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Duality of Lusztig and RTT integral forms of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$

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ABSTRACT

We show that the Lusztig integral form is dual to the RTT integral form of the type A quantum loop algebra with respect to the new Drinfeld pairing, by utilizing the shuffle algebra realization of the former and the PBWD bases of the latter obtained in [13].

1. Introduction

1.1. Summary

For a simple finite-dimensional Lie algebra $\mathfrak{g}$, the quantum function algebra is dual to the Lusztig form $\mathsf{U}_{\boldsymbol{v}}(\mathfrak{g})$ of the quantum group of $\mathfrak{g}$. For $\mathfrak{g}=\mathfrak{sl}_n$, this is reflected by the duality between the Lusztig and the RTT integral forms of $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ with respect to the Drinfeld-Jimbo pairing. In this short note, we establish an affine version of the above result for $\mathfrak{sl}_n$ replaced with $\widehat{\mathfrak{sl}}_n$ and the Drinfeld-Jimbo pairing replaced with the new Drinfeld pairing.

1.2. Outline of the paper

• In Section 2, we recall the quantum loop (quantum affine with the trivial central charge) algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as well as its two integral forms: $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ (naturally arising in the RTT presentation of [6]) and $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ (Lusztig form defined in the Drinfeld-Jimbo presentation). Both integral forms posses triangular decompositions, see Propositions 2.17, 2.28, generalizing the one for $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of Proposition 2.9. We also recall our constructions of the PBWD (Poincaré-Birkhoff-Witt-Drinfeld) bases for the "positive" and "negative" subalgebras of both integral forms established in [13], see Theorems 2.16, 2.31. Finally, in Section 2.4, we recall the new Drinfeld topological Hopf algebra structure and the new Drinfeld pairing on $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

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• In Section 3, we recall the shuffle algebra $S^{(n)}$, its two integral forms, and the shuffle algebra realizations of the "positive" subalgebras $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, see Theorem 3.4 (first established in [12]), and of $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n), \mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, see Theorem 3.7 and Remark 3.8, established in [13]. Finally, we enlarge $S^{(n)}$ to the extended shuffle algebra $S^{(n),\geq}$ by adjoining Cartan generators satisfying (3.9), thus obtaining the shuffle algebra realization (3.10) of $U^{\geq}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, and recall the formulas (3.11, 3.12) for the new Drinfeld coproduct on it, cf. [12, Proposition 3.5].

• In Section 4, we prove that the integral form $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is dual to $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ with respect to the new Drinfeld pairing, see Theorem 4.1, which constitutes the main result of this note. Our proof is crucially based on the shuffle realizations of Section 3 as well as utilizes the entire family of the PBWD bases of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of Theorem 2.16, see Remark 4.36.

1.3. Acknowledgments

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2. Quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and its integral forms

2.1. Quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$

Let $I=\{1,\ldots,n-1\}$, $(c_{ij})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{sl}_n$, and $\boldsymbol{v}$ be a formal variable. Following [2], define the quantum loop algebra of $\mathfrak{sl}_n$ (in the new Drinfeld presentation), denoted by $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, to be the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi^{\pm}_{i,\pm s}\}_{i\in I}^{r\in\mathbb{Z}, s\in\mathbb{N}}$ with the following defining relations:

$$[\psi^{\epsilon}_i(z), \psi^{\epsilon'}_j(w)] = 0,\ \psi^{\pm}_{i,0}\cdot\psi^{\mp}_{i,0} = 1, \tag{2.1}$$

$$(z-\boldsymbol{v}^{c_{ij}}w)e_i(z)e_j(w) = (\boldsymbol{v}^{c_{ij}}z-w)e_j(w)e_i(z), \tag{2.2}$$

$$(\boldsymbol{v}^{c_{ij}}z-w)f_i(z)f_j(w) = (z-\boldsymbol{v}^{c_{ij}}w)f_j(w)f_i(z), \tag{2.3}$$

$$(z-\boldsymbol{v}^{c_{ij}}w)\psi^{\epsilon}_i(z)e_j(w) = (\boldsymbol{v}^{c_{ij}}z-w)e_j(w)\psi^{\epsilon}_i(z), \tag{2.4}$$

$$(\boldsymbol{v}^{c_{ij}}z-w)\psi^{\epsilon}_i(z)f_j(w) = (z-\boldsymbol{v}^{c_{ij}}w)f_j(w)\psi^{\epsilon}_i(z), \tag{2.5}$$

$$[e_i(z), f_j(w)] = \frac{\delta_{ij}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\delta\left(\frac{z}{w}\right)\left(\psi^{+}_i(z)-\psi^{-}_i(z)\right), \tag{2.6}$$

$$\begin{aligned} & e_i(z)e_j(w) = e_j(w)e_i(z) \text{ if } c_{ij}=0, \\ & [e_i(z_1),[e_i(z_2),e_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} + [e_i(z_2),[e_i(z_1),e_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} = 0 \text{ if } c_{ij}=-1, \end{aligned} \tag{2.7}$$

$$\begin{aligned} & f_i(z)f_j(w) = f_j(w)f_i(z) \text{ if } c_{ij}=0, \\ & [f_i(z_1),[f_i(z_2),f_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} + [f_i(z_2),[f_i(z_1),f_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} = 0 \text{ if } c_{ij}=-1, \end{aligned} \tag{2.8}$$

where $[a,b]_x := ab - x\cdot ba$ and the generating series are defined as follows:

$$e_i(z) := \sum_{r\in\mathbb{Z}} e_{i,r}z^{-r},\ f_i(z) := \sum_{r\in\mathbb{Z}} f_{i,r}z^{-r},\ \psi^{\pm}_i(z) := \sum_{s\geq 0}\psi^{\pm}_{i,\pm s}z^{\mp s},\ \delta(z) := \sum_{r\in\mathbb{Z}} z^r.$$

Let $U^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n), U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n), U^{0}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ be the $\mathbb{C}(\boldsymbol{v})$-subalgebras of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated respectively by $\{f_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}, \{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}, \{\psi^{\pm}_{i,\pm s}\}_{i\in I}^{s\in\mathbb{N}}$. The following is standard (see e.g. [9, Theorem 2]):

Proposition 2.9. *(a) (Triangular decomposition of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$) The multiplication map*

$$m\colon U^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)\otimes_{\mathbb{C}(\boldsymbol{v})}U^0_{\boldsymbol{v}}(L\mathfrak{sl}_n)\otimes_{\mathbb{C}(\boldsymbol{v})}U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)\longrightarrow U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$$

is an isomorphism of $\mathbb{C}(\boldsymbol{v})$-vector spaces.
(b) The algebra $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ (resp. $U^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $U^0_{\boldsymbol{v}}(L\mathfrak{sl}_n)$) is isomorphic to the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ (resp. $\{f_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ and $\{\psi^\pm_{i,\pm s}\}_{i\in I}^{s\in\mathbb{N}}$) with the defining relations (2.2, 2.7) (resp. (2.3, 2.8) and (2.1)).

2.2. RTT integral form $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and its PBWD bases

Let $\{\alpha_i\}_{i=1}^{n-1}$ be the standard simple positive roots of $\mathfrak{sl}_n$, and Δ^+ be the set of positive roots: $\Delta^+=\{\alpha_j+\alpha_{j+1}+\ldots+\alpha_i\}_{1\le j\le i<n}$. Consider the following total ordering "$\le$" on Δ^+:

$$\alpha_j+\alpha_{j+1}+\ldots+\alpha_i\le\alpha_{j'}+\alpha_{j'+1}+\ldots+\alpha_{i'}\ \text{iff}\ j<j'\ \text{or}\ j=j',i\le i'. \tag{2.10}$$

This gives rise to the total ordering "$\le$" on $\Delta^+\times\mathbb{Z}$:

$$(\beta,r)\le(\beta',r')\ \text{iff}\ \beta<\beta'\ \text{or}\ \beta=\beta',r\le r'. \tag{2.11}$$

For any $1\le j\le i\le n-1$ and $r\in\mathbb{Z}$, we choose a *decomposition*

$$\underline{r}=\underline{r}(\alpha_j+\ldots+\alpha_i,r)=(r_j,\ldots,r_i)\in\mathbb{Z}^{i-j+1}\ \text{such that}\ r_j+\ldots+r_i=r. \tag{2.12}$$

A particular example of such a decomposition is

$$\underline{r}^{(0)}=\underline{r}^{(0)}(\alpha_j+\ldots+\alpha_i,r)=(r,0,\ldots,0). \tag{2.13}$$

Following [13, (2.11, 2.18)], define the elements $\widetilde{e}_{\beta,\underline{r}}\in U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $\widetilde{f}_{\beta,\underline{r}}\in U^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ via

$$\begin{aligned}\widetilde{e}_{\alpha_j+\alpha_{j+1}+\ldots+\alpha_i,\underline{r}}&:=(\boldsymbol{v}-\boldsymbol{v}^{-1})[\cdots[[e_{j,r_j},e_{j+1,r_{j+1}}]_{\boldsymbol{v}},e_{j+2,r_{j+2}}]_{\boldsymbol{v}},\cdots,e_{i,r_i}]_{\boldsymbol{v}},\\ \widetilde{f}_{\alpha_j+\alpha_{j+1}+\ldots+\alpha_i,\underline{r}}&:=(\boldsymbol{v}-\boldsymbol{v}^{-1})[\cdots[[f_{j,r_j},f_{j+1,r_{j+1}}]_{\boldsymbol{v}},f_{j+2,r_{j+2}}]_{\boldsymbol{v}},\cdots,f_{i,r_i}]_{\boldsymbol{v}}.\end{aligned} \tag{2.14}$$

In the special case $\underline{r}(\beta,r)=\underline{r}^{(0)}(\beta,r)$, see (2.13), we shall denote $\widetilde{e}_{\beta,\underline{r}},\widetilde{f}_{\beta,\underline{r}}$ simply by $\widetilde{e}_{\beta,r},\widetilde{f}_{\beta,r}$.

Define the *RTT integral form* $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\{\widetilde{e}_{\beta,r},\widetilde{f}_{\beta,r},\psi^\pm_{i,\pm s}\}_{i\in I,\beta\in\Delta^+}^{r\in\mathbb{Z},s\in\mathbb{N}}$. Let $\mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n),\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, and $\mathfrak{U}^0_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ be the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebras of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\{\widetilde{f}_{\beta,r}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}},\{\widetilde{e}_{\beta,r}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}}$, and $\{\psi^\pm_{i,\pm s}\}_{i\in I}^{s\in\mathbb{N}}$, respectively.

Remark 2.15. The name "*RTT integral form*" is motivated by the following two observations:
(a) Due to Theorem 2.16 below, we have $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)\otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]}\mathbb{C}(\boldsymbol{v})\simeq U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.
(b) Due to [7, Proposition 3.20], the subalgebra $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ coincides with the Υ-preimage of $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, where $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ is the RTT integral form of the quantum loop algebra of $\mathfrak{gl}_n$ [6] (cf. [7, §3(ii)]), while $\Upsilon\colon U_{\boldsymbol{v}}(L\mathfrak{sl}_n)\hookrightarrow\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)\otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]}\mathbb{C}(\boldsymbol{v})$ is the $\mathbb{C}(\boldsymbol{v})$-algebra embedding of [4].

As before, fix a decomposition $\underline{r}(\beta,r)$ for each pair $(\beta,r)\in\Delta^+\times\mathbb{Z}$. We order $\{\widetilde{e}_{\beta,\underline{r}(\beta,r)}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}}$ with respect to (2.11), while $\{\widetilde{f}_{\beta,\underline{r}(\beta,r)}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}}$ are ordered with respect to the opposite ordering on $\Delta^+\times\mathbb{Z}$. Finally, choose any total ordering of $\{\psi_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ defined via $\psi_{i,r}:=\begin{cases}\psi^+_{i,r}, & \text{if } r\ge 0\\ \psi^-_{i,r}, & \text{if } r<0\end{cases}$. Having specified these three total orderings, elements $F \cdot H \cdot E$ with F, E, H being ordered monomials in $\{\widetilde{f}_{\beta,\underline{r}(\beta,r)}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}}$, $\{\widetilde{e}_{\beta,\underline{r}(\beta,r)}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}}$, $\{\psi_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ (note that we allow negative powers of $\psi_{i,0}$), respectively, are called the *ordered PBWD monomials* (in the corresponding generators).

The following was established in [13, Theorems 2.15, 2.17, 2.19, 2.22], cf. [7, Theorem 3.24]:

Theorem 2.16. *Fix a decomposition $\underline{r}(\beta, r)$ for every pair $(\beta, r) \in \Delta^+ \times \mathbb{Z}$.*
(a1) The ordered PBWD monomials in $\{\widetilde{f}_{\beta,\underline{r}(\beta,r)}, \psi_{i,r}, \widetilde{e}_{\beta,\underline{r}(\beta,r)}\}_{i\in I,\beta\in\Delta^+}^{r\in\mathbb{Z}}$ form a basis of the free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-module $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.
(a2) The ordered PBWD monomials in $\{\widetilde{f}_{\beta,\underline{r}(\beta,r)}, \psi_{i,r}, \widetilde{e}_{\beta,\underline{r}(\beta,r)}\}_{i\in I,\beta\in\Delta^+}^{r\in\mathbb{Z}}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.
(b1) The ordered PBWD monomials in $\{\widetilde{e}_{\beta,\underline{r}(\beta,r)}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}}$ form a basis of the free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-module $\mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.
(b2) The ordered PBWD monomials in $\{\widetilde{e}_{\beta,\underline{r}(\beta,r)}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.
(c1) The ordered PBWD monomials in $\{\widetilde{f}_{\beta,\underline{r}(\beta,r)}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}}$ form a basis of the free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-module $\mathfrak{U}^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.
(c2) The ordered PBWD monomials in $\{\widetilde{f}_{\beta,\underline{r}(\beta,r)}\}_{\beta\in\Delta^+}^{r\in\mathbb{Z}}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $U^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.
(d1) The ordered PBWD monomials in $\{\psi_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ form a basis of the free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-module $\mathfrak{U}^{0}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.
(d2) The ordered PBWD monomials in $\{\psi_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $U^{0}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

This result together with Proposition 2.9 implies the triangular decomposition of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$:

Proposition 2.17. *The multiplication map*

$$m\colon \mathfrak{U}^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathfrak{U}^{0}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \longrightarrow \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$$

is an isomorphism of (free) $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-modules.

2.3. Lusztig integral form $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and its PBWD basis

To introduce the Lusztig integral form, we recall the Drinfeld-Jimbo realization of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. Let $\widetilde{I} = I \cup \{i_0\}$ be the vertex set of the extended Dynkin diagram and $(c_{ij})_{i,j\in\widetilde{I}}$ be the extended Cartan matrix. The *Drinfeld-Jimbo quantum loop algebra of* $\mathfrak{sl}_n$, denoted by $U^{\mathrm{DJ}}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, is the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i\in\widetilde{I}}$ with the following defining relations:

$$[K_i, K_j] = 0,\ K_i^{\pm 1} \cdot K_i^{\mp 1} = 1,\ \prod_{i\in\widetilde{I}} K_i = 1, \tag{2.18}$$

$$K_i E_j = \boldsymbol{v}^{c_{ij}} E_j K_i,\ K_i F_j = \boldsymbol{v}^{-c_{ij}} F_j K_i,\ [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{\boldsymbol{v} - \boldsymbol{v}^{-1}}, \tag{2.19}$$

$$E_i E_j = E_j E_i,\ F_i F_j = F_j F_i \text{ if } c_{ij} = 0, \tag{2.20}$$

$$[E_i, [E_i, E_j]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} = 0,\ [F_i, [F_i, F_j]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} = 0 \text{ if } c_{ij} = -1. \tag{2.21}$$

The following result is due to [2]:

Proposition 2.22. *There is a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $U^{\mathrm{DJ}}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \xrightarrow{\sim} U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, such that*

$$\begin{gathered} E_i \mapsto e_{i,0},\ F_i \mapsto f_{i,0},\ K_i^{\pm 1} \mapsto \psi^{\pm}_{i,0} \text{ for } i \in I, \\ E_{i_0} \mapsto (-\boldsymbol{v})^{-n} \cdot (\psi^+_{1,0} \cdots \psi^+_{n-1,0})^{-1} \cdot [\cdots [f_{1,1}, f_{2,0}]_{\boldsymbol{v}}, \cdots, f_{n-1,0}]_{\boldsymbol{v}}, \\ F_{i_0} \mapsto (-\boldsymbol{v})^{n} \cdot [e_{n-1,0}, \cdots, [e_{2,0}, e_{1,-1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}} \cdot \psi^+_{1,0} \cdots \psi^+_{n-1,0}. \end{gathered}$$

For $k \in \mathbb{N}$, set $[k]_{\boldsymbol{v}} := \frac{\boldsymbol{v}^k - \boldsymbol{v}^{-k}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}, [k]_{\boldsymbol{v}}! := \prod_{\ell=1}^{k} [\ell]_{\boldsymbol{v}}$. For $i \in \widetilde{I}, k \in \mathbb{N}$, define the divided powers

$$E_i^{(k)} := \frac{E_i^k}{[k]_{\boldsymbol{v}}!} \quad \text{and} \quad F_i^{(k)} := \frac{F_i^k}{[k]_{\boldsymbol{v}}!}. \tag{2.23}$$

Define the *Lusztig integral form* $\mathsf{U}_{\boldsymbol{v}}^{\mathrm{DJ}}(L\mathfrak{sl}_n)$ as the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}^{\mathrm{DJ}}(L\mathfrak{sl}_n)$ generated by $\{E_i^{(k)}, F_i^{(k)}, K_i^{\pm 1}\}_{i\in \widetilde{I}}^{k\in\mathbb{N}}$. In view of Proposition 2.22, it gives rise to the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ which shall be referred to as the *Lusztig integral form* of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

Let us now recall a more explicit description of $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. For $i \in I, r \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$, define

$$\begin{bmatrix} \psi_{i,0}^{+}; r \\ k \end{bmatrix} := \prod_{\ell=1}^{k} \frac{\psi_{i,0}^{+}\boldsymbol{v}^{r-\ell+1} - \psi_{i,0}^{-}\boldsymbol{v}^{-r+\ell-1}}{\boldsymbol{v}^{\ell}-\boldsymbol{v}^{-\ell}}. \tag{2.24}$$

We also define the pairwise commuting generators $\{h_{i,r}\}_{i\in I}^{r\neq 0}$ via

$$\psi_i^{\pm}(z) = \psi_{i,0}^{\pm} \cdot \exp\left(\pm(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{r>0} h_{i,\pm r} z^{\mp r}\right). \tag{2.25}$$

Finally, for $i \in I, r \in \mathbb{Z}, k \in \mathbb{N}$, we define the divided powers

$$\mathsf{e}_{i,r}^{(k)} := \frac{e_{i,r}^k}{[k]_{\boldsymbol{v}}!} \quad \text{and} \quad \mathsf{f}_{i,r}^{(k)} := \frac{f_{i,r}^k}{[k]_{\boldsymbol{v}}!}. \tag{2.26}$$

Let $\mathsf{U}_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n), \mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$, and $\mathsf{U}_{\boldsymbol{v}}^{0}(L\mathfrak{sl}_n)$ be the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebras of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\{\mathsf{f}_{i,r}^{(k)}\}_{i\in I}^{r\in\mathbb{Z},k\in\mathbb{N}}$, $\{\mathsf{e}_{i,r}^{(k)}\}_{i\in I}^{r\in\mathbb{Z},k\in\mathbb{N}}$, and $\{\psi_{i,0}^{\pm}, \frac{h_{i,\pm k}}{[k]_{\boldsymbol{v}}}, \left[\begin{smallmatrix}\psi_{i,0}^{+};r\\ k\end{smallmatrix}\right]\}_{i\in I}^{r\in\mathbb{Z},k\in\mathbb{Z}_{>0}}$, respectively.

Remark 2.27. The subalgebra $\mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \subset U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ was first considered in [8].

The following triangular decomposition of $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is due to [1, Proposition 6.1]:

Proposition 2.28. *(a)* $\mathsf{U}_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n), \mathsf{U}_{\boldsymbol{v}}^{0}(L\mathfrak{sl}_n), \mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ *are* $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$*-subalgebras of* $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.
(b) (Triangular decomposition of $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$*) The multiplication map*

$$m\colon \mathsf{U}_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathsf{U}_{\boldsymbol{v}}^{0}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \longrightarrow \mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$$

is an isomorphism of (free) $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$*-modules.*

Following (2.13, 2.14), define the elements $e_{\beta,r} \in U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ and $f_{\beta,r} \in U_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)$ via

$$\begin{aligned} e_{\alpha_j+\alpha_{j+1}+\ldots+\alpha_i,r} &:= [\cdots[[e_{j,r}, e_{j+1,0}]_{\boldsymbol{v}}, e_{j+2,0}]_{\boldsymbol{v}}, \cdots, e_{i,0}]_{\boldsymbol{v}}, \\ f_{\alpha_j+\alpha_{j+1}+\ldots+\alpha_i,r} &:= [\cdots[[f_{j,r}, f_{j+1,0}]_{\boldsymbol{v}}, f_{j+2,0}]_{\boldsymbol{v}}, \cdots, f_{i,0}]_{\boldsymbol{v}}. \end{aligned} \tag{2.29}$$

For $\beta \in \Delta^{+}, r \in \mathbb{Z}, k \in \mathbb{N}$, we define the divided powers

$$\mathsf{e}_{\beta,r}^{(k)} := \frac{e_{\beta,r}^k}{[k]_{\boldsymbol{v}}!} \quad \text{and} \quad \mathsf{f}_{\beta,r}^{(k)} := \frac{f_{\beta,r}^k}{[k]_{\boldsymbol{v}}!}. \tag{2.30}$$

Note that $\mathsf{e}_{\beta,r}^{(k)} \in \mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ and $\mathsf{f}_{\beta,r}^{(k)} \in \mathsf{U}_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)$ for any β, r, k as above, due to [11, Theorem 6.6].

Evoking (2.11), the monomials of the form $\overrightarrow{\prod\limits_{(\beta,r)\in\Delta^+\times\mathbb{Z}}} \mathsf{e}_{\beta,r}^{(k_{\beta,r})}$ and $\overleftarrow{\prod\limits_{(\beta,r)\in\Delta^+\times\mathbb{Z}}} \mathsf{f}_{\beta,r}^{(k_{\beta,r})}$ (with $k_{\beta,r}\in\mathbb{N}$ and only finitely many of them being nonzero) are called the *ordered PBWD monomials* of $\mathsf{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $\mathsf{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, respectively. The following result was established in [13]:

Theorem 2.31. *[13, Theorem 8.5] The ordered PBWD monomials form bases of the free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-modules $\mathsf{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $\mathsf{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, respectively.*

2.4. New Drinfeld Hopf algebra structure and Hopf pairing

Let us first recall the general notion of a Hopf pairing, following [10, §3]. Given two Hopf algebras A and B over a field k, the bilinear map

$$\varphi\colon A\times B\longrightarrow \mathsf{k}$$

is called a *Hopf pairing* if it satisfies the following properties (for any $a,a'\in A$ and $b,b'\in B$):

$$\varphi(a,bb')=\varphi(a_{(1)},b)\varphi(a_{(2)},b'),\ \varphi(aa',b)=\varphi(a,b_{(2)})\varphi(a',b_{(1)}), \tag{2.32}$$

$$\varphi(a,1_B)=\epsilon_A(a),\ \varphi(1_A,b)=\epsilon_B(b),\ \varphi(S_A(a),S_B(b))=\varphi(a,b), \tag{2.33}$$

where we use the Sweedler notation for the coproduct: $\Delta(x)=x_{(1)}\otimes x_{(2)}$.

Following [5, Theorem 2.1], we endow $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ with the new Drinfeld topological Hopf algebra structure by defining the coproduct Δ, the counit ϵ, and the antipode S as follows:

$$\Delta\colon \psi_i^\pm(z)\mapsto \psi_i^\pm(z)\otimes\psi_i^\pm(z),\ e_i(z)\mapsto e_i(z)\otimes 1+\psi_i^-(z)\otimes e_i(z),\ f_i(z)\mapsto 1\otimes f_i(z)+f_i(z)\otimes\psi_i^+(z),$$
$$\epsilon\colon e_i(z)\mapsto 0,\ f_i(z)\mapsto 0,\ \psi_i^\pm(z)\mapsto 1,$$
$$S\colon e_i(z)\mapsto -\psi_i^-(z)^{-1}e_i(z),\ f_i(z)\mapsto -f_i(z)\psi_i^+(z)^{-1},\ \psi_i^\pm(z)\mapsto\psi_i^\pm(z)^{-1}.$$

Thus, the $\mathbb{C}(\boldsymbol{v})$-subalgebras $U^{\leq}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $U^{\geq}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\{f_{i,r},\psi^+_{i,s},(\psi^+_{i,0})^{-1}\}_{i\in I}^{r\in\mathbb{Z},s\in\mathbb{N}}$ and $\{e_{i,r},\psi^-_{i,-s},(\psi^-_{i,0})^{-1}\}_{i\in I}^{r\in\mathbb{Z},s\in\mathbb{N}}$, respectively, are actually Hopf subalgebras of $(U_{\boldsymbol{v}}(L\mathfrak{sl}_n),\Delta,S,\epsilon)$.

The following is well-known (see e.g. [8, §9.3], cf. [12, Propositions 2.27, 2.30]):

Proposition 2.34. *The assignment*

$$\begin{gathered}\varphi(e_i(z),f_j(w))=\frac{\delta_{i,j}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\delta\left(\frac{z}{w}\right),\ \varphi(\psi_i^-(z),\psi_j^+(w))=\frac{\boldsymbol{v}^{c_{ij}}z-w}{z-\boldsymbol{v}^{c_{ij}}w},\\ \varphi(e_i(z),\psi_j^+(w))=0,\ \varphi(\psi_j^-(w),f_i(z))=0,\end{gathered} \tag{2.35}$$

gives rise to a non-degenerate Hopf algebra pairing $\varphi\colon U^{\geq}_{\boldsymbol{v}}(L\mathfrak{sl}_n)\times U^{\leq}_{\boldsymbol{v}}(L\mathfrak{sl}_n)\to\mathbb{C}(\boldsymbol{v})$.

3. Shuffle algebra $S^{(n)}$ and its integral forms

3.1. Shuffle algebra $S^{(n)}$

Let Σ_k denote the symmetric group in k elements, and set $\Sigma_{(k_1,\ldots,k_{n-1})}:=\Sigma_{k_1}\times\cdots\times\Sigma_{k_{n-1}}$ for $k_1,\ldots,k_{n-1}\in\mathbb{N}$. Consider an $\mathbb{N}^I$-graded $\mathbb{C}(\boldsymbol{v})$-vector space $\mathbb{S}^{(n)}=\bigoplus\limits_{\underline{k}=(k_1,\ldots,k_{n-1})\in\mathbb{N}^I}\mathbb{S}^{(n)}_{\underline{k}}$, where $\mathbb{S}^{(n)}_{(k_1,\ldots,k_{n-1})}$ consists of $\Sigma_{\underline{k}}$-symmetric rational functions in the variables $\{x_{i,r}\}_{i\in I}^{1\leq r\leq k_i}$. Define $\zeta_{i,j}(z):=\frac{z-\boldsymbol{v}^{-c_{ij}}}{z-1}$ for

$i,j \in I$. Let us introduce the bilinear *shuffle product* $\star$ on $\mathbb{S}^{(n)}$: given $F \in \mathbb{S}^{(n)}_{\underline{k}}$ and $G \in \mathbb{S}^{(n)}_{\underline{\ell}}$, define $F \star G \in \mathbb{S}^{(n)}_{\underline{k}+\underline{\ell}}$ via

$$(F \star G)(x_{1,1},\ldots,x_{1,k_1+\ell_1};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+\ell_{n-1}}) := \underline{k}!\cdot\underline{\ell}!\times \\ \mathrm{Sym}_{\Sigma_{\underline{k}+\underline{\ell}}}\left(F\left(\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}\right)G\left(\{x_{i',r'}\}_{i'\in I}^{k_{i'}<r'\le k_{i'}+\ell_{i'}}\right)\cdot\prod_{i\in I}^{i'\in I}\prod_{r\le k_i}^{r'>k_{i'}}\zeta_{i,i'}(x_{i,r}/x_{i',r'})\right). \tag{3.1}$$

Here, $\underline{k}! = \prod_{i\in I} k_i!$, while for $f \in \mathbb{C}(\{x_{i,1},\ldots,x_{i,m_i}\}_{i\in I})$ we define its *symmetrization* via

$$\mathrm{Sym}_{\Sigma_{\underline{m}}}(f)\left(\{x_{i,1},\ldots,x_{i,m_i}\}_{i\in I}\right) := \frac{1}{\underline{m}!}\cdot\sum_{(\sigma_1,\ldots,\sigma_{n-1})\in\Sigma_{\underline{m}}} f\left(\{x_{i,\sigma_i(1)},\ldots,x_{i,\sigma_i(m_i)}\}_{i\in I}\right).$$

This endows $\mathbb{S}^{(n)}$ with a structure of an associative unital algebra with the unit $\mathbf{1} \in \mathbb{S}^{(n)}_{(0,\ldots,0)}$.

We will be interested only in the subspace of $\mathbb{S}^{(n)}$ defined by the *pole* and *wheel conditions*:

- We say that $F \in \mathbb{S}^{(n)}_{\underline{k}}$ satisfies the *pole conditions* if

$$F = \frac{f(x_{1,1},\ldots,x_{n-1,k_{n-1}})}{\prod_{i=1}^{n-2}\prod_{r\le k_i}^{r'\le k_{i+1}}(x_{i,r}-x_{i+1,r'})},\ \text{where } f \in (\mathbb{C}(\boldsymbol{v})[\{x_{i,r}^{\pm 1}\}_{i\in I}^{1\le r\le k_i}])^{\Sigma_{\underline{k}}}. \tag{3.2}$$

- We say that $F \in \mathbb{S}^{(n)}_{\underline{k}}$ satisfies the *wheel conditions* if

$$F(\{x_{i,r}\}) = 0 \text{ once } x_{i,r_1} = \boldsymbol{v}x_{i+\epsilon,s} = \boldsymbol{v}^2 x_{i,r_2} \text{ for some } \epsilon, i, r_1, r_2, s, \tag{3.3}$$

where $\epsilon \in \{\pm 1\},\ i, i+\epsilon \in I,\ 1 \le r_1, r_2 \le k_i,\ 1 \le s \le k_{i+\epsilon}$.

Let $S^{(n)}_{\underline{k}} \subset \mathbb{S}^{(n)}_{\underline{k}}$ denote the subspace of all elements F satisfying these two conditions (3.2, 3.3) and set $S^{(n)} := \bigoplus_{\underline{k}\in\mathbb{N}^I} S^{(n)}_{\underline{k}}$. It is straightforward to check that the subspace $S^{(n)} \subset \mathbb{S}^{(n)}$ is $\star$-closed.

The resulting associative $\mathbb{C}(v)$-algebra $\left(S^{(n)},\star\right)$ shall be called the *shuffle algebra.*

3.2. Shuffle algebra realizations

The *shuffle algebra* $\left(S^{(n)},\star\right)$ is related to $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ via the following result of [13] (cf. [12]):

Theorem 3.4. *The assignment* $e_{i,r} \mapsto x^r_{i,1}$ $(i \in I, r \in \mathbb{Z})$ *gives rise to a* $\mathbb{C}(\boldsymbol{v})$*-algebra isomorphism* $\Psi\colon U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \xrightarrow{\sim} S^{(n)}$.

Remark 3.5. This result was manifestly used in [13] to establish parts (b2, c2) of Theorem 2.16.

The proof of Theorem 3.4 in [13] crucially utilized the *specialization maps* $\phi_{\underline{d}}$ [13, (3.12)], which we recall next. For a positive root $\beta = \alpha_j + \alpha_{j+1} + \ldots + \alpha_i$, define $j(\beta) := j, i(\beta) := i$, and let $[\beta]$ denote the integer interval $[j(\beta); i(\beta)]$. Consider a collection of the intervals $\{[\beta]\}_{\beta\in\Delta^+}$ each taken with a multiplicity $d_\beta \in \mathbb{N}$ and ordered with respect to the total ordering (2.10) (the order inside each group is irrelevant). Define $\underline{\ell} \in \mathbb{N}^I$ via $\sum_{i\in I}\ell_i\alpha_i = \sum_{\beta\in\Delta^+} d_\beta\beta$.

Let us now define the *specialization map* $\phi_{\underline{d}}$ (here, $\underline{d}$ denotes the collection $\{d_\beta\}_{\beta\in\Delta^+}$)

$$\phi_{\underline{d}}\colon S^{(n)}_{\underline{\ell}} \longrightarrow \mathbb{C}(\boldsymbol{v})[\{y^{\pm 1}_{\beta,s}\}^{1\le s\le d_\beta}_{\beta\in\Delta^+}]. \tag{3.6}$$

Split the variables $\{x_{i,r}\}_{i\in I}^{1\le r\le \ell_i}$ into $\sum_{\beta\in\Delta^+} d_\beta$ groups corresponding to the above intervals, and specialize those in the s-th copy of $[\beta]$ to $\boldsymbol{v}^{-j(\beta)}\cdot y_{\beta,s},\ldots,\boldsymbol{v}^{-i(\beta)}\cdot y_{\beta,s}$ in the natural order (the variable $x_{k,r}$ gets specialized to $\boldsymbol{v}^{-k}y_{\beta,s}$). For $F=\frac{f(x_{1,1},\ldots,x_{n-1,\ell_{n-1}})}{\prod_{i=1}^{n-2}\prod_{1\le r\le \ell_i}^{1\le r'\le \ell_{i+1}}(x_{i,r}-x_{i+1,r'})}\in S^{(n)}_{\underline{\ell}}$, we define $\phi_{\underline{d}}(F)$ as the corresponding specialization of f. Note that $\phi_{\underline{d}}(F)$ is independent of our splitting of the variables $\{x_{i,r}\}_{i\in I}^{1\le r\le \ell_i}$ into groups and is symmetric in $\{y_{\beta,s}\}_{s=1}^{d_\beta}$ for any $\beta\in\Delta^+$.

Following [13, Definition 8.6], an element $F\in S^{(n)}_{\underline{k}}$ is called **good** if the following holds:

- F is of the form (3.2) with $f\in\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\{x_{i,r}^{\pm1}\}_{i\in I}^{1\le r\le k_i}]$;
- $\phi_{\underline{d}}(F)$ is divisible by $(\boldsymbol{v}-\boldsymbol{v}^{-1})^{\sum_{\beta\in\Delta^+} d_\beta(i(\beta)-j(\beta))}$ for any $\underline{d}$ such that $\sum_{i\in I}k_i\alpha_i=\sum_{\beta\in\Delta^+}d_\beta\beta$.

Let $\mathsf{S}^{(n)}_{\underline{k}}\subset S^{(n)}_{\underline{k}}$ denote the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-submodule of all *good* elements. Set $\mathsf{S}^{(n)}:=\bigoplus_{\underline{k}\in\mathbb{N}^I}\mathsf{S}^{(n)}_{\underline{k}}$.

Theorem 3.7. *[13, Theorem 8.8] The $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $\Psi\colon U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)\xrightarrow{\sim}S^{(n)}$ of Theorem 3.4 gives rise to a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra isomorphism $\Psi\colon \mathsf{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)\xrightarrow{\sim}\mathsf{S}^{(n)}$.*

Remark 3.8. In [13, Theorem 3.34], we also established the shuffle realization of $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ by showing that the isomorphism Ψ of Theorem 3.4 gives rise to a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra isomorphism $\Psi\colon\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)\xrightarrow{\sim}\mathfrak{S}^{(n)}$, where $\mathfrak{S}^{(n)}$ denotes the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-submodule of all **integral** elements, see [13, Definition 3.31]. We skip the definition of the latter as it is not presently needed.

3.3. Extended shuffle algebra $S^{(n),\ge}$

For the purpose of the next section, define the *extended shuffle algebra* (cf. [12, §3.4]) $S^{(n),\ge}$ by adjoining pairwise commuting generators $\{\psi^-_{i,-s},(\psi^-_{i,0})^{-1}\}_{i\in I}^{s\in\mathbb{N}}$ with the following relations:

$$\psi_i^-(z)\star F=\left[F\left(\{x_{j,r}\}_{j\in I}^{1\le r\le k_j}\right)\cdot\prod_{j\in I}\prod_{r=1}^{k_j}\frac{\zeta_{i,j}(z/x_{j,r})}{\zeta_{j,i}(x_{j,r}/z)}\right]\star\psi_i^-(z) \tag{3.9}$$

for any $F\in S^{(n)}_{\underline{k}}$, where we set $\psi_i^-(z):=\sum_{s\ge0}\psi^-_{i,-s}z^s$, $\star$ denotes the multiplication in $S^{(n),\ge}$, and the ζ-factors in the right-hand side are all expanded in the non-negative powers of z.

Then, the isomorphism Ψ of Theorem 3.4 naturally extends to a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism

$$\Psi\colon U^{\ge}_{\boldsymbol{v}}(L\mathfrak{sl}_n)\xrightarrow{\sim}S^{(n),\ge}\ \text{ with }\ \psi^-_{i,-s}\mapsto\psi^-_{i,-s}. \tag{3.10}$$

Evoking the new Drinfeld Hopf algebra structure on $U^{\ge}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of Section 2.4, (3.10) induces the one on $S^{(n),\ge}$. The corresponding coproduct Δ is given by (cf. [12, Proposition 3.5]):

$$\Delta(\psi_i^-(z))=\psi_i^-(z)\otimes\psi_i^-(z), \tag{3.11}$$

$$\Delta(F)=\sum_{\underline{\ell}\in\mathbb{N}^I}^{\underline{\ell}\le\underline{k}}\frac{\left[\prod_{i\in I}\prod_{r>\ell_i}\psi_i^-(x_{i,r})\right]\star F(x_{i,r\le\ell_i}\otimes x_{i,s>\ell_i})}{\prod_{i,j\in I}\prod_{r\le\ell_i}^{s>\ell_j}\zeta_{j,i}(x_{j,s}/x_{i,r})} \tag{3.12}$$

for $F\in S^{(n)}_{\underline{k}}$, where $\underline{\ell}\le\underline{k}$ iff $\ell_i\le k_i$ for all i. We expand the right-hand side of (3.12) in the non-negative powers of $x_{j,s}/x_{i,r}$ for $s>\ell_j$ and $r\le\ell_i$, put the symbols $\psi^-_{i,-s}$ to the very left, then all powers of $x_{i,r}$ with $r\le\ell_i$, then the $\otimes$ sign, and finally all powers of $x_{i,r}$ with $r>\ell_i$.

4. Main result

The main result of this note is the duality of the integral forms $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ with respect to the $\mathbb{C}(\boldsymbol{v})$-valued new Drinfeld pairing φ on $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of Proposition 2.34:

Theorem 4.1. *(a)* $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n) = \{x \in U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n) | \varphi(x,y) \in \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}] \text{ for all } y \in \mathfrak{U}^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n)\}$.
(b) $\mathsf{U}^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n) = \{y \in U^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n) | \varphi(x,y) \in \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}] \text{ for all } x \in \mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)\}$.

Proof. We shall prove only part (a) as the proof of part (b) is completely analogous. Our proof is crucially based on the PBWD result for $\mathfrak{U}^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, Theorem 2.16(c1), the shuffle realization of $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, Theorem 3.7, and the shuffle realization (3.12) of the new Drinfeld coproduct. We will first establish Theorem 4.1(a) for $n=2$, and then generalize our arguments to $n>2$.

<u>*Case $n=2$*</u>. For $n=2$, we shall skip the first index i. Set $\widetilde{f}(z) := (\boldsymbol{v}-\boldsymbol{v}^{-1})f(z) = \sum_{r\in\mathbb{Z}} \widetilde{f}_r z^{-r}$. Then, Theorem 4.1(a) is equivalent to:

$$\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_2) = \left\{x \in U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_2) : \varphi\left(x, \widetilde{f}(z_1)\cdots\widetilde{f}(z_N)\right) \in \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][[z_1^{\pm1},\ldots,z_N^{\pm1}]]\ \forall\ N\right\}. \tag{4.2}$$

The algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ is $\mathbb{Z}$-graded via $\deg(e_r)=1, \deg(f_r)=-1, \deg(\psi^{\pm}_{\pm s})=0$ for any $r\in\mathbb{Z}, s\in\mathbb{N}$. In particular, $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_2) = \oplus_{k\in\mathbb{N}} U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_2)[k]$ with $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_2)[k]$ consisting of all degree k elements. Due to (2.35), the new Drinfeld pairing φ is *of degree zero*, that is

$$\varphi(x,y) = 0 \text{ for homogeneous elements } x,y \text{ with } \deg(x)+\deg(y)\neq 0. \tag{4.3}$$

Since $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_2)[1]$ is spanned by $\{e_r\}_{r\in\mathbb{Z}}$ and $\varphi(e_r,\widetilde{f}(z_1)) = z_1^r = \Psi(e_r)_{|x_1\mapsto z_1}$, we get

$$\varphi\left(x, \widetilde{f}(z_1)\right) = \Psi(x)_{|x_1\mapsto z_1} \quad \text{for any} \quad x \in U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_2)[1]. \tag{4.4}$$

Combining (4.4) with the shuffle formulas (3.11, 3.12) for the new Drinfeld coproduct Δ and the property (2.32), we obtain the general formula for the pairing with $\widetilde{f}(z_1)\cdots\widetilde{f}(z_N)$:

Lemma 4.5. *For $x \in U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_2)[k]$, we have*

$$\varphi\left(x, \widetilde{f}(z_1)\cdots\widetilde{f}(z_N)\right) = \delta_{k,N}\cdot\Psi(x)_{|x_r\mapsto z_r}\cdot\prod_{1\le r<s\le N}\zeta^{-1}(z_r/z_s) \tag{4.6}$$

with the factors $\zeta^{-1}(z_r/z_s)$ expanded in the non-negative powers of z_s/z_r.

Proof. Due to (4.3), we have $\varphi\left(x, \widetilde{f}(z_1)\cdots\widetilde{f}(z_N)\right) = 0$ if $k\neq N$. Henceforth, we will assume $k=N$. Set $F := \Psi(x) \in S^{(2)}_N$, so that $F = F(x_1,\ldots,x_N)$ is a symmetric Laurent polynomial.

Due to the property (2.32), we have

$$\varphi\left(x, \widetilde{f}(z_1)\cdots\widetilde{f}(z_N)\right) = \varphi\left(\Delta^{(N-1)}(x), \widetilde{f}(z_1)\otimes\cdots\otimes\widetilde{f}(z_N)\right), \tag{4.7}$$

where $\Delta^{(\ell)}\colon U^{\geq}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \to U^{\geq}_{\boldsymbol{v}}(L\mathfrak{sl}_n)^{\otimes(\ell+1)}$ $(\ell\in\mathbb{Z}_{>0})$ are defined inductively via

$$\Delta^{(1)} := \Delta \text{ and } \Delta^{(\ell)} := (\Delta\otimes \mathrm{Id}^{\otimes(\ell-1)})\circ\Delta^{(\ell-1)} \text{ for } \ell\geq 2.$$

Evoking the formulas (3.11, 3.12) and the property (4.3), we obtain

$$\varphi\left(\Delta^{(N-1)}(x), \widetilde{f}(z_1)\otimes\cdots\otimes\widetilde{f}(z_N)\right) = \varphi\left(\Psi^{-1}(G), \widetilde{f}(z_1)\otimes\cdots\otimes\widetilde{f}(z_N)\right), \tag{4.8}$$

where

$$G = \frac{(\prod_{r=2}^N \psi^-(x_r)\otimes\prod_{r=3}^N \psi^-(x_r)\otimes\cdots\otimes\psi^-(x_N)\otimes 1)\star F(x_1\otimes x_2\otimes\cdots\otimes x_N)}{\prod_{1\le r<s\le N}\zeta(x_s/x_r)}. \tag{4.9}$$

Recalling the properties (2.32, 4.3) and the formula (4.4), we get

$$\begin{gathered}\varphi\left(\psi^-(t_1)\cdots\psi^-(t_\ell)x, \widetilde{f}(z_1)\right) = \\ \prod_{r=1}^{\ell}\varphi\left(\psi^-(t_r),\psi^+(z_1)\right)\cdot\varphi\left(x,\widetilde{f}(z_1)\right) = \prod_{r=1}^{\ell}\frac{\zeta(t_r/z_1)}{\zeta(z_1/t_r)}\cdot\Psi(x)_{|x_1\mapsto z_1}\end{gathered} \tag{4.10}$$

for $x \in U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_2)[1]$, with the right-hand side expanded in the non-negative powers of t_r/z_1. Combining (4.7)–(4.10), we finally obtain

$$\varphi\left(x, \widetilde{f}(z_1)\cdots\widetilde{f}(z_N)\right) = \Psi(x)_{|x_r\mapsto z_r}\cdot\prod_{1\le r<s\le N}\zeta^{-1}(z_r/z_s). \tag{4.11}$$

This completes our proof of Lemma 4.5. □

Thus, the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-submodule of $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_2)$ defined by the right-hand side of (4.2) is $\mathbb{N}$-graded. Moreover, $x \in U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_2)[k]$ satisfies $\varphi(x, \widetilde{f}(z_1)\cdots\widetilde{f}(z_N)) \in \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][[z_1^{\pm1},\ldots,z_N^{\pm1}]]$ for all N if and only if $\Psi(x)\in\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][x_1^{\pm1},\ldots,x_k^{\pm1}]$. The latter is equivalent to the inclusion $x\in \mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_2)$, due to Theorem 3.7 (as all specialization maps ϕ_* of (3.6) are trivial for $n=2$).

This completes our proof of Theorem 4.1(a) in the smallest rank case $n = 2$.

<u>*Case* $n > 2$</u>. For any $1\le j\le i\le n-1$, define the series

$$\widetilde{f}_{j;i}(z_j,\ldots,z_i) := (\boldsymbol{v}-\boldsymbol{v}^{-1})[\cdots[[f_j(z_j), f_{j+1}(z_{j+1})]_{\boldsymbol{v}}, f_{j+2}(z_{j+2})]_{\boldsymbol{v}},\cdots, f_i(z_i)]_{\boldsymbol{v}}. \tag{4.12}$$

Note that $\widetilde{f}_{j;i}(z_j,\ldots,z_i)\in\mathfrak{U}_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)[[z_j^{\pm1},\ldots,z_i^{\pm1}]]$, and its coefficients encode $\widetilde{f}_{\alpha_j+\ldots+\alpha_i,\underline{r}}$ of (2.14) for all possible decompositions $\underline{r}\in\mathbb{Z}^{i-j+1}$. For $i=j$, we shall denote $\widetilde{f}_{j;j}(z)$ simply by $\widetilde{f}_j(z)$, so that $\widetilde{f}_{j;i}(z_j,\ldots,z_i) := (\boldsymbol{v}-\boldsymbol{v}^{-1})^{j-i}[\cdots[[\widetilde{f}_j(z_j),\widetilde{f}_{j+1}(z_{j+1})]_{\boldsymbol{v}},\widetilde{f}_{j+2}(z_{j+2})]_{\boldsymbol{v}}\cdots,\widetilde{f}_i(z_i)]_{\boldsymbol{v}}$. Similar to the $n=2$ case treated above, our primary goal is to compute the new Drinfeld pairing with products of these $\widetilde{f}_{j;i}(z_j,\ldots,z_i)$.

The algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is $\mathbb{Z}^I$-graded via $\deg(e_{i,r})=1_i, \deg(f_{i,r})=-1_i, \deg(\psi^{\pm}_{i,\pm s})=\underline{0}$ for all $i\in I, r\in\mathbb{Z}, s\in\mathbb{N}$, where $\underline{0}=(0,\ldots,0)$ and $1_i=(0,\ldots,1,\ldots,0)$ with 1 placed at the i-th spot. In particular, $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)=\oplus_{\underline{k}\in\mathbb{N}^I}U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)[\underline{k}]$ with $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)[\underline{k}]$ consisting of all degree $\underline{k}$ elements. Due to (2.35), the new Drinfeld pairing φ is *of degree zero*, that is

$$\varphi(x,y)=0 \text{ for homogeneous elements } x,y \text{ with } \deg(x)+\deg(y)\neq\underline{0}. \tag{4.13}$$

Similar to (4.4), we obtain

$$\varphi\left(x,\widetilde{f}_j(z_j)\right)=\Psi(x)_{|x_{j,1}\mapsto z_j}\quad\text{for any}\quad x\in U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)[1_j]. \tag{4.14}$$

The following result generalizes (4.14) and is proved completely analogously to Lemma 4.5:

Lemma 4.15. *For $x \in U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)[\underline{k}]$ and any collection $j_1, \ldots, j_N \in I$, we have*

$$\varphi\left(x, \widetilde{f}_{j_1}(z^{(1)}_{j_1}) \cdots \widetilde{f}_{j_N}(z^{(N)}_{j_N})\right) = \delta_{\underline{k}, 1_{j_1}+\ldots+1_{j_N}} \cdot \Psi(x)_{|x_{i,r} \mapsto z^{(*)}_{j_*}} \cdot \prod_{1 \le r < s \le N} \zeta^{-1}_{j_r, j_s}\left(z^{(r)}_{j_r}/z^{(s)}_{j_s}\right)$$

with the factors $\zeta^{-1}_{j_r,j_s}(z^{(r)}_{j_r}/z^{(s)}_{j_s})$ expanded in the non-negative powers of $z^{(s)}_{j_s}/z^{(r)}_{j_r}$.

Remark 4.16. The specialization $\Psi(x)_{|x_{i,r}\mapsto z^{(*)}_{j_*}}$ in Lemma 4.15 should be understood as follows. For each $i \in I$, there are k_i variables $\{x_{i,r}\}_{r=1}^{k_i}$ ("of color i") featuring in $\Psi(x)$. Since $k_i = \#\{1 \le t \le N | j_t = i\}$, say $j_{t_{i,1}} = \ldots = j_{t_{i,k_i}} = i$, then we specialize $x_{i,r} \mapsto z^{(t_{i,r})}_{j_{t_{i,r}}}$ in $\Psi(x)$.

In what follows, we use the convention that

$$\frac{1}{z-w} \text{ represents the series } \sum_{m=0}^{\infty} z^{-m-1} w^m. \tag{4.17}$$

For $1 \le j \le i \le n-1$, consider a graph $Q_{j,i}$ whose vertices are labeled by $j, j+1, \ldots, i$ and the vertices $k, k+1$ $(j \le k < i)$ are connected by a single edge. Let $\mathrm{Or}_{j,i}$ denote the set of all orientations π of $Q_{j,i}$. Evoking (4.17), for $\pi \in \mathrm{Or}_{j,i}$ and $j \le k < i$, define $\zeta^{-1}_{\pi,k}(z,w)$ via

$$\zeta^{-1}_{\pi,k}(z,w) := \begin{cases} (z-w) \cdot \frac{1}{z-\boldsymbol{v}w}, & \text{if } k \to k+1 \text{ in } \pi \\ \boldsymbol{v}(z-w) \cdot \frac{1}{w-\boldsymbol{v}z}, & \text{if } k \leftarrow k+1 \text{ in } \pi \end{cases}. \tag{4.18}$$

Simplifying all $[a,b]_{\boldsymbol{v}}$ as $ab - \boldsymbol{v}ba$ in (4.12), thus expressing the latter as a sum of 2^{i-j} terms, Lemma 4.15 implies the formula for the new Drinfeld pairing with $\widetilde{f}_{j;i}(z_j, \ldots, z_i)$:

Lemma 4.19. *For $x \in U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)[\underline{k}]$ and $1 \le j \le i < n$, we have*

$$\varphi\left(x, \widetilde{f}_{j;i}(z_j, \ldots, z_i)\right) = \frac{\delta_{\underline{k}, 1_j+\ldots+1_i}}{(\boldsymbol{v}-\boldsymbol{v}^{-1})^{i-j}} \cdot \Psi(x)_{|x_{k,1}\mapsto z_k} \cdot \sum_{\pi \in \mathrm{Or}_{j,i}} \prod_{j \le k < i} \zeta^{-1}_{\pi,k}(z_k, z_{k+1}). \tag{4.20}$$

Remark 4.21. The denominator of $\Psi(x)_{|x_{k,1}\mapsto z_k}$ is canceled by the numerators of $\zeta^{-1}_{*,*}$-factors.

Corollary 4.22. *If $\varphi\left(x, \widetilde{f}_{j;i}(z_j, \ldots, z_i)\right) \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][[z_j^{\pm 1}, \ldots, z_i^{\pm 1}]]$, then $\phi_{\underline{d}}(\Psi(x))$ is divisible by $(\boldsymbol{v}-\boldsymbol{v}^{-1})^{i-j}$, where $\phi_{\underline{d}}$ is the specialization map (3.6) with $\underline{d} = \{d_\beta\}, d_\beta = \delta_{\beta, \alpha_j+\ldots+\alpha_i}$.*

Proof. Due to (4.13), we may assume that $x \in U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)[1_j + \ldots + 1_i]$, so that

$$\Psi(x) = \frac{p(x_{j,1}, \ldots, x_{i,1})}{(x_{j,1}-x_{j+1,1})\cdots(x_{i-1,1}-x_{i,1})} \text{ with } p \in \mathbb{C}(v)[x^{\pm 1}_{j,1}, \ldots, x^{\pm 1}_{i,1}]. \tag{4.23}$$

First, let us assume that $p(x_{j,1}, \ldots, x_{i,1}) = x^{a_j}_{j,1} \cdots x^{a_i}_{i,1}$. Pick sufficiently small integers $r_{j+1}, \ldots, r_i \ll 0$, so that $a_k + r_k < 0$ for $j < k \le i$. Then, evaluating the coefficient of $\prod_{k=j+1}^{i} z_k^{-r_k}$ in the right-hand side of (4.20), we get a nonzero contribution only from $\pi \in \mathrm{Or}_{j,i}$ with $k \to k+1$ for all $j \le k < i$. Moreover, the corresponding contribution equals

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})^{j-i}\boldsymbol{v}^{\mathsf{A}} z_j^{\mathsf{B}} \cdot \left(z_j^{a_j}(\boldsymbol{v}^{-1}z_j)^{a_{j+1}} \cdots (\boldsymbol{v}^{j-i}z_j)^{a_i}\right) \tag{4.24}$$

with

$$\mathsf{A} = \sum_{j<k\le i} (j-k)(r_k - 1 + \delta_{k,i}),\ \mathsf{B} = \sum_{j<k\le i} (r_k - 1). \tag{4.25}$$

Note that A, B of (4.25) are actually independent of $a_j, \ldots, a_i$. Thus, for any x as above and the associated Laurent polynomial p of (4.23), comparing the coefficients of $\prod_{k=j+1}^{i} z_k^{-r_k}$ in (4.20) for sufficiently small $r_{j+1}, \ldots, r_i \ll 0$, we obtain

$$\begin{gathered}\varphi\left(x, (\boldsymbol{v}-\boldsymbol{v}^{-1})[\cdots[f_j(z), f_{j+1,r_{j+1}}]_{\boldsymbol{v}}, \cdots, f_{i,r_i}]_{\boldsymbol{v}}\right) = \\ (\boldsymbol{v}-\boldsymbol{v}^{-1})^{j-i} \cdot \boldsymbol{v}^{\mathsf{A}} z^{\mathsf{B}} \cdot p(z, \boldsymbol{v}^{-1}z, \ldots, \boldsymbol{v}^{j-i}z).\end{gathered} \tag{4.26}$$

Combining (4.26) and the definition of $\phi_{\underline{d}}$ with $\underline{d} = \{d_\beta\}, d_\beta = \delta_{\beta,\alpha_j+\ldots+\alpha_i}$, (3.6), we see that $\phi_{\underline{d}}(\Psi(x))$ is indeed divisible by $(\boldsymbol{v}-\boldsymbol{v}^{-1})^{i-j}$. This completes our proof of Corollary 4.22. □

Combining (4.20) with the shuffle formulas (3.11, 3.12) for the new Drinfeld coproduct Δ and the property (2.32), we obtain the formula for the pairing with $\prod_{r=1}^{N} \widetilde{f}_{j_r;i_r}(z_{j_r}^{(r)}, \ldots, z_{i_r}^{(r)})$:

Lemma 4.27. *For $x \in U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)[\sum_{r=1}^{N}\sum_{k=j_r}^{i_r} 1_k]$, we have*

$$\begin{gathered}\varphi\left(x, \widetilde{f}_{j_1;i_1}(z_{j_1}^{(1)}, \ldots, z_{i_1}^{(1)}) \cdots \widetilde{f}_{j_N;i_N}(z_{j_N}^{(N)}, \ldots, z_{i_N}^{(N)})\right) = \prod_{r<s} \prod_{j_r \le k \le i_r}^{j_s \le \ell \le i_s} \zeta_{k,\ell}^{-1}(z_k^{(r)}/z_\ell^{(s)}) \times \\ (\boldsymbol{v}-\boldsymbol{v}^{-1})^{\sum_{r=1}^{N}(j_r-i_r)} \cdot \Psi(x)_{|x_{i,r}\mapsto z_i^{(*)}} \cdot \prod_{r=1}^{N}\left(\sum_{\pi_r \in \mathrm{Or}_{j_r,i_r}} \prod_{j_r \le k < i_r} \zeta_{\pi_r,k}^{-1}(z_k^{(r)}, z_{k+1}^{(r)})\right)\end{gathered} \tag{4.28}$$

with the factors $\zeta_{k,\ell}^{-1}(z_k^{(r)}/z_\ell^{(s)})$ expanded in the non-negative powers of $z_\ell^{(s)}/z_k^{(r)}$.

Remark 4.29. The specialization $\Psi(x)_{|x_{i,r}\mapsto z_i^{(*)}}$ in (4.28) should be understood as follows. For each $i \in I$, there are $k_i = \#\{1 \le t \le N | j_t \le i \le i_t\}$ variables $\{x_{i,r}\}_{r=1}^{k_i}$ ("of color i") featuring in $\Psi(x)$. If $1 \le t_{i,1} < \ldots < t_{i,k_i} \le N$ denote the corresponding indices, such that $j_{t_{i,r}} \le i \le i_{t_{i,r}}$, then we specialize $x_{i,r} \mapsto z_i^{(t_{i,r})}$ in $\Psi(x)$, cf. Remark 4.16.

Since the proof of Lemma 4.27 is entirely analogous to that of Lemma 4.5, we leave details to the interested reader. Similar to Corollary 4.22, we obtain the following result:

Corollary 4.30. *If $\varphi\left(x, \widetilde{f}_{j_1;i_1}(z_{j_1}^{(1)}, \ldots, z_{i_1}^{(1)}) \cdots \widetilde{f}_{j_N;i_N}(z_{j_N}^{(N)}, \ldots, z_{i_N}^{(N)})\right)$ is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-valued Laurent polynomial in $\{z_i^{(r)}\}_{1\le r\le N}^{j_r \le i \le i_r}$, then $\phi_{\underline{d}}(\Psi(x))$ is divisible by $(\boldsymbol{v}-\boldsymbol{v}^{-1})^{\sum_{r=1}^{N}(i_r-j_r)}$, where $\phi_{\underline{d}}$ is the specialization map (3.6) with $\underline{d} = \{d_\beta\}$, $d_{\alpha_j+\ldots+\alpha_i} = \#\{1 \le r \le N | j_r = j, i_r = i\}$.*

This result, combined with Theorem 3.7, implies the inclusion "⊇" in Theorem 4.1(a):

Proposition 4.31. $\mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \supseteq \{x \in U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) | \varphi(x,y) \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ for all $y \in \mathfrak{U}_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)\}$.

Thus, it remains to establish the opposite inclusion "⊆" in Theorem 4.1(a):

Proposition 4.32. $\mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \subseteq \{x \in U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) | \varphi(x,y) \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ for all $y \in \mathfrak{U}_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)\}$.

Proof. Our proof will proceed in several steps by reducing to the setup in which (4.26) applies.

First, evoking the shuffle realization of the subalgebra $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, Theorem 3.7, and of the new Drinfeld coproduct, formula (3.12), we immediately obtain the following result:

Lemma 4.33. *For any $x \in \mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, we have $\Delta(x) = x'_{(1)}x_{(1)} \otimes x_{(2)}$ in the Sweedler notation (the right-hand side is an infinite sum) with $x_{(1)}, x_{(2)} \in \mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $x'_{(1)}$–a monomial in $\psi^{-}_{*,*}$.*

Combining Lemma 4.33 with (2.32), it thus suffices to show that given any $x \in \mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)[1_j + \ldots + 1_i]$, x'–a monomial in $\psi^{-}_{*,*}$, and $\underline{r} = (r_j, \ldots, r_i) \in \mathbb{Z}^{i-j+1}$, we have

$$\varphi\left(x'x, \widetilde{f}_{\alpha_j+\ldots+\alpha_i,\underline{r}}\right) \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]. \tag{4.34}$$

Evoking the property (2.32) once again, for the proof of (4.34) it suffices to establish

$$\varphi\left(x, \widetilde{f}_{\alpha_j+\ldots+\alpha_i,\underline{r}}\right) \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}] \tag{4.35}$$

for any $x \in \mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)[1_j + \ldots + 1_i]$ and any $\underline{r} = (r_j, \ldots, r_i) \in \mathbb{Z}^{i-j+1}$.

We shall prove (4.35) by induction in $i-j$. The base case $i = j$ is obvious. Given $x \in \mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)[1_j+\ldots+1_i]$, the validity of (4.35) for $\underline{r} = (r_j, \ldots, r_i)$ with sufficiently small $r_{j+1}, \ldots, r_i \ll 0$ is due to (4.26). We shall call such $\underline{r} \in \mathbb{Z}^{i-j+1}$ "x–sufficiently small". To establish (4.35) for a general $\underline{r}$, we shall apply the PBWD result of Theorem 2.16(c1) with the choice of decompositions $\underline{r}(\beta, r)$ such that $\underline{r}(\alpha_j + \ldots + \alpha_i, r)$ are all "x–sufficiently small". Then, combining Theorem 2.16(c1) with the $\mathbb{Z}^I$-grading on $U^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, we see that the element $\widetilde{f}_{\alpha_j+\ldots+\alpha_i,\underline{r}}$ can be written as a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$–linear combination of $\widetilde{f}_{\alpha_j+\ldots+\alpha_i,\underline{r}(\alpha_j+\ldots+\alpha_i,r)}$ $(r \in \mathbb{Z})$ and degree > 1 ordered monomials in $\widetilde{f}_{\alpha_{j'}+\ldots+\alpha_{i'},\underline{r}'}$ with $j \le j' \le i' \le i$ and $i' - j' < i - j$. By the above observation, $\varphi\left(x, \widetilde{f}_{\alpha_j+\ldots+\alpha_i,\underline{r}(\alpha_j+\ldots+\alpha_i,r)}\right) \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ for any $r \in \mathbb{Z}$. Finally, we claim that the pairing of x with degree > 1 monomials in $\widetilde{f}_{\alpha_{j'}+\ldots+\alpha_{i'},\underline{r}'}$ is $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-valued. To see this, apply the above arguments ((2.32) and Lemma 4.33) again, subsequently reducing to (4.35) with (j, i) replaced by (j', i'), which is established by the induction assumption.

This completes our proof of Proposition 4.32. □

Combining Propositions 4.31, 4.32, we get the proof of Theorem 4.1(a) for arbitrary n. □

Remark 4.36. The above proof of Theorem 4.1 is crucially based on our construction of the entire family of Poincaré-Birkhoff-Witt-Drinfeld bases of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ for all decompositions $\underline{r}$ (rather than picking the canonical one $\underline{r}^{(0)}$ of (2.13)).

Remark 4.37. The finite counterpart of Theorem 4.1, where $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is replaced with $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ and the new Drinfeld pairing φ is replaced with the Drinfeld-Jimbo pairing, is well-known, see e.g. [3, §3]. In [3], this duality is extended to the duality between the Cartan-extended subalgebras $\mathsf{U}'^{,\ge}(\mathfrak{sl}_n)$ and $\mathfrak{U}'^{,\le}(\mathfrak{sl}_n)$ (resp. $\mathsf{U}'^{,\le}(\mathfrak{sl}_n)$ and $\mathfrak{U}'^{,\ge}(\mathfrak{sl}_n)$)), where $'$ is used to indicate yet enlarged algebras by adding more Cartan elements, see [3, Theorem 3.1].

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