

ON PSEUDO-ANOSOV AUTOEQUIVALENCES

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ABSTRACT. Motivated by results of Thurston, we prove that any autoequivalence of a triangulated category induces a filtration by triangulated subcategories, provided the existence of Bridgeland stability conditions. The filtration is given by the exponential growth rate of masses under iterates of the autoequivalence, and only depends on the choice of a connected component of the stability manifold. We then propose a new definition of pseudo-Anosov autoequivalences, and prove that our definition is more general than the one previously proposed by Dimitrov, Haiden, Katzarkov, and Kontsevich. We construct new examples of pseudo-Anosov autoequivalences on the derived categories of quintic Calabi–Yau threefolds and quiver Calabi–Yau categories. Finally, we prove that certain pseudo-Anosov autoequivalences on quiver 3-Calabi–Yau categories act hyperbolically on the space of Bridgeland stability conditions.

1. INTRODUCTION

Let Σ_g be a closed orientable surface of genus $g \geq 2$. The mapping class group $\text{Mod}(\Sigma_g)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g . Thurston proved a far-reaching classification theorem for elements of $\text{Mod}(\Sigma_g)$, showing that each must be either periodic, reducible, or pseudo-Anosov [41]. Among these three types, pseudo-Anosov maps are dynamically the most interesting ones. Moreover, a generic element in $\text{Mod}(\Sigma_g)$ is pseudo-Anosov [38]. Pseudo-Anosov maps are pervasive throughout low-dimensional topology, geometry, and dynamics. For instance, a classical theorem of Thurston [40] says that the mapping torus constructed from a diffeomorphism $f: \Sigma_g \rightarrow \Sigma_g$ has a hyperbolic structure if and only if f is isotopic to a pseudo-Anosov map.

More recently, a striking series of papers by Gaiotto–Moore–Neitzke [19], Bridgeland–Smith [13], Haiden–Katzarkov–Kontsevich [22], and Dimitrov–Haiden–Katzarkov–Kontsevich [14], established connections between Teichmüller theory and the theory of Bridgeland stability conditions on triangulated categories. A series of analogies emerged, such as the one between geodesics (for flat metrics) and stable objects, with lengths of the former giving the masses of the later. Further correspondences are summarized in Table 1. Based on

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these connections, the categorical analogue of the notion of systoles on Riemann surfaces has been studied in [18, 21].

Surface	Triangulated category
diffeomorphisms	autoequivalences
closed curve C	object E
intersections $C_1 \cap C_2$	morphisms $\text{Hom}^\bullet(E_1, E_2)$
flat metrics	stability conditions
geodesics	stable objects
$\text{length}(C)$	$\text{mass}(E)$
$\text{slope}(C)$	$\text{phase}(E)$

TABLE 1. Analogies between smooth surfaces and triangulated categories.

Since generic elements in mapping class groups are of pseudo-Anosov type, it is then natural to ask, under the correspondence described in Table 1,

What is the categorical analogue of pseudo-Anosov maps?

In other words,

What properties are satisfied by *generic* autoequivalences on triangulated categories?

We study the categorifications of several different characterizations of pseudo-Anosov maps in this article. Recall that a map $f: \Sigma_g \rightarrow \Sigma_g$ is called *pseudo-Anosov* if there exists a pair of transverse measured foliations on Σ_g and a real number $\lambda > 1$ such that the foliations are preserved by f and their transverse measures are multiplied by λ and $1/\lambda$ respectively. The number λ is called the *stretch factor* of the pseudo-Anosov map. Thurston gives another characterization of pseudo-Anosov maps:

Theorem 1.1 (Thurston [41], Theorem 5). *For any diffeomorphism f of a surface Σ , there is a finite set of algebraic integers $1 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k$ such that for any homotopically nontrivial simple closed curve α , there is a λ_i such that for any Riemannian metric g on Σ ,*

$$\lim_{n \rightarrow \infty} (l_g(f^n \alpha))^{1/n} = \lambda_i.$$

Here l_g denotes the length of a shortest representative in the homotopy class. Moreover, f is isotopic to a pseudo-Anosov map if and only if $k = 1$ and $\lambda_1 > 1$. In this case, λ_1 is the stretch factor of the pseudo-Anosov map.

There is yet another characterization of pseudo-Anosov maps. Let (X, d) be a metric space and $f: X \rightarrow X$ be an isometry. The *translation length* of f is defined by

$$\tau(f) := \inf_{x \in X} \{d(x, f(x))\}.$$

The isometry f is said to be *hyperbolic* if $\tau(f) > 0$ and $\tau(f) = d(x, f(x))$ for some $x \in X$. Now let $f \in \text{Mod}(\Sigma_g)$ and recall that f acts on Teichmüller space $\text{Teich}(\Sigma_g)$ as an isometry for the Teichmüller metric. Moreover, f is pseudo-Anosov if and only if f acts hyperbolically on $\text{Teich}(\Sigma_g)$. In this case, the translation length and the stretch factor are related by $\tau(f) = \log \lambda(f)$.

Motivated by the definition of pseudo-Anosov maps and the correspondence in Table 1, Dimitrov, Haiden, Katzarkov, and Kontsevich [14] proposed a definition of “pseudo-Anosov autoequivalence” of a triangulated category. We recall their definition in Section 2.3 and will refer to such autoequivalences as “DHKK pseudo-Anosov autoequivalences” throughout this article. Some properties of these autoequivalences are studied in a recent work of Kikuta [28]. In particular, it was proved in [28] that DHKK pseudo-Anosov autoequivalences act hyperbolically on $\text{Stab}^\dagger(\mathcal{D})/\mathbb{C}$, and also satisfy the relation $\tau(f) = \log \lambda(f)$ between translation lengths and stretch factors.

However, a drawback of the definition of DHKK pseudo-Anosov autoequivalences, which we will discuss in Section 3.1, is that it is too restrictive in some cases. For instance, the set of DHKK pseudo-Anosov autoequivalences does not contain the autoequivalences induced by certain pseudo-Anosov maps constructed by Thurston [41] which act trivially on the homology of Σ_g .

We propose a new definition of *pseudo-Anosov autoequivalences* on triangulated categories. Our definition is motivated by the characterization given by Theorem 1.1. We show in Section 2.2 that any autoequivalence on a triangulated category $\mathcal{D}$ induces a filtration by triangulated subcategories, provided the following two assumptions on $\mathcal{D}$:

Assumption (A): There exists a Bridgeland stability condition on $\mathcal{D}$.

Assumption (B): Under Assumption (A), a connected component $\text{Stab}^\dagger(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ is fixed once and for all.

Throughout the article, we consider triangulated categories satisfying these two assumptions. The filtration of $\mathcal{D}$ that we associated to an autoequivalence is the categorical analogue of the growth rates of lengths $1 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k$ in Theorem 1.1. We then make the following definition.

Definition 1.2 (see Definition 2.13). Let $\mathcal{D}$ be a triangulated category satisfying Assumptions (A) and (B). An autoequivalence $\Phi: \mathcal{D} \rightarrow \mathcal{D}$ is said to be *pseudo-Anosov* if its

associated growth filtration has only one step:

$$0 \subset \mathcal{D}_\lambda = \mathcal{D} \text{ and } \lambda > 1.$$

We prove that this new definition is more general than the definition of DHKK pseudo-Anosov autoequivalences.

Theorem 1.3 (see Theorem 2.17). *Let $\mathcal{D}$ be a triangulated category satisfying Assumptions (A) and (B), and let $\Phi \in \text{Aut}(\mathcal{D})$ be an autoequivalence. If Φ is DHKK pseudo-Anosov, then it is pseudo-Anosov.*

Note that there is a distinguished choice of connected component of $\text{Stab}(\mathcal{D})$ for several examples of triangulated categories $\mathcal{D}$ that admit Bridgeland stability conditions. In this case, we choose $\text{Stab}^\dagger(\mathcal{D})$ in Assumption (B) to be the distinguished connected component. Examples of such categories $\mathcal{D}$ include:

- The bounded derived category of coherent sheaves on a smooth complex projective variety X , where X is a curve, surface, abelian threefold, Fano threefold, or a quintic threefold [9, 34, 10, 3, 6, 5, 32, 7, 31]. For these triangulated categories, there is a distinguished connected component of $\text{Stab}(\mathcal{D}^b\text{Coh}(X))$ which contains *geometric stability conditions*.
- $\mathcal{D}(\Gamma_N Q)$, the finite dimensional derived category of the N -Calabi–Yau Ginzburg dg algebra $\Gamma_N Q$ associated to an acyclic quiver Q , where $N \geq 2$ is an integer [20, 25, 26]. For these triangulated categories, there is a distinguished connected component of $\text{Stab}(\mathcal{D}(\Gamma_N Q))$ which contains stability conditions whose heart $P((0, 1])$ coincides with the canonical heart $\mathcal{H}_{\Gamma_N Q} \subset \mathcal{D}(\Gamma_N Q)$ associated to the quiver. See Section 3.1 for more details in the case of A_2 -quiver.

We construct new examples of pseudo-Anosov autoequivalences on the derived categories of coherent sheaves on quintic threefolds and on certain quiver Calabi–Yau categories. Note that it is not known whether there exists any DHKK pseudo-Anosov autoequivalence on a Calabi–Yau category of dimension greater than one.

Theorem 1.4 (see Theorem 3.2 and Proposition 3.7). *Let $\mathcal{D}(\Gamma_N A_2)$ be the N -Calabi–Yau category associated to the A_2 -quiver, where $N \geq 3$ is an odd integer. Then*

- *Any composition of spherical twists T_1 and T_2^{-1} that is neither T_1^a nor T_2^{-b} is a pseudo-Anosov autoequivalence of $\mathcal{D}(\Gamma_N A_2)$.*
- *There is no DHKK pseudo-Anosov autoequivalence of $\mathcal{D}(\Gamma_N A_2)$.*

We also prove that certain “palindromic” pseudo-Anosov autoequivalences of $\mathcal{D}(\Gamma_N A_2)$ act hyperbolically on $\text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))/\mathbb{C}$ and satisfy the relation between translation lengths and stretch factors (see Theorem 3.8 for the precise statement).

Theorem 1.5 (see Theorem 3.10). *Let X be a quintic Calabi–Yau hypersurface in $\mathbb{CP}^4$. Then*

$$\Phi := T_{\mathcal{O}} \circ (- \otimes \mathcal{O}(-1))$$

is a pseudo-Anosov autoequivalence on $\mathcal{D}^b(X)$. Here $T_{\mathcal{O}}$ denotes the spherical twist with respect to the structure sheaf $\mathcal{O}_X$.

In Section 4, we discuss two ways in which a pseudo-Anosov map in the classical sense induces a pseudo-Anosov autoequivalence on the Fukaya category of a surface.

In the last section, we propose several interesting open questions on categorical dynamical systems that are related to pseudo-Anosov autoequivalences. In particular, we discuss other possible ways to define the notion of pseudo-Anosov autoequivalences without Assumptions (A) and (B). We also introduce the notion of *irreducible* autoequivalence of a category in Definition 5.5. It is then a formal consequence that if an autoequivalence is irreducible and has positive mass growth for a nonzero object, then it is pseudo-Anosov in the sense of Definition 1.2.

In Appendix A, we show that the mass growth rates with respect to the complexity function (Example 2.3) and to stability conditions (Example 2.5) on the derived category of coherent sheaves on elliptic curves coincide.

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2. GROWTH FILTRATIONS AND PSEUDO-ANOSOV AUTOEQUIVALENCES

In this section, we develop the notion of growth filtration associated to any autoequivalence of a triangulated category $\mathcal{D}$ which satisfies Assumptions (A) and (B) in the introduction. We then use it to define the notion of pseudo-Anosov autoequivalences. Some examples will be worked out in Section 3.

2.1. Mass functions on triangulated categories. In order to define the mass growth of an object with respect to an autoequivalence, one needs to impose a *mass function* on the underlying triangulated category $\mathcal{D}$.

Definition 2.1. Let $\mathcal{D}$ be a triangulated category. A *mass function* on $\mathcal{D}$ is a non-negative function on the objects

$$\mu: \text{Ob}(\mathcal{D}) \rightarrow \mathbb{R}_{\geq 0}$$

such that

- (1) $\mu(A) + \mu(C) \geq \mu(B)$ for any exact triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ in $\mathcal{D}$.
- (2) $\mu(E) = \mu(E[1])$ for any $E \in \mathcal{D}$.
- (3) $\mu(E) \leq \mu(E \oplus F)$ for any $E, F \in \mathcal{D}$.
- (4) $\mu(0) = 0$.

One can find several natural examples of mass functions on triangulated categories.

Example 2.2. Suppose $\mathcal{D}$ is k -linear and of finite type, i.e. for any pair of objects $E, F \in \mathcal{D}$ the sum $\sum_{i \in \mathbb{Z}} \dim_k \text{Hom}_{\mathcal{D}}(E, F[i])$ is finite. Then for any nonzero object $E \in \mathcal{D}$,

$$\mu_E(F) := \sum_{i \in \mathbb{Z}} \dim_k \text{Hom}_{\mathcal{D}}(E, F[i])$$

defines a mass function on $\mathcal{D}$. Note that in this example we may have $\mu_E(F) = 0$ for a nonzero object $F \in \mathcal{D}$.

Example 2.3. Let G be a split generator of a triangulated category $\mathcal{D}$. Then the complexity function considered in [14] defined as

$$\delta_G(E) := \inf \left\{ k \in \mathbb{N} \left| \begin{array}{ccc} 0 & \xrightarrow{\quad} & \star \\ & \searrow & \nearrow \\ & G[n_1] & \end{array} \quad \cdots \quad \begin{array}{ccc} \star & \xrightarrow{\quad} & E \oplus \star \\ & \searrow & \nearrow \\ & G[n_k] & \end{array} \right. \right\}$$

defines a mass function on $\mathcal{D}$.

The main examples of mass functions that we will consider are the ones given by Bridgeland stability conditions. We recall the definition and some basic facts about Bridgeland stability conditions.

Definition 2.4 (Bridgeland [9]). Fix a triangulated category $\mathcal{D}$ and a group homomorphism $\text{cl}: K_0(\mathcal{D}) \rightarrow \Gamma$ to a finite rank abelian group Γ . A Bridgeland stability condition $\sigma = (Z, P)$ on $\mathcal{D}$ consists of:

- a group homomorphism $Z: \Gamma \rightarrow \mathbb{C}$, and
- a collection of full additive subcategories $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$ of $\mathcal{D}$,

such that:

- (1) $Z(E) := Z(\text{cl}([E])) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$ for any $0 \neq E \in P(\phi)$;
- (2) $P(\phi + 1) = P(\phi)[1]$;
- (3) If $\phi_1 > \phi_2$ and $A_i \in P(\phi_i)$ then $\text{Hom}(A_1, A_2) = 0$;
- (4) For any $0 \neq E \in \mathcal{D}$, there exists a (unique) collection of exact triangles

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & E_1 & \xrightarrow{\quad} & E_2 & \longrightarrow & \dots \longrightarrow E_{n-1} \xrightarrow{\quad} E \\ & & \swarrow & & \swarrow & & \swarrow \\ & & A_1 & & A_2 & & A_n \end{array}$$

such that $A_i \in P(\phi_i)$ and $\phi_1 > \phi_2 > \dots > \phi_n$. Denote $\phi_\sigma^+(E) := \phi_1$ and $\phi_\sigma^-(E) := \phi_n$. The *mass* of E with respect to σ is defined to be

$$m_\sigma(E) := \sum_{i=1}^n |Z(A_i)|.$$

- (5) (Support property [30]) There exists a constant $C > 0$ and a norm $\|\cdot\|$ on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$\|\text{cl}([E])\| \leq C|Z(E)|$$

for any $E \in P(\phi)$ and any $\phi \in \mathbb{R}$.

The group homomorphism $Z: \Gamma \rightarrow \mathbb{C}$ is called the *central charge*. Nonzero objects in $P(\phi)$ are called σ -*semistable object of phase* ϕ . The sequence in (4) is called the Harder–Narasimhan filtration of E and $A_1, \dots, A_n$ are called the semistable factors of E .

Denote by $\text{Stab}(\mathcal{D})$ (or more precisely $\text{Stab}_\Gamma(\mathcal{D})$) the set of Bridgeland stability conditions on $\mathcal{D}$ with respect to $\text{cl}: K_0(\mathcal{D}) \rightarrow \Gamma$. There is a useful topology on $\text{Stab}(\mathcal{D})$ introduced by Bridgeland, which is induced by the generalized metric [9, Proposition 8.1]:

$$d(\sigma_1, \sigma_2) := \sup_{E \neq 0} \left\{ |\phi_{\sigma_2}^-(E) - \phi_{\sigma_1}^-(E)|, |\phi_{\sigma_2}^+(E) - \phi_{\sigma_1}^+(E)|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\} \in [0, \infty].$$

The forgetful map

$$\text{Stab}(\mathcal{D}) \rightarrow \text{Hom}(\Gamma, \mathbb{C}), \quad \sigma = (Z, P) \mapsto Z$$

is a local homeomorphism [9, Theorem 7.1]. Hence $\text{Stab}(\mathcal{D})$ is naturally a finite dimensional complex manifold.

There are two natural group actions by $\text{Aut}(\mathcal{D})$ and $\widetilde{\text{GL}^+(2, \mathbb{R})}$ on the space of Bridgeland stability conditions which commute with each other [9, Lemma 8.2]. The group of autoequivalences $\text{Aut}(\mathcal{D})$ acts on $\text{Stab}(\mathcal{D})$ by isometries with respect to the generalized metric. To describe the action explicitly, let $\Phi \in \text{Aut}(\mathcal{D})$ be an autoequivalence and define

$$\sigma = (Z, P) \mapsto \Phi \cdot \sigma := (Z \circ \Phi^{-1}, P'),$$

where $P'(\phi) = \Phi(P(\phi))$.

Let $\pi: \widetilde{\text{GL}^+(2, \mathbb{R})} \rightarrow \text{GL}^+(2, \mathbb{R})$ be the universal cover. Recall that $\widetilde{\text{GL}^+(2, \mathbb{R})}$ is isomorphic to the group of pairs (T, f) where $f: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing map with $f(\phi+1) = f(\phi) + 1$, and $T \in \text{GL}^+(2, \mathbb{R})$ such that the induced maps on $S^1 \cong \mathbb{R}/2\mathbb{Z} \cong (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0}$ coincide. Given $g = (T, f) \in \widetilde{\text{GL}^+(2, \mathbb{R})}$, define its action on $\text{Stab}(\mathcal{D})$ as

$$\sigma = (Z, P) \mapsto \sigma \cdot g := (T^{-1} \circ Z, P''),$$

where $P''(\phi) = P(f(\phi))$. Note that the subgroup $\mathbb{C} \subset \widetilde{\text{GL}^+(2, \mathbb{R})}$ acts freely and isometrically on $\text{Stab}(\mathcal{D})$. For $z \in \mathbb{C}$, its action on $\text{Stab}(\mathcal{D})$ is given by

$$\sigma = (Z, P) \mapsto \sigma \cdot z := (\exp(-i\pi z)Z, P'''),$$

where $P'''(\phi) = P(\phi + \text{Re}z)$.

Each Bridgeland stability condition on $\mathcal{D}$ defines a mass function on $\mathcal{D}$.

Example 2.5. Let σ be a Bridgeland stability condition on $\mathcal{D}$. By Ikeda [24, Proposition 3.3], we have

$$m_\sigma(B) \leq m_\sigma(A) + m_\sigma(C)$$

for any exact triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ in $\mathcal{D}$. Hence the mass m_σ defines a mass function on $\mathcal{D}$ in the sense of Def. 2.1.

Remark 2.6. The three examples of mass functions above are all analogues of classical objects in Teichmüller theory. Example 2.2 is an analogue of the geometric intersection number between two curves, see [14, Lemma 2.19]. Example 2.3 is an analogue of word length of a curve, and Example 2.5 is an analogue of Riemannian length of a curve.

Let $\mathcal{D}'$ be a thick triangulated subcategory of $\mathcal{D}$, i.e. $\mathcal{D}'$ is closed under shifts, mapping cones, and direct summands. We denote the space of *relative* mass functions by

$$\mathcal{M}(\mathcal{D}, \mathcal{D}') := \{\mu \text{ mass function on } \mathcal{D} \mid \mu(E) = 0 \iff E \in \mathcal{D}'\},$$

and denote $\mathcal{M}(\mathcal{D}) := \mathcal{M}(\mathcal{D}, 0)$ the space of mass functions that vanish only at the zero object. This is the analogue of the space of length functions on Riemann surfaces in Teichmüller theory. We define a generalized metric on $\mathcal{M}(\mathcal{D}, \mathcal{D}')$ in a similar way as [9, Proposition 8.1].

Definition 2.7. Let $\mathcal{M}(\mathcal{D}, \mathcal{D}')$ be the space of mass functions on $\mathcal{D}$ that vanish on $\mathcal{D}'$. Then

$$d(\mu_1, \mu_2) := \sup_{E \notin \mathcal{D}'} \left\{ \left| \log \frac{\mu_1(E)}{\mu_2(E)} \right| \right\} \in [0, \infty]$$

defines a generalized metric on $\mathcal{M}(\mathcal{D}, \mathcal{D}')$.

We will only use $\mathcal{M}(\mathcal{D})$ in the main text of this article. In Appendix B we discuss some basic properties of mass functions: functoriality and induced relative mass functions.

Observe that $\text{Aut}(\mathcal{D})$ and $\mathbb{R}_{>0}$ act isometrically on the space of mass functions $\mathcal{M}(\mathcal{D})$, and the actions by $\mathbb{R}_{>0}$ are free. We also consider the quotient spaces $\text{Stab}(\mathcal{D})/\mathbb{C}$ and $\mathcal{M}(\mathcal{D})/\mathbb{R}_{>0}$. Note that the orbits of the $\mathbb{C}$ -action (resp. $\mathbb{R}_{>0}$ -action) in $\text{Stab}(\mathcal{D})$ (resp. $\mathcal{M}(\mathcal{D})$) are closed. Hence the quotient (generalized) metrics on $\text{Stab}(\mathcal{D})/\mathbb{C}$ and $\mathcal{M}(\mathcal{D})/\mathbb{R}_{>0}$ are given by

$$\bar{d}(\bar{\sigma}_1, \bar{\sigma}_2) := \inf_{z \in \mathbb{C}} d(\sigma_1, \sigma_2 \cdot z) \quad \text{and} \quad \bar{d}(\bar{\mu}_1, \bar{\mu}_2) := \inf_{r \in \mathbb{R}_{>0}} d(\mu_1, \mu_2 \cdot r).$$

Lemma 2.8. *The map $\text{Stab}(\mathcal{D}) \rightarrow \mathcal{M}(\mathcal{D})$ which sends σ to its mass m_σ descends to a map $\text{Stab}(\mathcal{D})/\mathbb{C} \rightarrow \mathcal{M}(\mathcal{D})/\mathbb{R}_{>0}$. Moreover, both maps are contractions between (generalized) metric spaces, therefore continuous.*

Proof. This follows easily from the definitions of the (generalized) metrics on $\text{Stab}(\mathcal{D})$ and $\mathcal{M}(\mathcal{D})$. $\square$

We denote the image of these two maps by $\mathcal{M}_{\text{Stab}}(\mathcal{D})$ and $\mathcal{M}_{\text{Stab}}(\mathcal{D})/\mathbb{R}_{>0}$ respectively. Note that the map $\text{Stab}(\mathcal{D})/\mathbb{C} \rightarrow \mathcal{M}(\mathcal{D})/\mathbb{R}_{>0}$ is not injective in general. For instance, let $\mathcal{D}$ be the derived category of $\mathbb{C}$ -linear representations of the A_2 -quiver. Let E_1 and E_2 be the simple representations with dimension vectors $(1, 0)$ and $(0, 1)$. Choose any Bridgeland stability condition $\sigma \in \text{Stab}(\mathcal{D})$ such that $0 < \phi(E_1) < \phi(E_2) < 1$ and $|Z(E_1)| = |Z(E_2)| = 1$. Such Bridgeland stability conditions induce the same mass function on $\mathcal{D}$, but they are not in the same $\mathbb{C}$ -orbit.

2.2. Growth filtrations of autoequivalences. In this section, we define the growth filtration associated to an autoequivalence on a triangulated category $\mathcal{D}$ which satisfies Assumptions (A) and (B) in the introduction.

Definition 2.9. Let μ be a mass function on a triangulated category $\mathcal{D}$, and Φ be an autoequivalence on $\mathcal{D}$. The mass growth of an object E with respect to both μ and Φ is defined to be

$$h_{\mu, \Phi}(E) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu(\Phi^n E).$$

The following lemma follows directly from the definition of mass functions.

Lemma 2.10. *Let μ and Φ be as above. Then*

$$\mathcal{D}_\lambda := \{E : h_{\mu, \Phi}(E) \leq \log \lambda\}$$

is a Φ -invariant thick triangulated subcategory of $\mathcal{D}$.

We now restrict our attention to the mass functions given by Bridgeland stability conditions. The following proposition is a special case of Ikeda [24, Proposition 3.10]. It can also be deduced from a result of Bridgeland [9, Proposition 8.1] stating that if a stability condition $\tau \in B_\epsilon(\sigma)$ is in a neighborhood of another stability condition σ for small enough $\epsilon > 0$, then there are constants $C_1, C_2 > 0$ such that

$$C_1 m_\tau(E) < m_\sigma(E) < C_2 m_\tau(E)$$

for any nonzero object $E \in \mathcal{D}$.

Proposition 2.11. *If σ and τ lie in the same connected component of $\text{Stab}(\mathcal{D})$, then $h_{m_\sigma, \Phi}(E) = h_{m_\tau, \Phi}(E)$ for any autoequivalence Φ and any object $E \in \mathcal{D}$.*

Definition 2.12. Let $\mathcal{D}$ be a triangulated category such that $\text{Stab}(\mathcal{D}) \neq \emptyset$. We fix a choice of a connected component $\text{Stab}^\dagger(\mathcal{D})$ of $\text{Stab}(\mathcal{D})$, and define the *growth filtration* associated to an autoequivalence $\Phi \in \text{Aut}(\mathcal{D})$ to be the filtration $\{\mathcal{D}_\lambda\}$ by Φ -invariant thick triangulated subcategories of $\mathcal{D}$ with respect to the mass functions given by the stability conditions in $\text{Stab}^\dagger(\mathcal{D})$.

The possible mass growth rates λ are the categorical analogue of the growth rates of lengths $1 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k$ in Thurston's theorem (Theorem 1.1). However, we do not know whether there are only finitely many possible mass growth rates for an autoequivalence in general. The algebraicity of λ is unclear in general as well. The problem of algebraicity of categorical entropy was also raised in [14, Question 4.1].

Note that one can also define the growth filtration (therefore define the notion of pseudo-Anosov autoequivalences as in the next section) using other mass functions, for instance the complexity functions in Example 2.3 or the Ext-distance functions in Example 2.2. See Section 5 and Appendix A for more discussions on these alternative definitions. In the main text, we will stick with the growth filtration defined in Definition 2.12.

2.3. Pseudo-Anosov autoequivalences. From now on, we consider triangulated categories $\mathcal{D}$ satisfying

Assumption (A): $\text{Stab}(\mathcal{D}) \neq \emptyset$, and

Assumption (B): we fix a connected component $\text{Stab}^\dagger(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$.

Then any autoequivalence $\Phi \in \text{Aut}(\mathcal{D})$ has an associated growth filtration $\{\mathcal{D}_\lambda\}$ of $\mathcal{D}$ as in Definition 2.12. Motivated by Thurston's Theorem 1.1, we propose the following definition of pseudo-Anosov autoequivalences.

Definition 2.13. An autoequivalence Φ is said to be *pseudo-Anosov* if the associated growth filtration has only one step:

$$0 \subset \mathcal{D}_\lambda = \mathcal{D}$$

with $\lambda > 1$. In other words, the mass growth of any nonzero object is the same positive constant $\log \lambda$. We call λ the *stretch factor* of Φ .

In a previous work of Dimitrov, Haiden, Katzarkov, and Kontsevich [14], another definition of pseudo-Anosov autoequivalences was proposed. We call such autoequivalences “DHKK pseudo-Anosov” in this article.

Definition 2.14 ([14], Definition 4.1 and [28], Definition 4.6). An autoequivalence $\Phi \in \text{Aut}(\mathcal{D})$ is said to be *DHKK pseudo-Anosov* if there exists a Bridgeland stability condition $\sigma \in \text{Stab}^\dagger(\mathcal{D})$ and an element $g \in \widetilde{\text{GL}^+(2, \mathbb{R})}$ such that

- $\Phi \cdot \sigma = \sigma \cdot g$.
- $\pi(g) = \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix}$ or $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \in \text{GL}^+(2, \mathbb{R})$ for some $|r| > 1$.

Here $\lambda := |r| > 1$ is called the *DHKK stretch factor* of Φ .

Remark 2.15. We use the slightly modified definition of DHKK pseudo-Anosov by Kikuta [28, Definition 4.6]. The only difference from the definition in [14] is that we require the stability condition σ to lie in the distinguished connected component $\text{Stab}^\dagger(\mathcal{D})$.

The definition of DHKK pseudo-Anosov autoequivalences is motivated by the original definition of pseudo-Anosov maps on Riemann surfaces – there exists a pair of transverse measured foliations that are preserved by the map and the transverse measures are stretched/contracted by the map. In Definition 2.14, the Bridgeland stability condition σ plays the role of a pair of measured foliations, and the two conditions on (Φ, σ, g) correspond to the preservation and stretching/contracting of the foliations by the map.

Remark 2.16. A DHKK pseudo-Anosov autoequivalence will preserve the set of semistable objects associated to the corresponding Bridgeland stability condition. On the other hand, it should act on the Grothendieck group of the category in a non-trivial stretching/contracting manner. These two requirements make it difficult to find DHKK pseudo-Anosov maps and indeed very few examples are known, see for instance [14, 28].

We show that the definition of pseudo-Anosov in Definition 2.13 is more general than DHKK pseudo-Anosov. In the next section, we provide examples of pseudo-Anosov autoequivalences that are not DHKK pseudo-Anosov.

Theorem 2.17. *Let $\mathcal{D}$ be a triangulated category and $\Phi \in \text{Aut}(\mathcal{D})$ an autoequivalence. If Φ is DHKK pseudo-Anosov, then it is pseudo-Anosov. Moreover, the stretch factor and the DHKK stretch factor coincide.*

Proof. Let Φ be a DHKK pseudo-Anosov autoequivalence on $\mathcal{D}$. By definition, there exists a Bridgeland stability condition $\sigma \in \text{Stab}^\dagger(\mathcal{D})$ and an element $g \in \widetilde{\text{GL}^+(2, \mathbb{R})}$ satisfying the conditions in Definition 2.14.

We claim that the mass growth of any nonzero object $E \in \mathcal{D}$ with respect to Φ and σ is $\log \lambda > 0$ where $\lambda > 1$ is the DHKK stretch factor. Let $A_1, A_2, \dots, A_k$ be the Harder–Narasimhan semistable factors of E with respect to the stability condition $\sigma \in \text{Stab}^\dagger(\mathcal{D})$. Since $\Phi \cdot \sigma = \sigma \cdot g$, the set of σ -semistable objects and the order of their phases are preserved by Φ . Hence the Harder–Narasimhan semistable factors of $\Phi^n E$ are given by $\Phi^n A_1, \Phi^n A_2, \dots, \Phi^n A_k$, so

$$m_\sigma(\Phi^n E) = \sum_{i=1}^k |Z_\sigma(\Phi^n A_i)|.$$

Thus, to prove the claim, it suffices to show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |Z_\sigma(\Phi^n A)| = \log \lambda$$

for any σ -semistable object $A \in \mathcal{D}$.

By the conditions in Definition 2.14, the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \log |Z_\sigma(\Phi^n A)|$ is either $\log \lambda$ or $-\log \lambda$. If there were a σ -semistable object $A \in \mathcal{D}$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \log |Z_\sigma(\Phi^n A)| = -\log \lambda$, then the central charges of the semistable objects $A, \Phi A, \Phi^2 A, \dots$ would tend to zero, which contradicts with the support property of stability conditions (Definition 2.4). This concludes the proof. $\square$

We conclude this section by recalling the definition and some properties of categorical entropy of an autoequivalence introduced by Dimitrov–Haiden–Katzarkov–Kontsevich [14]. This will be useful for the discussions in the next section.

Definition 2.18 ([14], Definition 2.5). Let $\mathcal{D}$ be a triangulated category with a split generator G and let $\Phi: \mathcal{D} \rightarrow \mathcal{D}$ be an endofunctor. The categorical entropy of Φ is defined to be

$$h_{\text{cat}}(\Phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \delta_G(\Phi^n G) \in [-\infty, \infty),$$

where δ_G is the complexity function considered in Example 2.3.

The relation between categorical entropy and mass growth with respect to Bridgeland stability conditions has been studied by Ikeda [24].

Theorem 2.19 ([24], Theorem 3.5). *Let $\Phi: \mathcal{D} \rightarrow \mathcal{D}$ be an endofunctor and σ be a Bridgeland stability condition on $\mathcal{D}$. Assume that $\mathcal{D}$ has a split generator $G \in \mathcal{D}$. Then*

$$\sup_{E \in \mathcal{D}} \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n} \log m_\sigma(\Phi^n E) \right\} = \limsup_{n \rightarrow \infty} \frac{1}{n} \log m_\sigma(\Phi^n G) \leq h_{\text{cat}}(\Phi).$$

In particular, if $\Phi \in \text{Aut}(\mathcal{D})$ is pseudo-Anosov, then its categorical entropy is positive since

$$h_{\text{cat}}(\Phi) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log m_\sigma(\Phi^n G) = \log \lambda > 0.$$

3. EXAMPLES

In this section, we construct examples of pseudo-Anosov autoequivalences on certain quiver Calabi–Yau categories as well as the derived category of coherent sheaves on quintic Calabi–Yau threefolds.

3.1. Examples from A_2 -quivers. We study the pseudo-Anosovness of autoequivalences on the Calabi–Yau category $\mathcal{D}(\Gamma_N A_2)$ of dimension $N \geq 3$ associated to an A_2 -quiver:

$$\bullet_1 \longrightarrow \bullet_2$$

The category $\mathcal{D}(\Gamma_N A_2)$ is defined to be the derived category of finite dimensional dg-modules over the Ginzburg N -Calabi–Yau dg-algebra of the A_2 -quiver. We recall some basic properties of $\mathcal{D}(\Gamma_N A_2)$ and refer to [20, 25, 26, 29] for the definition. By [25, Theorem 6.3], $\mathcal{D}(\Gamma_N A_2)$ is an N -Calabi–Yau triangulated category, i.e. for any pair of objects $E, F \in \mathcal{D}(\Gamma_N A_2)$, there are natural isomorphisms

$$\text{Hom}^\bullet(E, F) \cong \text{Hom}^\bullet(F, E[N])^\vee.$$

The Ginzburg dg algebra $\Gamma_N A_2$ is concentrated in negative degrees, therefore its derived category has a canonical t -structure given by the standard truncation functors. This induces a canonical t -structure on $\mathcal{D}(\Gamma_N A_2)$.

Proposition 3.1 ([2, Section 2], [27, Section 5.1]). *There is a canonical bounded t -structure on $\mathcal{D}(\Gamma_N A_2)$ with the heart $\mathcal{H}_{\Gamma_N A_2}$ such that the zeroth homology functor $H_0: \mathcal{D}(\Gamma_N A_2) \rightarrow \text{mod-}kA_2$ induces an equivalence of abelian categories:*

$$H_0: \mathcal{H}_{\Gamma_N A_2} \xrightarrow{\sim} \text{mod-}kA_2.$$

Here $\text{mod-}kA_2$ denotes the category of finite dimensional kA_2 -modules, where kA_2 is the path algebra of the A_2 -quiver.

Therefore, the heart $\mathcal{H}_{\Gamma_N A_2}$ is generated by the simple $\Gamma_N A_2$ -modules S_1, S_2 which correspond to the two vertices of the A_2 -quiver; and $\mathcal{D}(\Gamma_N A_2)$ also is generated by S_1, S_2 (and their shifts). By [25, Lemma 4.4], both S_1, S_2 are spherical objects in the N -Calabi–Yau category $\mathcal{D}(\Gamma_N A_2)$, namely, they satisfy

$$\mathrm{Hom}^\bullet(S_1, S_2) = \mathbb{C}[-1] \text{ and } \mathrm{Hom}^\bullet(S_i, S_i) = \mathbb{C} \oplus \mathbb{C}[-N] \text{ for } i = 1, 2.$$

For any spherical object S , one can associate to it an autoequivalence called the (Seidel–Thomas) *spherical twist* T_S [39]:

$$T_S(E) = \mathrm{Cone}(\mathrm{Hom}^\bullet(S, E) \otimes S \rightarrow E).$$

We denote the spherical twist associated to S_i by T_i .

The space of Bridgeland stability conditions on $\mathcal{D}(\Gamma_N A_2)$ has been studied in [12, 13, 23]. There is a distinguished connected component of $\mathrm{Stab}(\mathcal{D}(\Gamma_N A_2))$ which contains all stability conditions whose heart $P((0, 1])$ coincides with the canonical heart $\mathcal{H}_{\Gamma_N A_2}$ (see for instance [23, Section 5] and references therein). We choose $\mathrm{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))$ in Assumption (B) to be the distinguished connected component.

We prove a large class of autoequivalences on $\mathcal{D}(\Gamma_N A_2)$ are pseudo-Anosov.

Theorem 3.2. *Let $N \geq 3$ be an odd integer. Let $\Phi \in \mathrm{Aut}(\mathcal{D}(\Gamma_N A_2))$ be any composition of T_1 and T_2^{-1} (e.g. $T_1^{a_1} T_2^{-b_1} T_1^{a_2} T_2^{-b_2} \dots$ for $a_i, b_i \geq 0$) that is neither T_1^a nor T_2^{-b} . Then Φ is pseudo-Anosov.*

For the ease of notations, we make the following definition.

Definition 3.3. Let $\mathcal{D}$ be a triangulated category and $E, A_1, \dots, A_n$ be objects in $\mathcal{D}$. We write

$$E \in \{A_1, A_2, \dots, A_n\}$$

if there exists a sequence of exact triangles

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & \star & \xrightarrow{\quad} & \star & \xrightarrow{\quad} & \dots \xrightarrow{\quad} \star \xrightarrow{\quad} E. \\ & & \nwarrow \text{dashed} & \nearrow & \nwarrow \text{dashed} & \nearrow & \nwarrow \text{dashed} \nearrow \\ & & A_1 & & A_2 & & A_n \end{array}$$

Lemma 3.4. *Let $N \geq 3$ be an integer. In $\mathcal{D}(\Gamma_N A_2)$, we have:*

- $T_1(S_1) = S_1[1 - N]$ and $T_1(S_2) \in \{S_2, S_1\}$.
- $T_2(S_1) \in \{S_1, S_2[2 - N]\}$ and $T_2(S_2) = S_2[1 - N]$.
- $T_1^{-1}(S_1) = S_1[N - 1]$ and $T_1^{-1}(S_2) \in \{S_1[N - 2], S_2\}$.
- $T_2^{-1}(S_1) \in \{S_2, S_1\}$ and $T_2^{-1}(S_2) = S_2[N - 1]$.

Proof. The lemma follows directly from the definition of spherical objects, spherical twists, and the fact that $\mathrm{Hom}^\bullet(S_1, S_2) = \mathbb{C}[-1]$. For instance, $T_2^{-1}(S_1) \in \{S_2, S_1\}$ can be obtained by applying T_2^{-1} to the exact triangle $\mathrm{Hom}^\bullet(S_2, S_1) \otimes S_2 \rightarrow S_1 \rightarrow T_2(S_1) \rightarrow$. $\square$

The following lemma will be useful throughout this section.

Lemma 3.5. *Let $N \geq 3$ be an integer. Let $\sigma \in \mathrm{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))$ be a stability condition such that S_1, S_2 and their shifts are the only indecomposable semistable objects of σ , and their phases (with respect to σ) satisfy*

$$0 < \phi(S_1) < \phi(S_2) < 1.$$

Suppose that $E \in \{S_{i_1}^{\oplus q_1}[j_1], S_{i_2}^{\oplus q_2}[j_2]\}$ for some $i_1, i_2 \in \{1, 2\}$, $q_1, q_2 \in \mathbb{Z}_{>0}$, and $j_1, j_2 \in \mathbb{Z}$. Moreover, suppose that

$$\phi(S_{i_1}) + j_1 = \phi(S_{i_1}^{\oplus q_1}[j_1]) < \phi(S_{i_2}^{\oplus q_2}[j_2]) = \phi(S_{i_2}) + j_2.$$

Then

- (a) *If $i_1 \neq i_2$ or $j_2 - j_1 \neq 1$, then $E \in \{S_{i_2}^{\oplus q_2}[j_2], S_{i_1}^{\oplus q_1}[j_1]\}$.*
- (b) *If $i_1 = i_2$ and $j_2 - j_1 = 1$, then $E \in \{S_{i_2}^{\oplus q_2-t}[j_2], S_{i_1}^{\oplus q_1-t}[j_1]\}$ for some $t \geq 0$.*

In other words, one can exchange the order of $S_{i_1}^{\oplus q_1}[j_1]$ and $S_{i_2}^{\oplus q_2}[j_2]$ so that the phases are in decreasing order, with the caveat that some of the factors might disappear when $i_1 = i_2$ and $j_2 - j_1 = 1$.

Proof. If $\phi(S_{i_1}^{\oplus q_1}[j_1]) < \phi(S_{i_2}^{\oplus q_2}[j_2])$, then $\{S_{i_1}^{\oplus q_1}[j_1], S_{i_2}^{\oplus q_2}[j_2]\}$ is of one of the following three types:

- $\{S_i^{\oplus q_1}[j_1], S_i^{\oplus q_2}[j_1 + n]\}$ for some $i \in \{1, 2\}$ and $n > 0$,
- $\{S_2^{\oplus q_1}[j_1], S_1^{\oplus q_2}[j_1 + n]\}$ for some $n > 0$,
- $\{S_1^{\oplus q_1}[j_1], S_2^{\oplus q_2}[j_1 + n]\}$ for some $n \geq 0$.

Observe that $\mathrm{Hom}(S_{i_2}^{\oplus q_2}[j_2], S_{i_1}^{\oplus q_1}[j_1 + 1]) = 0$ holds for all three types, except for $i_1 = i_2$ and $j_2 - j_1 = 1$. Hence if $i_1 \neq i_2$ or $j_2 - j_1 \neq 1$, then $E \cong S_{i_1}^{\oplus q_1}[j_1] \oplus S_{i_2}^{\oplus q_2}[j_2]$. This proves part (a).

If $i := i_1 = i_2$ and $j := j_1 = j_2 - 1$, then there is an exact triangle

$$S_i^{\oplus q_1}[j] \rightarrow E \rightarrow S_i^{\oplus q_2}[j + 1] \rightarrow .$$

Recall that any object E in $\mathcal{D}$ has a unique filtration with respect to the heart of a bounded t-structure $\mathcal{A} := \mathcal{H}_{\Gamma_N A_2} = \langle S_1, S_2 \rangle_{\text{ext}}$ on $\mathcal{D}$:

$$\begin{array}{ccccccc}
 0 & \xrightarrow{\quad} & \star & \xrightarrow{\quad} & \star & \rightarrow \dots \rightarrow & \star & \xrightarrow{\quad} & E, \\
 & & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow \\
 & & H_{\mathcal{A}}^{-k_1}(E)[k_1] & & H_{\mathcal{A}}^{-k_2}(E)[k_2] & & H_{\mathcal{A}}^{-k_m}(E)[k_m] & &
 \end{array}$$

where $k_1 > k_2 > \dots > k_m$ and $H_{\mathcal{A}}^{-k_i}(E) \in \mathcal{A}$ for each i . The objects $H_{\mathcal{A}}^{-k_i}(E)$ are sometimes called the cohomological objects of E with respect to $\mathcal{A}$. Any exact triangle induces a long exact sequence of cohomological objects. Hence the exact triangle $S_i^{\oplus q_1}[j] \rightarrow E \rightarrow S_i^{\oplus q_2}[j+1] \rightarrow$ induces an exact sequence in $\mathcal{A}$:

$$0 \rightarrow H_{\mathcal{A}}^{-j-1}(E) \rightarrow S_i^{\oplus q_2} \rightarrow S_i^{\oplus q_1} \rightarrow H_{\mathcal{A}}^{-j}(E) \rightarrow 0.$$

Therefore we have

$$H_{\mathcal{A}}^{-j-1}(E) = S_i^{\oplus q_2-t} \quad \text{and} \quad H_{\mathcal{A}}^{-j}(E) = S_i^{\oplus q_1-t}$$

for some $t \geq 0$, since S_i is a simple object in $\mathcal{A}$. Thus there is an exact triangle

$$S_i^{\oplus q_2-t}[j+1] \rightarrow E \rightarrow S_i^{\oplus q_1-t}[j] \rightarrow .$$

This proves part (b). □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Since $\mathcal{H}_{\Gamma_N A_2} \cong \text{mod-}kA_2$ is a finite length abelian category with finitely many simple objects $\{S_1, S_2\}$ (see Proposition 3.1), any pair of points $(z_1, z_2) \in \mathbb{H}^2$ in the upper half-plane gives a Bridgeland stability condition on $\mathcal{D}(\Gamma_N A_2)$, such that the central charges of S_1 and S_2 are z_1 and z_2 , respectively ([11, Lemma 5.2]). We choose $z_1, z_2 \in \mathbb{H}$ such that $z_1 = e^{i\pi\phi_1}$ and $z_2 = e^{i\pi\phi_2}$, where $0 < \phi_1 < \phi_2 < 1$, and let $\sigma = (Z, P) \in \text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))$ denotes the corresponding stability condition. Then we have

$$0 < \phi(S_1) = \phi_1 < \phi(S_2) = \phi_2 < 1 \quad \text{and} \quad |Z(S_1)| = |Z(S_2)| = 1.$$

The only indecomposable objects in $\mathcal{H}_{\Gamma_N A_2} \cong \text{mod-}kA_2$ are the simple objects S_1, S_2 , and an object E given by the extension between these two objects:

$$0 \rightarrow S_2 \rightarrow E \rightarrow S_1 \rightarrow 0.$$

Since we choose z_1, z_2 so that $0 < \phi(S_1) < \phi(S_2) < 1$, the object E is not semistable with respect to σ . Therefore, the only indecomposable σ -semistable objects are S_1, S_2 , and their shifts.

Let us compute the mass growth of the split generator $S_1 \oplus S_2$ of $\mathcal{D}(\Gamma_N A_2)$ with respect to the autoequivalence Φ and the stability condition σ . By Lemma 3.4, since Φ is a composition of positive powers of T_1 and T_2^{-1} , for any $n \in \mathbb{N}$ we have

$$\Phi^n(S_1 \oplus S_2) \in \{S_{i_1}^{\oplus q_1}[(N-1)j_1], S_{i_2}^{\oplus q_2}[(N-1)j_2], \dots, S_{i_k}^{\oplus q_k}[(N-1)j_k]\}$$

for some $i_1, \dots, i_k \in \{1, 2\}$, $j_1, \dots, j_k \in \mathbb{Z}$, and $q_1, \dots, q_k \in \mathbb{N}$. By Lemma 3.5, one can reorder the sequence so that the objects are of decreasing phases (since $N \geq 3$, so case (b) in Lemma 3.5 does not happen in the sequence). This gives the Harder–Narasimhan filtration of $\Phi^n(S_1 \oplus S_2)$. By the choice of the stability condition σ , the mass of $\Phi^n(S_1 \oplus S_2)$ is simply the sum of the powers $q_1 + \dots + q_k$ in the sequence. Hence in order to compute the mass growth of $\Phi^n(S_1 \oplus S_2)$, one only needs to compute the increment of the number of S_1 's and S_2 's when Φ acts on S_1 and S_2 .

The spherical twist T_1 corresponds to the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

i.e. if there is a sequence with x_1 S_1 's and x_2 S_2 's ($S_i^{\oplus q}$ counts as q S_i 's), then after applying T_1 , the new sequence will have $(x_1 + x_2)$ S_1 's and x_2 S_2 's. This follows from the fact that $T_1(S_1) = S_1[1 - N]$ and $T_1(S_2) \in \{S_2, S_1\}$. Similarly, T_2^{-1} corresponds to the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let $\Phi = T_1^{a_1} T_2^{-b_1} T_1^{a_2} T_2^{-b_2} \dots T_1^{a_p} T_2^{-b_p}$ for some $a_i, b_i \geq 0$. Then the corresponding matrix of Φ is

$$M_\Phi := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{b_1} \dots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_p} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{b_p} \in \mathrm{SL}(2, \mathbb{Z}).$$

Since Φ is neither T_1^a nor T_2^{-b} , the corresponding matrix M_Φ has trace $t \geq 3$ and eigenvalues $\frac{t \pm \sqrt{t^2 - 4}}{2} \notin \mathbb{Q}$. Since $(1, 1)^T$ is not an eigenvector of the eigenvalue $\frac{t - \sqrt{t^2 - 4}}{2}$, one can conclude that the mass growth of $S_1 \oplus S_2$ is

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log m_\sigma(\Phi^n(S_1 \oplus S_2)) = \log \frac{t + \sqrt{t^2 - 4}}{2} =: \log \lambda > 0.$$

To prove that Φ is pseudo-Anosov, one needs to show that the mass growth of any nonzero object E is also $\log \lambda$. By Theorem 2.19, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z_\sigma(\Phi^n E)| \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log m_\sigma(\Phi^n E) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log m_\sigma(\Phi^n(S_1 \oplus S_2)) = \log \lambda.$$

On the other hand, since N is an odd number, the actions of T_1 and T_2^{-1} on the Grothendieck group $K_0(\mathcal{D}(\Gamma_N A_2))$ is given by the same matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, with respect to the basis $\{[S_1], [S_2]\} \subset K_0(\mathcal{D}(\Gamma_N A_2))$ (cf. Lemma 3.4).- By [9], one can deform σ if necessary so that the kernel of Z_σ does not contain any eigenvector of M_Φ . Hence the (exponential) growth of $|Z_\sigma(\Phi^n E)|$ is $\log \lambda$ unless $[E] \in K_0(\mathcal{D}(\Gamma_N A_2))$ is an eigenvector of the smaller eigenvalue

$$\lambda' = \frac{t - \sqrt{t^2 - 4}}{2},$$

which is impossible since $[E]$ is an integral vector. Hence

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z_\sigma(\Phi^n E)| = \limsup_{n \rightarrow \infty} \frac{1}{n} \log m_\sigma(\Phi^n E) = \log \lambda$$

for any nonzero object E , therefore Φ is a pseudo-Anosov autoequivalence. $\square$

Remark 3.6. The same argument also works for compositions of $T_1, T_2^{-1}, \dots, T_{2n-1}, T_{2n}^{-1}$ on the N -Calabi–Yau category associated to the A_{2n} -quiver ($N \geq 3$ odd):

$$\bullet_1 \longrightarrow \bullet_2 \longleftarrow \bullet_3 \longrightarrow \cdots \longleftarrow \bullet_{2n-1} \longrightarrow \bullet_{2n},$$

provided there is no integral vector in the span of the eigenvectors of eigenvalues other than the spectral radius. The reason is that the action of $T_1, T_2^{-1}, \dots, T_{2n-1}, T_{2n}^{-1}$ on the Grothendieck group $K_0(\mathcal{D}(\Gamma_N A_{2n}))$ is represented by non-negative matrices. For instance, one can show that $T_1 T_2^{-1} T_3 T_4^{-1}$ on $\mathcal{D}_4^N$ is pseudo-Anosov. This autoequivalence has a geometric interpretation as the total monodromy acting on the Fukaya category of the fiber of a Lefschetz fibration whose vanishing cycles are the $2n$ spherical objects $S_1, \dots, S_{2n}$.

On the other hand, the pseudo-Anosov autoequivalences we find in Theorem 3.2 are not DHKK pseudo-Anosov. In fact, we prove the following stronger statement.

Proposition 3.7. *Let $\mathcal{D}$ be a triangulated category. If for any $\sigma \in \text{Stab}^\dagger(\mathcal{D})$ there are only finitely many σ -stable objects (up to shifts), then $\mathcal{D}$ does not possess any DHKK pseudo-Anosov autoequivalence.*

Proof. Let $\Phi \in \text{Aut}(\mathcal{D})$ be a DHKK pseudo-Anosov autoequivalence and assume the triple (Φ, σ, g) satisfies the conditions in Definition 2.14. Then Φ preserves the set of all σ -stable objects. Since there are only finitely many σ -stable objects up to shifts, there exists some $N > 0$ such that Φ^N preserves all σ -stable objects up to shifts by even integers. Note that the classes of σ -stable objects generate $K_0(\mathcal{D})$. Therefore Φ^N acts trivially on $K_0(\mathcal{D})$. This contradicts with the stretching/contracting condition of DHKK pseudo-Anosov autoequivalences. $\square$

Recall that an isometry f of a metric space (X, d) is called *hyperbolic* if its *translation length* $\tau(f) := \inf_{x \in X} \{d(x, f(x))\}$ is positive and the infimum is achieved by some $x \in X$. It is proved by Kikuta [28] that any DHKK pseudo-Anosov autoequivalence Φ acts hyperbolically on $(\text{Stab}^\dagger(\mathcal{D})/\mathbb{C}, \bar{d})$, and $\tau(\Phi) = \log \lambda$. We note that it is more interesting to study the hyperbolicity on the quotient space $\text{Stab}^\dagger(\mathcal{D})/\mathbb{C}$ rather than $\text{Stab}^\dagger(\mathcal{D})$, because trivial examples like the shift functors $[k]$ on any triangulated category act hyperbolically on $\text{Stab}^\dagger(\mathcal{D})$.

We prove the hyperbolicity of the “palindromic” (see the statement of Theorem 3.8 for the definition) pseudo-Anosov autoequivalences constructed in Theorem 3.2 for the case $N = 3$. Note that the proof is less straightforward than it is for the DHKK pseudo-Anosov autoequivalences in [28]. One of the key ingredients in the proof, which is also used in the classical theory of pseudo-Anosov maps, is the Perron–Frobenius theorem. It states that a real square matrix with positive entries has a unique largest real eigenvalue λ and that its corresponding eigenvector can be chosen to have positive components. Moreover, the absolute value of any other eigenvalue is strictly less than λ .

Theorem 3.8. *Let $N \geq 3$ be an odd integer. Let $\Phi \in \text{Aut}(\mathcal{D}(\Gamma_N A_2))$ be a palindromic composition of T_1 and T_2^{-1} (i.e. $T_1^{a_1} T_2^{-b_1} T_1^{a_2} T_2^{-b_2} \dots T_1^{a_p}$ for $a_i, b_i \geq 0$ such that the sequences $(a_1, b_1, \dots, a_p)$ and $(a_p, b_{p-1}, a_{p-1}, \dots, b_1, a_1)$ coincide) that is neither T_1^a nor T_2^{-b} . Then Φ acts hyperbolically on $\mathcal{M}_{\text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))}/\mathbb{R}^+$. If $N = 3$, then Φ also acts hyperbolically on $\text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))/\mathbb{C}$. Moreover, in both cases the translation length $\tau(\Phi) = \log \lambda$, where λ is the stretch factor of Φ .*

We prove the following lemma before proving Theorem 3.8.

Lemma 3.9. *Let $\sigma \in \text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))$ be a stability condition such that S_1, S_2 and their shifts are the only indecomposable semistable objects of σ , and their phases satisfy*

$$0 < \phi(S_1) < \phi(S_2) < 1.$$

Suppose that

$$E \in \{S_{i_1}^{\oplus q_1}[j_1], S_{i_2}^{\oplus q_2}[j_2], \dots, S_{i_k}^{\oplus q_k}[j_k]\},$$

where $i_1, \dots, i_k \in \{1, 2\}$, $j_1, \dots, j_k \in \mathbb{Z}$, and $q_1, \dots, q_k \in \mathbb{N}$. Then

$$\min_{1 \leq \ell \leq k} \{j_\ell + \phi(S_{i_\ell})\} \leq \phi_\sigma^-(E) \leq \phi_\sigma^+(E) \leq \max_{1 \leq \ell \leq k} \{j_\ell + \phi(S_{i_\ell})\}$$

and

$$m_\sigma(E) \leq \sum_{\ell=1}^k q_\ell |Z_\sigma(S_{i_\ell})|.$$

Proof. Firstly, if $j_1 + \phi(S_{i_1}) > \dots > j_k + \phi(S_{i_k})$, then $\{S_{i_1}^{\oplus q_1}[j_1], S_{i_2}^{\oplus q_2}[j_2], \dots, S_{i_k}^{\oplus q_k}[j_k]\}$ gives the Harder–Narasimhan filtration of E , hence we have

$$\phi_{\sigma}^{-}(E) = j_k + \phi(S_{i_k}) = \min_{1 \leq \ell \leq k} \{j_{\ell} + \phi(S_{i_{\ell}})\}, \quad \phi_{\sigma}^{+}(E) = j_1 + \phi(S_{i_1}) = \max_{1 \leq \ell \leq k} \{j_{\ell} + \phi(S_{i_{\ell}})\},$$

and

$$m_{\sigma}(E) = \sum_{\ell=1}^k q_{\ell} |Z_{\sigma}(S_{i_{\ell}})|.$$

Now suppose $j_{\ell} + \phi(S_{i_{\ell}}) < j_{\ell+1} + \phi(S_{i_{\ell+1}})$ for some ℓ . By Lemma 3.5, one can exchange the order of $\{S_{i_{\ell}}^{\oplus q_{\ell}}[j_{\ell}], S_{i_{\ell+1}}^{\oplus q_{\ell+1}}[j_{\ell+1}]\}$ unless $i := i_{\ell} = i_{\ell+1}$ and $j := j_{\ell} = j_{\ell+1} - 1$:

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C. \\ & \swarrow \text{dashed} & \searrow & \swarrow \text{dashed} & \searrow \\ & S_i^{\oplus q_{\ell}}[j] & & S_i^{\oplus q_{\ell+1}}[j+1] & \end{array}$$

In this case, by part (b) of Lemma 3.5 and the Octahedral Axiom of triangulated categories, there are exact triangles

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & \star & \xrightarrow{\quad} & C. \\ & \swarrow \text{dashed} & \searrow & \swarrow \text{dashed} & \searrow \\ & S_i^{\oplus(q_{\ell+1}-t)}[j_{\ell}+1] & & S_i^{\oplus(q_{\ell}-t)}[j_{\ell}] & \end{array}$$

for some $t \geq 0$. In other words, one can still exchange the order of these two terms to make the phases decreasing, but the number of factors of $S_i[j_{\ell}]$ and $S_i[j_{\ell}+1]$ might decrease.

In conclusion, one can exchange the order of the factors in $\{S_{i_1}^{\oplus q_1}[j_1], S_{i_2}^{\oplus q_2}[j_2], \dots, S_{i_k}^{\oplus q_k}[j_k]\}$ so that the phases are in the decreasing order to get the Harder–Narasimhan filtration of E , with the caveat that some of the factors might disappear during the process when exchanging two terms of the type $\{S_i^{\oplus q_{\ell}}[j], S_i^{\oplus q_{\ell+1}}[j+1]\}$. Hence

$$\min_{1 \leq \ell \leq k} \{j_{\ell} + \phi(S_{i_{\ell}})\} \leq \phi_{\sigma}^{-}(E) \leq \phi_{\sigma}^{+}(E) \leq \max_{1 \leq \ell \leq k} \{j_{\ell} + \phi(S_{i_{\ell}})\}$$

and

$$m_{\sigma}(E) \leq \sum_{\ell=1}^k q_{\ell} |Z_{\sigma}(S_{i_{\ell}})|$$

as desired. $\square$

Now we prove Theorem 3.8.

Proof of Theorem 3.8. Recall the *stable translation length* of an isometry f on a metric space (X, d) is defined as

$$\bar{\tau}(f) := \lim_{n \rightarrow \infty} \frac{d(x, f^n(x))}{n}.$$

One can check by the triangle inequality that $\overline{\tau}(f)$ is independent of the choice of $x \in X$, and $\tau(f) \geq \overline{\tau}(f)$. The idea of proof is to find a Bridgeland stability condition $\sigma \in \text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))$ such that

- (a) $\log \lambda \geq \bar{d}(\bar{\sigma}, \Phi \bar{\sigma})$, and
- (b) $\lim_{n \rightarrow \infty} \frac{1}{n} \bar{d}(\bar{\sigma}, \Phi^n \bar{\sigma}) \geq \log \lambda$.

Here $\bar{d}$ is the metric on $\text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))/\mathbb{C}$ or $\mathcal{M}_{\text{Stab}^\dagger}(\mathcal{D}(\Gamma_N A_2))/\mathbb{R}^+$ defined in Section 2.1. Assuming that one can find such a stability condition σ , then we have

$$\log \lambda \geq \bar{d}(\bar{\sigma}, \Phi \bar{\sigma}) \geq \tau(\Phi) \geq \overline{\tau}(\Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \bar{d}(\bar{\sigma}, \Phi^n \bar{\sigma}) \geq \log \lambda.$$

Therefore $\tau(\Phi) = \overline{\tau}(\Phi) = \log \lambda > 0$ and the infimum $\tau(\Phi)$ is achieved by $\bar{d}(\bar{\sigma}, \Phi \bar{\sigma})$. Hence the action is hyperbolic.

We choose the stability condition σ as follows. Let M_Φ be the corresponding matrix of Φ we defined in the proof of Theorem 3.2:

$$M_\Phi := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{b_1} \cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_p}.$$

Since Φ is neither T_1^a nor T_2^{-b} , the matrix M_Φ is a *positive* matrix. By Perron–Frobenius theorem, the eigenvector $v = [v_1, v_2]^T$ of the larger eigenvalue of M_Φ^T can be chosen to have positive components $v_1, v_2 > 0$. We choose a Bridgeland stability condition $\sigma \in \text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))$ such that S_1, S_2 and their shifts are the only indecomposable semistable objects of σ , and their phases and central charges satisfy

$$0 < \phi(S_1) < \phi(S_2) < 1, \quad |Z(S_1)| = v_1 \quad \text{and} \quad |Z(S_2)| = v_2.$$

We claim that σ chosen in this way satisfies conditions (a) and (b) above for the metric space $\mathcal{M}_{\text{Stab}^\dagger}(\mathcal{D}(\Gamma_N A_2))/\mathbb{R}^+$, and for $\text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))/\mathbb{C}$ when $N = 3$.

Proof of (a): We first consider the metric space $(\text{Stab}^\dagger(\mathcal{D}(\Gamma_N A_2))/\mathbb{C}, \bar{d})$. Recall that

$$\begin{aligned} \bar{d}(\bar{\sigma}, \Phi \bar{\sigma}) &= \inf_{z \in \mathbb{C}} d(\sigma, \Phi \sigma \cdot z) \\ &= \inf_{\substack{x \in \mathbb{R} \\ y > 0}} \sup_{E \neq 0} \left\{ |\phi_\sigma^-(E) - \phi_\sigma^-(\Phi E) + x|, |\phi_\sigma^+(E) - \phi_\sigma^+(\Phi E) + x|, \left| \log \frac{ym_\sigma(\Phi E)}{m_\sigma(E)} \right| \right\}. \end{aligned}$$

Let $E \in \mathcal{D}(\Gamma_N A_2)$ be any nonzero object. We claim the following inequalities hold:

- (1) $\phi_\sigma^\pm(T_1(E)) \leq \phi_\sigma^\pm(E) \leq \phi_\sigma^\pm(T_1(E)) + (N - 1)$.
- (2) $\phi_\sigma^\pm(T_2^{-1}(E)) - (N - 1) \leq \phi_\sigma^\pm(E) \leq \phi_\sigma^\pm(T_2^{-1}(E))$.

We will only prove (1) as the proof of (2) is similar. Suppose the Harder–Narasimhan filtration of E is:

$$S_{i_1}^{\oplus q_1}[j_1], S_{i_2}^{\oplus q_2}[j_2], \dots, S_{i_k}^{\oplus q_k}[j_k],$$

where $i_1, \dots, i_k \in \{1, 2\}$, $j_1, \dots, j_k \in \mathbb{Z}$, $q_1, \dots, q_k \in \mathbb{N}$, and

$$\phi_\sigma^+(E) = j_1 + \phi(S_{i_1}) > \dots > j_k + \phi(S_{i_k}) = \phi_\sigma^-(E).$$

Then

$$T_1(E) \in \{T_1(S_{i_1})^{\oplus q_1}[j_1], \dots, T_1(S_{i_k})^{\oplus q_k}[j_k]\}.$$

Recall from Lemma 3.4 that $T_1(S_1) = S_1[1 - N]$ and $T_1(S_2) \in \{S_2, S_1\}$. Combining with Lemma 3.9, we have

$$\phi_\sigma^+(T_1(E)) \leq \max_{1 \leq \ell \leq k} \{j_\ell + \phi(S_{i_\ell})\} = \phi_\sigma^+(E)$$

and

$$\phi_\sigma^-(T_1(E)) \geq \min_{1 \leq \ell \leq k} \{j_\ell + \phi(S_{i_\ell})\} - (N - 1) = \phi_\sigma^-(E) - (N - 1).$$

Similarly, one can apply T_1^{-1} on E :

$$T_1^{-1}(E) \in \{T_1^{-1}(S_{i_1})^{\oplus q_1}[j_1], \dots, T_1^{-1}(S_{i_k})^{\oplus q_k}[j_k]\}.$$

Again by Lemma 3.4 and 3.9, we have

$$\phi_\sigma^+(T_1^{-1}(E)) \leq \max_{1 \leq \ell \leq k} \{j_\ell + \phi(S_{i_\ell})\} + (N - 1) = \phi_\sigma^+(E) + (N - 1)$$

and

$$\phi_\sigma^-(T_1^{-1}(E)) \geq \min_{1 \leq \ell \leq k} \{j_\ell + \phi(S_{i_\ell})\} = \phi_\sigma^-(E).$$

By applying T_1 to both inequalities, one gets

$$\phi_\sigma^-(T_1(E)) \leq \phi_\sigma^-(E) \leq \phi_\sigma^+(E) \leq \phi_\sigma^+(T_1(E)) + (N - 1).$$

This proves the inequalities in (1). The proof of inequalities in (2) is similar. Hence

$$\left| \phi_\sigma^\pm(E) - \phi_\sigma^\pm(T_1(E)) - \frac{N-1}{2} \right| \leq \frac{N-1}{2} \text{ and } \left| \phi_\sigma^\pm(E) - \phi_\sigma^\pm(T_1^{-1}(E)) + \frac{N-1}{2} \right| \leq \frac{N-1}{2}$$

for any nonzero object $E \in \mathcal{D}$.

Let $\Phi = T_1^{a_1} T_2^{-b_1} T_1^{a_2} T_2^{-b_2} \dots T_1^{a_p}$ for some $a_i, b_i \geq 0$. Define

$$s := \sum_{i=1}^p a_i + \sum_{i=1}^{p-1} b_i \text{ and } x := \sum_{i=1}^{p-1} b_i - \sum_{i=1}^p a_i.$$

By the above inequalities, we have

$$\left| \phi_\sigma^\pm(E) - \phi_\sigma^\pm(\Phi E) + \frac{x(N-1)}{2} \right| \leq \frac{s(N-1)}{2}$$

for any nonzero object E . Since Φ is neither T_1^a nor T_2^{-b} , the trace of M_Φ is at least $s + 1$. Hence

$$\log \lambda \geq \frac{s + 1 + \sqrt{(s + 1)^2 - 4}}{2}.$$

Therefore if $N = 3$, we have

$$\sup_{E \neq 0} \left\{ \left| \phi_{\sigma}^{\pm}(E) - \phi_{\sigma}^{\pm}(\Phi E) + \frac{x(N-1)}{2} \right| \right\} \leq \frac{s(N-1)}{2} = s < \frac{s+1 + \sqrt{(s+1)^2 - 4}}{2} \leq \log \lambda.$$

Hence in order to prove statement (a), i.e. that $\bar{d}(\bar{\sigma}, \Phi \bar{\sigma}) \leq \log \lambda$, it remains to show that

$$m_{\sigma}(\Phi E) \leq \lambda m_{\sigma}(E) \quad \text{and} \quad m_{\sigma}(\Phi^{-1} E) \leq \lambda m_{\sigma}(E)$$

for any nonzero object E . Suppose the Harder–Narasimhan filtration of E has x_1 S_1 's and x_2 S_2 's (up to shifts). Write $x = (x_1 \ x_2)^T$ as a vector. Then the mass of E is given by $m_{\sigma}(E) = v^T x$ since $|Z_{\sigma}(S_1)| = v_1$ and $|Z_{\sigma}(S_2)| = v_2$. By Lemma 3.9, we have

$$m_{\sigma}(\Phi E) \leq v^T M_{\Phi} x = \lambda v^T x = \lambda m_{\sigma}(E).$$

To show that $m_{\sigma}(\Phi^{-1} E) \leq \lambda m_{\sigma}(E)$, recall from Lemma 3.4 that

$$T_1^{-1}(S_1) = S_1[N-1] \quad \text{and} \quad T_1^{-1}(S_2) \in \{S_1[N-2], S_2\},$$

$$T_2(S_1) \in \{S_1, S_2[2-N]\} \quad \text{and} \quad T_2(S_2) = S_2[1-N].$$

Hence T_1^{-1} corresponds to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in the sense that if

$$F \in \{S_{i_1}^{\oplus q_1}[j_1], S_{i_2}^{\oplus q_2}[j_2], \dots, S_{i_k}^{\oplus q_k}[j_k]\},$$

where there are x_1 S_1 's and x_2 S_2 's among $\{S_{i_1}^{\oplus q_1}[j_1], S_{i_2}^{\oplus q_2}[j_2], \dots, S_{i_k}^{\oplus q_k}[j_k]\}$ (here $S_i^{\oplus q}$ counts as q S_i 's), then $T_1^{-1}(F)$ can be written as

$$T_1^{-1}(F) \in \{S_{c_1}^{\oplus p_1}[d_1], S_{c_2}^{\oplus p_2}[d_2], \dots, S_{c_{\ell}}^{\oplus p_{\ell}}[d_{\ell}]\},$$

where there are $x_1 + x_2$ S_1 's and x_2 S_2 's among $\{S_{c_1}^{\oplus p_1}[d_1], S_{c_2}^{\oplus p_2}[d_2], \dots, S_{c_{\ell}}^{\oplus p_{\ell}}[d_{\ell}]\}$.

Similarly, T_2 corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Hence the corresponding matrix of

$\Phi^{-1} = T_1^{-a_p} T_2^{b_{p-1}} \dots T_2^{b_1} T_1^{-a_1}$ is

$$M_{\Phi^{-1}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_p} \dots \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{b_1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_1} = M_{\Phi}$$

which coincides with M_{Φ} since Φ is assumed to be palindromic. Hence by Lemma 3.9,

$$m_{\sigma}(\Phi^{-1} E) \leq v^T M_{\Phi^{-1}} x = v^T M_{\Phi} x = \lambda v^T x = \lambda m_{\sigma}(E).$$

This concludes the proof of (a) for the metric space $(\text{Stab}^{\dagger}(\mathcal{D}(\Gamma_N A_2))/\mathbb{C}, \bar{d})$ for $N = 3$. The proof for $(\mathcal{M}_{\text{Stab}^{\dagger}(\mathcal{D}(\Gamma_N A_2))}/\mathbb{R}^+, \bar{d})$ is even simpler because we only need to deal with the mass term.

Proof of (b): We claim that

$$m_\sigma(\Phi^n S_1) = \lambda^n m_\sigma(S_1) = m_\sigma(\Phi^{-n} S_1)$$

for any $n \in \mathbb{N}$. By Lemma 3.4,

$$\Phi^n(S_1) \in \{S_{i_1}^{\oplus q_1}[(N-1)j_1], S_{i_2}^{\oplus q_2}[(N-1)j_2], \dots, S_{i_k}^{\oplus q_k}[(N-1)j_k]\}$$

for some $i_1, \dots, i_k \in \{1, 2\}$, $j_1, \dots, j_k \in \mathbb{Z}$, and $q_1, \dots, q_k \in \mathbb{N}$. By Lemma 3.5, one can reorder the sequence to get the Harder–Narasimhan filtration of $\Phi^n(S_1)$ since $N \geq 3$. Therefore

$$m_\sigma(\Phi^n S_1) = v^T M_\Phi^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda^n v^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda^n m_\sigma(S_1).$$

On the other hand, again by Lemma 3.4, we have

$$\Phi^{-n}(S_1) \in \{S_1^{\oplus p_1}[(N-1)i_1], S_2^{\oplus q_1}[(N-1)j_1+1], \dots, S_1^{\oplus p_k}[(N-1)i_k], S_2^{\oplus q_k}[(N-1)j_k+1]\}$$

for some $p_1, q_1, \dots, p_k, q_k \geq 0$ and $j_1, \dots, j_k \in \mathbb{Z}$. (The degree shifts of S_1 's are divisible by $N-1$, while the degree shifts of S_2 's are equal to 1 modulo $N-1$.) By Lemma 3.5, one can reorder the sequence to get the Harder–Narasimhan filtration of $\Phi^{-n}(S_1)$ since $N \geq 3$. Using the assumption that Φ is palindromic, we have

$$m_\sigma(\Phi^{-n} S_1) = v^T M_{\Phi^{-1}}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v^T M_\Phi^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda^n v^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda^n m_\sigma(S_1).$$

This proves the claim that $m_\sigma(\Phi^n S_1) = \lambda^n m_\sigma(S_1) = m_\sigma(\Phi^{-n} S_1)$. Therefore

$$\begin{aligned} \bar{d}(\bar{\sigma}, \Phi^n \bar{\sigma}) &\geq \inf_{y>0} \sup_{E \neq 0} \left\{ \left| \log \frac{ym_\sigma(\Phi^n E)}{m_\sigma(E)} \right| \right\} \\ &\geq \inf_{y>0} \max \left\{ \left| \log \frac{ym_\sigma(\Phi^n S_1)}{m_\sigma(S_1)} \right|, \left| \log \frac{ym_\sigma(S_1)}{m_\sigma(\Phi^{-n} S_1)} \right| \right\} = n \log \lambda. \end{aligned}$$

Hence we have $\lim_{n \rightarrow \infty} \frac{1}{n} \bar{d}(\bar{\sigma}, \Phi^n \bar{\sigma}) \geq \log \lambda$. $\square$

3.2. An example from quintic Calabi–Yau threefolds. We prove that the autoequivalence on the derived category of quintic Calabi–Yau threefolds X considered by the first author in [17] and Ouchi in [36] is pseudo-Anosov. The existence of Bridgeland stability conditions on $\mathcal{D}^b(X)$ has been established by Li [31] recently. There is a distinguished connected component of $\text{Stab}(\mathcal{D}^b(X))$ containing *geometric stability conditions* for which skyscraper sheaves are stable and of the same phase. We choose $\text{Stab}^\dagger(\mathcal{D})$ in Assumption (B) to be the distinguished connected component.

Theorem 3.10. *Let X be a quintic Calabi–Yau hypersurface in $\mathbb{CP}^4$. Then*

$$\Phi := T_{\mathcal{O}} \circ (- \otimes \mathcal{O}(-1))$$

is a pseudo-Anosov autoequivalence on $\mathcal{D}^b(X)$. Here $T_{\mathcal{O}}$ denotes the spherical twist with respect to the structure sheaf $\mathcal{O}_X$.

Proof. Take any $\sigma \in \text{Stab}^+(\mathcal{D}^b(X))$. Let $G \in \mathcal{D}^b(X)$ be a split generator and $E \in \mathcal{D}^b(X)$ be any nonzero object. By Theorem 2.19, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z_{\sigma}(\Phi^n E)| \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log m_{\sigma}(\Phi^n E) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log m_{\sigma}(\Phi^n G) \leq h_{\text{cat}}(\Phi).$$

Here $h_{\text{cat}}(\Phi)$ denotes the categorical entropy of $\Phi \in \text{Aut}(\mathcal{D}^b(X))$. By [17, Remark 4.3], we have

$$h_{\text{cat}}(\Phi) = \log \rho([\Phi]).$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the eigenvalues of $[\Phi] \in \text{Aut}(H^{\text{ev}}(X; \mathbb{C}))$ in which $\lambda_1 = \rho([\Phi]) > 1$, and let $v_1, v_2, v_3, v_4 \in H^{\text{ev}}(X; \mathbb{C})$ be the corresponding eigenvectors. By [9], one can deform the stability condition σ if needed so that $Z_{\sigma}(v_1) \neq 0$. Then we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z_{\sigma}(\Phi^n E)| = \log \rho([\Phi]) \iff [E] \notin \text{Span}\{v_2, v_3, v_4\}.$$

One can check by brute force computations that the span of v_2, v_3, v_4 does not contain any rational vectors, see Appendix C. This proves that the mass growth

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log m_{\sigma}(\Phi^n E) = \log \lambda_1 \approx 2.04 > 0$$

for any nonzero object E is the same positive number. Hence $\Phi = T_{\mathcal{O}} \circ (- \otimes \mathcal{O}(-1))$ is pseudo-Anosov. $\square$

Note that the same argument can be used to show that the autoequivalence $\Phi = T_{\mathcal{O}} \circ (- \otimes \mathcal{O}(-1))$ is pseudo-Anosov on $\mathcal{D}^b(X)$ for any Calabi–Yau manifold X of odd dimension, assuming the existence of Bridgeland stability conditions on $\mathcal{D}^b(X)$, and assuming that there is no integral vector in the span of the eigenvectors of eigenvalues other than the largest one.

3.3. Other examples. We show that on the derived category of curves, the notion of pseudo-Anosov autoequivalences coincides with DHKK pseudo-Anosov autoequivalences.

Proposition 3.11. *Let $\mathcal{D} = \mathcal{D}^b(C)$ be the derived category of a smooth projective curve C over $\mathbb{C}$. Then $\Phi \in \text{Aut}(\mathcal{D})$ is pseudo-Anosov if and only if it is DHKK pseudo-Anosov.*

Proof. By Kikuta [28, Proposition 4.13, 4.14], in the case when $\mathcal{D}$ is the derived category of a curve, $\Phi \in \text{Aut}(\mathcal{D})$ is DHKK pseudo-Anosov if and only if its categorical entropy $h_{\text{cat}}(\Phi) > 0$. On other hand, by Theorem 2.19 an autoequivalence Φ is pseudo-Anosov implies $h_{\text{cat}}(\Phi) > 0$. The proposition then follows from Theorem 2.17. $\square$

4. PSEUDO-ANOSOV MAPS AS PSEUDO-ANOSOV AUTOEQUIVALENCES

In this section we discuss two ways in which a pseudo-Anosov map can induce a pseudo-Anosov autoequivalence on the Fukaya category of the surface. The first is based on the $\mathbb{Z}$ -graded Fukaya category and stability conditions constructed from quadratic differentials, while the second is based on the $\mathbb{Z}/2$ -graded Fukaya category and the mass function defined as complexity with respect to a generator. In the second case we cannot quite prove that the induced autoequivalence is pseudo-Anosov, but provide strong evidence for this. Throughout, we assume that surfaces are oriented and maps orientation preserving.

4.1. $\mathbb{Z}$ -graded Fukaya category. Let S be a closed surface with a finite set $M \subset S$ of marked points and suppose $f: S \setminus M \rightarrow S \setminus M$ is pseudo-Anosov. By definition this means that there is a complex structure on S , a meromorphic quadratic differential φ on S (not identically zero) with at worst simple poles at each point $p \in M$ and holomorphic on $S \setminus M$, and a stretch factor $\lambda > 1$ such that f acts on φ by scaling $\operatorname{Re}(\sqrt{\varphi})$ by λ and $\operatorname{Im}(\sqrt{\varphi})$ by $1/\lambda$. Thus, if F^u is the horizontal measured foliation of φ and F^s is the vertical measured foliation of φ , then F^u and F^s are a pair of transverse measured foliations such that the underlying foliations are preserved by f and the measure for F^u (resp. F^s) is scaled by a factor λ (resp. $1/\lambda$) under the action of f . The definition in terms of a quadratic differential is particularly natural from Bers' point of view of the Thurston classification [8].

Let $Z \subset S$ be the set of zeros and poles of φ . Note that $M \subset Z$ and Z is non-empty unless S is the torus and φ is constant. In order to avoid having to deal with this special case, we assume $Z \neq \emptyset$ from now on, though this is not essential. Fix an arbitrary ground field $\mathbf{k}$, then there is a triangulated A_∞ -category $\mathcal{F} = \mathcal{F}(S \setminus Z, F^u, \mathbf{k})$ over $\mathbf{k}$, the $\mathbb{Z}$ -graded *wrapped Fukaya category* of the punctured surface $S \setminus Z$ with grading foliation F^u and coefficients in $\mathbf{k}$, see e.g. [22] for the construction. Moreover, according to the main theorem in [22], the quadratic differential φ gives a stability condition σ on $\mathcal{F}$ such that stable objects correspond to saddle connections and closed geodesic loops (with grading and $\mathbf{k}$ -linear local system) on the flat surface $(S, |\varphi|)$. The central charge is given by

$$Z([\gamma]) = \int_{\gamma} \sqrt{\varphi}$$

which is well-defined for

$$[\gamma] \in H_1(S, Z; \mathbb{Z}\sqrt{\varphi}) \cong K_0(\mathcal{F})$$

where $\mathbb{Z}\sqrt{\varphi}$ denotes the local system of integer multiples of choices of $\sqrt{\varphi}$.

Since the pseudo-Anosov map f preserves the subset Z and the grading foliation F^u , it induces an autoequivalence f_* of $\mathcal{F}$. From the description of the action of f on φ and

the correspondence between quadratic differentials and stability conditions is clear that f acts on σ like the element $\text{diag}(\lambda, 1/\lambda) \in \widetilde{GL^+(2, \mathbb{R})}$, i.e. f_* is DHKK pseudo-Anosov. This example was the motivation for the definition of pseudo-Anosov autoequivalence in [14].

4.2. $\mathbb{Z}/2$ -graded Fukaya category. The construction above has the perhaps unwanted feature that the category $\mathcal{F}$ depends on the choice of f in the mapping class group. Another approach is to consider a version of the Fukaya category of S which does not depend on a grading foliation and is thus only $\mathbb{Z}/2$ -graded. Such categories do not admit stability conditions, so we need to use the more general notion of pseudo-Anosov autoequivalence introduced here where the mass function on the triangulated category is the complexity with respect to a generator.

Let S be a closed surface of genus $g > 1$ and choose a symplectic form ω on S . The Fukaya category $\mathcal{F} = \mathcal{F}(S, \omega)$ of the surface S is an A_∞ -category over the Novikov field $\mathbf{k}((t^{\mathbb{R}}))$ where $\mathbf{k}$ is some coefficient field of characteristic zero. Objects of $\mathcal{F}$ can be taken to be immersed oriented curves L which do not bound an immersed one-gon (teardrop) together with a unitary rank one local system over $\mathbf{k}((t^{\mathbb{R}}))$. A self-contained elementary definition of $\mathcal{F}$ can be found in [1]. We assume that $\mathcal{F}$ has been formally completed so that its homotopy category $H^0(\mathcal{F})$ is triangulated and all idempotents split.

The mapping class group $\text{MCG}(S)$ of S does not act naturally on $\mathcal{F}$, at least not without making additional choices. Instead one considers the *symplectic mapping class group* of S which is the quotient $\text{Symp}(S)/\text{Ham}(S)$ of the group of symplectomorphisms by the normal subgroup of hamiltonian symplectomorphisms. This group surjects onto $\text{MCG}(S)$, i.e. every element of $\text{MCG}(S)$ is realized by some symplectomorphism.

While $\mathcal{F}$ does not admit a stability condition, we can still use mass functions of the form $\mu(E) := \delta(G, E)$ where G is a generator, e.g. a direct sum of certain $2g$ simple closed curves, and $\delta(G, E)$ denotes the complexity of E with respect to G as in [14]. These mass functions are all equivalent up to some constant and moreover equivalent to mass functions of the form

$$\mu(E) := \dim \text{Ext}^\bullet(G, E) = \dim \text{Ext}^0(G, E) + \dim \text{Ext}^1(G, E)$$

since $\mathcal{F}$ is smooth and proper.

Conjecture 4.1. Let f be a pseudo-Anosov symplectomorphism of a closed symplectic surface (S, ω) . Then the induced autoequivalence $\Phi = f_*$ of the Fukaya category $\mathcal{F}(S, \omega)$ is pseudo-Anosov with respect to the mass function $\mu(E) := \dim \text{Ext}^\bullet(G, E)$ where G is any generator of $\mathcal{F}(S, \omega)$. Moreover, the stretch factor $\lambda > 1$ of f coincides with the stretch factor of the autoequivalence f_* .

As in [14] one uses the fact from pseudo-Anosov theory that geometric intersection numbers $i(S_1, f^n S_2)$ grow exponentially with rate λ to conclude that $\dim \text{Ext}^\bullet(G, f_*^n E)$ has exponential growth rate λ , i.e. $h_{\mu, \Phi}(E) = \log \lambda$, for any object $E \in \mathcal{F}$ which comes from a simple closed curve. Here we take G to be a generator which is a direct sum of simple closed curves. Since simple closed curves split-generate, we also know that $h_{\mu, \Phi}(E) \leq \log \lambda$ for any $E \in \mathcal{F}$. The statement of the conjecture means precisely that equality holds for all non-zero objects E of $\mathcal{F}$.

The above discussion generalizes, with some changes, to the case of punctured surfaces S . In this case the symplectic form should be chosen to have infinite area, and in fact the definition of the (wrapped) Fukaya category from [22] can be used, except for replacing $\mathbb{Z}$ -gradings by $\mathbb{Z}/2$ -gradings. The category is smooth, but not proper, and the classification of indecomposable objects in terms of immersed curves from [22] could be of use.

5. OPEN QUESTIONS

We propose several open questions related to the study of pseudo-Anosov autoequivalences.

Alternative definitions of pseudo-Anosov autoequivalences. In Section 2 and 3, we study pseudo-Anosov autoequivalences with respect to the mass functions given by the Bridgeland stability conditions in a fixed connected component $\text{Stab}^\dagger(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$. One can use other mass functions, for instance the mass function δ_G given by the complexity with respect to a split generator G (Example 2.3) to define the mass growth of an object in a triangulated category, thereby defines another notion of “pseudo-Anosov autoequivalence” (with respect to the complexity function) following the same idea as in Section 2.3.

Definition 5.1. Let $\mathcal{D}$ be a triangulated category with a split generator G . We say an autoequivalence $\Phi \in \text{Aut}(\mathcal{D})$ is *pseudo-Anosov with respect to the complexity function* if there exists $\lambda > 1$ such that for any nonzero object E in $\mathcal{D}$,

$$\limsup_{n \rightarrow \infty} \frac{\log \delta_G(\Phi^n E)}{n} = \log \lambda > 0.$$

Note that the mass growth with respect to the complexity function is independent of the choice of the split generator: if G' is another split generator of $\mathcal{D}$, by [14, Proposition 2.3], we have $\delta_G(\Phi^n E) \leq \delta_{G'}(G') \delta_{G'}(\Phi^n E)$. One can also use the mass function μ_G given by the dimensions of $\text{Hom}^\bullet(G, -)$ in Example 2.2 to define yet another notion of pseudo-Anosov autoequivalences.

Definition 5.2. Let $\mathcal{D}$ be a triangulated category with a split generator G . We say an autoequivalence $\Phi \in \text{Aut}(\mathcal{D})$ is *pseudo-Anosov with respect to the Ext-distance function* if

there exists $\lambda > 1$ such that for any nonzero object E in $\mathcal{D}$,

$$\limsup_{n \rightarrow \infty} \frac{\log \mu_G(\Phi^n E)}{n} = \limsup_{n \rightarrow \infty} \frac{\log \sum_{i \in \mathbb{Z}} \dim \operatorname{Hom}_{\mathcal{D}}(G, E[i])}{n} = \log \lambda > 0.$$

It turns out that Definition 5.1 and 5.2 are equivalent, since there exist constants $C_1, C_2 > 0$ depending on G such that $C_1 \delta_G(E) \leq \mu_G(E) \leq C_2 \delta_G(E)$ holds for any nonzero object E [14, proof of Theorem 2.7]. It is then important to determine whether the definition via complexity function (or equivalently, Ext-distance function) coincides with our previous definition via stability conditions. More precisely:

Question 5.3. Let $\mathcal{D}$ be a triangulated category with a split generator and $\operatorname{Stab}(\mathcal{D}) \neq \emptyset$.

- Is it true that the property of being a pseudo-Anosov autoequivalence does not depend on the choice of a connected component of $\operatorname{Stab}(\mathcal{D})$?
- Does the notion of “pseudo-Anosov autoequivalence” with respect to the complexity function (or equivalently the Ext-distance function) coincide with our definition of pseudo-Anosov autoequivalence (Definition 2.13)?

We prove in Appendix A that both questions have affirmative answers if $\mathcal{D}$ is the bounded derived category of coherent sheaves on an elliptic curve (Theorem A.2). Note that it is proved by Ikeda [24, Theorem 3.14] that if a connected component $\operatorname{Stab}^\circ(\mathcal{D}) \subset \operatorname{Stab}(\mathcal{D})$ contains an algebraic stability condition, then the mass growths with respect to m_σ and δ_G coincide for any $\sigma \in \operatorname{Stab}^\circ(\mathcal{D})$. Hence both questions above also have affirmative answers for triangulated categories $\mathcal{D}$ with the property that each connected component of $\operatorname{Stab}(\mathcal{D})$ contains an algebraic stability condition, e.g. $\mathcal{D}^b \operatorname{Coh}(\mathbb{P}^1)$ [15, 34, 35]. Examples of triangulated categories admitting algebraic stability conditions include derived categories with full strong exceptional collections, see for instance [16, Remark 4.8, 4.9, 4.10], [33], [37].

Genericity of pseudo-Anosov autoequivalences.

Question 5.4. Let $\mathcal{D}$ be a Calabi–Yau category satisfying Assumptions (A) and (B). Are pseudo-Anosov autoequivalences “generic” in $\operatorname{Aut}(\mathcal{D})$?

Note that the statement is not true if one removes the Calabi–Yau condition. For instance, the derived category $\mathcal{D}^b(C)$ of coherent sheaves on a smooth projective curve C with genus $g(C) \neq 1$ does not have any pseudo-Anosov autoequivalences by Kikuta [28, Proposition 4.13] and Proposition 3.11.

Irreducible autoequivalences. In light of the Nielsen–Thurston classification of mapping class group elements, it is natural to consider the following definition.

Definition 5.5. Let $\Phi: \mathcal{D} \rightarrow \mathcal{D}$ be an autoequivalence of a triangulated category $\mathcal{D}$. We say Φ is

- *irreducible* if it has no Φ -invariant proper thick triangulated subcategory;
- *strongly irreducible* if there is no finite collection of proper thick triangulated subcategories that are permuted under Φ .

It follows from the definition that if an autoequivalence Φ is irreducible and $h_{m_\sigma, \Phi}(E) > 1$ for some object $E \in \mathcal{D}$ and $\sigma \in \text{Stab}^\dagger(\mathcal{D})$, then Φ is pseudo-Anosov. Indeed, if Φ is irreducible, then the associated growth filtration of Φ can only have one step:

$$0 \subset \mathcal{D}_\lambda = \mathcal{D}.$$

The condition that there exists an object with positive mass growth $h_{m_\sigma, \Phi}(E) > 1$ implies $\lambda > 1$. Hence Φ is pseudo-Anosov. Similarly, if an autoequivalence Φ is irreducible and has positive categorical entropy (Definition 2.18), then Φ is pseudo-Anosov with respect to the complexity function (Definition 5.1). An interesting question is whether the reverse implication holds.

Question 5.6. Are (DHKK) pseudo-Anosov autoequivalences (strongly) irreducible?

Question 5.7. Do pseudo-Anosov autoequivalences or irreducible autoequivalences with positive entropy on a triangulated category $\mathcal{D}$ act hyperbolically on the metric space $\text{Stab}^\dagger(\mathcal{D})/\mathbb{C}$ or $\mathcal{M}_{\text{Stab}^\dagger}(\mathcal{D})/\mathbb{R}_{>0}$?

Other examples of pseudo-Anosov autoequivalences. It would be interesting to find pseudo-Anosov autoequivalences on triangulated categories that are not Calabi–Yau as well. For instance, it is proved in [14, Theorem 2.17] that the Serre functor on the derived category of quiver representations has positive entropy if the quiver is not of extended Dynkin type. Moreover, the Serre functor on the derived category of representations of Kronecker quiver with at least three arrows is DHKK pseudo-Anosov.

Question 5.8. Is the Serre functor on the derived category of quiver representations pseudo-Anosov if the quiver is not of extended Dynkin type?

APPENDIX A. THE CASE OF ELLIPTIC CURVES

Let $\mathcal{D} = \mathcal{D}^b\text{Coh}(X)$ be the bounded derived category of coherent sheaves on an elliptic curve X . We prove in this section that the notion of pseudo-Anosov autoequivalences with respect to Bridgeland stability conditions (Definition 2.13), complexity function (Definition 5.1), and Ext-distance function (Definition 5.2) are all equivalent.

Recall that $\text{Stab}(\mathcal{D}) \cong \mathbb{C} \times \mathbb{H}$ is connected [9], hence the mass growth rates with respect to any Bridgeland stability condition on $\mathcal{D}$ are the same (Proposition 2.11). We choose the stability condition $\sigma \in \text{Stab}(\mathcal{D})$ such that

- $Z_\sigma(E) = -\deg(E) + i \cdot \text{rank}(E)$.
- For $0 < \phi \leq 1$, the (semi)stable objects $P_\sigma(\phi)$ are the slope-(semi)stable coherent sheaves whose central charges lie in the ray $\mathbb{R}_{>0} \cdot \exp(i\pi\phi)$.
- For other $\phi \in \mathbb{R}$, define $P_\sigma(\phi)$ via the property $P_\sigma(\phi + 1) = P_\sigma(\phi)[1]$.

Proposition A.1. *Fix an ample line bundle $\mathcal{O}(1)$ on the elliptic curve X and a split generator $G = \mathcal{O} \oplus \mathcal{O}(1)$ of $\mathcal{D}$. There exists a constant $C > 0$ such that*

$$\mu_G(E) := \sum_{i \in \mathbb{Z}} \dim \text{Hom}_{\mathcal{D}}(G, E[i]) < C \cdot m_\sigma(E)$$

holds for any object E in $\mathcal{D}$.

Proof. By the triangle inequality of μ_G (cf. Definition 2.1(1)), it suffices to show the inequality holds for slope-semistable coherent sheaves. By considering Jordan–Holder filtrations of semistable coherent sheaves, one can further reduce to proving the inequality only for stable vector bundles and indecomposable torsion sheaves.

Let E be an indecomposable torsion sheaf on X . Then $E \cong \mathcal{O}_X / \mathcal{O}_X(-sx_0)$ for some $x_0 \in X$ and $s > 0$. We have

$$\begin{aligned} \mu_G(E) &= \dim H^0(E) + \dim H^1(E) + \dim H^0(E(-1)) + \dim H^1(E(-1)) \\ &= 2s = 2m_\sigma(E). \end{aligned}$$

Let E be a stable vector bundle on the elliptic curve X of rank r and degree d . By [4, Lemma 15] and Hirzebruch–Riemann–Roch theorem,

- if $d > 0$, then $\dim H^0(E) = d$ and $\dim H^1(E) = 0$;
- if $d = 0$, then $\dim H^0(E) = \dim H^1(E)$ and both equal to 0 or 1;
- if $d < 0$, then $\dim H^0(E) = 0$ and $\dim H^1(E) = -d$.

Hence we have $\dim H^0(E) + \dim H^1(E) \leq |d| + 2$ for any stable vector bundle E . Thus

$$\begin{aligned} \mu_G(E) &= \dim H^0(E) + \dim H^1(E) + \dim H^0(E(-1)) + \dim H^1(E(-1)) \\ &\leq |d| + |d - r| + 4 \\ &< \sqrt{r^2 + d^2} + \sqrt{2(r^2 + d^2)} + 4\sqrt{r^2 + d^2} \\ &= (5 + \sqrt{2})m_\sigma(E). \end{aligned}$$

□

Theorem A.2. *Let $\mathcal{D} = \mathcal{D}^b\text{Coh}(X)$ be the bounded derived category of coherent sheaves on an elliptic curve X . Let $G = \mathcal{O} \oplus \mathcal{O}(1)$ be a split generator of $\mathcal{D}$. Then*

$$\limsup_{n \rightarrow \infty} \frac{\log m_\sigma(\Phi^n E)}{n} = \limsup_{n \rightarrow \infty} \frac{\log \delta_G(\Phi^n E)}{n} = \limsup_{n \rightarrow \infty} \frac{\log \mu_G(\Phi^n E)}{n}$$

holds for any nonzero object E in $\mathcal{D}$. Hence the notion of pseudo-Anosov autoequivalences with respect to these three mass functions are equivalent.

Proof. By Ikeda [24, Proposition 3.4], we have $m_\sigma(E) \leq m_\sigma(G)\delta_G(E)$ for any nonzero object E in $\mathcal{D}$. Combining with [14, proof of Theorem 2.7] (cf. Section 5) and Proposition A.1, we obtain

$$\limsup_{n \rightarrow \infty} \frac{\log \delta_G(\Phi^n E)}{n} = \limsup_{n \rightarrow \infty} \frac{\log \mu_G(\Phi^n E)}{n} \leq \limsup_{n \rightarrow \infty} \frac{\log m_\sigma(\Phi^n E)}{n} \leq \limsup_{n \rightarrow \infty} \frac{\log \delta_G(\Phi^n E)}{n}$$

for any nonzero object E . This concludes the proof. $\square$

APPENDIX B. BASIC PROPERTIES OF MASS FUNCTIONS

We defined the notion of mass functions on triangulated categories in Definition 2.1. A nice property that mass functions have but Bridgeland stability conditions do not have is that they are functorial.

Definition B.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor between triangulated categories. Let $\mu : \text{Ob}(\mathcal{D}) \rightarrow \mathbb{R}_{\geq 0}$ be a mass function on $\mathcal{D}$. Then one can define the pullback of the mass function via F :

$$F^* \mu : \text{Ob}(\mathcal{C}) \rightarrow \mathbb{R}_{\geq 0}, \quad E \mapsto \mu(F(E)),$$

which is clearly a mass function on $\mathcal{C}$.

Another nice property of mass functions is that one can induce relative mass functions from mass functions.

Definition B.2. Let $\mathcal{D}'$ be a thick triangulated subcategory of $\mathcal{D}$. Let $\mu : \text{Ob}(\mathcal{D}) \rightarrow \mathbb{R}_{\geq 0}$ be a mass function on $\mathcal{D}$. Define the induced mass function $i_*^{\mathcal{D}'} \mu : \text{Ob}(\mathcal{D}) \rightarrow \mathbb{R}_{\geq 0}$ by

$$(i_*^{\mathcal{D}'} \mu)(E) := \inf \left\{ \sum_{A_i \notin \mathcal{D}'} \mu(A_i) \left| \begin{array}{ccc} 0 & \xrightarrow{\quad} & \star \\ & \searrow & \nearrow \\ & A_1 & \end{array} \quad \cdots \quad \begin{array}{ccc} \star & \xrightarrow{\quad} & E \oplus \star \\ & \searrow & \nearrow \\ & A_n & \end{array} \right. \right\}.$$

While it is not clear to us how to pushforward a mass function from $\mathcal{D}$ to a quotient $\mathcal{D}/\mathcal{D}'$, the mass function $i_*^{\mathcal{D}'} \mu$ can serve as a replacement of this process. In particular, one can see that the induced mass function $i_*^{\mathcal{D}'} \mu$ vanishes on $\mathcal{D}'$. The proposition below shows the compatibility between the pullback of mass functions via quotient maps and the induced relative mass functions.

Proposition B.3. *Let $\mathcal{D}'$ be a thick triangulated subcategory of $\mathcal{D}$, and $q_{\mathcal{D}'} : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{D}'$ be the quotient map. Let $\mu : \text{Ob}(\mathcal{D}/\mathcal{D}') \rightarrow \mathbb{R}_{\geq 0}$ be a mass function on $\mathcal{D}/\mathcal{D}'$. Then*

$$i_*^{\mathcal{D}'} q_{\mathcal{D}'}^* \mu = q_{\mathcal{D}'}^* \mu.$$

Proof. Since $q_{\mathcal{D}'}^* \mu$ vanishes on $\mathcal{D}'$, for any sequence of exact triangles

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & \star & \xrightarrow{\quad} & \star & \xrightarrow{\quad} & \dots \xrightarrow{\quad} \star & \xrightarrow{\quad} & E \oplus \star, \\ & \nwarrow \text{dashed} & \nearrow & & \nwarrow \text{dashed} & \nearrow & & & \nwarrow \text{dashed} & \nearrow \\ & & A_1 & & & A_2 & & & & A_n \end{array}$$

we have

$$(q_{\mathcal{D}'}^* \mu)(E) \leq \sum_i (q_{\mathcal{D}'}^* \mu)(A_i) = \sum_{A_i \notin \mathcal{D}'} (q_{\mathcal{D}'}^* \mu)(A_i).$$

□

APPENDIX C. COMPUTATION IN THE QUINTIC CALABI–YAU THREEFOLD CASE

We use the notations in Section 3.2. Let $H = \mathcal{O}(1)$ be the ample generator of $\text{Pic}(X)$. The powers $1, H, H^2, H^3$ form a basis of $H^{\text{ev}}(X; \mathbb{Q})$. With respect to this basis, $[\Phi]$ corresponds to the following matrix

$$\begin{pmatrix} 6 & -\frac{20}{3} & 5 & -5 \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 \\ -\frac{1}{6} & \frac{1}{2} & -1 & 1 \end{pmatrix}.$$

Its eigenvalues and corresponding eigenvectors are:

$$\begin{aligned} \lambda_1 &= \frac{1}{4} \left(9 + 3\sqrt{5} + \sqrt{110 + 54\sqrt{5}} \right) \\ \lambda_2 &= \frac{1}{4} \left(9 - 3\sqrt{5} + i\sqrt{-110 + 54\sqrt{5}} \right) \\ \lambda_3 &= \frac{1}{4} \left(9 - 3\sqrt{5} - i\sqrt{-110 + 54\sqrt{5}} \right) \\ \lambda_4 &= \frac{1}{4} \left(9 + 3\sqrt{5} - \sqrt{110 + 54\sqrt{5}} \right) \end{aligned}$$

and

$$v_1 = \begin{pmatrix} \frac{-3}{122} \left(185 + 75\sqrt{5} + \sqrt{104150 + 47070\sqrt{5}} \right) \\ \frac{1}{61} (60 + 54\sqrt{5}) \\ \frac{-3}{61} \sqrt{575 + 426\sqrt{5}} \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} \frac{3}{122} \left(-185 + 75\sqrt{5} + i\sqrt{-104150 + 47070\sqrt{5}} \right) \\ \frac{1}{61} (60 - 54\sqrt{5}) \\ \frac{-3}{61} i \sqrt{-575 + 426\sqrt{5}} \\ 1 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} \frac{3}{122} \left(-185 + 75\sqrt{5} - i\sqrt{-104150 + 47070\sqrt{5}} \right) \\ \frac{\frac{1}{61}(60 - 54\sqrt{5})}{\frac{3}{61}i\sqrt{-575 + 426\sqrt{5}}} \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} \frac{-3}{122} \left(185 + 75\sqrt{5} - \sqrt{104150 + 47070\sqrt{5}} \right) \\ \frac{\frac{1}{61}(60 + 54\sqrt{5})}{\frac{3}{61}\sqrt{575 + 426\sqrt{5}}} \\ 1 \end{pmatrix}$$

The following lemma follows from straightforward but tedious computations.

Lemma C.1. *The span of $\{v_2, v_3, v_4\}$ over complex numbers $\mathbb{C}$ does not contain any rational vector.*

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