

MOVING BOUNDARY PROBLEMS

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ABSTRACT. Moving boundary problems are ubiquitous in nature, technology and engineering. Examples include the human heart and heart valves interacting with blood flow, biodegradable micro-beads swimming in water to clean up water pollution, a micro camera in the human intestine used for an early colon cancer detection, and design of next generation vascular stents to prop open the clogged arteries, and prevent heart attacks. These are time-dependent, dynamic processes, which involve the interaction between fluids and various structures. Analysis and numerical simulation of fluid-structure interaction (FSI) problems can provide insight into the “invisible” properties of flows and structures, and can be used to advance design of novel technologies and improve the understanding of many physical and biological phenomena. Mathematical analysis of FSI models is at the core of this understanding. In this paper we give a brief survey of the recent progress in the area of mathematical well-posedness for moving boundary problems describing fluid-structure interaction between incompressible, viscous fluids and elastic, viscoelastic, and rigid solids.

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1. INTRODUCTION

Moving boundary problems are time-dependent problems describing the motion of a quantity such as fluid in a domain which is moving due to e.g., the domain’s exterior boundary motion, or the motion of an immersed structure within the fluid,

Date: November 2019.

2000 Mathematics Subject Classification. Primary 76D05, 76D03; Secondary 74F10, 76D27.

Key words and phrases. Moving boundary problems, fluid-structure interaction.

The author was supported in part by NSF Grants DMS-1853340 and DMS-1613757.

or both. An immersed structure within the fluid defines an interior fluid domain boundary described by the fluid-structure interface location, which is, in general, time dependent. Moving boundary problems can be classified into two types. The first type is a moving boundary problem in which the domain boundary motion is given *a priori*, and the second is a moving boundary problem in which the motion of the domain boundary is not known *a priori*, but is one of the unknowns in the problem. In both cases the flow inside the domain is strongly affected by the motion of the domain boundary, while in the second case the motion of the domain boundary is simultaneously adjusted as it is being impacted by the flow inside the domain. In this case, two sets of coupling conditions need to be prescribed to capture this *two-way* coupling: the kinematic condition describing the coupling of the kinematic quantities such as a velocity, and the dynamic coupling condition describing the dynamic balance of forces. Depending on the problem at hand, this two-way coupling can be evaluated along the “current” location of the moving boundary, or along the fixed domain boundary. The coupling along a fixed domain boundary is said to be linear (i.e., it is linearized around the fixed, reference domain configuration), while the coupling along the current location of the moving boundary not known *a priori* is said to be nonlinear. See Fig. 1. Needless to say, the nonlinear coupling between

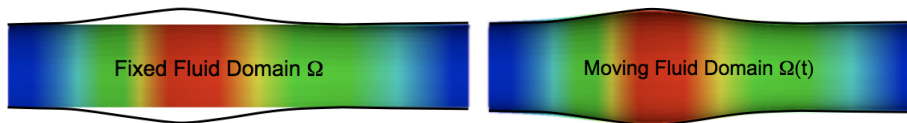


FIGURE 1. Fluid flow in a cylinder moving from left to right. The colors show the magnitude of pressure. The figure on the left shows the fluid domain in a *linearly coupled* fluid-structure interaction problem. The figure on the right shows the fluid domain in the *nonlinearly coupled* fluid-structure interaction problem.

the flow and boundary motion gives rise to an exceedingly complicated nonlinear moving boundary problem, for which the theory of existence and uniqueness of solutions, and their continuous dependence on data, has only recently become the focus of a systematic mathematical study. Thus, it is of major interest to develop a general framework for the study of solutions to moving boundary problems.

In this paper we survey some recent developments and open problems in this area. In particular, we focus on problems arising from the interaction between incompressible, viscous fluids and elastic, viscoelastic, or rigid solids (also referred to as “structures”). Fluid-structure interaction (FSI) problems are ubiquitous in nature, technology and engineering: from environmental science where pollutant concentration is studied in aquifers through poroelastic media, to biomedical engineering and cardiovascular medicine where, e.g., design of vascular stents for the treatment of coronary artery disease or heart valve replacement are studied.

Interestingly enough, even though the mathematical theory of the motion of bodies in a liquid is one of the oldest and most classical problems in fluid mechanics, mathematicians have only recently become interested in a systematic study of the basic problems related to fluid-structure interaction. One reason for this may be that problems of this type are notoriously difficult to study. In addition to the nonlinearity in the fluid and possibly the structure equations, the nonlinear coupling

between the fluid and structure motion may give rise to strong geometric nonlinearities. The mathematical study of existence of solutions to the coupled problems must account for the nonlinearities due to the strong energy exchange between the fluid and (elastic) structure motion in problems with nonlinear coupling, and employ novel compactness arguments to deal with the nonlinearities. Due to the fluid domain motion, the compactness results must hold for *a family of operators* defined on *time-dependent function spaces* associated with moving domains not known *a priori*. Additionally, the compactness arguments must account for the fact that the coupled problem involves two sets of equations of different type (parabolic vs. hyperbolic) accounting for the different physics in the problem. Crucial for the existence proofs and the compactness arguments is to make use of the parabolic regularizing effects (by the fluid viscosity) to keep the high frequency oscillations of the (hyperbolic) structure under control.

In existence proofs, and in numerical schemes, an additional difficulty is imposed by the incompressibility of the fluid. The main difficulty in existence proofs is related to the construction of divergence-free extensions of fluid velocity to a larger domain containing all the moving domains, and obtaining quantitative estimates of the extensions in terms of the changing geometry. Incompressibility is intimately related to the pressure, and pressure is a major component of the load, i.e., contact force, exerted by the fluid onto the solid. Designing constructive existence proofs and numerical schemes that approximate the load “correctly” is a key ingredient for the stability of constructive solution schemes. In particular, the fluid surrounding the structure affects the structure motion as an extra mass that the structure must displace when moving within a fluid. This has long been known in engineering as the “added mass effect”. Not accounting for the added mass effect can have negative impact on the stability of partitioned FSI schemes, and is a well-known problem in FSI problems for which the density of the structure is less than or equal to that of the fluid, i.e., for which the structure is “light” with respect to the fluid. The added mass is a leading order effect in biofluidic FSI problems, since biological tissues (structures) have density which is approximately the same as that of the surrounding fluid. A failure to account for this effect is associated with the lack of uniform energy estimates in constructive existence proofs for nonlinearly coupled FSI problems.

The question of **global-in-time existence** of solutions to moving boundary problems is affected by two open problems. One is inherited from the Navier-Stokes equations and the open outstanding question of global existence of strong solutions, and the other is related to the so called “no collision” paradox: global weak solution existence results for moving boundary problems are typically obtained until a possible fluid domain degeneracy occurs, such as, e.g., collapse of the cylindrical tube leading to the cross-sectional area of the tube approaching zero. The problems of finite time contact between elastic bodies in a viscous, incompressible fluid remains an open question, known as the “no-collision paradox”. As we shall see below, various questions related to the no-collision paradox are being investigated, including the possibility of finite time contact for “classical” models, investigation and design of mathematical models that would allow finite time contact, and the type of boundary conditions (no-slip versus slip) for which finite-time contact may occur.

Nonlinearities in the coupled FSI problem also affect the study of **uniqueness** of solutions. It is not surprising that uniqueness of weak solutions to the coupled FSI problems is still largely an outstanding open problem, since even in the case of classical 3D Navier-Stokes equations, the uniqueness of the Leray-Hopf weak solutions has not been resolved. However, recent advances in this area are significant, and we summarize those results below.

To explain the main challenges in more detail, we present a benchmark problem for FSI involving elastic structures, and a benchmark problem for FSI involving rigid solids, and provide a literature review of the recent results.

2. FSI WITH ELASTIC STRUCTURES

Although the development of numerical methods for fluid-structure interaction problems started almost 40 years ago (see e.g., [42, 131, 113, 114, 46, 73, 74, 41, 87, 117, 9, 8, 90, 47, 86, 73, 74, 75, 76, 78] and the references therein), the development of existence theory for FSI problems started less than 20 years ago. We state a benchmark problem in this field, and summarize some recent results and open problems.

Description of the Main Problem. To describe the interaction between a fluid and an elastic (or viscoelastic) structure across a moving interface mathematically, two types of coupling conditions have to be prescribed. This contrasts classical fluid dynamics problems defined on fixed domains where only one boundary condition, e.g., the no-slip condition, is sufficient to define the problem. As mentioned earlier, the two sets of coupling conditions describe: (1) how the kinematic quantities, such as velocity, are coupled (the *kinematic coupling condition*, e.g., no-slip), and (2) the elastodynamics of the fluid-structure interface (the *dynamic coupling condition*). While the precise form of the kinematic and dynamic coupling conditions depends on the particular application at hand, the most common coupling is done via the no-slip kinematic condition, stating that the fluid and structure velocities are continuous across the moving interface, and the dynamic coupling condition stating that the fluid-structure interface, namely the moving boundary, is driven by the jump in traction, i.e., normal stress, across the interface. For problems in which one expects small interface displacements and small displacement gradients, the coupling conditions may be evaluated at a fixed interface, without changing the fluid domain, namely, the fluid and structure may be *linearly coupled* [45, 10, 11, 93]. For problems where this may not be a good approximation of reality, the coupling conditions must be evaluated across the moving interface, giving rise to an additional nonlinearity in the problem, which is due to the change of geometry of the moving boundary, namely, the fluid and structure are *nonlinearly coupled*. In the latter case the fluid domain is a function of time, and additionally, it is not known *a priori* since it depends on the unknowns in the problem, namely, the displacement of the fluid-structure interface.

The geometric nonlinearity, associated with the fluid domain motion not known a priori, presents one of the major, new difficulties in studying this class of problems mathematically.

2.1. FSI benchmark problem with no-slip. The simplest example of a moving boundary problem with nonlinear coupling involving a deformable (elastic) structure, is a benchmark problem deriving from modeling blood flow in a segment of an artery. The fluid domain is a cylinder, with an elastic (viscoelastic) lateral boundary. For simplicity, we present the problem in 2D, although 3D versions of the problem have been studied in e.g., [103]. In this benchmark problem, we will be assuming that the lateral boundary is thin, with small thickness $h \ll 1$, and with the *reference configuration* Γ corresponding to a straight cylinder of length L and radius R :

$$\Gamma = \{(z, R) | z \in (0, L)\}.$$

In most literature involving FSI with thin structures, except for the recent results in [23, 106], the lateral boundary is assumed to displace only in the vertical (normal, transverse) direction, rendering longitudinal displacement negligible. By using η to denote the vertical component of displacement, the fluid domain $\Omega^\eta(t)$ can be described by:

$$\Omega^\eta(t) = \{(z, r) \in \mathbb{R}^2 | z \in (0, L), r \in (0, R + \eta(t, z))\},$$

where R is the radius of the reference cylinder. The *reference fluid domain* will be denoted by $\Omega = (0, L) \times (0, R)$.

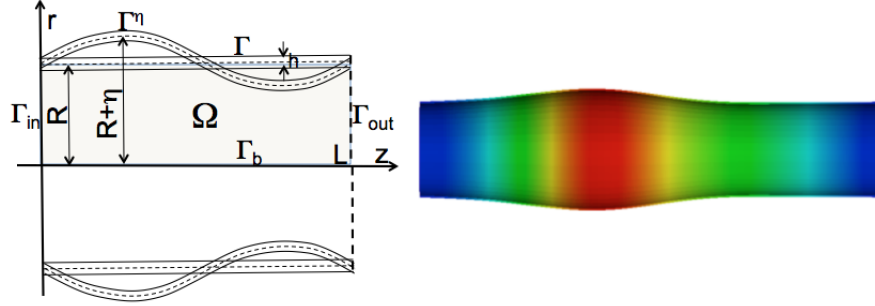


FIGURE 2. Left: A sketch of the fluid domain and the moving lateral boundary. Right: Pressure wave propagation in elastic tube.

The fluid flow is modeled by the Navier-Stokes equations for an incompressible, viscous fluid, defined on a moving domain $\Omega^\eta(t)$, not known *a priori*:

$$(2.1) \quad \left. \begin{aligned} \rho_f (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) &= \nabla \cdot \boldsymbol{\sigma} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \right\} \text{ in } \Omega^\eta(t), \quad t \in (0, T),$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, ρ_f is the fluid density, and $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_z, u_r)$ is the fluid velocity. For Newtonian fluids

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}),$$

where μ is the dynamic viscosity coefficient, and $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ is the symmetrized gradient of \mathbf{u} .

In this benchmark problem the flow is driven by the inlet and outlet dynamic pressure data, and the flow is normal to the inlet and outlet boundary $\Gamma_{in} =$

$\{0\} \times (0, R)$ and $\Gamma_{out} = \{L\} \times (0, R)$:

$$(2.2) \quad \left. \begin{aligned} p + \frac{\rho_f}{2}|u|^2 &= P_{in/out}(t), \\ u_r &= 0, \end{aligned} \right\} \quad \text{on } \Gamma_{in/out},$$

where $P_{in/out} \in L^2_{loc}(0, \infty)$ are given. At the bottom boundary $\Gamma_b = (0, L) \times \{0\}$ the symmetry boundary conditions are prescribed:

$$(2.3) \quad u_r = \partial_r u_z = 0, \quad \text{on } \Gamma_b.$$

See Fig. 2 left.

Different inlet/outlet boundary conditions have been used in numerical simulations, including Dirichlet data given in terms of the prescribed fluid velocity, and Neumann data given in terms of the prescribed normal stress. Prescribing the “correct” boundary conditions, especially at the outflow boundary, is very important in correctly capturing the physiological flow conditions within a sub-region of the cardiovascular system that is being modeled [117]. From the analysis point of view, it is now known that some of the “numerically convenient” outlet boundary conditions, such as the Neumann, or “do nothing” outlet boundary conditions, may produce instabilities [132] or ruin well-posedness by producing multiple solutions to the steady flow, as was recently shown in [77]. In this paper we consider dynamic pressure data (2.2) both at the inlet and at the outlet, which is a boundary condition “consistent” with the energy of the coupled problem.

Under the fluid loading, and possibly some external loading, the elastic cylinder deforms. See Fig 2. We denote by:

$$(2.4) \quad \Gamma^\eta(t) = \{(t, z, R + \eta(t, z)) | z \in (0, L)\}$$

the location of the deformed cylinder lateral boundary at time t . The elastic properties of the cylinder’s lateral wall can be described by an operator \mathcal{L}_e , so that the elastodynamics problem, in Lagrangian formulation, can be written as:

$$(2.5) \quad \rho_s h \partial_{tt} \eta + \mathcal{L}_e \eta = f, \quad \text{on } \Gamma, \quad t \in (0, T),$$

where ρ_s is the structure density, h is the thin structure thickness, and f is the vertical component of the external loading (force density) experienced by the elastic structure. The loading f in the coupled problem will come from the jump in the normal stress across the structure, i.e., from the fluid load experienced by the structure (assuming that the external loading is zero). The operator \mathcal{L}_e is associated with the elastic energy of the structure, such as the membrane or shell energy, see [102], and is typically continuous, positive-definite, and coercive on some Hilbert space χ .

The coupling. The fluid flow influences the motion of the structure through traction forces, i.e., by the normal stress exerted onto the structure at $\Gamma^\eta(t)$, while the structure influences the fluid through its inertial and elastic forces due to the structure motion and stretching/recoil. Additionally, the fluid and structure “feel” each other through the continuity of the fluid and structure velocities at the interface. Thus, the kinematic and dynamic coupling conditions, respectively, are:

$$(2.6) \quad (\partial_t \eta(t, z), 0) = \mathbf{u}(t, z, R + \eta(t, z)),$$

$$(2.7) \quad \rho_s h \partial_t^2 \eta + \mathcal{L}_e \eta = -J (\boldsymbol{\sigma} \mathbf{n})|_{(t, z, R + \eta(t, z))} \cdot \mathbf{e}_r,$$

where $J = \sqrt{1 + \left(\frac{\partial \eta}{\partial t}\right)^2}$, is the Jacobian of the transformation from Eulerian coordinates to Lagrangian coordinates, and \mathbf{e}_r is the unit vector in the vertical direction. Here, we have assumed that the external forcing onto the structure is zero. Generalizations to include external forcing due to the presence of another elastic structure or other types of forcing can be found in [104].

The geometric nonlinearity due to the fluid domain motion, described by the composite function $\mathbf{u}(t, z, R + \eta(t, z))$, is *generally* handled by introducing a family of mappings, parameterized by time, called the Arbitrary Lagrangian-Eulerian (ALE) mappings, discussed below in Sec. 3. In terms of ALE mappings, the trace of the fluid velocity on $\Gamma^\eta(t)$ is described by a composite function between the velocity and the ALE mapping.

Equations (2.1)–(2.7) define a nonlinear moving-boundary problem for the unknown functions \mathbf{u} and η . The problem is supplemented with initial conditions:

$$(2.8) \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \eta(0, \cdot) = \eta_0, \quad \partial_t \eta(0, \cdot) = v_0,$$

satisfying the compatibility conditions:

$$(2.9) \quad \begin{aligned} \mathbf{u}_0(z, R + \eta_0(z)) &= v_0(z) \mathbf{e}_r, \quad z \in (\Gamma), \\ \eta_0(0) &= \eta_0(L) = v_0(0) = v_0(L) = 0, \\ R + \eta_0(z) &> 0, \quad z \in [0, L]. \end{aligned}$$

Thus, the **benchmark nonlinear moving-boundary problem**, which exemplifies the main difficulties associated with studying moving boundary problems with nonlinear coupling, can be summarized as follows: Find $\mathbf{u} = (u_z(t, z, r), u_r(t, z, r))$, p , and $\eta(t, z)$ such that:

$$\begin{aligned} \left. \begin{aligned} \rho_f (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) &= \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, p) \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \right\} &\text{ in } \Omega^\eta(t), \quad t \in (0, T), \\ \left. \begin{aligned} \mathbf{u}|_{\Gamma^\eta(t)} &= \partial_t \eta \mathbf{e}_r, \\ \rho_s h \partial_t^2 \eta + \mathcal{L}_e \eta &= -J^\eta \boldsymbol{\sigma} \mathbf{n}|_{\Gamma^\eta(t)} \cdot \mathbf{e}_r, \end{aligned} \right\} &\text{ on } (0, T) \times \Gamma, \\ \left. \begin{aligned} u_r &= 0, \\ \partial_r u_z &= 0, \end{aligned} \right\} &\text{ on } (0, T) \times \Gamma_b, \\ \left. \begin{aligned} p + \frac{\rho_f}{2} |u|^2 &= P_{in/out}(t), \\ u_r &= 0, \end{aligned} \right\} &\text{ on } (0, T) \times \Gamma_{in/out}, \\ \eta(t, 0) = \partial_z \eta(t, 0) = \eta(t, L) = \partial_z \eta(t, L) &= 0 \quad \text{ on } (0, T) \\ \left. \begin{aligned} \mathbf{u}(0, \cdot) &= \mathbf{u}_0, \\ \eta(0, \cdot) &= \eta_0, \\ \partial_t \eta(0, \cdot) &= v_0. \end{aligned} \right\} &\text{ at } t = 0. \end{aligned}$$

The energy. This benchmark problem satisfies the following formal energy inequality:

$$(2.10) \quad \frac{d}{dt} E(t) + D(t) \leq C(P_{in}(t), P_{out}(t)),$$

where $E(t)$ denotes the sum of the kinetic energy of the fluid and of the structure, and the elastic energy of the membrane shell:

$$(2.11) \quad E(t) = \frac{\rho_f}{2} \|\mathbf{u}\|_{L^2(\Omega^\eta(t))}^2 + \frac{\rho_s h}{2} \|\partial_t \eta\|_{L^2(\Gamma)}^2 + \langle \mathcal{L}_e \eta, \eta \rangle,$$

where $\langle \mathcal{L}_e \eta, \eta \rangle$ corresponds to the elastic energy of the structure, which for the cylindrical Koiter shell allowing only radial displacement reads:

$$\langle \mathcal{L}_e \eta, \eta \rangle := \frac{1}{2} (C_0 \|\eta\|_{L^2(\Gamma)}^2 + C_1 \|\partial_z \eta\|_{L^2(\Gamma)}^2 + C_2 \|\partial_z^2 \eta\|_{L^2(\Gamma)}^2).$$

The term $D(t)$ captures dissipation due to fluid viscosity:

$$(2.12) \quad D(t) = \mu \|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega^\eta(t))}^2,$$

and $C(P_{in}(t), P_{out}(t))$ is a constant which depends only on the inlet and outlet pressure data, which are both functions of time.

2.2. FSI benchmark problem with Navier-slip. While the assumption on the continuity of normal velocity components is reasonable for impermeable boundaries:

$$(2.13) \quad (\partial_t \boldsymbol{\eta} - \mathbf{u}) \cdot \mathbf{n} = 0 \quad (\text{non-penetration condition}),$$

the continuity of the tangential velocity component in the no-slip condition is justified only when molecular viscosity is considered [99]. Navier contested the no-slip condition for Newtonian fluids [109] when he claimed that the tangential, slip velocity should be proportional to the shear stress. For moving boundary problems this means that the jump in the tangential components of the fluid and solid velocities at the moving boundary is proportional to the shear stress:

$$(2.14) \quad (\partial_t \boldsymbol{\eta} - \mathbf{u}) \cdot \boldsymbol{\tau} = \alpha \boldsymbol{\sigma} \mathbf{n} \cdot \boldsymbol{\tau} \quad (\text{Navier slip}),$$

where \mathbf{n} and $\boldsymbol{\tau}$ to denote the unit normal and tangent to the fluid domain boundary, respectively, $\boldsymbol{\sigma}$ is the fluid Cauchy stress tensor, and α is the proportionality constant known as the slip length; $1/\alpha$ has the units of friction.

The benchmark problem defined on the domain shown in Fig. 2, incorporating the Navier-slip condition as the kinematic coupling condition, can be summarized as follows: Find $(\mathbf{u}, p, \boldsymbol{\eta})$ such that the following holds

The fluid equations:

$$\left. \begin{aligned} \rho_F (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) &= \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, p), \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \right\} \text{ in } \Omega^\eta(t), \quad t \in (0, T);$$

The elastic structure (Navier-slip coupling on $(0, L) \times (0, T)$):

$$\begin{aligned} \rho_S h \partial_{tt} \boldsymbol{\eta}(t, z) + \mathcal{L}_e \boldsymbol{\eta}(t, z) &= -J(t, z) \boldsymbol{\sigma}(\boldsymbol{\varphi}(t, z)) \mathbf{n}(t, z), \\ \partial_t \boldsymbol{\eta}(t, z) \cdot \mathbf{n}(t, z) &= \mathbf{u}(\boldsymbol{\varphi}(t, z)) \cdot \mathbf{n}(t, z), \\ (\partial_t \boldsymbol{\eta}(t, z) - \mathbf{u}(\boldsymbol{\varphi}(t, z))) \cdot \boldsymbol{\tau}(t, z) &= \alpha \boldsymbol{\sigma}(\boldsymbol{\varphi}(t, z)) \mathbf{n}(t, z) \cdot \boldsymbol{\tau}(t, z), \end{aligned}$$

with

$$\boldsymbol{\eta}(t, 0) = \partial_z \boldsymbol{\eta}(t, 0) = \boldsymbol{\eta}(t, L) = \partial_z \boldsymbol{\eta}(t, L) = 0, \quad t \in (0, T);$$

Boundary data at the inlet/outlet boundary $\Gamma_{in/out} \times (0, T)$:

$$p + \frac{\rho_F}{2} |\mathbf{u}|^2 = P_i, \quad \mathbf{u} \cdot \boldsymbol{\tau} = 0.$$

Boundary data at the bottom, symmetry boundary $\Gamma_b \times (0, T)$:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \partial_{\mathbf{n}} u_\tau = 0.$$

with u_τ denoting the tangential component of velocity \mathbf{u} .

Initial conditions:

$$(2.15) \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \boldsymbol{\eta}(0, \cdot) = \boldsymbol{\eta}_0, \quad \partial_t \boldsymbol{\eta}(0, \cdot) = \mathbf{v}_0.$$

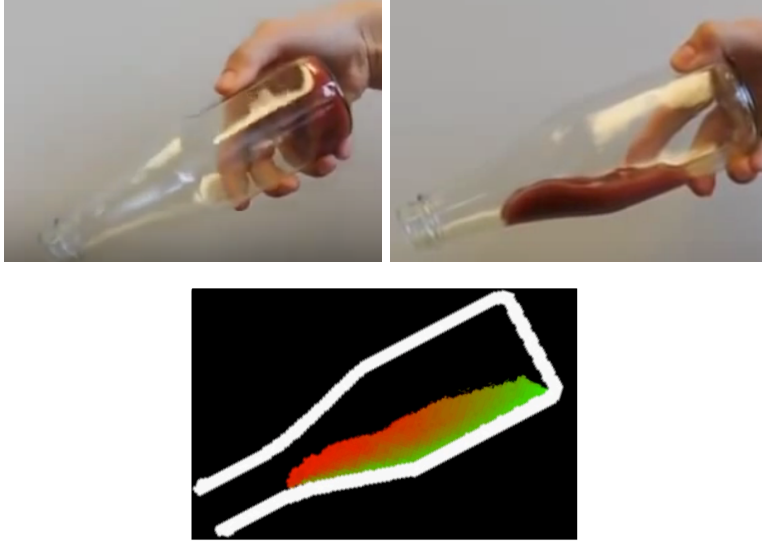


FIGURE 3. Hydrophilic vs. hydrophobic surfaces: no-slip vs. slip condition. Left: Classical Ketchup flow. Right: Ketchup flowing in a bottle treated with a no-stick coating. Bottom: Numerical simulation (by Čanić and Wang) of flow with the Navier slip boundary condition. The three snap-shots were taken at the same time after the bottled had been tilted downwards. The colors in the third panel denote magnitude of fluid velocity: red is high, green is low velocity.

The following **energy estimate holds**:

$$\begin{aligned}
 (2.16) \quad & \frac{1}{2} \frac{d}{dt} (\rho_F \|\mathbf{u}\|_{L^2(\Omega^\eta(t))}^2 + \rho_S h \|\partial_t \boldsymbol{\eta}\|_{L^2(\Gamma)}^2 + c \|\boldsymbol{\eta}\|_{\chi(\Gamma)}^2) \\
 & + \mu \|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega^\eta(t))}^2 + \frac{1}{\alpha} \|u_\tau - \partial_t \eta_\tau\|_{L^2(\Gamma^\eta(t))}^2 \leq \mathbf{C},
 \end{aligned}$$

where \mathbf{C} depends on the initial and boundary data, and constant c in front of the χ -norm of $\boldsymbol{\eta}$ is associated with the coercivity of the structure operator \mathcal{L}_e . The reference configuration of the lateral boundary Γ is $\Gamma = (0, L) \times \{R\}$.

The no-slip condition is reasonable for a great variety of problems for which the slip length α is indeed very close to zero. However, in many cases of practical significance no-slip is not adequate. Examples include flows over hydrophobic surfaces or surfaces treated with a no-stick coating, see Fig. 3, flows over “rough” surfaces such as those of, e.g., grooved vascular tissue scaffolds, and problems involving contact of smooth solids immersed in a viscous, incompressible fluid. More precisely, for flows over “rough” (rigid and fixed) surfaces it has been shown that the Navier slip boundary condition is the appropriate “effective boundary condition” [99, 100]. Instead of using the no-slip condition at the small groove scale, the effective Navier slip boundary condition is applied at the corresponding “groove-free” *smooth* boundary [99, 100]. Regarding contact of smooth bodies immersed in a viscous, incompressible fluid, recent studies have shown that contact is not possible if the no-slip boundary condition is considered [81, 82, 122]. A resolution to this no-collision paradox is to employ a different boundary condition, such as the

Navier slip boundary condition, which allows contact between smooth rigid bodies [112]. Problems of this type arise, e.g., in modeling elastic heart valve closure where different kinds of *ad hoc* “gap” conditions with the no-slip boundary condition have been used to get around this difficulty.

3. THE ARBITRARY LAGRANGIAN-EULERIAN (ALE) MAPPINGS.

To deal with the problems associated with the motion of the fluid domain, different approaches have been taken. One approach is to consider the entire moving boundary problem written in Lagrangian coordinates, as was done in [91, 30, 29]. This is possible to do when the fluid domain is contained in a “closed container” and no fluid escapes the fluid domain, which is not the case with the benchmark problem, considered above. Another approach is to map the problems from the moving domain onto a fixed reference domain, using a family of mappings, known as the Arbitrary Lagrangian-Eulerian (ALE) mappings.

The Arbitrary Lagrangian-Eulerian mapping is a family of (diffeomorphic) mappings, parameterized by t , such that

$$(3.1) \quad \mathcal{A}^\eta(t) : \Omega \rightarrow \Omega^\eta(t).$$

ALE mappings have been extensively used in numerical simulations of moving boundary problems, see e.g. [41, 42, 131]. Recently, they have proven to be useful in mathematical analysis as well [102, 23, 103]. In numerics, one of the reasons for the introduction of ALE mappings is the calculation of the the discretized time derivative $\partial_t \mathbf{u}$ since a finite difference approximation of the time derivative, e.g., $(\mathbf{u}^{n+1} - \mathbf{u}^n)/\Delta t$, contains the functions \mathbf{u}^{n+1} and \mathbf{u}^n which are defined on two different domains, one corresponding to the time $t^{n+1} = (n+1)\Delta t$, and the other to $t^n = n\Delta t$. A way to calculate the time derivative of the fluid velocity is then to map the fluid velocities at times t^{n+1} and t^n onto a fixed domain Ω via the ALE mappings corresponding to t^{n+1} and t^n , evaluate the time derivative there, and then map everything back to the physical domain $\Omega^n := \Omega^{\eta^n}(t^n)$ to solve the fluid equations on the “current” domain Ω^n . This introduces an extra advection term in the Navier-Stokes equations, describing the contribution due to the fluid domain motion, so that the Navier-Stokes equations in ALE form become:

$$(3.2) \quad \left. \begin{aligned} \rho_f (\partial_t \mathbf{u} + ((\mathbf{u} - \mathbf{w}^{\eta^n}) \cdot \nabla) \mathbf{u}) &= \nabla \cdot \sigma, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \right\} \text{ in } (t^n, t^{n+1}) \times \Omega^n,$$

where $\mathbf{w}^{\eta^n} = \partial_t \mathcal{A}^{\eta^n}(t^n)$ describes the fluid domain velocity, defined by the time derivative of the ALE mapping. In numerical solvers, the ALE mapping is often defined by the harmonic extension of the boundary data onto the fluid domain, i.e., as a solution to the following elliptic problem:

$$\begin{aligned} \Delta \mathcal{A}^\eta(t) &= 0 \text{ on } \Omega, \\ \mathcal{A}^\eta(t) &= \eta(t) \text{ on } \Gamma, \\ \mathcal{A}^\eta(t) &= 0 \text{ on } \partial\Omega \setminus \Gamma, \end{aligned}$$

calculated at every time step $t = t^n$. Other elliptic (elastic) operators have also been used.

In addition to numerical solvers, the ALE approach has recently been used to study the existence of solutions to moving boundary problems by either mapping the entire fluid problem onto the fixed domain Ω via an ALE mapping and analyzing the problem there, as in [102, 103, 104], or by mimicking the approach described

above, used in numerical simulations, and working on the “current” domain Ω^n , as in [23]. In the former case, an additional set of nonlinearities is introduced because the gradient operator ∇ in physical space is mapped into an operator ∇^η , defined on the fixed, reference domain. The mapped fluid velocity is no longer divergence-free in terms of the operator ∇ , which presents some difficulties when trying to use known results that hold for divergence-free functions.

Using ALE mappings in analysis requires assumptions on its regularity, which, of course, depends on the regularity of the fluid-structure interface η , not known *a priori*. In most works that use ALE mappings, an assumption on the regularity of the ALE mapping is made *a priori*, which is then justified *a posteriori*, after a proof showing existence of a solution with sufficient regularity of η is obtained.

For the benchmark problem with no-slip coupling presented above in Sec. 2.1, one can introduce a family of ALE mappings, parameterized by η , given explicitly by:

$$(3.3) \quad \mathcal{A}^\eta(t) : \Omega \rightarrow \Omega^\eta(t), \quad \mathcal{A}^\eta(t)(\tilde{z}, \tilde{r}) := \begin{pmatrix} \tilde{z} \\ (R + \eta(t, \tilde{z}))\tilde{r} \end{pmatrix}, \quad (\tilde{z}, \tilde{r}) \in \Omega,$$

where (\tilde{z}, \tilde{r}) denote the coordinates in the reference domain $\Omega = (0, L) \times (0, R)$. Mapping $\mathcal{A}^\eta(t)$ is a bijection, and its Jacobian is given by

$$(3.4) \quad \mathcal{J}_{\mathcal{A}^\eta} = |\det \nabla \mathcal{A}^\eta(t)| = |R + \eta(t, \tilde{z})|.$$

Composite functions with the ALE mapping will be denoted by

$$(3.5) \quad \mathbf{u}^\eta(t, \cdot) = \mathbf{u}(t, \cdot) \circ \mathcal{A}^\eta(t) \quad \text{and} \quad p^\eta(t, \cdot) = p(t, \cdot) \circ \mathcal{A}^\eta(t).$$

The derivatives of composite functions satisfy:

$$\partial_t \mathbf{u} = \partial_t \mathbf{u}^\eta - (\mathbf{w}^\eta \cdot \nabla^\eta) \mathbf{u}^\eta, \quad \nabla \mathbf{u} = \nabla^\eta \mathbf{u}^\eta,$$

where the ALE domain velocity, \mathbf{w}^η , and the transformed gradient, ∇^η , are given by:

$$(3.6) \quad \mathbf{w}^\eta = \partial_t \eta \tilde{r} \mathbf{e}_r, \quad \nabla^\eta = \begin{pmatrix} \partial_{\tilde{z}} - \tilde{r} \frac{\partial_z \eta}{R + \eta} \partial_{\tilde{r}} \\ \frac{1}{R + \eta} \partial_{\tilde{r}} \end{pmatrix}.$$

Note that

$$(3.7) \quad \nabla^\eta \mathbf{v} = \nabla \mathbf{v} (\nabla A_\eta)^{-1}.$$

The following notation will also be useful:

$$\sigma^\eta = -p^\eta \mathbf{I} + 2\mu \mathbf{D}^\eta(\mathbf{u}^\eta), \quad \mathbf{D}^\eta(\mathbf{u}^\eta) = \frac{1}{2}(\nabla^\eta \mathbf{u}^\eta + (\nabla^\eta)^\tau \mathbf{u}^\eta).$$

The resulting problem, defined entirely on the fixed, reference domain, in ALE framework now reads: find $\mathbf{u}(t, \tilde{z}, \tilde{r})$, $p(t, \tilde{z}, \tilde{r})$ and $\eta(t, \tilde{z})$ such that

$$(3.8) \quad \left. \begin{aligned} \rho_f (\partial_t \mathbf{u}^\eta + ((\mathbf{u}^\eta - \mathbf{w}^\eta) \cdot \nabla^\eta) \mathbf{u}^\eta) &= \nabla^\eta \cdot \sigma^\eta, \\ \nabla^\eta \cdot \mathbf{u}^\eta &= 0, \end{aligned} \right\} \text{in } (0, T) \times \Omega,$$

$$\left. \begin{aligned} \mathbf{u}^\eta &= \partial_t \eta \mathbf{e}_r, \\ \rho_s h \partial_t^2 \eta + \mathcal{L}_e \eta &= -J \sigma^\eta \mathbf{n} \cdot \mathbf{e}_r, \end{aligned} \right\} \text{on } (0, T) \times \Gamma,$$

$$\left. \begin{aligned} u_r^\eta &= 0, \\ \partial_r u_z^\eta &= 0 \end{aligned} \right\} \text{on } (0, T) \times \Gamma_b,$$

$$\left. \begin{aligned} p + \frac{\rho_f}{2} |\mathbf{u}^\eta|^2 &= P_{in/out}(t), \\ u_r^\eta &= 0, \end{aligned} \right\} \text{ on } (0, T) \times \Gamma_{in/out},$$

$$\mathbf{u}^\eta(0, \cdot) = \mathbf{u}_0^\eta, \eta(0, \cdot) = \eta_0, \partial_t \eta(0, \cdot) = v_0.$$

A review of the recent result on existence theory related to this benchmark problem is presented next.

4. RECENT RESULTS AND OPEN PROBLEMS IN FSI WITH ELASTIC STRUCTURES

The development of existence theory for moving boundary, fluid-structure interaction problems started in the late 1990's/early 2000's. The first existence results were obtained for the cases in which the structure is completely immersed in the fluid, and the structure was considered to be either a rigid body, or described by a finite number of modal functions. See e.g., [15, 31, 36, 37, 39, 48, 57, 122], and the references therein. The analysis of the coupling between the 2D or 3D Navier-Stokes equations and 2D or 3D linear elasticity started in the early 2000's with the works in which the coupling between the fluid and structure was assumed across a fixed fluid-structure interface (*linear coupling*) as in [45, 10, 11, 93], and then extended to problems with *nonlinear coupling* in the works [14, 69, 96, 68, 25, 70, 102, 103, 95, 34, 35, 91, 30, 29]. More precisely, concerning nonlinear FSI models, the first FSI existence result, locally in time, was obtained in [14], where a **strong** solution for an interaction between an incompressible, viscous fluid in 2D and a 1D viscoelastic string was obtained, assuming periodic boundary conditions. This result was extended by Lequeurre in [96], where the existence of a unique, local in time, strong solution for any data, and the existence of a global strong solution for small data, was proved in the case when the structure is modeled as a clamped viscoelastic beam. D. Coutand and S. Shkoller proved existence, locally in time, of a unique, regular solution for an interaction between a viscous, incompressible fluid in 3D and a 3D structure, immersed in the fluid, where the structure was modeled by the equations of linear [34], or quasi-linear [35] elasticity. In the case when the structure (solid) is modeled by a linear wave equation, I. Kukavica et al. proved the existence, locally in time, of a strong solution, assuming lower regularity for the initial data [91, 88]. A similar result for compressible flows can be found in [92]. In [116] Raymod et al. considered a FSI problem between a linear elastic solid immersed in an incompressible viscous fluid, and proved the existence and uniqueness of a strong solution. Most of the above mentioned existence results for strong solutions are local in time. In [89] a global existence result for small data was obtained by Ignatova et al. for a moving boundary FSI problem involving a damped linear wave equation with some additional damping terms in the coupling conditions, showing exponential decay in time of the solution. In the case when the structure is modeled as a 2D elastic shell interacting with a viscous, incompressible fluid in 3D, the existence, locally in time, of a unique regular solution was proved by Shkoller et al. in [30, 29]. We mention that the works of Shkoller et al., and Kukavica et al. were obtained in the context of Lagrangian coordinates, which were used for both the structure and fluid subproblems.

In the context of **weak** solutions, the first existence results came out in 2005 when Chambolle et al. showed the existence of a weak solution for a FSI problem between a $3D$ incompressible, viscous fluid and a $2D$ viscoelastic plate in [25]. Grandmont improved this result in [70] to hold for a $2D$ elastic plate. A constructive existence proof for the interaction between an incompressible, viscous fluid and a linearly elastic Koiter shell with transverse displacement was designed in [102]. The first constructive existence proof for moving boundary problems was presented by Ladyzhenskaya in 1970 where the interaction between an incompressible, viscous fluid and a *given moving boundary* was constructed using a time-discretization approach, known as Rothe's method, assuming high regularity of the given interface [94]. Muha and Čanić designed a Rothe's-type method in the context of moving boundaries that are *not known a priori* in 2013 [102]. To deal with a moving boundary not known *a priori* they introduced the time discretization via Lie operator splitting, which has been used in numerical schemes, described in [66]. The FSI problem studied in [102] is split into a fluid and a structure subproblem, with the coupling designed so that the resulting scheme is stable. This was achieved by using the results about the "added mass effect", published in [24], which showed the importance of implicit treatment of the fluid and structure inertia in loosely coupled partitioned schemes. The splitting of the coupled problem in [102] was then done in such a way that the fluid and structure inertia terms are kept implicitly together, which provided uniform energy estimate, not otherwise attainable using the "classical" Dirichlet-Neumann" partitioned schemes [115, 118]. A compactness argument, discussed below in Section 5 was used to show that subsequence of approximate solutions converge to a weak solution to the coupled problem. After 2013 similar approaches were used to prove existence of a weak solution for a nonlinear FSI problem involving a nonlinear Koiter shell [105], a multi-layered structure [104], and a Koiter shell with the Navier slip condition [106].

To complete the discussion of well-posedness, we mention here a result on continuous dependence of weak solutions on initial data, obtained in [79] for a fluid structure interaction problem with a free boundary type coupling condition.

In all these works existence of a weak solution was proved for as long as the elastic boundary does not touch "the bottom" (rigid) portion of the fluid domain boundary. Recently, Grandmont and Hillairet showed that contact between a rigid bottom of a fluid container, and a *viscoelastic beam*, is not possible in finite time [71]. The finite-time contact involving thin and thick elastic structures interacting with an incompressible, viscous fluid is still **open**.

We conclude this section with a few general remarks related to the geometric non-linearity in FSI problems for which the coupling across the current location of the moving interface is needed to describe the physical problem. The strong exchange of energy between the fluid and structure motion in the nonlinearly coupled problems gives rise to the various difficulties in the study of mathematical well-posedness. In particular, the functional spaces based on the finite energy considerations may not provide sufficient regularity of the moving interface to even define the trace of the fluid velocity at the fluid-structure interface, and additionally, may lead to various fluid domain degeneracies, as shown in Fig. 4. These problems are particularly evident when elastic structures are thin (modeled by the reduced membrane or shell equations) and the structure model accounts for both transverse and tangential components of displacement, and the structure is interacting in $3D$ with the flow of

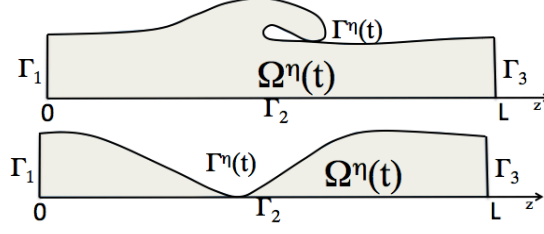


FIGURE 4. Two ways the fluid domain can degenerate. Top: loss of injectivity of the ALE mapping $\mathcal{A}^\eta(t)$. Bottom: loss of injectivity of the ALE mapping $\mathcal{A}^\eta(t)$ and loss of strict positivity of the Jacobian $\mathcal{A}^\eta(t)$.

an incompressible viscous fluid. In those cases, the weak solution techniques based on finite energy spaces are often times insufficient to guarantee even the Lipschitz regularity of the fluid-structure interface, see [23]. This is one of the reason why most literature on the existence of (weak) solutions to moving boundary problems involving thin elastic structures assumes only the transverse component of displacement to be different from zero [14, 96, 68, 25, 70, 102, 103, 95]. Recently, a 3D FSI problem allowing transverse and tangential displacements of a mesh-supported shell was studied in [23] where an existence of a weak solution in 3D was obtained under an extra assumption on the uniform Lipschitz property for the fluid-structure interface. FSI problems with elastic structures that are slightly more regular than the Koiter shell, such as, e.g., tripolar materials studied in [16, 119], do not suffer from this difficulty.

The issues related to non-zero transverse displacement cannot be avoided in FSI problems with the Navier-slip coupling. This is the reason why the existence result in 2D for a FSI problem involving a Koiter shell interacting with the flow of an incompressible, viscous fluid via the Navier-slip condition [106], holds only until the fluid domain remains regular, in the sense that degeneracies of the type shown in Fig. 4 do not occur. In problems with slip, compactness may be an additional problem since the regularizing effects by fluid viscosity are transferred to the structure only via the non-penetration condition holding in the normal direction to the boundary. Never the less, the friction effects in the tangential direction can be used to compensate for the lack of regularization provided by the fluid viscosity. More details about compactness for problems on moving domains are presented next.

5. COMPACTNESS

Compactness results similar to the classical Aubin-Lions-Simon lemma [7, 124] that hold for moving boundary problems are difficult to obtain because, among other things, the function spaces depend on time via the fluid domain motion, and the fluid domains are not known *a priori*. A compactness result in generalized Bochner spaces $L^2(0, T; H(t))$, where $H(t)$ is a family of Hilbert spaces which depend on time, is needed to deal with the fluid flow nonlinearities and with the geometric nonlinearities associated with the fluid domain motion. Such a compactness result should include conditions on the dynamic change of the fluid domain geometry that would contribute to the compactness argument.

To the best of our knowledge, there is no general compactness theory similar to Aubin-Lions-Simon lemma [7, 124] for spaces $L^2(0, T; H(t))$, where $H(t)$ depends on time. There are several compactness results for particular, specific problems for which the spatial domain depends on time, but they were proved using assumptions that hold for that particular problem at hand. The first result of this type was presented in Sather's PhD thesis [120], and then by Fujita and Sauer in [53], where they studied the Navier-Stokes equations in a given, non-cylindrical domain, namely, in a domain which depends on time, and whose motion is given *a priori*. Similarly, in [33], the authors used a compactness result to study a fluid-rigid body interaction problem, where an assumption on high regularity of the domain motion had to be used to obtain an existence result (see also [50, 122]). In the case of fluid-elastic structure interaction problems, the assumption on high regularity of the interface is typically not satisfied, thus different approaches need to be employed.

In the context of fluid-elastic structure interaction problems, we mention [39, 70] where the authors considered a fluid-elastic structure interaction problem between the flow of a viscous, incompressible fluid and an elastic/viscoelastic plate, in which a compactness argument based on Simon's theorem was used to show L^2 -strong convergence of approximate solutions. We also mention [85] where this approach was used in the case of a non-Newtonian fluid. A similar problem, but in a more general geometrical setting, was studied in [95], where compactness of a set of approximate weak solutions, based on a particular linearization and regularization of the problem, was proved by using a modification of the ideas from the proof of Aubin-Lions lemma. Both approaches used the fluid viscosity and kinematic coupling condition to control high frequency oscillations of the structure velocity. Recently, a version of Aubin-Lions lemma for a moving domain problem was proved in the context of *compressible fluids*, see [18], where an existence of a solution to an FSI problem between a compressible fluid and a linearly elastic shell was obtained. The lack of the fluid incompressibility constraint simplifies the compactness argument for the velocity field.

Compactness results in more general frameworks were studied in [110, 111], where the authors developed a functional framework based on the flow method and the Piola transform for problems in *smoothly* moving domains, where the flow causing domain motion was given *a priori*. In those works a version of the Aubin-Lions lemma was obtained within this framework. A different version of Aubin-Lions lemma, in a more general form, was also considered in [101]. The approach in [101] was based on negative Sobolev space-type estimates, defined on non-cylindrical, i.e., time-dependent domains. The latter approach did not require high degree of smoothness of the domain motion.

We also mention the results obtained in [12, 28, 101], where generalizations of the Aubin-Lions-Simon lemma in various types of nonlinear settings were obtained, and the work in [43] where a version of Aubin-Lions-Simon result was obtained in the context of finite element spaces. We also mention the works by Elliott et al. where compactness arguments were developed and used to study parabolic problems on moving surfaces [4, 5, 2, 6, 3].

Most of the works mentioned above were obtained for continuous time, i.e., the time variable was not discretized, and most of them were tailored for a particular application in mind. Working with discretized time brings some additional difficulties in terms of the uniform bounds for the time-shifts (translations in time).

In the time-discretized case, namely, for the approaches based on Rothe's semi-discretization method, the uniform bounds on the time-shifts need to be somewhat stronger to guarantee compactness, see Proposition 2 in [44]. In particular, the work in [44] addresses a version of Aubin-Lions-Simon result for piecewise constant functions in time, obtained using Rothe's method, but for a problem defined on a *fixed* Banach space.

The work presented in [107] concerns a generalization of the Aubin-Lions-Simon result involving Hilbert spaces that are *solution dependent*, and not necessarily known *a priori*. This is a significant step forward, since the result can be applied to a large class of moving boundary problems, including numerical solvers. To account for the time dependence of the function spaces associated with the motion of the fluid domains, the authors identified a new set of conditions, which quantify the dependence of the Hilbert spaces on time so that an extension of Aubin-Lions-Simon result can be applied to a sequence of approximate solutions constructed using Rothe's method.

More precisely, the compactness result in [107] is designed for problems which can be described in general as evolution problems:

$$(5.1) \quad \begin{aligned} \frac{d\mathbf{u}}{dt} &= A^t \mathbf{u}, t \in (0, T), \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned}$$

where $A^t : V(t) \rightarrow W(t)$ is a family of (nonlinear) spatial differential operators that depend on time as a parameter. For example, $\frac{d\mathbf{u}}{dt} = A^t \mathbf{u}$ may correspond to the Navier-Stokes equations for an incompressible, viscous fluid defined on a moving domain $\Omega(t)$. In this case, A^t is a spatial differential operator that associates to each \mathbf{u} the function $\nabla \cdot \boldsymbol{\sigma} - \mathbf{u} \cdot \nabla \mathbf{u}$, where $\boldsymbol{\sigma}$ is the fluid Cauchy stress tensor, and \mathbf{u} is divergence free, satisfying certain boundary conditions on $\Omega(t)$.

A way to "solve" this class of problems is to semi-discretize the problem in time by sub-dividing the time interval $(0, T)$ into N sub-intervals of width $\Delta t = T/N$, and introduce the piecewise constant approximate functions

$$(5.2) \quad \mathbf{u}_{\Delta t} = \mathbf{u}_{\Delta t}^n \quad \text{for } t \in ((n-1)\Delta t, n\Delta t], \quad n = 1, \dots, N,$$

which satisfy, e.g., a backward Euler approximation of the problem on (t^n, t^{n+1}) :

$$\frac{\mathbf{u}_{\Delta t}^{n+1} - \mathbf{u}_{\Delta t}^n}{\Delta t} = A^{t^{n+1}} \mathbf{u}_{\Delta t}^{n+1} \quad \text{or} \quad \frac{\mathbf{u}_{\Delta t}^{n+1} - \mathbf{u}_{\Delta t}^n}{\Delta t} = A^{t^n} \mathbf{u}_{\Delta t}^{n+1},$$

where the choice of $A^{t^{n+1}}$ or A^{t^n} depends on the problem at hand. For example, if the motion of the domain $\Omega(t)$ is specified *a priori*, $A^{t^{n+1}}$ is typically used, where $A^{t^{n+1}}$ describes an approximation of the spatial differential operator defined on the "current" domain $\Omega(t^{n+1})$. If the motion of the domain $\Omega(t)$ is not known *a priori*, but it depends on the solution of the underlying problem, then A^{t^n} is typically used, where A^{t^n} describes an approximation of the spatial differential operator defined on the "previous" domain $\Omega(t^n)$.

Functions $\mathbf{u}_{\Delta t}$ are defined for all $t \in (0, T)$ and they are piecewise constant on the interval $((n-1)\Delta t, n\Delta t]$, where the constant is defined by its value at $n\Delta t$. This approach to solving the evolution problem (5.1) is sometimes called the Rothe's method.

Rothe's method provides a constructive proof which uses semi-discretization of the continuous problem with respect to time to design approximate solutions $\{\mathbf{u}_{\Delta t}\}$

where $\Delta t = T/N$, for every $N \in \mathbb{N}$. It was first used by Ladyzhenskaya in [94], for a moving-boundary problem in which the motion of a *smooth* moving boundary was known *a priori*. The aim is to prove the existence of a sub-sequence of $\{\mathbf{u}_{\Delta t}\}$ which converges to a weak solution of (5.1) as $\Delta t \rightarrow 0$, or equivalently, as $N \rightarrow \infty$. Since the problem is nonlinear, weak convergence is not sufficient to show that the limit is a weak solution of the underlying problem. This is why compactness arguments need to be employed to conclude that there exists a sub-sequence $\{\mathbf{u}_{\Delta t}\}$, which is precompact in a certain generalized Bochner space $L^2(0, T; H(t))$. This will allow passage to the limit in nonlinear terms and show that the limit, as $\Delta t \rightarrow 0$, of approximate weak solutions satisfies the weak formulation of the continuous problem.

Employing this strategy to prove the existence of weak solutions to this class of problems is highly nontrivial, and is at the center of the current research in this area [102, 105, 104, 106]. The main source of difficulties is associated with the fact that for every $N \in \mathbb{N}$ and $n \in \{1, \dots, N\}$, the approximate weak solutions $\mathbf{u}_{\Delta t}^n$, which are functions of the spatial variables, belong to different solution spaces $V_{\Delta t}^n$, which are associated with the operators $A^{t^n} : V_{\Delta t}^n \rightarrow W_{\Delta t}^n$, and are defined on different domains $\Omega(t_{\Delta t}^n)$, thus $V_{\Delta t}^n = V(\Omega(t_{\Delta t}^n))$. We would like to find the conditions under which $\{\mathbf{u}_{\Delta t}\}$ is precompact in some $L^2(0, T; H(\Omega_{\Delta t}(t)))$, where the definition of $L^2(0, T; H(\Omega_{\Delta t}(t)))$ needs to be made precise. Namely, we want to find the conditions under which there exists a sub-sequence, also denoted by $\{\mathbf{u}_{\Delta t}\}$, which converges in $L^2(0, T; H(\Omega_{\Delta t}(t)))$ to a function in $L^2(0, T; H(\Omega(t)))$, as $\Delta t \rightarrow 0$.

There are two ways how to make the notion of convergence in $L^2(0, T; H(\Omega_{\Delta t}(t)))$ precise. One way is to introduce a family of *mappings*, which map the domains $\Omega(t_{\Delta t}^n)$ onto a fixed domain Ω , and work in the space $L^2(0, T; H(\Omega))$. The other approach is to *extend* the functions $\mathbf{u}_{\Delta t}^n$ onto a larger, fixed domain Ω^M , and work in the space $L^2(0, T; H(\Omega^M))$. In both cases, certain conditions describing the regularity in time of the domain motion need to be satisfied, in order for a compactness argument to hold. In [107] those conditions are identified, and a generalization of the Aubin-Lions-Simon compactness result was obtained, which can be used in both approaches, mentioned above.

The compactness result from [107] was applied to study existence of solutions to FSI with Koiter shell [102, 105], to FSI involving multi-layered structures [104], and to FSI with Koiter shell and Navier-slip coupling [106]. Since the result is based on the backward Euler time discretization approaches to the coupled FSI problem, the compactness result from [107] is a promising tool for proving convergence of numerical schemes that use the backward Euler scheme to discretize the problem in time [19, 20, 21, 22].

6. FSI WITH RIGID BODIES

The study of the motion of rigid bodies immersed in an incompressible, viscous fluid started almost fifty years ago with the works of Weinberger [130] and Sauer [121], who investigated the stationary problems. The study continued by the works of Galdi et al. who studied both the stationary and dynamic problems [58, 59, 61, 60], which paved a way for the most recent results in this field, which we survey below. Before we give a more detailed account of the recent developments, we

first present a mathematical formulation of a benchmark problem in fluid-structure interaction between a rigid body and an incompressible, viscous fluid.

6.1. FSI benchmark problem with rigid bodies. We consider an FSI problem between an incompressible, viscous fluid and a motion of a solid in a fluid container, as shown in Fig. 5.



FIGURE 5. A sketch of a falling rigid ball in a fluid container.

We denote by Ω the region corresponding to the fluid container, and by $S_0 \subset \Omega$ the solid (rigid structure) in the container, with the center of mass \mathbf{q}_0 . The motion of a rigid body is fully described by the translation of its center of mass, and by the (rigid) rotation about the center of mass, described by the functions

$$\mathbf{q} : [0, T] \rightarrow \mathbb{R}^3 \text{ and } \mathbf{Q} : [0, T] \rightarrow SO(3),$$

where $SO(3)$ is the group of rotations in 3D, and \mathbf{q} describes the translation of the center of mass. The trajectory of all the points in the rigid body is described by:

$$\mathbf{x} = \mathbf{B}(t, \mathbf{y}) = \mathbf{q}(t) + \mathbf{Q}(t) (\mathbf{y} - \mathbf{q}(0)), \quad \mathbf{y} \in S_0, \quad t \in [0, T].$$

At time t , the solid occupies the set

$$S(t) = \{\mathbf{x} \in \Omega_0 \mid \mathbf{x} = \mathbf{B}(t, \mathbf{y}), \mathbf{y} \in S_0 = \mathbf{B}(t, S_0),$$

defining the fluid domain at time t , $\Omega_F(t) = \Omega \setminus \overline{S(t)}$. As in the case of elastic structures, the fluid domain is not known *a priori* since it depends on the structural unknowns in the problem, namely the location of the rigid solid at time t .

As before, the fluid flow is described by the Navier-Stokes equations for an incompressible, viscous Newtonian fluid (2.1), while the equations of motion of the rigid body are given by a system of six ordinary differential equations (Euler equations) describing the conservation of linear and angular momentum:

$$(6.1) \quad \begin{aligned} m \frac{d^2 \mathbf{q}}{dt^2} &= \mathbf{f}, \\ \frac{d(\mathbf{J} \boldsymbol{\omega})}{dt} &= \mathbf{g}, \end{aligned}$$

where m is the mass of the rigid body, $\boldsymbol{\omega}$ corresponds to angular velocity, \mathbf{J} is the inertial tensor, and \mathbf{f} and \mathbf{g} are the total force and torque acting on the rigid body, respectively. The inertial tensor is defined by:

$$\mathbf{J} = \int_{S(t)} \rho_S (|\mathbf{x} - \mathbf{q}(t)|^2 \mathbf{I} - (\mathbf{x} - \mathbf{q}(t)) \otimes (\mathbf{x} - \mathbf{q}(t))) d\mathbf{x},$$

where ρ_S is the structure density.

The coupling. The fluid and structure are coupled through two sets of coupling conditions: the kinematic and dynamics coupling conditions. For the *kinematic*

coupling condition we take the no-slip, which says that the trace of the fluid velocity \mathbf{u} at the rigid body boundary is equal to the velocity \mathbf{u}_S of the rigid body itself:

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{u}_S(t, \mathbf{x}), \quad \mathbf{x} \in \partial S(t), \quad t \in (0, T),$$

where \mathbf{u}_S is given by:

$$\mathbf{u}_S = \frac{d\mathbf{x}}{dt} = \frac{d}{dt} (\mathbf{q}(t) + \mathbf{Q}(t) (\mathbf{y} - \mathbf{q}(0))) = \mathbf{q}'(t) + \boldsymbol{\omega}(t) \times (\mathbf{y} - \mathbf{q}(0)).$$

The *dynamic coupling condition* describes the balance of forces and torque. It says that the motion of the rigid structure in the fluid is driven by the contact force exerted by the fluid onto the structure. More precisely, the force and torque \mathbf{f} and \mathbf{g} in (6.1) are replaced by:

$$\mathbf{f} = - \int_{\partial S(t)} \boldsymbol{\sigma} \mathbf{n} ds(\mathbf{x}), \quad \mathbf{g} = - \int_{\partial S(t)} (\mathbf{x} - \mathbf{q}(t)) \times \boldsymbol{\sigma} \mathbf{n} ds(\mathbf{x}).$$

Thus, the **benchmark nonlinear moving-boundary problem**, describing fluid-structure interaction between an incompressible, viscous fluid and a rigid solid immersed in the fluid, can be summarized as follows: Find $(\mathbf{u}, p, \mathbf{q}, \boldsymbol{\omega})$ such that

$$\left. \begin{aligned} \rho_f (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) &= \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, p) \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \right\} \text{ in } \Omega_F(t), \quad t \in (0, T),$$

$$\mathbf{u} = \mathbf{q}'(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{q}(0)) \text{ on } (0, T) \times \partial S(t),$$

$$\left. \begin{aligned} m \frac{d^2 \mathbf{q}}{dt^2} &= - \int_{\partial S(t)} \boldsymbol{\sigma} \mathbf{n} ds(\mathbf{x}) \\ \frac{d(\mathbf{J} \boldsymbol{\omega})}{dt} &= - \int_{\partial S(t)} (\mathbf{x} - \mathbf{q}(t)) \times \boldsymbol{\sigma} \mathbf{n} ds(\mathbf{x}) \end{aligned} \right\} \text{ in } (0, T),$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial \Omega$$

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0, \text{ in } \Omega, \quad \mathbf{q}(0, \cdot) = \mathbf{q}_0, \quad \mathbf{q}'(0, \cdot) = \mathbf{a}_0, \quad \boldsymbol{\omega}(0, \cdot) = \boldsymbol{\omega}_0.$$

Weak solutions. A weak solution is a function (\mathbf{u}, \mathbf{B}) which satisfies the following two conditions:

1. The function $\mathbf{B}(t, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, which defines the time-dependent set $S(t) = \mathbf{B}(t, S)$ and the corresponding Eulerian velocity \mathbf{u}_S , is an orientation preserving isometry;
2. The function $\mathbf{u} \in L^2(0, T; V(t)) \cap L^\infty(0, T; L^2(\Omega))$ satisfies:

$$\begin{aligned} \int_0^T \int_{\Omega \setminus \partial S(t)} \{ \mathbf{u} \cdot \partial_t \boldsymbol{\psi} + (\mathbf{u} \otimes \mathbf{u}) : \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{\psi}) \} d\mathbf{x} dt - \int_{\Omega} \mathbf{u}(T) \boldsymbol{\psi}(T) d\mathbf{x} \\ = - \int_{\Omega} \mathbf{u}_0 \boldsymbol{\psi}(0) d\mathbf{x}, \end{aligned}$$

for all test functions $\boldsymbol{\psi} \in H^2(0, T; V(t))$, where

$$V(t) = \{ \mathbf{v} \in H_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0, \mathbf{D}(\mathbf{v}) = 0 \text{ in } S(t) \}.$$

7. RECENT RESULTS AND OPEN PROBLEMS IN FSI WITH RIGID SOLIDS

Existence of a unique, local-in-time (or small data) **strong** solution is now known in both two and three space dimensions, and for both the slip [1, 128] and the no-slip coupling [36, 67, 97, 126].

In terms of **weak** solutions of Leray-Hopf type, existence up to collision was obtained by Gérard-Varet and Hillairet in [64] for the slip coupling, and more recently by Chemetov and Nečasová in [26], where they showed global-in-time existence including collision, assuming the Navier slip condition prescribed at the solid boundary, and no-slip at the container boundary. Global existence with the no-slip coupling was established in [32, 37, 38, 80, 122].

The question of uniqueness of weak solution is still largely **open**. Even for the classical case of the 3D Navier-Stokes equations the uniqueness of the Leray-Hopf weak solution is an outstanding open problem (see e.g. [56]). However, there are classical results of weak-strong uniqueness type (see e.g. [56, 123, 127]) which state that the strong solution (defined in an appropriate way) is unique in a larger class of weak solutions. For the Navier-Stokes equations the weak solutions that satisfy Serrin's conditions are regular [123]. In the most recent paper by Muha, Nečasová and Radošević [108], these classical weak-strong uniqueness type results are extended to the case of a fluid-rigid body system under the condition that the rigid body does not touch the boundary of the container. Namely, in the case of contact it has been shown that weak solutions are not unique, see [49, 125], because there are multiple ways of extending the solution beyond the contact.

While these results discuss uniqueness of strong solutions, the results on uniqueness of weak solutions are sparse. The principal difficulty lies in the fact that different solutions are defined on different domains, so comparison between solutions is difficult. Usually, to handle this difficulty, the problem is mapped onto a fixed domain using a mapping that depends on the regularity of solutions, so strong solutions are easier to deal with. In 2015, however, uniqueness of weak solution for a fluid-rigid body system in the 2D case was obtained by Glass and Sueur in [65] for the no-slip case, and by Bravin in 2019 for the slip case [17], while uniqueness results of weak-strong type were recently published in [27, 40, 54]. In [40] the authors studied a rigid body with a cavity filled with fluid, while in [27] a rather high regularity for strong solutions was required for the uniqueness result to hold (the time derivative and second spatial derivatives of the fluid velocity were required to be in L^2). In [54] the authors studied a rigid body with a cavity filled with a compressible fluid, and showed a weak-strong uniqueness property using a relative entropy inequality. The most recent result by Muha, Nečasová and Radošević [108] generalizes these results, since they prove, for both slip and the no-slip case, a generalization of the well-known weak-strong uniqueness result for the Navier-Stokes equations to the fluid-rigid body system. More precisely, they prove that weak solutions which additionally satisfy the Prodi-Serrin $L^r - L^s$ condition are unique in the class of Leray-Hopf weak solutions.

8. FINITE-TIME CONTACT

As already addressed in the works related to global-in-time existence of weak solutions mentioned above, global existence of solutions to FSI problems involving incompressible, viscous fluids is affected by the possibility of contact: either the

contact between rigid bodies immersed in the fluid, a contact between elastic structures immersed in the fluid, or the contact between an elastic structure with the fluid container (rigid) boundary. While in the case of compressible fluids, contact of rigid bodies is possible in finite time [48], the incompressible, viscous case is different since contact in finite time between rigid bodies is not possible for the scenarios studied in [98, 32, 72, 80, 83, 62, 64, 63]. In a pioneering work in 2009 Nestupa and Penel showed that contact in finite time between rigid bodies immersed in a viscous, incompressible fluid is possible if the Navier-slip boundary condition is used, which became a precursor for a number of existence results involving a slip condition and FSI with rigid solids, described above.

Finite time contact involving elastic structures remains to be an outstanding **open** problem, although a recent result by Grandmont and Hillairet in 2016 [71], indicates that finite time contact with the no-slip condition and deformable structure is impossible. More precisely, Grandmont and Hillairet studied the interaction between a 1D *viscoelastic beam* and a 2D viscous, incompressible fluid, assuming the no-slip coupling, and showed (1) that contact in finite time is not possible, and (2) that strong solutions exist globally in time. Their result is the first no-contact result involving deformable solids, and the first global existence result for FSI problems with an incompressible, viscous fluid and deformable structures.

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