Blow-up Solutions for a Mean Field Equation on a Flat Torus

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ABSTRACT. We prove the existence of a family of blow-up solutions of a mean field equation on a flat torus. The solutions blow up at two points where the critical values (modulo translations) of Greenś function are attained. Moreover, the solutions we build are evenly symmetric about both axes.

1. INTRODUCTION

In this paper, we consider a mean field equation on the flat Torus T^2 with fundamental domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, that is,

$$\Delta u + \rho \left(\frac{e^u}{\int_{T^2} e^u} - 1 \right) = 0. \tag{1.1}$$

This equation arises from Onsager's vortex theory. Also, it appears as a limiting case of the Chern-Simons gauge theory (see [21]).

One may consider the more general form of the equation on a compact Riemannian surface M without boundary:

$$\Delta u + \rho \left(\frac{he^u}{\int_M he^u} - \frac{1}{|M|} \right) = 0, \tag{1.2}$$

where $h \in C^\infty(M)$ is a positive potential function and $|M|$ is the total area of the surface M.

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Equations of type (1.2) have been broadly studied by various authors. For details, see [2, 4, 8, 10, 16, 18, 20] and the references therein. Most recently, the second author and Moradifam in [13] proved the one-dimensional symmetry of the solutions if the problem is considered on the torus T_ε with fundamental domain $[-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}] \times [-\frac{1}{2}, \frac{1}{2}]$ provided that $\rho \le 8\pi$. They developed a brand new tool named "the sphere covering inequality" (see [12]) to study mean field equations and related problems. In particular, if we consider the case of T^2 with the fundamental domain being a square (the case when we take $\varepsilon = 1$), the solution must be a constant. They also proved the even symmetry of the solution if the potential h is evenly symmetric when $\rho \le 8\pi$. The even symmetry of the solutions can be extended to the case $\rho \le 16\pi$ when we assume further that the solution has critical points at the origin and the diagonal point $(-1/(2\varepsilon), \frac{1}{2})$.

It is well known that the solution set of equation (1.2) is compact if $\rho \ne 8\pi\mathbb{N}$. Brezis and Merle in [3] first showed that the sequence of the solutions of the precribed Gauss curvature problem, that is,

$$\begin{aligned} -\Delta u &= he^u \quad && \text{in } \Omega \subset \mathbb{R}^2, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

is bounded locally or goes to $-\infty$ uniformly over compact subsets, or there exists a blow-up set which consists of discrete points such that the solution goes to $+\infty$ in the blow-up set and goes to $-\infty$ otherwise. Moreover, in the latter case, it is shown that $h_n e^{u_n} \to \sum_{i=1}^m \alpha_i \delta_{a_i}$. Later, Li and Shafrir in [15] obtained the exact value of α_i, that is, $\alpha_i = 8\pi$. Below, we cite a theorem in [14] concerning the blow-up solutions of equation (1.1).

Proposition 1.1. *Let u_n be a blow-up sequence of solutions of the equations*

$$\begin{aligned} &\Delta u_n + \rho_n \left(\frac{e^{u_n}}{\int_M e^{u_n}} - \frac{1}{|M|} \right) = 0, \\ &\int_M e^{u_n} = 1. \end{aligned}$$

where $\rho_n \to 8\pi m$ with $m \in \mathbb{N}$. Assume that $\max_M |u_n| \to \infty$. Then, after passing to a subsequence, there exist m distinct points $a_1, a_2, \dots, a_m$, and m sequences of points $a_i^n \to a_i$, $1 \le i \le m$, such that the following hold:

(a) *$u_n \to -\infty$ uniformly on any compact subset of $M \setminus \{a_1, a_2, \dots, a_m\}$.*
(b) *$u_n(a_i^n) \to +\infty$ for each $1 \le i \le m$.*
(c) *In $C^2_{\mathrm{loc}}(M \setminus \{a_1, a_2, \dots, a_m\})$,*

$$u_n - \overline{u_n} \to 8\pi \sum_{i=1}^m G(\cdot, a_i),$$

where $\overline{u_n}$ is the average of u_n on M and $G(x,y)$ is Green's function of the Laplace-Beltrami operator of the manifold M; that is, $G(x,y)$ satisfies the equation

$$
\begin{aligned}
-\Delta_x G(x,y) &= \delta_y - \frac{1}{|M|} && \text{in } M,\\
\int_M G(x,y)\,\mathrm{d}x &= 0 && \text{for all } y \in M.
\end{aligned}
$$

Consequently,

$$
\frac{\rho_n}{\int_M e^{u_n}} e^{u_n} \to 8\pi \sum_{i=1}^{m} \delta_{a_i} \quad \text{in the sense of measure.}
$$

In this paper, we construct blow-up solutions whose behavior is predicted by Proposition 1.1. In particular, we consider the solutions which are evenly symmetric about the x_1 and x_2 axes and blow up at exactly two points ξ_1 and ξ_2 provided that $\rho_n \to 16\pi$, that is, the case when $m = 2$. The blow-up pair (ξ_1, ξ_2) is fixed to be one of the following three choices:

$$
\begin{aligned}
\xi_1 &= (0,0), \quad \xi_2 = \left(\frac{1}{2}, 0\right);\\
\xi_1 &= (0,0), \quad \xi_2 = \left(0, \frac{1}{2}\right);\\
\xi_1 &= (0,0), \quad \xi_2 = \left(\frac{1}{2}, \frac{1}{2}\right).
\end{aligned}
$$

Let us also mention about the case $m = 1$, that is, when we have only one blow-up point. Lin and Lucia in [17] proved the following result.

Proposition 1.2. *Let T^2 be the flat torus whose fundamental domain is the unit square. For any sequence (ρ_n, u_n) with*

$$
\Delta u_n + \rho_n \left(\frac{e^{u_n}}{\int_{T^2} e^{u_n}} - 1 \right) = 0,
$$

and $u_n \not\equiv 0$ for all n, and $\rho_n \searrow 8\pi$, we have

$$
\liminf_{\rho_n \to 8\pi} \|\nabla u_n\|_{L^2} = \infty.
$$

The existence of nontrivial solutions of equation (1.1) is proved by Struwe and Tarantello [24] using the mountain pass argument when $\rho \in (8\pi, 4\pi^2)$.

Combining their result with Proposition 1.2, we can conclude that the mountain pass solutions must blow up as $\rho \to 8\pi$. Ricciardi and Tarantello in [23] showed that the nontrivial one-dimensional solutions of (1.1) exist if and only if $\rho > 4\pi^2$. Note that the construction approach also works for $m = 1$. If we fix the blow-up point at the origin and assume that the solutions are symmetric about both x_1 and x_2 axes, the same type of blow-up solutions can be obtained using the Lyapunov-type reduction as in this paper.

The construction of blow-up solutions is partially motivated by the work of Chen and Lin (see [5]). They first built a class of approximate solutions by gluing the standard solutions (also called the bubbles) and Green's function together. The locations of the bubbles are given by the critical points of a function relating h and Green's function. They also discovered the equations that govern the scales of these bubbles. Furthermore, they proved that all possible blow-up solutions should belong to the above mentioned class. Finally, they were able to compute the topological degree of equation (1.2) which turns out to be a quantity that is only dependent on the topological index of the Riemann surface M. However, in their paper they made an essential assumption, that the function

$$f_h(p_1, p_2, \dots, p_m) = \sum_j \Big[\log h(p_j) + 4\pi R(p_j) + \sum_{l \neq j} 8\pi G(p_j, p_l) \Big],$$

where $(p_1, p_2, \dots, p_m) \in M^m$ and $R(x)$ denotes the diagonal of the regular part of Green's function, is a Morse function while proving the existence of the blow-up solutions. In the case of T^2 and $h \equiv 1$, f_h is no longer a Morse function because of the translation invariance of Green's function. Thus, the existence in this case is not clear solely by the result of Chen and Lin. Esposito and Figueroa in [10] successfully generalized Chen and Lin's existence result to "stable" critical points of f_h. In their notion, a critical set D of f_h is stable if any C^1 perturbation of f_h still admits a critical point in a neighborhood of D. Not surprisingly, the critical set of $G(x, y)$ is not stable if we consider the Green's function as a function on $T^2 \times T^2$.

This paper is organized as follows. In Section 2, we state the main result and some preliminaries. In Section 3, we construct an approximate solution and get some useful estimates. Section 4 is devoted to the proof of the invertibility of the linearized operator. In Section 5, we reduce the problem to one of finding the scale of bubbles. In Section 6, we solve the reduced problem, that is, find the scale λ in terms of the parameter ρ.

2. MAIN RESULT AND PRELIMINARIES

Before we state the main result, let us first introduce Green's function $G(x, y)$ of $-\Delta$ on T^2:

$$\Delta_x G(x, y) = \delta_y - 1 \quad \text{in } T^2,$$

and

$$\int_{T^2} G(x, y)\,\mathrm{d}x = 0 \qquad \text{for all } y \in T^2.$$

We also define the regular part of Green's function to be

$$R(x, y) = G(x, y) + \frac{1}{2\pi}\log|x - y|,$$

and abuse the notation a little bit to define $R(x) = R(x, x)$ as the diagonal of $R(x, y)$.

Actually, Green's function of T^2 is sometimes written as $G(z)$ where $z = x - y$. Here, we consider x, y, z as complex numbers. We cite the explicit formula of $G(z)$ in terms of the doubly periodic function from Chen and Oshita's paper [7], as follows.

Lemma 2.1. *Green's function $G(z)$ on T^2 is given as follows:*

$$G(z) := \operatorname{Im}\left(\frac{|z|^2 - z^2}{-4i} - \frac{z}{2} + \frac{i}{12}\right) - \frac{1}{2\pi}\left|(1 - e(z)) \times \prod_{n=1}^{\infty}(1 - e(ni + z))(1 - e(ni - z))\right|,$$

where $e(z) = e^{2\pi i z}$.

Furthermore, we also have a formula for the regular part $R(z)$:

$$R(z) := -\frac{1}{2\pi}\log\left|e\left(\frac{z^2}{4i} - \frac{z}{2} + \frac{i}{12}\right)\frac{1 - e(z)}{z} \times \prod_{n=1}^{\infty}(1 - e(ni + z))(1 - e(ni - z))\right| + \frac{|z|^2}{4}.$$

Chen, Lin, and Wang [6] obtained some significant information on the critical points of $G(z)$, and we also calculate the second derivatives of $R(z)$ at $z = 0$, as follows.

Lemma 2.2. *Green's function $G(z)$ has three non-degenerate critical points: $z = \frac{1}{2}$, $z = i/2$, and $z = \frac{1}{2} + i/2$. Among them, the first two critical points are saddle points, and the third one is a minimum.*

Furthermore, we also calculate the second derivatives of the regular part $R(z)$ of Green's function, that is,

$$\nabla^2 R(z)\big|_{z=0} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} - \frac{\pi}{6} + 4\pi q & 0 \\ 0 & -\frac{1}{2} + \frac{\pi}{6} - 4\pi q \end{bmatrix},$$

where

$$q = \sum_{n=1}^{\infty} \frac{e^{-2\pi n}}{(1-e^{-2\pi n})^2}.$$

Remark 2.3. Chen, Lin, and Wang [6] also showed that Green's function $G(z)$ is evenly symmetric about the x_1 and x_2 axes, and so also is the regular part $R(z)$. Thus, we have $\nabla R(z)|_{z=0} = \mathbf{0}$.

We seek for blow-up solutions that are evenly symmetric about x_1 and x_2 axes as $\rho \to 16\pi$.

Theorem 2.4. *Let $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$ small enough. Let $\rho = 16\pi + \varepsilon$. Assume that $\xi_1 = (0,0)$ and $\xi_2 = (\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ or $(\frac{1}{2}, \frac{1}{2})$. Then, for each ε, there exist a $\lambda > 0$ and a solution u_λ to equation (1.1) such that the following hold:*

$$\begin{aligned} &\varepsilon = (128\pi^2 + o(1))\lambda^2 \ln \frac{1}{\lambda}, \\ &u_\lambda(\xi_j) \to \infty \quad \text{for } j = 1,2, \\ &u_\lambda(x) \to -\infty \quad \text{for all } x \in T^2 \setminus \{\xi_1, \xi_2\} \text{ as } \varepsilon \to 0, \end{aligned}$$

and

$$u_\lambda(x_1, x_2) = u_\lambda(-x_1, x_2) = u_\lambda(x_1, -x_2).$$

Moreover, we have

$$\frac{\rho}{\int_{T^2} e^{u_\lambda}} e^{u_\lambda} \to 8\pi(\delta_{\xi_1} + \delta_{\xi_2}) \quad \text{in a sense of measure, as } \varepsilon \to 0.$$

Remark 2.5. Note that the blow-up solutions we found in the case of $\xi_2 = (\frac{1}{2}, 0)$ have the following property:

$$u_\lambda\left(x_1 + \frac{1}{2}, x_2\right) = u_\lambda(x_1, x_2).$$

In other words, the solutions become solutions of the mean field equation on the flat torus with fundamental domain $[-\frac{1}{4}, \frac{1}{4}] \times [-\frac{1}{2}, \frac{1}{2}]$, which blow up only at the origin. The same thing happens when $\xi_2 = (0, \frac{1}{2})$. When we take $\xi_2 = (\frac{1}{2}, \frac{1}{2})$, the solutions we constructed also correspond to the one-point blow-up solutions on a flat torus whose fundamental domain is a tilted square with length $\sqrt{2}/2$, since in this case we have

$$u_\lambda(x_1, x_2) = u_\lambda\left(x_1 + \frac{1}{2}, x_2 + \frac{1}{2}\right).$$

Remark 2.6. The same type of construction will also work for T_ε where ε is arbitrarily given. Much as in Remark 2.5, one can show the existence of the one-point blow-up solutions to (1.1) when T^2 is replaced by any rhombus torus (i.e., flat torus with its fundamental domain being a rhombus) as $\rho \to 8\pi$. We also would like to mention that Lin and Wang in [19] established the existence and uniqueness of solutions to a mean field equation with singular source on rhombus tori.

The proof of Theorem 2.4 relies on a Lyapunov-type reduction. We first construct an approximate solution which behaves like the standard bubble near the blow-up points ξ_1 and ξ_2 and behaves like Green's function away from these two points. Then, we carry out a finite dimensional variational reduction for which the main ingredient is an analysis, of independent interest, of bounded invertibility up to the dilations of the linearized operator in suitable L^∞-weighted spaces with certain symmetries. This method consequently reduces the original problem to a problem of finding an appropriate scale λ of the bubbles.

3. An Approximate Solution

In this section, we will construct an approximate solution of equation (1.1) and obtain some estimates of this approximate solution. Let $R_0 > 0$ be a fixed number such that $R_0 < \frac{1}{8}$. Let η be a standard cut-o function such that

$$\begin{aligned} &\eta(s) = 1 && \text{for } s \le 1; \\ &\eta(s) = 0 && \text{for } s \ge 2; \\ &0 < \eta(s) < 1 && \text{for } 1 < s < 2. \end{aligned}$$

We further assume that $|\eta'(s)| \le 2$. Let

$$\eta_{R_0,\xi}(x) = \eta\left(\frac{\operatorname{dist}(x,\xi)}{R_0}\right) \tag{3.1}$$

where $\operatorname{dist}(x,\xi)$ denotes the geodesic distance between x and ξ.

Given $\varepsilon \in (0, \varepsilon_0)$, we choose $\lambda > 0$ such that

$$64\pi^2\lambda^2 \ln\frac{1}{\lambda} < \varepsilon < 256\pi^2\lambda^2 \ln\frac{1}{\lambda}. \tag{3.2}$$

In other words, the above inequality can also be written as

$$\lambda_1(\varepsilon) < \lambda < \lambda_2(\varepsilon),$$

where one can solve $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ from (3.2).

Let $w_{\lambda,1}$ be the solution of the following equation:

$$\begin{aligned} &-\Delta w_{\lambda,1} = \frac{8\lambda^2}{(\lambda^2 + (\operatorname{dist}(x,\xi_1))^2)^2}\eta_{R_0,\xi_1} - m_1, \\ &\int_{T^2} w_{\lambda,1} = 0, \end{aligned}$$

where

$$m_1 = \int_{T^2} \frac{8\lambda^2}{(\lambda^2 + (\mathrm{dist}(x, \xi_1))^2)^2} \eta_{R_0,\xi_1}.$$

Let $w_{\lambda,2}$ be the solution of the following equation:

$$\begin{aligned} -\Delta w_{\lambda,2} &= \frac{8\lambda^2}{(\lambda^2 + (\mathrm{dist}(x, \xi_2))^2)^2} \eta_{R_0,\xi_2} - m_2, \\ \int_{T^2} w_{\lambda,2} &= 0, \end{aligned}$$

where

$$m_2 = \int_{T^2} \frac{8\lambda^2}{(\lambda^2 + (\mathrm{dist}(x, \xi_2))^2)^2} \eta_{R_0,\xi_2}.$$

By simple calculations, one can obtain the so-called "masses" of $w_{\lambda,1}$ and $w_{\lambda,2}$, that is,

$$m_1 = m_2 = 8\pi + O(\lambda^2).$$

We introduce $\tilde{w}_\lambda$ to be the sum of $w_{\lambda,1}$ and $w_{\lambda,2}$, that is, $\tilde{w}_\lambda = w_{\lambda,1} + w_{\lambda,2}$, and a constant related to λ

$$\bar{w}_\lambda = 2\ln\lambda + \ln 8 - 8\pi R(\xi, \xi) - 8\pi G(\xi_1, \xi_2).$$

Then, we are ready to provide an ansatz for solutions of equation (1.1): namely, $w_\lambda = \tilde{w}_\lambda + \bar{w}_\lambda$.

We then calculate the values of $w_{\lambda,1}$ and w_{λ_2} at the blow-up points ξ_1 and ξ_2:

$$\begin{aligned} (3.3) \quad w_{\lambda,1}(\xi_1) &= \int_{T^2} G(\xi_1, y) \left[\frac{8\lambda^2}{(\lambda^2 + |y - \xi_1|^2)^2} \eta_{R_0,\xi_1}(y) - m_1 \right] \mathrm{d}y \\ &= \int_{B(0,R_0)} \left[-\frac{1}{2\pi} \ln|y| + R(0, y) \right] \frac{8\lambda^2}{(\lambda^2 + |y|^2)^2} \,\mathrm{d}y + O(\lambda^2) \\ &= \int_{B(0,R_0/\lambda)} \left[-\frac{1}{2\pi} \ln\lambda - \frac{1}{2\pi} \ln|z| + R(\lambda z) \right] \frac{8}{(1 + |z|^2)^2} \,\mathrm{d}z + O(\lambda^2) \\ &= -4\ln\lambda + \frac{4\lambda^2 \ln\lambda}{\lambda^2 + R_0^2} - \frac{1}{2\pi} \int_{B(0,R_0/\lambda)} \frac{8\ln|z|}{(1 + |z|^2)^2} \,\mathrm{d}z + 8\pi R(0) \\ &\quad + \lambda^2 \int_{B(0,R_0/\lambda)} \frac{4(z\nabla^2 R(z)|_{z=0} z^T)}{(1 + |z|^2)^2} \,\mathrm{d}z + O(\lambda^2) \\ &= -4\ln\lambda + 8\pi R(0) - 4\pi\lambda^2 \ln\lambda + O(\lambda^2). \end{aligned}$$

For $|z| < R_0/\lambda$, we have

$$
\begin{aligned}
(3.4)\quad & w_{\lambda,1}(\xi_1+\lambda z) - w_{\lambda,1}(\xi_1) \\
&= \int_{T^2} [G(\xi_1+\lambda z, y) - G(\xi_1, y)] \frac{8\lambda^2}{(\lambda^2+|y-\xi_1|^2)^2} \eta_{R_0,\xi_1}\, \mathrm{d}y \\
&= \int_{B(0,R_0/\lambda)} -\frac{1}{2\pi}[\ln|z-z'| - \ln|z'|] \frac{8}{(1+|z'|^2)^2}\, \mathrm{d}z' \\
&\quad + \int_{B(0,R_0/\lambda)} [R(\lambda(z'-z)) - R(\lambda z')] \frac{8}{(1+|z'|^2)^2}\, \mathrm{d}z' + O(\lambda^3|z|) \\
&= \ln\left(\frac{1}{(1+|z|^2)^2}\right) + 2\pi\lambda^2(z_1^2+z_2^2) - 4\pi\lambda^2\left(\frac{1}{2} - \frac{\pi}{6} + 4\pi q\right)(z_1^2 - z_2^2) \\
&\quad + O(\lambda^3|z|^3) + O(\lambda^3 \ln\lambda |z|) + O(\lambda^3|z|).
\end{aligned}
$$

For $|z| \geq 2R_0/\lambda$, that is, $|x| \geq 2R_0$, we have

$$
\begin{aligned}
(3.5)\quad w_{\lambda,1}(x) &= \int_{T^2} G(x,x') \frac{8\lambda^2}{(\lambda^2+|x'|^2)^2} \eta_{R_0,\xi_1}\, \mathrm{d}x' \\
&= \int_{B(0,R_0/\lambda)} G(x, \xi_1+\lambda z') \frac{8}{(1+|z'|)^2}\, \mathrm{d}z' + \lambda^2 f_1(x) \\
&= \int_{B(0,R_0/\lambda)} \Big[G(\xi_1 - x) + \lambda \nabla G(\xi_1 - x)\cdot z' \\
&\quad + \frac{\lambda^2(z'\nabla^2 G(\xi_1-x) z'^T)}{2} \Big] \frac{8}{(1+|z'|^2)^2}\, \mathrm{d}z' + \lambda^2 f_1(x) \\
&= 8\pi G(\xi_1 - x) + 4\pi\lambda^2 \int_0^{R_0/\lambda} \mathrm{Tr}(\nabla^2 G(\xi_1 - x)) \frac{r^3}{(1+r^2)^2}\, \mathrm{d}r + \lambda^2 f_1(x) \\
&= 8\pi G(\xi_1 - x) - 4\pi\lambda^2 \ln\lambda + \lambda^2 f_1(x),
\end{aligned}
$$

where $f_1(x)$ is a C^1 function.

In particular, we can get

$$
w_{\lambda,1}(\xi_2) = 8\pi G(\xi_1 - \xi_2) - 4\pi\lambda^2 \ln\lambda + O(\lambda^2).
$$

Combining (3.3) and (3.4), we have for $|z| < R_0/\lambda$

$$
\begin{aligned}
(3.6)\quad w_{\lambda,1}(\xi_1+\lambda z) = {}& -4\ln\lambda + \ln\left(\frac{1}{(1+|z|^2)^2}\right) + 8\pi R(0) + 2\pi\lambda^2|z|^2 \\
& - 4\pi\lambda^2\left(\frac{1}{2} - \frac{\pi}{6} + 4\pi q\right)(z_1^2 - z_2^2) - 4\pi\lambda^2\ln\lambda \\
& + O(\lambda^3|z|^3) + O(\lambda^3|z|) + O(\lambda^3\ln\lambda|z|).
\end{aligned}
$$

Similarly, we can obtain the same result for $w_{\lambda,2}$ when $|z| < R_0/\lambda$ if we now shift the center of the coordinate system to ξ_2:

$$w_{\lambda,2}(\xi_2 + \lambda z) = -4\ln\lambda + \ln\left(\frac{1}{(1+|z|^2)^2}\right) + 8\pi R(0) + 2\pi\lambda^2|z|^2 - 4\pi\lambda^2\left(\frac{1}{2} - \frac{\pi}{6} + 4\pi q\right)(z_1^2 - z_2^2) - 4\pi\lambda^2\ln\lambda + O(\lambda^3|z|^3) + O(\lambda^3|z|) + O(\lambda^3\ln\lambda|z|).$$

We also have a result similar to (3.5) for $w_{\lambda,2}$ when $\operatorname{dist}(x, \xi_2) \geq 2R_0$,

$$w_{\lambda,2}(x) = 8\pi G(\xi_2 - x) - 4\pi\lambda^2\ln\lambda + \lambda^2 f_2(x), \tag{3.7}$$

where $f_2(x)$ is a C^1 function when $\operatorname{dist}(x, \xi_2) \geq 2R_0$.

In particular, we have $w_{\lambda,2}(\xi_1) = w_{\lambda,1}(\xi_2)$.

Thus, we obtain the following lemma concerning the values of w_λ near the blow-up points ξ_1 and ξ_2.

Lemma 3.1. *We have an inner approximation*

$$w_\lambda(\xi_k + \lambda z) = \ln\left(\frac{8}{\lambda^2(1+|z|^2)^2}\right) + 2\pi\lambda^2|z|^2 - 4\pi\lambda^2\left(\frac{1}{2} - \frac{\pi}{6} + 4\pi q\right)(z_1^2 - z_2^2) + 4\pi\lambda^2(z\nabla^2 G(\xi_2 - \xi_1)z^T) - 8\pi\lambda^2\ln\lambda + O(\lambda^3|z|^3) + O(\lambda^3\ln\lambda|z|) + O(\lambda^3|z|),$$

of w_λ inside the ball $z \in B(0, R_0/\lambda)$, and the above equation holds for both blow-up points if we set the center to ξ_k for $k = 1, 2$.

Remark 3.2. Note that here we have

$$\operatorname{Tr}(\nabla^2 G(\xi_2 - \xi_1)) = \Delta G(\xi_2 - \xi_1) = 1.$$

We then give an outer approximation, as follows.

Lemma 3.3. *When* $\min(\operatorname{dist}(x, \xi_1), \operatorname{dist}(x, \xi_2)) \geq 2R_0$, *we have*

$$w_\lambda(x) = 2\ln\lambda + \ln 8 - 8\pi R(0) - 8\pi G(\xi_1 - \xi_2) + 8\pi G(\xi_1 - x) + 8\pi G(\xi_2 - x) - 8\pi\lambda^2\ln\lambda + \lambda^2(f_1(x) + f_2(x)).$$

Comparing the approximate solution with the function constructed by gluing the inner approximation and the outer approximation together using an "intermediate layer" (a cut-o function such as η_{λ^α} for some $\alpha \in (0,1)$), we can estimate the approximate solution when $R_0 < \operatorname{dist}(x, \xi_k) < 2R_0$ as well as e^{w_λ}. (For details, see Lemma 3.1 in [11].) In particular, we have

$$e^{w_\lambda} \le \sum_{k=1}^{2} \frac{8\lambda^2}{(\lambda^2 + (\operatorname{dist}(x, \xi_k))^2)^2}[1 + \theta_\lambda(x)],$$

where θ_λ has the property that for some constant $C > 0$,

$$|\theta_\lambda(x)| \le C\lambda \sum_{k=1}^{2} \left[\frac{\operatorname{dist}(x, \xi_k)}{\lambda} + 1\right].$$

More precisely, when $|z| < R_0/\lambda$, we have

$$\begin{aligned} e^{w_\lambda(\xi_k+\lambda z)} = \frac{8}{\lambda^2(1+|z|^2)^2}\Big[& 1 + 2\pi\lambda^2|z|^2 \\ & - 4\pi\lambda^2\left(\frac{1}{2} - \frac{\pi}{6} + 4\pi q\right)(z_1^2 - z_2^2) \\ & + 4\pi\lambda^2(z\nabla^2 G(\xi_2 - \xi_1)z^T) - 8\pi\lambda^2 \ln\lambda \\ & + O(\lambda^3|z|^3) + O(\lambda^3 \ln\lambda|z|) + O(\lambda^3|z|)\Big]. \end{aligned} \tag{3.8}$$

When $\operatorname{dist}(x, \xi_k) \ge R_0$ for $k = 1, 2$, we have

$$e^{w_\lambda(x)} = O(\lambda^2). \tag{3.9}$$

Let us then estimate the error of the approximate solution by inserting the ansats w_λ into equation (1.1).

Lemma 3.4. *Let*

$$S_\rho(u) = \Delta u + \rho\left(\frac{e^u}{\int_{T^2} e^u} - 1\right).$$

Then, there exists a constant $C > 0$ such that

$$|S_\rho(w_\lambda)(\xi_k + \lambda z)| \le C\left[\lambda^2 \ln\frac{1}{\lambda} + \frac{\ln(1/\lambda)}{(1+|z|^2)^2} + \frac{|z|^2}{(1+|z|^2)^2}\right],$$

for $|z| < R_0/\lambda$ and $k = 1, 2$,

$$|S_\rho(w_\lambda)(x)| \le C\lambda^2 \ln\frac{1}{\lambda},$$

for $\operatorname{dist}(x, \xi_k) \ge R_0$ and $k = 1, 2$.

Furthermore, we have that $S_\rho(w_\lambda)$ is evenly symmetric about both axes.

Proof. We first use (3.8) and (3.9) to estimate the integral of e^{w_λ}, that is,

$$
\begin{aligned}
(3.10)\quad \int_{T^2} e^{w_\lambda} &= 2\int_{B(\xi_1,R_0)} e^{w_\lambda} + \int_{T^2\setminus(B(\xi_1,R_0)\cup B(\xi_2,R_0))} e^{w_\lambda} \\
&= 2\int_{B(0,R_0/\lambda)} \frac{8}{(1+|z|^2)^2}\Big[1 + 2\pi\lambda^2|z|^2 \\
&\quad - 4\pi\lambda^2\left(\frac{1}{2} - \frac{\pi}{6} + 4\pi q\right)(z_1^2 - z_2^2) \\
&\quad + 4\pi\lambda^2(z\nabla^2 G(\xi_2-\xi_1)z^T) - 8\pi\lambda^2\ln\lambda + O(\lambda^3|z|^3) \\
&\quad + O(\lambda^3|z|^3) + O(\lambda^3\ln\lambda|z|) + O(\lambda^2)\Big] \\
&\quad + \int_{T^2\setminus(B(\xi_1,R_0)\cup B(\xi_2,R_0))} e^{w_\lambda} \\
&= 2(8\pi - 128\pi^2\lambda^2\ln\lambda + O(\lambda^2)),
\end{aligned}
$$

where we use Remark 2.3 and Remark 3.2.

When $|z| < R_0/\lambda$, we have

$$
\begin{aligned}
&S_\rho(w_\lambda)(\xi_k + \lambda z) \\
&\quad = \Delta w_\lambda(\xi_k+\lambda z) + \rho\left(\frac{e^{w_\lambda(\xi_k+\lambda z)}}{16\pi - 256\pi^2\lambda^2\ln\lambda + O(\lambda^2)} - 1\right) \\
&\quad = 16\pi + O(\lambda^2) - \frac{8}{\lambda^2(1+|z|^2)^2} \\
&\qquad + \frac{(16\pi+\varepsilon)e^{w_\lambda(\xi_k+\lambda z)}}{16\pi - 256\pi^2\lambda^2\ln\lambda + O(\lambda^2)} - (16\pi+\varepsilon) \\
&\quad = -\varepsilon + O(\lambda^2) + \frac{\varepsilon + 256\pi^2\lambda^2\ln\lambda + O(\lambda^2)}{16\pi - 256\pi^2\lambda^2\ln\lambda + O(\lambda^2)}\cdot\frac{8}{\lambda^2(1+|z|^2)^2} \\
&\qquad + O\left(\frac{\ln(1/\lambda)}{(1+|z|^2)^2}\right) + O\left(\frac{|z|^2}{(1+|z|^2)^2}\right).
\end{aligned}
$$

We know from (3.2) that $\varepsilon = O(\lambda^2\ln\lambda)$; then, we have

$$
|S_\rho(w_\lambda)(\xi_k+\lambda z)| \le C\left[\lambda^2\ln\frac{1}{\lambda} + \frac{\ln(1/\lambda)}{(1+|z|^2)^2} + \frac{|z|^2}{(1+|z|^2)^2}\right]
$$

for $|z| < R_0/\lambda$ and $k = 1, 2$.

Similarly, using (3.10) we can estimate the outer error:

$$
S_\rho(w_\lambda)(x) = -\varepsilon + O(\lambda^2) + \frac{16\pi+\varepsilon}{16\pi + O(\lambda^2\ln\lambda)}O(\lambda^2)
$$

since (3.9) holds for all $\mathrm{dist}(x,\xi_k) \ge R_0$ and $k = 1, 2$.

The rest of the lemma follows from the last identity. ◻

Equation (1.1) has a variational structure; that is, critical points of the energy functional

$$J_\rho(u) = \frac{1}{2}\int_{T^2} |\nabla u|^2 - \rho \ln\left(\int_{T^2} e^u\right) + \rho \int_{T^2} u$$

correspond to the solutions of equation (1.1). Our next goal is to estimate the energy functional of the approximate solution w_λ.

Lemma 3.5. *The energy of w_λ is*

$$\begin{aligned} J_\rho(w_\lambda) = &-64\pi^2[R(0) + G(\xi_1 - \xi_2)] - 16\pi \ln(2\pi) \\ &- 16\pi + 2\varepsilon \ln \lambda + 128\pi^2\lambda^2 \ln \lambda \\ &- \varepsilon[\ln(2\pi) - 8\pi R(0) - 8\pi G(\xi_1 - \xi_2)] + O(\lambda^2). \end{aligned}$$

Proof. From (3.10), we can compute

$$\begin{aligned} -\rho \ln\left(\int_{T^2} e^{w_\lambda}\right) &= -(16\pi + \varepsilon)\ln(16\pi - 256\pi^2\lambda^2 \ln \lambda + O(\lambda^2)) \\ &= -16\pi(\ln(16\pi) - 16\pi\lambda^2 \ln \lambda + O(\lambda^2)) \\ &\quad - \varepsilon(\ln(16\pi) - 16\pi\lambda^2 \ln \lambda + O(\lambda^2)). \end{aligned} \tag{3.11}$$

Also, we can easily compute

$$\begin{aligned} \rho \int_{T^2} w_\lambda &= (16\pi + \varepsilon)\bar{w}_\lambda \\ &= 16\pi(2\ln \lambda + \ln 8 - 8\pi R(0) - 8\pi G(\xi_1 - \xi_2)) \\ &\quad + \varepsilon(2\ln \lambda + \ln 8 - 8\pi R(0) - 8\pi G(\xi_1 - \xi_2)). \end{aligned} \tag{3.12}$$

Then, the only term remaining is the following:

$$\begin{aligned} &\frac{1}{2}\int_{T^2} |\nabla w_\lambda|^2 = \frac{1}{2}\langle -\Delta w_\lambda, w_\lambda\rangle = \frac{1}{2}\langle -\Delta w_\lambda, \tilde{w}_\lambda\rangle \\ &= \frac{1}{2}\int_{T^2}\left[\frac{8\lambda^2}{(\lambda^2 + |y - \xi_1|^2)^2}\eta_1 + \frac{8\lambda^2}{(\lambda^2 + |y - \xi_2|)^2}\eta_2\right](w_{\lambda,1} + w_{\lambda,2})\,\mathrm{d}y \\ &= \int_{T^2} \frac{8\lambda^2}{(\lambda^2 + |y - \xi_1|)^2}\eta_1 w_{\lambda,1}\,\mathrm{d}y + \int_{T^2} \frac{8\lambda^2}{(\lambda^2 + |y - \xi_1|)^2}\eta_1 w_{\lambda,2}\,\mathrm{d}y \\ &= J_1 + J_2, \end{aligned} \tag{3.13}$$

where we denote η_{R_0,ξ_k} as η_k for simplicity, $k = 1, 2$.

Let us use (3.6) to compute J_1 first:

$$\begin{aligned}
J_1 &= \int_{B(0,R_0/\lambda)} \frac{8w_{\lambda,1}(\xi_1+\lambda z)}{(1+|z|^2)^2}\,dz + O(\lambda^2)\\
&= [-4\ln\lambda + 8\pi R(0) - 4\pi\lambda^2\ln\lambda + O(\lambda^2)]\int_{B(0,R_0/\lambda)} \frac{8}{(1+|z|^2)^2}\,dz\\
&\quad + 2\pi\lambda^2\int_{B(0,R_0/\lambda)} \frac{8|z|^2}{(1+|z|^2)^2}\,dz\\
&\quad - 2\int_{B(0,R_0/\lambda)} \ln(1+|z|^2)\frac{8}{(1+|z|^2)^2}\,dz + O(\lambda^2)\\
&= -32\pi\ln\lambda + 64\pi^2R(0) - 16\pi - 64\pi^2\lambda^2\ln\lambda + O(\lambda^2).
\end{aligned}$$

Then, we use (3.7) to compute J_2:

$$\begin{aligned}
J_2 &= \int_{B(0,R_0/\lambda)} \frac{8w_{\lambda,2}(\xi_1+\lambda z)}{(1+|z|^2)^2}\,dz + O(\lambda^2)\\
&= \int_{B(0,R_0/\lambda)} \frac{8(8\pi G(\xi_2-\xi_1-\lambda z) - 4\pi\lambda^2\ln\lambda + \lambda^2 f_2(\xi_1+\lambda z)}{(1+|z|^2)^2}\,dz + O(\lambda^2)\\
&= 8\pi G(\xi_2-\xi_1)\int_{B(0,R_0/\lambda)} \frac{8}{(1+|z|^2)^2}\,dz\\
&\quad + 4\pi\lambda^2\int_{B(0,R_0/\lambda)} \frac{8(z\nabla^2G(\xi_2-\xi_1)z^T)}{(1+|z|^2)^2}\,dz - 32\pi^2\lambda^2\ln\lambda + O(\lambda^2)\\
&= 64\pi^2G(\xi_2-\xi_1) - 64\pi^2\lambda^2\ln\lambda + O(\lambda^2).
\end{aligned}$$

Therefore, by (3.11), (3.12), and (3.13) we have

$$\begin{aligned}
J_\rho(w_\lambda) = -64\pi^2[R(0) + G(\xi_1-\xi_2)] &- 16\pi\ln(2\pi)\\
&- 16\pi + 2\varepsilon\ln\lambda + 128\pi^2\lambda^2\ln\lambda\\
&- \varepsilon[\ln(2\pi) - 8\pi R(0) - 8\pi G(\xi_1-\xi_2)] + O(\lambda^2). \qquad \square
\end{aligned}$$

4. THE LINEARIZED OPERATOR

In this section, we will establish a solvability theory for the linearized operator under a suitable orthogonality condition.

Let us introduce an operator

$$\mathcal{L}(u) = \Delta u + \frac{\rho}{\int_{T^2} e^{w_\lambda}} e^{w_\lambda} u.$$

The above operator is connected with the linearized operator of S_ρ through the following:

$$S'_\rho(w_\lambda)(u) = \mathcal{L}\left(u - \frac{\int_{T^2} e^{w_\lambda} u}{\int_{T^2} e^{w_\lambda}}\right).$$

Let

$$L(u) = \lambda^2 \mathcal{L}(u).$$

If we shift the center of the coordinate system to ξ_k and blow up the torus T^2 by the scale λ to T^2_λ, then the linearized operator $\mathcal{L}$ scaled by λ^2 formally approaches a linear operator $\tilde{L}$ in $\mathbb{R}^2$, that is,

$$\tilde{L}(u) = \Delta_z u + \frac{8}{(1+|z|^2)^2} u,$$

where $z = (x - \xi_k)/\lambda$.

The operator $\tilde{L}$ can be obtained by linearizing the equation $\Delta u + e^u = 0$ at the radial solution $v(z) = \ln(8/(1+|z|^2)^2)$. An important fact we are going to exploit in developing the solvability theory is the non-degeneracy of v modulo the invariance of the equations under translations and dilations, that is,

$$\zeta \mapsto v(z - \zeta); \quad s \mapsto v\left(\frac{z}{s}\right) - 2\ln s.$$

Thus, we set

$$\begin{aligned} \varphi_k(z) &= \frac{\partial}{\partial \zeta_k} v(z+\zeta)\big|_{\zeta=0}, \quad k = 1, 2, \\ \varphi_0(z) &= \frac{\partial}{\partial s}\left[v\left(\frac{z}{s}\right) - 2\ln s\right]\bigg|_{s=1}. \end{aligned}$$

Direct computation shows that

$$\varphi_k = \frac{-4z_k}{1+|z|^2}, \quad \text{for } k = 1, 2$$

and

$$\varphi_0 = \frac{2(|z|^2 - 1)}{1+|z|^2}.$$

It is shown that the only bounded solutions of $\tilde{L}(u) = 0$ in $\mathbb{R}^2$ are precisely the linear combinations of the φ_k, $k = 0, 1, 2$ (see Baraket and Pacard's paper [1] for a detailed proof). Let us define

$$\varphi_{i,j} := \varphi_i\left(\frac{x - \xi_j}{\lambda}\right)$$

as a function on T_λ^2 without ambiguity, $i = 0, 1, 2$ and $j = 1, 2$.

Moreover, let us pick a large but fixed number $R_1 > 0$. We introduce another type of cut-o function:

$$\begin{aligned} \chi_R(s) &= 1 \quad \text{for } s \le R, \\ \chi_R(s) &= 0 \quad \text{for } s \ge R+1, \\ 0 < \chi_R &< 1 \quad \text{for } R < s < R+1. \end{aligned}$$

We further assume that

$$|\chi_R'(s)| \le 2.$$

Let $\chi_{R,\xi}(z) = \chi_R(|z-\xi|)$. Denote ξ_j/λ as $\xi_j^{'}$ for short.

Next, let us introduce some functional set-ups of the problem.

Let

$$C_s^{k,\alpha}(T_\lambda^2) = \{u \in C^{k,\alpha}(T_\lambda^2) \mid u(z_1, z_2) = u(z_1, -z_2) = u(-z_1, z_2)\},$$

where k is any positive integer and $\alpha \in (0,1)$, and one can also define $L_s^p(T_\lambda^2)$, $1 \le p \le \infty$, respectively.

We consider the following norms:

$$\|\psi\|_\infty = \sup_{z \in T_\lambda^2} |\psi(z)|,$$

$$\|\psi\|_* = \sup_{z \in T_\lambda^2} \Big(\sum_{j=1}^{2} (1 + \operatorname{dist}(z, \xi_j'))^{-3} + \lambda^2 \Big)^{-1} |\psi(z)|.$$

Let

$$C = \{u \in L^\infty(T_\lambda^2) \mid u(z_1, z_2) = u(z_1, -z_2) = u(-z_1, z_2),\ \|u\|_* < \infty\},$$

and

$$C_* = \Big\{u \in L^\infty(T_\lambda^2) \mid u(z_1, z_2) = u(z_1, -z_2) = u(-z_1, z_2), \\ \|u\|_* < \infty,\ u \perp \varphi_{0,j}\chi_{R_1,\xi_j'}\Big\}.$$

Given $h \in C$, we consider the linear problem of finding a function $\phi \in C_*$ and scalars c_j, $j = 1, 2$ such that

$$L(\phi) = h + \sum_{j=1}^{2} c_j \chi_{R_1,\xi_j'} \varphi_{0,j} \quad \text{in } T_\lambda^2. \tag{4.1}$$

We observe that the orthogonality condition in the definition of C_* is only with respect to the approximate kernel generated by dilation. Furthermore, we can easily find that the elements in C_* are also perpendicular to the approximate kernels that are generated by translations, that is,

$$u \perp \varphi_{i,j}\chi_{R_1,\xi_j'}, \quad \text{for all } i = 0,1,1 \text{ and } j = 1,2, \ u \in C_*.$$

The main result in this section shows the solvability of (4.1) and an *a priori* estimate which is uniform in small λ in the functional settings of the enlarged flat torus T_λ^2.

Proposition 4.1. *There exist positive constants λ_0 and C, such that, for any $\lambda \in (0,\lambda_0)$, there is a unique solution to problem* (4.1). *Moreover, if $h \in C^\alpha(T_\lambda^2)$ then*

$$\|\phi\|_\infty \le C\|h\|_*. \tag{4.2}$$

The proof of this result consists of two steps. The first step is to establish a uniform *a priori* estimate for the problem (4.1) under the additional orthogonality conditions of ϕ generated by translations. More precisely, we consider the problem

$$L(\phi) = h \qquad \text{in } T_\lambda^2, \tag{4.3}$$

$$\int_{T_\lambda^2} \chi_{R_1,\xi_j'}\varphi_{i,j}\phi = 0 \quad \text{for all } i = 0,1,2, \ j = 1,2. \tag{4.4}$$

Lemma 4.2. *Assume that $h \in C^\alpha(T_\lambda^2)$. Then, there exist positive numbers λ_0 and C, such that for any $\lambda \in (0,\lambda_0)$ and any solution to* (4.3)–(4.4)*, one has*

$$\|\phi\|_\infty \le C\|h\|_*.$$

Proof. We will adopt the same technique introduced by del Pino, Kowalczyk, and Musso in their paper [22] to prove the invertibility of the linearized operator of the mean field equation in bounded domain but with Dirichlet boundary condition.

We prove this lemma by contradiction. Assume then there exist sequences $\lambda_n \to 0$, h_n with $\|h_n\|_* \to 0$ and $\|\phi_n\|_\infty = 1$ such that

$$L(\phi_n) = h_n \qquad \text{in } T_\lambda^2,$$

$$\int_{T_{\lambda_n}^2} \chi_{R_1,\xi_j'}\varphi_{i,j}\phi_n = 0 \quad \text{for all } i = 0,1,2, j = 1,2.$$

The contradiction is obtained via several major steps. The key step is to construct a positive supersolution in order to show that the operator L satisfies the maximum principle in T_λ^2 outside large balls centered at the points ξ_j'. Let us introduce a sort of "projection" of the radial solution $f_0(r) = (r^2-1)/(r^2+1)$ in $\mathbb{R}^2$ of

$$\Delta f_0 + \frac{8}{(1+r^2)^2} f_0 = 0,$$

onto a proper functional space on T_λ^2, that is, to some function $\tilde f(z)$ such that

$$\begin{aligned} -\Delta_z \tilde f &= \frac{8}{(1+|z|^2)^2} f_0(|z|) - \lambda^2 m, \\ \int_{T_\lambda^2} \tilde f(z)\,\mathrm{d}z &= 0, \\ m &= \int_{T_\lambda^2} \frac{8}{(1+|z|^2)^2} f_0(|z|)\,\mathrm{d}z, \end{aligned}$$

where $z = x/\lambda \in T_\lambda^2$.

One can estimate the mass m by direct computations

$$\begin{aligned} &\int_{B(0,1/(2\lambda))} \frac{8}{(1+|z|^2)^2} f_0(|z|)\,\mathrm{d}z \\ &\qquad \le m \le \int_{B(0,\sqrt{2}/(2\lambda))} \frac{8}{(1+|z|^2)^2} f_0(|z|)\,\mathrm{d}z. \end{aligned} \tag{4.5}$$

Then, it can be easily seen from (4.5) that $m < 0$ and $m = O(\lambda^2)$.

To construct a positive supersolution, we need to show that the function $\tilde f(z)$ is uniformly bounded. Let us transfer the function back into a function on T^2. Using the same notation here without ambiguity, we know that the function $\tilde f(x)$ satisfies the equations

$$-\Delta_x \tilde f = \frac{8\lambda^2}{(\lambda^2+|x|^2)^2} \cdot \frac{|x/\lambda|^2 - 1}{|x/\lambda|^2+1} - m \quad \text{in } T^2$$

and

$$\int_{T^2} \tilde f(x)\,\mathrm{d}x = 0.$$

Then, for any $x \in T^2$, we can compute the value of $\tilde f(x)$:

$$\begin{aligned} \tilde f(x) &= \int_{T^2} G(x,y) \frac{8\lambda^2}{(\lambda^2+|y|^2)^2} \cdot \frac{|y/\lambda|^2-1}{|y/\lambda|^2+1}\,\mathrm{d}y \\ &= \int_{T^2} -\frac{1}{2\pi} \ln|x-y| \frac{8\lambda^2}{(\lambda^2+|y|^2)^2} \cdot \frac{|y/\lambda|^2-1}{|y/\lambda|^2+1}\,\mathrm{d}y \\ &\qquad + \int_{T^2} R(x,y) \frac{8\lambda^2}{(\lambda^2+|y|^2)^2} \cdot \frac{|y/\lambda|^2-1}{|y/\lambda|^2+1}\,\mathrm{d}y \\ &= I_1 + I_2. \end{aligned}$$

For the estimation of I_2, we know that $R(x,y)$ and $(|y/\lambda|^2-1)/(|y/\lambda|^2+1)$ are both uniformly bounded for all $x, y \in T^2$. Therefore, we have

$$|I_2| \le C \int_{T^2} \frac{8\lambda^2}{(\lambda^2+|y|^2)^2}\,\mathrm{d}y \le C. \tag{4.6}$$

However, it is a little more technical to estimate I_1:

$$(4.7)\qquad \begin{aligned} I_1 = &\int_{B(0,R_0)} -\frac{1}{2\pi}\ln|x-y|\frac{8\lambda^2}{(\lambda^2+|y|^2)^2}\cdot\frac{|y/\lambda|^2-1}{|y/\lambda|^2+1}\,\mathrm{d}y \\ &+\int_{T^2\setminus B(0,R_0)} -\frac{1}{2\pi}\ln|x-y|\frac{8\lambda^2}{(\lambda^2+|y|^2)^2}\cdot\frac{|y/\lambda|^2-1}{|y/\lambda|^2+1}\,\mathrm{d}y. \end{aligned}$$

We know that the latter part in (4.7) can be bounded by $C\lambda^2$ for some constant C, since the singularity of singular part of Green's function is somewhat mild, and the integral

$$\int_{T^2} |\ln|x-y||\,\mathrm{d}y$$

is uniformly bounded for any $x \in T^2$.

Thus, the only remaining part we need to estimate is the first part in I_1, that is,

$$(4.8)\qquad \begin{aligned} I_1 &= \int_{B(0,R_0)} -\frac{1}{2\pi}\ln|x-y|\frac{8\lambda^2}{(\lambda^2+|y|^2)^2}\cdot\frac{|y/\lambda|^2-1}{|y/\lambda|^2+1}\,\mathrm{d}y + O(\lambda^2) \\ &= \int_{\mathbb{R}^2} -\frac{1}{2\pi}\ln|x-y|\frac{8\lambda^2}{(\lambda^2+|y|^2)^2}\cdot\frac{|y/\lambda|^2-1}{|y/\lambda|^2+1}\,\mathrm{d}y \\ &\quad+\int_{\mathbb{R}^2\setminus B(0,R_0)} \frac{1}{2\pi}\ln|x-y|\frac{8\lambda^2}{(\lambda^2+|y|^2)^2}\cdot\frac{|y/\lambda|^2-1}{|y/\lambda|^2+1}\,\mathrm{d}y + O(\lambda^2) \\ &= I_{1,1} + I_{1,2} + O(\lambda^2). \end{aligned}$$

Let us first compute $I_{1,1}$. More precisely, if we consider $I_{1,1}$ as a function of x, one can easily check that $I_{1,1}$ is a radial function. Furthermore, we have

$$-\Delta I_{1,1} = \frac{8\lambda^2}{(\lambda^2+|x|^2)^2}\cdot\frac{|x/\lambda|^2-1}{|x/\lambda|^2+1}.$$

We also know from direct computations that

$$I_{1,1}(0) = \int_{\mathbb{R}^2} -\frac{1}{2\pi}\ln|y|\frac{8\lambda^2}{(\lambda^2+|y|^2)^2}\cdot\frac{|y/\lambda|^2-1}{|y/\lambda|^2+1}\,\mathrm{d}y = -2.$$

It is easy to check that

$$-\Delta\left(\frac{-2}{|x/\lambda|^2+1}\right) = \frac{8\lambda^2}{(\lambda^2+|x|^2)^2}\cdot\frac{|x/\lambda|^2-1}{|x/\lambda|^2+1}.$$

Since any radial harmonic function in $\mathbb{R}^2$ is constant, we can conclude that

$$(4.9)\qquad I_{1,1}(x) = \frac{-2}{|x/\lambda|^2+1}.$$

For $I_{1,2}$, we notice that

$$\frac{8\lambda^2}{(\lambda^2+|y|^2)^2} \le C\frac{\lambda^2}{|y|^4}$$

and

$$\frac{1}{2} \le \frac{|y/\lambda|^2-1}{|y/\lambda|^2+1} \le 1 \quad \text{for } |y| > R_0.$$

Then, we have

$$I_{1,2} \le C\lambda^2 \int_{\mathbb{R}^2\setminus B(0,R_0)} \frac{|\ln|x-y||}{|y|^4}\,\mathrm{d}y \le C\lambda^2 \tag{4.10}$$

for a uniform $C > 0$ with respect to any $x \in T^2$.

Finally, combining (4.6), (4.8), (4.9), and (4.10), we have

$$|\tilde{f}| \le C$$

for some uniform $C > 0$.

Now, we let $\tilde{v}(z) = \tilde{f}(z) + C$, such that $\tilde{v} > \frac{1}{2}$ for all $z \in T_\lambda^2$. We are ready to define a comparison function in T_λ^2,

$$V(z) = \sum_{j=1}^{2} \tilde{v}(a|z-\xi_j'|) \quad \text{for } z \in T_\lambda^2.$$

We observe that

$$-\Delta V = \sum_{j=1}^{2} \frac{8a^2(a^2|z-\xi_j'|^2-1)}{(1+a^2|z-\xi_j'|^2)^3} - a^2\lambda^2 m,$$

so that for $|z-\xi_j'| > 10/a$ for $j = 1, 2$,

$$-\Delta V \ge 2\sum_{j=1}^{2} \frac{a^2}{(1+a^2|z-\xi_j'|^2)^2} \ge \sum_{j=1}^{2} \frac{a^{-2}}{|z-\xi_j'|^4}.$$

On the other hand, in the same region,

$$e^{w_\lambda} V \le C\sum_{j=1}^{2} \frac{1}{|z-\xi_j'|^4}.$$

Hence, if a is a fixed small constant and $R_2' > 0$ is chosen su ciently large depending on the choice of a, then we have $L(V) < 0$ in $\tilde{T}_\lambda^2 := T_\lambda^2 \setminus \bigcup_{j=1}^{2} B(\xi_j', R_2')$.

Here, we are able to find a positive supersolution V on $\tilde{T}^2_\lambda$. Then, we conclude that the operator L satisfies the maximum principle, that is, if $L(u) \leq 0$ in $\tilde{T}^2_\lambda$ and $u \geq 0$ on $\partial \tilde{T}^2_\lambda$, then $u \geq 0$ in $\tilde{T}^2_\lambda$.

Let us fix such a number $R_2' > 0$, which we may take larger whenever it is needed; that is, we can take $R_2' = R_2/\lambda$ for some positive R_2. Now, let us consider the "inner norm"

$$\|\phi\|_i = \sup_{\bigcup_{j=1}^2 B(\xi_j', R_2')} |\phi|.$$

Then, the second step in this proof is to show the following claim is true: there is a constant C such that if $L(\phi) = h$ in T^2_λ then

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*].$$

We need suitable barrier functions to prove the above claim.

Let g be the solution of the problem

$$\begin{aligned} -\Delta g &= \frac{2}{|z|^3} + 2\lambda^2 \quad && \text{for } z \in T^2_\lambda \setminus B(0, R_2'), \\ g(z) &= 0 && \text{on } \partial B(0, R_2'). \end{aligned}$$

If we introduce Green's function $\tilde{G}(x, y)$ of the manifold $T^2 \setminus B(0, R_2)$ with Dirichlet boundary condition, that is,

$$\begin{aligned} -\Delta_x \tilde{G}(x, y) &= \delta_y, \quad && x \in T^2 \setminus B(0, R_2), \\ \tilde{G}(x, y) &= 0, && x \in \partial B(0, R_2), \end{aligned}$$

then we can show the uniform boundedness of the function $g(z)$ by applying the same technique as in the proof of the uniform boundedness of the function $\tilde{f}(z)$, with $G(x, y)$ being replaced by $\tilde{G}(x, y)$. Furthermore, we define two more auxiliary functions using g:

$$g_j(z) = g(z - \xi_j'), \quad \text{for } j = 1, 2.$$

Let us introduce a barrier

$$\tilde{\phi}(z) = 2\|\phi\|_i V(z) + \|h\|_* \sum_{j=1}^{2} g_j(z).$$

Then, it is easy to check that $L(\tilde{\phi}) \leq h$ in $\tilde{T}^2_\lambda$ and $\tilde{\phi} \geq \phi$ on $\partial \tilde{T}^2_\lambda$. Hence, we have $\phi \leq \tilde{\phi}$ in $\tilde{T}^2_\lambda$. Similarly, one can also show that $\phi \geq -\tilde{\phi}$ in $\tilde{T}^2_\lambda$, and the claim follows.

In the last step, we go back to the contradiction argument. The claim in the second step shows that for some $\kappa > 0$, we have $\|\phi_n\|_i \geq \kappa$ since $\|\phi_n\|_\infty = 1$. Let us set $\hat{\phi}_n(z) = \phi_n(\xi_j' + z)$ where the index j is such that $\sup_{|z-\xi_j'|<R_2'} |\phi_n| \geq \kappa$. Without loss of generality, we can assume the index j is the same for all n. Elliptic estimates readily imply that $\hat{\phi}_n$ converges uniformly over any compact subset to a bounded solution $\hat{\phi} \neq 0$ of a problem in $\mathbb{R}^2$

$$\Delta\phi + \frac{8}{(1+|z|^2)^2}\phi = 0.$$

This implies that $\hat{\phi}$ is a linear combination of the functions φ_k, $k = 0, 1, 2$. However, the orthogonal conditions that all ϕ_n satisfy imply that $\hat{\phi} \equiv 0$. The result of the lemma then follows from the contradiction. □

We are now ready to provide a complete proof of the main result of this section.

Proof of Proposition 4.1. We first establish the validity of the *a priori* estimate (4.2). Lemma 4.2 yields

$$\|\phi\|_\infty \leq C\Big[\|h\|_* + \sum_{j=1}^{2} |c_j|\Big], \tag{4.11}$$

so it su ces to estimate the values of the constants $|c_j|$. Let us consider the cuto function $\eta_{R_3',\xi_j'}$ introduced in (3.1) where $R_3' = R_3/\lambda$ for some $R_3 > 0$. We multiply equation (4.1) by the test function $\varphi_{0,j}\eta_{R_3',\xi_j'}$, and integrate

$$\langle L(\phi), \eta_{R_3',\xi_j'}\varphi_{0,j}\rangle = \langle h, \eta_{R_3',\xi_j'}\varphi_{0,j}\rangle + c_j \int_{T_\lambda^2} \chi_{R_1,\xi_j'}|\varphi_{0,j}|^2. \tag{4.12}$$

On the other hand, we have

$$\langle L(\phi), \eta_{R_3',\xi_j'}\varphi_{0,j}\rangle = \langle \phi, L(\eta_{R_3',\xi' j}\varphi_{0,j})\rangle. \tag{4.13}$$

Now, we have

$$L(\eta_{R_3',\xi_j'}\varphi_{0,j}) = \Delta\eta_{R_3',\xi_j'}\varphi_{0,j} + 2\nabla\eta_{R_3',\xi_j'}\nabla\varphi_{0,j} + \lambda\left[O\left(\frac{r}{(1+r^2)^2}\right) + O\left(\frac{1}{(1+r^2)^2}\right)\right],$$

with $r = |z - \xi_j'|$. Since

$$\Delta\eta_{R_3',\xi_j'} = O(\lambda^2), \quad \nabla\eta_{R_3',\xi_j'} = O(\lambda)$$

and $\varphi_{0,j} = O(1)$, $\nabla\varphi_{0,j} = O(r^{-3})$, we have

$$L(\eta_{R_3',\xi_j'}\varphi_{0,j}) = O(\lambda^2) + \lambda\left[O\left(\frac{r}{(1+r^2)^2}\right) + O\left(\frac{1}{(1+r^2)^2}\right)\right]. \tag{4.14}$$

Therefore, we have

$$|\langle\phi, L(\eta_{R_3',\xi_j'}\varphi_{0,j})\rangle| \leq C\lambda\|\phi\|_\infty.$$

Combining this above estimate with (4.11), (4.12), and (4.13), we obtain

$$|c_j| \leq C[\|h\|_* + \lambda|c_k|] \quad \text{for } j \neq k.$$

It follows that $|c_j| \leq C\|h\|_*$. Furthermore, from (4.11) we know that (4.2) is true.

It only remains to verify the solvability assertion. The Fredholm alternative tells us that problem (4.1) has a unique solution if and only the associated homogeneous problem has only trivial solution. The homogeneous problem is equivalent to equation (4.1) with $h = 0$. From the *a priori* estimate we just proved, we know that the homegeneous problem only admits a trivial solution. This finishes the proof. □

If we add another orthogonal condition to ϕ and consider the problem

$$L(\phi) = h + \sum_{j=1}^{2} c_j\chi_{R_1,\xi_j'}\varphi_{0,j} + c_0 \quad \text{in } T_\lambda^2, \tag{4.15}$$

$$\phi \perp \varphi_{0,j}\chi_{R_1,\xi_j'}, \tag{4.16}$$

$$\phi \perp e^{w_\lambda}, \tag{4.17}$$

where $h \in C$, then we have the following corollary.

Corollary 4.3. *Assume the conditions in Proposition 4.1 hold; then, problem (4.15)–(4.17) has a unique solution. Also, if $h \in C^\alpha(T_\lambda^2)$ then $\|\phi\|_\infty \leq C\|h\|_*$.*

Proof. Following the same argument as in the proof of Proposition 4.1, we test (4.15) with $\varphi_{0,j}\eta_{R_3',\xi_j'}$:

$$\begin{aligned}&\langle L(\phi), \eta_{R_3',\xi_j'}\varphi_{0,j}\rangle \\ &\quad = \langle h, \eta_{R_3',\xi_j'}\varphi_{0,j}\rangle + c_j\int_{T_\lambda^2}\chi_{R_1,\xi_j'}|\varphi_{0,j}|^2 + c_0\int_{T_\lambda^2}\chi_{R_1,\xi_j'}\varphi_{0,j}.\end{aligned} \tag{4.18}$$

Integrating (4.15), we have

$$\int_{T_\lambda^2} h + \sum_{j=1}^{2}\int_{T_\lambda^2} c_j\chi_{R_1,\xi_j'}\varphi_{0,j} + \frac{c_0}{\lambda^2} = 0. \tag{4.19}$$

Combining (4.18) and (4.19) with Proposition 4.1, we can obtain

$$\frac{|c_0|}{\lambda^2} \le C\|h\|_*.$$

Hence, we have

$$\|\phi\| \le C\left[\|h\|_* + \frac{c_0}{\lambda^2}\right] \le C\|h\|_*. \qquad \square$$

The result of Corollary 4.3 implies that the unique solution $\phi = T(h)$ of the problem (4.15)–(4.17) defines a continuous linear map from Banach space C to $L_s^\infty(T_\lambda^2)$.

5. REDUCE TO A ONE-DIMENSIONAL PROBLEM

In this section, we reduce the infinite-dimensional problem of finding ϕ for

$$S_\rho(w_\lambda + \phi) = 0 \tag{5.1}$$

to a one-dimensional problem of finding appropriate scale λ while ρ is given.

We now expand $S_\rho(w_\lambda + \phi)$ as

$$S_\rho(w_\lambda + \phi) = S_\rho(w_\lambda) + \mathcal{L}\left(\phi - \frac{\int_{T^2} e^{w_\lambda}\phi}{\int_{T^2} e^{w_\lambda}}\right) + N(\phi),$$

where

$$N(\phi) = \left[\frac{\rho}{\int_{T^2} e^{w_\lambda+\phi}} e^{\phi} - \frac{\rho}{\int_{T^2} e^{w_\lambda}} - \left(\phi - \frac{\int_{T^2} e^{w_\lambda}\phi}{\int_{T^2} e^{w_\lambda}}\right)\right] e^{w_\lambda}. \tag{5.2}$$

Since the lefthand side of equation (5.1) is invariant if we add a constant to ϕ, we can further assume that $\int_{T^2} e^{w_\lambda}\phi = 0$.

We abuse the notation here to denote ϕ as a function in C_*. Moreover, we consider problem (5.1) in the dilated coordinates; that is, w_λ, $S_\rho(w_\lambda)$ and $N(\phi)$ are now considered to be functions on T_λ^2.

To exploit the reduction procedure, we solve the following nonlinear intermediate problem first:

$$L(\phi) = -\lambda^2[S_\rho(w_\lambda) + N(\phi)] + \sum_{j=1}^{2} c_j \chi_{R_1,\xi_j'} \varphi_{0,j} + c_0 \quad \text{in } T_\lambda^2, \tag{5.3}$$

$$\phi \in C_*, \tag{5.4}$$

$$\int_{T_\lambda^2} e^{w_\lambda}\phi = 0. \tag{5.5}$$

We will use the solvability theory just established in the previous section to show an existence result to (5.3)–(5.5). We assume that the conditions in Proposition 4.1 hold.

Lemma 5.1. *The problem* (5.3)–(5.5) *has a unique solution* ϕ *which satisfies* $\|\phi\|_\infty \le C\lambda$.

Proof. We first rewrite (5.3)–(5.5) into a fixed point form:

$$\phi = T(-\lambda^2[S_\rho(w_\lambda) + N(\phi)]) \equiv A(\phi).$$

For some constant $C > 0$ su ciently large, let us consider the region

$$\mathcal{F} \equiv \{\phi \in C_* \mid \phi \perp e^{w_\lambda},\ \|\phi\|_\infty \le C\lambda\}.$$

From Corollary 4.3, we have

$$\|A(\phi)\|_\infty \le C\lambda^2[\|S_\rho(w_\lambda)\|_* + \|N(\phi)\|_*].$$

By Lemma 3.4, we have the following estimate:

$$\|S_\rho(w_\lambda)\|_* \le C\frac{1}{\lambda}.$$

Also, the definition of N in (5.2) immediately implies that

$$\lambda^2\|N(\phi)\|_* \le C\lambda^2 \ln\frac{1}{\lambda}.$$

It is also immediate that N satisfies the contraction condition

$$\lambda^2\|N(\phi_1) - N(\phi_2)\|_* \le C\|\phi_1^2 - \phi_2^2\|_\infty + C\lambda\|\phi_1 - \phi_2\|_\infty \le C\lambda\|\phi_1 - \phi_2\|_\infty.$$

Hence, we get

$$\|A(\phi)\|_\infty \le C\lambda,$$
$$\|A(\phi_1) - A(\phi_2)\|_\infty \le C\lambda\|\phi_1 - \phi_2\|_\infty,$$

for su ciently small λ.

Therefore, the operator A is a contraction mapping of $\mathcal{F}$ if $\lambda \in (0, \lambda_0)$ where λ_0 is a constant small enough. The existence of a unique fixed point is guaranteed. This concludes the proof. ∎

Lemma 5.2. *For all* ϕ *found in Lemma* 5.1*, we have that* $c_1 = c_2$ *and* $c_0 = -2\lambda^2\mathcal{A}c_1$*, where the* c_i*,* $i = 1, 2$ *are coe cients in* (5.3) *and* $\mathcal{A} = \int_{\mathbb{R}^2}\chi_{R_1}\varphi_0(z)$.

Proof. By integrating equation (5.3), we have

$$\mathcal{A}(c_1 + c_2) = -\frac{c_0}{\lambda^2}. \tag{5.6}$$

Since the problem (5.3)–(5.5) is invariant if we switch the center of the coordinate system to ξ_2, we have

$$\langle L(\phi), \eta_{R_3',\xi_1'}\varphi_{0,1}\rangle = \langle L(\phi), \eta_{R_3',\xi_2'}\varphi_{0,2}\rangle. \tag{5.7}$$

Then, the lemma follows if we combine (5.6) and (5.7). ☐

We also need to estimate the dependence of ϕ as a function on T^2 on the parameter λ.

Lemma 5.3. *The fixed point ϕ found in Lemma 5.1 satisfies* $\left\|\frac{\partial \phi}{\partial \lambda}\right\|_\infty \le C$.

Proof. We study the problem (5.3)–(5.5) on the original flat torus T^2:

$$\mathcal{L}(\phi) = -[S_\rho(w_\lambda) + N(\phi)] + \sum_{j=1}^{2} \frac{c_j}{\lambda^2}\chi_{R_1,\xi_j'}\left(\frac{x}{\lambda}\right)\varphi_0\left(\frac{x-\xi_j}{\lambda}\right) + \frac{c_0}{\lambda^2},$$
$$\int_{T^2} \phi\chi_{R_1,\xi_j'}\left(\frac{x}{\lambda}\right)\varphi_0\left(\frac{x-\xi_j}{\lambda}\right) = 0,$$
$$\int_{T^2} e^{w_\lambda}\phi = 0,$$

and ϕ is symmetric about both axes. We di erentiate the above equation with respect to λ:

$$\mathcal{L}\left(\frac{\partial\phi}{\partial\lambda}\right) + \frac{\partial\left(\dfrac{\rho}{\int_{T^2} e^{w_\lambda}} e^{w_\lambda}\right)}{\partial\lambda}\phi$$
$$= -\left[\frac{\partial S_\rho(w_\lambda)}{\partial\lambda} + \frac{\partial N(\phi)}{\partial\lambda}\right] + \sum_{j=1}^{2}\frac{\partial c_j'}{\partial\lambda}\chi_{R_1,\xi_j'}\left(\frac{x}{\lambda}\right)\varphi_0\left(\frac{x-\xi_j}{\lambda}\right)$$
$$+ \sum_{j=1}^{2} c_j'\left(-\frac{|x-\xi_j|}{\lambda^2}\right)\chi_{R_1}'\left(\frac{|x-\xi_j|}{\lambda}\right)\varphi_0\left(\frac{x-\xi_j}{\lambda}\right)$$
$$+ \sum_{j=1}^{2} c_j'\chi_{R_1,\xi_j'}\frac{\partial\varphi_0\left(\frac{x-\xi_j}{\lambda}\right)}{\partial\lambda} + \frac{\partial c_0'}{\partial\lambda},$$

where $c'_j = c_j/\lambda^2$ for $j = 0, 1, 2$.

Again, we scale the torus to T^2_λ; then, we have

$$(5.8)\qquad L\left(\frac{\partial \phi}{\partial \lambda}\right) = -\lambda^2 \left[\frac{\partial \left(\frac{\rho}{\int_{T^2} e^{w_\lambda}} e^{w_\lambda} \right)}{\partial \lambda} \phi + \frac{\partial S_\rho(w_\lambda)}{\partial \lambda} + \frac{\partial N(\phi)}{\partial \lambda} - \frac{\partial c'_0}{\partial \lambda} \right] + \sum_{j=1}^{2} c'_j(-|x-\xi_j|)\chi'_{R_1}\left(\frac{|x-\xi_j|}{\lambda}\right)\varphi_{0,j} + \sum_{j=1}^{2} \lambda^2 c'_j \chi_{R_1,\xi'_j} \frac{\partial \varphi_{0,j}}{\partial \lambda} + \sum_{j=1}^{2} \lambda^2 \frac{\partial c'_j}{\partial \lambda} \chi_{R_1,\xi'_j}\varphi_{0,j},$$

and

$$\int_{T^2_\lambda} \frac{\partial \phi}{\partial \lambda} \chi_{R_1,\xi'_j}\varphi_{0,j} - \int_{T^2_\lambda} \phi \left[\left(-\frac{|x-\xi_j|}{\lambda^2} \right) \chi'_{R_1}\left(\frac{|x-\xi_j|}{\lambda} \right) \varphi_{0,j} + \chi_{R_1,\xi'_j} \frac{\partial \varphi_{0,j}}{\partial \lambda} \right].$$

Exploiting the same argument as in the proof of Proposition 4.1, we have

$$|c_j| \le C\lambda, \quad \text{for } j = 1, 2.$$

Therefore, by integrating (5.8), we have

$$(5.9)\qquad \left|\frac{\partial c'_0}{\partial \lambda}\right| \le C, \quad \left|\frac{\partial c'_j}{\partial \lambda}\right| \le \frac{C}{\lambda^2},$$

with again $j = 1, 2$.

Furthermore, direct calculations lead to

$$(5.10)\qquad \lambda^2 \left\| \frac{\partial \left(\frac{\rho}{\int_{T^2} e^{w_\lambda}} e^{w_\lambda} \right)}{\partial \lambda} \phi \right\|_* \le C, \quad \lambda^2 \left\| \frac{\partial S_\rho(w_\lambda)}{\partial \lambda} \right\|_* \le C.$$

The derivation of the above two estimates needs a careful check on the asymptotic behavior of $\partial w_\lambda/\partial\lambda$. One can proceed as in the way we calculate the asymptotic behavior of the approximate solution w_λ in Section 3. We omit the detailed calculations here.

It is also easy to check that

$$\left\| \sum_{j=1}^{2} c_j'(-x)\chi_{R_1}'\left(\frac{|x-\xi_j|}{\lambda}\right)\varphi_{0,j} + \sum_{j=1}^{2} c_j\chi_{R_1,\xi_j'}\frac{\partial\varphi_{0,j}}{\partial\lambda}\right\|_* \le C.$$

We now use the orthogonal condition

$$\int_{T_\lambda^2} e^{w_\lambda}\phi = 0$$

together with (3.8) and (5.10) to derive that

$$\lambda^2\left\|\frac{\partial N(\phi)}{\partial\lambda}\right\|_* \le C + C\lambda\left\|\frac{\partial\phi}{\partial\lambda}\right\|\infty.$$

Set b_j as follows:

$$\begin{aligned} &b_j\int_{T_\lambda^2}\chi_{R_1,\xi_j'}|\varphi_{0,j}|^2 \\ &\quad = \int_{T_\lambda^2}\phi\left[\left(-\frac{|x-\xi_j|}{\lambda^2}\right)\chi_{R_1}'\left(\frac{|x-\xi_j|}{\lambda}\right)\varphi_{0,j} + \chi_{R_1,\xi_j'}\frac{\partial\varphi_{0,j}}{\partial\lambda}\right]. \end{aligned}$$

We can easily verify that $|b_j| \le C$. Define $\tilde h$ as follows:

$$\begin{aligned} \tilde h = -\lambda^2 &\left[\frac{\partial\left(\frac{\rho}{\int_{T^2} e^{w_\lambda}}e^{w_\lambda}\right)}{\partial\lambda}\phi + \frac{\partial S_\rho(w_\lambda)}{\partial\lambda} + \frac{\partial N(\phi)}{\partial\lambda} - \frac{\partial c_0'}{\partial\lambda}\right] \\ &+ \sum_{j=1}^{2} c_j'(-|x-\xi_j|)\chi_{R_1}'\left(\frac{|x-\xi_j|}{\lambda}\right)\varphi_{0,j} \\ &+ \sum_{j=1}^{2}\lambda^2 c_j'\chi_{R_1,\xi_j'}\frac{\partial\varphi_{0,j}}{\partial\lambda} - \sum_{j=1}^{2} b_j L(\eta_{R_3',\xi_j'}\varphi_{0,j}). \end{aligned}$$

If $\tilde{\psi}$ is the unique solution to the problem

$$L(\tilde{\psi}) = \tilde{h} + \sum_{j=1}^{2} \mathrm{d}_j \chi_{R_1,\xi_j'} \varphi_{0,j},$$
$$\tilde{\psi} \perp \chi_{R_1,\xi_j'} \varphi_{0,j}, \quad \tilde{\psi} \in C_*,$$

then we can express $\partial\phi/\partial\lambda$ in terms of $\tilde{\psi}$, that is,

$$\frac{\partial \phi}{\partial \lambda} = \tilde{\psi} + \sum_{j=1}^{2} b_j \eta_{R_3',\xi_j'} \varphi_{0,j}. \tag{5.11}$$

Finally, combining (5.9)–(5.11) and (4.14) and applying Lemma 4.2, we have

$$\left\| \frac{\partial \phi}{\partial \lambda} \right\|_\infty \leq C.$$

□

6. Solving the Reduced Problem

In this section, we shall solve $S_\rho(w_\lambda + \phi) = 0$.

Lemma 6.1. *The energy of the* $w_\lambda + \phi$ *satisfies*

$$J_\rho(w_\lambda + \phi) = J_\rho(w_\lambda) + O(\lambda^2),$$

where ϕ *is found through the fixed point argument in Section 5.*

Proof. Expanding $J_\rho(w_\lambda + \phi)$ yields

$$J_\rho(w_\lambda + \phi) = J_\rho(w_\lambda) - \langle S_\rho(w_\lambda + \theta\phi), \phi \rangle_{T^2},$$

for some $\theta \in (0, 1)$.

Let us try to estimate $S_\rho(w_\lambda + \theta\phi)$:

$$S_\rho(w_\lambda + \theta\phi) = S_\rho(w_\lambda) + \theta\Delta\phi + O(\lambda e^{w_\lambda}). \tag{6.1}$$

By the fact that $\|\phi\|_\infty \leq C\lambda$ and Lemma 3.4, we have

$$\langle S_\rho(w_\lambda), \phi \rangle_{T^2} = o(\lambda^2).$$

It is easy to check that

$$\int_{T^2} |e^{w_\lambda} \phi| \leq C\lambda.$$

We only need to estimate the inner product of ϕ and the remaining term in (6.1):

$$\begin{aligned}
|\langle \Delta\phi, \phi\rangle_{T^2}| &= \left| \langle \mathcal{L}(\phi), \phi\rangle_{T^2} - \frac{\rho}{\int_{T^2} e^{w_\lambda}} \langle e^{w_\lambda}\phi, \phi\rangle_{T^2} \right| \\
&= |\langle \mathcal{L}(\phi), \phi\rangle_{T^2}| + O(\lambda^2) \\
&= \left| \sum_{j=1}^{2} c_j' \left\langle \chi_{R_1}\left(\frac{|x-\xi_j|}{\lambda}\right)\varphi_0\left(\frac{x-\xi_j}{\lambda}\right), \phi \right\rangle_{T^2} + c_0' \int_{T^2} \phi \right| + O(\lambda^2) \\
&= O(\lambda^2).
\end{aligned}$$

Therefore, we have

$$J_\rho(w_\lambda + \phi) = J_\rho(w_\lambda) + O(\lambda^2). \qquad \square$$

If we consider $J_\rho(w_\lambda + \phi)$ as a function of λ, then Lemma 3.5 and Lemma 6.1 imply that

$$\begin{aligned}
J_\rho(w_\lambda + \phi) = &-64\pi^2[R(0) + G(\xi_1 - \xi_2)] - 16\pi \ln(2\pi) - 16\pi + 2\varepsilon \ln \lambda \\
&+ 128\pi^2\lambda^2 \ln \lambda - \varepsilon[\ln(2\pi) - 8\pi R(0) - 8\pi G(\xi_1 - \xi_2)] + O(\lambda^2).
\end{aligned}$$

By the standard degree theory, we have the following lemma concerning the critical point of $J_\rho(w_\lambda + \phi)$.

Lemma 6.2. *The energy function $J_\rho(w_\lambda + \phi)$ is a C^1 function with respect to λ for $\lambda \in (\lambda_1, \lambda_2)$, and hence it has a local maximum point λ_*. Furthermore, we have*

$$\varepsilon = (128\pi^2 + o(1))\lambda_*^2 \ln \frac{1}{\lambda_*}, \quad \text{as } \varepsilon \to 0,$$

where $\rho = 16\pi + \varepsilon$ and $\varepsilon \in (0, \varepsilon_0)$.

Finally, we finish the proof of Theorem 2.4 by showing the following lemma.

Lemma 6.3. *When $\lambda = \lambda_*$, we have $c_1' = c_2' = c_0' = 0$, where*

$$S_\rho(w_\lambda + \phi) = \sum_{j=1}^{2} c_j' \chi_{R_1}\left(\frac{|x-\xi_j|}{\lambda}\right)\varphi_0\left(\frac{x-\xi_j}{\lambda}\right) + c_0'.$$

Proof. Since λ_* is a critical point of the function $J_\rho(w_\lambda + \phi)$, we have

$$\left.\frac{\partial J_\rho(w_\lambda + \phi)}{\partial \lambda}\right|_{\lambda=\lambda_*} = \left.\left\langle S_\rho(w_\lambda + \phi), \frac{\partial(w_\lambda + \phi)}{\partial \lambda}\right\rangle_{T^2}\right|_{\lambda=\lambda_*} = 0.$$

Computations show that

$$
\begin{aligned}
&\left\langle S_\rho(w_\lambda+\phi), \frac{\partial(w_\lambda+\phi)}{\partial\lambda}\right\rangle_{T^2} \\
&\quad = \int_{T^2}\left[\sum_{j=1}^{2} c_j'\chi_{R_1}\left(\frac{|x-\xi_j|}{\lambda}\right)\varphi_0\left(\frac{x-\xi_j}{\lambda}\right)+c_0'\right]\left(\frac{\partial w_\lambda}{\partial\lambda}+\frac{\partial\phi}{\partial\lambda}\right) \\
&\quad = \sum_{j=1}^{2} c_j\int_{T_\lambda^2}\chi_{R_1,\xi_j'}\varphi_{0,j}(z)\left(\frac{\varphi_{0,j}(z)}{\lambda}+O(1)\right)\mathrm{d}z \\
&\qquad + c_0\sum_{j=1}^{2}\int_{T_\lambda^2}\chi_{R_1,\xi_j'}\left(\frac{\varphi_{0,j}(z)}{\lambda}+O(1)\right)\mathrm{d}z \\
&\qquad + c_0'\int_{T^2\setminus(B(\xi_1,R_0))\cup B(\xi_2,R_0))}\left(\frac{2}{\lambda}+O(1)\right)\mathrm{d}x \\
&\quad = \left(\frac{\mathcal{B}}{\lambda}+O(1)\right)\sum_{j=1}^{2}c_j+\left(\frac{\mathcal{A}}{\lambda}+O(1)\right)c_0+\left(\frac{2(1-2\pi R_0^2)}{\lambda}+O(1)\right)c_0',
\end{aligned}
$$

where $\mathcal{B}$ is the constant

$$\mathcal{B} = \int_{\mathbb{R}^2}\chi_{R_1}|\varphi_0(z)|^2.$$

We know from Lemma 5.2 and the above calculations that

$$\left(\frac{\mathcal{B}}{\lambda}-\frac{2\mathcal{A}(1-2\pi R_0^2)}{\lambda}+O(1)\right)c_1 = 0.$$

It is easy to see that by choosing R_1 su ciently large, we have

$$\mathcal{B}-2\mathcal{A}(1-2\pi R_0^2)\neq 0.$$

Therefore, we have $c_1 = c_2 = c_0 = 0$. Finally, we obtain ϕ_* associated with λ_* such that $S_\rho(w_{\lambda_*}+\phi_*) = 0$.

The exact blow-up solution $w_{\lambda_*}+\phi_*$ of equation (1.1) is thus constructed. ☐

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