Nonlinearity

PAPER

Propagation acceleration in reaction diffusion equations with anomalous diffusions

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **34** (2021) 1544–1576 https://doi.org/10.1088/1361-6544/abe17c

Propagation acceleration in reaction diffusion equations with anomalous diffusions

Jérôme Coville[1], **Changfeng Gui**[2,*] and **Mingfeng Zhao**[3]

[1] UR 546 Biostatistique et Processus Spatiaux, INRAE, Domaine St Paul Site Agroparc, F-84000 Avignon, France

[2] One USA Circle, Department of Mathematics, University of Texas at San Antonio, San Antonio, TX 78249, United States of America

[3] Center for Applied Mathematics, Tianjin University, Tianjin, 300072, People's Republic of China

E-mail: jerome.coville@inrae.fr, changfeng.gui@utsa.edu and mingfeng.zhao@tju.edu.cn

Received 20 March 2020, revised 11 December 2020
Accepted for publication 29 January 2021
Published 3 March 2021

Abstract

In this paper we consider the propagation speed in a reaction diffusion system with an anomalous Lévy process diffusion, modeled by a nonlocal equation with a fractional Laplacian and a generalized monostable or ignition nonlinearity. Given a typical Heaviside initial datum, we show that the speed of interface propagation displays an algebraic rate behavior in time, in contrast to the known linear rate in the classical model of Brownian motion and the exponential rate in the KPP model with the anomalous diffusion, and depends on the sensitive balance between the anomaly of the diffusion process and the strength of monostable reaction. In particular, for the combustion model with a fractional Laplacian $(-\Delta)^s$, we show that the speed of propagation transits continuously from being linear in time, when a traveling wave solution exists for $s \in (1/2, 1)$, to being algebraic in time with a power reciprocal to $2s$, when no traveling wave solution exists for $s \in (0, 1/2)$.

Keywords: traveling wave, propagation speed, interface propagation, reaction diffusion, fractional Laplacian

Mathematics Subject Classification numbers: 35B51, 35K55, 35K57, 35R09, 35R11.

(Some figures may appear in colour only in the online journal)

*Author to whom any correspondence should be addressed.

1. Introduction

The study of propagation phenomena is a classical topic in analysis as it provides a robust way to understand some pattern formations that arise in a wide range of context ranging from population dynamics in ecology [19, 27], to combustion [25] and phase transition [6]. Concretely, this often leads to analyse the asymptotic properties of the solution $u(t, x)$ of the parabolic problem used to model the phenomenon considered. When this model is a reaction diffusion equation, this lead then to the study of the properties of the solutions of

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) & \text{for } t > 0, x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

with respect to the nonlinearity f and the initial data u_0. In this particular situation, when f is a smooth bistable, ignition or monostable nonlinearity, say f Lipschitz such that $f(0) = f(1) = 0$, $f'(1) < 0$, it is known that the solution of the equation (1) can exhibit some phase transition behaviour. More precisely, for a Heaviside type initial datum u_0, i.e. $u_0(x) = \mathbb{1}_{H_e}(x)$ where H_e denotes a half-space $\{x \in \mathbb{R}^N \mid x \cdot e < 0\}$, then the solution $u(t, x)$ of (1) converges locally uniformly as $t \to +\infty$ to 1 and the 'invasion process' resulting of this initial datum can be characterized by particular solutions of (1) called planar front $\varphi(x \cdot e - ct)$ [6, 18, 27, 36], where (φ, c) solves here the following equations

$$\begin{cases} \varphi''(z) + c\varphi'(z) + f(\varphi(z)) = 0 & \text{for } z \in \mathbb{R}, \\ \lim_{z\to-\infty} \varphi(z) = 1, \\ \lim_{z\to+\infty} \varphi(z) = 0. \end{cases} \tag{2}$$

In particular, for any $\lambda \in (0, 1)$ the super level set $E_\lambda(t) := \{x \in \mathbb{R}^N | u(t, x) \geqslant \lambda\}$ grows at a constant speed. That is there exists $x^+(\lambda), x^-(\lambda)$ in $\mathbb{R}^N$ and a family of half-space $H^+(t)$ defined by

$$H^+(t) := \{x \in \mathbb{R}^N \mid x \cdot e - ct \leqslant 0\}$$

such that E_λ satisfies

$$x^-(\lambda) + H^+(t) \subset E_\lambda(t) \subset x^+(\lambda) + H^+(t).$$

Thanks to the comparison principle satisfied by such semi-linear equations (1), clearly this particular phase transition behaviour appears also for other type of initial data $u_0 \geqslant \mathbb{1}_{H_e}$ that have some decay as $x \cdot e \to -\infty$. For those initial data, we may then wonder if the above description of the behaviour of super level set E_λ still holds true and if not how can we characterize it. As shown in [2, 24, 26, 32, 34], the above characterization may not hold in general and in some situation an accelerated transition may occur. Indeed when $N = 1$ and for a monostable nonlinearity f of KPP type, that is $f \in C([0, 1])$ such that $f(0) = f(1) = 0$, $f > 0, f'(0) > 0, f'(1) < 0$, and such that $f(s) \leqslant f'(0)s$, then for $u_0(x) > 0$, Hamel and Roques have obtained in [24] a sharp description of the speed of the level line of the solution of the corresponding Cauchy problem. In particular, they show that when u_0 is such that $u_0(x) \sim \frac{1}{x^\alpha}$, as $x \to +\infty$, then the level lines of the solution move exponentially fast. That is, for any $\lambda \in (0, 1)$ there exists points $x(t) \in E_\lambda(t)$ such that $x(t) \sim e^{f'(0)t}$. More generally, they prove that

Theorem 1.1. *Let u_0 be a C^2 non-increasing initial data on some semi-infinite interval $[\xi_0, +\infty)$ and such that*

$$\partial_{xx}u_0(x) = o(u_0(x)) \quad \text{as } x \to +\infty.$$

Then, for any $\lambda \in (0, 1)$, $\varepsilon \in (0, f'(0))$, $\mu > 0$ and $\nu > 0$, there exists $T_{\lambda,\varepsilon,\mu,\nu} \geqslant t_\lambda$ such that

$$\Gamma_\lambda(t) \subset u_0^{-1}([\mu e^{-(f'(0)+\varepsilon)t}, \nu e^{-(f'(0)-\varepsilon)t}]),$$

where Γ_λ denotes

$$\Gamma_\lambda(t) := \{x \in \mathbb{R} \mid u(t, x) = \lambda\}.$$

From this result, we can see the clear dependence of the speed of the level sets of the solution $u(t, x)$ with respect to the decay behavior of u_0. Similar sharp descriptions of the speed of the level sets have been obtain for more general monostable type of nonlinearity, see for example [2, 26, 32, 34]. On the other hand, thanks to the work of Fife and McLeod [18], and Alfaro [2] we can see that accelerated transitions will never occur when the nonlinearity considered is bistable or of ignition type.

In this spirit, in this paper we are interested in propagation acceleration phenomena that are caused by anomalous diffusions such as super diffusions, which plays important roles in various physical, chemical, biological and geological processes. (See, e.g., [35] for a brief summary and references therein.) A typical feature of such anomalous diffusions is related to Lévy stochastic processes which may possesses discontinuous 'jumps' in their paths and have long range dispersal, while the standard diffusion is related to the Brownian motion. Analytically, certain Lévy processes (α stable) may be modeled by their infinitesimal generators which are fractional Laplace operators $(-\Delta)^s u$ with $0 < s < 1$, whose Fourier transformation $\widehat{(-\Delta)^s u}$ is $(2\pi|\xi|)^{2s}\hat{u}$. (See [30].)

More precisely, we consider the following one-dimensional reaction–diffusion equation involving the fractional Laplacian:

$$\begin{cases} \partial_t u + (-\Delta)^s u = f(u), & t > 0, \ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3}$$

where

(a) $(-\Delta)^s$ $(0 < s < 1)$ denotes the fractional Laplacian operator:

$$(-\Delta)^s u(x) = C_{1,s}\text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} \mathrm{d}y,$$

where $C_{1,s}$ is a positive normalization constant in the sense that $\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\hat{u}(\xi)$. For simplicity, in the whole article, let us assume that $C_{1,s} = 1$ after a suitable normalization.

(b) f is a C^3 function on $[0, 1]$.

(c) $u_0(x)$ is the initial condition.

The precise assumptions on f and u_0 will be given later on.

Along with other types of nonlocal models (integro-differential or integro-difference) such nonlocal fractional reaction diffusion model (3) has received a lot of attention lately. Contrary to the standard reaction diffusion equation (1), accelerated transitions can be observed for Heaviside type initial data [10, 17, 22, 28, 31] in the anomalous reaction diffusion systems.

The mechanism that triggers the acceleration in this situation is then intrinsically different from that in the classical reaction diffusion and seems governed by subtle interplay between the long range jumps in the diffusion processes and the strength of the pushes and pullings in the reaction part, mathematically, i.e. the tails of the kernel and the properties of nonlinearity f considered. Namely, when f is of bistable type then planar wave exists for all $s \in (0, 1)$ [11, 23, 33] and the solution to (3) with a reasonable Heaviside initial data u_0 will converge to a planar front (see [1]). On the other hand, for the same initial data but for a KPP type nonlinearity, the solution will accelerate exponentially fast [10, 17], that is, for $x(t) \in \Gamma(t)$ we have $x(t) \sim e^{f'(0)t}$.

For more general monostable nonlinearities f, including those of ignition type, the picture is less clear and only results on the existence/non-existence of planar front have been obtained. More precisely, when f is an ignition nonlinearity then a planar front can only exist in the range $s \in \left(\frac{1}{2}, 1\right)$ (see [22, 29]). Whereas for a general monostable nonlinearity f, i.e. $f(t) \sim t^p(1-t)$), the existence of a planar front only occurs when $p > 2$ and in the range $s \in (\frac{p}{2(p-1)}, 1)$ (see [22]). In the later case, this suggests that as in the KPP case, an accelerated transition will then occur for any $s \in (0, 1)$ when $1 < p < 2$. A natural question is then, as in the KPP case, does the level sets move with an exponential speed when $1 < p < 2$?

One objective of this paper is to answer this question and give a more detailed characterization of the speed of the level set for general monostable nonlinearities f.

1.1. Main results

Let us now describe more precisely the assumptions we made and state our main results.

Assumption 1 (Degenerate monostable nonlinearity). The nonlinearity $f : [0, 1] \to [0, \|f\|_\infty]$ is of class C^1, and is of the monostable type, in the sense that

$$f(0) = f(1) = 0, \qquad f(u) > 0, \quad \text{for all } u \in (0, 1). \tag{4}$$

The steady state 0 is degenerate, in the sense that, there exist some constants $r > 0$ and $\beta > 1$ such that

$$f(u) \leqslant ru^\beta, \quad \text{for all } u \in [0, 1]. \tag{5}$$

The steady state 1 is stable, in the sense that

$$f'(1) < 0. \tag{6}$$

Assumption 2 (Front like initial datum). The initial data $u_0 : \mathbb{R} \longrightarrow [0, 1]$ is of class C^1 and satisfies

(a) $0 \leqslant u_0(x) \leqslant 1$ for all $x \in \mathbb{R}$.
(b) $\liminf\limits_{x \to -\infty} u_0(x) > 0$.
(c) $u_0(x) \equiv 0$ on $[a, +\infty)$ for some a.

Under this two assumptions, we first prove that

Theorem 1.2. *For any $0 < s < 1$, assume that the nonlinearity f satisfies assumption 1, and the initial data $u_0(x)$ satisfies assumption 2. Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$ and consider the super level set $E_\lambda(t) = \{x \in \mathbb{R} | u(t, x) > \lambda\}$ of the solution $u(t, x)$. Let us define*

$$x_\lambda(t) = \sup E_\lambda(t).$$

If we assume further that $\frac{\beta}{2s(\beta-1)} > 1$*, then for any* $\lambda \in (0,1)$ *there exist some constants* $T_\lambda > 0$ *and* $C(\lambda) > 0$ *such that*

$$E_\lambda(t) \subseteq (-\infty, x_\lambda(t)), \quad \text{and} \quad x_\lambda(t) \leqslant C(\lambda) t^{\frac{\beta}{2s(\beta-1)}}, \qquad \forall t > T_\lambda.$$

When $\beta > 2$ and $\frac{\beta}{2s(\beta-1)} \leqslant 1$, the existence of the traveling wave to the problem (3) provided $\frac{\beta}{2s(\beta-1)} \leqslant 1$ was proved by Gui and Huan in [22] meaning that for the solution $u(t,x)$ to (3) with some front-like data, if we look at the level set of $u(t,x)$, then the spatial variable x may linearly depend on the time variable t. In this sense, our condition $\frac{\beta}{2s(\beta-1)} > 1$ is sharp. In addition, we can observe from our results that when $\beta > 1$ then the level set of the solution $u(t,x)$ to the equation (3) moves at most at a polynomial rate, i.e. $x_\lambda(t) \sim t^\gamma$ with $\gamma := \sup\left\{1; \frac{1}{2s} + \frac{1}{\beta-1}\right\}$. These results are in sharp contrasts with the results of Cabre *et al* [10] for the KPP case. In particular, they highlight the fact that the exponential acceleration is strongly related to the non-degeneracy of the nonlinearity f and only occurs when f is such that $f'(0) > 0$, a situation that allows an exponential growth at low density.

Next, we prove a first lower bound of the speed of the level set. Namely, we show that

Theorem 1.3 (A rough lower bound). *For any* $0 < s < 1$*, assume that the nonlinearity* f *satisfies* $f(u) \geqslant 0$ *for all* $u \in [0,1]$*, and the initial data* $u_0(x)$ *satisfies assumption* 2*. Let* $u(t,x)$ *be the solution to the problem* (3) *with the initial data* $u_0(x)$ *and consider the super level set* $E_\lambda(t) = \{x \in \mathbb{R} | u(t,x) > \lambda\}$ *of the solution* $u(t,x)$*. Let us define*

$$x_\lambda(t) = \sup E_\lambda(t).$$

Then for any $\lambda \in (0,1)$*, there exist some constants* $T'_\lambda > 0$ *and* $C'(\lambda) > 0$ *such that*

$$x_\lambda(t) \geqslant C'(\lambda) t^{\frac{1}{2s}}, \quad \forall t > T'_\lambda.$$

Combining the latter with the upper bound obtained in theorem 1.3, as a immediate corollary we then get.

Corollary 1.1. *For any* $0 < s < 1$*, assume that the nonlinearity* f *satisfies assumption* 1*, and the initial data* $u_0(x)$ *satisfies assumption* 2*. Let* $u(t,x)$ *be the solution to the problem* (3) *with the initial data* $u_0(x)$ *and consider the super level set* $E_\lambda(t) = \{x \in \mathbb{R} | u(t,x) > \lambda\}$ *of the solution* $u(t,x)$*. Let us define*

$$x_\lambda(t) = \sup E_\lambda(t).$$

If we assume further that $\frac{\beta}{2s(\beta-1)} > 1$*, then for any* $\lambda \in (0,1)$*, there exist some constants* $T_\lambda > 0$*,* $C(\lambda) > 0$ *and* $C'(\lambda) > 0$ *such that*

$$C'(\lambda) t^{\frac{1}{2s}} \leqslant x_\lambda(t) \leqslant C(\lambda) t^{\frac{\beta}{2s(\beta-1)}}, \quad \forall t > T_\lambda.$$

Although these first estimates on the speed seem rather crude these are still quite interesting, in particular in the case $0 < s < \frac{1}{2}$, as they give a very simple way of showing the non-existence of the traveling wave solution to the problem (3) with any general non negative function f and in particular for the Fisher–KPP nonlinearity. These results also highlight a fundamental difference between nonlocal model versus local model when considering an ignition type nonlinearity. Indeed, when the nonlinearity f is of ignition type, we can easily deduce from the work of Alfaro [2] that accelerated transitions never occur in the classical reaction diffusion model (1) whereas they do in the nonlocal reaction diffusion (3) when $s \in (0, 1/2)$. This is also a clear evidence that in the nonlocal setting, unlike in the local setting (1) the two types of nonlinearities: bistable and ignition type are not alike in the sense that the dynamic obtained are completely different. In this nonlocal setting, a condition on the decay of the tail of the kernel appears then of crucial importance in order to guarantee the existence of traveling front. Namely, from our results we can see that, when f is non negative the kernel must satisfy some first moment integrability condition to expect to observe traveling front solutions. This finite first moment condition suggests that a similar condition should hold true as well for convolution type nonlocal models studied in [15] as these two models share many similarities. That is, in such convolution type models, for a traveling front to exist the kernel need to satisfy a first moment condition.

Let us look now more deeply at the consequences of these first estimates on the speed for the combustion model and for supercritical fractional Laplacians (that is, $0 < s < \frac{1}{2}$). In this situation, from the above estimates we can in fact derive a sharp estimate on the speed of propagation. Namely, we show

Corollary 1.2 (Combustion model for supercritical and critical fractional Laplacians). *For any $0 < s \leqslant \frac{1}{2}$, assume that the initial data $u_0(x)$ satisfies assumption 2, and the nonlinearity f is a combustion type nonlinearity, in the sense that either f is a fully degenerate monostable nonlinearity, i.e. f monostable satisfying assumption 1 and such that for all $k \in \mathbb{N}$ $f^{(k)}(0) = 0$ or f is of ignition type, that is there exists some $\theta \in (0, 1)$ such that*

$$f(1) = 0 = f(u), \quad \text{for all } u \in [0, \theta], \quad \text{and} \quad f(u) > 0 \quad \text{for all } u \in (\theta, 1). \tag{7}$$

Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$ and consider the super level set $E_\lambda(t) = \{x \in \mathbb{R} | u(t, x) > \lambda\}$ of the solution $u(t, x)$. Let us define

$$x_\lambda(t) = \sup\ E_\lambda(t).$$

Then for any $\varepsilon > 0$, and for any $\lambda \in (0, 1)$, there exist some constants $T_{\lambda,\varepsilon} > 0$, $C(\lambda, \varepsilon) > 0$ and $C'(\lambda) > 0$ such that

$$C'(\lambda)t^{\frac{1}{2s}} \leqslant x_\lambda(t) \leqslant C(\lambda, \varepsilon)t^{\frac{1}{2s}+\varepsilon}, \qquad \forall t > T_{\lambda,\varepsilon}.$$

The proof of this corollary is quite straightforward. Indeed, the ignition type model can be thought as some limit case of the degenerated monostable situation. In particular, for any combustion nonlinearity f we may find a constant $C_0 > 0$ such that for all $\beta > 1$ we have

$$f(u) \leqslant f_\beta(u) := C_0 u^\beta(1 - u).$$

Recall that since we assume that the fractional Laplacian is either super-critical or critical (i.e. $s \in (0, \frac{1}{2}]$) then we can check that for all $\beta > 1$ the condition below is satisfied:

$$\frac{\beta}{2s(\beta - 1)} = \frac{1}{2s} + \frac{1}{2s(\beta - 1)} > 1.$$

Thus by using a standard comparison argument and corollary 1.1, we may deduce that for any $\beta > 1$ there exists $C(\beta)$ and T_β such that for all $t \geqslant T_\beta$

$$x_\lambda(t) \leqslant C(\lambda, \beta)t^{\frac{1}{2s}+\frac{1}{2s(\beta-1)}}.$$

The results of corollary 1.2 follows then by picking β so large that we have $\frac{1}{2s(\beta-1)} \leqslant \varepsilon$.

Note that this estimate is sharp in the sense it gives the right asymptotic for the speed of the level set, i.e. we get $x_\lambda(t) \sim t^{\frac{1}{2s}}$ as $t \to \infty$. It also provides a useful information for the critical case $s = \frac{1}{2}$, where we see that the level set moves asymptotically with a constant speed although there is no existence of a traveling front in this situation.

Lastly, in the spirit of [3], let us obtain a finer lower bound on the speed for general degenerate monostable nonlinearities f, i.e. $\exists\, \beta \in (1, +\infty)$, such that $\lim_{u\to 0} \frac{f(u)}{u^\beta} = l > 0$.

Theorem 1.4 (A finer lower bound). *For any $0 < s < 1$, assume that the nonlinearity f satisfies assumption 1 and $f(u) \geqslant r_1 u^\beta$ as $u \to 0^+$ for some small $r_1 > 0$, and the initial data $u_0(x)$ satisfies assumption 2. Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$ and consider the super level set $E_\lambda(t) = \{x \in \mathbb{R} | u(t, x) > \lambda\}$ of the solution $u(t, x)$. Let us define*

$$x_\lambda(t) = \sup\ E_\lambda(t).$$

If we assume further that $\frac{1}{2s(\beta-1)} > 1$, then for any $\lambda \in (0, 1)$, there exist some constants $T_\lambda > 0$, $C(\lambda) > 0$ and $C'(\lambda) > 0$ such that

$$C'(\lambda) t^{\frac{1}{2s(\beta-1)}} \leqslant x_\lambda(t) \leqslant C(\lambda) t^{\frac{\beta}{2s(\beta-1)}}, \quad \forall t > T_\lambda.$$

Notice that $\frac{1}{2s} < \frac{1}{2s(\beta-1)}$ if and only if $1 < \beta < 2$. Hence when $\frac{1}{2} < s < 1$ and $1 < \beta < 2$, the lower bound in theorem 1.4 is better than the one in theorem 1.3. From these estimates we can then deduce the following generic estimate:

Theorem 1.5 (A generic bound). *For any $0 < s < 1$, assume that the nonlinearity f satisfies assumption 1 and $f(u) \geqslant r_1 u^\beta$ as $u \to 0^+$ for some small $r_1 > 0$, and the initial data $u_0(x)$ satisfies assumption 2. Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$ and consider the super level set $E_\lambda(t) = \{x \in \mathbb{R} | u(t, x) > \lambda\}$ of the solution $u(t, x)$. Let us define*

$$x_\lambda(t) = \sup\ E_\lambda(t).$$

Then for any $\lambda \in (0, 1)$, there exist some constants $T_\lambda > 0$, $C(\lambda) > 0$ and $C'(\lambda) > 0$ such that

$$C'(\lambda) t^{\max\left\{\frac{1}{2s(\beta-1)};\frac{1}{2s}\right\}} \leqslant x_\lambda(t) \leqslant C(\lambda) t^{\frac{1}{2s(\beta-1)}+\frac{1}{2s}}, \quad \forall t > T_\lambda.$$

This last results clearly indicate that the speed of the level sets is the result of a fine interplay between the diffusion process intimately linked to the quantity $t^{1/2s}$ and the reaction term f which, as we will see in the proof, is strongly linked to the quantity $t^{\frac{1}{2s(\beta-1)}}$.

To have a more synthetic view of our results, we summarize them in the following schematic picture (figure 1):

1.2. Further comments

Before going to the proofs of our results, we would like to make some further comments. First, we would like to emphasize that similar results were previously obtained in [3] in the context of integro-differential equation

$$\begin{cases} \partial_t u(t, x) = J \star u(t, x) - u(t, x) + f(u(t, x)) & \text{for } t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), \end{cases} \tag{8}$$

where $J \star u$ stands for the standard convolution and J is a positive probability density with a finite first moment, i.e. $J \in L^1(\mathbb{R})$ such that $J \geqslant 0$, $\int_{\mathbb{R}} J(z)\,\mathrm{d}z = 1$, $\int_{\mathbb{R}} J(z)|z|\,\mathrm{d}z < +\infty$. The two equations (3) and (8) shares some similarities, and in particular the equation (3) may be viewed

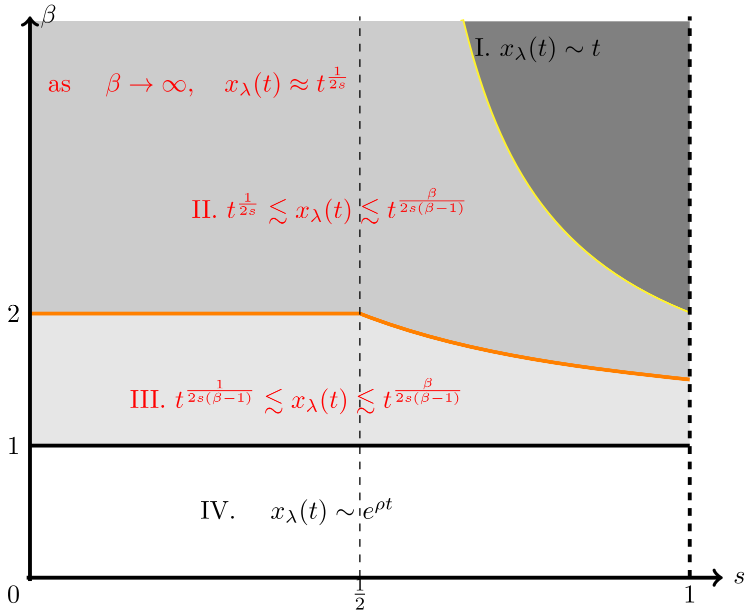


Figure 1. (I) In the dark gray region above the top yellow curve, it holds that $\beta > 2$ and $\frac{\beta}{2s(\beta-1)} \leqslant 1$. The existence of the traveling front implies the linear dependence $x_\lambda(t) \sim t$ ([22]). (II) In the gray region bounded by the yellow and orange curves, the front propagation rate satisfies the algebraic relation $t^{\frac{1}{2s}} \lesssim x_\lambda(t) \lesssim t^{\frac{\beta}{2s(\beta-1)}}$ (theorems 1.2 and 1.3). As $\beta \to \infty$, the rate of propagation assumes the optimal algebraic dependence $x_\lambda(t) \sim t^{\frac{1}{2s}}$ as the reaction nonlinearity approaches the type in the combustion model. (III) In the light gray region bounded by the orange curve and the black line $\beta = 1$, it holds that $\frac{1}{2s(\beta-1)} > 1$ and $\frac{1}{2s} \leqslant \frac{1}{2s(\beta-1)}$, the front propagation rate satisfies the algebraic dependence $t^{\frac{1}{2s(\beta-1)}} \lesssim x_\lambda(t) \lesssim t^{\frac{\beta}{2s(\beta-1)}}$ (theorems 1.2 and 1.4). (IV) In the white region below and including $\beta = 1$ and above $\beta = 0$, the front propagation has exponential rate $x_\lambda(t) \sim e^{\rho t}$ for some $\rho > 0$ ([10], theorem 1.1).

as a reformulation of the equation (8) but with a non integrable singular kernel. However, the treatment of the singularity is of crucial importance here and induces some technical difficulties, for which the ideas developed to analyze (8) do not straightforwardly apply. Indeed, in our setting, the challenge of the singularity is intrinsic and related to the physical nature of the fractional Laplacian. The approach here is hence not just an adaptations of the proofs given in [3], and we have to deal with the singularity carefully. In particular, we go a step further in our understanding of the mechanism triggering acceleration by describing the situation for $s \in (0, \frac{1}{2})$, a situation which is not treated in [3] at all. We believe that some of the techniques developed here will be also useful to study propagation phenomena in the equation (8) for kernels that do not satisfy this first moment condition. In particular, the analysis presented here should provide the ground for a deeper understanding of nonlocal combustion problems modeled by the equation (8) studied in [15], by ensuring that the existence of traveling front is conditioned to a first moment property satisfied by the kernel. Works in this direction are currently underway.

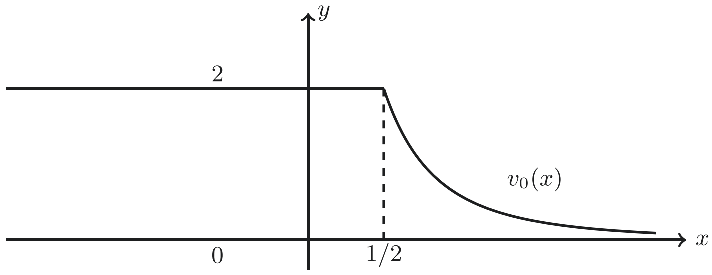
We also want to stress that although our results give some good insights on the speed of the level sets, apart from situations involving combustion nonlinearities where a precise asymptotic is known, there is still a gap in our estimates and the right behavior, that we believe is $t^{\frac{\beta}{2s(\beta-1)}}$, has not been capture yet. Using new approaches, recent progress have been made on the understanding of acceleration phenomena in various situations, namely for semilinear equation like (1) with a nonlinear diffusion instead of the classical diffusion [4, 5, 21] as well as for the equation (8) with Fisher–KPP type nonlinearity [8, 20]. The different approaches developed in these works may be of some help in this task. Works in this direction are also under considerations.

The paper is organized as follows. In section 2, we prove theorem 1.2 and obtain the upper bound on the speed of the level set. Then in section 3, we obtain the generic lower bound on this speed, theorem 1.3. Finally, in the last section, section 4, we prove the a refine estimate of this speed when a degenerate monostable nonlinearity f is considered, theorem 1.4.

2. Upper bound on the speed of the super level sets

Construction of a supersolution: for some constant $p > 0$ which will be determined later on, let us define

$$v_0(x) = \begin{cases} 2, & \text{if } x \leqslant \dfrac{1}{2}, \\ \dfrac{1}{x^p}, & \text{if } x > \dfrac{1}{2}. \end{cases}$$



For any $\gamma > 0$, let $w(t, x)$ be the solution to the following initial-value problem:

$$\begin{cases} \dfrac{\mathrm{d}w(t,x)}{\mathrm{d}t} = \gamma[w(t,x)]^{\beta}, \\ w(0,x) = v_0(x). \end{cases}$$

Since $\beta > 1$, we can solve the above problem and obtain

$$w(t,x) = \frac{1}{\left[[v_0(x)]^{1-\beta} - \gamma(\beta-1)t\right]^{\frac{1}{\beta-1}}}.$$

The definition of $v_0(x)$ tells us that $w(t, x)$ is well defined for $t \in \left[0, \frac{1}{2^{p(\beta-1)}\gamma(\beta-1)}\right)$ if $x \leqslant 1/2$; and $w(t, x)$ is well defined for $t \in \left[0, \frac{x^{p(\beta-1)}}{\gamma(\beta-1)}\right)$ if $x > 1/2$. When t is fixed, the function $w(t, x)$ is decreasing with respect to x.

Let $x_0(t) = [1 + \gamma(\beta - 1)t]^{\frac{1}{p(\beta-1)}}$ for all $t > 0$, then we know that $x_0(t) > 1$, $w(t, x_0(t)) = 1$, $w(t, x) > 1$ for all $x < x_0(t)$, and $w(t, x) < 1$ for all $x > x_0(t)$. Let us consider the function

$$m(t,x) = \begin{cases} 1, & \text{if } x \leqslant x_0(t), \\ w(t,x), & \text{if } x > x_0(t). \end{cases}$$

Then $m(t, x)$ is well defined for all $t \geqslant 0$ and all $x \in \mathbb{R}$, and $0 < m(t, x) \leqslant 1$ for all $t \geqslant 0$ and all $x \in \mathbb{R}$. Let us start by showing some *a priori* estimate on w that we will constantly used in our construction.

Claim 2.1. If $p + 1 \geqslant p\beta$, then there exists some constant $C(p, \beta) > 0$ such that

$$|\partial_x w(t,x)| + |\partial^2_{xx} w(t,x)| \leqslant C(p,\beta), \quad \forall t > 0,\ \forall x > x_0(t) - \frac{1}{2}.$$

Proof. Since $x_0(t) > 1$ for all $t > 0$, for any $x > x_0(t) - \frac{1}{2}$ we then have $v_0(x) = \frac{1}{x^p}$ and

$$w(t,x) = [x^{p(\beta-1)} - \gamma(\beta-1)t]^{\frac{1}{1-\beta}}.$$

Now, a direct computation then shows that

$$\begin{aligned} \partial_x w(t,x) &= \frac{1}{1-\beta} \cdot [[v_0(x)]^{1-\beta} - \gamma(\beta-1)t]^{\frac{1}{1-\beta}-1} \cdot (1-\beta)[v_0(x)]^{-\beta} \cdot v_0'(x), \\ &= -p[w(t,x)]^{\beta} \cdot x^{p\beta-p-1}. \end{aligned}$$

Since $\beta > 1$, $p + 1 \geqslant p\beta$, $0 < w(t, x)$ is decreasing, and $x > x_0(t) - \frac{1}{2} > \frac{1}{2}$, then we have

$$|\partial_x w(t,x)| \leqslant \frac{p}{2^{p\beta-p-1}} w^{\beta}\left(x_0(t) - \frac{1}{2}\right).$$

Let us now estimate $w^{\beta}\left(x_0(t) - \frac{1}{2}\right)$. By definition of w and $x_0(t)$, we have

$$\begin{aligned} w^{\beta-1}\left(x_0(t) - \frac{1}{2}\right) &= \frac{1}{\left(x_0(t) - \frac{1}{2}\right)^{p(\beta-1)} - \gamma(\beta-1)t} \\ &= \frac{1}{1 + \left(x_0(t) - \frac{1}{2}\right)^{p(\beta-1)} - x_0^{p(\beta-1)}}. \end{aligned}$$

Recall that for $\alpha \leqslant 1$ and $\delta > 0$ the map $s \mapsto \frac{1}{1+(s-\delta)^{\alpha}-s^{\alpha}}$ is decreasing for $s \geqslant \delta$, therefore since $x_0(t) > 1$ and $p(\beta - 1) \leqslant 1$ we have

$$w^{\beta-1}\left(x_0(t) - \frac{1}{2}\right) \leqslant \frac{1}{1 + \left(1 - \frac{1}{2}\right)^{p(\beta-1)} - 1} = 2^{p(\beta-1)}.$$

As a consequence, we then obtain that

$$|\partial_x w(t,x)| \leqslant p2^{p+1}.$$

On the other hand, we have

$$\partial^2_{xx} w(t,x) = -p\partial_x \left[[w(t,x)]^{\beta} \cdot x^{p\beta-p-1}\right],$$
$$= -p\beta[w(t,x)]^{\beta-1} \cdot \partial_x w(t,x) \cdot x^{p\beta-p-1}$$
$$- p[w(t,x)]^{\beta} \cdot (p\beta - p - 1) \cdot x^{p\beta-p-2}.$$

Since $\beta > 1$, $p+2 > p+1 \geqslant p\beta$, $x > x_0(t) - \frac{1}{2} > \frac{1}{2}$ and $0 < w(t,x) \leqslant w\left(x_0(t) - \frac{1}{2}, t\right)$ we then have

$$|\partial^2_{xx} w(t,x)| \leqslant 2^{p+2}[p^2\beta + p\beta(p+1-p\beta)].$$

□

Next, let us obtain a first estimate on the fractional Laplacian of m.

Claim 2.2. If $p+1 \geqslant p\beta$, then there exists some constant $C(s,p,\beta) > 1$ such that

$$(-\Delta)^s m(t,x) \geqslant -C(s,p,\beta), \quad \forall t > 0,\ \forall x > x_0(t).$$

Proof. By the definition of $(-\Delta)^s m(t,x)$, we can obtain that

$$-(-\Delta)^s m(t,x) = \frac{1}{2}\int_{\mathbb{R}} \frac{m(t,x+h) + m(t,x-h) - 2m(t,x)}{|h|^{1+2s}} \mathrm{d}h,$$
$$= \frac{1}{2}\int_{|h|\geqslant\frac{1}{2}} \frac{m(t,x+h) + m(t,x-h) - 2m(t,x)}{|h|^{1+2s}} \mathrm{d}h$$
$$+ \frac{1}{2}\int_{|h|<\frac{1}{2}} \frac{m(t,x+h) + m(t,x-h) - 2m(t,x)}{|h|^{1+2s}} \mathrm{d}h,$$
$$= I_1 + I_2.$$

Since $0 < m(t,x) \leqslant 1$ for all $t > 0$ and all $x \in \mathbb{R}$, then there exists some $C_1(s,p,\beta) > 0$ such that

$$I_1 \leqslant \frac{1}{2}\int_{|h|\geqslant\frac{1}{2}} \frac{4}{|h|^{1+2s}} \mathrm{d}h =: C_1(s,p,\beta).$$

To estimate I_2, let us observe that by definition of m we have for all $t > 0$, $m(t,x) \leqslant w(t,x)$ for all x and $m(t,x) = w(t,x)$ for $x \geqslant x_0(t)$. Therefore for all $t > 0$ and $x > x_0(t)$ we then have

$$I_2 \leqslant \frac{1}{2}\int_{|h|<\frac{1}{2}} \frac{w(t,x+h) + w(t,x-h) - 2w(t,x)}{|h|^{1+2s}} \mathrm{d}h.$$

By claim 2.1, we see that there exists C_0 such that for all $|h| \leqslant \frac{1}{2}$ and $x \geqslant x_0(t)$

$$|w(t,x+h) + w(t,x-h) - 2w(t,x)| \leqslant C(p,\beta)|h|^2,$$

and therefore

$$I_2 \leqslant \frac{C}{2}\int_{|h|<\frac{1}{2}} |h|^{1-2s} \mathrm{d}h =: C_2\,(s, \mathrm{d}h =: C_2(s,p,\beta).$$

□

Let us now prove that for the right choice of parameter p and γ, the function m satisfies for all $t > 0$ and $x > x_0(t)$

$$\partial_t m(t,x) + (-\Delta)^s m(t,x) - f(m(t,x)) \geqslant 0.$$

To this end, we start by showing that

Claim 2.3. Assume that $p + 1 \geqslant p\beta$ and let C_1 be the positive constant defined in claim 2.2. For any fixed $\gamma > \gamma_0 := r + 2C_1$, let us define

$$x_\gamma(t) = \left[\left(\frac{\gamma - r}{C_1}\right)^{\frac{\beta-1}{\beta}} + \gamma(\beta-1)t\right]^{\frac{1}{p(\beta-1)}}.$$

Then $x_0(t) < x_\gamma(t)$ for all $t > 0$, and for any (t, x) such that $x_0(t) < x \leqslant x_\gamma(t)$, we have

$$\partial_t m(t,x) + (-\Delta)^s m(t,x) - f(m(t,x)) \geqslant 0.$$

Proof. Since $\gamma - r \geqslant 2C_1 > 2$ and since $\beta > 1$, by definition of $x_\gamma(t)$ we have $x_\gamma(t) > [1 + \gamma(\beta-1)t]^{\frac{1}{p(\beta-1)}} = x_0(t)$. Now for any (t, x) such that $x_0(t) < x \leqslant x_\gamma(t)$, by assumption 1 and claim 2.2, it follows that

$$\begin{aligned}
&\partial_t m(t,x) + (-\Delta)^s m(t,x) - f(m(t,x))\\
&\quad = \partial_t w(t,x) + (-\Delta)^s m(t,x) - f(w(t,x)),\\
&\quad = \gamma[w(t,x)]^\beta - f(w(t,x)) + (-\Delta)^s m(t,x),\\
&\quad \geqslant \gamma[w(t,x)]^\beta - r[w(t,x)]^\beta - C_1,\\
&\quad \geqslant (\gamma - r)[w(t,x)]^\beta - C_1.
\end{aligned}$$

Since $w(t, x)$ is decreasing with respect to x and since $x_0(t) < x \leqslant x_\gamma(t)$ we then obtain

$$\partial_t m(t,x) + (-\Delta)^s m(t,x) - f(m(t,x)) \geqslant (\gamma - r)[w(t, x_\gamma(t))]^\beta - C_1 = 0.$$

□

Let us now verify that for γ well chosen m is a supersolution to (3) for $t > 0$ and $x > x_\gamma(t)$. First, we introduce the notations $q := p(\beta - 1)$ and $\sigma := \gamma(\beta - 1)t$. For some constant $K > 2$ which will be determined later on, let us write

$$\begin{aligned}
-(-\Delta)^s m(t,x) &= \int_{-\infty}^{\frac{x_0(t)-x}{K}} \frac{m(t,x+z) - m(t,x)}{|z|^{1+2s}}\mathrm{d}z + \mathrm{P.V.}\int_{\frac{x_0(t)-x}{K}}^{+\infty} \frac{m(t,x+z) - m(t,x)}{|z|^{1+2s}}\mathrm{d}z,\\
&=: I_1 + I_2.
\end{aligned}$$

Since $\beta > 1$, by using the definitions of $x_\gamma(t)$ and $x_0(t)$ we can find γ which may depend on K such that $\gamma > \gamma_0$ (where γ_0 is defined in claim 2.3), and $x_0(t) - x < -K$ for all $x > x_\gamma(t)$. Recall that by definition of m we have $0 < m(t, x) \leqslant 1$ for all $t > 0$ and $x \in \mathbb{R}$. Therefore we have

$$I_1 \leqslant \int_{-\infty}^{\frac{x_0(t)-x}{K}} \frac{1}{|z|^{1+2s}}\mathrm{d}z = \frac{1}{2s}\cdot\left[\frac{K}{x - x_0(t)}\right]^{2s}.$$

By choosing $q < 1$ (that is, $p(\beta - 1) < 1$), we then have $x^q \leqslant [x_0(t)]^q + (x - x_0(t))^q$, that is, $[x^q - [x_0(t)]^q]^{\frac{1}{q}} \leqslant x - x_0(t)$, which in turn implies that

$$\frac{1}{[x - x_0(t)]^{2s}} \leqslant \frac{1}{[x^q - [x_0(t)]^q]^{\frac{2s}{q}}}.$$

Since $x > x_0(t) + K > 2$, it follows that $(x^q - 1)^{\frac{1}{q}} > 1$ and so we get $\left[v_0((x^q - 1)^{\frac{1}{q}})\right]^{1-\beta} = x^q - 1$, which yields to

$$\begin{aligned} w(t, (x^q - 1)^{\frac{1}{q}}) &= \frac{1}{\left[\left[v_0((x^q - 1)^{\frac{1}{q}})\right]^{1-\beta} - \gamma(\beta - 1)t\right]^{\frac{1}{\beta-1}}}, \\ &= \frac{1}{[x^q - 1 - \gamma(\beta - 1)t]^{\frac{1}{\beta-1}}}, \\ &= \frac{1}{[x^q - [x_0(t)]^q]^{\frac{1}{\beta-1}}}. \end{aligned}$$

Recall that $\frac{2s}{q} \cdot (\beta - 1) = \frac{2s}{p}$ and $x - x_0(t) > K$, so we then obtain

$$I_1 \leqslant \frac{1}{2s} \cdot K^{2s} \cdot \left[w(t, (x^q - 1)^{\frac{1}{q}})\right]^{\frac{2s}{p}}.$$

Let us now estimate $w(t, (x^q - 1)^{\frac{1}{q}})$ in terms of $w(t, x)$. Let us first observe that for $x > x_\gamma(t) > 1$ and $q = p(\beta - 1)$, we have

$$\begin{aligned} \frac{w(t, (x^q - 1)^{\frac{1}{q}})}{w(t, x)} &= \frac{1}{w(t, x)} \cdot \frac{1}{[x^q - 1 - \gamma(\beta - 1)t]^{\frac{1}{\beta-1}}}, \\ &= \frac{1}{w(t, x)} \cdot \frac{1}{\left[x^q - 1 - [v_0(x)]^{1-\beta} + [w(t, x)]^{1-\beta}\right]^{\frac{1}{\beta-1}}}, \\ &= \frac{1}{w(t, x)} \cdot \frac{1}{\left[x^q - 1 - x^{-p(1-\beta)} + [w(t, x)]^{1-\beta}\right]^{\frac{1}{\beta-1}}}, \\ &= \frac{1}{w(t, x)} \cdot \frac{1}{\left[-1 + [w(t, x)]^{1-\beta}\right]^{\frac{1}{\beta-1}}}, \\ &= \frac{1}{\left[1 - [w(t, x)]^{\beta-1}\right]^{\frac{1}{\beta-1}}}. \end{aligned}$$

On another hand, by using that $w(t, x)$ is decreasing with respect to x, we straightforwardly see that for $x > x_\gamma(t)$ we have

$$[w(t, x)]^{\beta-1} < [w(t, x_\gamma(t))]^{\beta-1} = \left[\frac{C_1}{\gamma - r}\right]^{\frac{\beta-1}{\beta}}.$$

So by enlarging γ if necessary such that $\left[\frac{C_1}{\gamma-r}\right]^{\frac{\beta-1}{\beta}} \leqslant \frac{1}{2}$, we get from the above equality that for $x > x_\gamma(t)$

$$\frac{w(t,(x^q-1)^{\frac{1}{q}})}{w(t,x)} \leqslant 2^{\frac{1}{\beta-1}}.$$

As a consequence, we get

$$I_1 \leqslant C_2 K^{2s} \cdot [w(t,x)]^{\frac{2s}{p}},$$

where $C_2 := \frac{2^{\frac{2s}{q}-1}}{s}$.

Let us now estimate I_2. Since $x_0(t) - x < -K$ we can check that

$$\begin{aligned} I_2 &= \int_{\frac{x_0(t)-x}{K}}^{-1} \frac{m(t,x+z)-m(t,x)}{|z|^{1+2s}}\mathrm{d}z + \text{P.V.}\int_{-1}^{1} \frac{m(t,x+z)-m(t,x)}{|z|^{1+2s}}\mathrm{d}z \\ &\quad + \int_{1}^{+\infty} \frac{m(t,x+z)-m(t,x)}{|z|^{1+2s}}\mathrm{d}z, \\ &=: I_3 + I_4 + I_5. \end{aligned}$$

Let us first focus on I_3. Observe that since $x_0(t) > 1$ and $K > 2$, then for all $\frac{x_0(t)-x}{K} \leqslant z \leqslant -1$ we have $x + z > \frac{x}{2} > 1$. So by making the change of variables $z = xu$ we get

$$\begin{aligned} I_3 &= \int_{\frac{x_0(t)-x}{K}}^{-1} \frac{w(t,x+z)-w(t,x)}{|z|^{1+2s}}\mathrm{d}z, \\ &= \int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{w(t,x+xu)-w(t,x)}{|xu|^{1+2s}} \cdot x\mathrm{d}u, \\ &= xw(t,x)\int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{1}{|xu|^{1+2s}} \cdot \left[\frac{w(t,x+xu)}{w(t,x)} - 1\right]\mathrm{d}u, \\ &= xw(t,x)\int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{1}{|xu|^{1+2s}} \cdot \left[\frac{[(x+xu)^{p(\beta-1)} - (\beta-1)\gamma t]^{\frac{1}{1-\beta}}}{[x^{p(\beta-1)} - (\beta-1)\gamma t]^{\frac{1}{1-\beta}}} - 1\right]\mathrm{d}u, \\ &= xw(t,x)\int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{1}{|xu|^{1+2s}} \cdot \left[\frac{1}{\left[\frac{(1+u)^q-1}{1-\frac{\sigma}{x^q}} + 1\right]^{\frac{p}{q}}} - 1\right]\mathrm{d}u. \end{aligned}$$

At this point to continue our estimate of I_3, we prove the following claim.

Claim 2.4. For $q < 1$, there exists some $K(q) > 0$ such that for all $t > 0$, all $K \geqslant K(q)$, all $x > x_0(t)$, and all $u \in \left[\frac{x_0(t)-x}{Kx}, 0\right]$, we have

$$\frac{(1+u)^q-1}{1-\frac{\sigma}{x^q}} - 1 \geqslant -\frac{1}{2}.$$

Proof. The proof goes identically with the one in [3]. □

Now by claim 2.4 and by using Lagrange's mean value theorem, there exists some constant $C_3 > 0$ such that

$$\frac{1}{\left[\frac{(1+u)^q-1}{1-\frac{\sigma}{x^q}}+1\right]^{\frac{p}{q}}}-1 \leqslant -\frac{p}{q}[1+C_3]\cdot\frac{(1+u)^q-1}{1-\frac{\sigma}{x^q}},$$

$$\leqslant \frac{p}{q}[1+C_3]\cdot x^q\cdot\frac{1-(1+u)^q}{x^q-\sigma}.$$

By using the equality $[w(t,x)]^{1-\beta} = x^q - \sigma$ and since $0 < q < 1$, from the above estimate we get

$$\frac{1}{\left[\frac{(1+u)^q-1}{1-\frac{\sigma}{x^q}}+1\right]^{\frac{p}{q}}}-1 \leqslant \frac{p}{q}[1+C_3]\cdot x^q[w(t,x)]^{\beta-1}[1-(1+u)^q],$$

$$\leqslant \frac{p}{q}[1+C_3]\cdot x^q[w(t,x)]^{\beta-1}\cdot|u|^q,$$

$$\leqslant C_4\cdot x^q[w(t,x)]^{\beta-1}\cdot|u|^q.$$

So we have

$$I_3 \leqslant xw(t,x)\int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}}\frac{1}{|xu|^{1+2s}}\cdot C_4\cdot x^q[w(t,x)]^{\beta-1}\cdot|u|^q\mathrm{d}u,$$

$$\leqslant C_4[w(t,x)]^{\beta}\int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}}\frac{1}{|xu|^{1+2s}}\cdot|xu|^q\cdot x\mathrm{d}u,$$

$$\leqslant C_4[w(t,x)]^{\beta}\int_{\frac{x_0(t)-x}{K}}^{-1}\frac{1}{|z|^{1+2s-q}}\mathrm{d}z.$$

By choosing q such that $2s - q > 0$, that is, $2s > q = p(\beta - 1)$ we then achieve

$$I_3 \leqslant C_4[w(t,x)]^{\beta}\int_{-\infty}^{-1}\frac{1}{|z|^{1+2s-q}}\mathrm{d}z = C_5[w(t,x)]^{\beta}.$$

Let us now estimate I_4. Clearly $x + z > 1$ for any $z \in [-1, 1]$ since $x > 2$. Therefore by using definition of m and since $x > x_\gamma(t) > x_0(t) + K > 2$ we have $m(t, x + z) = w(t, x + z)$ for all $x > x_\gamma(t)$ and $z \in [-1, 1]$. Thus, we have

$$I_4 = \frac{1}{2}\int_{-1}^{1}\frac{m(t,x+z)+m(t,x-z)-2m(t,x)}{|z|^{1+2s}}\mathrm{d}z,$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{w(t,x+z)+w(t,x-z)-2w(t,x)}{|z|^{1+2s}}\mathrm{d}z.$$

By claim 2.1 and since $p + 1 \geqslant p\beta$ and $x > x_\gamma(t) > x_0(t) + 2 > x_0(t) - \frac{1}{2}$, we know that

$$|\partial^2_{xx}w(t,x)| \leqslant p^2\beta[w(t,x)]^{2\beta-1} + (p+1-p\beta)[w(t,x)]^{\beta}.$$

Since $\beta > 1$ and $0 < w(t, x) \leqslant 1$, there exists then $C_6 > 0$ such that

$$|\partial^2_{xx}m(t,x)| \leqslant C_6[w(t,x)]^{\beta},$$

and so we get

$$I_4 \leqslant \frac{1}{2}\int_{-1}^{1} \frac{C_6[w(t,x)]^\beta |z|^2}{|z|^{1+2s}}\mathrm{d}z = C_7[w(t,x)]^\beta.$$

The estimate I_5 is straightforward. Indeed, since $m(t,x)$ is decreasing with respect to x, then for all $z \geqslant 0$ and $x \in \mathbb{R}$, $m(t,x+z) - m(t,x) \leqslant 0$ and thus

$$I_5 \leqslant 0.$$

By compiling all the estimates, we then get for $x \geqslant x_\gamma(t)$

$$\begin{aligned}
&\partial_t m(t,x) + (-\Delta)^s m(t,x) - f(m(t,x)) \\
&\quad = \partial_t w(t,x) + (-\Delta)^s m(t,x) - f(w(t,x)), \\
&\quad = \gamma[w(t,x)]^\beta - f(w(t,x)) + (-\Delta)^s m(t,x), \\
&\quad \geqslant \gamma[w(t,x)]^\beta - r[w(t,x)]^\beta - I_1 - I_2, \\
&\quad \geqslant (\gamma - r)[w(t,x)]^\beta - I_1 - I_3 - I_4 - I_5, \\
&\quad \geqslant [w(t,x)]^\beta \left[\gamma - rC_2K^{2s} \cdot [w(t,x)]^{\frac{2s}{p}-\beta} - C_5 - C_7\right].
\end{aligned}$$

Let us take $p = \frac{2s}{\beta}$ (in this case, we have $p+1 \geqslant p\beta$ and $2s > p(\beta-1)$) and fix $K > \sup\left\{2, K\left(\frac{2s(\beta-1)}{\beta}\right)\right\}$ so that claim 2.4 holds true. Then by choosing γ large enough, we then achieve for all $x > x_\gamma(t)$

$$\partial_t m(t,x) + (-\Delta)^s m(t,x) - f(m(t,x)) \geqslant 0.$$

In summary, we conclude that for $p = \frac{2s}{\beta}$ and γ large enough, $m(t,x)$ is a supersolution to the problem (3) for all $x > x_0(t)$ which moreover satisfies $u(t,x) < 1 < m(t,x)$ for all $t > 0$ and $x \leqslant x_0(t)$. By a straightforward application of the comparison principle we therefore get

$$u(t,x) \leqslant m(t,x), \quad \forall t > 0,\ \forall x \in \mathbb{R}.$$

So for any $\lambda \in (0,1)$ and any $x \in \Gamma_\lambda(t)$, then

$$\lambda \leqslant u(t,x) \leqslant m(t,x) = \frac{1}{\left[x^{\frac{2s(\beta-1)}{\beta}} - \gamma(\beta-1)t\right]^{\frac{1}{\beta-1}}},$$

and since $\beta > 1$, it follows:

$$x \leqslant \left[\left(\frac{1}{\lambda}\right)^{\beta-1} + \gamma(\beta-1)t\right]^{\frac{\beta}{2s(\beta-1)}}.$$

Hence when $T_\lambda \gg 1$, we have

$$x_\lambda(t) \leqslant C(\lambda)\cdot t^{\frac{\beta}{2s(\beta-1)}}.$$

3. Lower bound on the speed of the super level sets

Proof of Theorem 1.3. By assumption 2 on the initial data $u_0(x)$, we can construct a non-increasing function $\widehat{u}_0(x)$ such that $\widehat{u}_0(x) \leqslant u_0(x)$ for all $x \in \mathbb{R}$ and

$$\widehat{u}_0(x) = \begin{cases} c_0, & \text{if } x \leqslant -R_0 - 1, \\ 0, & \text{if } x \geqslant -R_0, \end{cases}$$

for some small $0 < c_0 \ll 1$ and some large $R_0 \gg 1$.

Let $v(t,x)$ be the solution of the following problem:

$$\begin{cases} \partial_t v + (-\Delta)^s v = 0, & t > 0,\ x \in \mathbb{R}, \\ v(0,x) = \widehat{u}_0(x), & x \in \mathbb{R}. \end{cases}$$

Since $f(u) \geqslant 0$ for all $u \in [0,1]$, it is easy to see that $v(t,x)$ is a subsolution to the problem (3) and by the comparison principle, we have

$$v(t,x) \leqslant u(t,x), \quad \forall t > 0,\ x \in \mathbb{R}.$$

Let $p_s(t,x)$ be the heat kernel for $(-\Delta)^s$, then we have

$$v(t,x) = \int_{\mathbb{R}} \widehat{u}_0(x-y) p_s(t,y) \mathrm{d}y, \quad \forall t > 0,\ x \in \mathbb{R}.$$

For the heat kernel associated to $(-\Delta)^s$, it is well known that there exists some constant $1 > C_1 > 0$ such that

$$\frac{C_1}{t^{\frac{1}{2s}} \left[1 + |t^{-\frac{1}{2s}} x|^{1+2s}\right]} \leqslant p_s(t,x) \leqslant \frac{C_1^{-1}}{t^{\frac{1}{2s}} \left[1 + |t^{-\frac{1}{2s}} x|^{1+2s}\right]}, \quad \forall t > 0,\ x \in \mathbb{R}.$$

So we get

$$\begin{aligned} v(t,x) &\geqslant \int_{\mathbb{R}} \widehat{u}_0(x-y) \cdot \frac{C_1}{t^{\frac{1}{2s}} \left[1 + |t^{-\frac{1}{2s}} y|^{1+2s}\right]} \mathrm{d}y, \\ &\geqslant \int_{x+R_0+1}^{+\infty} \frac{c_0 \cdot C_1}{t^{\frac{1}{2s}} \left[1 + |t^{-\frac{1}{2s}} y|^{1+2s}\right]} \mathrm{d}y, \\ &\geqslant \int_{t^{-\frac{1}{2s}}(x+R_0+1)}^{+\infty} \frac{c_0 \cdot C_1}{1 + |z|^{1+2s}} \mathrm{d}z. \end{aligned} \tag{9}$$

In particular, we have

$$u(1,x) \geqslant \int_{x+R_0+1}^{+\infty} \frac{c_0 \cdot C_1}{1 + |z|^{1+2s}} \mathrm{d}z.$$

Thus for $x > 1$ we may find a constant $C_2 > 0$ such that

$$u(1,x) \geqslant \frac{C_2}{x^{2s}}.$$

As a result, we can find a small enough $d > 0$ such that

$$u(1,x) \geqslant v(1,x) \geqslant \widetilde{u}_0(x) := \begin{cases} d & \text{for } x \leqslant 1, \\ \dfrac{d}{x^{2s}} & \text{for } x \geqslant 1. \end{cases} \tag{10}$$

Hence, from the comparison principle and up to a shift in time, we only need to get the lower estimate for the case for which $u(t,x)$ is the solution to the problem (3) with the initial data $\widetilde{u}_0$. Since $\widetilde{u}_0(x)$ is non-increasing, then $u(t,x)$ is decreasing with respect to x. Let $\lambda_0 := \int_{\frac{1}{2}}^{+\infty} \frac{c_0 \cdot C_1}{1+|z|^{1+2s}}\mathrm{d}z$, $x_B(t) := \frac{t^{\frac{1}{2s}}}{4}$, and $x_{\lambda_0}(t)$ be such that $u(t, x_{\lambda_0}(t)) = \lambda_0$. From (9), then there exist $T_{\lambda_0} \gg 1$ such that for all $t \geqslant T_{\lambda_0}$ we have

$$v(t,x_B(t)) \geqslant \int_{\frac{1}{4}+t^{-\frac{1}{2s}}(R_0+1)}^{+\infty} \frac{c_0 \cdot C_1}{1+|z|^{1+2s}}\mathrm{d}z \geqslant \int_{\frac{1}{2}}^{+\infty} \frac{c_0 \cdot C_1}{1+|z|^{1+2s}}\mathrm{d}z = \lambda_0.$$

As above the non-increasing behaviour of $\hat{u}_0(x)$ implies that $v(t,x)$ is decreasing with respect to x. Since $u(t,x) \geqslant v(t,x)$ for all $t \geqslant 0$ and all $x \geqslant 0$, and since $u(t,x)$ and $v(t,x)$ are decreasing with respect to x, we then get $\frac{t^{\frac{1}{2s}}}{4} = x_B(t) \leqslant x_{\lambda_0}(t)$. The above argument holding as well for any $0 < \lambda \leqslant \lambda_0$, it provides the lower estimate.

It remains to obtain a similar bound for a given $\lambda_0 < \lambda < 1$. To obtain the bound we can argue as in [3]. To do so let us first prove an invasion lemma on the solution of the Cauchy problem (3). Namely,

Proposition 3.1. *For any* $0 < s < 1$, *assume that the nonlinearity f satisfies assumption* 1, *and the initial data* $u_0(x)$ *satisfies assumption* 2. *Assume* $\beta > 1$ *and* $\frac{\beta}{2s(\beta-1)} > 1$, *and let* $u(t,x)$ *be the solution to the problem* (3) *with the initial data* $u_0(x)$. *Then, for any* $A \in \mathbb{R}$,

$$\lim_{t\to\infty} u(t,x) = 1 \quad \textit{uniformly in } (-\infty, A], \tag{11}$$

and, for any $\lambda \in (0,1)$,

$$\lim_{t\to\infty} \frac{x_\lambda(t)}{t} = +\infty. \tag{12}$$

Let us postpone for a moment the proof of proposition 3.1 and finish the proof of theorem 1.3. Let us denote by $w(t,x)$ the solution of (3) starting from a non-increasing w_0 such that

$$w_0(x) = \begin{cases} \lambda_0 & \text{if } x \leqslant -1, \\ 0 & \text{if } x \geqslant 0. \end{cases} \tag{13}$$

It follows from proposition 3.1 that there is a time $\tau_\lambda > 0$ such that

$$w(\tau_\lambda, x) > \lambda, \quad \forall x \leqslant 0. \tag{14}$$

On the other hand, since $\widehat{u}_0(x)$ is non-decreasing, then $v(t,x)$ is also decreasing with respect to x. Since $u(t,x_B(t)) \geqslant \lambda_0$, then

$$u(t,x) \geqslant v(t,x) \geqslant \lambda_0, \quad \forall x \leqslant x_B(t).$$

So it follows from (13) that

$$u(T,x) \geqslant w_0(x - x_{\lambda_0}(T)), \quad \forall T \geqslant 0, \forall\ x \in \mathbb{R}.$$

The comparison principle then yields

$$u(T+\tau,x) \geqslant w(\tau, x - x_{\lambda_0}(T)), \quad \forall T \geqslant 0,\ \forall \tau \geqslant 0,\ \forall x \in \mathbb{R}.$$

In view of (14), this implies that

$$u(T+\tau_\lambda,x) > \lambda, \quad \forall T \geqslant 0, \forall\ x \leqslant x_{\lambda_0}(T).$$

Hence, for $t \geqslant \tau_\lambda$, the above implies

$$x_\lambda(t) \geqslant x_{\lambda_0}(t-\tau_\lambda) = \frac{(t-\tau_\lambda)^{\frac{1}{2s}}}{4} \geqslant \underline{C}t^{\frac{1}{2s}},$$

provided $t \geqslant T'_\lambda$, with $T'_\lambda > \tau_\lambda$ large enough. This concludes the proof of the lower estimate.

In summary, we conclude that for any $\lambda \in (0,1)$ there exist some constants $T'_\lambda > 0$ and $C'(\lambda) > 0$ such that

$$x_\lambda(t) \geqslant C'(\lambda)t^{\frac{1}{2s}}, \quad \forall t > T'_\lambda.$$

□

Let us now complete our argument by proving proposition 3.1. To do so, let us first establish the following result.

Proposition 3.2 (Speeds of a sequence of bistable traveling waves). *For any $0 < s < 1$, assume that the nonlinearity f satisfies assumption 1, and the initial data $u_0(x)$ satisfies assumption 2. Assume $\beta > 1$ and $\frac{\beta}{2s(\beta-1)} > 1$. Let $(g_n) = (g_{\theta_n})$ be a sequence of bistable nonlinearities such that $g_n \leqslant g_{n+1} \leqslant f$ and $g_n \to f$. Let (c_n, U_n) be the associated sequence of traveling waves. Then*

$$\lim_{n\to\infty} c_n = +\infty.$$

Proof. Since $g_{n+1} \geqslant g_n$ it follows from standard sliding techniques [7, 12–14, 16] that $c_{n+1} \geqslant c_n$. Assume now by contradiction that $c_n \nearrow \bar{c}$ for some $\bar{c} \in \mathbb{R}$. Observe that since $g_n \to f$ and $\int_0^1 f(s)\,\mathrm{d}s > 0$, we have $c_n \geqslant c_0 > 0$ for n large enough, says, $n \geqslant n_0$. As a consequence, for all $n \geqslant n_0$, U_n is smooth and since any translation of U_n is a still a solution, without loss of generality, we can assume the normalization $U_n(0) = 1/2$. Now, thanks to Helly's theorem [9] and up to extraction, U_n converges to a monotone function $\bar{U}$ such that $\bar{U}(0) = \frac{1}{2}$. Also, since $c_n < \bar{c}$, from the equation we get an uniform bound on U'_n, U''_n and up to extraction, U_n also converges in $C^2_{\mathrm{loc}}(\mathbb{R})$, and the limit has to be $\bar{U}$. As a result, $\bar{U}$ is monotone, smooth and solves

$$\begin{cases} (-\Delta^s)\bar{U} + \bar{c}\bar{U}' + f(\bar{U}) = 0 & \text{on } \mathbb{R}, \\ \bar{U}(-\infty) = 1, \quad \bar{U}(0) = \dfrac{1}{2}, \quad \bar{U}(\infty) = 0. \end{cases}$$

In other words, we have constructed a monostable traveling wave under assumption that $\beta > 1$ and $\frac{\beta}{2s(\beta-1)} > 1$, which is a contradiction with the result in [22]. □

Equipped with this technical result we can establish proposition 3.1.

Proof. First, we prove (11) for the particular case where the initial datum u_0 is a smooth non-increasing function such that

$$u_0(x) = \begin{cases} d_0 & \text{for } x \leqslant -1, \\ 0 & \text{for } x \geqslant 0, \end{cases} \tag{15}$$

for an arbitrary $0 < d_0 < 1$. Since u_0 is non-increasing, we deduce from the comparison principle that, for all $t > 0$, the function $u(t, x)$ is still decreasing in x.

Let us now extend f by 0 outside the interval $[0, 1]$. From [1] and proposition 3.2, there exists $0 < \theta < d_0$ and a Lipschitz bistable function $g \leqslant f$—i.e. $g(0) = g(\theta) = g(1) = 0$, $g(s) < 0$ in $(0, \theta)$, $g(s) > 0$ in $(\theta, 1)$, and $g'(0) < 0$, $g'(1) < 0$, $g'(\theta) > 0$—such that there exists a smooth decreasing function U_θ and $c_\theta > 0$ verifying

$$(-\Delta)^s U_\theta + c_\theta U_\theta' + g(U_\theta) = 0 \quad \text{on } \mathbb{R},$$
$$U_\theta(-\infty) = 1, \qquad U_\theta(\infty) = 0.$$

Let us now consider $v(t, x)$ the solution of the Cauchy problem

$$\partial_t v(t, x) = -(-\Delta)^s v(t, x) + g(v(t, x)) \quad \text{for } t > 0,\ x \in \mathbb{R},$$
$$v(0, x) = u_0(x).$$

Since $g \leqslant f$, v is a subsolution of the Cauchy problem (3) and by the comparison principle, $v(t, x) \leqslant u(t, x)$ for all $t > 0$ and $x \in \mathbb{R}$.

Now, thanks to the global asymptotic stability result [1, theorem 3.1], since $d_0 > \theta$, then we know that there exists $\xi \in \mathbb{R}$, $C_0 > 0$ and $\kappa > 0$ such that for all $t \geqslant 0$

$$\|v(t, \cdot) - U_\theta(\cdot - c_\theta t + \xi)\|_{L^\infty} \leqslant C_0 \mathrm{e}^{-\kappa t}.$$

Therefore, for all $t > 0$ and $x \in \mathbb{R}$, we have

$$u(t, x) \geqslant v(t, x) \geqslant U_\theta(x - c_\theta t + \xi) - C_0 \mathrm{e}^{-\kappa t}.$$

Since $c_\theta > 0$, by sending $t \to \infty$, we get $1 \geqslant \liminf_{t\to\infty} u(t, x) \geqslant \lim_{t\to\infty}[U_\theta(x - c_\theta t + \xi) - C_0\mathrm{e}^{-\kappa t}] = 1$. As a result, for all $x \in \mathbb{R}$, we have $1 \geqslant \limsup_{t\to\infty} u(t, x) \geqslant \liminf_{t\to\infty} u(t, x) = 1$, which implies that $u(t, x) \to 1$ as $t \to \infty$. Since $u(t, x)$ is decreasing in x, then the convergence is uniform on any set $(-\infty, A]$. This concludes the proof of (11) for our particular initial datum.

For a generic initial data satisfying assumption 2, we can always, up to a shift in space, construct a smooth non-increasing $\widetilde{u_0}$ satisfying (15) and $\widetilde{u_0} \leqslant u_0$. Since the solution $\widetilde{u}(t, x)$ of the Cauchy problem starting from $\widetilde{u_0}$ satisfies (11), so does $u(t, x)$ thanks to the comparison principle. □

4. Another better lower bound on the speed of the super level sets

Here we prove another lower bound on the speed of $x_\lambda(t)$ when $1 < \beta < 2$ and $\frac{1}{2s(\beta-1)} > 1$ (notice that $\frac{1}{2s(\beta-1)} > \frac{1}{2s}$ if and only if $1 < \beta < 2$). In the whole of this section, let us assume the conditions in theorem 1.4 hold. As above to measure the acceleration, we use a subsolution that fills the space with a superlinear speed. The construction of this subsolution is an adaptation of the one proposed by Alfaro and Coville [3] for a nonlocal diffusion with an integrable kernel. It essentially contains three steps.

Step one. It consists in using the diffusion to gain an algebraic tail at time $t = 1$.

From the proof of Theorem 1.3, we have shown that we can find a small enough $d > 0$ such that

$$u(1,x) \geqslant v(1,x) \geqslant v_0(x) := \begin{cases} d & \text{for } x \leqslant 1, \\ \dfrac{d}{x^{2s}} & \text{for } x \geqslant 1. \end{cases} \tag{16}$$

Hence, from the comparison principle and up to a shift in time, it is enough to prove the lower estimate for $u(t,x)$ which is the solution starting from the initial data v_0, which we do below.

Step two. Here we construct explicitly the subsolution that we are considering.

Following Alfaro–Coville [3] let us consider the function $g(y) := y(1-By)$, with $B > \frac{1}{2d}$, it is easy to see that $g(y) \leqslant 0$ if and only if $0 \leqslant y \leqslant \frac{1}{B}$, and $g(y) \leqslant g(\frac{1}{2B}) = \frac{1}{4B} < d$ for all $y \in \mathbb{R}$.

As in the previous subsection, for any $\gamma > 0$, let $w(\cdot,x)$ denotes the solution to the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t}(t,x) = \gamma[w(t,x)]^{\beta}, \\ w(0,x) = v_0(x). \end{cases}$$

That is

$$w(t,x) = \frac{1}{\left[[v_0(x)]^{1-\beta} - \gamma(\beta-1)t\right]^{\frac{1}{\beta-1}}},$$

where v_0 is defined in (16).

Notice that $w(t,x)$ is not defined for all times. When $x \leqslant 1$, $w(t,x)$ is defined for $t \in [0, \frac{1}{d^{\beta-1}\gamma(\beta-1)})$, whereas for $x > 1$, $w(t,x)$ is defined for $t \in \left[0, T(x) := \frac{x^{2s(\beta-1)}}{d^{\beta-1}\gamma(\beta-1)}\right)$. Let us define

$$x_B(t) := d^{\frac{1}{2s}}\left[(2B)^{\beta-1} + \gamma(\beta-1)t\right]^{\frac{1}{2s(\beta-1)}}. \tag{17}$$

Since $B > \frac{1}{2d}$ and $\beta > 1$, then $x_B(t) > 1$ and $w(t, x_B(t)) = \frac{1}{2B}$. For $x < 1$ and $0 < t < \frac{1}{d^{\beta-1}\gamma(\beta-1)}$, since $v_0(x') = d$ for all $x' \leqslant 1$, then we have $\partial_x w(t,x) = \partial_{xx} w(t,x) = 0$. For $x > 1$ and $0 < t < T(x)$, we compute

$$\begin{aligned} \partial_x w(t,x) &= \frac{1}{1-\beta} \cdot [[v_0(x)]^{1-\beta} - \gamma(\beta-1)t]^{\frac{1}{1-\beta}-1} \cdot (1-\beta)[v_0(x)]^{-\beta} \cdot v_0'(x), \\ &= -2d^{1-\beta} s[w(t,x)]^{\beta} \cdot x^{2s\beta-2s-1}, \\ &< 0. \\ \partial_{xx} w(t,x) &= -2d^{1-\beta}s \cdot \left[\beta[w(t,x)]^{\beta-1} \cdot \partial_x w(t,x) \cdot x^{2s\beta-2s-1}\right. \\ &\quad \left. + [w(t,x)]^{\beta} \cdot (2s\beta - 2s - 1)x^{2s\beta-2s-2}\right], \\ &= -2d^{1-\beta}s \cdot \left[\beta[w(t,x)]^{\beta-1}\right. \\ &\quad \cdot \left(-2d^{1-\beta}s[w(t,x)]^{\beta} \cdot x^{2s\beta-2s-1}\right) \cdot x^{2s\beta-2s-1} \\ &\quad \left. + [w(t,x)]^{\beta} \cdot (2s\beta-2s-1)x^{2s\beta-2s-2}\right], \\ &= 2d^{1-\beta}s[w(t,x)]^{\beta} \cdot x^{2s\beta-2s-2} \\ &\quad \cdot [2d^{1-\beta}s\beta \cdot [w(t,x)]^{\beta-1} x^{2s\beta-2s} + 2s + 1 - 2s\beta], \\ &= 2d^{1-\beta}s[w(t,x)]^{\beta} \cdot x^{2s\beta-2s-2} \cdot \left[2d^{1-\beta}s\beta \cdot \left[[v_0(x)]^{1-\beta}\right.\right. \end{aligned}$$

$$
\begin{aligned}
&- \gamma(\beta-1)t]^{-1}x^{2s\beta-2s} + 2s + 1 - 2s\beta\big],\\
&> 2d^{1-\beta}s[w(t,x)]^{\beta}\cdot x^{2s\beta-2s-2}\\
&\quad\cdot[2d^{1-\beta}s\beta\cdot[v_0(x)]^{\beta-1}x^{2s\beta-2s}+2s+1-2s\beta],\\
&> 2d^{1-\beta}s[w(t,x)]^{\beta}\cdot x^{2s\beta-2s-2}\\
&\quad\cdot\left[2d^{1-\beta}s\beta\cdot\left(\frac{d}{x^{2s}}\right)^{\beta-1}x^{2s\beta-2s}+2s+1-2s\beta\right],\\
&> 2d^{1-\beta}s[w(t,x)]^{\beta}\cdot x^{2s\beta-2s-2}\cdot(2s+1),\\
&> 0.
\end{aligned}
$$

In the first inequality of the computation of $\partial_{xx}w(t,x)$, we used the condition $\beta > 1$ and $\gamma > 0$. Hence, for any $t > 0$, the function $w(t,\cdot)$ is decreasing and convex with respect to the variable x.

Let us now define the continuous function

$$
m(t,x) := \begin{cases} \dfrac{1}{4B} & \text{for } x \leqslant x_B(t),\\ g(w(t,x)) & \text{for } x_B(t) < x. \end{cases}
$$

Note that from the definition of $m(t,x)$ we can see that the function $m(t,\cdot)$ is $C^{1,1}(\mathbb{R})$ for all $t > 0$, and $m(\cdot,x)$ is $C^{1,1}((0,+\infty))$ for all $x \in \mathbb{R}$. In addition, we have

$$
\begin{aligned}
&\partial_x m(t,x) = \partial_x w(t,x)(1-2Bw(t,x))^{+} \quad \text{and}\\
&\partial_t m(t,x) = \gamma w^{\beta}(t,x)(1-2Bw(t,x))^{+}.
\end{aligned}
$$

Let us make now some useful observations:

- when $x > x_B(0) = (2dB)^{\frac{1}{2s}} > 1$, we have $m(0,x) = g(w(0,x)) = g(v_0(x)) \leqslant v_0(x)$;
- when $x < 1$, we have $m(0,x) = \frac{1}{4B} < d = v_0(x)$;
- when $1 \leqslant x \leqslant x_B(0) = (2\mathrm{d}B)^{\frac{1}{2s}}$, we have $v_0(x) = \frac{d}{x^{2s}} \geqslant \frac{d}{2\mathrm{d}B} = \frac{1}{2B} > \frac{1}{4B} = m(0,x)$.

Hence $m(0,x) \leqslant v_0(x)$ for all $x \in \mathbb{R}$.
Let us now show that $m(t,x)$ is a subsolution for an appropriate choice of γ and B.

By the definition of $m(t,x)$, we have $\partial_t m(t,x) = \gamma w^{\beta}(t,x)(1-2Bw(t,x))^{+}$, therefore we get

$$
\partial_t m(t,x) \leqslant \begin{cases} 0 & \text{for } x \leqslant x_B(t)-1,\\ \gamma w^{\beta}(t,x) & \text{for } x_B(t)-1 < x. \end{cases} \tag{18}
$$

Since f satisfies $f(u) \geqslant r_1 u^{\beta}$ as $u \to 0^{+}$, then there exists a small $r_2 > 0$ such that $f(u) \geqslant r_2 u^{\beta}(1-u)$ for all $0 \leqslant u \leqslant 1$. When $x \leqslant x_B(t)$, then $m(t,x) = \frac{1}{4B}$ and since $w(t,x_B(t)) = \frac{1}{2B}$, then

$$
\begin{aligned}
f(m(t,x)) &\geqslant r_2[m(t,x)]^{\beta}[1-m(t,x)],\\
&\geqslant r_2\left[\frac{1}{2}w(t,x_B(t))\right]^{\beta}\left(1-\frac{1}{4B}\right) = \frac{r_2}{2^{\beta}}\left(1-\frac{1}{4B}\right)[w(t,x_B(t))]^{\beta}.
\end{aligned}
$$

Similarly, when $x > x_B(t)$, since $0 \leqslant g(y) \leqslant \frac{1}{4B}$ and $w(t,x) \leqslant \frac{1}{2B}$, then

$$\begin{aligned} f(m(t,x)) &\geqslant r_2[w(t,x)(1 - Bw(t,x))]^{\beta}[1 - g(w(t,x))], \\ &\geqslant r\left[w(t,x)\left(1 - B\cdot\frac{1}{2B}\right)\right]^{\beta}\left[1 - \frac{1}{4B}\right] = \frac{r}{2^{\beta}}\left(1 - \frac{1}{4B}\right)[w(t,x)]^{\beta}. \end{aligned}$$

In summary, we have

$$f(m(t,x)) \geqslant \begin{cases} C_0[w(t,x_B(t))]^{\beta} & \text{for } x \leqslant x_B(t), \\ C_0[w(t,x)]^{\beta} & \text{for } x > x_B(t), \end{cases} \tag{19}$$

where $C_0 := \frac{r}{2^{\beta}}\left(1 - \frac{1}{4B}\right)$.

Let us now derive *a priori* estimates on the fractional diffusion term $(-\Delta)^s m(t,x)$ on the three regions $x \leqslant x_B(t) - 1$, $x_B(t) - 1 < x < x_B(t) + 1$ and $x > x_B(t) + 1$. For simplicity of the presentation, we dedicated a subsection to each region and let us start with the region $x \leqslant x_B(t) - 1$.

• When $x \leqslant x_B(t) - 1$:

In this region of space, we claim that

Claim 4.1.

(a) If $\frac{1}{2} < s < 1$, then there exists $C_3 > 0$ such that for all $x \leqslant x_B(t) - 1$, we have

$$(-\Delta)^s m(t,x) \leqslant -C_3 v_0'(x_B(t))[v_0(x_B(t))]^{-\beta}[w(t,x_B(t))]^{\beta}.$$

(b) If $0 < s \leqslant \frac{1}{2}$, for large enough $B \gg 1$, then there exists $C_3 > 0$ such that for all $x \leqslant x_B(t) - 1$, we have

$$(-\Delta)^s m(t,x) \leqslant \frac{C_3}{B^2}.$$

Note that the singularity here play a major role and the estimate strongly depends on the value of s.

Proof. For $x \leqslant x_B(t) - 1$, since $m(t,y) = \frac{1}{4B} = m(t,x_B(t))$ for all $y \leqslant x_B(t)$, we then have

$$(-\Delta)^s m(t,x) = \int_{x_B(t)}^{+\infty} \frac{m(t,x_B(t)) - m(t,y)}{|x-y|^{1+2s}}\,\mathrm{d}y.$$

We now treat separately the following two situations: $\frac{1}{2} < s < 1$, $0 < s \leqslant \frac{1}{2}$.

Case I: $\frac{1}{2} < s < 1$. By using the fundamental theorem of calculus we get

$$\begin{aligned} -(-\Delta)^s m(t,x) &= \int_{x_B(t)}^{+\infty}\int_0^1 \frac{(y - x_B(t))}{|x-y|^{1+2s}}\partial_x m(t, x_B(t) + \tau(y - x_B(t)))\,\mathrm{d}\tau\,\mathrm{d}y, \\ &= \int_0^{\infty}\int_0^1 \frac{z}{|x - x_B(t) - z|^{1+2s}}\partial_x m(t, x_B(t) + \tau z)\,\mathrm{d}\tau\,\mathrm{d}z. \end{aligned}$$

Now, since $w(t,\cdot)$ is a positive, decreasing and convex function, for any $z, \tau > 0$, we have

$$\begin{aligned} \partial_x m(t, x_B(t) + \tau z) &= \partial_x w(t, x_B(t) + \tau z)\left(1 - 2Bw(t, x_B(t) + \tau z)\right), \\ &\geqslant \partial_x w(t, x_B(t)). \end{aligned}$$

So, we obtain that

$$-(-\Delta)^s m(t,x) \geqslant \partial_x w(t,x_B(t)) \int_0^\infty \frac{z}{|x-x_B(t)-z|^{1+2s}} \mathrm{d}z.$$

In addition, since $x < x_B(t) - 1$ we have $x - x_B(t) - z < -1 - z < 0$ for any $z > 0$, which then enforces $|x - x_B(t) - z| \geqslant |1 + z| > 0$. By using that $\frac{1}{2} < s < 1$, we therefore get

$$\int_0^\infty \frac{z}{|x-x_B(t)-z|^{1+2s}} \mathrm{d}z < C_3 := \int_0^\infty \frac{z}{|1+z|^{1+2s}} \mathrm{d}z < +\infty.$$

As a result

$$\begin{aligned} -(-\Delta)^s m(t,x) &\geqslant C_3 \partial_x w(t,x_B(t)) \\ &= -C_3 v_0'(x_B(t))[v_0(x_B(t))]^{-\beta}[w(t,x_B(t))]^{\beta}, \quad \forall x \leqslant x_B(t) - 1. \end{aligned}$$

Case II: $0 < s \leqslant \frac{1}{2}$. In this situation, the previous argumentation fails and we argue as follows. Since $w(t,y) \geqslant 0$ for all t and all y, for any constant $R > 1$ which will be determined later on, we have

$$\begin{aligned} (-\Delta)^s m(t,x) &= \int_{x_B(t)}^{+\infty} \frac{m(t,x_B(t)) - m(t,y)}{|x-y|^{1+2s}} \mathrm{d}y, \\ &= \int_{x_B(t)}^{+\infty} \frac{w(t,x_B(t)) - B[w(t,x_B(t))]^2 - w(t,y) + B[w(t,y)]^2}{|x-y|^{1+2s}} \mathrm{d}y, \\ &= \int_{x_B(t)}^{+\infty} \frac{[w(t,x_B(t)) - w(t,y)][1 - B[w(t,x_B(t)) + w(t,y)]]}{|x-y|^{1+2s}} \mathrm{d}y, \\ &\leqslant \int_{x_B(t)}^{+\infty} \frac{w(t,x_B(t)) - w(t,y)}{|x-y|^{1+2s}} \mathrm{d}y \qquad \text{Since } w(t,y) \geqslant 0, \\ &\leqslant \int_{x_B(t)}^{+\infty} \frac{w(t,x_B(t)) - w(t,y)}{|y-x_B(t)+1|^{1+2s}} \mathrm{d}y, \\ &\leqslant \int_0^{+\infty} \frac{w(t,x_B(t)) - w(t,x_B(t)+z)}{|1+z|^{1+2s}} \mathrm{d}z, \\ &\leqslant \int_0^{R} \frac{w(t,x_B(t)) - w(t,x_B(t)+z)}{|1+z|^{1+2s}} \mathrm{d}z \\ &\quad + \int_R^{+\infty} \frac{w(t,x_B(t)) - w(t,x_B(t)+z)}{|1+z|^{1+2s}} \mathrm{d}z, \\ &\leqslant I_1 + I_2. \end{aligned}$$

Let us now estimate I_1 and I_2. Since $w(t,y) \geqslant 0$ for all t and all y, for I_2 we have

$$\begin{aligned} I_2 &\leqslant \int_R^{+\infty} \frac{w(t,x_B(t))}{|1+z|^{1+2s}} \mathrm{d}z, \\ &\leqslant \frac{1}{2B} \cdot \int_R^{+\infty} \frac{1}{z^{1+s}} \mathrm{d}z, \\ &\leqslant \frac{1}{2B} \cdot \frac{1}{s} \cdot \frac{1}{R^s}. \end{aligned} \tag{20}$$

On the other hand, by using the fundamental theorem of calculus and the convexity of $w(t, y)$ with respect to y, we get for I_1

$$I_1 = \int_0^R \int_0^1 \frac{-\partial_x w(t, x_B(t) + \tau z) \cdot z}{|1+z|^{1+2s}} \mathrm{d}\tau \mathrm{d}z, \leqslant \int_0^R \int_0^1 \frac{-\partial_x w(t, x_B(t)) \cdot z}{|1+z|^{1+2s}} \mathrm{d}\tau \mathrm{d}z,$$

$$\leqslant -\partial_x w(t, x_B(t)) \int_0^R \frac{z}{|1+z|^{1+2s}} \mathrm{d}z.$$

Thus, by using the definition of $\partial_x w(t, x_B(t))$, $R > 1$ and since $|y|^{2s} > |y|^s$ in $(1, R)$ we get

$$\begin{aligned} I_1 &\leqslant -\partial_x w(t, x_B(t)) \int_0^R \frac{z}{|1+z|^{1+2s}} \mathrm{d}z, \\ &\leqslant -\partial_x w(t, x_B(t)) \int_1^{2R} \frac{1}{y^{2s}} \mathrm{d}y, \\ &\leqslant -\partial_x w(t, x_B(t)) \int_1^{2R} \frac{1}{y^{s}} \mathrm{d}y, \\ I_1 &\leqslant 2d^{1-\beta} s \left(\frac{1}{2B}\right)^{\beta} \cdot [x_B(t)]^{2s\beta-2s-1} \cdot \frac{1}{1-s} \cdot (2R)^{1-s}, \end{aligned}$$

which using that $x_B(t) \geqslant d^{\frac{1}{2s}} (2B)^{\frac{1}{2s}}$ and $2s\beta - 2s - 1 < 0$ enforces that

$$\begin{aligned} I_1 &\leqslant 2d^{1-\beta} s \left(\frac{1}{2B}\right)^{\beta} \cdot \left[d^{\frac{1}{2s}} (2B)^{\frac{1}{2s}}\right]^{2s\beta-2s-1} \cdot \frac{1}{1-s} \cdot (2R)^{1-s}, \\ &\leqslant C_{3,1} B^{-\left(1+\frac{1}{2s}\right)} R^{1-s}. \end{aligned}$$

Combining the latter estimate with (20), we then get

$$(-\Delta)^s m(t, x) \leqslant I_1 + I_2 \leqslant C_{3,1} B^{-\left(1+\frac{1}{2s}\right)} R^{1-s} + \frac{1}{2B} \cdot \frac{1}{s} \cdot \frac{1}{R^s}.$$

By taking R such that $C_{3,1} B^{-\left(1+\frac{1}{2s}\right)} R^{1-s} = \frac{1}{2B} \cdot \frac{1}{s} \cdot \frac{1}{R^s}$, that is, $R = \frac{1}{2sC_{3,1}} B^{\frac{1}{2s}}$, we then achieve

$$(-\Delta)^s m(t, x) \leqslant \frac{C_3}{B^2}.$$

□

Let us now obtain some estimate in the region $x \geqslant x_B(t) + 1$.

• When $x \geqslant x_B(t) + 1$:

In this region, we claim that

Claim 4.2.

(a) If $\frac{1}{2} < s < 1$, then there exists positive constant C_4 such that for all $x \geqslant x_B(t) + 1$, we have

$$(-\Delta)^s m(t, x) \leqslant -C_4 v_0'(x)[v_0(x)]^{-\beta} [w(t, x)]^{\beta}.$$

(b) If $0 < s \leqslant \frac{1}{2}$, for large enough $B \gg 1$, then there exists positive constant C_4 such that for all $x \geqslant x_B(t) + 1$, we have

$$(-\Delta)^s m(t, x) \leqslant -C_4 \partial_x w(t, x) + C_4 [w(t, x)]^{1-2s+2s\beta} x^{2s(2s\beta-2s-1)}.$$

Proof. First, we have

$$
\begin{aligned}
(-\Delta)^s m(t,x) &= \mathrm{P.V.}\int_{\mathbb{R}} \frac{m(t,x)-m(t,y)}{|x-y|^{1+2s}}\mathrm{d}y,\\
&= \int_{-\infty}^{x_B(t)} \frac{m(t,x)-m(t,y)}{|x-y|^{1+2s}}\mathrm{d}y + \mathrm{P.V.}\int_{x_B(t)}^{+\infty} \frac{m(t,x)-m(t,y)}{|x-y|^{1+2s}}\mathrm{d}y,\\
&=: I_1 + I_2.
\end{aligned}
$$

For I_1, since $\partial_x m(t,x) = \partial_x w(t,x)(1-2Bw(t,x))^+$, then $m(t,\cdot)$ is decreasing and since $x \geqslant x_B(t)+1$ we get

$$
I_1 \leqslant 0. \tag{21}
$$

For I_2, we have

$$
\begin{aligned}
I_2 &= \mathrm{P.V.}\int_{x_B(t)-x}^{\infty} \frac{m(t,x)-m(t,x+z)}{|z|^{1+2s}}\mathrm{d}z,\\
&= \int_{x_B(t)-x}^{-\frac{1}{2}} \frac{m(t,x)-m(t,x+z)}{|z|^{1+2s}}\mathrm{d}z + \mathrm{P.V.}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{m(t,x)-m(t,x+z)}{|z|^{1+2s}}\mathrm{d}z\\
&\quad + \int_{\frac{1}{2}}^{+\infty} \frac{m(t,x)-m(t,x+z)}{|z|^{1+2s}}\mathrm{d}z,\\
&=: I_3 + I_4 + I_5.
\end{aligned}
$$

Let us estimate this three integrals. Again by using that $m(t,\cdot)$ is decreasing and $x \geqslant x_B(t)+1$ we trivially have

$$
I_3 \leqslant 0. \tag{22}
$$

For I_4, we observe that since the function $m(t,\cdot)$ is smooth (at least C^2) on $(x-\frac{1}{2}, x+\frac{1}{2})$, then we have

$$
I_4 = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{m(t,x+z)+m(t,x-z)-2m(t,x)}{|z|^{1+2s}}\mathrm{d}z \leqslant C_{4,1}|\partial_{xx}m(t,x)|.
$$

A direct computation of $\partial_{xx}m(t,x)$ shows that $|\partial_{xx}m(t,x)| \leqslant C_{4,2}|\partial_x w(t,x)|$, which implies that

$$
I_4 \leqslant -C_{4,3}\partial_x w(t,x), \tag{23}
$$

since w is decreasing in x. To complete our proof we need to estimate I_5 and to do so we treat separately the following two situations: $\frac{1}{2} < s < 1$, $0 < s \leqslant \frac{1}{2}$.

Case I: $\frac{1}{2} < s < 1$. In this situation, by using the fundamental theorem of calculus, we obtain

$$
I_5 = -\int_{\frac{1}{2}}^{+\infty}\int_0^1 \frac{z\partial_x w(t,x+\tau z)\,(1-2Bw(t,x+\tau z))}{|z|^{1+2s}}\,\mathrm{d}z\,\mathrm{d}\tau,
$$

which by the convexity and the monotonicity of $w(t,\cdot)$ and since $\frac{1}{2} < s < 1$, then yields

$$
\begin{aligned}
I_5 &\leqslant -\int_{\frac{1}{2}}^{\infty}\int_0^1 \frac{z}{|z|^{1+2s}}\partial_x w(t,x+\tau z)\,(1-2Bw(t,x+\tau z))\ \mathrm{d}z\,\mathrm{d}\tau,\\
&\leqslant -\partial_x w(t,x)\int_{\frac{1}{2}}^{\infty}\frac{z}{|z|^{1+2s}}\,\mathrm{d}z,\\
&\leqslant -\partial_x w(t,x)\frac{1}{2^{1-2s}(2s-1)}.
\end{aligned}
$$

By combining the latter with (21)–(23), we can therefore find a constant $C_4 > 0$ such that

$$
(-\Delta)^s m(t,x) \leqslant -C_4\partial_x w(t,x) = -C_4 v_0'(x) v_0^{-\beta}(x) w^{\beta}(t,x).
$$

Case II: $0 < s \leqslant \frac{1}{2}$. In this situation, again the previous argumentation fails and we argue as follows. For any $R > 1$, let us rewrite I_5 as follows

$$
\begin{aligned}
I_5 &= \int_{\frac{1}{2}}^{R}\frac{m(t,x)-m(t,x+z)}{|z|^{1+2s}}\mathrm{d}z + \int_R^{+\infty}\frac{m(t,x)-m(t,x+z)}{|z|^{1+2s}}\mathrm{d}z,\\
&=: I_6 + I_7.
\end{aligned}
$$

An easy computation shows that since $m(t,x) \leqslant w(t,x)$, and $R > 1$ we get

$$
\begin{aligned}
I_7 &\leqslant \int_R^{+\infty}\frac{m(t,x)}{|z|^{1+2s}}\mathrm{d}z \leqslant w(t,x)\cdot\int_R^{+\infty}\frac{1}{z^{1+2s}}\mathrm{d}z,\\
&\qquad\qquad \leqslant w(t,x)\cdot\int_R^{+\infty}\frac{1}{z^{1+s}}\mathrm{d}z = \frac{1}{s}w(t,x)\cdot\frac{1}{R^s}. \qquad (24)
\end{aligned}
$$

On the other hand by using the fundamental theorem of calculus, we obtain the following for I_6

$$
I_6 = -\int_{\frac{1}{2}}^{R}\int_0^1 \frac{z\partial_x w(t,x+\tau z)\,(1-2Bw(t,x+\tau z))}{|z|^{1+2s}}\,\mathrm{d}z\,\mathrm{d}\tau,
$$

which by using the convexity and the monotonicity of $w(t,\cdot)$ can be estimate as follows

$$
\begin{aligned}
I_6 &\leqslant -\int_{\frac{1}{2}}^{R}\int_0^1 \frac{z}{|z|^{1+2s}}\partial_x w(t,x+\tau z)\,(1-2Bw(t,x+\tau z))\ \mathrm{d}z\,\mathrm{d}\tau,\\
&\leqslant -\partial_x w(t,x)\int_{\frac{1}{2}}^{R}\frac{z}{|z|^{1+2s}}\,\mathrm{d}z,\\
&\leqslant -\partial_x w(t,x)\left(\int_{\frac{1}{2}}^{1}\frac{1}{z^{2s}}\,\mathrm{d}z + \int_1^R\frac{1}{z^{2s}}\,\mathrm{d}z\right),\\
&\leqslant -\partial_x w(t,x)\left(\int_{\frac{1}{2}}^{1}\frac{1}{z^{2s}}\,\mathrm{d}z + \int_1^R\frac{1}{z^{s}}\,\mathrm{d}z\right) \leqslant -\partial_x w(t,x)\left(C(s)+\frac{1}{s}R^{1-s}\right). \qquad (25)
\end{aligned}
$$

Combining (25) with (24) and using the definition of $\partial_x w(t,x)$ we thus obtain

$$
I_5 \leqslant \frac{1}{s}w(t,x)\cdot\frac{1}{R^s} + C_{6,2}[w(t,x)]^{\beta}\cdot x^{2s\beta-2s-1}R^{1-s} + C(s)\partial_x w(t,x).
$$

By taking R such that $C_{6,2}[w(t,x)]^{\beta}\cdot x^{2s\beta-2s-1}R^{1-s}=\frac{1}{s}w(t,x)\cdot\frac{1}{R^s}$, that is, $R=\frac{1}{sC_{6,2}}\cdot[w(t,x)]^{1-\beta}\cdot x^{-2s\beta+2s+1}$, we then get

$$I_5\leqslant C_{7,2}[w(t,x)]^{1-2s+2s\beta}x^{2s(2s\beta-2s-1)}+C(s)\partial_x w(t,x),$$

which combined with (21)–(23) then yields

$$(-\Delta)^s m(t,x)\leqslant -C_4\partial_x w(t,x)+C_4[w(t,x)]^{1-2s+2s\beta}x^{2s(2s\beta-2s-1)}.$$

□

Lastly, let us estimate $-(-\Delta)^s m(t,x)$ in the region $x_B(t)-1<x<x_B(t)+1$.

• When $x_B(t)-1\leqslant x\leqslant x_B(t)+1$:

In this last region, we claim that

Claim 4.3.

(a) If $\frac{1}{2}<s<1$, then there exists positive constant C_5 such that for all $x_B(t)-1\leqslant x\leqslant x_B(t)+1$, we have

$$(-\Delta)^s m(t,x)\leqslant -C_5 v_0'(x)[v_0(x)]^{-\beta}[w(t,x)]^{\beta}.$$

(b) If $0<s\leqslant\frac{1}{2}$, for large enough $B\gg 1$, then there exists positive constant C_5 such that for all $x_B(t)-1\leqslant x\leqslant x_B(t)+1$, we have

$$(-\Delta)^s m(t,x)\leqslant -C_5\partial_x w(t,x)+C_5[w(t,x)]^{1-2s+2s\beta}x^{2s(2s\beta-2s-1)}.$$

Proof. Again let us rewrite the fractional Laplacian in the following way:

$$\begin{aligned}(-\Delta)^s m(t,x)&=\mathrm{P.V.}\int_{\mathbb{R}}\frac{m(t,x)-m(t,y)}{|x-y|^{1+2s}}\mathrm{d}y,\\&=\int_{-\infty}^{x-1}\frac{m(t,x)-m(t,y)}{|x-y|^{1+2s}}\mathrm{d}y+\mathrm{P.V.}\int_{x-1}^{+\infty}\frac{m(t,x)-m(t,y)}{|x-y|^{1+2s}}\mathrm{d}y,\\&=:I_1+I_2.\end{aligned}$$

Again, by using the monotone character of $m(t,\cdot)$, we have

$$I_1\leqslant 0, \tag{26}$$

and for I_2, we have

$$\begin{aligned}I_2&=\mathrm{P.V.}\int_{-1}^{\infty}\frac{m(t,x)-m(t,x+z)}{|z|^{1+2s}}\mathrm{d}z,\\&=-\frac{1}{2}\int_{-1}^{1}\frac{m(t,x+z)+m(t,x-z)-2m(t,x)}{|z|^{1+2s}}\mathrm{d}z+\int_{1}^{\infty}\frac{m(t,x)-m(t,x+z)}{|z|^{1+2s}}\mathrm{d}z,\\&=I_3+I_4.\end{aligned}$$

Observe that again to estimate I_2 we break the integral into two part and we can easily see that the contribution of I_4 can be estimated as in the proof of the previous claim so we will not repeat it. If fact, here the only change with respect to the situation $x>x_B(t)+1$, is the contribution of I_3 since unlike the previous case, the function m is not any more a C^2 smooth function on the domain of integration and we need then more precise estimate. So let us now look more closely at I_3.

For I_3, thanks to the definition of m, we can get

$$I_3 = -\frac{1}{2}\int_{-1}^{1}\int_{0}^{1} z\frac{\partial_x m(t,x+\tau z) - \partial_x m(t,x-\tau z)}{|z|^{1+2s}}\mathrm{d}\tau\mathrm{d}z,$$
$$= -\frac{1}{2}\int_{-1}^{1}\int_{0}^{1} \frac{z}{|z|^{1+2s}}\left[\partial_x w(t,x+\tau z)(1-2Bw(t,x+\tau z))^{+}\right.$$
$$\left. - \partial_x w(t,x-\tau z)(1-2Bw(t,x-\tau z))^{+}\right]\,\mathrm{d}\tau\mathrm{d}z.$$

Let us rewrite the bracket inside the integral as follows:

$$\partial_x w(t,x+\tau z)(1-2Bw(t,x+\tau z))^{+} - \partial_x w(t,x-\tau z)(1-2Bw(t,x-\tau z))^{+}$$
$$= [\partial_x w(t,x+\tau z) - \partial_x w(t,x-\tau z)]\cdot(1-2Bw(t,x+\tau z))^{+}$$
$$+\ \partial_x w(t,x-\tau z)[(1-2Bw(t,x+\tau z))^{+} - (1-2Bw(t,x-\tau z))^{+}].$$

Then we can decompose I_3 into two integrals $I_3 = I_5 + I_6$ with

$$I_5 := \frac{1}{2}\int_{-1}^{1}\int_{0}^{1}\frac{z}{|z|^{1+2s}}[\partial_x w(t,x+\tau z) - \partial_x w(t,x-\tau z)]$$
$$\times (1-2Bw(t,x+\tau z))^{+}\,\mathrm{d}\tau\mathrm{d}z,$$
$$I_6 := \frac{1}{2}\int_{-1}^{1}\int_{0}^{1}\frac{z}{|z|^{1+2s}}\partial_x w(t,x-\tau z)\left[(1-2Bw(t,x+\tau z))^{+}\right.$$
$$\left. - (1-2Bw(t,x-\tau z))^{+}\right]\,\mathrm{d}\tau\mathrm{d}z.$$

Now since $w(t,x)$ is smooth, using the fundamental theorem of calculus, we get

$$I_5 = -\frac{1}{2}\int_{-1}^{1}\int_{0}^{1}\int_{-1}^{1}\frac{2\tau z^2}{|z|^{1+2s}}\partial_{xx} w(t,x+\sigma\tau z)(1-2Bw(t,x+\tau z))^{+}\,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}z.$$

Since $w(t,\cdot)$ is convex, then

$$I_5 \leqslant 0.$$

For I_6, by using the convexity of $w(t,\cdot)$ and the uniform Lipschitz continuity of the function $(1-2Bw(t,x))^{+}$ for $x \in (x_B(t)-1, x_B(t)+1)$, then we have

$$I_6 \leqslant -\frac{1}{2}\partial_x w(t,x-1)\int_{-1}^{1}\int_{0}^{1}\frac{|z|}{|z|^{1+2s}}|(1-2Bw(t,x+\tau z))^{+}$$
$$-(1-2Bw(t,x-\tau z))^{+}|\,\mathrm{d}\tau\mathrm{d}z,$$
$$\leqslant -C\partial_x w(t,x-1)\int_{-1}^{1}\frac{z^2}{|z|^{1+2s}}\mathrm{d}z,$$
$$\leqslant -C_{5,2}\partial_x w(t,x-1).$$

A direct computation gives us some $C_{5,1} > 0$ such that $\partial_x w(t, x-1) \geqslant C_{5,1}\partial_x w(t,x)$, which implies that

$$I_6 \leqslant -C_{5,3}\partial_x w(t,x).$$

Hence

$$I_3 \leqslant -C_{5,3}\partial_x w(t,x). \tag{27}$$

□

By collecting (18), claims 4.1, 4.2, 4.3 and (19), now we can show that $m(t,x)$ is a subsolution for some appropriate choices of B and γ.

If $\frac{1}{2} < s < 1$, by (18), claims 4.1, 4.2, 4.3 and (19), we have

$$(\partial_t m + (-\Delta)^s m - f(m))(t,x) \leqslant \begin{cases} -[w(t,x_B(t))]^\beta\,[C_0 + h(t,x_B(t))] & \text{for } x \leqslant x_B(t)-1, \\ -[w(t,x)]^\beta\,[C_0 + h(t,x) - \gamma] & \text{for } x > x_B(t)-1, \end{cases}$$

where $h(t,x) = C_6 v_0'(x)[v_0(x)]^{-\beta}$ with $C_6 \geqslant \max\{C_3, C_4, C_5\}$.

We now choose $\gamma \leqslant \frac{C_0}{2}$. In view of the above inequalities, to complete the construction of the subsolution $m(t,x)$, it suffices to find a condition on B so that $h(t,x) \geqslant -\frac{C_0}{2}$ for all $t > 0$ and all $x \in \mathbb{R}$. From the definitions of $h(t,x)$ and $v_0(x)$, it suffices to achieve

$$x^{(\beta-1)2s-1} \leqslant \frac{C_0 d^{\beta-1}}{4sdC_6}, \quad \text{for all } t > 0,\ x \geqslant x_B(t)-1.$$

Since $(\beta-1)2s < 1$, this reduces to the following condition on $x_B(0)$:

$$x_B(0) \geqslant \left(\frac{C_0 d^{\beta-1}}{4sdC_6}\right)^{\frac{1}{1-2s(\beta-1)}} + 1.$$

From (17) we have $x_B(0) = (2Bd)^{\frac{1}{2s}}$. Hence, in view of the definition of C_0, the above inequality holds by selecting $B \geqslant B_0$, with $B_0 > 0$ large enough.

If $0 < s \leqslant \frac{1}{2}$, by (18), claim 4.1, and (19), we have for $x \leqslant x_B(t) - 1$

$$\begin{aligned}(\partial_t m + (-\Delta)^s m - f(m))(t,x) &\leqslant \frac{C_3}{B^2} - C_0[w(t,x_B(t))]^\beta, \\ &\leqslant \frac{C_3}{B^2} - C_0\left(\frac{1}{2B}\right)^\beta, \\ &\leqslant B^{-2}[C_3 - 2^{-\beta}C_0 B^{2-\beta}].\end{aligned}$$

Since $\beta < 2$, then there exists some $B_1 \gg 1$ such that $C_3 - 2^{-\beta}C_0 B^{2-\beta} < 0$ for all $B \geqslant B_1$. Hence in this case, we have

$$(\partial_t m + (-\Delta)^s m - f(m))(t,x) < 0.$$

When $x > x_B(t) - 1$, by (18), claims 4.2, 4.3 and (19), we have

$$\begin{aligned}
&(\partial_t m + (-\Delta)^s m - f(m))(t,x) \\
&\quad \leqslant \gamma[w(t,x)]^\beta - C_6\partial_x w(t,x) + C_6[w(t,x)]^{1-2s+2s\beta}x^{2s(2s\beta-2s-1)} - C_0 w^\beta(t,x), \\
&\quad \leqslant \gamma[w(t,x)]^\beta + C_7[w(t,x)]^\beta \cdot x^{2s\beta-2s-1} + C_6[w(t,x)]^{1-2s+2s\beta}x^{2s(2s\beta-2s-1)} \\
&\qquad - C_0[w(t,x)]^\beta, \\
&\quad \leqslant [w(t,x)]^\beta[\gamma + C_7 x^{2s\beta-2s-1} + C_6[w(t,x)]^{1-2s+2s\beta-\beta}x^{2s(2s\beta-2s-1)} - C_0].
\end{aligned}$$

It is easy to see that $x_B(t) \geqslant d^{\frac{1}{2s}}(2B)^{\frac{1}{2s}}$ and since $2s\beta - 2s - 1 < 0$ we have for B large enough

$$C_7 x^{2s\beta-2s-1} \leqslant \frac{C_0}{3}.$$

Note that since $\beta > 1$ and $0 < s \leqslant \frac{1}{2}$, we have $1 - 2s + 2s\beta - \beta \leqslant 0$ and therefore since $w(t,x) \geqslant w(0,x) = v_0(x) = \frac{d}{x^{2s}}$, we have

$$\begin{aligned}
C_6[w(t,x)]^{1-2s+2s\beta-\beta}x^{2s(2s\beta-2s-1)} &\leqslant C_8 x^{-2s(1-2s+2s\beta-\beta)} \cdot x^{2s(2s\beta-2s-1)}, \\
&\leqslant C_8 x^{2s(\beta-2)}.
\end{aligned}$$

Using that $\beta < 2$ and since $x \geqslant x_B(t) - 1 \geqslant d^{\frac{1}{2s}}(2B)^{\frac{1}{2s}} - 1$, so when B is large enough, we have $C_8 x^{2s(\beta-2)} \leqslant \frac{C_0}{3}$, which implies that

$$C_6[w(t,x)]^{1-2s+2s\beta-\beta}x^{2s(2s\beta-2s-1)} \leqslant \frac{C_0}{3}.$$

So by taking $\gamma = \frac{C_0}{3}$ we achieve

$$(\partial_t m + (-\Delta)^s m - f(m))(t,x) \leqslant 0.$$

In summary, for any $0 < s < 1$, after some good choices of γ and B, then the function $m(t,x)$ indeed is a subsolution.

Step three. It consists in using the subsolution to prove the lower estimate in theorem 1.4.

Fix $\gamma > 0$ and $B_0 > 0$ as in the previous step such that $m(t,x)$ is a subsolution. From the comparison principle we get $m(t,x) \leqslant u(t,x)$, for all $t > 0$ and $x \in \mathbb{R}$. Recall that $m(t, x_{B_0}(t)) = \frac{1}{4B_0}$ and that $u(t,\cdot)$ is nonincreasing (since initial datum v_0 is nonincreasing) such that

$$u(t,x) \geqslant \frac{1}{4B_0}, \quad \forall x \leqslant x_{B_0}(t). \tag{28}$$

In particular, for any $0 < \lambda \leqslant \frac{1}{4B_0}$, the 'largest' element $x_\lambda(t)$ of the super level set $\Gamma_\lambda(t)$ has to satisfy

$$x_\lambda(t) \geqslant x_{B_0}(t) \geqslant d^{\frac{1}{\alpha-1}}[\gamma(\beta-1)t]^{\frac{1}{2s(\beta-1)}},$$

which provides the lower estimate.

It now remains to obtain a similar bound for a given $\frac{1}{4B_0} < \lambda < 1$. Such estimate can be obtained by redoing the argument in section 3.

Acknowledgments

J Coville carried out this work in the framework of Archimède Labex (ANR-11-LABX-0033) and of the A*MIDEX project (ANR-11-IDEX-0001-02), funded by the 'Investissements d'Avenir' French Government program managed by the French National Research Agency (ANR). He has also received funding from the ANR DEFI Project NONLOCAL (ANR-14-CE25-0013). C Gui is supported by NSF US National Science Foundation Grant Nos. DMS-1601885 and DMS-1901914 and Simons Foundation Award No. 617072. M Zhao is supported by the National Natural Science Foundation of China Grant No. 11801404.

ORCID iDs

Jérôme Coville https://orcid.org/0000-0002-7364-366X
Changfeng Gui https://orcid.org/0000-0001-5903-6188

References

[1] Achleitner F *et al* 2015 Traveling waves for a bistable equation with nonlocal diffusion *Adv. Differ. Equ.* **20** 887–936
[2] Alfaro M 2017 Slowing Allee effect versus accelerating heavy tails in monostable reaction diffusion equations *Nonlinearity* **30** 687
[3] Alfaro M and Coville J 2017 Propagation phenomena in monostable integro-differential equations: acceleration or not? *J. Differ. Equ.* **263** 5727–58
[4] Alfaro M and Giletti T 2020 Interplay of nonlinear diffusion, initial tails and Allee effect on the speed of invasions *Ann. Sc. Norm. Super. PisaCl. Sci.* **XXI** 1223–55
[5] Alfaro M and Giletti T 2019 When fast diffusion and reactive growth both induce accelerating invasions *Commun. Pure Appl. Anal.* **18** 3011–34
[6] Aronson D G and Weinberger H F 1978 Multidimensional nonlinear diffusion arising in population genetics *Adv. Math.* **30** 33–76
[7] Berestycki H and Nirenberg L 1991 On the method of moving planes and the sliding method *Bol. Soc. Bras. Math.* **22** 1–37
[8] Bouin E, Garnier J, Henderson C and Patout F 2018 Thin front limit of an integro-differential Fisher–KPP equation with fat-tailed kernels *SIAM J. Math. Anal.* **50** 3365–94
[9] Brunk H D, Ewing G M and Utz W R 1956 Some Helly theorems for monotone functions *Proc. Am. Math. Soc.* **7** 776
[10] Cabré X and Roquejoffre J-M 2013 The influence of fractional diffusion in Fisher–KPP equations *Commun. Math. Phys.* **320** 679–722
[11] Cabré X and Sire Y 2015 Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions *Trans. Am. Math. Soc.* **367** 911–41
[12] Chen X 1997 Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations *Adv. Differ. Equ.* **2** 125–60
[13] Coville J 2006 On uniqueness and monotonicity of solutions of non-local reaction diffusion equation *Ann. Math.* **185** 461–85
[14] Coville J 2007 Maximum principles, sliding techniques and applications to nonlocal equations *Electron. J. Differ. Equ.* **68** 23 (electronic)
[15] Coville J 2007 Travelling fronts in asymmetric nonlocal reaction diffusion equations: the bistable and ignition cases (CCSD-Hal E-print)
[16] Coville J and Dupaigne L 2005 Propagation speed of travelling fronts in non local reaction–diffusion equations *Nonlinear Anal. Theory Methods Appl.* **60** 797–819
[17] Engler H 2010 On the speed of spread for fractional reaction–diffusion equations *Int. J. Differ. Equ.* **2010** 1–16

[18] Fife P C and McLeod J B 1977 The approach of solutions of nonlinear diffusion equations to travelling front solutions *Arch. Ration. Mech. Anal.* **65** 335–61
[19] Fisher R A 1937 The wave of advance of advantageous genes *Ann. Eugenics* **7** 335–69
[20] Garnier J 2011 Accelerating solutions in integro-differential equations *SIAM J. Math. Anal.* **43** 1955–74
[21] Garnier J, Hamel F, Hamel F and Roques L 2017 Transition fronts and stretching phenomena for a general class of reaction-dispersion equations *Discrete Contin. Dyn. Syst.* A **37** 743–56
[22] Gui C and Huan T 2015 Traveling wave solutions to some reaction diffusion equations with fractional Laplacians *Calc. Var.* **54** 251–73
[23] Gui C and Zhao M 2015 Traveling wave solutions of Allen–Cahn equation with a fractional Laplacian *Ann. Inst. Henri Poincare* **32** 785–812
[24] Hamel F and Roques L 2010 Fast propagation for KPP equations with slowly decaying initial conditions *J. Differ. Equ.* **249** 1726–45
[25] Kanel J I 1961 Certain problem of burning-theory equations *Dokl. Akad. Nauk SSSR* **136** 277–80
[26] Kay A L, Sherratt J A and McLeod J B 2001 Comparison theorems and variable speed waves for a scalar reaction–diffusion equation *Proc. R. Soc. Edinburgh* **131** 1133–61
[27] Kolmogorov A N, Petrovsky I G and Piskunov N S 1937 Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique *Bull. Moskov. Gos. Univ. Mat. Mekh* **1** 125
[28] Mancinelli R, Vergni D and Vulpiani A 2003 Front propagation in reactive systems with anomalous diffusion *Physica* **185** 175–95
[29] Mellet A, Roquejoffre J-M and Sire Y 2009 Generalized fronts for one-dimensional reaction–diffusion equations *Discrete Contin. Dyn. Syst.* **26** 303–12
[30] Metzler R and Joseph K 2000 The random walk's guide to anomalous diffusion: a fractional dynamics approach *Phys. Rep.* **339** 77
[31] Nec Y, Nepomnyashchy A A and Golovin A A 2008 Front-type solutions of fractional Allen–Cahn equation *Physica* **237** 3237–51
[32] Needham D J and Barnes A N 1999 Reaction–diffusion and phase waves occurring in a class of scalar reaction–diffusion equations *Nonlinearity* **12** 41
[33] Palatucci G, Savin O and Valdinoci E 2013 Local and global minimizers for a variational energy involving a fractional norm *Ann. Math.* **192** 673–718
[34] Sherratt J A 2005 An analysis of vegetation stripe formation in semi-arid landscapes *J. Math. Biol.* **51** 183–97
[35] Volpert V A, Nec Y and Nepomnyashchy A A 2013 Fronts in anomalous diffusion-reaction systems *Phil. Trans. R. Soc.* A **371** 20120179
[36] Weinberger H F 1982 Long-time behavior of a class of biological models *SIAM J. Math. Anal.* **13** 353–96