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Stability of Turing bifurcation in a weighted networked reaction–diffusion system[☆]

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ABSTRACT

By introducing a weighted networked structure to the classical reaction–diffusion system, we investigate the Turing bifurcation which changes the trivial equilibrium to the nontrivial equilibrium. We show the existence of Turing bifurcation if the diffusion rate is large. By a weakly nonlinear analysis, we induce the amplitude equation of Turing bifurcation. By analyzing the amplitude equation, we show that the Turing bifurcation is stable.

1. Introduction

Turing bifurcation is one of the central issues in reaction–diffusion systems due to its wide applications to physics, chemistry, biology and neurophysiology [1]. In recent years the application of big data has introduced networked structure to reaction–diffusion system, which can help to improve the accuracy of the real-world model. To be more specific, the direction of Gaussian diffusion for classical reaction–diffusion systems is isotropic, while the direction of diffusion for data-driven diffusion systems is anisotropy [2]. In order to study the anisotropy diffusion, we introduce a weighted networked structure to the classical reaction–diffusion system.

The weighted network can be described by the weighted graph. A graph $G = (V, E)$ includes vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. If vertex y is adjacent to vertex x, we denote $y \sim x$. A graph is weighted

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if each adjacent x and y is assigned a weight function ω. Here $\omega: V \times V \to [0, \infty)$ is a positive function satisfying that $\omega(x, y) = \omega(y, x)$ and $\omega(x, y) > 0$ if and only if $x \sim y$. Based on the weighted function, we propose out some definitions confined on weighed graph as follows:

$$d_\omega x := \sum_{y\sim x,\ y\in V} \omega(x, y) \tag{1.1}$$

$$\int_V f d_\omega (\text{or simply} \int_V f) := \sum_{x\in V} f(x) d_\omega x \tag{1.2}$$

$$D_{\omega,y} f(x) := (f(y) - f(x)) \sqrt{\frac{\omega(x, y)}{d_\omega x}} \tag{1.3}$$

$$\nabla_\omega f(x) := \left(D_{\omega,y} f(x)\right)_{y\sim x,\ y\in V} \tag{1.4}$$

$$\Delta_\omega f(x) := \sum_{y\sim x} (f(y) - f(x)) \frac{\omega(x, y)}{d_\omega x} \tag{1.5}$$

where (1.1)–(1.5) are called graph differential, graph integral, graph directional derivative, graph gradient, and graph Laplacian, respectively.

In this paper we consider a reaction–diffusion system defined on weighted networks:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\omega u + u - av + buv - u^3, & (x, t) \in V \times (0, +\infty), \\ \frac{\partial v}{\partial t} = d\Delta_\omega v + u - cv, & (x, t) \in V \times (0, +\infty), \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x), & x \in V. \end{cases} \tag{1.6}$$

Here a, b, c and d are positive constants. The Hopf bifurcation of system (1.6) with a unweighted network has been shown to exist in [3]. The Turing bifurcation of system (1.6) without a networked structure has been proved to exist in [4]. The reaction–diffusion systems with graph Laplacian have been considered in many papers [5–12], where various techniques have been proposed to investigate the existence and qualitative properties of solutions. Our aim here is to extend the Turing bifurcation result of the classical reaction–diffusion systems to graph Laplacian diffusion systems. Our main theorem is as follows:

Theorem 1.1. *Suppose that*

$$c < d,\ a \le \frac{(c + d)^2}{4d} \tag{1.7}$$

hold, then system (1.6) has the following dynamical properties:

(i) System (1.6) undergoes a Turing bifurcation at the trivial equilibrium $(0, 0)$ when $a = \frac{(d+c)^2}{4d}$.
(ii) Turing bifurcation is stable, i.e., when the bifurcation parameter a crosses $\frac{(d+c)^2}{4d}$, the stable trivial equilibrium $(0, 0)$ changes to a stable nontrivial equilibrium.

2. Preliminaries

Lemma 2.1 (*Lemma 2.1 of [10]*)**.** *For any pair of functions $f: V \to \mathbb{R}$ and $g: V \to \mathbb{R}$, the graph Laplacian Δ_ω satisfies that*

$$2\int_V f(-\Delta_\omega) g = \int_V \nabla_\omega f \cdot \nabla_\omega g = 2\int_V g(-\Delta_\omega) f. \tag{2.1}$$

In particular, in the case $f = g$, we have

$$2\int_V f(-\Delta_\omega) f = \int_V |\nabla_\omega f|^2. \tag{2.2}$$

Lemma 2.2. *Consider the eigenvalue problem*

$$\begin{cases} -\Delta_\omega \phi(x) = \lambda \phi(x), & x \in V, \\ \int_V \phi^2(x) = 1, \end{cases} \tag{2.3}$$

there exists a series of eigenvalues

$$\{\lambda_k\}_{k=1}^n: \ \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n,$$

whose associated eigenfunctions are $\{\phi_k\}_{k=1}^n$. *Moreover,*

$$\lambda_1 = \inf_{u \in L^2(V)} \frac{1}{2} \int_V |\nabla_\omega u|^2 \equiv 0. \tag{2.4}$$

Proof. Let λ be an eigenvalue of (2.3) with corresponding eigenfunction $\phi(x)$. We multiply the first equation of (2.3) by $\phi(x)$ and integrate the product over V. By using Lemma 2.1, we have $\lambda = \frac{1}{2}\int_V |\nabla_\omega \phi|^2$. We introduce a functional

$$F(u) = \frac{1}{2} \int_V |\nabla_\omega u|^2, \tag{2.5}$$

where the domain $D(F)$ of F is $D(F) := \{u: \ u \in L^2(V), \|u\|_{L^2(V)} = 1\}$.

Step 1: Determine λ_1. By virtue of the definition of F and (1.4), for any $u \in D(F)$, we have

$$\lim_{\|u\|_{L^2(V)} \to \infty} F(u) = \infty.$$

Hence F is a weakly lower semicontinuous in $D(F)$, that is, there exists $\phi_1 \in D(F)$ such that

$$\lambda_1 := F(\phi_1) = \inf_{u \in D(F)} F(u). \tag{2.6}$$

Here λ_1 and ϕ_1 is a solution of the eigenvalue problem (2.3). Moreover, it easy to verify that $\lambda_1 = 0$ and $\phi_1(x) = 1$, that is, λ_1 is simple.

Step 2: Determine λ_2. By using the weakly lower semicontinuous property of F, the second minimization problem admits a solution, i.e., there is $\phi_2(x) \in L^2(V)$ with $\|\phi_2\|_{L^2(V)} = 1$ and $\phi_2 \perp \phi_1$ such that

$$\lambda_2 := F(\phi_2) = \inf_{u \in L^2(V)} \{F(u): \ \|u\|_{L^2(V)} = 1, u \perp \phi_1\}, \tag{2.7}$$

where $u \perp \phi_1$ means that u and ϕ_1 are orthogonal, that is, $\int_V u\phi_1 = 0$. By the definition of (2.7), we have $\lambda_2 > \lambda_1$.

Repeating the above steps, we can construct a sequence $\{\lambda_k\}_{k=1}^n$ whose associated eigenfunctions are $\{\phi_k\}_{k=1}^n$. Owing to the fact that V is finite, we induce that n is finite. We thus complete the proof. □

3. Proof of the main theorem

Proof of Theorem 1.1. *Step (i):* By setting $\mathbf{U} := (u, v)^T$, we rewrite system (1.6) to the following form:

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{L}\mathbf{U} + \mathbf{R}(\mathbf{U}), \tag{3.1}$$

where

$$\mathbf{L} = \begin{pmatrix} \Delta_\omega + 1 & -a \\ 1 & d\Delta_\omega - c \end{pmatrix}, \text{ and } \mathbf{R}(\mathbf{U}) = \begin{pmatrix} buv - u^3 \\ 0 \end{pmatrix}. \tag{3.2}$$

In view of Lemma 2.2, we can assume the solution of the linearization of system (3.1) possessing the following form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{\sigma t}\begin{pmatrix} \sum_{i=1}^{n} u_i\phi_i \\ \sum_{i=1}^{n} v_i\phi_i \end{pmatrix}, \tag{3.3}$$

where ϕ_i is the eigenfunction of the corresponding eigenvalue λ_i. Substituting (3.3) into the linearization of system (3.1), we have

$$\sigma e^{\sigma t}\begin{pmatrix} \sum_{i=1}^{n} u_i\phi_i \\ \sum_{i=1}^{n} v_i\phi_i \end{pmatrix} = -e^{\sigma t}\begin{pmatrix} \sum_{i=1}^{n} u_i\lambda_i\phi_i \\ d\sum_{i=1}^{n} v_i\lambda_i\phi_i \end{pmatrix} + e^{\sigma t}\begin{pmatrix} 1 & -a \\ 1 & -c \end{pmatrix}\begin{pmatrix} \sum_{i=1}^{n} u_i\phi_i \\ \sum_{i=1}^{n} v_i\phi_i \end{pmatrix}. \tag{3.4}$$

In terms of the orthogonality of ϕ_i for $i = 1, 2, \ldots, n$, (3.4) implies to

$$\sigma e^{\sigma t}\begin{pmatrix} u_i\phi_i \\ v_i\phi_i \end{pmatrix} = -e^{\sigma t}\begin{pmatrix} u_i\lambda_i\phi_i \\ dv_i\lambda_i\phi_i \end{pmatrix} + e^{\sigma t}\begin{pmatrix} 1 & -a \\ 1 & -c \end{pmatrix}\begin{pmatrix} u_i\phi_i \\ v_i\phi_i \end{pmatrix}. \tag{3.5}$$

Since $e^{\sigma t}$ and ϕ_i are nonzero functions, we then have

$$\begin{pmatrix} \sigma + \lambda_i - 1 & a \\ -1 & \sigma + d\lambda_i + c \end{pmatrix}\begin{pmatrix} u_i \\ v_i \end{pmatrix} = 0. \tag{3.6}$$

Since there exists a nontrivial solution of (3.6), the eigenvalues σ are roots of the characteristic polynomials

$$\sigma^2 + g(\lambda_i)\sigma + h(\lambda_i) = 0, \tag{3.7}$$

where $g(\lambda_i) = (1 + d)\lambda_i - 1 + c$ and $h(\lambda_i) = d\lambda_i^2 + (c - d)\lambda_i - c + a$.

System (3.7) has a zero eigenvalue if and only if $h(\lambda_i) = 0$. Since (1.7) holds, $h(\lambda_i)$ has a single minimum at (λ_c, a_c), where

$$\lambda_c = \frac{d - c}{2d},\ a_c = \frac{(d + c)^2}{4d}, \tag{3.8}$$

such that $h(\lambda_i) = 0$. Since (3.7) possessing a zero eigenvalue means that Turing bifurcation occurs, system (1.6) undergoes a Turing bifurcation at the trivial equilibrium $(0, 0)$ when $a = \frac{(d+c)^2}{4d}$.

Step (ii): We rewrite the solution of system (3.1) as a weakly nonlinear expansion depending upon ε, which is the dimensionless distance from the bifurcation parameter threshold a_c. Here we set $\varepsilon^2 = \frac{a_c - a}{a_c}$. Since the slow mode of Turing pattern bifurcation is the active mode, we introduce the slow time $T = \varepsilon^2 t$ and expand both u and v as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \varepsilon^2\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + \varepsilon^3\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} + \cdots. \tag{3.9}$$

We rewrite the linear operator $\mathbf{L}$ in the following form:

$$\mathbf{L} = \mathbf{L_c} + (a_c - a)\mathbf{M}, \tag{3.10}$$

where

$$\mathbf{L_c} = \begin{pmatrix} \Delta_\omega + 1 & -a_c \\ 1 & d\Delta_\omega - c \end{pmatrix},\ \mathbf{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.11}$$

Substituting (3.9) into the system (3.1) and collecting same powers of ε, we obtain at orders $\varepsilon^j (j = 1, 2, 3)$ the sequences of equations as follows:

$$\begin{aligned}
O(\varepsilon):&\quad \mathbf{L_c}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = 0, \\
O(\varepsilon^2):&\quad \mathbf{L_c}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -bu_1v_1 \\ 0 \end{pmatrix}, \\
O(\varepsilon^3):&\quad \mathbf{L_c}\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \frac{\partial}{\partial T}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} -a_cv_1 - b(u_1v_2 + u_2v_1) + u_1^3 \\ 0 \end{pmatrix}.
\end{aligned} \tag{3.12}$$

According to (3.8), $(u_1, v_1)^T$ is the eigenvector corresponding to system (3.6) with $\sigma = 0$. Therefore, at $O(\varepsilon)$ the solution is given in the form

$$(u_1, v_1)^T = \rho A(T)\phi_c, \quad \text{with } \rho := (\rho_1, \rho_2)^T = (\frac{c+d}{2}, 1)^T,$$

where λ_c is given in (3.8) and ϕ_c is the associated eigenfunction, $A(T)$ is the amplitude of the solution and is still unknown at this level. The form of $A(T)$ will be determined by the perturbational term of the higher order.

Next, we turn to $O(\varepsilon^2)$. The equation is written in the form

$$\mathbf{L_c}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = -bA^2\phi_c^2 \begin{pmatrix} \frac{c+d}{2} \\ 0 \end{pmatrix}. \tag{3.13}$$

After some direct computation, the solution of system (3.13) is

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = A^2 \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + A^2\phi_c^2 \begin{pmatrix} \rho_3 \\ \rho_4 \end{pmatrix},$$

where $\rho_3 = \frac{bc(c+d)}{2(a_c-c)}$ and $\rho_4 = \frac{b(c+d)}{2(a_c-c)}$.

Now $O(\varepsilon^3)$. The equation is written in the form

$$\begin{aligned} \mathbf{L_c}\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} &= \frac{dA}{dT}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}\phi_c + \begin{pmatrix} -a_c\rho_2 A - 2b\rho_1\rho_2 A^3 \\ 0 \end{pmatrix}\phi_c \\ &\quad + \begin{pmatrix} \rho_1^3 - b\rho_1\rho_4 - b\rho_2\rho_3 \\ 0 \end{pmatrix} A^3\phi_c^3. \end{aligned} \tag{3.14}$$

We recall the Fredholm solubility condition. The adjoint operator of $\mathbf{L_c}$ is denoted by $\mathbf{L_c^+}$. The nontrivial kernel of the operator $\mathbf{L_c^+}$ is $(1, \frac{c+d}{2d})^T\phi_c$. After multiplying (3.14) by $(1, \frac{c+d}{2d})^T\phi_c$ and integrating the product over V, we obtain

$$\frac{dA}{dT} = \sigma A - LA^3, \tag{3.15}$$

where

$$\sigma = -\frac{c+d}{2(1+d)}, \text{ and } L = \frac{2bd}{1+d}. \tag{3.16}$$

(3.15) is called the amplitude equation of Turing bifurcation. Owing to $L > 0$, the bifurcation likes a supercritical Pitchfork bifurcation. Hence we conclude that Turing bifurcation is stable, which completes the proof.

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