$

Next, performing similar calculations as to how we obtained (7.9), (7.10), and (7.14), one has

(7.15)
$$\sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \sup_{0\leq t\leq T} \left\| \widehat{\partial^\alpha h}(t,\bar{k}) \right\| d\Sigma(\bar{k})$$

$$+ \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \left(\int_0^T \left\| \widehat{\partial^\alpha h} \right\|_D^2 dt \right)^{1/2} d\Sigma(\bar{k})$$

$$\lesssim \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \left\| \widehat{\partial^\alpha f}_0 \right\| d\Sigma(\bar{k}) + \sup_{0<t<T} E(e^{\lambda t^p} g_\pm)$$

$$+ \sqrt{\lambda p} \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \left(\int_0^T t^{p-1} \left\| \widehat{\partial^\alpha h} \right\|^2 dt \right)^{1/2} d\Sigma(\bar{k}).$$

We recall that above equation and in the below equation we use the notation $\|\cdot\| = \|\cdot\|_{L^2_{x_1,v}}$ and $\|\cdot\|_D = \|\cdot\|_{L^2_{x_1,v,D}}$. Further, the last term on the right-hand can be estimated by the time-velocity splitting technique in the completely same way as for we treated (6.3) in the torus case. Thus, from (7.15) we have

$$\sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \sup_{0\leq t\leq T} \left\| \widehat{\partial^\alpha h}(t,\bar{k}) \right\| d\Sigma(\bar{k}) + \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \left(\int_0^T \left\| \widehat{\partial^\alpha h} \right\|_D^2 dt \right)^{1/2} d\Sigma(\bar{k})$$

$$\lesssim \sum_{|\alpha|\leq 1} \int_{\mathbb{Z}^2} \left\| w_{q,\vartheta} \widehat{\partial^\alpha f}_0 \right\| d\Sigma(\bar{k}) + E(w_{q,\vartheta} \widehat{g_\pm}) + \sup_{0\leq t\leq T} E(e^{\lambda t^p} \widehat{g_\pm}),$$

which proves (2.14). This also concludes the proof of the time decay rates of solutions.

Positivity and uniqueness. The uniqueness of the initial boundary value problem (1.10), (1.11), (1.14), or (1.15) can be proved by applying the similar method as the previous "energy estimates" part and which is now quite standard. Also, the local solution that we extend here from Section 8 is unique. Therefore we omit these analogous details. Noticing that on the boundary

$$F(t,\pm 1,\bar{x},v) = \mu + \mu^{\frac{1}{2}} g_\pm(t,\bar{x},v) \geq 0,$$

the positivity of the solution to the Landau equation is guaranteed by the maximum principle; cf. [45]. For the non-cutoff Boltzmann case, the positivity of the solution can also be proved by using the same argument as in [43, p. 833]. This completes the proofs of Theorems 2.3 and 2.4. □

PROOF OF THEOREM 2.5. We shall show that under the assumptions (2.18) or (2.19), the regularity of the initial data and the boundary data can propagate from the boundary into the interior of the channel along the tangential direction. In fact,

let $|\alpha| \leq 1$, then we may derive the following trilinear estimates with an extra Fourier multiplier $\langle \bar{k} \rangle^{2m}$:

$$
\begin{aligned}
(7.16) \quad & \int_{\mathbb{Z}^2} \left(\int_0^T |(\mathscr{F}_{\bar{x}}\{\Gamma(\partial^\alpha f, g)\}, \langle \bar{k} \rangle^{2m} w_{q,\vartheta}^2 \widehat{h})| dt \right)^{1/2} d\Sigma(\bar{k}) \\
& \leq C_\eta \left(\| w_{q,\vartheta} \partial^\alpha f \|_{L^1_{\bar{k},m} L^\infty_T L^2_{x_1,v}} \| w_{q,\vartheta} g \|_{L^1_{\bar{k},m} L^2_T H^1_{x_1} L^2_{v,D}} \right. \\
& \quad \left. + \| w_{q,\vartheta} \partial^\alpha f \|_{L^1_{\bar{k},m} L^2_T L^2_{x_1} L^2_{v,D}} \| w_{q,\vartheta} g \|_{L^1_{\bar{k},m} L^\infty_T H^1_{x_1} L^2_v} \right) \\
& \quad + \eta \| w_{q,\vartheta} h \|_{L^1_{\bar{k},m} L^2_T L^2_{x_1} L^2_{v,D}},
\end{aligned}
$$

and

$$
\begin{aligned}
(7.17) \quad & \int_{\mathbb{Z}^2} \left(\int_0^T |(\mathscr{F}_{\bar{x}}\{\Gamma(f, \partial^\alpha g)\}, \langle \bar{k} \rangle^{2m} w_{q,\vartheta}^2 \widehat{h})| dt \right)^{1/2} d\Sigma(\bar{k}) \\
& \leq C_\eta \left(\| w_{q,\vartheta} \partial^\alpha g \|_{L^1_{\bar{k},m} L^\infty_T L^2_{x_1,v}} \| w_{q,\vartheta} f \|_{L^1_{\bar{k},m} L^2_T H^1_{x_1} L^2_{v,D}} \right. \\
& \quad \left. + \| w_{q,\vartheta} \partial^\alpha g \|_{L^1_{\bar{k},m} L^2_T L^2_{x_1} L^2_{v,D}} \| w_{q,\vartheta} f \|_{L^1_{\bar{k},m} L^\infty_T H^1_{x_1} L^2_v} \right) \\
& \quad + \eta \| w_{q,\vartheta} h \|_{L^1_{\bar{k},m} L^2_T L^2_{x_1} L^2_{v,D}}.
\end{aligned}
$$

Moreover, regarding the macroscopic dissipation, for the inflow boundary condition (1.14), analogous to it holds for $|\alpha| \leq 1$ that

$$
\begin{aligned}
(7.18) \quad & \| \partial^\alpha [a, b, c] \|_{L^1_{\bar{k},m} L^2_T L^2_{x_1,v}} \\
& \lesssim \sum_{|\alpha| \leq 1} \left(\| \{\mathbf{I} - \mathbf{P}\} \partial^\alpha f \|_{L^1_{\bar{k},m} L^2_T L^2_{x_1} L^2_{v,D}} \right. \\
& \quad \left. + \| \partial^\alpha f \|_{L^1_{\bar{k}} L^\infty_T L^2_{x_1,v}} + \| \partial^\alpha f_0 \|_{L^1_{\bar{k},m} L^2_{x_1,v}} \right) + E(\langle \bar{k} \rangle^m \widehat{g}_\pm) \\
& \quad + \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \left(\int_0^T \| (\langle \bar{k} \rangle^m \widehat{\partial^\alpha H}, \mu^{1/4})_{L^2_v} \|^2_{L^2_{x_1}} dt \right)^{1/2} d\Sigma(\bar{k}) \\
& \quad + \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \langle \bar{k} \rangle^m \Upsilon_T^+(\widehat{\partial^\alpha f}) d\Sigma(\bar{k}) + \int_{\mathbb{Z}^2} \langle \bar{k} \rangle^m \Upsilon_T^- \left(\frac{\widehat{H}}{|v_1|} \right) d\Sigma(\bar{k}).
\end{aligned}
$$

For the specular reflection boundary condition (1.15), for $|\alpha| \leq 1$, it similarly holds that

$$
\begin{aligned}
\|\partial^\alpha &[a, b, c]\|_{L^1_{\overline{k},m} L^2_T L^2_{x_1}, v} \\
&\lesssim \sum_{|\alpha| \leq 1} \|\{\mathbf{I} - \mathbf{P}\} \partial^\alpha f\|_{L^1_{\overline{k},m} L^2_T L^2_{x_1} L^2_{v,D}} + \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_{L^1_{\overline{k},m} L^\infty_T L^2_{x_1}, v} \\
&\quad + \sum_{|\alpha| \leq 1} \|\partial^\alpha f_0\|_{L^1_{\overline{k},m} L^2_{x_1}, v} \\
&\quad + \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \left(\int_0^T \|(\langle \overline{k} \rangle^m \widehat{\partial^\alpha H}, \mu^{1/4})_{L^2_v}\|^2_{L^2_{x_1}} \, dt \right)^{1/2} d\Sigma(\overline{k}).
\end{aligned}
\tag{7.19}
$$

We point out that the proofs of these last two macroscopic estimates follow directly as in the proofs of Theorem 5.2.

In what follows, we will only explain the regularity propagation properties for the inflow boundary condition. The corresponding results for the specular reflection boundary condition can be obtained similarly to how we obtained (7.14) since the mode multiplier $\langle \overline{k} \rangle^{2m}$ doesn't influence the symmetry of $f(x_1, \overline{x}, v_1, \overline{v})$. Indeed, let $|\alpha| \leq 1$; by taking the complex inner product of (7.1) and $\langle \overline{k} \rangle^{2m} w_{q,\vartheta} \widehat{\partial^\alpha f}$ with respect to (x_1, v), we obtain

$$
\begin{aligned}
&\left(\partial_t \widehat{\partial^\alpha f}, w_{q,\vartheta} \langle \overline{k} \rangle^{2m} \widehat{\partial^\alpha f} \right) + \left(v_1 \partial_{x_1} \widehat{\partial^\alpha f}, w_{q,\vartheta} \langle \overline{k} \rangle^{2m} \widehat{\partial^\alpha f} \right) \\
&\quad + \left(i \overline{k} \cdot \overline{v} \widehat{\partial^\alpha f}, w_{q,\vartheta} \langle \overline{k} \rangle^{2m} \widehat{\partial^\alpha f} \right) + \left(L \widehat{\partial^\alpha f}, w_{q,\vartheta} \langle \overline{k} \rangle^{2m} \widehat{\partial^\alpha f} \right) \\
&\quad = \left(\mathscr{F}_{\overline{x}} \{ \partial^\alpha \Gamma(f, f) \}, w_{q,\vartheta} \langle \overline{k} \rangle^{2m} \widehat{\partial^\alpha f} \right),
\end{aligned}
$$

for any $0 \leq t \leq T$ and $\overline{k} \in \mathbb{Z}^2$. We use Lemma A.3 to observe that the equation above further implies that

$$
\begin{aligned}
\int_{\mathbb{Z}^2} &\sup_{0 \leq t \leq T} \| w_{q,\vartheta} \langle \overline{k} \rangle^m \widehat{\partial^\alpha f}(t, \overline{k}) \| d\Sigma(\overline{k}) \\
&+ \sqrt{2\delta_0} \int_{\mathbb{Z}^2} \left(\int_0^T \| w_{q,\vartheta} \langle \overline{k} \rangle^m \widehat{\partial^\alpha f} \|^2_D \, dt \right)^{1/2} d\Sigma(\overline{k}) \\
&- C \int_{\mathbb{Z}^2} \left(\int_0^T \| \langle \overline{k} \rangle^m \widehat{\partial^\alpha f} \|^2_D \, dt \right)^{1/2} d\Sigma(\overline{k}) \\
&+ \int_{\mathbb{Z}^2} \langle \overline{k} \rangle^m \Upsilon^+_{T,w} (\widehat{\partial^\alpha f}) d\Sigma(\overline{k}) - \int_{\mathbb{Z}^2} \langle \overline{k} \rangle^m \Upsilon^-_{T,w} (\widehat{\partial^\alpha f}) d\Sigma(\overline{k}) \\
&\lesssim \int_{\mathbb{Z}^2} \| w_{q,\vartheta} \widehat{\partial^\alpha f_0} \| d\Sigma(\overline{k}) \\
&\quad + \int_{\mathbb{Z}^2} \left(\int_0^T \left| (\mathscr{F}_{\overline{x}} \{ \partial^\alpha \Gamma(f, f) \}, w^2_{q,\vartheta} \langle \overline{k} \rangle^{2m} \widehat{\partial^\alpha f}) \right| dt \right)^{1/2} d\Sigma(\overline{k}).
\end{aligned}
\tag{7.20}
$$

Similarly, using Lemmas A.1 and A.2, it also holds without any velocity weight that

$$\int_{\mathbb{Z}^2} \sup_{0 \le t \le T} \| \langle \bar{k} \rangle^m \widehat{\partial^\alpha f}(t, \bar{k}) \| d\Sigma(\bar{k})$$

$$+ \sqrt{2\delta_0} \int_{\mathbb{Z}^2} \left(\int_0^T \| \langle \bar{k} \rangle^m \{ \mathbf{I} - \mathbf{P} \} \widehat{\partial^\alpha f} \|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{k})$$

$$(7.21) \quad + \int_{\mathbb{Z}^2} \langle \bar{k} \rangle^m \Upsilon_T^+ (\widehat{\partial^\alpha f}) d\Sigma(\bar{k}) - \int_{\mathbb{Z}^2} \langle \bar{k} \rangle^m \Upsilon_T^- (\widehat{\partial^\alpha f}) d\Sigma(\bar{k})$$

$$\lesssim \int_{\mathbb{Z}^2} \| \widehat{\partial^\alpha f}_0 \| d\Sigma(\bar{k})$$

$$+ \int_{\mathbb{Z}^2} \left(\int_0^T \left| (\mathscr{F}_{\bar{x}} \{ \partial^\alpha \Gamma(f, f) \}, w_{q,\vartheta}^2 \langle \bar{k} \rangle^{2m} \widehat{\partial^\alpha f}) \right| dt \right)^{1/2} d\Sigma(\bar{k}).$$

In (7.20) and (7.21) we notice that, as proved previously, the boundary terms vanish for the specular reflection boundary condition. Also, the boundary term with negative signs for incoming boundary velocities can be bounded by $E(w_{q,\vartheta} \langle \bar{k} \rangle^m \widehat{g}_\pm)$. The trilinear terms can be further bounded as in (7.16) and (7.17). Therefore, applying all of those estimates, a linear combination of (7.20) and (7.21) together with (7.18) or (7.19) implies that

$$\sum_{|\alpha| \le 1} \int_{\mathbb{Z}^2} \sup_{0 \le t \le T} \| w_{q,\vartheta} \langle \bar{k} \rangle^m \widehat{\partial^\alpha f}(t, \bar{k}) \| d\Sigma(\bar{k})$$

$$+ \sum_{|\alpha| \le 1} \int_{\mathbb{Z}^2} \left(\int_0^T \| w_{q,\vartheta} \langle \bar{k} \rangle^m \widehat{\partial^\alpha f} \|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{k})$$

$$\lesssim \sum_{|\alpha| \le 1} \int_{\mathbb{Z}^2} \| w_{q,\vartheta} \langle \bar{k} \rangle^m \widehat{\partial^\alpha f}_0 \| d\Sigma(\bar{k}) + E(w_{q,\vartheta} \langle \bar{k} \rangle^m \widehat{g}_\pm),$$

provided that $\epsilon_0 > 0$ in (2.18) is suitably small. Note that in the above estimate for the specular reflection case there is no boundary term $E(w_{q,\vartheta} \langle \bar{k} \rangle^m \widehat{g}_\pm)$ on the right-hand side, so in that case the smallness assumption (2.18) is actually just (2.19). Then the above estimate implies both of the estimates (2.20) and (2.21) for solutions corresponding to the inflow and specular reflection boundary conditions, respectively. This then completes the proof of Theorem 2.5.   □

8 Local-in-Time Existence

In this section, we are concerned with the local-in-time existence of solutions with mild regularity to the problem (**PT**) in the torus case and to the problem (**PC**) in the finite channel case. For brevity of presentation, we give the full details of the proof only for the non-cutoff Boltzmann equation with the specular reflection

boundary condition in the finite channel. The main idea is motivated by [2, theorem 4.2, p. 541] and [65, lemma 5.1, p. 4098]. Our approach used here can also be adopted to treat the local-in-time existence for the other cases mentioned in this paper. Then the analogous local existence theorem to Theorem 8.1 is true in the other cases that arise in this paper as in Theorems 2.1–2.5. For brevity of the presentation we omit these details. However, we remark that in the case of the torus domain, it is possible to carry out a straightforward proof of local-in-time existence and uniqueness based on the standard approximation argument in terms of the known existence results for regular initial data.

THEOREM 8.1 (Local existence). *Let all the conditions of Theorem 2.4 be satisfied. Then there are $\epsilon_0 > 0$, $T_0 > 0$, and $C_0 > 0$ such that if $F_0(x_1, \bar{x}, v) = \mu + \mu^{1/2} f_0(x_1, \bar{x}, v) \geq 0$ and*

$$\sum_{|\alpha| \leq 1} \|\partial^\alpha f_0\|_{L^1_{\frac{1}{k}} L^2_{x_1,v}} \leq \epsilon_0,$$

then the specular reflection boundary problem for the non-cutoff Boltzmann equation in the finite channel (1.10) *and* (1.11) *with* (1.15) *admits a unique solution*

$$f(t, x, v), \quad 0 \leq t \leq T_0, \ x \in \Omega = I \times \mathbb{T}^2, \ v \in \mathbb{R}^3,$$

with

(8.1)
$$f, \ \nabla_x f \in L^1_{\frac{1}{k}} L^\infty_{T_0} L^2_{x_1,v} \cap L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}$$

satisfying

$$F(t, x_1, \bar{x}, v) = \mu + \mu^{1/2} f(t, x_1, \bar{x}, v) \geq 0,$$
$$f(t, -x_1, \bar{x}, -v_1, \bar{v}) = f(t, x_1, \bar{x}, v_1, \bar{v}),$$

and the uniform estimate

(8.2)
$$\sum_{|\alpha| \leq 1} \left(\|\partial^\alpha f\|_{L^1_{\frac{1}{k}} L^\infty_{T_0} L^2_{x_1,v}} + \|\partial^\alpha f\|_{L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}} \right)$$
$$\leq C_0 \sum_{|\alpha| \leq 1} \|\partial^\alpha f_0\|_{L^1_{\frac{1}{k}} L^2_{x_1,v}}.$$

To prove Theorem 8.1, we start from the following linear inhomogeneous problem:

(8.3)
$$\begin{cases} \partial_t g + v_1 \partial_{x_1} g + \bar{v} \cdot \nabla_{\bar{x}} g + \mathscr{L}_1 g - \Gamma(\mathbb{h}, g) = -\mathscr{L}_2 \mathbb{h}, \\ g(0, x, v) = g_0(x, v), \\ g(-1, \bar{x}, v_1, \bar{v})|_{v_1 > 0} = g(-1, \bar{x}, -v_1, \bar{v}), \\ g(1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} = g(1, \bar{x}, -v_1, \bar{v}), \end{cases}$$

for a given function $\mathbb{h} = \mathbb{h}(t, x, v)$, where we have denoted the linear operators

$$\mathscr{L}_1 f = -\mu^{-\frac{1}{2}} Q(\mu, \mu^{\frac{1}{2}} f), \quad \mathscr{L}_2 f = -\mu^{-\frac{1}{2}} Q(\mu^{\frac{1}{2}} f, \mu).$$

For the later use, we recall that thanks to [3, lemma 2.15 and proposition 2.16, pp. 937–939], it holds that

(8.4)
$$\langle \mathscr{L}_1 g, g \rangle \gtrsim |g|_D^2 - C_1 \|\langle v \rangle^{\gamma/2} g\|_{L_v^2}^2,$$
$$|\langle \mathscr{L}_2 g, h \rangle| \lesssim \|\mu^{1/10^3} g\|_{L_v^2} \|\mu^{1/10^3} h\|_{L_v^2},$$

where $C_1 > 0$ is a universal constant. The solvability of (8.3) is guaranteed by the following lemma.

LEMMA 8.2. *There are $\epsilon_0 > 0$, $T_1 > 0$, and $C_1 > 0$ such that if*

$$g_0, \nabla_x g_0 \in L_{\bar{k}}^1 L_{x_1,v}^2, \quad \mathbb{h}, \nabla_x \mathbb{h} \in L_{\bar{k}}^1 L_{T_0}^\infty L_{x_1,v}^2 \cap L_{\bar{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2,$$

for $T_0 \in (0, T_1]$, and it holds that

(8.5)
$$g_0(x_1, \bar{x}, v_1, \bar{v}) = g_0(-x_1, \bar{x}, -v_1, \bar{v}),$$
$$\mathbb{h}(t, x_1, \bar{x}, v_1, \bar{v}) = \mathbb{h}(t, -x_1, \bar{x}, -v_1, \bar{v}),$$

and

(8.6)
$$\sum_{|\alpha| \leq 1} \left\{ \|\partial^\alpha \mathbb{h}\|_{L_{\bar{k}}^1 L_{T_0}^\infty L_{x_1,v}^2} + \|\partial^\alpha \mathbb{h}\|_{L_{\bar{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2} \right\} \leq \epsilon_0,$$

then the initial boundary value problem (8.3) admits a unique weak solution

$$g(t, x, v), \quad 0 \leq t \leq T_0, \ x \in \Omega = I \times \mathbb{T}^2, \ v \in \mathbb{R}^3,$$

satisfying

(8.7)
$$g(t, x_1, \bar{x}, v_1, \bar{v}) = g(t, -x_1, \bar{x}, -v_1, \bar{v})$$

and

(8.8)
$$\sum_{|\alpha| \leq 1} \|\partial^\alpha g\|_{L_{\bar{k}}^1 L_{T_0}^\infty L_{x_1,v}^2} + \sum_{|\alpha| \leq 1} \|\partial^\alpha g\|_{L_{\bar{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2}$$
$$\leq C_0 \left(\sum_{|\alpha| \leq 1} \|\partial^\alpha g_0\|_{L_{\bar{k}}^1 L_{x_1,v}^2} + \sqrt{T_1} \sum_{|\alpha| \leq 1} \|\partial^\alpha \mathbb{h}\|_{L_{\bar{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2} \right).$$

PROOF. We will divide the proof into four steps.

Step 1. First of all, we remark that since the operators $\mathscr{L}_1$, $\mathscr{L}_2$, and $\Gamma(\mathbb{h}, \cdot)$ involve a singular kernel and nonlocal derivatives, then it may be too hard to prove the local-in-time existence of solutions to the linear problem (8.3) directly by applying the standard ODE theory. However, because

$$L_{\bar{k}}^1 L_{T_0}^\infty H_{x_1}^1 L_v^2 \subset L^\infty((0, T_0) \times \mathbb{T}^2; H_{x_1}^1 L_v^2) \subset L_{T_0}^\infty L_{\bar{x}}^2 H_{x_1}^1 L_v^2,$$

we may expect to achieve the proof through an approximation procedure by first constructing approximate solutions in the function space

$$L^\infty(0, T_0; L_{\bar{x}}^2 H_{x_1}^1 L_v^2).$$

The advantage of using the above function space in this proof is that, compared to $L_k^1 L_{T_0}^\infty H_{x_1}^1 L_v^2$, it is more convenient to work with the weak formulation of (8.3) in the L^2 framework because $L_{\bar{x}}^2 H_{x_1}^1 L_v^2$ is a Hilbert space. For this purpose, we will smooth out the data $\mathbb{h}$ and g_0 in (8.3) with respect to the spatial variable as

$$(8.9) \qquad \mathbb{h}_\varepsilon = \mathbb{h} *_x \chi_\varepsilon, \quad g_{0,\varepsilon} = g_0 *_x \chi_\varepsilon.$$

Here $*_x$ is the standard convolution with respect to x, and χ_ε with $\varepsilon > 0$ suitably small is a smooth mollifier for $x \in \Omega = I \times \mathbb{T}^2$ such that $0 \le \chi_\varepsilon \le 1$, and

$$\chi_\varepsilon(x_1, \bar{x}) = \begin{cases} 1 & \text{for } |x_1| < 1 - \varepsilon, \\ 0 & \text{for } 1 - \frac{\varepsilon}{2} < |x_1| < 1, \end{cases}$$

and χ_ε is even in $x_1 \in (-1, 1)$. As $\mathbb{h}$ and g_0 are spatially periodic in $\bar{x}$, so are $\mathbb{h}_\varepsilon$ and $g_{0,\varepsilon}$. Moreover, it is obvious to see that the symmetric property (8.5) also holds true for $\mathbb{h}_\varepsilon$ and $g_{0,\varepsilon}$. Now, in order to solve (8.3), let us first consider the corresponding linear inhomogeneous problem with $\mathbb{h}$ and g_0 replaced by data $\mathbb{h}_\varepsilon$ and $g_{0,\varepsilon}$ defined in (8.9), and set $g_\varepsilon = g_\varepsilon(t, x, v)$ to be the corresponding solution, namely, g_ε satisfies

$$(8.10) \qquad \begin{cases} \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon + \mathscr{L}_1 g_\varepsilon - \Gamma(\mathbb{h}_\varepsilon, g_\varepsilon) = -\mathscr{L}_2 \mathbb{h}_\varepsilon, \\ g_\varepsilon(0, x, v) = g_{0,\varepsilon}(x, v), \\ g_\varepsilon(-1, \bar{x}, v_1, \bar{v})|_{v_1 > 0} = g_\varepsilon(-1, \bar{x}, -v_1, \bar{v}), \\ g_\varepsilon(1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} = g_\varepsilon(1, \bar{x}, -v_1, \bar{v}). \end{cases}$$

Step 2. Next, our goal is to solve (8.10). We define a linear operator $\mathscr{G}$ as

$$(8.11) \qquad \mathscr{G} = -\partial_t + \left(v_1 \partial_{x_1} + \bar{v} \cdot \nabla_{\bar{x}} + \mathscr{L}_1 - \Gamma(\mathbb{h}_\varepsilon, \cdot) \right)^*,$$

where the adjoint operator $(\cdot)^*$ is taken with respect to the complex inner product in $L_{x,v}^2$. Note that $\mathscr{L}_1$ is self-adjoint and the adjoint of $\Gamma(\mathbb{h}, \cdot)$ can be indeed defined as $\Gamma^*(\mathbb{h}, \cdot)$, which is given by [43, p. 843]. We then *claim* that $\mathscr{G}$ is injective over the following function space:

$$(8.12) \qquad \begin{aligned} \mathbb{W}_1 = \{ g : g \in H^1(0, T_0; \mathscr{S}(\Omega \times \mathbb{R}_v^3)) \text{ such that } g(T_0, x, v) \equiv 0, \\ g(t, x_1, \bar{x}, v_1, \bar{v}) = g(t, -x_1, \bar{x}, -v_1, \bar{v}), \text{ and} \\ g(t, \pm 1, \bar{x}, v_1, \bar{v})|_{v_1 \ne 0} = g(t, \pm 1, \bar{x}, -v_1, \bar{v}) \}. \end{aligned}$$

Here $\mathscr{S}(\Omega \times \mathbb{R}_v^3))$ is the standard Schwartz class by which we mean that $g \in \mathscr{S}(\Omega \times \mathbb{R}_v^3))$ is C^∞ in all variables on the interior and decays faster than any polynomial in the $\mathbb{R}_v^3$ variables and is periodic in $\mathbb{T}^2$. To show this claim, we take

$g_\varepsilon \in \mathbb{W}_1$ and compute for $\alpha = (\alpha_1, \overline{\alpha})$ with $\alpha_1 \leq 1$ and $|\overline{\alpha}| \leq 6$:

$$
\mathscr{R}(\partial^\alpha \mathscr{G} g_\varepsilon, \partial^\alpha g_\varepsilon)
$$

(8.13)
$$
= -\frac{1}{2} \frac{d}{dt} \|\partial^\alpha g_\varepsilon\|^2 + (\mathscr{L}_1 \partial^\alpha g_\varepsilon, \partial^\alpha g_\varepsilon) + \mathscr{R}(v_1 \partial_{x_1} \partial^\alpha g_\varepsilon, \partial^\alpha g_\varepsilon)
$$
$$
- \sum_{\alpha' \leq \alpha} C_{\alpha'}^\alpha \mathscr{R}(\Gamma^*(\partial^{\alpha'} \mathbb{h}_\varepsilon, \partial^{\alpha - \alpha'} g_\varepsilon), \partial^\alpha g_\varepsilon).
$$

The constant $C_{\alpha'}^\alpha$ is the multinomial coefficient. Here and below we have used the notation $\|\cdot\| = \|\cdot\|_{L^2_{x,v}}$ to be the L^2 norm in x and v, and similarly the inner product is $(\cdot, \cdot) = (\cdot, \cdot)_{L^2_{x,v}}$. We note that the analogous property to (7.13) for g_ε can be used to show that the boundary term satisfies

$$
\mathscr{R}(v_1 \partial_{x_1} \partial^\alpha g_\varepsilon, \partial^\alpha g_\varepsilon) = 0.
$$

We will perform our estimates by iterating upon $|\alpha|$.

From this identity with $|\alpha| = 0$, we apply (8.4) to obtain for $0 < t < T_0$ that

$$
\|g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|g_\varepsilon\|_D^2 \, d\tau
$$
$$
\lesssim \int_t^{T_0} |\mathscr{R}(\mathscr{G} g_\varepsilon, g_\varepsilon)| \, d\tau + \int_t^{T_0} |\mathscr{R}(g_\varepsilon, \Gamma(\mathbb{h}_\varepsilon, g_\varepsilon))| \, d\tau
$$
$$
+ T_0 \sup_{t \leq \tau \leq T_0} \|g_\varepsilon(\tau)\|^2.
$$

We also split into $\langle v \rangle \geq R$ and $\langle v \rangle \leq R$ for some large $R \geq 1$ to handle the lower bound in (8.4), which results in the last term above. It further follows from Lemma 4.1 with $w_{q,\vartheta} = 1$ that

$$
\|g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|g_\varepsilon\|_D^2 \, d\tau
$$
$$
\lesssim \left(\sup_{t \leq \tau \leq T_0} \|g_\varepsilon(\tau)\| \right) \int_t^{T_0} \|\mathscr{G} g_\varepsilon(\tau)\| \, d\tau
$$
$$
+ C_\eta \int_t^{T_0} \|g_\varepsilon\|_D^2 \|\mathbb{h}_\varepsilon\|_{L^\infty_x L^2_v}^2 \, d\tau + C_\eta \int_t^{T_0} \|g_\varepsilon\|^2 \|\mathbb{h}_\varepsilon\|_{L^\infty_x L^2_{v,D}}^2 \, d\tau
$$
$$
+ \eta \int_t^{T_0} \|g_\varepsilon\|_D^2 \, d\tau + T_0 \sup_{t \leq \tau \leq T_0} \|g_\varepsilon(\tau)\|^2,
$$

for any $0 < t < T_0$. Above and in the rest of this section we use the additional norm notation $\|\cdot\|_D^2 = \|\cdot\|_{L^2_{x,v,D}}^2$. At the same time, with the Sobolev embedding $H^1_{x_1} \supset L^\infty(I)$, we note that using (8.6) we have

(8.14)
$$
\sup_{t \leq \tau \leq T_0} \|\mathbb{h}_\varepsilon\|_{L^\infty_x L^2_v}^2 \lesssim \|\mathbb{h}_\varepsilon\|_{L^{\frac{1}{k}}_k L^\infty_{T_0} H^1_{x_1} L^2_v}^2 \lesssim \|\mathbb{h}\|_{L^{\frac{1}{k}}_k L^\infty_{T_0} H^1_{x_1} L^2_v}^2 \leq C\epsilon_0
$$

and

$$(8.15) \quad \int_t^{T_0} \|\mathbb{h}_\varepsilon\|^2_{L_x^\infty L_{v,D}^2} d\tau \lesssim \|\mathbb{h}_\varepsilon\|^2_{L_{\frac{1}{k}}^1 L_{T_0}^2 H_{x_1}^1 L_{v,D}^2}$$

$$\lesssim \|\mathbb{h}\|^2_{L_{\frac{1}{k}}^1 L_{T_0}^2 H_{x_1}^1 L_{v,D}^2} \le C\epsilon_0,$$

where the constant $C > 0$ above is uniform and does not depend on $\varepsilon > 0$.

Therefore, for $\epsilon_0 > 0$ small enough in (8.6) and $T_0 > 0$ sufficiently small we have

$$(8.16) \quad \|g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|g_\varepsilon\|^2_D d\tau \lesssim \int_t^{T_0} \|\mathscr{G}g_\varepsilon\| d\tau,$$

for any $0 < t < T_0$. Then it is direct that the estimate (8.16) implies that $\mathscr{G}$ defined in (8.11) is injective over $\mathbb{W}_1$. Hence, the claim is proved.

For the time integral term on the right-hand side of (8.16), we should note that for $g_\varepsilon \in \mathbb{W}_1$, and for any α, it holds that

$$\int_t^{T_0} \|\partial^\alpha \mathscr{G}g_\varepsilon\| d\tau$$

$$(8.17) \quad \lesssim \int_t^{T_0} \left(\|\partial_\tau \partial^\alpha g_\varepsilon\| + \|v \cdot \partial^\alpha \nabla_x g_\varepsilon\| + \|\langle v \rangle^{(\gamma/2+s)^+} \partial^\alpha g_\varepsilon\|_{L_x^2 H_v^2} \right) d\tau$$

$$+ \int_t^{T_0} \left(\sum_{\alpha' \le \alpha} \|\langle v \rangle^{(\gamma/2+s)^+} \partial^{\alpha-\alpha'} g_\varepsilon\|_{L_x^2 H_v^2} \|\partial^{\alpha'} \mathbb{h}_\varepsilon\|_{L_x^\infty L_v^2} \right) d\tau$$

$$\le C_\varepsilon < \infty.$$

Here we have used [65, prop. 6.10, p. 4107] in order to estimate $\mathscr{L}_1 \partial^{\alpha-\alpha'} g_\varepsilon$ and $\Gamma(\partial^{\alpha'} \mathbb{h}_\varepsilon, \partial^{\alpha-\alpha'} g_\varepsilon)$.

In particular, to obtain (8.17) we used that for any α' we have

$$(8.18) \quad \sup_{t \le \tau \le T_0} \|\partial^{\alpha'} \mathbb{h}_\varepsilon\|^2_{L_x^\infty L_v^2} \le C_\varepsilon \|\mathbb{h}\|^2_{L_{\frac{1}{k}}^1 L_{T_0}^\infty H_{x_1}^1 L_v^2},$$

and for later use we additionally have that

$$(8.19) \quad \int_t^{T_0} \|\partial^{\alpha'} \mathbb{h}_\varepsilon\|^2_{L_x^\infty L_{v,D}^2} d\tau \le C_\varepsilon \|\mathbb{h}\|^2_{L_{\frac{1}{k}}^1 L_{T_0}^2 H_{x_1}^1 L_{v,D}^2},$$

where $C_\varepsilon > 0$ above may depend on ε. The terms in the upper bound above are finite due to (8.6).

Now we will perform an estimate similar to (8.16) but including derivatives: ∂^α. A key point is that the estimates (8.18) and (8.19) depend crucially upon $\varepsilon > 0$. However, we need a fixed uniform smallness of $\epsilon_0 > 0$ in (8.6), and we cannot allow $\epsilon_0 \to 0$ as $\varepsilon \to 0$. Therefore we will carefully iterate the derivatives such that we only use (8.6) when we use the uniform-in-$\varepsilon > 0$ estimates (8.14) and (8.15) with no derivatives on $\mathbb{h}_\varepsilon$. When we have derivatives on $\mathbb{h}_\varepsilon$ such as in (8.18)

and (8.19), then we instead use the previous steps in the iteration to close our estimates.

In the case $|\alpha| = 1$, using (8.13) and taking one derivative we can again apply (8.4) to obtain for $0 < t < T_0$ that

$$
\|\partial^\alpha g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau
$$
$$
\lesssim \int_t^{T_0} |\mathscr{R}(\partial^\alpha \mathscr{G} g_\varepsilon, \partial^\alpha g_\varepsilon)| \, d\tau + \int_t^{T_0} |\mathscr{R}(\partial^\alpha g_\varepsilon, \Gamma(\hbar_\varepsilon, \partial^\alpha g_\varepsilon))| \, d\tau
$$
$$
+ \int_t^{T_0} |\mathscr{R}(g_\varepsilon, \Gamma(\partial^\alpha \hbar_\varepsilon, \partial^\alpha g_\varepsilon))| \, d\tau + T_0 \sup_{t \le \tau \le T_0} \|\partial^\alpha g_\varepsilon(\tau)\|^2.
$$

It further follows from Lemma 4.1 with $w_{q,\vartheta} = 1$ and Sobolev embedding that

$$
\|\partial^\alpha g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau
$$
$$
\lesssim \sup_{t \le \tau \le T_0} \|\partial^\alpha g_\varepsilon(\tau)\| \int_t^{T_0} \|\partial^\alpha \mathscr{G} g_\varepsilon(\tau)\| \, d\tau
$$
$$
+ C_\eta \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \|\hbar_\varepsilon\|_{L_x^\infty L_v^2}^2 \, d\tau + C_\eta \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|^2 \|\hbar_\varepsilon\|_{L_x^\infty L_{v,D}^2}^2 \, d\tau
$$
$$
+ C_\eta \int_t^{T_0} \|g_\varepsilon\|_D^2 \|\partial^\alpha \hbar_\varepsilon\|_{L_x^\infty L_v^2}^2 \, d\tau + C_\eta \int_t^{T_0} \|g_\varepsilon\|^2 \|\partial^\alpha \hbar_\varepsilon\|_{L_x^\infty L_{v,D}^2}^2 \, d\tau
$$
$$
+ \eta \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau + T_0 \sup_{t \le \tau \le T_0} \|\partial^\alpha g_\varepsilon(\tau)\|^2,
$$

which holds for any $0 < t < T_0$. We conclude from (8.6), (8.14), and (8.15) and the uniform smallness of $\epsilon_0 > 0$ that

$$
\|\partial^\alpha g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau
$$
$$
\lesssim \int_t^{T_0} \|\partial^\alpha \mathscr{G} g_\varepsilon(\tau)\| \, d\tau + \int_t^{T_0} \|g_\varepsilon\|_D^2 \|\partial^\alpha \hbar_\varepsilon\|_{L_x^\infty L_v^2}^2 \, d\tau
$$
$$
+ \int_t^{T_0} \|g_\varepsilon\|^2 \|\partial^\alpha \hbar_\varepsilon\|_{L_x^\infty L_{v,D}^2}^2 \, d\tau.
$$

Next using (8.18) and (8.19) (in this next step we are not using the smallness of $\epsilon_0 > 0$), we take a suitable linear combination of the last estimate with (8.16) to

obtain that

$$(8.20) \quad \begin{aligned} &\sum_{|\alpha|\leq 1} \|\partial^\alpha g_\varepsilon(t)\|^2 + \lambda \sum_{|\alpha|\leq 1} \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau \\ &\qquad \lesssim \sum_{|\alpha|\leq 1} \int_t^{T_0} \|\partial^\alpha \mathscr{G} g_\varepsilon(\tau)\| d\tau. \end{aligned}$$

In particular, we multiply (8.16) by a suitable large constant and then add it to the previous line to obtain (8.20).

In general, for $\alpha = (\alpha_1, \bar{\alpha})$ and $\bar{\alpha} = (\bar{\alpha}_2, \bar{\alpha}_3)$ we will prove that

$$(8.21) \quad \begin{aligned} &\sum_{\substack{\alpha_1 \leq 1, |\bar{\alpha}|\leq 6, \\ |\alpha|\leq m}} \|\partial^\alpha g_\varepsilon(t)\|^2 + \lambda \sum_{\substack{\alpha_1 \leq 1, |\bar{\alpha}|\leq 6, \\ |\alpha|\leq m}} \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau \\ &\qquad \lesssim \sum_{\substack{\alpha_1 \leq 1, |\bar{\alpha}|\leq 6, \\ |\alpha|\leq m}} \int_t^{T_0} \|\partial^\alpha \mathscr{G} g_\varepsilon(\tau)\| d\tau \end{aligned}$$

holds for $m \in \{2, \ldots, 7\}$. The $m = 0$ case was shown in (8.16) and the $m = 1$ case was shown in (8.20).

We assume that (8.21) holds for $|\alpha| = m$ and then prove that it also holds for $|\alpha| = m + 1$. Consider any fixed α with $|\alpha| = m + 1$; again using (8.13) and (8.4), we obtain for $0 < t < T_0$ that

$$\begin{aligned} &\|\partial^\alpha g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau \\ &\qquad \lesssim \int_t^{T_0} |\mathscr{R}(\partial^\alpha \mathscr{G} g_\varepsilon, \partial^\alpha g_\varepsilon)| d\tau + \int_t^{T_0} |\mathscr{R}(\partial^\alpha g_\varepsilon, \Gamma(\mathbb{h}_\varepsilon, \partial^\alpha g_\varepsilon))| d\tau \\ &\qquad\quad + \int_t^{T_0} \sum_{\alpha' < \alpha} |\mathscr{R}(\partial^{\alpha'} g_\varepsilon, \Gamma(\partial^{\alpha-\alpha'} \mathbb{h}_\varepsilon, \partial^\alpha g_\varepsilon))| d\tau \\ &\qquad\quad + T_0 \sup_{t \leq \tau \leq T_0} \|\partial^\alpha g_\varepsilon(\tau)\|^2. \end{aligned}$$

It similarly follows from Lemma 4.1 and the Sobolev embedding that

$$
\begin{aligned}
\|\partial^\alpha g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau \\
\lesssim \sup_{t \le \tau \le T_0} \|\partial^\alpha g_\varepsilon(\tau)\| \int_t^{T_0} \|\partial^\alpha \mathscr{G} g_\varepsilon(\tau)\| d\tau \\
+ C_\eta \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \|\mathbb{h}_\varepsilon\|_{L_x^\infty L_v^2}^2 \, d\tau \\
+ C_\eta \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|^2 \|\mathbb{h}_\varepsilon\|_{L_x^\infty L_{v,D}^2}^2 \, d\tau \\
+ C_\eta \int_t^{T_0} \sum_{\alpha' < \alpha} \Big(\|\partial^{\alpha'} g_\varepsilon\|^2 \|\partial^{\alpha-\alpha'} \mathbb{h}_\varepsilon\|_{L_x^\infty L_{v,D}^2}^2 \\
+ \|\partial^{\alpha'} g_\varepsilon\|_D^2 \|\partial^{\alpha-\alpha'} \mathbb{h}_\varepsilon\|_{L_x^\infty L_v^2}^2 \Big) d\tau \\
+ \eta \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau + T_0 \sup_{t \le \tau \le T_0} \|\partial^\alpha g_\varepsilon(\tau)\|^2,
\end{aligned}
$$

which holds for any $0 < t < T_0$. Again using (8.6), (8.14), and (8.15) and the uniform smallness of $\epsilon_0 > 0$, we obtain

$$
\begin{aligned}
\|\partial^\alpha g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|\partial^\alpha g_\varepsilon\|_D^2 \, d\tau \\
\lesssim \int_t^{T_0} \|\partial^\alpha \mathscr{G} g_\varepsilon(\tau)\| d\tau \\
+ \int_t^{T_0} \sum_{\alpha' < \alpha} \Big(\|\partial^{\alpha'} g_\varepsilon\|^2 \|\partial^{\alpha-\alpha'} \mathbb{h}_\varepsilon\|_{L_x^\infty L_{v,D}^2}^2 + \|\partial^{\alpha'} g_\varepsilon\|_D^2 \|\partial^{\alpha-\alpha'} \mathbb{h}_\varepsilon\|_{L_x^\infty L_v^2}^2 \Big) d\tau.
\end{aligned}
$$

Next using (8.18) and (8.19) (and not relying on the smallness of $\epsilon_0 > 0$), since $\alpha' < \alpha$ we take a suitable linear combination of the above last estimate with (8.21) for $|\alpha| = m$ to obtain that (8.21) also holds for $|\alpha| = m + 1$. In particular, again we multiply (8.21) for $|\alpha| = m$ by a suitable large constant and then add it to the previous line to obtain (8.21) for $|\alpha| = m + 1$, which proves (8.21).

Furthermore, we define the codomain of the mapping $\mathscr{G}$ on $\mathbb{W}_1$ as

$$
\mathbb{W}_2 = \{w : w = \mathscr{G} g, \ g \in \mathbb{W}_1\} \subset L^1(0, T_0; L^2(\Omega \times \mathbb{R}^3)),
$$

and we define the functional

$$
\mathscr{M} : \mathbb{W}_2 \to \mathbb{C},
$$

$$
w_\varepsilon = \mathscr{G} \mathbb{h}_\varepsilon \mapsto (\mathbb{h}_\varepsilon(0), g_{0,\varepsilon}) - \int_0^{T_0} (\mathscr{L}_2 \mathbb{h}_\varepsilon, \mathbb{h}_\varepsilon) dt = \mathscr{M}(w_\varepsilon),
$$

where $h_\varepsilon \in \mathbb{W}_1$ is uniquely determined by $w_\varepsilon \in \mathbb{W}_2$ as $\mathscr{G} : \mathbb{W}_1 \to \mathbb{W}_2$ is bijective. By (8.16) we have that

$$
(8.22) \quad
\begin{aligned}
&|\mathscr{M}(w_\varepsilon)| \\
&\leq \|h_\varepsilon(0)\| \|g_{0,\varepsilon}\| + C \|\mathbb{h}_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \|h_\varepsilon\|_{L^1(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \\
&\leq C_{T_0}(\varepsilon) \big(\|g_{0,\varepsilon}\| + \|\mathbb{h}_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))}\big) \|\mathscr{G} h_\varepsilon\|_{L^1(0,T_0;L^2(\Omega \times \mathbb{R}^3))}.
\end{aligned}
$$

This implies that $\mathscr{M} : \mathbb{W}_2 \to \mathbb{C}$ can be extended to be a bounded linear functional on $L^1(0, T_0; L^2(\Omega \times \mathbb{R}^3))$. So, using the Hahn-Banach theorem, there is $g_\varepsilon \in L^\infty(0, T_0; L^2(\Omega \times \mathbb{R}^3))$ such that

$$
(8.23) \quad \mathscr{M}(w_\varepsilon) = \int_0^{T_0} (g_\varepsilon(t), w_\varepsilon(t)) dt \quad \forall w_\varepsilon \in L^1(0, T_0; L^2(\Omega \times \mathbb{R}^3)),
$$

and it holds that

$$
\|g_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \leq C_{T_0}(\varepsilon) \big(\|g_{0,\varepsilon}\| + \|\mathbb{h}_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))}\big).
$$

Recall (8.12). In terms of (8.23), for any $h_\varepsilon \in \mathbb{W}_1$, it follows that

$$
(8.24) \quad \mathscr{M}(\mathscr{G} h_\varepsilon) = \int_0^{T_0} (g_\varepsilon(t), \mathscr{G} h_\varepsilon) dt = (h_\varepsilon(0), g_{0,\varepsilon}) - \int_0^{T_0} (\mathscr{L}_2 \mathbb{h}_\varepsilon, h_\varepsilon) dt,
$$

which implies that $g_\varepsilon \in L^\infty(0, T_0; L^2(\Omega \times \mathbb{R}^3))$ is a weak solution to (8.10). Moreover, from the energy estimate (8.16), it is straightforward to verify $g_\varepsilon \in L^2_{T_0} L^2_x L^2_{v,D}$.

Step 3. In this step, we are going to show that the obtained weak solution g_ε possesses higher regularity such as

$$
(8.25) \quad \partial^\alpha g_\varepsilon \in L^\infty_{T_0} L^2_x L^2_v \cap L^2_{T_0} L^2_x L^2_{v,D}
$$

for $\alpha \in \mathscr{A}$ with

$$
\mathscr{A} = \{\alpha : \alpha = (\alpha_1, \bar{\alpha}), \alpha_1 \leq 1, |\bar{\alpha}| \leq 6\}.
$$

The proof uses an induction on $n = |\alpha|$ with the help of the weak formulation in (8.10). The main issue to be dealt with arises from the fact that ∂^α does not commute with $\mathscr{G}$ since the linear operator $\Gamma(\mathbb{h}_\varepsilon, \cdot)$ depends on the function $\mathbb{h}_\varepsilon(t, x, v)$.

Indeed, for $|\alpha| = 1$, thanks to (8.17), we can still obtain a bounded linear functional on $L^1(0, T_0; L^2(\Omega \times \mathbb{R}^3))$, given by

$$
\mathscr{M}_\alpha : \mathbb{W}_2 \to \mathbb{C},
$$

$$
w_\varepsilon = \mathscr{G} h_\varepsilon \mapsto -(\partial^\alpha h_\varepsilon(0), g_{0,\varepsilon}) - \int_0^{T_0} (\mathscr{L}_2 \partial^\alpha \mathbb{h}_\varepsilon, h_\varepsilon) dt
$$

$$
+ \int_0^{T_0} (\Gamma(\partial^\alpha \mathbb{h}_\varepsilon, g_\varepsilon), h_\varepsilon) dt.
$$

The above uses g_ε from (8.23). Note further that for $|\alpha| = 1$ we have

$$(8.26) \quad \mathscr{M}(\partial^\alpha \mathscr{G} h_\varepsilon) = \mathscr{M}(\mathscr{G} \partial^\alpha h_\varepsilon) - \int_0^{T_0} (g_\varepsilon, \Gamma^*(\partial^\alpha \mathbb{h}_\varepsilon, h_\varepsilon)) dt = -\mathscr{M}_\alpha(\mathscr{G} h_\varepsilon).$$

Moreover, similar to obtaining (8.22), after integrating by parts on the initial data term and using (8.16), one has

$$\begin{aligned}
|\mathscr{M}_\alpha(w_\varepsilon)| \\
&\leq \|h_\varepsilon(0)\| \|\partial^\alpha g_{0,\varepsilon}\| + C \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \|h_\varepsilon\|_{L^1(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \\
&\quad + C \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L^\infty(0,T_0;H_x^2 L_v^2)} \|g_\varepsilon\|_{L_{T_0}^2 L_x^2 L_{v,D}^2(\mathbb{R}^3)} \|h_\varepsilon\|_{L_{T_0}^2 L_x^2 L_{v,D}^2(\mathbb{R}^3)} \\
&\quad + C \|g_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L_{T_0}^2 H_x^2 L_{v,D}^2(\mathbb{R}^3)} \|h_\varepsilon\|_{L_{T_0}^2 L_x^2 L_{v,D}^2(\mathbb{R}^3)} \\
&\leq C_{T_0} \|\mathscr{G} h_\varepsilon\|_{L_{T_0}^1 L_{x,v}^2} \left(\|\partial^\alpha g_{0,\varepsilon}\| + \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \right. \\
&\quad + \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L^\infty(0,T_0;H_x^2 L_v^2)} \|g_\varepsilon\|_{L_{T_0}^2 L_x^2 L_{v,D}^2(\mathbb{R}^3)} \\
&\quad \left. + \|g_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L_{T_0}^2 H_x^2 L_{v,D}^2(\mathbb{R}^3)} \right).
\end{aligned}$$

Therefore, as before, there is $g_\varepsilon^\alpha \in L^\infty(0, T_0; L^2(\Omega \times \mathbb{R}^3))$ such that

$$(8.27) \quad \mathscr{M}_\alpha(w_\varepsilon) = \int_0^{T_0} (g_\varepsilon^\alpha(t), w_\varepsilon(t)) dt \quad \forall w_\varepsilon \in L^1(0, T_0; L^2(\Omega \times \mathbb{R}^3)),$$

and similarly to before

$$\begin{aligned}
\|g_\varepsilon^\alpha\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \\
&\leq C_{T_0} \left(\|\partial^\alpha g_{0,\varepsilon}\| + \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \right. \\
&\quad + \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L^\infty(0,T_0;H_x^2 L_v^2)} \|g_\varepsilon\|_{L_{T_0}^2 L_x^2 L_{v,D}^2(\mathbb{R}^3)} \\
&\quad \left. + \|g_\varepsilon\|_{L^\infty(0,T_0;L^2(\Omega \times \mathbb{R}^3))} \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L_{T_0}^2 H_x^2 L_{v,D}^2(\mathbb{R}^3)} \right).
\end{aligned}$$

Note that (8.27) then reads as

$$(8.28) \quad \begin{aligned}
&- \int_0^{T_0} (\partial_t h_\varepsilon, g_\varepsilon^\alpha) + \int_0^{T_0} (v \cdot \nabla_x h_\varepsilon, g_\varepsilon^\alpha) + \int_0^{T_0} (\mathscr{L}_1 h_\varepsilon, g_\varepsilon^\alpha) \\
&\quad - \int_0^{T_0} (\Gamma(\mathbb{h}_\varepsilon, g_\varepsilon^\alpha), h_\varepsilon) \\
&= -(\partial^\alpha h_\varepsilon(0), g_{0,\varepsilon}) - \int_0^{T_0} (\mathscr{L}_2 \partial^\alpha \mathbb{h}_\varepsilon, g_\varepsilon^\alpha) + \int_0^{T_0} (\Gamma(\partial^\alpha \mathbb{h}_\varepsilon, g_\varepsilon), g_\varepsilon^\alpha),
\end{aligned}$$

meaning that g_ε^α is a weak solution to the following problem:

$$\begin{cases}
\partial_t g_\varepsilon^\alpha + v \cdot \nabla_x g_\varepsilon^\alpha + \mathscr{L}_1 g_\varepsilon^\alpha - \Gamma(\mathbb{h}_\varepsilon, g_\varepsilon^\alpha) = -\mathscr{L}_2 \partial^\alpha \mathbb{h}_\varepsilon + \Gamma(\partial^\alpha \mathbb{h}_\varepsilon, g_\varepsilon), \\
g_\varepsilon^\alpha(0, x, v) = \partial^\alpha g_{0,\varepsilon}(x, v), \\
g_\varepsilon(t, -1, \bar{x}, v_1, \bar{v})|_{v_1 > 0} = g_\varepsilon(t, -1, \bar{x}, -v_1, \bar{v}), \\
g_\varepsilon(t, 1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} = g_\varepsilon(t, 1, \bar{x}, -v_1, \bar{v}).
\end{cases}$$

Then using (8.23) with (8.26) and (8.27), we observe that

$$\int_0^{T_0} (g_\varepsilon^\alpha, \mathcal{G} h_\varepsilon) dt = -\int_0^{T_0} (g_\varepsilon, \partial^\alpha \mathcal{G} h_\varepsilon) dt$$

for any h_ε, so that $g_\varepsilon^\alpha = \partial^\alpha g_\varepsilon$ with $|\alpha| = 1$ in the weak sense. This proves (8.25) with $|\alpha| = 1$.

Now we assume that (8.25) is true for $\alpha \in \mathscr{A}$ with $|\alpha| = m \geq 1$. Letting $\alpha \in \mathscr{A}$ with $|\alpha| = m + 1$, as shown before, one sees that

$$\mathcal{M}_\alpha : \mathbb{W}_2 \to \mathbb{C},$$

$$w_\varepsilon = \mathcal{G} h_\varepsilon \mapsto (-1)^{|\alpha|} (\partial^\alpha h_\varepsilon(0), g_{0,\varepsilon}) - \int_0^{T_0} (\mathcal{L}_2 \partial^\alpha \mathbb{h}_\varepsilon, h_\varepsilon) dt$$

$$+ \sum_{\alpha > \alpha' \geq 0} \widetilde{C}_{\alpha'}^\alpha \int_0^{T_0} (\Gamma(\partial^{\alpha - \alpha'} \mathbb{h}_\varepsilon, \partial^{\alpha'} g_\varepsilon), h_\varepsilon) dt,$$

can be extended to be a bounded linear functional on $L^\infty(0, T_0; L^2(\Omega \times \mathbb{R}^3))$. Here, since $|\alpha'| \leq m$, the induction assumption has been used. Then we also have

$$(8.29) \quad \begin{aligned} \mathcal{M}(\partial^\alpha \mathcal{G} h_\varepsilon) &= \mathcal{M}(\mathcal{G} \partial^\alpha h_\varepsilon) - \sum_{\alpha > \alpha' \geq 0} C_{\alpha'}^\alpha \int_0^{T_0} (g_\varepsilon, \Gamma^*(\partial^{\alpha - \alpha'} \mathbb{h}_\varepsilon, \partial^{\alpha'} h_\varepsilon)) dt \\ &= (-1)^{|\alpha|} \mathcal{M}_\alpha(\mathcal{G} h_\varepsilon). \end{aligned}$$

In (8.29) and above $C_{\alpha'}^\alpha$ and $\widetilde{C}_{\alpha'}^\alpha$ are the constants that arise from the multinomial formula, and they depend only on α and α'.

Then, similarly, there is $g_\varepsilon^\alpha \in L^\infty(0, T_0; L^2(\Omega \times \mathbb{R}^3))$ such that

$$\mathcal{M}_\alpha(w_\varepsilon) = \int_0^{T_0} (g_\varepsilon^\alpha(t), w_\varepsilon(t)) dt \quad \forall w_\varepsilon \in L^1(0, T_0; L^2(\Omega \times \mathbb{R}^3)),$$

and again we have

$$\begin{aligned} \| g_\varepsilon^\alpha \|_{L^\infty(0, T_0; L^2(\Omega \times \mathbb{R}^3))} \\ \leq C_{T_0} \Big(\| \partial^\alpha g_{0,\varepsilon} \| + \| \partial^\alpha \mathbb{h}_\varepsilon \|_{L^\infty(0, T_0; L^2(\Omega \times \mathbb{R}^3))} \\ + \sum_{\alpha > \alpha' \geq 0} \| \partial^{\alpha - \alpha'} \mathbb{h}_\varepsilon \|_{L^\infty(0, T_0; H_x^2 L_v^2)} \| \partial^{\alpha'} g_\varepsilon \|_{L_{T_0}^2 L_x^2 L_{v,D}^2(\mathbb{R}^3)} \\ + \sum_{\alpha > \alpha' \geq 0} \| \partial^{\alpha'} g_\varepsilon \|_{L^\infty(0, T_0; L^2(\Omega \times \mathbb{R}^3))} \| \partial^{\alpha - \alpha'} \mathbb{h}_\varepsilon \|_{L_{T_0}^2 H_x^2 L_{v,D}^2(\mathbb{R}^3)} \Big). \end{aligned}$$

Consequently, similar to the previous arguments, one sees that g_ε^α with $\alpha \in \mathscr{A}$ and $|\alpha| = m + 1$ is a weak solution to the following problem:

$$
\begin{cases}
\partial_t g_\varepsilon^\alpha + v \cdot \nabla_x g_\varepsilon^\alpha + \mathscr{L}_1 g_\varepsilon^\alpha - \Gamma(\mathbb{h}_\varepsilon, g_\varepsilon^\alpha) \\
\quad = -\mathscr{L}_2 \partial^\alpha \mathbb{h}_\varepsilon + \sum_{\alpha > \alpha' \geq 0} \Gamma(\partial^{\alpha-\alpha'} \mathbb{h}_\varepsilon, \partial^{\alpha'} g_\varepsilon), \\
g_\varepsilon^\alpha(0, x, v) = \partial^\alpha g_{0,\varepsilon}(x, v), \\
g_\varepsilon(-1, \bar{x}, v_1, \bar{v})|_{v_1 > 0} = g_\varepsilon(-1, \bar{x}, -v_1, \bar{v}), \\
g_\varepsilon(1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} = g_\varepsilon(1, \bar{x}, -v_1, \bar{v}),
\end{cases}
$$

and furthermore as in (8.29), $g_\varepsilon^\alpha = \partial^\alpha g_\varepsilon$ holds true in the weak sense as

$$
\int_0^{T_0} (g_\varepsilon^\alpha, \mathscr{G} h_\varepsilon) dt = (-1)^{m+1} \int_0^{T_0} (g_\varepsilon, \partial^\alpha \mathscr{G} h_\varepsilon) dt
$$

for any h_ε. Hence, one has that $\partial^\alpha g_\varepsilon \in L^\infty(0, T_0; L^2(\Omega \times \mathbb{R}^3))$, and it is also straightforward to further verify that $\partial^\alpha g_\varepsilon \in L_{T_0}^2 L_x^2 L_{v,D}^2$. This justifies (8.25) for $\alpha \in \mathscr{A}$ with $|\alpha| = m + 1$. Thus, by induction, (8.25) is proved. Note that in the above steps many of the estimates depended crucially on fixed $\varepsilon > 0$.

Step 4. We are going to show that there is $g = g(t, x, v)$, $0 < t < T_0$, $x \in \Omega$, $v \in \mathbb{R}^3$ with $g(t, x_1, \bar{x}, v_1, \bar{v}) = g(t, -x_1, \bar{x}, -v_1, \bar{v})$ such that

$$
g, \nabla_x g \in L_{\frac{1}{k}}^1 L_{T_0}^\infty L_{x_1}^2 L_v^2 \cap L_{\frac{1}{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2,
$$

and for any $|\alpha| \leq 1$, $\partial^\alpha g_\varepsilon$ strongly converges to $\partial^\alpha g$ in the above space as $\varepsilon \to 0^+$.

Firstly, starting from the approximate solution g_ε with the estimate (8.25) in the regular Sobolev space, we need to prove that for any $|\alpha| \leq 1$, we have that

$$
\partial^\alpha g_\varepsilon \in L_{\frac{1}{k}}^1 L_{T_0}^\infty L_{x_1}^2 L_v^2 \cap L_{\frac{1}{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2.
$$

Indeed, letting $|\alpha| \leq 1$, to prove the estimates on $\partial^\alpha g_\varepsilon$ in $L_{\frac{1}{k}}^1 L_{T_0}^\infty L_{x_1}^2 L_v^2$, we split the whole Fourier space $\mathbb{Z}_{\frac{2}{k}}^2$ as $\{\bar{k} \neq 0\} \cup \{\bar{k} = 0\}$, and write

$$
\begin{aligned}
\|\partial^\alpha g_\varepsilon\|_{L_{\frac{1}{k}}^1 L_{T_0}^\infty L_{x_1}^2 L_v^2} &= \int_{\mathbb{Z}^2} \sup_{0 \leq t \leq T_0} \|\widehat{\partial^\alpha g_\varepsilon}\| d\Sigma(\bar{k}) \\
(8.30) \qquad &= \int_{\bar{k} \neq 0} \sup_{0 \leq t \leq T_0} \|\widehat{\partial^\alpha g_\varepsilon}\| d\Sigma(\bar{k}) + \sup_{0 \leq t \leq T_0} \left\| \int_{\mathbb{T}^2} \partial^\alpha g_\varepsilon d\bar{x} \right\|.
\end{aligned}
$$

Above, and in the rest of Step 4, we use the notation $\|\cdot\| = \|\cdot\|_{L_{x_1}^2 L_v^2}$.

Estimate on the first term on the right-hand side of (8.30): To treat the sup norm in time we may rewrite it as

$$
(8.31) \qquad \sup_{0 \leq t \leq T_0} |\cdot| = \lim_{p \to \infty} \|\cdot\|_{L_{T_0}^p} = \liminf_{p \to \infty} \|\cdot\|_{L_{T_0}^p}.
$$

Let $\widehat{\Delta_{\bar{x}}^{b} g} := |\bar{k}|^{b} \hat{g}$ for $b \in \mathbb{R}$. Then it holds that

$$
\begin{aligned}
\int_{\bar{k} \neq 0} & \sup_{0 \leq t \leq T_0} \| \widehat{\partial^{\alpha} g_{\varepsilon}} \| d\Sigma(\bar{k}) \\
(8.32) \qquad &= \int_{\bar{k} \neq 0} \frac{1}{|\bar{k}|^{5/2}} \sup_{0 \leq t \leq T_0} \| \widehat{\Delta_{\bar{x}}^{5/2} \partial^{\alpha} g_{\varepsilon}} \| d\Sigma(\bar{k}) \\
&= \int_{\bar{k} \neq 0} \frac{1}{|\bar{k}|^{5/2}} \liminf_{p \to \infty} \Big\| \| \widehat{\Delta_{\bar{x}}^{5/2} \partial^{\alpha} g_{\varepsilon}} \| \Big\|_{L_{T_0}^p} d\Sigma(\bar{k}) \\
&\leq \liminf_{p \to \infty} \int_{\bar{k} \neq 0} \frac{1}{|\bar{k}|^{5/2}} \Big\| \| \widehat{\Delta_{\bar{x}}^{5/2} \partial^{\alpha} g_{\varepsilon}} \| \Big\|_{L_{T_0}^p} d\Sigma(\bar{k}),
\end{aligned}
$$

where Fatou's lemma has been used in the last inequality. Let $p \geq 2$ with the conjugate p' such that $1/p + 1/p' = 1$, then we can derive from Hölder's inequality that

$$
\begin{aligned}
\int_{\bar{k} \neq 0} & \frac{1}{|\bar{k}|^{5/2}} \Big\| \| \widehat{\Delta_{\bar{x}}^{5/2} \partial^{\alpha} g_{\varepsilon}} \| \Big\|_{L_{T_0}^p} d\Sigma(\bar{k}) \\
&\leq \left(\int_{\bar{k} \neq 0} \frac{1}{|\bar{k}|^{5 p'/2}} d\Sigma(\bar{k}) \right)^{1/p'} \left(\int_{\bar{k} \neq 0} \Big\| \| \widehat{\Delta_{\bar{x}}^{5/2} \partial^{\alpha} g_{\varepsilon}} \| \Big\|_{L_{T_0}^p}^{p} d\Sigma(\bar{k}) \right)^{1/p} \\
&\leq C_{p'} \cdot T_0^{1/p} \sup_{0 \leq t \leq T_0} \left(\int_{\mathbb{Z}^2} \| \widehat{\Delta_{\bar{x}}^{5/2} \partial^{\alpha} g_{\varepsilon}} \|^p d\Sigma(\bar{k}) \right)^{1/p}.
\end{aligned}
$$

Applying the inequality $\| \cdot \|_{\ell^p(\mathbb{Z}^2)} \leq \| \cdot \|_{\ell^2(\mathbb{Z}^2)}$ for any $2 \leq p \leq \infty$, we see that

$$
\left(\int_{\mathbb{Z}^2} \| \widehat{\Delta_{\bar{x}}^{5/2} \partial^{\alpha} g_{\varepsilon}} \|^p d\Sigma(\bar{k}) \right)^{1/p} \leq \left(\int_{\mathbb{Z}^2} \| \widehat{\Delta_{\bar{x}}^{5/2} \partial^{\alpha} g_{\varepsilon}} \|^2 d\Sigma(\bar{k}) \right)^{1/2},
$$

which by Plancherel's identity the last upper bound is further bounded as

$$
\big\| \Delta_{\bar{x}}^{5/2} \partial^{\alpha} g_{\varepsilon} \big\|_{L_{x,v}^2} \leq \sum_{\beta \in \mathscr{A}} \big\| \partial^{\beta} g_{\varepsilon} \big\|_{L_{x,v}^2}.
$$

Plugging those estimates above into (8.32) and taking the lim inf as $p \to \infty$ gives

$$
\begin{aligned}
(8.33) \qquad &\int_{\bar{k} \neq 0} \sup_{0 \leq t \leq T_0} \| \widehat{\partial^{\alpha} g_{\varepsilon}} \| d\Sigma(\bar{k}) \\
&\leq C \sum_{\beta \in \mathscr{A}} \sup_{0 \leq t \leq T_0} \big\| \partial^{\beta} g_{\varepsilon} \big\|_{L_{x,v}^2} \leq C_{\varepsilon} < \infty.
\end{aligned}
$$

Estimate on the second term on the right-hand side of (8.30): We first apply Minkowski's inequality to obtain

$$
\sup_{0 \leq t \leq T_0} \left\| \int_{\mathbb{T}^2} \partial^{\alpha} g_{\varepsilon} \, d\bar{x} \right\| \leq \sup_{0 \leq t \leq T_0} \int_{\mathbb{T}^2} \| \partial^{\alpha} g_{\varepsilon} \| d\bar{x} \leq \int_{\mathbb{T}^2} \sup_{0 \leq t \leq T_0} \| \partial^{\alpha} g_{\varepsilon} \| d\bar{x}.
$$

By (8.31) and Fatou's lemma, one has

$$
(8.34) \qquad \int_{\mathbb{T}^2} \sup_{0 \leq t \leq T_0} \|\partial^\alpha g_\varepsilon\| d\bar{x} = \int_{\mathbb{T}^2} \liminf_{p \to \infty} \big\| \|\partial^\alpha g_\varepsilon\| \big\|_{L^p_{T_0}} d\bar{x}
$$
$$
\leq \liminf_{p \to \infty} \int_{\mathbb{T}^2} \big\| \|\partial^\alpha g_\varepsilon\| \big\|_{L^p_{T_0}} d\bar{x}.
$$

Similarly to before, for any $p \geq 2$ with the conjugate p', by Hölder's inequality it holds that

$$
\int_{\mathbb{T}^2} \big\| \|\partial^\alpha g_\varepsilon\| \big\|_{L^p_{T_0}} d\bar{x} \leq \left(\int_{\mathbb{T}^2} 1^{p'} d\bar{x} \right)^{1/p'} \left(\int_{\mathbb{T}^2} \big\| \|\partial^\alpha g_\varepsilon\| \big\|^p_{L^p_{T_0}} d\bar{x} \right)^{1/p}
$$
$$
= C_{p'} \left(\int_{\mathbb{T}^2} \int_0^{T_0} \|\partial^\alpha g_\varepsilon\|^p dt \, d\bar{x} \right)^{1/p},
$$

which implies that

$$
\liminf_{p \to \infty} \int_{\mathbb{T}^2} \big\| \|\partial^\alpha g_\varepsilon\| \big\|_{L^p_{T_0}} d\bar{x} \leq \liminf_{p \to \infty} C_{p'} \left(\int_{\mathbb{T}^2} \int_0^{T_0} \|\partial^\alpha g_\varepsilon\|^p dt \, d\bar{x} \right)^{1/p}
$$
$$
= C \sup_{0 \leq t \leq T_0} \sup_{\bar{x} \in \mathbb{T}^2} \|\partial^\alpha g_\varepsilon\|.
$$

By the embedding $H^2(\mathbb{T}^2_{\bar{x}}) \subset L^\infty(\mathbb{T}^2_{\bar{x}})$, plugging the above estimate into (8.34), we have

$$
(8.35) \qquad \int_{\mathbb{T}^2} \sup_{0 \leq t \leq T_0} \|\partial^\alpha g_\varepsilon\| d\bar{x} \leq C \sum_{\beta \in \mathscr{A}} \sup_{0 \leq t \leq T_0} \big\| \partial^\beta g_\varepsilon \big\|_{L^2_{x,v}} \leq C_\varepsilon < \infty.
$$

Therefore, by (8.35) and (8.33), it follows from (8.30) that $\partial^\alpha g_\varepsilon \in L^1_{\bar{k}} L^\infty_{T_0} L^2_{x_1} L^2_v$ for $|\alpha| \leq 1$. We have the following estimate:

$$
\sum_{|\alpha| \leq 1} \|\partial^\alpha g_\varepsilon\|_{L^1_{\bar{k}} L^2_{T_0} L^2_{x_1} L^2_v}
$$
$$
= \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \left(\int_0^{T_0} \|\widehat{\partial^\alpha g_\varepsilon}\|^2 dt \right)^{1/2} d\Sigma(\bar{k})
$$
$$
= \sum_{|\alpha| \leq 1} \int_{\bar{k} \neq 0} \frac{1}{|\bar{k}|^{3/2}} \left(\int_0^{T_0} \|\widehat{\Delta^{3/2}_{\bar{x}} \partial^\alpha g_\varepsilon}\|^2 dt \right)^{1/2} d\Sigma(\bar{k})
$$
$$
+ \sum_{|\alpha| \leq 1} \left(\int_0^{T_0} \left\| \int_{\mathbb{T}^2} \partial^\alpha g_\varepsilon d\bar{x} \right\|^2 dt \right)^{1/2}
$$
$$
\lesssim \left(\int_0^{T_0} \sum_{\alpha_1 \leq 1, |\bar{\alpha}| \leq 6} \|\partial^\alpha g_\varepsilon(t)\|^2_{L^2_{x,v}} dt \right)^{1/2} \leq C_\varepsilon < \infty.
$$

We have shown that $\partial^\alpha g_\varepsilon \in L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}$ for $|\alpha| \leq 1$ where the bounds above depend upon $\varepsilon > 0$.

Secondly, we will now show that g_ε and $\nabla_x g_\varepsilon$ are bounded in $L^1_{\frac{1}{k}} L^\infty_{T_0} L^2_{x_1} L^2_v \cap L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}$ uniformly for any $\varepsilon > 0$. In fact, turning to (8.10), we may perform similar energy estimates to those for obtaining (7.8) using (8.4) to deduce that

$$
\sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \sup_{0 \leq t \leq T_0} \|\widehat{\partial^\alpha g_\varepsilon}(t, \bar{k})\| \, d\Sigma(\bar{k})
$$
$$
+ \sqrt{2\delta_0} \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \left(\int_0^{T_0} \|\widehat{\partial^\alpha g_\varepsilon}\|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{k})
$$
$$
\lesssim \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \|\widehat{\partial^\alpha g_{0,\varepsilon}}\| \, d\Sigma(\bar{k})
$$
$$
+ \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \left(\int_0^{T_0} \left| \left(\widehat{\partial^\alpha \Gamma(\mathbb{h}_\varepsilon, g_\varepsilon)}, \widehat{\partial^\alpha g_\varepsilon} \right) \right| dt \right)^{1/2} d\Sigma(\bar{k})
$$
$$
+ \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \left(\int_0^{T_0} |(\mathscr{L}_2 \widehat{\partial^\alpha \mathbb{h}_\varepsilon}, \widehat{\partial^\alpha g_\varepsilon})| \, dt \right)^{1/2} d\Sigma(\bar{k})
$$
$$
+ T_0 \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \sup_{0 \leq t \leq T_0} \|\widehat{\partial^\alpha g_\varepsilon}(t, \bar{k})\| \, d\Sigma(\bar{k}).
$$

The above upper bound is then further bounded by

$$
\lesssim \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \|\partial^\alpha g_{0,\varepsilon}\| \, d\Sigma(\bar{k})
$$
$$
+ C_\eta \sum_{|\alpha| \leq 1} \|\partial^\alpha g_\varepsilon\|_{L^1_{\frac{1}{k}} L^\infty_{T_0} L^2_{x_1} L^2_v} \sum_{|\alpha| \leq 1} \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}}
$$
$$
+ C_\eta \sum_{|\alpha| \leq 1} \|\partial^\alpha g_\varepsilon\|_{L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}} \sum_{|\alpha| \leq 1} \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L^1_{\frac{1}{k}} L^\infty_{T_0} L^2_{x_1} L^2_v}
$$
$$
+ C_\eta \sum_{|\alpha| \leq 1} \|\partial^\alpha \mathbb{h}_\varepsilon\|_{L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}} + \eta \sum_{|\alpha| \leq 1} \|\partial^\alpha g_\varepsilon\|_{L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}}
$$
$$
+ T_0 \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \sup_{0 \leq t \leq T_0} \|\widehat{\partial^\alpha g_\varepsilon}(t, \bar{k})\| \, d\Sigma(\bar{k}).
$$

On the other hand, it is straightforward to check that

$$\sum_{|\alpha|\le 1}\|\partial^\alpha\mathbb{h}_\varepsilon\|_{L_{\bar{k}}^1 L_{T_0}^\infty L_{x_1,v}^2} + \sum_{|\alpha|\le 1}\|\partial^\alpha\mathbb{h}_\varepsilon\|_{L_{\bar{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2}$$

$$\lesssim \sum_{|\alpha|\le 1}\|\partial^\alpha\mathbb{h}\|_{L_{\bar{k}}^1 L_{T_0}^\infty L_{x_1,v}^2} + \sum_{|\alpha|\le 1}\|\partial^\alpha\mathbb{h}\|_{L_{\bar{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2} \le C\epsilon_0,$$

and similarly

$$\sum_{|\alpha|\le 1}\int_{\mathbb{Z}^2}\|\widehat{\partial^\alpha g}_{0,\varepsilon}\|d\Sigma(\bar{k}) \le C \sum_{|\alpha|\le 1}\int_{\mathbb{Z}^2}\|\widehat{\partial^\alpha g}_0\|d\Sigma(\bar{k}).$$

The two bounds above are independent of $\varepsilon > 0$. Consequently, we obtain the following uniform-in-ε estimate:

$$\begin{aligned}
(8.36)\quad & \sum_{|\alpha|\le 1}\int_{\mathbb{Z}^2}\sup_{0\le t\le T}\|\widehat{\partial^\alpha g}_\varepsilon(t,\bar{k})\|d\Sigma(\bar{k}) \\
& + \sum_{|\alpha|\le 1}\int_{\mathbb{Z}^2}\left(\int_0^T\|\widehat{\partial^\alpha g}_\varepsilon\|_D^2\,dt\right)^{1/2}d\Sigma(\bar{k}) \\
& \lesssim \sum_{|\alpha|\le 1}\int_{\mathbb{Z}^2}\|\widehat{\partial^\alpha g}_0\|d\Sigma(\bar{k}) + \sum_{|\alpha|\le 1}\|\partial^\alpha\mathbb{h}\|_{L_{\bar{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2}.
\end{aligned}$$

This holds on a short time interval $T_0 > 0$.

Lastly, we prove that for $|\alpha| \le 1$, the sequence $(\partial^\alpha g_\varepsilon)_{\varepsilon>0}$ is Cauchy in the space $L_{\bar{k}}^1 L_{T_0}^\infty L_{x_1}^2 L_v^2 \cap L_{\bar{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2$ as $\varepsilon \to 0^+$. To do so, it suffices to show that

$$\begin{aligned}
(8.37)\quad & \sum_{|\alpha|\le 1}\int_{\mathbb{Z}^2}\sup_{0\le t\le T_0}\|\widehat{\partial^\alpha g}_{\varepsilon_1} - \widehat{\partial^\alpha g}_{\varepsilon_2}\|d\Sigma(\bar{k}) \\
& + \sum_{|\alpha|\le 1}\int_{\mathbb{Z}^2}\left(\int_0^{T_0}\|\widehat{\partial^\alpha g}_{\varepsilon_1} - \widehat{\partial^\alpha g}_{\varepsilon_2}\|_D^2\,dt\right)^{1/2}d\Sigma(\bar{k}) \\
& \lesssim \sum_{|\alpha|\le 1}\int_{\mathbb{Z}^2}\|\widehat{\partial^\alpha g}_{0,\varepsilon_1} - \widehat{\partial^\alpha g}_{0,\varepsilon_2}\|d\Sigma(\bar{k}) \\
& + \sum_{|\alpha|\le 1}\|\partial^\alpha\mathbb{h}_{\varepsilon_1} - \partial^\alpha\mathbb{h}_{\varepsilon_2}\|_{L_{\bar{k}}^1 L_{T_0}^2 L_{x_1}^2 L_{v,D}^2}.
\end{aligned}$$

Above the implicit constant is uniform in $\varepsilon > 0$. Indeed, in terms of

$$\begin{cases}
\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon + \mathscr{L}_1 g_\varepsilon - \Gamma(\mathbb{h}_\varepsilon, g_\varepsilon) = -\mathscr{L}_2\mathbb{h}_\varepsilon, \\
g_\varepsilon(0, x, v) = g_{0,\varepsilon}(x, v), \\
g_\varepsilon(t, -1, \bar{x}, v_1, \bar{v})|_{v_1>0} = g_\varepsilon(t, -1, \bar{x}, -v_1, \bar{v}), \\
g_\varepsilon(t, 1, \bar{x}, v_1, \bar{v})|_{v_1<0} = g_\varepsilon(t, 1, \bar{x}, -v_1, \bar{v}),
\end{cases}$$

the estimate (8.37) can be derived by the energy estimates similar to obtaining (8.36), and we omit the details for brevity. Therefore, there is a function g with $g, \nabla_x g \in L^1_{\frac{1}{k}} L^\infty_{T_0} L^2_{x_1} L^2_v \cap L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}$ such that $\partial^\alpha g_\varepsilon \to \partial^\alpha g$ ($|\alpha| \leq 1$) in $L^1_{\frac{1}{k}} L^\infty_{T_0} L^2_{x_1} L^2_v \cap L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}$ as $\varepsilon \to 0^+$. Moreover, it is direct using similar estimates to verify that the limit function g is a unique solution to (8.3) satisfying the symmetric property (8.7) and the estimate (8.8). This completes the proof of Lemma 8.2. $\qquad\square$

PROOF OF THEOREM 8.1. We construct the approximate solution sequence, which is denoted by $(f^n(t, x, v))_{n=0}^\infty$ for the problem (1.10), (1.11), and (1.15) using the following iterative scheme:

$$\begin{cases} \partial_t f^{n+1} + v_1 \partial_{x_1} f^{n+1} + \bar{v} \cdot \nabla_{\bar{x}} f^{n+1} + \mathscr{L}_1 f^{n+1} - \Gamma(f^n, f^{n+1}) = -\mathscr{L}_2 f^n, \\ f^{n+1}(0, x, v) = f_0(x, v), \\ f^{n+1}(-1, \bar{x}, v_1, \bar{v})|_{v_1 > 0} = f^{n+1}(-1, \bar{x}, -v_1, \bar{v}), \\ f^{n+1}(1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} = f^{n+1}(1, \bar{x}, -v_1, \bar{v}), \end{cases}$$

for $n = 0, 1, \ldots$, where we have set $f^0(t, x, v) \equiv f_0(x, v)$. With Lemma 8.2 in hand, it is a standard procedure to apply the induction argument to show that there are $\epsilon_0 > 0$ and $T_0 > 0$ such that if

$$\|f_0\|_{L^1_{\frac{1}{k}} L^2_{x_1, v}} + \|\nabla_x f_0\|_{L^1_{\frac{1}{k}} L^2_{x_1, v}} \leq \epsilon_0,$$

then the approximate solution sequence $(f^n(t, x, v))_{n=0}^\infty$ is well-defined in the space $L^1_{\frac{1}{k}} L^\infty_{T_0} L^2_{x_1} L^2_v \cap L^1_{\frac{1}{k}} L^2_{T_0} L^2_{x_1} L^2_{v,D}$; it is also a Cauchy sequence in this function space. Then the limit function $f(t, x, v)$ such that (8.1) holds true by similar estimates is indeed a unique solution of (1.10), (1.11), and (1.15) satisfying the desired symmetric property and the estimate (8.2). For the proof of positivity, we can use the argument from [43, p. 833]; the details are omitted for brevity. The proof of Theorem 8.1 is complete. $\qquad\square$

Appendix: Basic Estimates

In this appendix, we collect some known basic estimates for the linearized Landau operator or Boltzmann operator L as in (1.12). We will use the following estimates, which were proven in [45, cor. 1 and lemma 5, p. 400]:

LEMMA A.1 (Landau case). *Let L be defined as in (1.12). Suppose $\gamma \geq -3$; then there exist two generic constants $\delta_0, C > 0$, such that*

$$\delta_0 |\{\mathbf{I} - \mathbf{P}\}g|_D^2 \leq (Lg, g)_{L^2_v},$$

$$|g|_D^2 \geq C \left\{ \left| \langle v \rangle^{\frac{\gamma}{2}} \{\mathbf{P}_v \partial_j g\} \right|_2^2 + \left| \langle v \rangle^{\frac{\gamma+2}{2}} \{(\mathbf{I} - \mathbf{P}_v) \partial_j g\} \right|_2^2 + \left| \langle v \rangle^{\frac{\gamma+2}{2}} g \right|_2^2 \right\},$$

where $\mathbf{P}_v$ is the projection defined as

$$\mathbf{P}_v h_j = \sum_{m=1}^3 h_m v_m \frac{v_j}{|v|^2}, \quad j \in \{1, 2, 3\},$$

for any vector-valued function $h(v) = [h_1(v), h_2(v), h_3(v)]$.

For the Boltzmann case (1.5) with (1.12), recalling also Remark 3.3, we have the following linearized estimate from [43, eq. (2.13), p. 784].

LEMMA A.2 (Boltzmann case). *Let L be defined as* (1.12). *For $0 < s < 1$ and $\gamma > -3$. Then there exists a uniform quantitative constant $C_0 > 0$ such that*

$$\frac{1}{C_0}|\{\mathbf{I} - \mathbf{P}\}g|_D^2 \leq (Lg, g)_{L_v^2} \leq C_0|\{\mathbf{I} - \mathbf{P}\}g|_D^2,$$

This lemma is also independently contained in [3]. From [3] and also similarly in [43, eq. (2.15), p. 784], recalling again Remark 3.3, one has that

$$\delta_0\left\{|g|_{H_{\gamma/2}^s}^2 + \left|\langle v\rangle^{\frac{\gamma+2s}{2}}g\right|_2^2\right\} \leq |g|_D^2 \leq C|g|_{H_{s+\gamma/2}^s}^2,$$

where $\delta_0, C > 0$. Here the weighted fractional Sobolev norm $|g|_{H_\ell^s}^2 = |\langle v\rangle^\ell g|_{H^s}^2$ is given by

$$|g|_{H_\ell^s}^2 = |\langle v\rangle^\ell g|_{L_v^2}^2 + \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} du\, \frac{[\langle v\rangle^\ell g(v) - \langle u\rangle^\ell g(u)]^2}{|v - u|^{3+2s}}\, \chi_{|v-u|\leq 1}.$$

This norm turns out to be equivalent to

$$|g|_{H_\ell^s}^2 = \int_{\mathbb{R}^3} dv\, \left|(1 - \Delta_v)^{\frac{s}{2}}(w^\ell(v)g(v))\right|^2.$$

We refer to [3] for additional details.

Lastly, we give the following uniform weighted estimates for both the linear Landau and Boltzmann operator as follows.

LEMMA A.3 (Weighted estimates). *Let L be given by* (1.12), *we have the estimate*

$$\left(Lg, w_{q,\vartheta}^2 g\right)_{L_v^2} \geq \delta_q|w_{q,\vartheta}g|_D^2 - C|g|_{L^2(B_R)}^2,$$

where $\delta_q, C > 0$, and B_R denotes the closed ball in $\mathbb{R}_v^3$ with center 0 and radius a constant $R > 0$. Here (q, ϑ) are given as in (**H**) *in* (1.26) *for the Boltzmann case and the Landau case.*

The lemma above is proven in [76, lemma 9, p. 323] for the Landau case. For the Boltzmann case, such an estimate is proven in [43, lemma 2.6, p. 783, Equation (2.10)] with polynomial weights. To prove the estimate with exponential weights as in (1.25) with (1.26) it will follow directly from using the estimate [43, lemma 2.6, p. 783, eq. (2.10)] or [3], combined with the techniques for handling the exponential weight in the proof of [76, lemma 9, p. 323] following the restrictions in (1.26). See also [29].

Bibliography

[1] Alexandre, R.; Desvillettes, L.; Villani, C.; Wennberg, B. Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal.* **152** (2000), no. 4, 327–355. doi:10.1007/s002050000083

[2] Alexandre, R.; Morimoto, Y.; Ukai, S.; Xu, C.-J.; Yang, T. Global existence and full regularity of the Boltzmann equation without angular cutoff. *Comm. Math. Phys.* **304** (2011), no. 2, 513–581. doi:10.1007/s00220-011-1242-9

[3] Alexandre, R.; Morimoto, Y.; Ukai, S.; Xu, C.-J.; Yang, T. The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential. *J. Funct. Anal.* **262** (2012), no. 3, 915–1010. doi:10.1016/j.jfa.2011.10.007

[4] Alexandre, R.; Morimoto, Y.; Ukai, S.; Xu, C.-J.; Yang, T. Local existence with mild regularity for the Boltzmann equation. *Kinet. Relat. Models* **6** (2013), no. 4, 1011–1041. doi:10.3934/krm.2013.6.1011

[5] Alexandre, R.; Villani, C. On the Boltzmann equation for long-range interactions. *Comm. Pure Appl. Math.* **55** (2002), no. 1, 30–70. doi:10.1002/cpa.10012

[6] Alonso, R.; Morimoto, Y.; Sun, W.; Yang, T. Non-cutoff Boltzmann equation with polynomial decay perturbation. Preprint, 2018. arXiv:1812.05299 [math.AP]

[7] Ambrose, D. M.: The radius of analyticity for solutions to a problem in epitaxial growth on the torus. Preprint, 2018. arXiv:1807.01740 [math.AP]

[8] Bahouri, H.; Chemin, J.-Y.; Danchin, R. *Fourier analysis and nonlinear partial differential equations.* Grundlehren der mathematischen Wissenschaften, 343. Springer, Heidelberg, 2011. doi:10.1007/978-3-642-16830-7

[9] Bobylev, A.; Gamba, I. M.; Zhang, C. On the rate of relaxation for the Landau kinetic equation and related models. *J. Stat. Phys.* **168** (2017), no. 3, 535–548. doi:10.1007/s10955-017-1814-y

[10] Briant, M.; Guo, Y. Asymptotic stability of the Boltzmann equation with Maxwell boundary conditions. *J. Differential Equations* **261** (2016), no. 12, 7000–7079. doi:10.1016/j.jde.2016.09.014

[11] Caflisch, R. E. The Boltzmann equation with a soft potential. I. Linear, spatially-homogeneous. *Comm. Math. Phys.* **74** (1980), no. 1, 71–95.

[12] Caflisch, R. E. The Boltzmann equation with a soft potential. II. Nonlinear, spatially-periodic. *Comm. Math. Phys.* **74** (1980), no. 2, 97–109.

[13] Cao, Y.; Kim, C.; Lee, D. Global strong solutions of the Vlasov-Poisson-Boltzmann system in bounded domains. *Arch. Ration. Mech. Anal.* **233** (2019), no. 3, 1027–1130. doi:10.1007/s00205-019-01374-9

[14] Carrapatoso, K.; Mischler, S. Landau equation for very soft and Coulomb potentials near Maxwellians. *Ann. PDE* **3** (2017), no. 1, Art. 1, 65 pp. doi:10.1007/s40818-017-0021-0

[15] Cercignani, C. Existence and uniqueness in the large for boundary value problems in kinetic theory. *J. Mathematical Phys.* **8** (1967), no. 8, 1653–1656. doi:10.1063/1.1705405

[16] Cercignani, C. Existence, uniqueness, and convergence of the solutions of models in kinetic theory. *J. Mathematical Phys.* **9** (1968), no. 4, 633–639. doi:10.1063/1.1664621

[17] Cercignani, C. On the initial-boundary value problem for the Boltzmann equation. *Arch. Rational Mech. Anal.* **116** (1992), no. 4, 307–315. doi:10.1007/BF00375670

[18] Cercignani, C.; Illner, R.; Pulvirenti, M. *The mathematical theory of dilute gases.* Applied Mathematical Sciences, 106. Springer, New York, 1994. doi:10.1007/978-1-4419-8524-8

[19] Chemin, J.-Y.; Lerner, N. Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes. *J. Differential Equations* **121** (1995), no. 2, 314–328. doi:10.1006/jdeq.1995.1131

[20] Chen, H.; Hu, X.; Li, W.-X.; Zhan, J. Gevrey smoothing effect for the spatially inhomogeneous Boltzmann equations without cut-off. Preprint, 2018. arXiv:1805.12543 [math.AP]

[21] Constantin, P.; Córdoba, D.; Gancedo, F.; Rodríguez-Piazza, L.; Strain, R. M. On the Muskat problem: global in time results in 2D and 3D. *Amer. J. Math.* **138** (2016), no. 6, 1455–1494. doi:10.1353/ajm.2016.0044

[22] Constantin, P.; Córdoba, D.; Gancedo, F.; Strain, R. M. On the global existence for the Muskat problem. *J. Eur. Math. Soc. (JEMS)* **15** (2013), no. 1, 201–227. doi:10.4171/JEMS/360

[23] Desvillettes, L.; Villani, C. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.* **159** (2005), no. 2, 245–316. doi:10.1007/s00222-004-0389-9

[24] DiPerna, R. J.; Lions, P.-L. On the Fokker-Planck-Boltzmann equation. *Comm. Math. Phys.* **120** (1988), no. 1, 1–23.

[25] DiPerna, R. J.; Lions, P.-L. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)* **130** (1989), no. 2, 321–366. doi:10.2307/1971423

[26] Duan, R.; Huang, F.; Wang, Y.; Yang, T. Global well-posedness of the Boltzmann equation with large amplitude initial data. *Arch. Ration. Mech. Anal.* **225** (2017), no. 1, 375–424. doi:10.1007/s00205-017-1107-2

[27] Duan, R.; Liu, S. Compressible Navier-Stokes approximation for the Boltzmann equation in bounded domains. Preprint, 2018. arXiv:1806.09796 [math.AP]

[28] Duan, R.; Liu, S.; Xu, J. Global well-posedness in spatially critical Besov space for the Boltzmann equation. *Arch. Ration. Mech. Anal.* **220** (2016), no. 2, 711–745. doi:10.1007/s00205-015-0940-4

[29] Duan, R.; Liu, S.; Yang, T.; Zhao, H. Stability of the nonrelativistic Vlasov-Maxwell-Boltzmann system for angular non-cutoff potentials. *Kinet. Relat. Models* **6** (2013), no. 1, 159–204. doi:10.3934/krm.2013.6.159

[30] Duan, R.; Sakamoto, S. Solution to the Boltzmann equation in velocity-weighted Chemin-Lerner type spaces. *Kinet. Relat. Models* **11** (2018), no. 6, 1301–1331. doi:10.3934/krm.2018051

[31] Duan, R.; Wang, Y. The Boltzmann equation with large-amplitude initial data in bounded domains. *Adv. Math.* **343** (2019), 36–109. doi:10.1016/j.aim.2018.11.007

[32] Esposito, R.; Guo, Y.; Kim, C.; Marra, R. Non-isothermal boundary in the Boltzmann theory and Fourier law. *Comm. Math. Phys.* **323** (2013), no. 1, 177–239. doi:10.1007/s00220-013-1766-2

[33] Esposito, R.; Guo, Y.; Kim, C.; Marra, R. Stationary solutions to the Boltzmann equation in the hydrodynamic limit. *Ann. PDE* **4** (2018), no. 1, Art. 1, 119 pp. doi:10.1007/s40818-017-0037-5

[34] Esposito, R.; Lebowitz, J. L.; Marra, R. Hydrodynamic limit of the stationary Boltzmann equation in a slab. *Comm. Math. Phys.* **160** (1994), no. 1, 49–80.

[35] Esposito, R.; Lebowitz, J. L.; Marra, R. The Navier-Stokes limit of stationary solutions of the nonlinear Boltzmann equation. *J. Statist. Phys.* **78** (1995), no. 1-2, 389–412. doi:10.1007/BF02183355

[36] Fan, Y.; Lei, Y.; Liu, S.; Zhao, H. The non-cutoff Vlasov-Maxwell-Boltzmann system with weak angular singularity. *Sci. China Math.* **61** (2018), no. 1, 111–136. doi:10.1007/s11425-016-9083-x

[37] Gancedo, F.; García-Juárez, E.; Patel, N.; Strain, R. M. On the Muskat problem with viscosity jump: global in time results. *Adv. Math.* **345** (2019), 552–597. doi:10.1016/j.aim.2019.01.017

[38] Glassey, R. T. *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1996. doi:10.1137/1.9781611971477

[39] Golse, F.; Imbert, C.; Mouhot, C.; Vasseur, A. F. Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **19** (2019), no. 1, 253–295.

[40] Grad, H. Principles of the kinetic theory of gases. *Handbuch der Physik (herausgegeben von S. Flügge), Bd. 12, Thermodynamik der Gase*, 205–294. Springer, Berlin-Göttingen-Heidelberg, 1958.

[41] Grad, H. Asymptotic theory of the Boltzmann equation. II. *Rarefied gas dynamics (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962), Vol. I*, 26–59. Academic Press, New York, 1963.

[42] Granero-Belinchón, R.; Magliocca, M. Global existence and decay to equilibrium for some crystal surface models. *Discrete Contin. Dyn. Syst.* **39** (2019), no. 4, 2101–2131. doi:10.3934/dcds.2019088

[43] Gressman, P. T.; Strain, R. M. Global classical solutions of the Boltzmann equation without angular cut-off. *J. Amer. Math. Soc.* **24** (2011), no. 3, 771–847. doi:10.1090/S0894-0347-2011-00697-8

[44] Gualdani, M. P.; Mischler, S.; Mouhot, C. Factorization of non-symmetric operators and exponential H-theorem. *Mém. Soc. Math. Fr. (N.S.)* **153** (2017), 137 pp.

[45] Guo, Y. The Landau equation in a periodic box. *Comm. Math. Phys.* **231** (2002), no. 3, 391–434. doi:10.1007/s00220-002-0729-9

[46] Guo, Y. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal.* **169** (2003), no. 4, 305–353. doi:10.1007/s00205-003-0262-9

[47] Guo, Y. The Boltzmann equation in the whole space. *Indiana Univ. Math. J.* **53** (2004), no. 4, 1081–1094. doi:10.1512/iumj.2004.53.2574

[48] Guo, Y. Boltzmann diffusive limit beyond the Navier-Stokes approximation. *Comm. Pure Appl. Math.* **59** (2006), no. 5, 626–687. doi:10.1002/cpa.20121

[49] Guo, Y. Decay and continuity of the Boltzmann equation in bounded domains. *Arch. Ration. Mech. Anal.* **197** (2010), no. 3, 713–809. doi:10.1007/s00205-009-0285-y

[50] Guo, Y.; Kim, C.; Tonon, D.; Trescases, A. Regularity of the Boltzmann equation in convex domains. *Invent. Math.* **207** (2017), no. 1, 115–290. doi:10.1007/s00222-016-0670-8

[51] Guo, Y.; Liu, S. The Boltzmann equation with weakly inhomogeneous data in bounded domain. *J. Funct. Anal.* **272** (2017), no. 5, 2038–2057. doi:10.1016/j.jfa.2016.08.017

[52] Hamdache, K. Initial-boundary value problems for the Boltzmann equation: global existence of weak solutions. *Arch. Rational Mech. Anal.* **119** (1992), no. 4, 309–353. doi:10.1007/BF01837113

[53] He, L.-B.; Jiang, J.-C. On the global dynamics of the inhomogeneous Boltzmann equations without angular cutoff: Hard potentials and Maxwellian molecules. Preprint, 2017. arXiv:1710.00315 [math.AP]

[54] Hérau, F.; Tonon, D.; Tristani, I. Regularization estimates and Cauchy theory for inhomogeneous Boltzmann equation for hard potentials without cut-off. Preprint, 2017. arXiv:1710.01098 [math.AP]

[55] Imbert, C.; Mouhot, C.; Silvestre, L. Decay estimates for large velocities in the Boltzmann equation without cutoff. Preprint, 2018. arXiv:1804.06135 [math.AP]

[56] Imbert, C.; Silvestre, L. The weak Harnack inequality for the Boltzmann equation without cutoff. *J. Eur. Math. Soc.* **22** (2020), no. 2, 507–592.

[57] Kim, C. Formation and propagation of discontinuity for Boltzmann equation in non-convex domains. *Comm. Math. Phys.* **308** (2011), no. 3, 641–701. doi:10.1007/s00220-011-1355-1

[58] Kim, C.; Lee, D. The Boltzmann equation with specular boundary condition in convex domains. *Comm. Pure Appl. Math.* **71** (2018), no. 3, 411–504. doi:10.1002/cpa.21705

[59] Kim, J.; Guo, Y.; Hwang, H. J. A L^2 to L^∞ approach for the Landau equation. Preprint, 2016. arXiv:1610.05346 [math.AP]

[60] Lei, Z.; Lin, F. Global mild solutions of Navier-Stokes equations. *Comm. Pure Appl. Math.* **64** (2011), no. 9, 1297–1304. doi:10.1002/cpa.20361

[61] Lions, P.-L. On Boltzmann and Landau equations. *Philos. Trans. Roy. Soc. London Ser. A* **346** (1994), no. 1679, 191–204. doi:10.1098/rsta.1994.0018

[62] Liu, J.-G.; Strain, R. M. Global stability for solutions to the exponential PDE describing epitaxial growth. *Interfaces Free Bound.* **21** (2019), no. 1, 61–86. doi:10.4171/IFB/417

[63] Liu, S.; Yang, X. The initial boundary value problem for the Boltzmann equation with soft potential. *Arch. Ration. Mech. Anal.* **223** (2017), no. 1, 463–541. doi:10.1007/s00205-016-1038-3

[64] Liu, T.-P.; Yang, T.; Yu, S.-H. Energy method for Boltzmann equation. *Phys. D* **188** (2004), no. 3-4, 178–192. doi:10.1016/j.physd.2003.07.011

[65] Morimoto, Y.; Sakamoto, S. Global solutions in the critical Besov space for the non-cutoff Boltzmann equation. *J. Differential Equations* **261** (2016), no. 7, 4073–4134. doi:10.1016/j.jde.2016.06.017

[66] Mouhot, C. De Giorgi-Nash-Moser and Hörmander theories: new interplay. *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, 2467–2493, World Sci. Publ., Hackensack, NJ, 2018.

[67] Mouhot, C.; Strain, R. M. Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff. *J. Math. Pures Appl. (9)* **87** (2007), no. 5, 515–535. doi:10.1016/j.matpur.2007.03.003

[68] Nishimura, K. Global existence for the Boltzmann equation in $L_v^r L_t^\infty L_x^\infty$ spaces. Preprint, 2018. arXiv:1806.01692 [math.AP]

[69] Pao, Y.-p. Boundary-value problems for the linearized and weakly nonlinear Boltzmann equation. *J. Mathematical Phys.* **8** (1967), 1893–1898. doi:10.1063/1.1705434

[70] Patel, N.; Strain, R. M. Large time decay estimates for the Muskat equation. *Comm. Partial Differential Equations* **42** (2017), no. 6, 977–999. doi:10.1080/03605302.2017.1321661

[71] Shizuta, Y.; Asano, K. Global solutions of the Boltzmann equation in a bounded convex domain. *Proc. Japan Acad. Ser. A Math. Sci.* **53** (1977), no. 1, 3–5.

[72] Silvestre, L. A new regularization mechanism for the Boltzmann equation without cut-off. *Comm. Math. Phys.* **348** (2016), no. 1, 69–100. doi:10.1007/s00220-016-2757-x

[73] Sohinger, V.; Strain, R. M. The Boltzmann equation, Besov spaces, and optimal time decay rates in $\mathbb{R}_x^n$. *Adv. Math.* **261** (2014), 274–332. doi:10.1016/j.aim.2014.04.012

[74] Strain, R. M. Optimal time decay of the non cut-off Boltzmann equation in the whole space. *Kinet. Relat. Models* **5** (2012), no. 3, 583–613. doi:10.3934/krm.2012.5.583

[75] Strain, R. M.; Guo, Y. Almost exponential decay near Maxwellian. *Comm. Partial Differential Equations* **31** (2006), no. 1-3, 417–429. doi:10.1080/03605300500361545

[76] Strain, R. M.; Guo, Y. Exponential decay for soft potentials near Maxwellian. *Arch. Ration. Mech. Anal.* **187** (2008), no. 2, 287–339. doi:10.1007/s00205-007-0067-3

[77] Strain, R. M.; Zhu, K. The Vlasov-Poisson-Landau system in $\mathbb{R}_x^3$. *Arch. Ration. Mech. Anal.* **210** (2013), no. 2, 615–671. doi:10.1007/s00205-013-0658-0

[78] Ukai, S. On the existence of global solutions of mixed problem for non-linear Boltzmann equation. *Proc. Japan Acad.* **50** (1974), 179–184.

[79] Ukai, S. Solutions of the Boltzmann equation. *Patterns and waves*, 37–96. Studies in Mathematics and Its Applications, 18. North-Holland, Amsterdam, 1986. doi:10.1016/S0168-2024(08)70128-0

[80] Ukai, S.; Yang, T. The Boltzmann equation in the space $L^2 \cap L_\beta^\infty$: global and time-periodic solutions. *Anal. Appl. (Singap.)* **4** (2006), no. 3, 263–310. doi:10.1142/S0219530506000784

[81] Vidav, I. Spectra of perturbed semigroups with applications to transport theory. *J. Math. Anal. Appl.* **30** (1970), 264–279. doi:10.1016/0022-247X(70)90160-5

[82] Villani, C. On the Cauchy problem for Landau equation: sequential stability, global existence. *Adv. Differential Equations* **1** (1996), no. 5, 793–816.

[83] Villani, C. On a new class of weak solutions to the spatially homogeneous Boltz-
     mann and Landau equations. *Arch. Rational Mech. Anal.* **143** (1998), no. 3, 273–307.
     doi:10.1007/s002050050106

[84] Villani, C. A review of mathematical topics in collisional kinetic theory. *Handbook of mathe-
     matical fluid dynamics, Vol. I*, 71–305. North-Holland, Amsterdam, 2002. doi:10.1016/S1874-
     5792(02)80004-0

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