# The Sphere Covering Inequality and Its Dual

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**Abstract**

We present a new proof of the sphere covering inequality in the spirit of comparison geometry, and as a by-product we find another sphere covering inequality that can be viewed as the dual of the original one. We also prove sphere covering inequalities on surfaces satisfying general isoperimetric inequalities, and discuss their applications to elliptic equations with exponential nonlinearities in dimension 2. The approach in this paper extends, improves, and unifies several inequalities about solutions of elliptic equations with exponential nonlinearities.


## 1 Introduction

Second-order nonlinear elliptic equations with exponential nonlinearity of the form

$$\Delta u + e^u = f(x) \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{1.1}$$

arise in many important problems in mathematics, mathematical physics, and biology. Such equations have been extensively studied in the context of Moser-Trudinger inequalities, Chern-Simons self-dual vortices, Toda systems, conformal geometry, statistical mechanics of two-dimensional turbulence, self-gravitating cosmic strings, theory of elliptic functions, and hyperelliptic curves and free boundary models of cell motility; see [2–4, 6–16, 18, 20–25, 32–38, 42] and the references cited therein.

The sphere covering inequality was recently introduced in [28], and has been applied to solve various problems about symmetry and uniqueness of solutions of elliptic equations with exponential nonlinearity in dimension $n = 2$. In particular, it was applied to prove a longstanding conjecture of Chang-Yang ( [20]) concerning the best constant in Moser-Trudinger-type inequalities [28], and has led to several symmetry and uniqueness results for mean field equations, Onsager vortices, Sinh-Gordon equation, cosmic string equation, Toda systems, and rigidity of Hawking mass in general relativity [26–31, 39, 41].

THEOREM 1.1 (The sphere covering inequality). *Let $\Omega_0 \subset \mathbb{R}^2$ be a simply connected domain. Assume $u_1 \in C^2(\bar{\Omega}_0)$ such that*

$$\Delta u_1 + e^{2u_1} \geq 0 \ \ on\ \Omega_0, \quad \int_{\Omega_0} e^{2u_1}\, dx \leq 4\pi. \tag{1.2}$$

*Let $\Omega \subset \Omega_0$ be a bounded open set. Assume $u_2 \in C^2(\bar{\Omega})$ such that*

$$\Delta u_2 + e^{2u_2} \geq \Delta u_1 + e^{2u_1} \ \ in\ \Omega, \quad u_2 > u_1 \ \ in\ \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

*Then*

$$\int_{\Omega} e^{2u_1}\, dx + \int_{\Omega} e^{2u_2}\, dx \geq 4\pi. \tag{1.3}$$

In this paper, we present an approach that completes, simplifies, and improves the sphere covering inequality and several other inequalities about solutions of the elliptic equations with exponential nonlinearities. In particular, we will prove the following generalization of the sphere covering inequality with a method different from the one in [28].

THEOREM 1.2. *Let $\Omega_0 \subset \mathbb{R}^2$ be a simply connected domain. Assume $u_1 \in C^2(\bar{\Omega}_0)$ such that*

$$\Delta u_1 + e^{2u_1} \geq 0 \ \ on\ \Omega_0, \quad \int_{\Omega_0} e^{2u_1} dx \leq 4\pi. \tag{1.4}$$

*Let $\Omega \subset \Omega_0$ be a bounded open set. Assume $u_2 \in C^2(\bar{\Omega})$ and $0 < \lambda \leq 1$ such that*

$$\Delta u_2 + \lambda e^{2u_2} \geq \Delta u_1 + e^{2u_1} \ \ in\ \Omega, \quad u_2 > u_1 \ \ in\ \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

*Then*

$$\int_{\Omega} e^{2u_1}\, dx + \int_{\Omega} e^{2u_2}\, dx \geq \frac{4\pi}{\lambda}. \tag{1.5}$$

We shall also prove the following inequality, which can be viewed as the dual of the sphere covering inequality.

THEOREM 1.3. *Let $\Omega_0 \subset \mathbb{R}^2$ be a simply connected domain. Assume $u_1 \in C^2(\bar{\Omega}_0)$ such that*

$$\Delta u_1 + e^{2u_1} \geq 0 \ \ on\ \Omega_0, \quad \int_{\Omega_0} e^{2u_1} dx \leq 4\pi. \tag{1.6}$$

*Let $\Omega \subset \Omega_0$ be a bounded open set. Assume $u_2 \in C^2(\bar{\Omega})$ such that*

$$\Delta u_2 + e^{2u_2} \leq \Delta u_1 + e^{2u_1} \ \ in\ \Omega, \quad u_2 < u_1 \ \ in\ \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

*Then*

$$\int_\Omega e^{2u_1}\,dx + \int_\Omega e^{2u_2}\,dx \geq 4\pi. \tag{1.7}$$

We will develop an approach for Theorem 1.2 that is different from the one in [28] and modify it to prove Theorem 1.3. Our method has the general spirit of comparison geometry. Under the assumption of Theorem 1.2, let $g = e^{2u_1}|dx|^2$; here $|dx|^2$ is the Euclidean metric. Then the Gauss curvature $K \leq 1$ and the area $\mu(\Omega_0) \leq 4\pi$, where $\mu(E)$ is the measure of $E$ associated with the metric $g$. It follows from [1] and [19, lemma 4.2] that the following isoperimetric inequality holds on $(\Omega_0, g)$ for any domain $E$ in $\Omega_0$ with smooth boundary

$$4\pi\mu(E) - \mu^2(E) \leq s^2(\partial E), \tag{1.8}$$

where $s$ is the one-dimensional measure associated with $g$. Using $g$ as a background metric, we can rewrite the differential inequality between $u_1$ and $u_2$ into a differential inequality involving $u = u_2 - u_1$. Applying ideas from [1, 40] to the resulting differential inequality on $(\Omega_0, g)$ gives us an inequality that will imply Theorem 1.2. Indeed, the proof of Theorem 1.2 is based on the following more general result.

THEOREM 1.4. *Let $(M, g)$ be a simply connected, smooth Riemann surface. Assume $K \leq 1$ and $\mu(M) \leq 4\pi$; here $K$ is the Gauss curvature and $\mu$ is the measure of $(M, g)$. Let $\Omega$ be a domain with compact closure and nonempty boundary, and $\lambda$ be a constant. If $u \in C^2(\bar{\Omega})$ such that $u > 0$ in $\Omega$ and*

$$-\Delta_g u + 1 \leq \lambda e^{2u}, \quad u|_{\partial\Omega} = 0. \tag{1.9}$$

*Then*

$$4\pi \int_\Omega e^{2u}\,d\mu - \lambda\left(\int_\Omega e^{2u}\,d\mu\right)^2 \leq 4\pi\mu(\Omega) - \mu^2(\Omega). \tag{1.10}$$

*In particular, if $0 < \lambda \leq 1$, then*

$$\int_\Omega e^{2u}\,d\mu + \mu(\Omega) \geq \frac{4\pi}{\lambda}. \tag{1.11}$$

It is interesting that in this comparison theorem, what is compared is not the area itself, but the quantity $4\pi\mu(\Omega) - \mu^2(\Omega)$, which is exactly the quantity that appeared in the isoperimetric inequality.

The proof of Theorem 1.3 also follows from the following more general result.

THEOREM 1.5. *Let $(M, g)$ be a simply connected, smooth Riemann surface. Assume $K \leq 1$ and $\mu(M) \leq 4\pi$; here $K$ is the Gauss curvature and $\mu$ is the measure of $(M, g)$. Let $\Omega$ be a domain with compact closure and nonempty boundary, and $\lambda$ be a constant. If $u \in C^2(\bar{\Omega})$ such that $u < 0$ in $\Omega$ and*

$$-\Delta_g u + 1 \geq \lambda e^{2u}, \quad u|_{\partial\Omega} = 0. \tag{1.12}$$

*Then*

$$4\pi\int_\Omega e^{2u}\,d\mu-\lambda\left(\int_\Omega e^{2u}\,d\mu\right)^2\geq 4\pi\mu(\Omega)-\mu^2(\Omega). \tag{1.13}$$

*In particular, if* $\lambda=1$*, then*

$$\int_\Omega e^{2u}\,d\mu+\mu(\Omega)\geq 4\pi. \tag{1.14}$$

In Section 2, we present proofs of Theorems 1.2, 1.3, 1.4, and 1.5. In Section 3, we will prove sphere covering inequalities on surfaces satisfying general isoperimetric inequalities and shall discuss their applications to elliptic equations with exponential nonlinearities.

## 2 Differential Inequalities on Surface with Curvature at Most 1

In this section we will prove Theorems 1.4 and 1.5. The main point is that the approach in [1,40] can be performed on a simply connected surface with curvature at most 1.

PROOF OF THEOREM 1.4. By approximation and replacing $\lambda$ with $\lambda+\varepsilon$, $\varepsilon$ a small positive number, we can assume $u$ is a Morse function. For $t>0$, let

$$\alpha(t)=\int_{\{u>t\}}e^{2u}\,d\mu,\quad \beta(t)=\int_{\{u>t\}}d\mu.$$

By the co-area formula we get

$$\begin{aligned}\alpha(t)=\int_{\{u>t\}}e^{2u}\frac{|\nabla u|}{|\nabla u|}d\mu&=\int_t^\infty d\tau\int_{\{u=\tau\}}\frac{e^{2u}}{|\nabla u|}ds\\&=\int_t^\infty\left(e^{2\tau}\int_{\{u=\tau\}}\frac{ds}{|\nabla u|}\right)d\tau\end{aligned}$$

and

$$\beta(t)=\int_{\{u>t\}}\frac{|\nabla u|}{|\nabla u|}d\mu=\int_t^\infty d\tau\int_{\{u=\tau\}}\frac{ds}{|\nabla u|}.$$

It follows that

$$\alpha'(t)=-e^{2t}\int_{\{u=t\}}\frac{ds}{|\nabla u|}\quad\text{and}\quad\beta'(t)=-\int_{\{u=t\}}\frac{ds}{|\nabla u|}.$$

In particular,

$$\alpha'(t)=e^{2t}\beta'(t).$$

On the other hand, by the differential inequality we have

$$\int_{\{u>t\}}(-\Delta u)d\mu+\beta(t)\leq\lambda\alpha(t),$$

hence

$$\int_{\{u=t\}}|\nabla u|ds\leq\lambda\alpha-\beta.$$

Multiplying both sides by $-\alpha'(t)$ we get

$$e^{2t}\int_{\{u=t\}}\frac{ds}{|\nabla u|}\int_{\{u=t\}}|\nabla u|ds \le -\lambda\alpha\alpha' + e^{2t}\beta\beta',$$

which implies

$$e^{2t}s(\{u=t\})^2 \le -\lambda\alpha\alpha' + e^{2t}\beta\beta'.$$

Applying the isoperimetric inequality on $(M,g)$ (see [1] and [19, lemma 4.2]) we get

$$e^{2t}(4\pi\beta-\beta^2) \le -\lambda\alpha\alpha' + e^{2t}\beta\beta'.$$

Consequently

$$4\pi(e^{2t})'\beta-(e^{2t})'\beta^2 \le -\lambda(\alpha^2)' + e^{2t}(\beta^2)'.$$

In other words,

$$4\pi\left[(e^{2t}\beta)'-\alpha'\right] \le -\lambda(\alpha^2)' + (e^{2t}\beta^2)',$$

and hence

$$4\pi(\alpha-e^{2t}\beta)-\lambda\alpha^2+e^{2t}\beta^2 \text{ is increasing.}$$

Thus

$$4\pi(\alpha(0)-\beta(0))-\lambda\alpha(0)^2+\beta(0)^2 \le 0.$$

In other words,

$$4\pi\int_\Omega e^{2u}\,d\mu-\lambda\left(\int_\Omega e^{2u}\,d\mu\right)^2 \le 4\pi\mu(\Omega)-\mu^2(\Omega).$$

When $0<\lambda\le 1$, we have

$$4\pi\int_\Omega e^{2u}\,d\mu-\lambda\left(\int_\Omega e^{2u}\,d\mu\right)^2 \le 4\pi\mu(\Omega)-\lambda\mu^2(\Omega)$$

and

$$4\pi\left(\int_\Omega e^{2u}\,d\mu-\mu(\Omega)\right) \le \lambda\left(\int_\Omega e^{2u}\,d\mu+\mu(\Omega)\right)\left(\int_\Omega e^{2u}\,d\mu-\mu(\Omega)\right).$$

Hence

$$\int_\Omega e^{2u}\,d\mu+\mu(\Omega) \ge \frac{4\pi}{\lambda}. \qquad \square$$

We will derive Theorem 1.5 by flipping all the inequalities.

PROOF OF THEOREM 1.5. Again we can assume $u$ is a Morse function. For $t<0$, let

$$\alpha(t)=\int_{\{u<t\}}e^{2u}\,d\mu, \quad \beta(t)=\int_{\{u<t\}}d\mu.$$

Then

$$\alpha'(t)=e^{2t}\int_{\{u=t\}}\frac{ds}{|\nabla u|}, \quad \beta'(t)=\int_{\{u=t\}}\frac{ds}{|\nabla u|}, \quad \alpha'(t)=e^{2t}\beta'(t).$$

On the other hand, since

$$\Delta u - 1 \le -\lambda e^{2u},$$

integrating on $\{u < t\}$ we get

$$\int_{\{u=t\}} |\nabla u| ds - \beta(t) \le -\lambda\alpha(t).$$

We have

$$\begin{aligned} e^{2t}\left(4\pi\beta - \beta^2\right) \le e^{2t} s^2(\{u = t\}) &\le e^{2t} \int_{\{u=t\}} \frac{ds}{|\nabla u|} \int_{\{u=t\}} |\nabla u| ds \\ &\le e^{2t}\beta\beta' - \lambda\alpha\alpha'. \end{aligned}$$

Hence

$$e^{2t}(\beta^2)' - \lambda(\alpha^2)' \ge 4\pi(e^{2t})'\beta - (e^{2t})'\beta^2.$$

It follows that

$$(e^{2t}\beta^2)' - \lambda(\alpha^2)' \ge 4\pi[(e^{2t}\beta)' - \alpha'],$$

and

$$4\pi(\alpha - e^{2t}\beta) + e^{2t}\beta^2 - \lambda\alpha^2 \text{ is increasing.}$$

In particular,

$$4\pi(\alpha(0) - \beta(0)) + \beta(0)^2 - \lambda\alpha(0)^2 \ge 0.$$

In other words,

$$4\pi \int_\Omega e^{2u}\, d\mu - \lambda\left(\int_\Omega e^{2u}\, d\mu\right)^2 \ge 4\pi\mu(\Omega) - \mu^2(\Omega).$$

When $\lambda = 1$, we have

$$4\pi\left(\mu(\Omega) - \int_\Omega e^{2u}\, d\mu\right) \le \left(\mu(\Omega) + \int_\Omega e^{2u}\, d\mu\right)\left(\mu(\Omega) - \int_\Omega e^{2u}\, d\mu\right).$$

Hence

$$\int_\Omega e^{2u}\, d\mu + \mu(\Omega) \ge 4\pi.$$ □

Theorem 1.2 easily follows from Theorem 1.4.

PROOF OF THEOREM 1.2. Let $g = e^{2u_1}|dx|^2$, then

$$K = -e^{-2u_1}\Delta u_1 \le 1$$

and $\mu(\Omega_0) \le 4\pi$. Let $u = u_2 - u_1$. We have

$$-\Delta u \le e^{2u_1}(\lambda e^{2u} - 1).$$

Hence

$$-\Delta_g u \le \lambda e^{2u} - 1.$$

Note that $u > 0$ in $\Omega$ and $u|_{\partial\Omega} = 0$. Thus by Theorem 1.4 we have

$$\int_\Omega e^{2u}\, d\mu + \mu(\Omega) \ge \frac{4\pi}{\lambda}.$$

In other words,

$$\int_\Omega e^{2u_1}\,dx + \int_\Omega e^{2u_2}\,dx \geq \frac{4\pi}{\lambda}. \qquad \square$$

By exactly the same argument as above, Theorem 1.3 follows from Theorem 1.5.

*Example* 2.1. Fix $0 < r < 1$. We take the stereographic projection of the unit sphere $S^2$ with respect to the north pole to plane

$$x_3 = -\sqrt{1-r^2} = -h_1;$$

then the standard metric on $S^2$ is written as

$$g_1 = \frac{4(1+h_1)^2}{\left(|x|^2 + (1+h_1)^2\right)^2}|dx|^2 = e^{2u_1}|dx|^2.$$

For $R > 1$, we do stereographic projection of $R \cdot S^2$ with respect to the north pole to the plane

$$x_3 = \sqrt{R^2 - r^2} = h_2;$$

then the metric on $R \cdot S^2$

$$g_2 = \frac{4R^2(R-h_2)^2}{\left(|x|^2 + (R-h_2)^2\right)^2}|dx|^2 = e^{2u_2}|dx|^2.$$

Note that for $|x| < r$, $u_2(x) > u_1(x)$,

$$\Delta u_1 + e^{2u_1} = 0, \quad \Delta u_2 + R^{-2}e^{2u_2} = 0,$$

$$\int_{B_r} e^{2u_1}dx = 2\pi(1-h_1), \quad \int_{B_r} e^{2u_2}dx = 2\pi R(R+h_2).$$

We have

$$\int_{B_r} e^{2u_1}\,dx + \int_{B_r} e^{2u_2}\,dx > 4\pi R^2.$$

This is an example for Theorem 1.2 with $\lambda = R^{-2}$.

*Example* 2.2. For $0 < r < R < 1$, we take the stereographic projection of $S^2$ with respect to the north pole to the plane

$$x_3 = \sqrt{1-r^2} = h_1;$$

then the metric on $S^2$ is written as

$$g_1 = \frac{4(1-h_1)^2}{\left(|x|^2 + (1-h_1)^2\right)^2}|dx|^2 = e^{2u_1}|dx|^2.$$

We also do stereographic projection of $R \cdot S^2$ with respect to the north pole to the plane

$$x_3 = -\sqrt{R^2 - r^2} = -h_2,$$

then the metric on $R \cdot S^2$ is written as

$$g_2 = \frac{4R^2(R+h_2)^2}{\left(|x|^2 + (R+h_2)^2\right)^2}|dx|^2 = e^{2u_2}|dx|^2.$$

Note that for $|x| < r$, $u_2(x) < u_1(x)$,

$$\Delta u_1 + e^{2u_1} = 0, \quad \Delta u_2 + e^{2u_2} = -\left(R^{-2} - 1\right)e^{2u_2},$$

$$\int_{B_r} e^{2u_1}\, dx = 2\pi(1+h_1), \quad \int_{B_r} e^{2u_2}\, dx = 2\pi R(R-h_2) > 2\pi(1-h_1).$$

Hence

$$\int_{B_r} e^{2u_1}\, dx + \int_{B_r} e^{2u_2}\, dx > 4\pi.$$

This is an example of Theorem 1.3.

*Example* 2.3. Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, and $u$ a smooth function on $\overline{\Omega}$ such that

$$-\Delta u + 1 \le e^{2u} \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0 \text{ in } \Omega.$$

It follows from Theorem 1.4 that

$$\int_\Omega e^{2u}\, dx + |\Omega| \ge 4\pi. \tag{2.1}$$

Because of the usual isoperimetric inequality on $\mathbb{R}^2$, the assumption $\mu(M) \le 4\pi$ in Theorem 1.4 is not needed in our situation. Here we will give an example where $\int_\Omega e^{2u}\, dx + |\Omega|$ is arbitrarily close to $4\pi$.

For $0 < h < 1$, denote $r = \sqrt{1-h^2}$. Take the stereographic projection of $S^2$ with respect to the north pole to the plane $x_3 = -h$; then the metric on $S^2$ is written as

$$g_1 = \frac{4(1+h)^2}{\left[|x|^2 + (1+h)^2\right]^2}|dx|^2 = e^{2u_1}|dx|^2.$$

We have

$$-\Delta u_1 = e^{2u_1} \text{ in } B_r, \quad u_1|_{\partial B_r} = 0, \quad 1 \le e^{2u_1} \le \frac{4}{(1+h)^2}.$$

Let

$$R = \frac{2}{1+h} \quad \text{and} \quad h_2 = \sqrt{R^2 - r^2}.$$

Take the stereographic projection of $R \cdot S^2$ with respect to the north pole to the plane $x_3 = h_2$; then the metric on $R \cdot S^2$ is written as

$$g_2 = \frac{4R^2(R-h_2)^2}{\left[|x|^2 + (R-h_2)^2\right]^2}|dx|^2 = e^{2u_2}|dx|^2.$$

We have

$$-\Delta u_2 = \frac{(1+h)^2}{4} e^{2u_2} \ \text{ in } B_r, \quad u_2|_{\partial B_r} = 0, \quad u_2 > u_1 \ \text{ in } B_r.$$

Let $u = u_2 - u_1$; then $u > 0$ in $B_r$ and $u|_{\partial B_r} = 0$. Moreover,

$$-\Delta u = \frac{(1+h)^2}{4} e^{2u_2} - e^{2u_1} = \frac{(1+h)^2}{4} e^{2u_1} e^{2u} - e^{2u_1}.$$

It follows that

$$-\Delta u + 1 \le -\Delta u + e^{2u_1} = \frac{(1+h)^2}{4} e^{2u_1} e^{2u} \le e^{2u}.$$

On the other hand,

$$\int_{B_r} e^{2u}\, dx + |B_r| = \int_{B_r} e^{2u_2 - 2u_1}\, dx + |B_r| \to 4\pi$$

as $h \uparrow 1^-$.

The above example shows that one cannot get any improvements to (1.11) by assuming $K \le a < 1$. Indeed, $a = 0$ in the example above.

## 3 Differential Equation on Surface Satisfying General Isoperimetric Inequalities

In this section we present sphere covering type inequalities on surfaces satisfying general isoperimetric inequalities (see (3.1) below) and discuss their applications to elliptic equations with exponential nonlinearities. We find the following definition particularly useful.

DEFINITION 3.1. Let $M$ be a smooth surface and $g$ be a metric on $M$. If for some $0 < \theta \le 1$ and $\kappa \in \mathbb{R}$ we have

$$4\pi\theta\mu(E) - \kappa\mu^2(E) \le s^2(\partial E) \tag{3.1}$$

for any compact smooth domain $E \subset M$ (here $s$ is the one-dimensional measure associated with $g$ and $\mu$ is the two-dimensional measure), then we say $(M, g)$ satisfies the $(\theta, \kappa)$-isoperimetric inequality.

THEOREM 3.2. *Let $(M, g)$ be a smooth Riemann surface satisfying the $(\theta, \kappa)$-isoperimetric inequality for some $\theta \in (0, 1]$ and $\kappa \in \mathbb{R}$. If $\Omega \subset M$ is an open domain with compact closure and $u \in C^\infty(\overline{\Omega})$ such that*

$$-\Delta_g u + \kappa = \lambda e^{2u} + f, \quad u|_{\partial\Omega} = 0, \quad u > 0 \ \text{ in } \Omega. \tag{3.2}$$

*Denote*

$$\Theta = \frac{1}{2\pi} \int_\Omega f^+\, d\mu; \tag{3.3}$$

*then*

$$4\pi(\theta-\Theta)\int_\Omega e^{2u}\,d\mu-\lambda\left(\int_\Omega e^{2u}\,d\mu\right)^2\leq 4\pi\theta\mu(\Omega)-\kappa\mu^2(\Omega). \tag{3.4}$$

*In particular, if* $\Theta=0$ *and* $0<\lambda\leq\kappa$*, then*

$$\int_\Omega e^{2u}\,d\mu+\mu(\Omega)\geq\frac{4\pi\theta}{\lambda}. \tag{3.5}$$

*Remark* 3.3. It is worth pointing out that as long as the $(\theta,\kappa)$-isoperimetric inequality is valid, the smoothness of $u$ and metric $g$ is not essential to our argument. In particular, $f$ can be replaced by a signed measure. This is useful in some singular Liouville-type equations. We will not elaborate this point further but refer the reader to [2, 3, 5] and the references therein.

PROOF. By approximation we can assume $u$ is a Morse function. For $t>0$, let

$$\alpha(t)=\int_{\{u>t\}}e^{2u}\,d\mu,\quad \beta(t)=\int_{\{u>t\}}d\mu.$$

As in the proof of Theorem 1.4, we have

$$\alpha'(t)=-e^{2t}\int_{\{u=t\}}\frac{ds}{|\nabla u|},\quad \beta'(t)=-\int_{\{u=t\}}\frac{ds}{|\nabla u|}.$$

Hence

$$\alpha'(t)=e^{2t}\beta'(t).$$

On the other hand,

$$\int_{\{u>t\}}(-\Delta u)d\mu+\kappa\beta(t)=\lambda\alpha(t)+\int_{\{u>t\}}f\,d\mu,$$

hence

$$\int_{\{u=t\}}|\nabla u|ds\leq\lambda\alpha-\kappa\beta+2\pi\Theta.$$

We have

$$e^{2t}\int_{\{u=t\}}\frac{ds}{|\nabla u|}\int_{\{u=t\}}|\nabla u|ds\leq-\lambda\alpha\alpha'+\kappa e^{2t}\beta\beta'-2\pi\Theta\alpha',$$

which implies

$$e^{2t}s^2(\{u=t\})\leq-\lambda\alpha\alpha'+\kappa e^{2t}\beta\beta'-2\pi\Theta\alpha'.$$

Using (3.1) we get

$$e^{2t}(4\pi\theta\beta-\kappa\beta^2)\leq-\lambda\alpha\alpha'+\kappa e^{2t}\beta\beta'-2\pi\Theta\alpha'.$$

It follows that

$$4\pi\theta[(e^{2t}\beta)'-\alpha']\leq-\lambda(\alpha^2)'+\kappa(e^{2t}\beta^2)'-4\pi\Theta\alpha'.$$

Integrating for $t$ from 0 to $\infty$, we get

$$4\pi\theta(\alpha(0)-\beta(0))\leq\lambda\alpha(0)^2-\kappa\beta(0)^2+4\pi\Theta\alpha(0).$$

In other words,

$$4\pi(\theta-\Theta)\int_\Omega e^{2u}\,d\mu-\lambda\left(\int_\Omega e^{2u}\,d\mu\right)^2\leq 4\pi\theta\mu^2(\Omega)-\kappa\mu(\Omega). \qquad \square$$

If we flip the inequalities as in the proof of Theorem 1.5, we get the following:

THEOREM 3.4. *Let $(M,g)$ be a smooth Riemann surface satisfying the $(\theta,\kappa)$-isoperimetric inequality for some $\theta\in(0,1]$ and $\kappa\in\mathbb{R}$. Assume $\Omega\subset M$ is an open domain with compact closure and $u\in C^\infty(\bar{\Omega})$ such that*

$$-\Delta_g u+\kappa=\lambda e^{2u}+f,\quad u|_{\partial\Omega}=0,\quad u<0\ \text{ in }\Omega. \tag{3.6}$$

*Denote*

$$\Theta=\frac{1}{2\pi}\int_\Omega f^-\,d\mu; \tag{3.7}$$

*then*

$$4\pi\theta\mu(\Omega)-\kappa\mu^2(\Omega)\leq 4\pi(\theta+\Theta)\int_\Omega e^{2u}\,d\mu-\lambda\left(\int_\Omega e^{2u}\,d\mu\right)^2. \tag{3.8}$$

*In particular, if $\Theta=0$ and $\lambda=\kappa>0$, then*

$$\int_\Omega e^{2u}\,d\mu+\mu(\Omega)\geq\frac{4\pi\theta}{\lambda}. \tag{3.9}$$

Next we discuss some known and new applications of Theorems 3.2 and 3.4.

*Example* 3.5 ([1,40]). Let $(M,\widetilde{g})$ be a simply connected, smooth Riemann surface with curvature $\widetilde{K}\leq 1$. If $E$ is a compact simply connected domain in $M$ with nonempty smooth boundary, then we can find $u\in C^\infty(E)$ such that

$$-\Delta_{\widetilde{g}}u=\widetilde{K}\text{ on }E,\quad u|_{\partial E}=0.$$

Let $g=e^{-2u}\widetilde{g}$; then the curvature of $g$ is 0. By the Riemann mapping theorem and the Taylor series argument for holomorphic functions in [17], $(E,g)$ satisfies the $(1,0)$-isoperimetric inequality. On the other hand,

$$-\Delta_g u\leq e^{2u}\text{ on }E,\quad u|_{\partial E}=0.$$

If we let

$$\Omega=\{p\in E:u(p)>0\},$$

then

$$-\Delta_g u\leq e^{2u}\ \text{ on }\Omega,\quad u|_{\partial\Omega}=0,\quad u>0\ \text{ in }\Omega.$$

Theorem 3.2 tells us

$$4\pi\int_\Omega e^{2u}\,d\mu-\left(\int_\Omega e^{2u}\,d\mu\right)^2\leq 4\pi\mu(\Omega).$$

Hence

$$4\pi\left(\int_E e^{2u}\,d\mu-\mu(E)\right)\le 4\pi\left(\int_\Omega e^{2u}\,d\mu-\mu(\Omega)\right)\le\left(\int_\Omega e^{2u}\,d\mu\right)^2 \le\left(\int_E e^{2u}\,d\mu\right)^2,$$

i.e.,

$$4\pi\widetilde{\mu}(E)-\widetilde{\mu}^2(E)\le 4\pi\mu(E)\le s^2(\partial E)=\widetilde{s}^2(\partial E).$$

This is exactly the argument given in [1,40].

If we assume further that $\widetilde{\mu}(M)\le 4\pi$, then following [19, lemma 4.2] we know, for $E$ to be a compact domain with boundary smooth but not necessarily simply connected, there still holds

$$4\pi\widetilde{\mu}(E)-\widetilde{\mu}^2(E)\le\widetilde{s}^2(\partial E).$$

In other words, $(1,1)$-isoperimetric inequality is true for $(M,\widetilde{g})$. As a consequence, Theorem 1.4 follows from Theorem 3.2.

*Example* 3.6. Let $(M,\widetilde{g})$ be a simply connected smooth Riemann surface with curvature $\widetilde{K}$. Assume $a\ge 0$ and

$$\Theta=\frac{1}{2\pi}\int_M(\widetilde{K}-a)^+\,d\widetilde{\mu}<1. \tag{3.10}$$

Then for any compact, simply connected domain $E$ in $M$ with nonempty smooth boundary, we have

$$4\pi(1-\Theta)\widetilde{\mu}(E)-a\widetilde{\mu}^2(E)\le\widetilde{s}^2(\partial E). \tag{3.11}$$

In particular, if we assume further that $\widetilde{\mu}(M)\le\frac{4\pi(1-\Theta)}{a}$, then $(M,\widetilde{g})$ satisfies the $(1-\Theta,a)$-isoperimetric inequality. In fact, this is even true when $\widetilde{g}$ is singular; see [2,3,5] and the references therein.

Indeed, as in the previous example, we can find $u\in C^\infty(E)$ such that

$$-\widetilde{\Delta}u=\widetilde{K}\ \text{ on } E,\quad u|_{\partial E}=0.$$

Let $g=e^{-2u}\widetilde{g}$; then $(E,g)$ satisfies the $(1,0)$-isoperimetric inequality and

$$-\Delta_g u=ae^{2u}+(\widetilde{K}-a)e^{2u}\ \text{ on } E.$$

Note that

$$\frac{1}{2\pi}\int_E\left[(\widetilde{K}-a)e^{2u}\right]^+\,d\mu=\frac{1}{2\pi}\int_E(\widetilde{K}-a)^+\,d\widetilde{\mu}\le\Theta<1.$$

Let

$$\Omega=\{p\in E: u(p)>0\};$$

then

$$-\Delta_g u=ae^{2u}+(\widetilde{K}-a)e^{2u}\ \text{ on }\Omega,\quad u|_{\partial\Omega}=0,\quad u>0\ \text{ in }\Omega.$$

Theorem 3.2 implies

$$4\pi(1-\Theta)\int_\Omega e^{2u}\,d\mu - a\left(\int_\Omega e^{2u}\,d\mu\right)^2 \le 4\pi\mu(\Omega).$$

Hence

$$\begin{aligned}4\pi(1-\Theta)\int_E e^{2u}\,d\mu - 4\pi\mu(E) &\le 4\pi(1-\Theta)\int_\Omega e^{2u}\,d\mu - 4\pi\mu(\Omega)\\ &\le a\left(\int_\Omega e^{2u}\,d\mu\right)^2 \le a\left(\int_E e^{2u}\,d\mu\right)^2.\end{aligned}$$

In other words,

$$4\pi(1-\Theta)\widetilde{\mu}(E) - a\widetilde{\mu}^2(E) \le 4\pi\mu(E) \le s^2(\partial E) = \widetilde{s}^2(\partial E).$$

*Example* 3.7 ([1–3, 5]). Let $\Omega_0 \subset \mathbb{R}^2$ be a simply connected domain and $u, h \in C^\infty(\Omega_0)$ with $h(x) > 0$ for any $x \in \Omega_0$. We write

$$-\Delta u = he^{2u} + f \tag{3.12}$$

and

$$\Theta = \frac{1}{2\pi}\int_{\Omega_0}\left(f - \frac{1}{2}\Delta\log h\right)^+ dx. \tag{3.13}$$

If $\Theta < 1$ and

$$\int_{\Omega_0} he^{2u}\,dx \le 4\pi(1-\Theta), \tag{3.14}$$

then $(\Omega_0, he^{2u}|dx|^2)$ satisfies the $(1-\Theta, 1)$-isoperimetric inequality. As pointed out earlier in Remark 3.3, the regularity assumption of $u$ and $h$ can be weakened, and we refer the reader to [2, 3, 5].

PROOF. For convenience we denote $g = he^{2u}|dx|^2$; then its curvature

$$\begin{aligned}K = h^{-1}e^{-2u}\left(-\Delta u - \frac{1}{2}\Delta\log h\right) &= h^{-1}e^{-2u}\left(he^{2u} + f - \frac{1}{2}\Delta\log h\right)\\ &= 1 + h^{-1}e^{-2u}\left(f - \frac{1}{2}\Delta\log h\right).\end{aligned}$$

Hence

$$\frac{1}{2\pi}\int_{\Omega_0}(K-1)^+\,d\mu = \frac{1}{2\pi}\int_{\Omega_0}\left(f - \frac{1}{2}\Delta\log h\right)^+ dx = \Theta < 1$$

and

$$\mu(\Omega_0) = \int_{\Omega_0} he^{2u}\,dx \le 4\pi(1-\Theta).$$

It follows from Example 3.6 that $(\Omega_0, g)$ satisfies the $(1-\Theta, 1)$-isoperimetric inequality □

If we replace the reference metric from Euclidean metric to an arbitrary one, we end up with the following formulation.

LEMMA 3.8. *Let $(M, g)$ be a simply connected Riemann surface with curvature $K$ and $u \in C^{\infty}(M)$. We write*

$$-\Delta_g u = e^{2u} + f \tag{3.15}$$

*and*

$$\Theta = \frac{1}{2\pi} \int_M (f + K)^+ \, d\mu. \tag{3.16}$$

*If $\Theta < 1$ and*

$$\int_M e^{2u} \, d\mu \leq 4\pi(1 - \Theta), \tag{3.17}$$

*then $(M, e^{2u} g)$ satisfies the $(1 - \Theta, 1)$-isoperimetric inequality.*

PROOF. Let $\widetilde{g} = e^{2u} g$; then

$$\widetilde{K} = e^{-2u}(K - \Delta u) = 1 + e^{-2u}(f + K).$$

In particular,

$$\frac{1}{2\pi} \int_M (\widetilde{K} - 1)^+ \, d\widetilde{\mu} = \frac{1}{2\pi} \int_M (f + K)^+ \, d\mu = \Theta < 1$$

and

$$\widetilde{\mu}(M) = \int_M e^{2u} \, d\mu \leq 4\pi(1 - \Theta).$$

It follows from Example 3.6 that $(M, \widetilde{g})$ satisfies the $(1 - \Theta, 1)$-isoperimetric inequality. □

Note that Lemma 3.8 also follows from Example 3.7 and the Riemann mapping theorem. With Lemma 3.8 at hand, we can deduce easily a variation of the sphere covering inequality.

PROPOSITION 3.9. *Let $(M, g)$ be a simply connected Riemann surface and $u_1 \in C^{\infty}(M)$. We write*

$$-\Delta_g u_1 = e^{2u_1} + f \tag{3.18}$$

*and*

$$\Theta = \frac{1}{2\pi} \int_M (f + K)^+ \, d\mu. \tag{3.19}$$

*Here $K$ is the curvature of $(M, g)$. Assume $\Theta < 1$ and*

$$\int_M e^{2u_1} \, d\mu \leq 4\pi(1 - \Theta). \tag{3.20}$$

*Let* $\Omega \subset M$ *be a domain with a compact closure and nonempty boundary. Assume* $u_2 \in C^\infty(\bar{\Omega})$ *and* $0 < \lambda \leq 1$ *such that*

$$\Delta_g u_2 + \lambda e^{2u_2} \geq \Delta_g u_1 + e^{2u_1} \ \ in\ \Omega, \quad u_2 > u_1 \ \ in\ \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

*Then*

$$\int_\Omega e^{2u_1}\, d\mu + \int_\Omega e^{2u_2}\, d\mu \geq \frac{4\pi(1-\Theta)}{\lambda}. \tag{3.21}$$

Note that Theorem 1.2 is a special case of Proposition 3.9.

PROOF OF PROPOSITION 3.9. Let $\tilde{g} = e^{2u_1} g$; then it follows from Lemma 3.8 that $(M, \tilde{g})$ satisfies the $(1-\Theta, 1)$-isoperimetric inequality. Let $u = u_2 - u_1$; then on $\Omega$ we have

$$-\Delta_g u \leq e^{2u_1}(\lambda e^{2u} - 1).$$

Hence

$$-\Delta_{\tilde{g}} u \leq \lambda e^{2u} - 1.$$

Moreover, $u > 0$ in $\Omega$ and $u|_{\partial\Omega} = 0$; it follows from Theorem 3.2 that

$$\int_\Omega e^{2u}\, d\tilde{\mu} + \tilde{\mu}(\Omega) \geq \frac{4\pi(1-\Theta)}{\lambda}.$$

In other words,

$$\int_\Omega e^{2u_1}\, d\mu + \int_\Omega e^{2u_2}\, d\mu \geq \frac{4\pi(1-\Theta)}{\lambda}.$$

□

Using Theorem 3.4 with the same proof, we also have a dual inequality generalizing Theorem 1.3.

PROPOSITION 3.10. *Let* $(M, g)$ *be a simply connected Riemann surface with curvature* $K$ *and* $u_1 \in C^\infty(M)$. *We write*

$$-\Delta_g u_1 = e^{2u_1} + f \tag{3.22}$$

*and*

$$\Theta = \frac{1}{2\pi} \int_M (f + K)^+\, d\mu. \tag{3.23}$$

*Assume* $\Theta < 1$ *and*

$$\int_M e^{2u_1}\, d\mu \leq 4\pi(1-\Theta). \tag{3.24}$$

*Let* $\Omega \subset M$ *be a domain with compact closure and nonempty boundary. Assume* $u_2 \in C^\infty(\bar{\Omega})$ *such that*

$$\Delta_g u_2 + e^{2u_2} \leq \Delta_g u_1 + e^{2u_1} \ \ in\ \Omega, \quad u_2 < u_1 \ \ in\ \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

*Then*

$$\int_\Omega e^{2u_1}\, d\mu + \int_\Omega e^{2u_2}\, d\mu \geq 4\pi(1-\Theta). \tag{3.25}$$

*Example* 3.11. Let $(M, g)$ be a simply connected Riemann surface with curvature $K$ and $u_1, h \in C^\infty(M)$ with $h > 0$. We write

$$-\Delta_g u_1 = he^{2u_1} + f \tag{3.26}$$

and

$$\Theta = \frac{1}{2\pi}\int_M \left(f + K - \frac{1}{2}\Delta_g \log h\right)^+ d\mu. \tag{3.27}$$

Assume $\Theta < 1$ and

$$\int_M he^{2u_1}\, d\mu \le 4\pi(1-\Theta). \tag{3.28}$$

Let $\Omega \subset M$ be a domain with compact closure and nonempty boundary. Assume $u_2 \in C^\infty(\bar{\Omega})$ and $0 < \lambda \le 1$ such that

$$\Delta_g u_2 + \lambda h e^{2u_2} \ge \Delta_g u_1 + he^{2u_1} \text{ in } \Omega, \quad u_2 > u_1 \text{ in } \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

Then

$$\int_\Omega he^{2u_1}\, d\mu + \int_\Omega he^{2u_2}\, d\mu \ge \frac{4\pi(1-\Theta)}{\lambda}. \tag{3.29}$$

PROOF. Let

$$v_1 = u_1 + \frac{1}{2}\log h, \quad v_2 = u_2 + \frac{1}{2}\log h;$$

then

$$-\Delta_g v_1 = e^{2v_1} + f - \frac{1}{2}\Delta \log h. \tag{3.30}$$

Moreover,

$$\Delta_g v_2 + \lambda e^{2v_2} \ge \Delta_g v_1 + e^{2v_1} \text{ in } \Omega, \quad v_2 > v_1 \text{ in } \Omega, \quad v_2|_{\partial\Omega} = v_1|_{\partial\Omega}.$$

Then we can apply Proposition 3.9 to get the desired conclusion. □

By a straightforward modification we can also deal with the case where $h$ changes sign.

*Example* 3.12. Let $(M, g)$ be a simply connected Riemann surface with curvature $K$ and $u_1, h, H \in C^\infty(M)$ with $h \le H$ and $H > 0$. We write

$$-\Delta_g u_1 = He^{2u_1} + f \tag{3.31}$$

and

$$\Theta = \frac{1}{2\pi}\int_M \left(f + K - \frac{1}{2}\Delta_g \log H\right)^+ d\mu. \tag{3.32}$$

Assume $\Theta < 1$ and

$$\int_M He^{2u_1}\, d\mu \le 4\pi(1-\Theta). \tag{3.33}$$

Let $\Omega \subset M$ be a domain with compact closure and nonempty boundary. Assume $u_2 \in C^\infty(\bar{\Omega})$ such that

$$\Delta_g u_2 + he^{2u_2} \geq \Delta_g u_1 + he^{2u_1} \text{ in } \Omega, \quad u_2 > u_1 \text{ in } \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

Then

$$\int_\Omega He^{2u_1}\,d\mu + \int_\Omega He^{2u_2}\,d\mu \geq 4\pi(1-\Theta). \tag{3.34}$$

PROOF. We have

$$\Delta_g u_2 \geq \Delta_g u_1 - h(e^{2u_2} - e^{2u_1}) \geq \Delta_g u_1 - H(e^{2u_2} - e^{2u_1}).$$

Hence

$$\Delta_g u_2 + He^{2u_2} \geq \Delta_g u_1 + He^{2u_1} \text{ in } \Omega.$$

Then we can apply Example 3.11. □

Next we turn to solutions of semilinear equations with equal weights; see [5,29].

PROPOSITION 3.13. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open simply connected domain. Assume $u_1, u_2 \in C^\infty(\bar{\Omega})$ such that*

$$\Delta u_1 + e^{2u_1} \geq 0 \tag{3.35}$$

*and*

$$\Delta u_1 + e^{2u_1} \geq \Delta u_2 + e^{2u_2} \text{ in } \Omega, \quad u_1 + c > u_2 \text{ in } \Omega, \quad u_1|_{\partial\Omega} + c = u_2|_{\partial\Omega}.$$

*Here $c$ is a constant. If*

$$\int_\Omega e^{2u_1}\,dx = \int_\Omega e^{2u_2}\,dx = \rho, \tag{3.36}$$

*then $\rho \geq 4\pi$.*

PROOF. Note that $c > 0$. If $\rho \leq 4\pi$, we will show $\rho = 4\pi$. Indeed, let $g = e^{2u_1}|dx|^2$; then $K \leq 1$ and $\mu(\Omega) = \rho \leq 4\pi$. If we write $u = u_2 - u_1 - c$, then

$$-\Delta_g u + 1 \geq e^{2c} \cdot e^{2u}.$$

Moreover, $u < 0$ in $\Omega$ and $u|_{\partial\Omega} = 0$. It follows from Theorem 1.5 that

$$4\pi \int_\Omega e^{2u}\,d\mu - e^{2c}\left(\int_\Omega e^{2u}\,d\mu\right)^2 \geq 4\pi\mu(\Omega) - \mu^2(\Omega).$$

Hence

$$0 \geq (1 - e^{-2c})\rho(4\pi - \rho),$$

and we get $\rho \geq 4\pi$. □

PROPOSITION 3.14. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, simply connected domain. Assume $u_1, u_2 \in C^\infty(\bar{\Omega})$ such that*

$$\Delta u_1 + e^{2u_1} = \Delta u_2 + e^{2u_2} \geq 0 \ \ in\ \Omega, \quad u_1|_{\partial\Omega} + c = u_2|_{\partial\Omega}.$$

*Here $c$ is a constant. If $u_1$ is not identically equal to $u_2$ and*

$$\int_\Omega e^{2u_1}\,dx = \int_\Omega e^{2u_2}\,dx = \rho, \tag{3.37}$$

*then $\rho \geq 4\pi$.*

PROOF. If $\rho \leq 4\pi$, we will show $\rho = 4\pi$. Indeed, let $g = e^{2u_1}|dx|^2$; then $K \leq 1$ and $\mu(\Omega) = \rho \leq 4\pi$. If we write $u = u_2 - u_1 - c$, then

$$-\Delta_g u + 1 = e^{2c} \cdot e^{2u}. \tag{3.38}$$

Let

$$\Omega^+ = \{x \in \Omega : u(x) > 0\}, \quad \Omega^- = \{x \in \Omega : u(x) < 0\};$$

then it follows from the unique continuation property that $|\Omega \setminus (\Omega^+ \cup \Omega^-)| = 0$. On $\Omega^+$, by Theorem 1.4 we have

$$4\pi \int_{\Omega^+} e^{2u}\,d\mu - e^{2c}\left(\int_{\Omega^+} e^{2u}\,d\mu\right)^2 \leq 4\pi\mu(\Omega^+) - \mu^2(\Omega^+).$$

On $\Omega^-$, by Theorem 1.5 we have

$$4\pi \int_{\Omega^-} e^{2u}\,d\mu - e^{2c}\left(\int_{\Omega^-} e^{2u}\,d\mu\right)^2 \geq 4\pi\mu(\Omega^-) - \mu^2(\Omega^-).$$

Using

$$\mu(\Omega^+) + \mu(\Omega^-) = \rho, \quad \int_{\Omega^+} e^{2u}\,d\mu + \int_{\Omega^-} e^{2u}\,d\mu = e^{-2c}\rho,$$

and subtracting the two inequalities we get

$$(4\pi - \rho)\left(\int_{\Omega^+} e^{2u}\,d\mu - \int_{\Omega^-} e^{2u}\,d\mu\right) \leq (4\pi - \rho)(\mu(\Omega^+) - \mu(\Omega^-)).$$

In other words,

$$(4\pi - \rho)\left(\int_{\Omega^+} e^{2u}\,d\mu - \mu(\Omega^+) + \mu(\Omega^-) - \int_{\Omega^-} e^{2u}\,d\mu\right) \leq 0.$$

Since $u$ is not identically equal to 0, we see

$$\int_{\Omega^+} e^{2u}\,d\mu - \mu(\Omega^+) + \mu(\Omega^-) - \int_{\Omega^-} e^{2u}\,d\mu > 0.$$

Hence $\rho \geq 4\pi$. □

We can replace the Euclidean domain with a Riemann surface.

*Example* 3.15. Let $(M, g)$ be a simply connected, compact Riemann surface with nonempty boundary and $u_1 \in C^\infty(M)$. We write

$$-\Delta_g u_1 = e^{2u_1} + f \tag{3.39}$$

and

$$\Theta = \frac{1}{2\pi}\int_M (f + K)^+ \, d\mu. \tag{3.40}$$

Here $K$ is the curvature of $g$. Assume $u_2 \in C^\infty(M)$ such that

$$\Delta_g u_1 + e^{2u_1} \geq \Delta_g u_2 + e^{2u_2} \ \text{ in } M,$$
$$u_1 + c > u_2 \ \text{ in } M, \quad u_1|_{\partial M} + c = u_2|_{\partial M}.$$

Here $c$ is a constant. If

$$\int_M e^{2u_1}\, d\mu = \int_M e^{2u_2}\, d\mu = \rho, \tag{3.41}$$

then

$$\rho \geq 4\pi(1 - \Theta). \tag{3.42}$$

PROOF. Without loss of generality we can assume $\Theta < 1$ and $\rho \leq 4\pi(1-\Theta)$. Let $\tilde{g} = e^{2u_1} g$, then Lemma 3.8 implies $(M, \tilde{g})$ satisfies $(1 - \Theta, 1)$-isoperimetric inequality. If we write $u = u_2 - u_1 - c$, then

$$-\tilde{\Delta} u + 1 \geq e^{2c} \cdot e^{2u} \ \text{ on } M.$$

Moreover, $u < 0$ in $M$ and $u|_{\partial M} = 0$. It follows from Theorem 3.4 that

$$4\pi(1-\Theta)\int_M e^{2u}\, d\tilde{\mu} - e^{2c}\left(\int_M e^{2u}\, d\tilde{\mu}\right)^2 \geq 4\pi(1-\Theta)\tilde{\mu}(M) - \tilde{\mu}^2(M).$$

Hence

$$0 \geq (1 - e^{-2c})\rho(4\pi(1-\Theta) - \rho).$$

Since $c > 0$, we get $\rho \geq 4\pi(1-\Theta)$. □

Using the argument in Example 3.11 we get the following:

*Example* 3.16. Let $(M, g)$ be a simply connected, compact Riemann surface with nonempty boundary and curvature $K$ and $u_1, h \in C^\infty(M)$ with $h > 0$. We write

$$-\Delta_g u_1 = h e^{2u_1} + f \tag{3.43}$$

and

$$\Theta = \frac{1}{2\pi}\int_M \left(f + K - \frac{1}{2}\Delta_g \log h\right)^+ d\mu. \tag{3.44}$$

Assume $u_2 \in C^\infty(M)$ such that

$$\Delta_g u_1 + h e^{2u_1} \geq \Delta_g u_2 + h e^{2u_2} \ \text{ in } M,$$
$$u_1 + c > u_2 \ \text{ in } M, \quad u_1|_{\partial M} + c = u_2|_{\partial M}.$$

Here $c$ is a constant. If

$$\int_M he^{2u_1}\,d\mu = \int_M he^{2u_2}\,d\mu = \rho, \tag{3.45}$$

then

$$\rho \geq 4\pi(1-\Theta). \tag{3.46}$$

In the same spirit as the proof of Proposition 3.14 but using both Theorem 3.2 and 3.4 instead, we have the following:

*Example* 3.17. Let $(M, g)$ be a simply connected, compact Riemann surface with nonempty boundary and curvature $K$, and $u_1, u_2, h \in C^\infty(M)$ with $h > 0$. Assume

$$-\Delta u_1 - he^{2u_1} = -\Delta u_2 - he^{2u_2} = f \text{ in } M, \tag{3.47}$$

$$u_1|_{\partial M} + c = u_2|_{\partial M}. \tag{3.48}$$

Here $c$ is a constant. We denote

$$\Theta = \frac{1}{2\pi}\int_M \left(f + K - \frac{1}{2}\Delta_g \log h\right)^+ d\mu. \tag{3.49}$$

If $u_1$ is not identically equal to $u_2$ and

$$\int_M he^{2u_1}\,d\mu = \int_M he^{2u_2}\,d\mu = \rho, \tag{3.50}$$

then

$$\rho \geq 4\pi(1-\Theta). \tag{3.51}$$

**Acknowledgments.** C. Gui is partially supported by National Science Foundation grants DMS-1601885 and DMS-1901914 and Simons Foundation grant 617072. A. Moradifam is supported by National Science Foundation grants DMS-1715850 and DMS-1953620.

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Received January 2019.