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SEMI-WAVES WITH Λ-SHAPED FREE BOUNDARY FOR NONLINEAR STEFAN PROBLEMS: EXISTENCE

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ABSTRACT. We show that for a monostable, bistable or combustion type of nonlinear function $f(u)$, the Stefan problem

$$\begin{cases} u_t - \Delta u = f(u),\ u > 0 & \text{for } x \in \Omega(t) \subset \mathbb{R}^{n+1}, \\ u = 0 \text{ and } u_t = \mu |\nabla_x u|^2 & \text{for } x \in \partial\Omega(t), \end{cases}$$

has a traveling wave solution whose free boundary is Λ-shaped, and whose speed is κ, where κ can be any given positive number satisfying $\kappa > \kappa_*$, and κ_* is the unique speed for which the above Stefan problem has a planar traveling wave solution. To distinguish it from the usual traveling wave solutions, we call it a semi-wave solution. In particular, if $\alpha \in (0, \pi/2)$ is determined by $\sin\alpha = \kappa_*/\kappa$, then for any finite set of unit vectors $\{\nu_i : 1 \le i \le m\} \subset \mathbb{R}^n$, there is a Λ-shaped semi-wave with traveling speed κ, with traveling direction $-e_{n+1} = (0, ..., 0, -1) \in \mathbb{R}^{n+1}$, and with free boundary given by a hypersurface in $\mathbb{R}^{n+1}$ of the form

$$x_{n+1} = \phi(x_1, ..., x_n) = \Phi^*(x_1, ..., x_n)) + O(1) \text{ as } |(x_1, ..., x_n)| \to \infty,$$

where

$$\Phi^*(x_1, ..., x_n) := -\left[\max_{1\le i\le m} \nu_i \cdot (x_1, ..., x_n)\right] \cot\alpha$$

is a solution of the eikonal equation $|\nabla\Phi| = \cot\alpha$ on $\mathbb{R}^n$.

1. Introduction

We study semi-wave solutions to the following Stefan problem with a nonlinear source term,

$$\begin{cases} u_t - \Delta u = f(u),\ u > 0 & \text{for } x \in \Omega(t), \\ u = 0 \text{ and } u_t = \mu |\nabla_x u|^2 & \text{for } x \in \partial\Omega(t), \end{cases} \tag{1.1}$$

where $\Omega(t) \subset \mathbb{R}^{n+1}$ ($n \ge 1$) is unbounded with boundary $\partial\Omega(t)$, μ is a given positive constant, and the source term f is a C^1 function satisfying $f(0) = 0$. In this problem, both $u(t, x)$ and $\Omega(t)$ are unknowns. The range of t in (1.1) will be $[0, \infty)$ when we consider a solution of (1.1) with a given initial value pair $(u(0, x), \Omega(0))$.

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When we look for a semi-wave solution of (1.1), the range of t is assumed to be the entire real line $\mathbb{R}^1$.

A solution pair $(u(t,x),\Omega(t))$ of (1.1) is called a semi-wave solution (or simply a semi-wave), if there exist a vector $\nu\in\mathbb{S}^n$, a positive constant κ, a function $v(x)$ and a domain (unbounded) $\Omega\subset\mathbb{R}^{n+1}$, so that

$$u(t,x)=v(x+t\kappa\nu),\ \Omega(t)=\Omega+t\kappa\nu.$$

In such a case, necessarily v satisfies

$$\begin{cases}-\Delta v+\kappa\nu\cdot\nabla v=f(v),\ v>0 & \text{for } x\in\Omega,\\ v=0 \text{ and } \kappa\nu\cdot\nabla v=\mu|\nabla v|^2 & \text{for } x\in\partial\Omega.\end{cases}$$

Here v, κ and Ω are all unknowns, and v is often referred to as the semi-wave profile, with κ the wave speed (in the direction $-\nu$).

Without loss of generality, we will assume $\nu=e_{n+1}=(0,...,0,1)\in\mathbb{R}^{n+1}$, and then the above problem becomes

$$\begin{cases}-\Delta v+\kappa v_{n+1}=f(v),\ v>0 & \text{for } x\in\Omega,\\ v=0 \text{ and } \kappa v_{n+1}=\mu|\nabla v|^2 & \text{for } x\in\partial\Omega.\end{cases}\tag{1.2}$$

Here v_{n+1} denotes the partial derivative $\partial_{x_{n+1}}v$.

In problem (1.1), if $\Omega(0)$ is bounded, then it is easy to show that $\Omega(t)$ remains bounded for every fixed $t>0$. Such an initial value problem (with $\Omega(0)$ bounded) has been studied extensively in recent years, starting with the paper [5], where the one space dimension case with a special monostable type of f was considered, and a spreading-vanishing dichotomy was proved for the long-time asymptotic behavior of the solution. In [5] problem (1.1) (in space dimension one with bounded $\Omega(0)$) was used to describe the spreading of a new or invading species, with the free boundary representing the spreading front. It was shown that when the species spread successfully, the asymptotic spreading speed is determined by the speed of the associated semi-wave. The research of [5] was extended in [6] to cover three types of nonlinearities, namely monostable type, bistable type and combustion type.

Recall that the nonlinear term f is always assumed to be C^1 in $[0,\infty)$ and $f(0)=0$. It is said to be of **monostable type** if it satisfies additionally

$$f(0)=f(1)=0,\ f'(0)>0>f'(1),\ (1-u)f(u)>0 \text{ for } u\in(0,1)\cup(1,\infty);$$

it is of **bistable type** if there exists $\theta\in(0,1)$ such that

$$\begin{cases} f(0)=f(\theta)=f(1)=0,\ f(u)<0 \text{ for } u\in(0,\theta)\cup(1,\infty),\ f(u)>0 \text{ for } u\in(\theta,1),\\ f'(0)<0,\ f'(1)<0,\ \int_0^1 f(u)du>0;\end{cases}$$

and it is of **combustion type** if there exists $\theta\in(0,1)$ such that

$$f(u)=0 \text{ for } u\in[0,\theta]\cup\{1\},\ (1-u)f(u)>0 \text{ for } u\in(\theta,1)\cup(1,\infty).$$

For these three types of nonlinearities, in one space dimension, it is shown in [6] that there is a unique semi-wave, whose speed is the asymptotic spreading speed of the species.

Lemma 1.1 (Theorem 6.2 of [6])**.** *Suppose that f is one of the three types of nonlinearities described above. Then there exists a constant $\kappa_*>0$ and a function*

$\psi \in C^2([0,+\infty))$ *satisfying*

$$
(1.3) \qquad \begin{cases} -\psi''(s) + \kappa_* \psi'(s) = f(\psi) \text{ for } s > 0, \\ \psi(0) = 0, \quad \psi'(0) = \frac{\kappa_*}{\mu}. \end{cases}
$$

Moreover, the pair (ψ, κ_*) *is unique, and* $\psi' > 0$ *in* $[0,+\infty)$, $\lim_{s\to+\infty} \psi(s) = 1$.

Clearly, for any unit vector $e \in \mathbb{R}^{n+1}$, $\psi(e \cdot x)$ is a solution of (1.2) with $\Omega = \{x \in \mathbb{R}^{n+1} : e \cdot x > 0\}$. The free boundary of this type of solutions is a hyperplane and the associated solutions

$$
u(t,x) := \psi(e \cdot x + t\kappa_*)
$$

of (1.1) are called planar semi-waves. Note that these planar semi-waves have the same speed κ_*.

In high space dimensions, (1.1) with bounded $\Omega(0)$ was studied in [4, 7, 8]. In this case the regularity of the free boundary is a highly nontrivial question, but the initial value problem always has a unique weak solution ([4]). The main feature of the behavior of the solution pair $(u(t,x), \Omega(t))$ in such a case is summarised in the following result of [8]:

(i) $\Omega(t)$ is expanding: $\overline{\Omega(0)} \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$.
(ii) $\partial\Omega(t) \setminus$ (convex hull of $\overline{\Omega(0)}$) is smooth ($C^{2,\alpha}$ if $f(u)$ is $C^{1,\alpha}$ near $u = 0$).
(iii) $\Omega_\infty := \cup_{t>0} \Omega(t)$ is either the entire space $\mathbb{R}^{n+1}$, or it is a bounded set.
(iv) When Ω_∞ is bounded, $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\Omega(t))} = 0$.
(v) When $\Omega_\infty = \mathbb{R}^{n+1}$, for all large t, $\partial\Omega(t)$ is a smooth closed hypersurface in $\mathbb{R}^{n+1}$, and there exists a continuous function $M(t)$ such that

$$
\partial\Omega(t) \subset \{x : M(t) - \frac{\pi}{2} d_0 \le |x| \le M(t)\},
$$

where d_0 is the diameter of $\Omega(0)$.
(vi) By a simple comparison argument and the result in [9], one sees that, when f is of monostable, bistable or combustion type,

$$
\lim_{t\to\infty} \frac{M(t)}{t} = \kappa_*,
$$

where $\kappa_* > 0$ is given in Lemma 1.1.

When $\Omega(0)$ is an unbounded domain, the initial value problem of (1.1) was considered in [3], and a unique weak solution exists for all time $t > 0$. The long-time asymptotic behaviour of the problem turns out to be strikingly different from the case $\Omega(0)$ is bounded, and is much more complicated. For example, some key steps in the approach to derive the regularity of the free boundary used in [8] do not work anymore in general, and the long-time asymptotic behaviour of the solution appears to differ considerably depending on the geometry of the unbounded domain $\Omega(0)$.

In [3], for a Fisher-KPP type monostable $f(u)$, the case $\Omega(0)$ lying between two parallel circular cones was examined. More precisely, let Λ^ϕ denote the cone with axis $e_{n+1} \in \mathbb{R}^{n+1}$, vertex the origin and half opening angle $\phi \in (0, \pi)$, given by

$$
\Lambda^\phi := \left\{ x \in \mathbb{R}^{n+1} \setminus \{0\} : \frac{x}{|x|} \cdot e_{n+1} > \cos\phi \right\}.
$$

Suppose that there exist $\xi_1 > \xi_2$ such that

$$
\Lambda^\phi + \xi_1 e_{n+1} \subset \Omega(0) \subset \Lambda^\phi + \xi_2 e_{n+1}.
$$

It is proved in [3] that if $\phi \in (\pi/2, \pi)$, then for any given small $\epsilon > 0$, there exists $T = T(\epsilon) > 0$ such that, for all $t > T$,

$$\Lambda^\phi - \left(\frac{\kappa_*}{\sin\phi} - \epsilon\right) te_{n+1} \subset \Omega(t) \subset \Lambda^\phi - \left(\frac{\kappa_*}{\sin\phi} + \epsilon\right) te_{n+1}.$$

This indicates that as $t \to \infty$, the free boundary $\partial\Omega(t)$ propagates to infinity in the direction $-e_{n+1}$ at roughly the speed $\kappa_*/\sin\phi$, with its shape approximated by the boundary of the circular cone Λ^ϕ.[1] It is conjectured that as $t \to \infty$, $(u(t,x), \Omega(t))$ converges to a conical shaped semi-wave of (1.1) in such a case, though little is known about such semi-waves of (1.1) so far.

In this paper, we study the existence of semi-waves with conical shaped free boundary, which we call Λ-shaped semi-waves (to be explained in more detail below). Their uniqueness and stability will be considered in a subsequent work.

We now explain our main results. For $x \in \mathbb{R}^{n+1}$, we will write $x = (y, x_{n+1})$ with $y \in \mathbb{R}^n$. For a given angle $\alpha \in (0, \pi/2)$, we consider the eikonal equation

$$|\nabla\Phi(y)| = \cot\alpha, \quad y \in \mathbb{R}^n. \tag{1.4}$$

We are only interested in concave solutions of (1.4). If Φ is a concave viscosity solution of (1.4), we will call the hypersurface in $\mathbb{R}^{n+1}$ given by the equation

$$x_{n+1} = \Phi(y), \;\; y \in \mathbb{R}^n$$

a concave eikonal surface with angle α (to $-e_{n+1}$ in $\mathbb{R}^{n+1}$). It is easily verified that the circular conical surface $x_{n+1} = -(\cot\alpha)\,|y|$ is such a surface. If $\{\nu_i : 1 \le i \le k\}$ are k distinct vectors in $\mathbb{S}^{n-1}$ and $\{\gamma_i : 1 \le i \le k\}$ are k arbitrary real numbers, then

$$x_{n+1} = \min_{1\le i\le k}[-(\cot\alpha)\,\nu_i \cdot y + \gamma_i]$$

is also a concave eikonal surface with angle α. The set of all viscosity solutions to (1.4) which are concave and continuous is described by the following result of [19]:

Lemma 1.2. *Let $\Phi \in C(\mathbb{R}^n)$. Then Φ is a concave viscosity solution of* (1.4) *if and only if there exists a lower semi-continuous map $\gamma : \mathbb{S}^{n-1} \to (-\infty, +\infty]$ such that*

$$\Phi(y) = \inf_{\nu\in\mathbb{S}^{n-1}}[-(\cot\alpha)\,\nu \cdot y + \gamma(\nu)].$$

Before introducing our results, we need another related equation, namely

$$\operatorname{div}\left(\frac{\nabla\Phi}{\sqrt{1+|\nabla\Phi|^2}}\right) = \kappa_* - \frac{\kappa_*/\sin\alpha}{\sqrt{1+|\nabla\Phi|^2}}, \;\; y \in \mathbb{R}^n, \tag{1.5}$$

where $\kappa_* > 0$ is from Lemma 1.1.

As explained in Section 2 below, a solution Φ to (1.5) generates a traveling wave of the forced mean curvature flow with speed $\kappa_*/\sin\alpha$ and force κ_* in the direction of $-e_{n+1}$ in $\mathbb{R}^{n+1}$.

For the semi-wave problem (1.2), it turns out that smooth concave sub-solutions of (1.5) play an important role. A very general class of such sub-solutions of (1.5) has been obtained in Proposition 4.3 of [19]. Here we describe one special case. Let $\alpha \in (0, \pi/2)$ and $\{\nu_i : i \in I\}$ be a finite or countable infinite set of unit vectors

[1]If $\phi \in (0, \pi/2)$, then the asymptotic profile of $\partial\Omega(t)$ is very different from $\partial\Lambda^\phi$; see Theorem 1.2 in [3] for a detailed description.

in $\mathbb{R}^n$. Let $\{\lambda_i : i \in I\}$ be a corresponding set of positive numbers such that $\sum_{i\in I} \lambda_i < \infty$. Then, by Proposition 4.3 of [19],

$$\Phi^*(y) := -\frac{2}{\kappa_* \sin\alpha} \ln\left(\sum_{i\in I} \lambda_i e^{\frac{\kappa_* \cos\alpha}{2}\, \nu_i\cdot y}\right) \tag{1.6}$$

is a smooth concave sub-solution to (1.5). Clearly, with $\gamma_i := -\frac{2\ln\lambda_i}{\kappa_* \sin\alpha}$, we have

$$\Phi^*_\infty(y) := \inf_{i\in I}\,[-(\cot\alpha)\,\nu_i \cdot y + \gamma_i] \geq \Phi^*(y) \tag{1.7}$$

and

$$\Phi^*(y) \geq \Phi^*_c(y) := -(\cot\alpha)\,|y| - \gamma^* \text{ with } \gamma^* := \frac{2\ln(\sum_{i\in I}\lambda_i)}{\kappa_* \sin\alpha}.$$

We are now ready to state a consequence of the main result of this paper.

Theorem 1.3. *Suppose the conditions in Lemma* 1.1 *are satisfied, and* $f \in C^{1,\sigma}([0,\delta])$ *for some* $\delta > 0$ *small and* $\sigma \in (0,1)$*. Then given any angle* $\alpha \in (0, \pi/2)$*, there exists a classical solution* $v(x)$ *to* (1.2) *with* $\kappa = \kappa^*/\sin\alpha$*, whose free boundary lies between the eikonal surface* $x_{n+1} = \Phi^*_\infty(y)$ *and the circular conical surface* $x_{n+1} = \Phi^*_c(y)$*.*

Our main result is the following.

Theorem 1.4. *Suppose the conditions in Lemma* 1.1 *are satisfied, and* $f \in C^{1,\sigma}([0,\delta])$ *for some* $\delta > 0$ *small and* $\sigma \in (0,1)$*. Let* $x_{n+1} = \Phi_\infty(y)$ *be a concave continuous eikonal surface with angle* $\alpha \in (0,\pi/2)$*, and* Φ *a smooth concave sub-solution of* (1.5) *satisfying* $\Phi(y) \leq \Phi_\infty(y)$ *in* $\mathbb{R}^n$*. Then* (1.2) *has a classical solution* $v(x)$ *satisfying*

$$\psi(d(x,\Gamma)) \geq v(y, x_{n+1}) \geq \psi(x_{n+1} - \Phi_\infty(y)), \tag{1.8}$$

where $\psi(s)$ *is given in Lemma* 1.1 *extended by* 0 *for* $s < 0$*, and* $d(x,\Gamma)$ *denotes the signed distance from* $x = (y, x_{n+1})$ *to* $\Gamma := \{x_{n+1} = \Phi(y)\}$*, with* $d(x,\Gamma) > 0$ *for* x *lying above* Γ*.*

From (1.8) we see that the free boundary $\partial\{v > 0\}$ of v is sandwiched between the eikonal surface $x_{n+1} = \Phi_\infty(y)$ (from above) and the surface $x_{n+1} = \Phi(y)$ (from below). Very often, the sub-solution Φ satisfies

$$\Phi(y) \geq \Phi_c(y) := -\cot\alpha\ |y| - \gamma$$

for some $\gamma \in \mathbb{R}^1$. In such a case clearly the free boundary of the semi-wave profile v lies between the eikonal surface $x_{n+1} = \Phi_\infty(y)$ and the circular conical surface $x_{n+1} = \Phi_c(y)$. We will use the term "Λ-shaped" to roughly describe the free boundary (i.e., the front) of such a semi-wave, and call v a semi-wave (profile) with Λ-shaped free boundary, or a Λ-shaped semi-wave (profile). In the literature, some surfaces of this type are called V-shaped, conical shaped, or "pyramidal shaped"; see, for example, [2, 11–14, 16, 20–26].

We now follow [19] to explain the generality of semi-waves that can be obtained by Theorem 1.4. Let us observe that the conical surface

$$\Gamma_c := \{x_{n+1} = -(\cot\alpha)|y|\}$$

is a 1-homogeneous concave eikonal surface with angle α. For any finite set of unit vectors $\{\nu_i : i \in I_0\} \subset \mathbb{R}^n$, clearly

$$\Gamma_\infty := \{x_{n+1} = \inf_{i\in I_0}\,[-(\cot\alpha)\,\nu_i \cdot y]\}$$

is a 1-homogeneous concave eikonal surface with angle α.

By Proposition 2.3 of [19], Φ_∞ is a continuous concave 1-homogeneous viscosity solution of (1.4) if and only if there is a finite or countable infinite set of unit vectors $\{\nu_i : i \in I\} \subset \mathbb{R}^n$ such that

$$\Phi_\infty(y) = \inf_{i\in I}[-(\cot\alpha)\,\nu_i \cdot y].$$

We may now choose a positive sequence $\{\lambda_i : i \in I\}$ such that $\sum_{i\in I}\lambda_i = 1$ and define Φ^* by (1.6). We then have

$$\Phi^*(y) \geq -(\cot\alpha)\,|y|.$$

Let $\Phi^*_\infty(y)$ be the eikonal solution given in (1.7). By Theorem 1.4, there exists a classical solution $v(x)$ to (1.2) with $\kappa = \kappa^*/\sin\alpha$, whose free boundary lies between the eikonal surface $x_{n+1} = \Phi^*_\infty(y)$ and the surface $x_{n+1} = \Phi^*(y)$. By the proof of Theorem 1.1 in [19], as $|y| \to \infty$,

$$\Phi^*(y) = \Phi_\infty(y) + o(|y|),\ \Phi^*_\infty(y) = \Phi_\infty(y) + o(|y|).$$

Therefore, if we use $x_{n+1} = \phi(y)$ to denote the equation for the free boundary of v, then

$$\phi(y) = \Phi_\infty(y) + o(|y|) \text{ as } |y| \to \infty. \tag{1.9}$$

In other words, the following conclusion holds.

Corollary 1.5. *For any given angle $\alpha \in (0, \pi/2)$, and any given continuous concave 1-homogeneous eikonal surface with angle α, expressed by $x_{n+1} = \Phi_\infty(y)$, there exists a classical semi-wave solution of* (1.1) *with speed $\frac{\kappa^*}{\sin\alpha}$, whose profile $v(x)$ has free boundary $x_{n+1} = \phi(y)$ with ϕ satisfying* (1.9)*; namely, near infinity the free boundary behaves like the given 1-homogeneous eikonal surface $x_{n+1} = \Phi_\infty(y)$ with angle α.*

Remark 1.6. In Corollary 1.5, if the set $\{\nu_i : i \in I\}$ is finite, say $|I| = k$, then apart from (1.9), we have additionally

$$0 \leq \Phi^*_\infty(y) - \phi(y) \leq \frac{2\ln k}{c_0 \sin\alpha},$$

which is a simple consequence of the estimate (8) in Theorem 1.2 of [19] and the estimate (1.8) in Theorem 1.4 here.

We note that when $f \equiv 0$, (1.1) reduces to the well-known one phase Stefan problem, which arises from the melting of ice in contact with water, and has been extensively studied; see, for example, [1, 10, 15, 18] and the references therein. In the setting of (1.1), $\Omega(t)$ is the water region, which is surrounded by ice, and $u(t, x)$ represents the temperature of water at location x and time t. However, in such a case, the expansion of $\Omega(t)$ is at a rate below $O(t)$, and therefore no semi-wave is expected to exist.

2. Some preparations

2.1. **Forced mean curvature flow and Fermi coordinates.** A hypersurface $\Gamma \subset \mathbb{R}^{n+1}$ is a travelling wave of the mean curvature flow with a force κ_* in the $-e_{n+1}$ direction if it satisfies, for some $\kappa > 0$, the equation

$$H(x) = \kappa_* - \kappa e_{n+1} \cdot \nu(x) \quad \text{for } x \in \Gamma, \tag{2.1}$$

where ν is the upward unit normal vector of Γ and H is the mean curvature of Γ with respect to ν. The constant κ is called the speed of the traveling wave in the $-e_{n+1}$ direction. See [19] for more details.

If such a hypersurface Γ is the graph of a function $\Phi \in C^2(\mathbb{R}^n)$, namely

$$\Gamma := \{(y, x_{n+1}) : x_{n+1} = \Phi(y),\ y \in \mathbb{R}^n\},$$

then Φ satisfies the equation

$$\operatorname{div}\left(\frac{\nabla\Phi}{\sqrt{1+|\nabla\Phi|^2}}\right) = \kappa_* - \frac{\kappa}{\sqrt{1+|\nabla\Phi|^2}},\quad y \in \mathbb{R}^n. \tag{2.2}$$

In the following we will only consider concave solutions of (2.2), and so Γ is a concave hypersurface. Then the distance to Γ from all points above Γ (i.e., $x = (y, x_{n+1})$ with $x_{n+1} > \Phi(y)$) forms a smooth and convex function.

In order to see the relationship between (2.2) and (1.2), it is better to write (1.2) in Fermi coordinates with respect to the hypersurface Γ. See [17, Section 2] for a detailed description of the Fermi coordinates.

Let (y, z) be the Fermi coordinates of a point $x \in \mathbb{R}^{n+1}$ with respect to the graph Γ of Φ, where $y \in \mathbb{R}^n$ is determined by the property that the point $(y, \Phi(y)) \in \Gamma$ is closest to x, and z denotes the signed distance of x to $(y, \Phi(y))$ (positive if x is above Γ). In other words, given a point $x \in \mathbb{R}^{n+1}$, if the nearest point on Γ to x is $(y, \Phi(y))$ and z is its signed distance to Γ, then

$$x = (y, \Phi(y)) + z\nu(y),$$

where

$$\nu(y) := \frac{(-\nabla\Phi(y), 1)}{\sqrt{1+|\nabla\Phi(y)|^2}}$$

is the upward unit normal vector of Γ at $(y, \Phi(y))$.

For each $z \in [0, \infty)$, let Γ_z be the hypersurface consisting of points whose signed distance to Γ equals z. In particular, Γ_0 is Γ itself. Denote by Δ_{Γ_z} the Laplace-Beltrami operator of Γ_z under the induced metric. The mean curvature of Γ_z at (y, z) is denoted by $H(y, z)$, which satisfies

$$H(y, z) = \sum_{i=1}^{n} \frac{\kappa_i}{1 - z\kappa_i}, \tag{2.3}$$

where κ_1,..., κ_n are the principal curvatures of Γ_0 at $(y, 0)$.

Let us note that in the Fermi coordinates the Euclidean Laplacian is expressed in the following way:

$$\Delta_{\mathbb{R}^{n+1}} = \Delta_{\Gamma_z} - H(y, z)\partial_z + \partial_{zz}.$$

2.2. **Construction of super-solutions.** We now explain how a super-solution to (1.2) can be constructed by making use of a concave sub-solution of (2.2). We will always assume that $\kappa > \kappa_*$, and hence there exists $\alpha \in (0, \pi/2)$ such that

$$\kappa = \frac{\kappa^*}{\sin\alpha},$$

and so (2.2) coincides with (1.5).

Let v be a solution of (1.2). In the Fermi coordinates, it satisfies

$$\begin{cases} -\Delta_{\Gamma_z} v + H(y,z)v_z - v_{zz} + \kappa e_n \cdot \nabla v = f(v) & \text{for } (y,z) \in \Omega := \{v > 0\}, \\ v = 0 \text{ and } \kappa e_n \cdot \nabla v = \mu |\nabla v|^2 & \text{for } (y,z) \in \partial\Omega. \end{cases} \tag{2.4}$$

In this form, the relationship between (1.2) and (1.5) becomes clearer.

Suppose Φ is a smooth concave sub-solution of (1.5), that is,

$$\operatorname{div}\left(\frac{\nabla\Phi(y)}{\sqrt{1+|\nabla\Phi(y)|^2}}\right) \geq \kappa_* - \kappa e_{n+1}\cdot\nu(y) \quad \text{for } y \in \mathbb{R}^n. \tag{2.5}$$

Since Γ is concave,

$$H(y,0) = \operatorname{div}\left(\frac{\nabla\Phi(y)}{\sqrt{1+|\nabla\Phi(y)|^2}}\right) \leq 0.$$

Moreover, as all principal curvatures of Γ are nonpositive, by (2.3), we have

$$H(y,0) \leq H(y,z) \leq 0, \quad \forall z > 0. \tag{2.6}$$

Lemma 2.1. *Define $u^*(x) := \psi(z)$ in the Fermi coordinates; then u^* is a super-solution of* (1.2)*, namely*

$$\begin{cases} -\Delta_{\mathbb{R}^{n+1}} u^* + \kappa e_{n+1}\cdot\nabla u^* \geq f(u^*) & \text{in } \{u^* > 0\}, \\ \kappa e_{n+1}\cdot\nabla u^* \geq \mu|\nabla u^*|^2 & \text{on } \partial\{u^* > 0\}. \end{cases}$$

Proof. In $\{u^* > 0\}$, we have

$$\begin{aligned} \Delta_{\mathbb{R}^{n+1}} u^* &= -\psi'(z)H(y,z) + \psi''(z) \\ &\leq -\psi'(z)H(y,0) + \kappa_*\psi'(z) - f(\psi(z)) \quad \big[\text{by (2.6) and (1.3)}\big] \\ &\leq \kappa\psi'(z)e_{n+1}\cdot\nu(y) - f(\psi(z)) \quad \big[\text{by (2.5)}\big] \\ &= \kappa e_{n+1}\cdot\nabla u^* - f(u^*). \end{aligned}$$

On $\partial\{u^* > 0\}$, which is exactly Γ, since $H(y,0) \leq 0$, by (2.5) we get

$$e_{n+1}\cdot\nu(y) \geq \frac{\kappa_*}{\kappa}, \tag{2.7}$$

and thus

$$\begin{aligned} \kappa e_{n+1}\cdot\nabla u^* &= \kappa\psi'(0)e_{n+1}\cdot\nu(y) \\ &= \frac{\kappa\mu}{\kappa_*}\psi'(0)^2 e_{n+1}\cdot\nu(y) \quad \big[\text{by (1.3)}\big] \\ &\geq \mu\psi'(0)^2 \quad \big[\text{by (2.7)}\big] \\ &= \mu|\nabla u^*|^2. \end{aligned}$$

Hence u^* is a super-solution of (1.2). □

Remark 2.2. From the definitions of the Fermi coordinates and u^*, it is easily seen that for any $x \in \mathbb{R}^{n+1}$ lying above the concave surface

$$\Gamma := \{x = (y, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = \Phi(y),\ y \in \mathbb{R}^n\},$$

we have

$$u^*(x) = \psi(d(x,\Gamma)), \quad \text{where } d(x,\Gamma) \text{ denotes the distance from } x \text{ to } \Gamma.$$

3. Existence of semi-waves with Λ-shaped free boundary

The approach here is somewhat similar in spirit to work on traveling wave solutions with curved fronts ([16, 20–22], etc.), though our techniques are very different.

Suppose that the conditions in Theorem 1.4 are satisfied. We may now use Φ to construct a sup-solution u^* of (1.2) as described in subsection 2.2. Let us recall from Remark 2.2 that, for every $x \in \Omega^* := \{(y, x_{n+1}) : x_{n+1} > \Phi(y)\}$,

$$u^*(x) = \psi(d(x,\Gamma)) \ \text{ with } \Gamma := \partial\Omega^* = \{x_{n+1} = \Phi(y)\}. \tag{3.1}$$

Lemma 3.1. *The super-solution u^* defined in* (3.1) *has the following properties:*

(i) *$\partial\{u^* > 0\} = \Gamma$ is smooth and u^* is smooth up to the boundary in $\{u^* > 0\} = \Omega^*$;*

(ii) *u^* is monotone increasing in all the directions $\nu \in \mathbb{S}^n \cap \mathcal{C}$, where*

$$\mathcal{C} := \{x = (y, x_{n+1}) : x_{n+1} > (\cot\alpha)\,|y|\}.$$

Proof. Part (i) is obvious from the definition. We now consider part (ii). Since $\Phi(y)$ is concave and

$$\operatorname{div}\left(\frac{\nabla\Phi}{\sqrt{1+|\nabla\Phi|^2}}\right) \geq \kappa_* - \frac{\kappa_*/\sin\alpha}{\sqrt{1+|\nabla\Phi|^2}}, \quad y \in \mathbb{R}^n,$$

we have

$$\kappa_* - \frac{\kappa_*/\sin\alpha}{\sqrt{1+|\nabla\Phi|^2}} \leq 0 \quad \text{for } y \in \mathbb{R}^n,$$

and hence $|\nabla\Phi(y)| \leq \cot\alpha$ for all $y \in \mathbb{R}^n$. Now a simple geometric consideration shows that the distance function $d(x,\Gamma)$ is increasing in the directions $\nu \in \mathbb{S}^n \cap \mathcal{C}$ for $x \in \Omega^*$. The conclusion in (ii) now follows from the monotonicity of the function ψ. □

3.1. Solution of an initial value problem.

We consider the solution U to the original Stefan problem (1.1) with initial value

$$(u(0,x), \Omega(0)) = (u^*(x), \Omega^*).$$

As described in [4] and [3], U is the unique weak solution of the following singular Cauchy problem:

$$\begin{cases} \partial_t\alpha(U) - \Delta U = f(U) & \text{in } (0,+\infty)\times\mathbb{R}^{n+1}, \\ U(0,x) = u^*(x) & \text{in } \mathbb{R}^{n+1}, \end{cases} \tag{3.2}$$

where

$$\alpha(\xi) := \begin{cases} \xi, & \text{if } \xi > 0 \\ -\mu^{-1}, & \text{otherwise,} \end{cases}$$

and $U(t,\cdot)$, u^* are understood as extended to the entire $\mathbb{R}^{n+1}$ by 0. More precisely, the weak solution of (3.2) is defined as the limit of the solutions U_m $(m = 1, 2, ...)$ to the regularised approximation problem

$$\begin{cases} \partial_t[\alpha_m(U_m)] - \Delta U_m = f(U_m) & \text{in } (0,+\infty)\times\mathbb{R}^{n+1}, \\ U_m(0,x) = u^*(x) & \text{in } \mathbb{R}^{n+1}, \end{cases} \tag{3.3}$$

where α_m is a sequence of smooth functions with the following properties:

$$\begin{cases} \lim_{m\to\infty} \alpha_m(\xi) = \alpha(\xi) \text{ uniformly in any compact subset of } \mathbb{R}^1 \setminus \{0\}, \\ \lim_{m\to\infty} \alpha_m(0) = -\mu^{-1}, \alpha'_m(\xi) \geq 1 \text{ for all } \xi \in \mathbb{R}^1,\ m \geq 1, \\ \xi - \mu^{-1} \leq \alpha_m(\xi) \leq \xi \text{ for all } \xi \in \mathbb{R}^1,\ m \geq 1. \end{cases}$$

By the maximum principle and standard parabolic regularity theory, $U_m(t,x) > 0$ for $t > 0$ and $x \in \mathbb{R}^{n+1}$, and U_m is a classical solution.

Lemma 3.2. *For each $t \geq 0$, $U_m(t,\cdot)$ is monotone increasing in every direction $\nu \in \mathbb{S}^n \cap \mathcal{C}$.*

Proof. For any $\nu \in \mathbb{S}^n \cap \mathcal{C}$, by differentiating (3.3) in the direction of ν, we get

$$\begin{cases} \partial_t[\alpha'_m(U_m)\partial_\nu U_m] - \Delta(\partial_\nu U_m) = f'(U_m)\partial_\nu U_m & \text{in } (0,+\infty) \times \mathbb{R}^{n+1}, \\ \partial_\nu U_m(0,x) = \partial_\nu u^*(x) \gneqq 0 & \text{in } \mathbb{R}^{n+1}. \end{cases}$$

Hence by the properties of α_m and the maximal principle, we deduce that $\partial_\nu U_m(t,x) > 0$ for any $t > 0$ and $x \in \mathbb{R}^{n+1}$. □

Lemma 3.3. *For any $\nu \in \mathbb{S}^n \cap \mathcal{C}$ and $t \geq 0$, $U(t,\cdot)$ is monotone nondecreasing in the direction ν; moreover, for $t > 0$ and $x \in \Omega(t)$, $\partial_\nu U(t,x) > 0$.*

Proof. By [4] and [3], as $m \to +\infty$, for any $t > 0$, $U_m(t,\cdot)$ converges to $U(t,\cdot)$ in $W^2_{loc}(\mathbb{R}^{n+1})$. Hence $U(t,\cdot)$ is monotone nondecreasing in every direction $\nu \in \mathbb{S}^n \cap \mathcal{C}$. Since U is smooth inside its positivity set $\{U > 0\} = \{(t,x) : x \in \Omega(t), t > 0\}$, we have $\partial_\nu U(t,x) \geq 0$ for $t > 0$ and $x \in \Omega(t)$. Moreover, $v := \partial_\nu U$ satisfies

$$v_t - \Delta v = f'(U)v,\ v \geq 0 \text{ for } x \in \Omega(t),\ t > 0.$$

By the strong maximum principle, we deduce that either $v > 0$ or $v \equiv 0$ (recall that $\Omega(t_1) \subset \Omega(t_2)$ if $0 \leq t_1 < t_2$). If the latter alternative happens, then $U(t,\cdot)$ takes a positive constant value in $\Omega(t)$ along any line parallel to ν, which leads to a contradiction as one can choose such a line which intersects $\partial\Omega(t)$. Hence $U_\nu(t,x) > 0$ for $t > 0$ and $x \in \Omega(t)$. □

Lemma 3.4. *Suppose that $f \in C^{1,\sigma}([0,\delta])$ for some $\delta > 0$ small and $\sigma \in (0,1)$. Then for any $t > 0$, $\partial\Omega(t)$ is a $C^{2,\sigma}$ hypersurface. Furthermore, U is a classical solution of* (1.1).

Proof. In view of Lemma 3.3, we can argue as on page 989 of [8] to deduce that $\partial\Omega(t)$ is a Lipschitz graph in the x_{n+1} direction, and its variation in t is $\frac{1}{2}$-Hölder continuous. The analysis in Sections 3.2 and 3.3 of [8] can then be applied to conclude that $\partial\Omega(t)$ is $C^{2,\sigma}$ and U is a classical solution. □

Remark 3.5. From Lemma 3.3 we see that $\partial\Omega(t)$ is uniformly Lipschitz for $t > 0$. It follows from the proof of Lemma 3.4 that $\partial\Omega(t)$ is uniformly $C^{2,\sigma}$ for $t > 0$. This fact will be used below.

3.2. **Existence of a semi-wave.** For simplicity, we will write

$$\kappa := \frac{\kappa^*}{\sin\alpha},\ \ \beta := \cot\alpha.$$

Define

$$V(t,x) := U(t, x - \kappa t e_{n+1}).$$

We will show that $v(x) := \lim_{t\to\infty} V(t,x)$ exists, and $u(t,x) := v(x + t\kappa e_{n+1})$ is a semi-wave solution of (1.1).

Define, for $t \geq 0$,

$$D(t) := \{V(t,\cdot) > 0\} = \{U(t,\cdot) > 0\} + \kappa t e_{n+1} = \Omega(t) + \kappa t e_{n+1}.$$

Lemma 3.6. *V is non-increasing in t.*

Proof. Set

$$\overline{U}(t,x) := u^*(x + t\kappa e_{n+1}).$$

Since u^* is a super-solution of (1.2), it is easily checked by using Theorem 2.10 of [3] that $\overline{U}$ is a super-solution of (1.1) with initial function $u^*(x)$. We may then apply Theorem 2.4 of [3] to conclude that

$$U(t,x) \leq \overline{U}(t,x) = u^*(x + t\kappa e_{n+1}) \text{ for } t > 0,\ x \in \mathbb{R}^{n+1}.$$

It follows that

$$V(t,x) \leq u^*(x) \text{ for } t > 0,\ x \in \mathbb{R}^{n+1}.$$

For any fixed $s > 0$ we now regard $V(s+t,x)$ as the solution of (1.1) with initial function $V(s,x)$. Since $V(s,x) \leq u^*(x)$ we can apply Theorem 2.4 in [3] again to obtain

$$V(s+t,x) \leq V(t,x) \text{ for } t > 0,\ x \in \mathbb{R}^{n+1}.$$

Thus $V(t,x)$ is nonincreasing in t. □

By Lemma 1.2 the eikonal solution Φ_∞ can be expressed as

$$\Phi_\infty(y) = \inf_{\nu \in \mathbb{S}^{n-1}} [-\beta\nu \cdot y + \gamma(\nu)],$$

where $\gamma : \mathbb{S}^{n-1} \to (-\infty, +\infty]$ is a lower semi-continuous function. If we denote $S_\gamma := \{\nu \in \mathbb{S}^{n-1} : \gamma(\nu) < +\infty\}$, then clearly

$$\Phi_\infty(y) = \inf_{\nu \in S_\gamma} [-\beta\nu \cdot y + \gamma(\nu)].$$

Lemma 3.7. *For any $t \geq 0$ and $x = (y, x_{n+1}) \in \mathbb{R}^{n+1}$,*

$$V(t,x) \geq u_*(x) := \psi\left(\frac{\kappa^*}{\kappa}\big[x_{n+1} - \Phi_\infty(y)\big]\right) = \sup_{\nu \in S_\gamma} \psi\left(\frac{\kappa^*}{\kappa}\big[x_{n+1} + \beta\nu \cdot y - \gamma(\nu)\big]\right).$$

Hence, for $t > 0$,

$$\Omega^* = \{u^* > 0\} = D(0) \supset D(t) \supset \{u_* > 0\} = \{x_{n+1} > \Phi_\infty(y)\} := \Omega_\infty.$$

Proof. Since $V(t,x) \geq 0$ and $u_*(x) = 0$ for $x \in \{x_{n+1} \leq \Phi_\infty(y)\}$ the required inequality holds trivially in $\{x_{n+1} \leq \Phi_\infty(y)\}$.

It remains to prove the desired inequality for $x \in \Omega_\infty = \{x_{n+1} > \Phi_\infty(y)\}$. Note that

$$\Omega_\infty = \cup_{\nu \in S_\gamma} D_\nu \text{ with } D_\nu := \{x_{n+1} > -\beta\, \nu \cdot y + \gamma(\nu)\}.$$

From $\Phi_\infty(y) \geq \Phi(y)$ in $\mathbb{R}^n$ we deduce $D_\nu \subset \Omega_\infty \subset \Omega^*$ for $\nu \in S_\gamma$. Since D_ν is a half space whose boundary ∂D_ν lies above Γ due to $\Phi_\infty(y) \geq \Phi(y)$, one easily verifies that, for $x = (y, x_{n+1}) \in D_\nu$,

$$d(x, \partial D_\nu) = \frac{\kappa^*}{\kappa}\big[x_{n+1} + \beta\nu \cdot y - \gamma(\nu)\big] \leq d(x, \Gamma).$$

Therefore,

$$u_\nu(x) := \psi\left(\frac{\kappa^*}{\kappa}\big[x_{n+1} + \beta\nu \cdot y - \gamma(\nu)\big]\right) \leq \psi\left(d(x,\Gamma)\right) = u^*(x) \text{ in } D_\nu.$$

By Theorem 2.4 of [3], we obtain $U_\nu(t,x) \le U(t,x)$ for $t>0$ and $x\in\mathbb{R}^{n+1}$, where U_ν denotes the unique weak solution of (1.1) with initial function $u_\nu(x)$. But as D_ν is a half space, from the choice of $u_\nu(x)$ it is easily seen that

$$U_\nu(t,x) = u_\nu(x+t\kappa e_{n+1}) = \psi\left(\frac{\kappa^*}{\kappa}\big[x_{n+1}+\beta\nu\cdot y-\gamma(\nu)\big]+\kappa^* t\right) \text{ for } t>0, x\in D_\nu.$$

Hence, for $t>0$ and $x\in D_\nu$,

$$V(t,x) = U(t,x-\kappa t e_{n+1}) \ge U_\nu(t,x-\kappa t e_{n+1}) = \psi\left(\frac{\kappa^*}{\kappa}\big[x_{n+1}+\beta\nu\cdot y-\gamma(\nu)\big]\right).$$

Now for fixed $x\in\Omega_\infty$, define

$$I_x := \{\nu\in S_\gamma : x\in D_\nu\},\quad I_x^c := S_\gamma\setminus I_x.$$

Then the above analysis shows that

$$V(t,x) \ge \sup_{\nu\in I_x}\psi\left(\frac{\kappa^*}{\kappa}\big[x_{n+1}+\beta\nu\cdot y-\gamma(\nu)\big]\right).$$

For $\nu\in I_x^c$, we have $x_{n+1}+\beta\nu\cdot y-\gamma(\nu)<0$ and hence

$$\psi\left(\frac{\kappa^*}{\kappa}\big[x_{n+1}+\beta\nu\cdot y-\gamma(\nu)\big]\right)=0.$$

Therefore

$$V(t,x) \ge \sup_{\nu\in I_x\cup I_x^c}\psi\left(\frac{\kappa^*}{\kappa}\big[x_{n+1}+\beta\nu\cdot y-\gamma(\nu)\big]\right) \text{ for } x\in\Omega_\infty.$$

As $I_x\cup I_x^c = S_\gamma$, the required inequality is proved for $t>0$ and $x\in\Omega_\infty$. □

Proof of Theorem 1.4. By Lemma 3.6 and Lemma 3.7, the limit

$$v(x) := \lim_{t\to+\infty} V(t,x) \tag{3.4}$$

exists. Moreover, it satisfies $u_*\le v\le u^*$. Since $f\in C^{1,\sigma}([0,\delta])$, by Remark 3.5 we see that the $C^{1+\frac{\sigma}{2},2+\sigma}$ norm of $V(t,x)$ is uniformly bounded in $\{V>0\}$, and $\partial D(t)$ is uniformly a $C^{2,\sigma}$ hypersurface for $t>0$. If we define

$$D_\infty := \cap_{t>0} D(t),$$

Then ∂D_∞ is $C^{2,\sigma}$ and $v\in C^{2,\sigma}(\overline{D}_\infty)$ with $\{v>0\}=D_\infty$.

For $t>0$ and $x\in D_\infty\subset D(t)$, by its definition, $V(t,x)$ satisfies

$$V_t-\Delta V+\kappa\partial_{n+1}V = f(V). \tag{3.5}$$

Letting $t\to\infty$ we obtain

$$-\Delta v+\kappa\partial_{n+1}v = f(v) \text{ for } x\in D_\infty.$$

From

$$V=0 \text{ and } V_t = \mu|\nabla_x V|^2-\kappa V_{n+1} \text{ for } x\in\partial D(t),$$

we deduce

$$v=0=\mu|\nabla v|^2-\kappa v_{n+1} \text{ for } x\in\partial D_\infty.$$

Hence v is a classical solution of (1.2). We have thus proved Theorem 1.4. □

Remark 3.8. By Lemma 3.3 and the proof of Theorem 1.4, we see that $\partial_\nu v(x)\ge 0$ for $x\in D_\infty$ and $\nu\in\mathbb{S}^n\cap\mathcal{C}$. Applying the strong maximum principle we further deduce $\partial_\nu v(x)>0$ for $x\in D_\infty$ and $\nu\in\mathbb{S}^n\cap\mathcal{C}$.

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