MEAN FIELD EQUATIONS ON TORI: EXISTENCE AND UNIQUENESS OF EVENLY SYMMETRIC BLOW-UP SOLUTIONS

Daniele Bartolucci

Department of Mathematics, University of Rome *"Tor Vergata"*
Via della ricerca scientifica n.1, 00133 Roma, Italy

Changfeng Gui and Yeyao Hu

Department of Mathematics, The University of Texas at San Antonio
San Antonio, TX 78249, USA

Aleks Jevnikar

Scuola Normale Superiore, Piazza dei Cavalieri 3
56126 Pisa, Italy

Wen Yang

Division of Mathematics, Wuhan Institute of Physics and Mathematics
Chinese Academy of Science, Wuhan, Hubei 430071, China

ABSTRACT. We are concerned with the blow-up analysis of mean field equations. It has been proven in [6] that solutions blowing-up at the same non-degenerate blow-up set are unique. On the other hand, the authors in [18] show that solutions with a degenerate blow-up set are in general non-unique. In this paper we first prove that evenly symmetric solutions on an arbitrary flat torus with a degenerate two-point blow-up set are unique. In the second part of the paper we complete the analysis by proving the existence of such blow-up solutions using a Lyapunov-Schmidt reduction method. Moreover, we deduce that all evenly symmetric blow-up solutions come from one-point blow-up solutions of the mean field equation on a "half" torus.

1. **Introduction.** In this paper, we consider the mean field equation on a flat torus $T := \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, i.e.,

$$\Delta u + \rho\left(\frac{e^u}{\int_T e^u} - \frac{1}{|T|}\right) = 0, \tag{1.1}$$

$$\int_T e^u = 1, \tag{1.2}$$

where ρ is a real parameter, Im $\frac{\omega_2}{\omega_1} > 0$ and $|T|$ denotes the total area of the torus. For convenience, in this paper we always assume that $|T| = 1$.

In the past decade there has been an extensive study of the mean field equation on a general compact Riemann surface M without boundary:

$$\Delta_M u + \rho \left(\frac{he^u}{\int_M he^u} - \frac{1}{|M|} \right) = 0, \tag{1.3}$$

where Δ_M denotes the corresponding Laplace-Beltrami operator on (M, g), $h \in C^\infty(M)$ is a non-negative potential function and $|M|$ is the total area of the surface M. To simplify our notation, we shall always assume $|M| = 1$. Equation (1.3) and its counterpart on bounded planar domains arise in several areas of mathematics and physics and there are by now many results concerning existence ([2, 9, 10, 11, 15, 24, 29, 30, 41, 42]), uniqueness of solutions ([4, 12, 13, 14, 26, 47, 48, 49, 60, 67]) and blow-up analysis ([3, 5, 16, 17, 19, 28, 31, 57, 58]). On one hand, they are derived as a mean field limit in the statistical mechanics description of two dimensional turbulent Euler flows ([20, 21]) and selfgravitating systems ([54, 56, 74]). On the other hand, (1.3) is related to conformal metrics on surfaces with or without conical singularities ([55, 71]) and to gauge field theories ([75]) possibly coupled with Einstein's general relativity ([33, 63, 70]). Recently they have attracted a lot of attention from the analytical point of view due to the close connection to the Chern-Simons-Higgs theory. The relativistic Abelian Chern-Simons gauge field theory was proposed by Jackiw and Weinberg [53] and Hong et al. [52] independently to investigate the physics of high temperature super-conductivity. The energy minimizers of these models satisfy self-dual equations while the Bogomol'nyi-type system of first-order differential equations could be reduced to a single second-order elliptic equation:

$$\Delta u + \frac{1}{\epsilon^2} e^u (1 - e^u) = 4\pi \sum_{i=1}^N \delta_{q_i} \quad \text{in} \quad \mathbb{R}^2, \tag{1.4}$$

where δ_{q_i} denotes the dirac measure at q_i. Equation (1.4) can be considered on flat tori or on the entire $\mathbb{R}^2$. Tarantello in [68] showed that one type of solutions to (1.4) converge to the solution of a mean field equation of type (1.3) after subtracting $2 \log \epsilon$ and a combination of the Green's function at the singular source q_j when the Chern-Simons coupling constant ϵ tends to 0. Latterly, Lin and Yan in [61] proved the local uniqueness of the blow-up solutions to (1.4). More recently that argument has been used by Bartolucci, Lee together with our third and fourth authors in [6] to establish the local uniqueness of the blow-up solutions to (1.3).

To state our main result we need some definitions first. Let $h(x)$ be a non-negative smooth function which vanishes only at a finite number of points and let $\vec{p} = (p_1, \cdots, p_m) \in M^m$ be such that

$$\{p_1, \cdots, p_m\} \cap \{x \in M \mid h(x) = 0\} = \emptyset.$$

We set

$$G_i^*(x) = 8\pi R(x, p_i) + 8\pi \sum_{j \neq i} G(x, p_j), \quad i = 1, \cdots, m, \tag{1.5}$$

where $G(x, y)$ is the Green's function:

$$-\Delta_M G(x, y) = \delta_y - 1 \text{ in } M, \quad \int_M G(x, y) dH^2(y) = 0,$$

and $R(x, y)$ denotes its regular part. We define

$$l(\vec{p}) = \sum_{i=1}^m \left[\Delta_M \log h(p_i) + 8\pi m - K(p_i) \right] h(p_i) e^{G_i^*(p_i)}, \tag{1.6}$$

where $K(x)$ stands for the Gaussian curvature at $x \in M$. Next, we will denote by $V^M(q, r)$ the pre image of the Euclidean ball of radius r, $B(q, r) \subset \mathbb{R}^2$, in a suitably defined isothermal coordinates system. For the case $m \geq 2$ we fix a sufficiently small constant $r_0 \in (0, \frac{1}{2})$ and a family of open sets M_j satisfying $M_l \cap M_i = \emptyset$ if $l \neq i$, $\bigcup_{i=1}^m \overline{M}_j = M$, $V^M(p_i, 2r_0) \subset M_i$, $i = 1, \cdots, m$. Then let us set

$$D(\vec{p}) = \lim_{r \to 0} \sum_{i=1}^m h(p_i) e^{G_i^*(p_i)} \left(\int_{M_i \setminus V^M(p_i, r_i)} e^{\Phi_i(x, \vec{p})} dH^2(x) - \frac{\pi}{r_i^2} \right), \tag{1.7}$$

where $M_1 = M$ if $m = 1$, $r_i = r\sqrt{8h(p_i)e^{G_i^*(p_i)}}$ and

$$\Phi_i(x, \vec{p}) = \sum_{l=1}^m 8\pi G(x, p_l) - G_i^*(p_i) + \log\left(\frac{h(x)}{h(p_i)}\right). \tag{1.8}$$

For $(x_1, \cdots, x_m) \in M \times \cdots \times M$ we define

$$f_m(x_1, \cdots, x_m) = \sum_{i=1}^m \left[\log\left(h(x_i)\right) + 4\pi R(x_i, x_i)\right] + 4\pi \sum_{i \neq j} G(x_i, x_j). \tag{1.9}$$

If a sequence of solutions of (1.3) is not uniformly bounded from above, then it is well known that (see [57]), passing to a subsequence if necessary, it holds,

$$\rho_n \frac{e^{u_n}}{\int_M e^{u_n}} \rightharpoonup 8\pi \sum_{i=1}^m \delta_{p_i}, \quad \rho_n \to 8\pi m, \quad \text{as } n \to +\infty,$$

weakly in the sense of measures in M, for some $m \in \mathbb{N}$. The points $\{p_1, \cdots, p_m\}$ are said to be the blow-up points ([19]). From [28, 62] we know that the blow-up points are critical points of $f_m(x_1, \cdots, x_m)$. Then, Bartolucci et al. [6] proved the following theorem.

Theorem A. ([6]) *Let $u_n^{(1)}$ and $u_n^{(2)}$ be two sequences of solutions to (1.3) with $\rho_n^{(1)} = \rho_n^{(2)} = \rho_n$ and blowing-up at the points p_j, for $j = 1, \cdots, m$, where $\vec{p} = (p_1, \cdots, p_m)$ is a non-degenerate critical point of f_m, i.e.*

$$det(D_M^2 f_m(\vec{p})) \neq 0. \tag{1.10}$$

Assume that either,

(1) $l(\vec{p}) \neq 0$, *or,*

(2) $l(\vec{p}) = 0$ *and* $D(\vec{p}) \neq 0$.

Then there exists an integer constant N_0 sufficiently large such that $u_n^{(1)} = u_n^{(2)}$ for all $n \geq N_0$.

A natural question is whether the assumptions of the latter theorem are necessary or not. It turns out that if we drop the non-degeneracy condition (1.10) the uniqueness property does not hold anymore in general, as the authors in [18] exhibit multiple one-peak solutions blowing-up at a degenerate critical point of f_1 on a bounded domain. On the contrary, we will prove that evenly symmetric solutions on a flat torus with $h \equiv 1$ and with a degenerate two-point blow-up set are unique.

Theorem 1.1. *Let $u_n^{(1)}$ and $u_n^{(2)}$ be two sequences of solutions to (1.1)-(1.2) with $\rho_n^{(1)} = \rho_n^{(2)} = \rho_n$ blowing-up at $p_1 = \vec{0}$ and at $p_2 = \frac{\omega_1}{2}$ or $\frac{\omega_2}{2}$ or $\frac{\omega_1 + \omega_2}{2}$. Assume that $u_n^{(i)}$ is evenly symmetric, i.e. $u_n^{(i)}(z) = u_n^{(i)}(-z)$ for all n and $i = 1, 2$. Then, there exists an N_0 sufficiently large such that $u_n^{(1)} = u_n^{(2)}$ for all $n \geq N_0$.*

We point out that in the latter setting we have $l(\vec{p}) = 32\pi e^{G_1^*(p_1)} \neq 0$, see also the explicit expression of the Green function and its regular part in (2.5) and (2.6), respectively. On the other hand, the blow-up set (p_1, p_2) is a degenerate critical point of f_2 defined in (1.9) due to the translation invariance. We exploit the ideas developed in [6] and [61] and take advantage of the evenly symmetric property to bypass the non-degeneracy assumption and also significantly simplify the highly non-trivial and technical original proofs. More precisely, assuming by contradiction the existence of two distinct blow-up solutions $u_n^{(i)}$ of (1.4) we consider their normalized difference

$$\xi_n = \frac{u_n^{(1)} - u_n^{(2)}}{\|u_n^{(1)} - u_n^{(2)}\|_{L^\infty(M)}}.$$

The starting point in analyzing ξ_n relies on the description of the blow-up solutions carried out by Chen and Lin in [28]. Moreover, we exploit the evenly symmetric property to deduce an estimate on the distance between the local maximum point and the blow-up point. The latter estimate will be crucially used in all the forthcoming arguments. Next, one can show that after a suitable scaling, ξ_n converges to an entire solution $\xi(x)$ of the linearized problem associated to the Liouville equation:

$$\Delta v + e^v = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{1.11}$$

Solutions of (1.11) with finite mass are completely classified by Chen and Li [34] and take the following form:

$$v(z) = v_{\mu,a}(z) = \log\left(\frac{8e^\mu}{(1 + e^\mu |z - a|^2)^2}\right), \quad \mu \in \mathbb{R}, \quad a = (a_1, a_2) \in \mathbb{R}^2. \tag{1.12}$$

Baraket and Pacard in [1] showed that the kernel of the linearized operator at $v_{0,0}$

$$L(\phi) = \Delta\phi + \frac{8}{(1 + |z|^2)^2}\phi \tag{1.13}$$

is spanned by three functions:

$$\varphi_0(z) = \frac{1 - |z|^2}{|z|^2 + 1} = \left.\frac{\partial v_{\mu,a}}{\partial \mu}\right|_{(\mu,a)=(0,0)},$$

$$\varphi_1(z) = \frac{z_1}{|z|^2 + 1} = -\frac{1}{4}\left.\frac{\partial v_{\mu,a}}{\partial a_1}\right|_{(\mu,a)=(0,0)}, \quad \varphi_2(z) = \frac{z_2}{|z|^2 + 1} = -\frac{1}{4}\left.\frac{\partial v_{\mu,a}}{\partial a_2}\right|_{(\mu,a)=(0,0)}.$$

Thus, we have

$$\xi(z) = \sum_{i=0}^{2} b_i \varphi_i(z) \tag{1.14}$$

for some constants $b_i \in \mathbb{R}$. The idea is then to use suitable Pohozaev identities to prove that $b_i = 0$ for each i. In particular, the evenly symmetric property is crucially used to guarantee that the elements of the kernel corresponding to the translation invariance vanishes. Finally, after showing that $\xi \not\equiv 0$ one gets a contradiction and thus necessarily $u_n^{(1)} \equiv u_n^{(2)}$.

Let us conclude this part by giving some comments on the recent study of the local uniqueness property. It turns out that one can also derive such property for the spike solution of Schrödinger equation. In [72], Wei showed that the single interior spike solution of a singularly perturbed semilinear Neumann problem is locally unique at a non-degenerate peak point. Stimulated by the works of Wei [72]

and Cao, Noussair and Yan [23], various authors have contributed many papers to this subject, see, e.g., [22, 40, 45, 50, 51, 73].

In the second part of the paper we complete the analysis by proving the existence of such evenly symmetric blow-up solutions using a Lyapunov-Schmidt reduction method.

Theorem 1.2. *Let $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$ small enough and let $\rho = 16\pi + \epsilon$. Let $p_1 = \vec{0}$ and $p_2 = \frac{\omega_1}{2}$ or $\frac{\omega_2}{2}$ or $\frac{\omega_1 + \omega_2}{2}$. Then, for each ϵ there exist a $\lambda > 0$ and a solution u_λ to equation (1.1) such that*

$$\epsilon = (32\pi + o(1)) \lambda e^{-\lambda},$$

$$u_\lambda(p_i) \to +\infty \text{ for } i = 1, 2, \quad u_\lambda(x) \to -\infty \text{ for all } x \in T \setminus \{p_1, p_2\}$$

as $\epsilon \to 0$, and

$$u_\lambda(z) = u_\lambda(-z).$$

Moreover, we have

$$\frac{\rho}{\int_T e^{u_\lambda}} e^{u_\lambda} \to 8\pi(\delta_{p_1} + \delta_{p_2}) \text{ in a sense of measure, as } \epsilon \to 0.$$

Motivated by the computation of the topological degree, Chen and Lin in [29] constructed blowing-up solutions under the assumption on f_m in (1.9) being a Morse function (see also [44] for a generalization of the latter result under a weaker assumption on the critical points of f_m being "stable"). However, as already pointed out the function f_m is not a Morse function in our setting. We adopt the strategy introduced by Cheng, the second and third author of the present paper in [37] where rectangular tori are considered to prove Theorem 1.2. In particular, we will extend the latter argument to general flat tori. We start by constructing an approximate blowing-up solution to (1.1). Then, we study the solvability of the linearized operator in a suitable functional setting. Finally, we reduce the problem to the one-dimensional problem of finding the appropriate scale of the bubbles.

Based on the local uniqueness of blow-up solutions, we can further show that the evenly symmetric two-point blow-up solutions are one-point blow-up solutions of the mean field equation on a "half" torus. Indeed, it is not difficult to see that $v_\lambda(x) = u_\lambda(x + p_2)$ is also an evenly symmetric solution of (1.1) which also blows-up at p_1, p_2 ($p_1 = 0$ and p_2 is one of the half periods). By Theorem 1.1, we get

$$u_\lambda(z) = u_\lambda(z + p_2).$$

In particular, taking $p_2 = \frac{\omega_1}{2}$, the solution we build becomes the solution to (1.1) on a flat torus $T_{\frac{1}{2}} := \mathbb{R}^2/\mathbb{Z}\frac{\omega_1}{2} + \mathbb{Z}\omega_2$ which blows-up at the origin. Similar phenomena can be observed for other choices of p_2. In conclusion we have the following result:

Corollary 1.3. *All evenly symmetric two-point blow-up solutions of (1.1) form one-point blow-up solutions of the mean field equation on a "half" torus.*

As a byproduct of Corollary 1.3, we actually establish that blow-up can always occur on an arbitrary flat torus as ρ converges to any multiple of 8π despite the degeneracy nature of the problem. One can build solutions by gluing many copies of one-point blow-up solutions constructed in this paper together. We would also like to mention Struwe and Tarantello's result [66] on the existence of two-dimensional non-trivial solutions if $\rho \in (8\pi, \lambda_1(T))$ where $\lambda_1(T)$ is the first eigenvalue of $-\Delta$ on T and Ricciardi and Tarantello's result [64] on the existence of one-dimensional solutions if $\rho > \lambda_1(T)$. We comment further that our result implies the existence of

two-dimensional non-trivial solutions at least for ρ close to and beyond $8\pi m$ even if $\lambda_1(T) \leq 8\pi$ where m is any positive integer.

The paper is organized as follows. In Section 2 we revisit some a priori estimates of blow-up solutions proved in [28] by Chen and Lin and we present an estimate on the distance between the local maximum point and the blow-up point. In Section 3 we provide the proof of the uniqueness property stated in Theorem 1.1. Finally, in Section 4 we construct blowing-up solutions and prove Theorem 1.2.

2. **Preliminaries.** In this section we recall some a priori estimates obtained by Chen and Lin in [28] for blow-up solutions of (1.1). Suppose that u_n is a sequence of blow-up solutions of (1.1)-(1.2) which blow up at p_1 and p_2, i.e.

$$\Delta u_n + \rho_n \left(e^{u_n} - 1\right) = 0 \text{ in } T, \quad \int_T e^{u_n} = 1, \qquad (2.1)$$

where $\rho_n \to 16\pi$ as $n \to \infty$. Let

$$\lambda_n = \max_T u_n, \qquad (2.2)$$

and

$$\lambda_{n,i} = \max_{B(p_i,\delta)} u_n = u_n(x_{n,i}) \text{ for } i = 1, 2, \qquad (2.3)$$

where $\delta > 0$ is a small fixed constant and $B(p_i, \delta)$ denotes a geodesic ball of radius δ on T centered at p_i. We recall that M_1 and M_2 are two open sets dividing T into two disjoint parts and $p_i \in M_i$ for $i = 1, 2$. Furthermore, r_0 is chosen as right after (1.6) to guarantee that $B(p_i, 2r_0) \subset M_i$ for $i = 1, 2$.

Remark 2.1. To simplify our notation, since T is a flat torus we shall treat $x \in T$ as a point in $\mathbb{R}^2$. Then, the notation $B(p, \delta)$ stands for the set of points $x \in T$ with $d(x, p) < \delta$, where the metric is the one inherited from the Euclidean metric of $\mathbb{R}^2$, i.e.

$$d(x,y) := \min_{z \in \{z \mid z = y + \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2\}} |x - z|.$$

Let us introduce the Green's function $G(x, y)$ of $-\Delta$ on T,

$$-\Delta_x G(x,y) = \delta_y(x) - 1 \text{ in } T, \qquad \int_T G(x,y)dx = 0 \text{ for all } y \in T. \qquad (2.4)$$

In particular, we have the explicit formula of $G(x, y)$ in terms of doubly periodic functions (see [35]):

$$G(x,y) = G(z) := \text{Im} \left(\frac{|z|^2 - \bar\omega_1 z^2/\omega_1}{2(\omega_1\bar\omega_2 - \bar\omega_1\omega_2)} - \frac{z}{2\omega_1} + \frac{\omega_2}{12\omega_1} \right) - \frac{1}{2\pi} \left| \log \left(1 - e\left(\frac{z}{\omega_1}\right) \right) \right|$$

$$- \frac{1}{2\pi} \log \left| \prod_{n=1}^{\infty} \left(1 - e\left(\frac{n\omega_2 + z}{\omega_1}\right) \right) \left(1 - e\left(\frac{n\omega_2 - z}{\omega_1}\right) \right) \right|,$$

$$(2.5)$$

where $z = x - y$ and x, y, z are numbers in the complex plane; $|T| = \text{Im } \bar\omega_1\omega_2 = 1$. It is easy to verify that $G(z) = G(-z)$. In particular, Chen, Lin and Wang in [32] showed that $G(z)$ is evenly symmetric about both axes if T is a rectangular torus, i.e. $G(z) = G(-z) = G(\bar z)$. We also define the regular part of the Green's function:

$$R(x,y) = R(z) := G(x,y) + \frac{1}{2\pi} \log \left(d(x,y) \right), \qquad (2.6)$$

where $d(x, y)$ is defined in the Remark 2.1.

Let $U_{n,i}$ be the standard bubble at $x_{n,i}$, i.e.

$$U_{n,i}(x) = \log\left(\frac{e^{\lambda_{n,i}}}{(1 + \frac{\rho_n}{8}e^{\lambda_{n,i}}(d(x,x_{n,i}))^2)^2}\right), \quad i = 1, 2. \tag{2.7}$$

Chen and Lin in [28] obtained some sharp estimates on the error term $\eta_{n,i}$, which is defined as follows

$$\eta_{n,i}(x) = u_n(x) - U_{n,i}(x) - (G_i^*(x) - G_i^*(x_{n,i})), \quad x \in B(x_{n,i}, \delta). \tag{2.8}$$

For $x \in B(x_{n,i}, \delta)$, they proved

$$\begin{aligned}
\eta_{n,i}(x) = &-\frac{128\pi}{\rho_n}e^{-\lambda_{n,i}}\left[\log\left(R_{n,i}|x - x_{n,i}| + 2\right)\right]^2 \\
&+ O\left(\log\left(R_{n,i}|x - x_{n,i}| + 2\right)e^{-\lambda_{n,i}}\right) \\
&+ O\left(\lambda_{n,i}e^{-\lambda_{n,i}}\right) = O(\lambda_{n,i}^2 e^{-\lambda_{n,i}}), \quad i = 1, 2,
\end{aligned} \tag{2.9}$$

where $R_{n,i} = \sqrt{\rho_n e^{\lambda_{n,i}}/8}$. It has also been proved in [57] that there are constants $c > 0$ and $c_\delta > 0$ such that,

$$|\lambda_n - \lambda_{n,i}| \le c \text{ for } i = 1, 2, \quad |u_n(x) + \lambda_n| \le c_\delta \text{ for } x \in T \setminus \bigcup_{i=1}^{2} B(p_i, \delta). \tag{2.10}$$

More precisely, see [28, Section 3], we have

$$e^{\lambda_{n,i}}e^{G_i^*(x_{n,i})} = e^{\lambda_{n,1}}e^{G_1^*(x_{n,1})}\left(1 + O\left(e^{-\frac{\lambda_{n,1}}{2}}\right)\right), \quad i = 1, 2. \tag{2.11}$$

In particular, see [28, Theorem 1.4], the following estimate holds,

$$\begin{aligned}
&\lambda_{n,i} + \int_T u_n(x)dx + 2\log\left(\frac{\rho_n}{8}\right) + G_i^*(x_{n,i}) \\
&= -\frac{32\pi}{\rho_n}\lambda_{n,i}^2 e^{-\lambda_{n,i}} + O\left(\lambda_{n,i}e^{-\lambda_{n,i}}\right), \quad i = 1, 2.
\end{aligned} \tag{2.12}$$

Notice that in [28, Lemma 5.5], the Pohozaev identity is used to derive that

$$\nabla G_i^*\big|_{x=x_{n,i}} = O\left(\lambda_{n,i}e^{-\lambda_{n,i}}\right), \quad i = 1, 2. \tag{2.13}$$

Together with the non-degeneracy assumption on the critical point $\vec{p}$, Bartolucci et al. [6] concluded that

$$|x_{n,i} - p_i| = O(\lambda_{n,i}e^{-\lambda_{n,i}}), \quad i = 1, 2. \tag{2.14}$$

However, in our case, the critical point $\vec{p} = (p_1, p_2)$ where $p_2 - p_1$ is one of three half-periods (namely $\omega_1/2$, $\omega_2/2$ and $\frac{\omega_1 + \omega_2}{2}$) is a degenerate critical point of $G(x, y)$. Thus, we need a different way to get the above estimate on the distance between the local maximum point and the blow-up point. By imposing the symmetry condition on u_n, we are able to show that (2.14) holds in our setting as well.

Proposition 2.2. *Suppose that $\{u_n\}$ is a sequence of blow-up solutions of (2.1), satisfying $\rho_n \to 16\pi^+$ and $u_n(z) = u_n(-z)$. Then $\{u_n\}$ blows-up at two points $p_1 = 0$ and p_2 which is any half-period. Furthermore, if $x_{n,i}$, $i = 1, 2$ is the local maximum point as defined in (2.3), then we have*

$$d(x_{n,i}, p_i) = O(\lambda_{n,i}e^{-\lambda_{n,i}}).$$

Proof. Ma and Wei in [62] proved that the blow-up points $(p_1, \cdots, p_m)$ of solutions to the corresponding Dirichlet problem of mean field type must be a critical point of the m-vortex Hamiltonian f_m, and they also pointed out that the same conclusion would also hold for (1.3). Chen and Lin in [28, Estimate B] obtained a similar conclusion in the manifold setting by using the Pohozaev identity. Therefore, the only possible two-point blow-up would happen at the critical points of $G(x,y)$. By assumption of the symmetry of solutions, the blow-up points must be one of the three cases stated in Theorem 1.2.

Then, it suffices to prove the estimate on $d(x_{n,i}, p_i)$, $i = 1, 2$. Without loss of generality, let us consider the case $i = 1$. By (2.8), we can write $u(x) = U_{n,1}(x) + G_1^*(x) - G_1^*(x_{n,1}) + \eta_{n,1}(x)$ for $x \in B(x_{n,1}, \delta)$. Since $x_{n,1} \to 0$, we are always able to choose n sufficiently large such that $-x_{n,1} \in B(x_{n,1}, \delta)$. Thus, using the fact that $u(x_{n,1}) = u(-x_{n,1})$ together with (2.9) and (2.13), we conclude that $|x_{n,1}| = O\left(\lambda_{n,1} e^{-\lambda_{n,1}}\right)$. Note that G_1^* is a smooth function.   □

Remark 2.3. Recently, Chen, Kuo, Lin and Wang in [36] showed that $G(z)$ might have an extra pair of "non-trivial" critical points other than the three half periods points for a class of flat tori. Moreover, the "non-trivial" critical points are always non-degenerate. Based on these observations, one should be able to construct two distinct families of blow-up solutions which are not evenly symmetric. It is also possible to prove "local uniqueness" for solutions that blow-up at the origin and one of the "non-trivial" critical points.

Let us define the local masses corresponding to the blow-up of u_n at p_i, $i = 1, 2$:

$$\rho_{n,i} = \rho_n \int_{B(p_i, \delta)} e^{u_n} dx, \quad i = 1, 2. \tag{2.15}$$

We have the following estimate on $\rho_{n,i}$, $i = 1, 2$, see [28, Section 3]

$$\rho_{n,i} - 8\pi = 16\pi \lambda_{n,i} e^{-\lambda_{n,i}} + O\left(e^{-\lambda_{n,i}}\right), \quad i = 1, 2. \tag{2.16}$$

For the total mass, see [28, Theorem 1.1], we have

$$\rho_n - 16\pi = 16\pi \sum_{i=1}^{2} \lambda_{n,i} e^{-\lambda_{n,i}} + O(e^{-\lambda_{n,i}}) = \frac{\lambda_{n,1} e^{-\lambda_{n,1}}}{e^{G_1^*(p_1)}} l(\vec{p}) + O\left(e^{-\lambda_{n,1}}\right), \tag{2.17}$$

where $l(\vec{p})$ is the quantity defined in (1.6). In particular, we recall $l(\vec{p}) = 32\pi e^{G_1^*(p_1)} \neq 0$. We would like to remark that a more refined estimate involving $D(\vec{q})$ on the total mass has been derived in [6, Theorem 1.3] which is crucial in the case where $l(\vec{p})$ vanishes.

We will also need the asymptotic behaviour of u_n outside the union of the balls $B(p_i, \delta)$, $i = 1, 2$. In particular, we consider the "outer" error defined as follows

$$\omega_n(x) = u_n(x) - \sum_{i=1}^{2} \rho_{n,i} G(x, x_{n,i}) - \int_T u_n dx. \tag{2.18}$$

It is already proved in [28, Estimate A] that

$$\omega_n = O(e^{-\lambda_n/2}) \quad \text{in} \quad C^1\left(T \setminus \bigcup_{i=1}^{2} B(p_i, \delta)\right). \tag{2.19}$$

3. **Uniqueness of blow-up solutions.** In this section we will prove Theorem 1.1 by contradiction. Suppose that (2.1) has two distinct solutions $u_n^{(1)}$ and $u_n^{(2)}$ which blow-up at p_i, $i = 1, 2$. Let us use $x_{n,i}^{(\ell)}$, $\lambda_n^{(\ell)}$, $\lambda_{n,i}^{(\ell)}$, $U_{n,i}^{(\ell)}$, $\eta_{n,i}^{(\ell)}$, $R_{n,i}^{(\ell)}$, $\rho_{n,i}^{(\ell)}$ and $\omega_n^{(\ell)}$ to denote $x_{n,i}$, λ_n, $\lambda_{n,i}$, $U_{n,i}$, $\eta_{n,i}$, $R_{n,i}$, $\rho_{n,i}$, ω_n, as defined in Section 2, corresponding to $u_n^{(\ell)}$, $\ell = 1, 2$, respectively.

As in [6, 61] we consider the normalized difference of the two solutions

$$\xi_n(x) = \frac{u_n^{(1)}(x) - u_n^{(2)}(x)}{\|u_n^{(1)} - u_n^{(2)}\|_{L^\infty(T)}}. \tag{3.1}$$

Roughly speaking, our aim is to show that the projections of ξ_n, $n \to +\infty$, on the three kernel functions (introduced in (1.14)) of the linearized operator (1.13) are zero and then derive a contradiction by showing that $\xi_n \not\equiv 0$, $n \to +\infty$. The plan is the following:

(1) study the asymptotic behavior of ξ_n inside and outside the blow-up disks,
(2) use a suitable Pohozaev identity to show the projection of ξ_n on the radial part kernel vanishes,
(3) exploit the evenly symmetric property to show the projections of ξ_n on the kernels related to translations are zero and finally prove Theorem 1.1.

3.1. **Some useful estimates.** We start by studying the asymptotic behavior of ξ_n. This part follows closely [6] jointly with Proposition 2.2, so we skip the computations and refer the reader to [6] for full details.

Lemma 3.1. ([6]) *There exists a constant $C > 0$ such that*

$$|\lambda_{n,i}^{(1)} - \lambda_{n,i}^{(2)}| \le C \left(\frac{1}{\lambda_{n,1}^{(1)}} + \frac{1}{\lambda_{n,1}^{(2)}} \right), \ i = 1, 2. \tag{3.2}$$

Moreover,

$$\|u_n^{(1)} - u_n^{(2)}\|_{L^\infty(T)} = O \left(|\lambda_{n,1}^{(1)} - \lambda_{n,1}^{(2)}| + \sum_{\ell=1}^2 \lambda_{n,1}^{(\ell)} e^{-\frac{\lambda_{n,1}^{(\ell)}}{2}} \right). \tag{3.3}$$

It is easy to see that ξ_n satisfies

$$\Delta \xi_n + f_n^*(x) = \Delta \xi_n + \rho_n c_n(x) \xi_n(x) = 0, \tag{3.4}$$

where

$$f_n^*(x) = \frac{\rho_n}{\|u_n^{(1)} - u_n^{(2)}\|_{L^\infty(T)}} \left(e^{u_n^{(1)}(x)} - e^{u_n^{(2)}(x)} \right), \tag{3.5}$$

and

$$c_n(x) = \frac{e^{u_n^{(1)}(x)} - e^{u_n^{(2)}(x)}}{u_n^{(1)}(x) - u_n^{(2)}(x)} = e^{u_n^{(1)}} \left(1 + O(\|u_n^{(1)} - u_n^{(2)}\|_{L^\infty(T)}) \right). \tag{3.6}$$

Next, in the following lemma we give the description of ξ_n both inside the bubbling disc $B(p_i, \delta)$, $i = 1, 2$, and away from the blow-up points p_i, $i = 1, 2$.

Lemma 3.2. ([6]) *Let*

$$\xi_{n,i}(z) = \xi_n \left(e^{-\frac{\lambda_{n,i}^{(1)}}{2}} z + x_{n,i}^{(1)} \right), \ |z| < \delta e^{-\frac{\lambda_{n,i}^{(1)}}{2}}, \ i = 1, 2,$$

then there exist constants $b_{i,0}, b_{i,1}, b_{i,2}$ such that

$$\xi_{n,i}(z) \to b_{i,0} \psi_{i,0}(z) + b_{i,1} \psi_{i,1}(z) + b_{i,2} \psi_{i,2}(z)$$

in $C_{loc}(\mathbb{R}^2)$, where

$$\psi_{i,0}(z) = \frac{1 - 2\pi|z|^2}{1 + 2\pi|z|^2}, \ \psi_{i,1}(z) = \frac{\sqrt{2\pi}z_1}{1 + 2\pi|z|^2}, \ \psi_{i,2}(z) = \frac{\sqrt{2\pi}z_2}{1 + 2\pi|z|^2}.$$

Furthermore $b_{1,0} = b_{2,0} = b_0$ for some constant b_0, and

$$\xi_n(x) = -b_0 + o(1), \quad \forall x \in T \setminus \bigcup_{i=1}^{2} B(p_i, e^{-\frac{\lambda_{n,i}^{(1)}}{2}} R),$$

for some $R > 0$ sufficiently large.

For the proof of Lemma 3.1 and Lemma 3.2, we refer the readers to [6, Lemma 3.1-Lemma 3.4].

3.2. **Radial part kernel.** We prove here that the projection of ξ_n on the radial part kernel vanishes. Since by Lemma 3.2 we have $b_{1,0} = b_{2,0} = b_0$, we need to show that $b_0 = 0$. Then, for $i = 1, 2$, let

$$\phi_{n,i}(y) = \frac{\rho_n}{2}\left(R(x_{n,i}^{(1)}, y) - R(x_{n,i}^{(1)}, x_{n,i}^{(1)}) + G(x_{n,l}^{(1)}, y) - G(x_{n,l}^{(1)}, x_{n,i}^{(1)})\right), \qquad (3.7)$$

where $l \neq i$, and

$$v_{n,i}^{(\ell)}(y) = u_n^{(\ell)}(y) - \phi_{n,i}(y), \quad \ell = 1, 2. \tag{3.8}$$

To show $b_0 = 0$ we need the following Pohozaev identity from [6, Lemma 3.6].

Lemma 3.3. ([6]) *For any fixed $r \in (0, \delta)$, we have*

$$\frac{1}{2}\int_{\partial B(x_{n,i}^{(1)}, r)} r\langle \nabla(v_{n,i}^{(1)} + v_{n,i}^{(2)}), \nabla\xi_n \rangle - \int_{\partial B(x_{n,i}^{(1)}, r)} r\langle \nu, \nabla(v_{n,i}^{(1)} + v_{n,i}^{(2)})\rangle\langle \nu, \nabla\xi_n \rangle$$

$$= \int_{\partial B(x_{n,i}^{(1)}, r)} \frac{r\rho_n}{\|v_{n,i}^{(1)} - v_{n,i}^{(2)}\|_{L^\infty(T)}} \left(e^{v_{n,i}^{(1)} + \phi_{n,i}} - e^{v_{n,i}^{(2)} + \phi_{n,i}}\right) \tag{3.9}$$

$$- \int_{B(x_{n,i}^{(1)}, r)} \frac{\rho_n\left(e^{v_{n,i}^{(1)} + \phi_{n,i}} - e^{v_{n,i}^{(2)} + \phi_{n,i}}\right)}{\|v_{n,i}^{(1)} - v_{n,i}^{(2)}\|_{L^\infty(\tilde{T})}}\left(2 + \langle \nabla\phi_{n,i}, x - x_{n,i}^{(1)}\rangle\right).$$

Next, we can follow the computations in [6, Lemma 4.2-Lemma 4.3] jointly with Proposition 2.2 to get the estimate on both sides of (3.9). For the left hand side of (3.9), we have

$$\text{LHS of (3.9)} = -4A_{n,i} - \frac{256b_0 e^{-\lambda_{n,1}^{(1)} + G_i^*(p_i)}}{\rho_n e^{G_1^*(p_1)}}\int_{M_i \setminus B(p_i, r)} e^{\Phi_i(x, \vec{p})}dx$$

$$+ o(e^{-\frac{\lambda_{n,i}^{(1)}}{2}}\sum_{l=1}^{2}|A_{n,l}|) + o(e^{-\lambda_{n,i}^{(1)}}), \quad i = 1, 2, \tag{3.10}$$

for fixed $r \in (0, r_0)$, where

$$A_{n,i} = \int_{M_i} \frac{\rho_n}{\|u_n^{(1)} - u_n^{(2)}\|_{L^\infty(T)}}\left(e^{u_n^{(1)}} - e^{u_n^{(2)}}\right),$$

Φ_i is defined in (1.8) and r_0 is introduced after (1.6). For the right hand side of (3.9), we have

$$
\begin{aligned}
\text{RHS of (3.9)} = &- e^{-\lambda_{n,1}^{(1)}} \left(\frac{128 b_0 e^{G_i^*(p_i)}}{\rho_n e^{G_1^*(p_1)}} \frac{\pi}{r^2} + \frac{512\pi^2 b_0 e^{G_i^*(p_i)}}{\rho_n e^{G_1^*(p_1)}} \right) \\
&- e^{-\lambda_{n,1}^{(1)}} \frac{128 b_0}{\rho_n e^{G_1^*(p_1)}} \int_{M_i \backslash B(p_i, r_i)} e^{G_i^*(p_i) + \Phi_i(x,\vec{p})} dx \\
&- e^{-\lambda_{n,1}^{(1)}} \left(\lambda_{n,1}^{(1)} + \log \left(\frac{\rho_n e^{G_1^*(p_1)}}{8 e^{G_i^*(p_i)}} r^2 \right) - 2 \right) \Pi e^{G_i^*(p_i)} \\
&+ O(e^{-\lambda_{n,1}^{(1)}})(r + R^{-1}) + o(e^{-\lambda_{n,i}^{(1)}})(\log r + \log R) \\
&+ O(\sum_l |A_{n,l}|(R^{-1} e^{-\frac{\lambda_{n,i}^{(1)}}{2}} + e^{-\lambda_{n,i}^{(1)}}(\lambda_{n,i}^{(1)} + \log r))) \\
&+ o(e^{-2\lambda_{n,i}^{(1)}} r^{-2}),
\end{aligned}
\tag{3.11}
$$

for any $r \in (0,1)$ and R sufficiently large. Here

$$
\Pi = \frac{512\pi^2 \left((\int_T \xi_n) - \frac{\|u_n^{(1)} - u_n^{(2)}\|_{L^\infty(T)}}{2} (\int_T \xi_n)^2 \right)}{\rho_n e^{G_1^*(p_1)}}.
$$

With (3.10) and (3.11), we are now able to show $b_0 = 0$.

Lemma 3.4. *It holds $b_0 = b_{1,0} = b_{2,0} = 0$.*

Proof. By (3.9)-(3.11), together with (2.10) and (2.12), we have

$$
- 4 A_{n,i} + O(e^{-\frac{\lambda_{n,1}^{(1)}}{2}} \sum_{l=1}^2 |A_{n,l}|) + o(e^{-\lambda_{n,1}^{(1)}})
$$

$$
= -2 A_{n,i} + O(\lambda_{n,i}^{(1)} e^{-\lambda_{n,i}^{(1)}}) + O(r^{-2} e^{-\lambda_{n,i}^{(1)}}) + o(e^{-\lambda_{n,i}^{(1)}} \log R),
$$

which implies that

$$
A_{n,i} = o(e^{-\frac{\lambda_{n,1}^{(1)}}{2}}), \quad i = 1, 2.
\tag{3.12}
$$

For any $r > 0$, let $r_i = r\sqrt{8 e^{G_i^*(p_i)}}$, $i = 1, 2$. For each point p_i, we choose $r = r_i$ in (3.9). By (3.9)-(3.12), we have

$$
\begin{aligned}
&\sum_{i=1}^2 \left[-4 A_{n,i} - \frac{256 b_0 e^{-\lambda_{n,1}^{(1)} + G_i^*(p_i)}}{\rho_n e^{G_1^*(p_1)}} \int_{M_i \backslash B(p_i, r_i)} e^{\Phi_i(x,\vec{p})} dx \right] \\
&= \sum_{i=1}^2 \left[-e^{-\lambda_{n,1}^{(1)}} \left(\frac{128 b_0 e^{G_i^*(p_i)}}{\rho_n e^{G_1^*(p_1)}} \frac{\pi}{r_i^2} + \frac{512\pi^2 b_0 e^{G_i^*(p_i)}}{\rho_n e^{G_1^*(p_1)}} \right) \right. \\
&\quad - e^{-\lambda_{n,1}^{(1)}} \frac{128 b_0}{\rho_n e^{G_1^*(p_1)}} \int_{M_i \backslash B(p_i, r_i)} e^{G_i^*(p_i) + \Phi_i(x,\vec{p})} dx \\
&\quad \left. - e^{-\lambda_{n,1}^{(1)}} \left(\lambda_{n,1}^{(1)} + \log \left(\frac{\rho_n e^{G_1^*(p_1)}}{8 e^{G_i^*(p_i)}} r_i^2 \right) - 2 \right) \Pi e^{G_i^*(p_i)} \right] \\
&\quad + o(e^{-\lambda_{n,1}^{(1)}}(r^{-2} + \log R + \log r)) + O(e^{-\lambda_{n,1}^{(1)}}(r + R^{-1})),
\end{aligned}
\tag{3.13}
$$

for any $r \in (0,1)$, $R > 1$ sufficiently large. Then using the fact that $A_{n,1} + A_{n,2} = 0$, we deduce that

$$
-\frac{256 b_0 e^{-\lambda_{n,1}^{(1)}}}{\rho_n e^{G_1^*(p_1)}} \sum_{i=1}^{2} e^{G_i^*(p_i)} \int_{M_i \backslash B(p_i, r_i)} e^{\Phi_i(x,\vec{p})} dx
$$

$$
= -e^{-\lambda_{n,1}^{(1)}} \frac{128 b_0}{\rho_n e^{G_1^*(p_1)}} \sum_{i=1}^{2} e^{G_i^*(p_i)} \frac{\pi}{r_i^2} - e^{-\lambda_{n,1}^{(1)}} \frac{32\pi b_0 l(\vec{p})}{\rho_n e^{G_1^*(p_1)}}
$$

$$
- e^{-\lambda_{n,1}^{(1)}} \frac{128 b_0}{\rho_n e^{G_1^*(p_1)}} \sum_{i=1}^{2} e^{G_i^*(p_i)} \int_{M_i \backslash B(p_i, r_i)} e^{\Phi_i(y,\vec{p})} dy
$$

$$
- e^{-\lambda_{n,1}^{(1)}} \frac{\Pi}{16\pi} l(\vec{p}) \left(\lambda_{n,1}^{(1)} + \log(\rho_n e^{G_1^*(p_1)} r^2) - 2 \right)
$$

$$
+ o(e^{-\lambda_{n,1}^{(1)}}(r^{-2} + \log R + \log r)) + O(e^{-\lambda_{n,1}^{(1)}}(r + R^{-1})). \qquad (3.14)
$$

By Lemma 3.2, we have $\int_T \xi_n = -b_0 + o(1)$. We divide (3.14) by $\lambda_{n,1}^{(1)} e^{-\lambda_{n,1}^{(1)}}$ and derive that $l(\vec{p})b_0 = o(1)$. Therefore, we conclude that $b_0 = 0$ since $l(\vec{p}) \neq 0$. □

3.3. The conclusion.

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We already know that the projections on the radial part kernel are zero, i.e. $b_{i,0} = 0$ for $i = 1, 2$, see Lemma 3.4. Let us show now that the projections $b_{i,k}$, $i, k = 1, 2$, on the kernel related to translations are zero. Using the fact that both $u_n^{(1)}$ and $u_n^{(2)}$ are evenly symmetric, we can see that $u_n^{(1)}(x + p_2)$ and $u_n^{(2)}(x + p_2)$ are also evenly symmetric. As a consequence, the projection of the normalized difference on the kernel related to translations vanishes automatically, i.e.

$$
b_{i,k} = 0, \ i, k = 1, 2. \qquad (3.15)
$$

We will conclude now by showing that $\xi_n \not\equiv 0$, $n \to +\infty$. Let x_n^* be a maximum point of ξ_n, then we have

$$
|\xi_n(x_n^*)| = 1. \qquad (3.16)
$$

Therefore, by Lemma 3.2 and Lemma 3.4, we find that $\lim_{n \to \infty} x_n^* = p_i$ for some i. Moreover, in view of Lemma 3.4 and the fact that $b_{i,k} = 0$ for $k = 1, 2$, we deduce that

$$
\lim_{n \to \infty} e^{\frac{\lambda_{n,i}^{(1)}}{2}} s_n = \infty, \qquad (3.17)
$$

where $s_n = |x_n^* - x_{n,i}^{(1)}|$. We set $\tilde{\xi}_n(x) = \xi_n(s_n x + x_{n,i}^{(1)})$, then (2.9) and (3.4) imply that $\tilde{\xi}_n$ satisfies

$$
\Delta \tilde{\xi}_n + \rho_n s_n^2 c_n(s_n x + x_{n,i}^{(1)}) \tilde{\xi}_n = \Delta \tilde{\xi}_n + \frac{\rho_n s_n^2 e^{\lambda_{n,1}^{(1)}}(1 + O(s_n|x|) + o(1))}{(1 + \frac{\rho_n}{8} e^{\lambda_{n,1}^{(1)}} s_n^2 |x|^2)^2} \tilde{\xi}_n.
$$

On the other hand, we have

$$
\left| \tilde{\xi}_n \left(\frac{x_n^* - x_{n,i}^{(1)}}{s_n} \right) \right| = |\xi_n(x_n^*)| = 1. \qquad (3.18)
$$

In view of (3.17) and $|\tilde{\xi}_n| \leq 1$, we see that if $\tilde{\xi}_n \to \tilde{\xi}_0$ on any compact subset of $\mathbb{R}^2 \setminus \{0\}$, then $\Delta\tilde{\xi}_0 = 0$ in $\mathbb{R}^2 \setminus \{0\}$. Since $\tilde{\xi}_0$ is also bounded, then we can conclude that $\tilde{\xi}_0$ is smooth and harmonic on entire $\mathbb{R}^2$. Hence $\tilde{\xi}_0$ is a constant. Since $\frac{|x_n^* - x_{n,i}^{(1)}|}{s_n} = 1$ and in view of (3.18), we obtain that either $\tilde{\xi}_0 = 1$ or $\tilde{\xi}_0 = -1$. In particular, we have that $|\tilde{\xi}_n| \geq \frac{1}{2}$ for $s_n \leq |x - x_{n,i}^{(1)}| \leq 2s_n$, which contradicts to the second conclusion of Lemma 3.2 because of the facts that $\lim_{n\to\infty} e^{\frac{\lambda_{n,i}^{(1)}}{2}} s_n = \infty$, $\lim_{n\to\infty} s_n = 0$ and $b_0 = b_{j,0} = 0$. This finishes the proof of Theorem 1.2. $\qquad \square$

4. **Existence of blow-up solutions.** In this section we will use a Lyapunov-type reduction method to construct blow-up solutions to (1.1). Since the proof of Theorem 1.2 follows along the same line as the arguments used in [46, Theorem 2.1] and [37, Theorem 2.3], we shall give the key steps and refer the readers to the above two papers for details.

4.1. **Approximate solution.** We start with an approximate solution of the equation (1.1) and obtain some estimates of this approximate solution. Let $R_0 > 0$ be a small fixed number and η be a cut-off function such that

$$\eta(s) = \begin{cases} 1 & \text{for } s \leq 1, \\ 0 & \text{for } s \geq 2, \end{cases} \quad 0 \leq \eta(x) \leq 1, \quad |\eta'(s)| \leq 2.$$

Let

$$\eta_{t,a}(x) = \eta\left(\frac{d(x,a)}{t}\right), \quad \forall a \in T \text{ and } t > 0. \tag{4.1}$$

Given $\epsilon \in (0, \epsilon_0)$, for later purposes we choose $\lambda > 0$ such that

$$16\pi\lambda e^{-\lambda} < \epsilon < 64\pi\lambda e^{-\lambda}, \tag{4.2}$$

or equivalently,

$$\lambda_1(\epsilon) < \lambda < \lambda_2(\epsilon), \tag{4.3}$$

where

$$16\pi\lambda_1(\epsilon)e^{-\lambda_1(\epsilon)} = \epsilon \quad \text{and} \quad 64\pi\lambda_2(\epsilon)e^{-\lambda_2(\epsilon)} = \epsilon.$$

Since our ansatz will resemble a bubble function around each blow-up point we start by letting $w_{\lambda,i}$ be the solution of the following localized equation:

$$-\Delta w_{\lambda,i} = \frac{16\pi e^\lambda}{(1 + 2\pi e^\lambda (d(x,p_i))^2)^2}\eta_{R_0,p_i} - m_0, \quad \int_T w_{\lambda,i} = 0, \tag{4.4}$$

where

$$m_0 = \int_T \frac{16\pi e^\lambda}{(1 + 2\pi e^\lambda (d(x,p_i))^2)^2}\eta_{R_0,p_i} dx = 8\pi + O(e^{-\lambda}), \quad i = 1,2, \tag{4.5}$$

where we used the fact that $|T| = 1$.

In order to have a good approximation we need the following estimates. Let us calculate the value of $w_{\lambda,i}(p_i)$, $i = 1, 2$:

$$w_{\lambda,i}(p_i) = \int_T G(p_i, y)\left[\frac{16\pi e^\lambda}{(1 + 2\pi e^\lambda |y|^2)^2}\eta_{R_0,p_i} - m_0\right] dy$$

$$
\begin{aligned}
&= \int_{B(0,R_0)} \left[-\frac{1}{2\pi} \log|y| + R(0,y) \right] \frac{16\pi e^\lambda}{(1 + 2\pi e^\lambda |y|^2)^2} dy + O(e^{-\lambda}) \\
&= \int_{B(0,\, e^{\frac{\lambda}{2}} R_0)} \left[\frac{\lambda}{4\pi} - \frac{1}{2\pi} \log|z| + R(0, e^{-\frac{\lambda}{2}} z) \right] \frac{16\pi dz}{(1 + 2\pi |z|^2)^2} + O(e^{-\lambda}) \\
&= \int_{B(0,\, e^{\frac{\lambda}{2}} R_0)} \left[\frac{\lambda}{4\pi} - \frac{1}{2\pi} \log|z| + R(0,0) \right] \frac{16\pi dz}{(1 + 2\pi |z|^2)^2} \\
&\quad + e^{-\lambda} \int_{B(0,\, e^{\frac{\lambda}{2}} R_0)} \frac{4\pi |z|^2}{(1 + 2\pi |z|^2)^2} dz + O(e^{-\lambda}) \\
&= 2\lambda + 2\log(2\pi) + 8\pi R(0,0) + \lambda e^{-\lambda} + O(e^{-\lambda}).
\end{aligned}
\tag{4.6}
$$

We can also estimate the value of $w_{\lambda,i}$ near p_i:

$$
\begin{aligned}
&w_{\lambda,i}(p_i + e^{-\frac{\lambda}{2}} z) - w_{\lambda,i}(p_i) \\
&= \int_T \left[G(p_i + e^{-\frac{\lambda}{2}} z, y) - G(p_i, y) \right] \frac{16\pi e^\lambda}{(1 + 2\pi e^\lambda |y - p_i|^2)^2} \eta_{R_0, p_i} dy \\
&= \int_{B(0,\, e^{\frac{\lambda}{2}} R_0)} -\frac{1}{2\pi} \left[\log|z - z'| - \log|z'| \right] \frac{16\pi dz'}{(1 + 2\pi |z'|^2)^2} \\
&\quad + \int_{B(0,\, e^{\frac{\lambda}{2}} R_0)} \left[R\left(e^{\frac{\lambda}{2}}(z' - z) \right) - R\left(e^{\frac{\lambda}{2}} z' \right) \right] \frac{16\pi dz'}{(1 + 2\pi |z'|^2)^2} + O\left(e^{-\frac{3}{2}\lambda} |z| \right) \\
&= \log\left(\frac{1}{(1 + 2\pi |z|^2)^2} \right) + e^{-\lambda} Q(z_1, z_2) + O\left(e^{-\frac{3}{2}\lambda} |z|^3 \right) + O\left(\lambda e^{-\frac{3}{2}\lambda} |z| \right)
\end{aligned}
\tag{4.7}
$$

for $|z| < e^{\frac{\lambda}{2}} R_0$, where $Q(z_1, z_2)$ is a quadratic form depending only on $\nabla^2 R(z)|_{z=0}$ with the property $\Delta Q = 8\pi$. Note here we use the fact that $\Delta R = 1$. For $|z| \geq 2e^{\frac{\lambda}{2}} R_0$, i.e., $d(x, p_i) \geq 2R_0$, we have

$$
\begin{aligned}
w_{\lambda,i}(x) = w_{\lambda,i}(p_i + e^{-\frac{\lambda}{2}} z) &= \int_T G(x, x') \frac{16\pi e^\lambda}{(1 + 2\pi e^\lambda |x' - p_i|^2)^2} \eta_{R_0, p_i} dx' \\
&= \int_{B(0,\, e^{\frac{\lambda}{2}} R_0)} G(x, p_i + e^{-\frac{\lambda}{2}} z') \frac{16\pi dz'}{(1 + 2\pi |z'|^2)^2} + O(e^{-\lambda}) \\
&= \int_{B(0,\, e^{\frac{\lambda}{2}} R_0)} \left[G(p_i - x) + e^{-\frac{\lambda}{2}} \nabla G(p_i - x) \cdot z' + e^{-\lambda} \frac{|z'|^2}{4} \right] \frac{16\pi dz'}{(1 + 2\pi |z'|^2)^2} + O(e^{-\lambda}) \\
&= 8\pi G(p_i - x) + \pi e^{-\lambda} \int_0^{e^{\frac{\lambda}{2}} R_0} \frac{8\pi r^3}{(1 + 2\pi r^2)^2} dr + O(e^{-\lambda}) \\
&= 8\pi G(p_i - x) + \lambda e^{-\lambda} + O(e^{-\lambda}),
\end{aligned}
\tag{4.8}
$$

Letting $\overline{w}_\lambda = -\lambda + \log\left(\frac{4}{\pi} \right) - 8\pi R(0) - 8\pi G(p_1, p_2)$ we are now ready to provide an ansatz for the solution of (1.1)

$$
w_\lambda = \sum_{i=1}^2 w_{\lambda,i} + \overline{w}_\lambda.
\tag{4.9}
$$

Combining (4.6)-(4.8), we obtain the following lemma concerning the asymptotic behavior of w_λ near the blow-up points p_1 and p_2:

Lemma 4.1. *For* $z \in B(0, e^{\frac{\lambda}{2}} R_0)$, *we have*

$$w_\lambda(p_i + e^{-\frac{\lambda}{2}} z) = \log\left(\frac{16\pi e^\lambda}{(1 + 2\pi |z|^2)^2}\right) + e^{-\lambda} Q(z_1, z_2) + 4\pi e^{-\lambda} z \nabla^2 G(p_1 - p_2) z^T$$

$$+ 2\lambda e^{-\lambda} + O(e^{-\frac{3\lambda}{2}} |z|^3) + O(\lambda e^{-\frac{3\lambda}{2}} |z|) + O(e^{-\lambda}).$$

While for x away from the blow-up points,

Lemma 4.2. *For* $x \in T \setminus (B(p_1, R_0) \cup B(p_2, R_0))$, *we have*

$$w_\lambda(x) = -\lambda + \log(4/\pi) - 8\pi R(0) - 8\pi G(p_1 - p_2)$$

$$+ 8\pi \sum_{i=1}^{2} G(p_i - x) + 2\lambda e^{-\lambda} + O(e^{-\lambda}).$$

Finally, we need to estimate the approximate solution in the neck region. It turns out that we can estimate w_λ and e^{w_λ} when $R_0 < d(x, p_k) < 2R_0$ by comparing w_λ with a function constructed by gluing the inner approximation and the outer approximation by using the cut-off function η_{λ^α} for some $\alpha \in (0, 1)$. It is readily checked that the "error" term could be controlled, please see [46, Lemma 3.1] for more details.

By using Lemmas 4.1, 4.2 and [46, Lemma 3.1] we can now estimate e^{w_λ}. We have

$$e^{w_\lambda} \leq \sum_{i=1}^{2} \frac{16\pi e^\lambda}{(1 + 2\pi e^\lambda (d(x, p_i))^2)^2} [1 + \theta_\lambda(x)] \tag{4.10}$$

where θ_λ has the property that for some constant $C > 0$,

$$|\theta_\lambda(x)| \leq C e^{-\frac{\lambda}{2}} \sum_{i=1}^{2} [e^{\frac{\lambda}{2}} d(x, p_i) + 1].$$

More precisely, when $|z| \leq R_0 e^{\frac{\lambda}{2}}$, we have

$$e^{w_\lambda(p_i + e^{-\frac{\lambda}{2}} z)} = \frac{16\pi e^\lambda}{(1 + 2\pi |z|^2)^2} [1 + e^{-\lambda} Q(z_1, z_2) + 4\pi e^{-\lambda} z \nabla^2 G(p_1 - p_2) z^T$$

$$+ 2\lambda e^{-\lambda} + O(e^{-\frac{3\lambda}{2}} |z|^3) + O(\lambda e^{-\frac{3\lambda}{2}} |z|) + O(e^{-\lambda})]. \tag{4.11}$$

When $d(x, p_i) \geq R_0$ for $i = 1, 2$, we have

$$e^{w_\lambda(x)} = O(e^{-\lambda}). \tag{4.12}$$

Finally, by exploiting (4.2), (4.11) and (4.12), we can follow the same computations in [37, Lemma 3.3] to obtain the estimate of the error of the ansatz.

Lemma 4.3. ([37]) *Let* $S_\rho(u) = \Delta u + \rho\left(\frac{e^u}{\int_T e^u} - 1\right)$. *Then there exists a constant* $C > 0$ *such that*

$$|S_\rho(w_\lambda(p_i + e^{-\frac{\lambda}{2}} z))| \leq C \left[\lambda e^{-\lambda} + \frac{\lambda}{(1 + 2\pi |z|^2)^2} + \frac{|z|^2}{(1 + 2\pi |z|^2)^2}\right] \quad \text{for } |z| < e^{\frac{\lambda}{2}} R_0,$$

and

$$|S_\rho(w_\lambda)(x)| \leq C\lambda e^{-\lambda} \quad \text{for } x \in T \setminus (B(p_1, R_0) \cup B(p_2, R_0)).$$

Furthermore, we have that $S_\rho(w_\lambda)$ *is evenly symmetric.*

We conclude this subsection by considering the energy of the approximate solution w_λ. Indeed, it is known that equation (1.1) has a variational structure, i.e., any critical point of the energy functional

$$J_\rho(u) = \frac{1}{2} \int_T |\nabla u|^2 - \rho \log \left(\int_T e^u \right) + \rho \int_T u \qquad (4.13)$$

corresponds to a solution of (1.1). Again by (4.2), (4.11) and (4.12) and by direct computations as in [37, Lemma 3.4] we can obtain the following expansion on the energy of the approximate solution w_λ.

Lemma 4.4. ([37]) *The energy of w_λ is*

$$\begin{aligned} J_\rho(w_\lambda) = & -64\pi^2[R(0) + G(p_1 - p_2)] - 16\pi \log(2\pi) - 16\pi - \epsilon\lambda \\ & - 32\pi\lambda e^{-\lambda} - \epsilon[2\log(2\pi) - 8\pi R(0) - 8\pi G(p_1 - p_2)] + O(e^{-\lambda}). \end{aligned}$$

4.2. **The linearized operator.** In this subsection, we shall establish the solvability theory for the linearized operator of S_ρ under suitable orthogonality condition. Let us introduce the operator

$$\mathcal{L}(u) = \Delta u + \frac{\rho}{\int_T e^{w_\lambda}} e^{w_\lambda} u. \qquad (4.14)$$

Observe that

$$S'_\rho(w_\lambda)(u) = \mathcal{L}\left(u - \frac{\int_T e^{w_\lambda} u}{\int_T e^{w_\lambda}} \right). \qquad (4.15)$$

Let

$$L(u) = e^{-\lambda} \mathcal{L}(u). \qquad (4.16)$$

If we shift the blow-up point p_i to the center and rescale the torus T to T_λ by the factor $e^{-\frac{\lambda}{2}}$, then formally the operator L converges to the operator $\tilde{L}$ in $\mathbb{R}^2$:

$$\tilde{L}(u) = \Delta_z u + \frac{16\pi}{(1 + 2\pi|z|^2)^2} u, \qquad (4.17)$$

where $z = e^{\frac{\lambda}{2}}(x - p_i)$. We point out that the operator $\tilde{L}$ can be obtained by linearizing the Liouville equation $\Delta u + e^u = 0$ at the radial solution $v(z) = \log\left(\frac{16\pi}{(1 + 2\pi|z|^2)^2} \right)$. A key fact that we are going to exploit is the non-degeneracy of v modulo the invariance of the Liouville equation under dilations and translations, i.e.,

$$\zeta \mapsto v(z - \zeta), \quad s \mapsto v(sz) + 2 \log s.$$

Thus, we let

$$\begin{aligned} \psi_0(z) &= \frac{\partial}{\partial s} [v(sz) + 2 \log s] \big|_{s=1}, \\ \psi_k(z) &= \frac{\partial}{\partial \zeta_j} v(z - \zeta) \big|_{\zeta=0}, \quad k = 1, 2. \end{aligned} \qquad (4.18)$$

Here ψ_k's coincide with $\psi_{i,k}$'s defined in Lemma 3.2. It is shown in [1] that the only bounded solutions of $\tilde{L}(u) = 0$ in $\mathbb{R}^2$ are precisely the linear combinations of ψ_k, $k = 0, 1, 2$. With a little abuse of notation, let $\psi_{i,k} := \psi_k(e^{\frac{\lambda}{2}}(x - p_i))$ denote a function on T_λ for $i = 1, 2$ and $k = 0, 1, 2$.

Next, we introduce the functional setting for the problem. To this end, we start by letting $\tilde{\eta}_R$ be the following cut-off function:

$$\tilde{\eta}_R(s) = \begin{cases} 1 & \text{for } s \leq R, \\ 0 & \text{for } s \geq R + 1, \end{cases} \quad 0 \leq \tilde{\eta}_R \leq 1, \quad |\tilde{\eta}'_R(s)| \leq 2.$$

Let $\tilde{\eta}_{R,p} = \tilde{\eta}_R(|z - p|)$. We also write p_i' to denote $e^{\frac{\lambda}{2}} p_i$. Then, we set

$$L_e^\infty(T_\lambda) = \left\{ u \in L^\infty(T_\lambda) \mid u(z) = u(-z) \right\}.$$

We introduce the following norms

$$\|\psi\|_\infty = \sup_{z \in T_\lambda} |\psi|, \qquad \|\psi\|_* = \sup_{z \in T_\lambda} \left(\sum_{j=1}^2 (1 + d(z, p_j'))^{-3} + e^{-\lambda} \right)^{-1} |\psi(z)|.$$

The choice of the latter norms is motivated by the construction of suitable barrier functions in the proof of some uniform estimates, see Step 2 in Lemma 4.5. Let

$$\mathcal{C} = \left\{ u \in L_e^\infty(T_\lambda) \mid \|u\|_* < \infty \right\},$$

and

$$\mathcal{C}_* = \left\{ u \in L_e^\infty(T_\lambda) \mid u \perp \psi_{i,0} \tilde{\eta}_{R_1, p_i'}, \ i = 1, 2 \right\},$$

where $R_1 > 0$ is a large but fixed number. We notice here that the orthogonality condition in the definition of $\mathcal{C}_*$ is only taken with respect to the elements of the approximate kernel generated by dilations. However, it is not difficult to see that the elements in $\mathcal{C}_*$ are also perpendicular to the approximate kernel that are generated by translations, i.e.,

$$u \perp \psi_{i,k} \tilde{\eta}_{R_1, p_i'}, \quad \forall i = 1, 2, \ k = 0, 1, 2, \quad u \in \mathcal{C}_*.$$

The main goal in this subsection is to prove a solvability result and an a priori estimate (uniform in λ) concerning the operator L given in (4.16) in the functional settings defined above under suitable orthogonality conditions. To this end, let us start with the following uniform a priori estimate for an auxiliary problem with the additional orthogonality conditions of ϕ under translations.

Lemma 4.5. *Let $h \in \mathcal{C} \cap C^\alpha(T_\lambda)$. Then, there exist $\lambda_0, C > 0$ such that for any $\lambda > \lambda_0$ and any $\phi \in \mathcal{C}_*$ such that*

$$L(\phi) = h \text{ in } T_\lambda, \quad \int_{T_\lambda} \tilde{\eta}_{R_1, p_i'} \psi_{i,k} \phi = 0, \quad \forall i = 1, 2, \ k = 0, 1, 2, \tag{4.19}$$

it holds

$$\|\phi\|_\infty \leq C \|h\|_*.$$

Proof. The proof follows the strategy first introduced by del Pino, Kowalczyk and Musso in [39] in dealing with a singularly perturbed Liouville-type equation on a bounded domain with Dirichlet boundary condition. The argument was then suitably adapted to the flat torus case in [37] so we will state just the main steps referring to [37, Lemma 4.2] for a detailed proof.

The proof is obtained by contradiction assuming that there exist sequences $\lambda_n \to +\infty$, h_n with $\|h_n\|_* \to 0$ and ϕ_n with $\|\phi_n\|_\infty = 1$ satisfying (4.19). The contradiction is obtained after the following steps.

Step 1. The first step is to construct a positive supersolution V in order to show that the operator L satisfies the maximum principle on the torus outside the bubbling disks $\tilde{T}_\lambda = T_\lambda \setminus \cup_{i=1}^2 B(p_i', R_2')$, for $R_2' > 0$ sufficiently large, i.e. if $L(u) \leq 0$ in $\tilde{T}_\lambda$ and $u \geq 0$ on $\partial \tilde{T}_\lambda$, then $u \geq 0$ in $\tilde{T}_\lambda$. This is done by defining a suitable projection of the radial solution $f_0(r) = \frac{r^2 - 1}{r^2 + 1}$ in $\mathbb{R}^2$ of

$$\Delta f_0 + \frac{8}{(1 + r^2)^2} f_0 = 0,$$

to a function space on T_λ in order to satisfy the periodic boundary conditions.

Step 2. The second step is to prove that there exists a constant $C > 0$ such that if $L(\phi) = h$ in T_λ, then

$$\|\phi\|_\infty \le C[\|\phi\|_{in} + \|h\|_*],$$

where $\|\cdot\|_{in}$ denotes the "inner" norm of a function on T_λ in the bubbling disks, i.e.

$$\|\phi\|_{in} = \sup_{\cup_{j=1}^2 B(p_j', R_2')} |\phi|.$$

One can use suitable barrier functions in $\tilde{T}_\lambda$ jointly with the maximum principle of Step 1 to derive the above claim.

Step 3. In the final step we will employ a convergence argument to finally deduce a contradiction. By assumptions and by Step 2 we get $\|\phi_n\|_{in} \ge \delta > 0$. Therefore, one can see that ϕ_n in a bubbling disk locally converge to a bounded non-zero solution of

$$\tilde{L}(\hat{\phi}) = 0$$

given in (4.17). Hence, $\hat{\phi}$ is a linear combination of ψ_k, $k = 0, 1, 2$, defined in (4.18). On the other hand, the orthogonal conditions on ϕ_n imply $\hat{\phi} \equiv 0$, yielding a contradiction.   □

We can now prove the solvability and a priori estimate of the following problem (4.20).

Proposition 4.6. *Let $h \in \mathcal{C}$. Then, there exist $\lambda_0, C > 0$, such that for all $\lambda > \lambda_0$, there exist a unique $\phi \in \mathcal{C}_*$ and numbers c_i, $i = 1, 2$ such that*

$$L(\phi) = h + \sum_{i=1}^2 c_i \tilde{\eta}_{R_1, p_i'} \psi_{i,0} \quad in \;\; T_\lambda. \tag{4.20}$$

Moreover, if $h \in C^\alpha(T_\lambda)$, then

$$\|\phi\|_\infty \le C\|h\|_*. \tag{4.21}$$

Proof. We start by proving the a priori estimate (4.21). One can apply Lemma 4.5 to get

$$\|\phi\|_\infty \le C \left[\|h\|_* + \sum_{i=1}^2 |c_i| \right].$$

We can reason exactly as in [37, Proposition 4.1] and after multiplying the equation (4.20) by the test function $\tilde{\eta}_{R_3', p_i'} \psi_{i,0}$, $R_3' > 0$ sufficiently large, derive $|c_i| \le C\|h\|_*$. Thus, (4.21) holds true.

The existence of a solution to (4.20) follows from the Fredholm alternative since we know that equation (4.20) has a unique solution if and only if the associated homogeneous problem (i.e. with $h \equiv 0$) has only the trivial solution. By the a priori estimate we conclude that this is the case and the proof is concluded.   □

Finally, we can reason as in the proof of Proposition 4.6 and deduce the following main result of this subsection, in which one more orthogonal condition to ϕ is imposed, see [37, Corollary 4.3].

Proposition 4.7. ([37]) *Let $h \in \mathcal{C}$. Then, there exist $\lambda_0, C > 0$, such that for all $\lambda > \lambda_0$, there exist a unique $\phi \in \mathcal{C}_*$ and numbers c_i, $i = 0, 1, 2$ such that*

$$L(\phi) = h + \sum_{i=1}^{2} c_i \tilde{\eta}_{R_1, p_i'} \psi_{i,0} + c_0 \ in \ T_\lambda, \quad \phi \perp e^{w_\lambda}. \tag{4.22}$$

Moreover, if $h \in C^\alpha(T_\lambda)$, then

$$\|\phi\|_\infty \le C\|h\|_*.$$

By means of the latter result we can define a continuous linear map $T : \mathcal{C} \to L_e^\infty(T_\lambda)$ given by

$$h \mapsto T(h) := \phi,$$

where ϕ is the unique solution of problem (4.22) obtained in Proposition 4.7.

4.3. **Finite-dimensional reduction and proof of Theorem 1.2.** We are ready to reduce the infinite dimensional problem of finding ϕ such that

$$S_\rho(w_\lambda + \phi) = 0 \tag{4.23}$$

to a one-dimensional problem of finding appropriate scale λ with given ρ. To this end we first expand $S_\rho(w_\lambda + \phi)$ as

$$S_\rho(w_\lambda + \phi) = S_\rho(w_\lambda) + \mathcal{L}\left(\phi - \frac{\int_T e^{w_\lambda} \phi}{\int_T e^{w_\lambda}}\right) + N(\phi), \tag{4.24}$$

where

$$N(\phi) = \left[\frac{\rho}{\int_T e^{w_\lambda + \phi}} e^\phi - \frac{\rho}{\int_T e^{w_\lambda}} - \left(\phi - \frac{\int_T e^{w_\lambda} \phi}{\int_T e^{w_\lambda}}\right)\right] e^{w_\lambda}. \tag{4.25}$$

Since $S_\rho(w_\lambda + \phi)$ is invariant under adding a constant to ϕ, we can further assume that

$$\int_T e^{w_\lambda} \phi = 0.$$

By slightly abuse of notation we still denote ϕ as a function in $\mathcal{C}_*$. Moreover, we consider problem (4.23) in the dilated coordinates, i.e. w_λ, $S_\rho(w_\lambda)$ and $N(\phi)$ are now treated as functions on T_λ.

In order to obtain a solution to (4.23) we first exploit the solvability of the linearized operator established in subsection 4.2 to solve the following intermediate problem.

Lemma 4.8. *There exist $\lambda_0, C > 0$, such that for all $\lambda > \lambda_0$, there exist a unique $\phi \in \mathcal{C}_*$ and numbers c_i, $i = 0, 1, 2$ such that*

$$L(\phi) = -e^{-\lambda} \left[S_\rho(w_\lambda) + N(\phi)\right] + \sum_{i=1}^{2} c_i \tilde{\eta}_{R_1, p_i'} \psi_{i,0} + c_0 \ in \ T_\lambda, \quad \phi \perp e^{w_\lambda}. \tag{4.26}$$

Moreover, it holds

$$\|\phi\|_\infty \le C e^{-\frac{\lambda}{2}}. \tag{4.27}$$

Proof. Let $T : \mathcal{C} \to L_e^\infty(T_\lambda)$ be the continuous linear map defined after Proposition 4.7. Then, we rewrite (4.26) as

$$\phi = A(\phi) := T\left(-e^{-\lambda}\left[S_\rho(w_\lambda) + N(\phi)\right]\right)$$

in the subspace $\mathcal{F} = \{\phi \in \mathcal{C}_* \mid \phi \perp e^{w_\lambda}, \|\phi\|_\infty \le C e^{-\frac{\lambda}{2}}\}$. By using Proposition 4.7 and the estimate of the error $S_\rho(w_\lambda)$ in Lemma 4.3 it is not difficult to show that

the operator A is a contraction map in the space $\mathcal{F}$. Thus, there exists a unique fixed point which is a solution to (4.26). □

To conclude and get a solution to (4.23) we are left with showing that $c_0 = c_1 = c_2 = 0$ in (4.26). To this end, we have the following properties.

Lemma 4.9. *Let ϕ and c_i, $i = 0, 1, 2$ be given as in Lemma 4.8. Then, it holds:*

(1) $c_1 = c_2$ *and* $c_0 = -2e^{-\lambda} \mathcal{A} c_1$, *where* $\mathcal{A} = \int_{\mathbb{R}^2} \tilde{\eta}_{R_1} \psi_0(z) dz$.

(2) $\|\frac{\partial \phi}{\partial \lambda}\|_\infty \leq C e^{-\frac{\lambda}{2}}$.

Proof. (1) From the invariance of problem (4.26) by the change $z \mapsto z + p_2'$ we have

$$\langle L(\phi), \tilde{\eta}_{R_1, p_1'} \psi_{1,0} \rangle = \langle L(\phi), \tilde{\eta}_{R_1, p_2'} \psi_{2,0} \rangle.$$

Moreover, integrating the equation in (4.26) on T_λ, we have

$$\mathcal{A}(c_1 + c_2) + e^\lambda c_0 = 0 \tag{4.28}$$

and (1) holds true.

(2) The idea is to differentiate (4.26) with respect to λ, write

$$L\left(\frac{\partial \phi}{\partial \lambda}\right) = \tilde{h} + \sum_{i=1}^{2} \tilde{c}_i \tilde{\eta}_{R_1, p_i'} \psi_{i,0},$$

for some suitable $\tilde{h}$, $\tilde{c}_i$ and finally exploit Proposition 4.6. We can follow the same computations as in [37, Lemma 5.3] so we omit the details. □

Moreover, we have the following property concerning the energy functional J_ρ defined in (4.13).

Lemma 4.10. *Let ϕ be given as in Lemma 4.8 and λ_1, λ_2 be as in (4.3). Then, $J_\rho(w_\lambda + \phi)$ is a C^1 function with respect to λ for $\lambda \in (\lambda_1, \lambda_2)$ and hence it has a local maximum point λ_*. Furthermore, we have*

$$\epsilon = (32\pi + o(1)) \lambda e^{-\lambda},$$

as $\epsilon \to 0$, where $\rho = 16\pi + \epsilon$.

Proof. We first make the following expansion

$$J_\rho(w_\lambda + \phi) = J_\rho(w_\lambda) + \langle S_\rho(w_\lambda + \theta\phi), \phi \rangle,$$

for some $\theta \in (0, 1)$. Then,

$$S_\rho(w_\lambda + \theta\phi) = S_\rho(w_\lambda) + \theta\Delta\phi + O(e^{-\frac{\lambda}{2}} e^{w_\lambda})$$

Exploiting $\|\phi\|_\infty \leq C e^{-\frac{\lambda}{2}}$, the estimate of the error $S_\rho(w_\lambda)$ in Lemma 4.3 and reasoning as in [37, Lemma 6.1] for the term $\Delta\phi$ it is easy to show that

$$J_\rho(w_\lambda + \phi) = J_\rho(w_\lambda) + O(e^{-\lambda}).$$

Then, by letter estimate and by Lemma 4.4 we have

$$\begin{aligned} J_\rho(w_\lambda) = & -64\pi^2[R(0) + G(p_1 - p_2)] - 16\pi \log(2\pi) - 16\pi - \epsilon\lambda \\ & - 32\pi\lambda e^{-\lambda} - \epsilon \left[2 \log(2\pi) - 8\pi R(0) - 8\pi G(p_1 - p_2)\right] + O(e^{-\lambda}). \end{aligned}$$

The proof of Lemma 4.10 follows then easily. □

Finally, we can prove now the main result of this section. It turns out that the scale λ_* given by the Lemma 4.10 is the right choice to solve the problem (4.23).

Proof of Theorem 1.2. We have to prove that for $\lambda = \lambda_*$, $c_0 = c_1 = c_2 = 0$ in (4.26). Since λ_* is a critical point of $J_\rho(w_\lambda + \phi)$ we have

$$0 = \frac{\partial J_\rho(w_\lambda + \phi)}{\partial \lambda}|_{\lambda=\lambda_*} = \left\langle S_\rho(w_\lambda + \phi), \frac{\partial(w_\lambda + \phi)}{\partial \lambda} \right\rangle |_{\lambda=\lambda_*}.$$

On the other hand, by using Lemma 4.9 and by direct computations as in [37, Lemma 6.3] it is not difficult to get

$$\left\langle S_\rho(w_\lambda + \phi), \frac{\partial(w_\lambda + \phi)}{\partial \lambda} \right\rangle = \left(Ce^{\frac{\lambda}{2}} + O(1) \right) c_1$$

and hence it follows $c_0 = c_1 = c_2 = 0$. Thus,

$$S_\rho(w_{\lambda_*} + \phi_*) = 0$$

and $w_{\lambda_*} + \phi_*$ is the desired blowing-up solution to (1.1). $\square$

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E-mail address: bartoluc@mat.uniroma2.it

E-mail address: changfeng.gui@utsa.edu

E-mail address: huyeyao@gmail.com

E-mail address: aleks.jevnikar@sns.it

E-mail address: wyang@wipm.ac.cn