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OPTIMAL REAL-TIME DETECTION OF A DRIFTING BROWNIAN COORDINATE

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Consider the motion of a Brownian particle in three dimensions, whose two spatial coordinates are standard Brownian motions with zero drift, and the remaining (unknown) spatial coordinate is a standard Brownian motion with a (known) nonzero drift. Given that the position of the Brownian particle is being observed in real time, the problem is to detect as soon as possible and with minimal probabilities of the wrong terminal decisions, which spatial coordinate has the nonzero drift. We solve this problem in the Bayesian formulation, under any prior probabilities of the nonzero drift being in any of the three spatial coordinates, when the passage of time is penalised linearly. Finding the exact solution to the problem in three dimensions, including a rigorous treatment of its nonmonotone optimal stopping boundaries, is the main contribution of the present paper. To our knowledge this is the first time that such a problem has been solved in the literature.

1. Introduction. Imagine the motion of a Brownian particle in three dimensions, whose two spatial coordinates are standard Brownian motions with zero drift, and the remaining (unknown) spatial coordinate is a standard Brownian motion with a (known) nonzero drift. Given that the position X of the Brownian particle is being observed in real time, the problem is to detect as soon as possible and with minimal probabilities of the wrong terminal decisions, which spatial coordinate has the nonzero drift. The purpose of the present paper is to derive the solution to this problem in the Bayesian formulation, under any prior probabilities of the nonzero drift being in any of the three spatial coordinates, when the passage of time is penalised linearly.

The loss to be minimised over sequential decision rules is expressed as the linear combination of the expected running time and the probabilities of the wrong terminal decisions. This problem formulation of sequential testing dates back to [24] and has been extensively studied to date (see [10] and the references therein). The linear combination represents the Lagrangian and once the optimisation problem has been solved in this form it will also lead to the solution of the constrained problem where upper bounds are imposed on the probabilities of the wrong terminal decisions. The central focus of the present paper is on the Lagrangian and the methods needed to solve the problem in this form. The constrained problem itself will not be considered in the present paper as this extension is somewhat lengthy and more routine.

Standard arguments show that the initial optimisation problem can be reduced to an optimal stopping problem for the posterior probability process Π of the nonzero drift being in the spatial coordinates given X. A canonical example of X in one dimension is Brownian

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motion having one among two constant drifts (see [13] and [21]). In this case Π is a one-dimensional Markov/diffusion process. This problem has also been solved in finite horizon (see [7]). Books [22], Section 4.2, and [17], Section 21, contain expositions of these results and provide further details and references. Signal-to-noise ratio in these problems (defined as the difference between the two drifts divided by the diffusion coefficient) is constant. Sequential testing problems for X in one dimension where the signal-to-noise ratio is not constant were studied more recently in [8] and [10]. In these problems Π is no longer Markovian, however, the process (Π, X) is a two-dimensional Markov/diffusion process with the infinitesimal generator of *parabolic* type.

Another canonical example of X in one dimension is Brownian motion having one among three or more constant drifts (see [23] for a discrete time analogue). This problem has been studied more recently in [25] (see also [3] for a Poisson process analogue). The Markov/diffusion process Π is two-dimensional and its infinitesimal generator is also of *parabolic* type.

Related sequential testing problems for X in three or more dimensions when each coordinate process of X can have a nonzero drift have been studied in [12] and [1]. These problems contain an element of optimal control as well in deciding which coordinate process should be observed at any given time. The former paper contains a review of other related papers (such as [18]) and the latter paper shows that the Markov/diffusion process Π is one-dimensional even if one admits infinitely many coordinate processes of X in the problem formulation.

In contrast to all the sequential testing problems studied to date we will see below that the two-dimensional Markov/diffusion process Π in the sequential testing problem of the present paper has the infinitesimal generator of *elliptic* type. Moreover, we will also see that the optimal stopping boundaries are *nonmonotone* as functions of the coordinate variables. This fact itself presents a formidable challenge as to our knowledge no rigorous treatment of nonmonotone optimal stopping boundaries (curves) has been exposed in the probabilistic literature as yet. Finding the exact solution to the problem for X in three dimensions, including a rigorous treatment of its nonmonotone optimal stopping boundaries, is the main contribution of the present paper. To our knowledge this is the first time that such a problem has been solved in the literature. The analogous problem for X in four/more dimensions introduces additional challenges for a rigorous treatment of "nonmonotone" optimal stopping boundaries (surfaces) and this is left for future research.

2. Outline of the paper. The exposition of the material is organised as follows. In Section 3 we derive the optimal stopping problem for $\Pi = (\Pi_0, \Pi_1, \Pi_2)$ where Π^i is the posterior probability process of the nonzero drift being in the spatial coordinate i given X for $i = 0, 1, 2$. Due to $\sum_{i=0}^{2} \Pi^i = 1$ clearly only two coordinates of Π matter and this is utilised by passing to the posterior probability ratio process $\Phi = (\Phi^1, \Phi^2)$ defined by $\Phi^i = \Pi^i/\Pi^0$ for $i = 1, 2$. The processes Π and Φ stand in one-to-one correspondence and we study the optimal stopping problem in terms of Φ throughout. The previous considerations take place under the probability measure $\mathsf{P}_\pi = \sum_{i=0}^{2} \pi_i \mathsf{P}_i$ where π_i is the prior probability of the nonzero drift being in the spatial coordinate i for $i = 0, 1, 2$. In Section 4 we show that a measure change from P_π to P_0 simplifies the setting upon verifying that the posterior probability ratio process Φ^i coincides (up to the initial point) with the likelihood ratio process L^i of P_i and P_0 given X for $i = 1, 2$. This provides an explicit link between the process Φ and the observed process X.

In Section 5 we show that the process Φ solves a coupled system of linear stochastic differential equations (of the geometric Brownian motion type) driven by two independent Brownian motions. This enables us to conclude that Φ is a Markov/diffusion process and derive a closed form expression for its infinitesimal generator which is a second-order partial differential equation of elliptic type. The optimal stopping problem for Φ is Bolza formulated and in Section 6 we disclose its Lagrange and Mayer formulations (see [17], Section 6, for the terminology). The Lagrange formulation is expressed in terms of the local time of Φ on three straight lines which makes the optimal stopping problem more intuitive.

The observed process X is three-dimensional and in Section 7 we consider the same optimal stopping problem when X is two-dimensional. In this case Φ is a one-dimensional Markov/ diffusion process so that standard arguments enable us to solve the optimal stopping problem in a closed form. The reduction of dimension three to dimension two corresponds to either Φ^1 or Φ^2 becoming 0 which is a natural boundary point for both processes (cf. [6]). The one-dimensional results of Section 7 are used in Section 8 to derive existence of the optimal stopping set and derive basic properties of the value function. We show that the optimal stopping set consists of three convex sets separated by the three straight lines that support the local time of Φ in the Lagrange formulation of the optimal stopping problem. Using symmetry arguments combined with the one-dimensional results of Section 7 we also derive the asymptotic behaviour of the optimal stopping boundaries at zero and infinity.

In Section 9 we derive a directional smooth fit between the value function and the loss function at the optimal stopping boundary. The proof of the smooth fit makes use of the asymptotic behaviour of the optimal stopping boundary at infinity to counterbalance the lack of the global smoothness of the underlying loss function in the optimal stopping problem. In Section 10 we show that the optimal stopping boundaries are nonmonotone in either direction of the state space of Φ and prove the existence of a "belly" which determines their curvature/shape. These arguments rely on the general hint from [16], Remark 13, on establishing the absence of jumps of the optimal stopping boundaries and make use of Hopf's boundary point lemma to derive a contradiction with the directional smooth fit.

In Section 11 we disclose the free-boundary problem which stands in one-to-one correspondence with the optimal stopping problem and establish the fact that the value function and the optimal stopping boundaries solve the free-boundary problem uniquely. In Section 12 we show that the optimal stopping boundaries can be characterised as the unique solution to a coupled system of nonlinear Fredholm integral equations. These equations can be used to find the optimal stopping boundaries numerically (using Picard iteration).

3. Formulation of the problem. In this section we formulate the sequential testing problem under consideration. The initial formulation of the problem will be revaluated under a change of measure in the next section.

1. We consider a Bayesian formulation of the problem where it is assumed that one observes a sample path of the three-dimensional Brownian motion $X = (X^0, X^1, X^2)$, whose two coordinates X^j and X^k are standard Brownian motions with zero drift, and the remaining (unknown) coordinate X^i is a standard Brownian motion having a (known) nonzero drift μ with a probability $\pi_i \in [0, 1]$ for $i = 0, 1, 2$ where $\pi_0+\pi_1+\pi_2 = 1$ and $i \neq j \neq k$ belong to $\{0, 1, 2\}$. The problem is to detect which coordinate is drifting as soon as possible and with minimal probabilities of the wrong terminal decisions. This real-time detection problem belongs to the class of sequential testing problems as discussed in Section 1 above.

2. Standard arguments imply that the previous setting can be realised on a probability space $(\Omega, \mathcal{F}, \mathsf{P}_\pi)$ with the probability measure P_π decomposed as follows

$$\mathsf{P}_\pi = \pi_0 \mathsf{P}_0 + \pi_1 \mathsf{P}_1 + \pi_2 \mathsf{P}_2 \tag{3.1}$$

for $\pi = (\pi_0, \pi_1, \pi_2) \in [0, 1]^3$ satisfying $\pi_0 + \pi_1 + \pi_2 = 1$ where P_i is the probability measure under which the observed process X has the ith coordinate equal to a standard Brownian motion with drift μ, and the remaining two coordinates are standard Brownian motions with zero drift for $i = 0, 1, 2$, with the three coordinates being independent. This can be formally achieved by introducing an unobservable random variable θ taking values $0, 1, 2$ with probabilities π_0, π_1, π_2 in $[0, 1]$ satisfying $\pi_0+\pi_1+\pi_2 = 1$ and being independent from three (mutually independent) standard Brownian motions B^0, B^1, B^2 so that $X = (X^0, X^1, X^2)$ after starting at a point in $\mathbb{R}^3$ solves the system of stochastic differential equations

$$dX_t^i = \mu I(\theta = i)\, dt + dB_t^i \tag{3.2}$$

for $i = 0, 1, 2$. Due to stationary and independent increments of Brownian motion it is clear that the starting point of X plays no role in the sequel so we will leave it unspecified.

3. Being based upon the continued observation of X, the problem is to test sequentially the hypotheses $H_0 : \theta = 0$, $H_1 : \theta = 1$, $H_2 : \theta = 2$ with a minimal loss. For this, we are given a sequential decision rule (τ, d_τ), where τ is a stopping time of X (i.e., a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s \mid 0 \le s \le t)$ of X for $t \ge 0$), and d_τ is an $\mathcal{F}_\tau^X$-measurable random variable taking values in the set $\{0, 1, 2\}$. After stopping the observation of X at time τ, the terminal decision function d_τ takes value i if and only if the hypothesis H_i is to be accepted for $i = 0, 1, 2$. With a constant $c > 0$ given and fixed, the problem then becomes to compute the risk function

$$\begin{aligned} V(\pi) = \inf_{(\tau, d_\tau)} \mathsf{E}_\pi\big[\tau + c\big(I(\theta = 0, d_\tau \ne 0) \\ + I(\theta = 1, d_\tau \ne 1) + I(\theta = 2, d_\tau \ne 2)\big)\big] \end{aligned} \tag{3.3}$$

for $\pi = (\pi_0, \pi_1, \pi_2) \in [0, 1]^3$ with $\pi_0+\pi_1+\pi_2 = 1$ and find the optimal decision rule $(\tau_*, d_{\tau_*}^*)$ at which the infimum in (3.3) is attained. Note that $\mathsf{E}_\pi(\tau)$ in (3.3) is the expected waiting time until the terminal decision is made, and $\mathsf{P}_\pi(\theta = i, d_\tau \ne i)$ are probabilities of the wrong terminal decisions for $i = 0, 1, 2$. Clearly, each probability $\mathsf{P}_\pi(\theta = i, d_\tau \ne i)$ could be further decomposed into the sum of two probabilities $\mathsf{P}_\pi(\theta = i, d_\tau = j)$ and $\mathsf{P}_\pi(\theta = i, d_\tau = k)$ for $i = 0, 1, 2$ and $i \ne j \ne k$ in $\{0, 1, 2\}$, and each of the six resulting probabilities could have a different constant/weight placed in front of them, however, since the constrained problems are not considered in the present paper as explained in the introduction, we only focus on the canonical setting of a single constant/weight c given in (3.3) above.

4. To tackle the sequential testing problem (3.3) we consider the *posterior probability process* $\Pi = ((\Pi_t^0, \Pi_t^1, \Pi_t^2))_{t \ge 0}$ of $H = (H_0, H_1, H_2)$ given X that is defined by

$$\Pi_t^i = \mathsf{P}_\pi(\theta = i \mid \mathcal{F}_t^X) \tag{3.4}$$

for $i = 0, 1, 2$ and $t \ge 0$. Noting that for any decision rule (τ, d_τ) we have

$$\sum_{i=0}^{2} \mathsf{P}_\pi(\theta = i, d_\tau \ne i) = \sum_{i=0}^{2} \mathsf{E}_\pi\big[\Pi_\tau^i I(d_\tau \ne i)\big] = \sum_{i=0}^{2} \mathsf{E}_\pi\big[(1-\Pi_\tau^i) I(d_\tau = i)\big] \tag{3.5}$$

where in the final equality we use that $\Pi_\tau^0+\Pi_\tau^1+\Pi_\tau^2 = 1$, it follows that

$$\begin{aligned} &\mathsf{E}_\pi\big[\tau + c\big(I(\theta = 0, d_\tau \ne 0) + I(\theta = 1, d_\tau \ne 1) + I(\theta = 2, d_\tau \ne 2)\big)\big] \\ &\quad \ge \mathsf{E}_\pi\big[\tau + c\big((1-\Pi_\tau^0) \wedge (1-\Pi_\tau^1) \wedge (1-\Pi_\tau^2)\big)\big] \end{aligned} \tag{3.6}$$

where equality is attained at the decision rule $(\tau, \tilde{d}_\tau)$ with $\tilde{d}_\tau$ defined as follows

$$\begin{aligned} \tilde{d}_\tau &= 0 \quad \text{if } (1-\Pi_\tau^0) \le (1-\Pi_\tau^1) \wedge (1-\Pi_\tau^2) \\ &= 1 \quad \text{if } (1-\Pi_\tau^1) \le (1-\Pi_\tau^0) \wedge (1-\Pi_\tau^2) \\ &= 2 \quad \text{if } (1-\Pi_\tau^2) \le (1-\Pi_\tau^0) \wedge (1-\Pi_\tau^1). \end{aligned} \tag{3.7}$$

This shows that the problem (3.3) is equivalent to the optimal stopping problem

$$V(\pi) = \inf_{\tau} \mathsf{E}_\pi\big[\tau + M(\Pi_\tau)\big] \tag{3.8}$$

where the infimum is taken over all stopping times τ of X, and the function M is given by

$$M(\pi) = c\big((1-\pi_0)\wedge(1-\pi_1)\wedge(1-\pi_2)\big) \tag{3.9}$$

for $\pi = (\pi_0, \pi_1, \pi_2) \in [0,1]^3$ with $\pi_0+\pi_1+\pi_2 = 1$. For this reason we focus on solving the optimal stopping problem (3.8) in what follows.

4. Measure change. In this section we show that changing the probability measure P_π for $\pi \in [0,1]^3$ with $\pi_0+ \pi_1+\pi_2 = 1$ to P_0 provides important simplifications of the setting which make the subsequent analysis more transparent. The change of measure argument is presented in Lemma 1 below. This is then followed by a reformulation of the optimal stopping problem (3.8) under the new probability measure P_0 in Proposition 2 below.

1. To connect the process Π in (3.8) to the observed process X we consider the *likelihood ratio process* $L = ((L_t^1, L_t^2))_{t\ge 0}$ defined by

$$L_t^i = \frac{d\mathsf{P}_{i,t}}{d\mathsf{P}_{0,t}} \tag{4.1}$$

where $\mathsf{P}_{i,t}$ and $\mathsf{P}_{0,t}$ denote the restrictions of P_i and P_0 to $\mathcal{F}_t^X$ for $t \ge 0$ and $i = 1, 2$. Using that $((X_t^0 - \mu t, X_t^i))_{t\ge 0}$ and $((X_t^0, X_t^i - \mu t))_{t\ge 0}$ define standard two-dimensional Brownian motions under P_0 and P_i respectively, by the Cameron–Martin–Girsanov theorem (see, e.g., [11], Theorem 5.1, p. 191) one finds that

$$L_t^i = e^{\mu(X_t^i - X_t^0)} \tag{4.2}$$

for $t \ge 0$ and $i = 1, 2$. A direct calculation indicated below shows that the *posterior probability ratio process* $\Phi = ((\Phi_t^1, \Phi_t^2))_{t\ge 0}$ defined by

$$\Phi_t^i = \frac{\Pi_t^i}{\Pi_t^0} \tag{4.3}$$

can be expressed in terms of L (and hence X as well) as follows

$$\Phi_t^i = \Phi_0^i L_t^i \tag{4.4}$$

for $t \ge 0$ where $\Phi_0^i = \pi_i/\pi_0$ for $i = 1, 2$. Recalling that $\Pi_t^0 + \Pi_t^1 + \Pi_t^2 = 1$ and formally setting $\Phi_t^0 \equiv 1$ it is easily seen that (4.3) is equivalent to

$$\Pi_t^i = \frac{\Phi_t^i}{1 + \Phi_t^1 + \Phi_t^2} \tag{4.5}$$

for $t \ge 0$ and $i = 0, 1, 2$.

2. To derive (4.3)–(4.5) one may use a standard rule for the Radon–Nikodym derivatives based on (3.1) that gives

$$\begin{aligned} \Pi_t^0 &= \mathsf{P}_\pi\big(\theta = 0 \,|\, \mathcal{F}_t^X\big) = \sum_{i=0}^{2} \pi_i\, \mathsf{P}_i\big(\theta = 0 \,|\, \mathcal{F}_t^X\big) \frac{d\mathsf{P}_{i,t}}{d\mathsf{P}_{\pi,t}} \\ &= \pi_0 \frac{d\mathsf{P}_{0,t}}{d\mathsf{P}_{\pi,t}} = \frac{1}{1 + \frac{\pi_1}{\pi_0}\frac{d\mathsf{P}_{1,t}}{d\mathsf{P}_{0,t}} + \frac{\pi_2}{\pi_0}\frac{d\mathsf{P}_{2,t}}{d\mathsf{P}_{0,t}}} \end{aligned} \tag{4.6}$$

$$\Pi_t^1 = \mathsf{P}_\pi\big(\theta=1\,|\,\mathcal{F}_t^X\big) = \sum_{i=0}^{2} \pi_i\,\mathsf{P}_i\big(\theta=1\,|\,\mathcal{F}_t^X\big)\,\frac{d\mathsf{P}_{i,t}}{d\mathsf{P}_{\pi,t}} \tag{4.7}$$

$$= \pi_1\,\frac{d\mathsf{P}_{1,t}}{d\mathsf{P}_{\pi,t}} = \frac{\frac{\pi_1}{\pi_0}\frac{d\mathsf{P}_{1,t}}{d\mathsf{P}_{0,t}}}{1+\frac{\pi_1}{\pi_0}\frac{d\mathsf{P}_{1,t}}{d\mathsf{P}_{0,t}}+\frac{\pi_2}{\pi_0}\frac{d\mathsf{P}_{2,t}}{d\mathsf{P}_{0,t}}}$$

$$\Pi_t^2 = \mathsf{P}_\pi\big(\theta=2\,|\,\mathcal{F}_t^X\big) = \sum_{i=0}^{2} \pi_i\,\mathsf{P}_i\big(\theta=2\,|\,\mathcal{F}_t^X\big)\,\frac{d\mathsf{P}_{i,t}}{d\mathsf{P}_{\pi,t}} \tag{4.8}$$

$$= \pi_2\,\frac{d\mathsf{P}_{2,t}}{d\mathsf{P}_{\pi,t}} = \frac{\frac{\pi_2}{\pi_0}\frac{d\mathsf{P}_{2,t}}{d\mathsf{P}_{0,t}}}{1+\frac{\pi_1}{\pi_0}\frac{d\mathsf{P}_{1,t}}{d\mathsf{P}_{0,t}}+\frac{\pi_2}{\pi_0}\frac{d\mathsf{P}_{2,t}}{d\mathsf{P}_{0,t}}}$$

where $\mathsf{P}_{\pi,t}$ denotes the restriction of P_π to $\mathcal{F}_t^X$ for $\pi=(\pi_0,\pi_1,\pi_2)\in[0,1]^3$ with $\pi_0+\pi_1+\pi_2=1$ and $t\ge 0$. It is then easily verified that (4.6)–(4.8) imply (4.3)–(4.5) as claimed.

3. Previous arguments suggest that changing the probability measure P_π to P_0 appears to be of canonical interest in the optimal stopping problem (3.8). In the sequel we let $\mathsf{P}_{\pi,\tau}$ denote the restriction of P_π to $\mathcal{F}_\tau^X$ where τ is a stopping time of X.

LEMMA 1. *The following identity holds*

$$\frac{d\mathsf{P}_{\pi,\tau}}{d\mathsf{P}_{0,\tau}} = \frac{\pi_0}{\Pi_\tau^0} \tag{4.9}$$

for all stopping times τ of X and all $\pi=(\pi_0,\pi_1,\pi_2)\in[0,1]^3$ with $\pi_0+\pi_1+\pi_2=1$.

PROOF. Using the same arguments as in (4.6) above we find that

$$\Pi_\tau^0 = \mathsf{P}_\pi\big(\theta=0\,|\,\mathcal{F}_\tau^X\big) = \sum_{i=0}^{2} \pi_i\,\mathsf{P}_i\big(\theta=0\,|\,\mathcal{F}_\tau^X\big)\,\frac{d\mathsf{P}_{i,\tau}}{d\mathsf{P}_{\pi,\tau}} = \pi_0\,\frac{d\mathsf{P}_{0,\tau}}{d\mathsf{P}_{\pi,\tau}} \tag{4.10}$$

for any τ and π as above. From (4.10) we see that (4.9) holds and the proof is complete. □

4. We now show that the optimal stopping problem (3.8) admits a transparent reformulation under the probability measure P_0 in terms of the process $\Phi=(\Phi^1,\Phi^2)$ defined in (4.3) above. Recall that Φ^i starts at π_i/π_0 and this dependence on the initial point will be indicated by a superscript π_i/π_0 to Φ replacing its coordinate superscript i for $i=1,2$ when needed.

PROPOSITION 2. *The value function V from* (3.8) *satisfies the identity*

$$V(\pi) = \pi_0\,\hat{V}\left(\frac{\pi_1}{\pi_0},\frac{\pi_2}{\pi_0}\right) \tag{4.11}$$

where the value function $\hat{V}$ is given by

$$\hat{V}\left(\frac{\pi_1}{\pi_0},\frac{\pi_2}{\pi_0}\right) = \inf_\tau \mathsf{E}_0\left[\int_0^\tau \big(1+\Phi_t^{\pi_1/\pi_0}+\Phi_t^{\pi_2/\pi_0}\big)\,dt + \hat{M}\big(\Phi_\tau^{\pi_1/\pi_0},\Phi_\tau^{\pi_2/\pi_0}\big)\right] \tag{4.12}$$

for $\pi=(\pi_0,\pi_1,\pi_2)\in[0,1]^3$ with $\pi_0+\pi_1+\pi_2=1$ where

$$\hat{M}(\varphi_1,\varphi_2) = c\big((\varphi_1+\varphi_2)\wedge(1+\varphi_1)\wedge(1+\varphi_2)\big) \tag{4.13}$$

for $(\varphi_1,\varphi_2)\in[0,\infty)^2$ and the infimum in (4.12) *is taken over all stopping times τ of X.*

PROOF. For $\pi = (\pi_0, \pi_1, \pi_2) \in [0, 1]^3$ with $\pi_0+\pi_1+\pi_2 = 1$ given and fixed, it is enough to show that the following identity holds

$$\mathsf{E}_\pi\big[\tau + M(\Pi_\tau)\big] = \pi_0\,\mathsf{E}_0\bigg[\int_0^\tau \big(1+\Phi_t^{\pi_1/\pi_0}+\Phi_t^{\pi_2/\pi_0}\big)\,dt + \hat{M}\big(\Phi_\tau^{\pi_1/\pi_0}, \Phi_\tau^{\pi_2/\pi_0}\big)\bigg] \tag{4.14}$$

for all bounded stopping times τ of X. For this, suppose that such a stopping time τ is given and fixed, and note by (4.5)–(4.9) that

$$\begin{aligned}\mathsf{E}_\pi\big[\tau + M(\Pi_\tau)\big] &= \pi_0\,\mathsf{E}_0\bigg[\frac{\tau}{\Pi_\tau^0} + \frac{M(\Pi_\tau)}{\Pi_\tau^0}\bigg] \\ &= \pi_0\,\mathsf{E}_0\big[\tau\big(1+\Phi_\tau^{\pi_1/\pi_0}+\Phi_\tau^{\pi_2/\pi_0}\big) \\ &\quad + \hat{M}\big(\Phi_\tau^{\pi_1/\pi_0}, \Phi_\tau^{\pi_2/\pi_0}\big)\big].\end{aligned} \tag{4.15}$$

Setting $M_t = 1+\Phi_t^{\pi_1/\pi_0}+\Phi_t^{\pi_2/\pi_0}$ for $t \geq 0$ we see by (4.2) and (4.4) that $M = (M_t)_{t\geq 0}$ is a continuous martingale under P_0 so that integration by parts gives

$$tM_t = \int_0^t M_s\,ds + \int_0^t s\,dM_s \tag{4.16}$$

where the final term defines a continuous martingale under P_0 for $t \geq 0$. Hence by the optional sampling theorem we obtain

$$\mathsf{E}_0(\tau M_\tau) = \mathsf{E}_0\bigg(\int_0^\tau M_t\,dt\bigg). \tag{4.17}$$

Inserting this back into (4.15) we obtain (4.14) as claimed and the proof is complete. □

5. It is clear from (4.2) and (4.4) that $\Phi = (\Phi^1, \Phi^2)$ is a strong Markov/diffusion process. We will formally verify this fact in the next section by deriving a coupled system of stochastic differential equations (driven by two independent Brownian motions) that Φ solves. Denoting the probability law of $\Phi^\varphi = (\Phi^{\varphi_1}, \Phi^{\varphi_2})$ under P_0 by $\mathsf{P}_\varphi^0 = \mathsf{P}_{\varphi_1,\varphi_2}^0$ (where we move 0 from the subscript to a superscript for notational reasons) we see that the optimal stopping problem (4.12) can be rewritten as follows

$$\hat{V}(\varphi_1, \varphi_2) = \inf_\tau \mathsf{E}_{\varphi_1,\varphi_2}^0\bigg[\int_0^\tau \big(1+\Phi_t^1+\Phi_t^2\big)\,dt + c\big(\big(\Phi_\tau^1+\Phi_\tau^2\big) \wedge \big(1+\Phi_\tau^1\big) \wedge \big(1+\Phi_\tau^2\big)\big)\bigg] \tag{4.18}$$

for $(\varphi_1, \varphi_2) \in [0, \infty)^2$ with $\mathsf{P}_{\varphi_1,\varphi_2}((\Phi_0^1, \Phi_0^2) = (\varphi_1, \varphi_2)) = 1$ where the infimum in (4.18) is taken over all stopping times τ of Φ. In this way we have reduced the initial sequential testing problem (3.3) to the optimal stopping problem (4.18) for the strong Markov/diffusion process Φ. We will see in the next section that this optimal stopping problem is inherently/fully two-dimensional with the infinitesimal generator of Φ being of elliptic type.

5. Elliptic PDE. In this section we derive a coupled system of stochastic differential equations (driven by two independent Brownian motions) that $\Phi = (\Phi^1, \Phi^2)$ solves. From this system we derive a closed-form expression for the infinitesimal generator of Φ that can be recognised as a partial differential operator of elliptic type. We also show that a diffeomorphic transformation of logarithmic type maps the process Φ (and its state space $(0, \infty)^2$) to a process Z (and its state space $\mathbb{R}^2$) whose coordinate processes Z^1 and Z^2 are independent Brownian motions with a nonzero and zero drift respectively.

1. From (4.2) and (4.4) we see that

$$\Phi_t^1 = \varphi_1 e^{\mu(B_t^1 - B_t^0) - \mu^2 t} \quad \& \quad \Phi_t^2 = \varphi_2 e^{\mu(B_t^2 - B_t^0) - \mu^2 t} \tag{5.1}$$

under P_0 for $t \geq 0$ where φ_1 and φ_2 belong to $[0, \infty)$. Hence by Itô's formula we find that

$$d\Phi_t^1 = \mu \Phi_t^1 (dB_t^1 - dB_t^0) \tag{5.2}$$

$$d\Phi_t^2 = \mu \Phi_t^2 (dB_t^2 - dB_t^0) \tag{5.3}$$

under P_0 with $\Phi_0^1 = \varphi_1$ and $\Phi_0^2 = \varphi_2$ in $[0, \infty)$. This shows that Φ^1 and Φ^2 are two correlated geometric Brownian motions.

2. A well-known (and easily verifiable) fact states that if $\tilde{B}^1$ and $\tilde{B}^2$ are two correlated standard Brownian motions satisfying $\mathsf{E}(\tilde{B}_t^1 \tilde{B}_t^2) = \rho t$ for $t \geq 0$ with $\rho \in (-1, 1)$, then $(\tilde{B}^1 + \tilde{B}^2)/\sqrt{2(1+\rho)}$ and $(\tilde{B}^1 - \tilde{B}^2)/\sqrt{2(1-\rho)}$ are two independent standard Brownian motions. Applying this implication to $\tilde{B}^1 := (B^1 - B^0)/\sqrt{2}$ and $\tilde{B}^2 := (B^2 - B^0)/\sqrt{2}$ with $\rho = 1/2$ it follows that

$$W^1 := \frac{\tilde{B}^1 + \tilde{B}^2}{\sqrt{3}} = \frac{B^1 + B^2 - 2B^0}{\sqrt{6}} \quad \& \quad W^2 := \frac{\tilde{B}^1 - \tilde{B}^2}{1} = \frac{B^1 - B^2}{\sqrt{2}} \tag{5.4}$$

are two independent standard Brownian motions. From (5.4) we see that

$$\tilde{B}^1 := \frac{B^1 - B^0}{\sqrt{2}} = \frac{\sqrt{3}W^1 + W^2}{2} \quad \& \quad \tilde{B}^2 := \frac{B^2 - B^0}{\sqrt{2}} = \frac{\sqrt{3}W^1 - W^2}{2}. \tag{5.5}$$

3. Making use of (5.5) in (5.2)+(5.3) we obtain

$$d\Phi_t^1 = \frac{\mu}{\sqrt{2}} \Phi_t^1 (\sqrt{3} dW_t^1 + dW_t^2) \tag{5.6}$$

$$d\Phi_t^2 = \frac{\mu}{\sqrt{2}} \Phi_t^2 (\sqrt{3} dW_t^1 - dW_t^2) \tag{5.7}$$

with $\Phi_0^1 = \varphi_1$ and $\Phi_0^2 = \varphi_2$ in $[0, \infty)$. This is a coupled system of stochastic differential equations (driven by two independent standard Brownian motions W^1 and W^2) that Φ^1 and Φ^2 solve (strongly) and this solution is pathwise unique (see, e.g., [20], pp. 128–131). Moreover, the solution $\Phi = (\Phi^1, \Phi^2)$ is both a strong Markov process (see, e.g., [20], pp. 158–163) and a strong Feller process (see, e.g., [20], pp. 170–173). Making use of (5.5) in (5.1) we see that

$$\Phi_t^1 = \varphi_1 e^{\frac{\mu}{\sqrt{2}}(\sqrt{3}W_t^1 + W_t^2) - \mu^2 t} \quad \& \quad \Phi_t^2 = \varphi_2 e^{\frac{\mu}{\sqrt{2}}(\sqrt{3}W_t^1 - W_t^2) - \mu^2 t} \tag{5.8}$$

under P_0 for $t \geq 0$ where φ_1 and φ_2 belong to $[0, \infty)$. Often we will write $\Phi_t^{\varphi_1}$ and $\Phi_t^{\varphi_2}$ for $t \geq 0$ to indicate dependence of Φ^1 and Φ^2 on the initial points φ_1 and φ_2 in $[0, \infty)$.

4. Knowing that $\Phi = (\Phi^1, \Phi^2)$ solves the system (5.6)+(5.7) and making use of Itô's calculus we find that the infinitesimal generator of Φ is given by

$$\mathbb{L}_\Phi = \mu^2 (\varphi_1^2 \partial^2_{\varphi_1\varphi_1} + \varphi_1\varphi_2 \partial^2_{\varphi_1\varphi_2} + \varphi_2^2 \partial^2_{\varphi_2\varphi_2}) \tag{5.9}$$

for φ_1 and φ_2 in $(0, \infty)$ (see, e.g., (2.7) in [16]). A standard classification of partial differential equations shows that $\mathbb{L}_\Phi$ is of elliptic type (see, e.g., (2.12) in [16]).

5. Defining a diffeomorphic transformation of $(0, \infty)^2$ to $\mathbb{R}^2$ by

$$D(\varphi_1, \varphi_2) = (\log(\varphi_1\varphi_2), \log(\varphi_1/\varphi_2)) \tag{5.10}$$

for $(\varphi_1, \varphi_2) \in (0, \infty)^2$, and setting

$$Z = (Z^1, Z^2) = D(\Phi_1, \Phi_2) = (\log(\Phi_1\Phi_2), \log(\Phi_1/\Phi_2)) \tag{5.11}$$

we see from (5.8) that

$$Z_t^1 = Z_0^1 - 2\mu^2 t + \sqrt{6}\mu W_t^1 \quad \& \quad Z_t^2 = Z_0^2 + \sqrt{2}\mu W_t^2 \tag{5.12}$$

under P_0 for $t \geq 0$ with $Z_0^1 = \log(\varphi_1\varphi_2)$ and $Z_0^2 = \log(\varphi_1/\varphi_2)$. This establishes a one-to-one correspondence between the process Φ in $(0,\infty)^2$ and the process Z in $\mathbb{R}^2$. Although the latter process Z may be viewed as a canonical building block which further clarifies the underlying setting, we will mainly study the optimal stopping problem (4.18) by means of the former process Φ in the sequel.

6. Lagrange and Mayer formulations. The optimal stopping problem (4.18) is Bolza formulated. In this section we derive its Lagrange and Mayer reformulations which are helpful in the subsequent analysis of the problem.

1. We first consider the Lagrange reformulation of the optimal stopping problem (4.18). For this, note that the loss function $\hat{M}$ from (4.13) that appears on the right-hand side of (4.18) is not smooth at the three straight lines

$$c_0 = \{ (\varphi_1, \varphi_2) \in [0,\infty)^2 \mid \varphi_1 = 1 \ \& \ \varphi_2 \in [0,1] \} \tag{6.1}$$

$$c_1 = \{ (\varphi_1, \varphi_2) \in [0,\infty)^2 \mid \varphi_1 \in [0,1] \ \& \ \varphi_2 = 1 \} \tag{6.2}$$

$$c_2 = \{ (\varphi_1, \varphi_2) \in [0,\infty)^2 \mid \varphi_1 = \varphi_2 \in [1,\infty) \} \tag{6.3}$$

ordered clockwise (see Figure 1). Note moreover that $\hat{M}$ is linear off the three straight lines and given by

$$\begin{aligned} \hat{M}(\varphi_1,\varphi_2) &= c(\varphi_1+\varphi_2) \ \text{ for } \ (\varphi_1,\varphi_2) \in \Delta_0 \\ &= c(1+\varphi_1) \ \text{ for } \ (\varphi_1,\varphi_2) \in \Delta_1 \\ &= c(1+\varphi_2) \ \text{ for } \ (\varphi_1,\varphi_2) \in \Delta_2 \end{aligned} \tag{6.4}$$

where $\Delta_0 := [0,1]^2$ is a subset of the state space surrounded by c_0 and c_1 (from the right and above), $\Delta_1 := \{ (\varphi_1,\varphi_2) \in [0,\infty)^2 \mid \varphi_2 \geq \varphi_1 \geq 1 \}$ is a subset of the state space surrounded

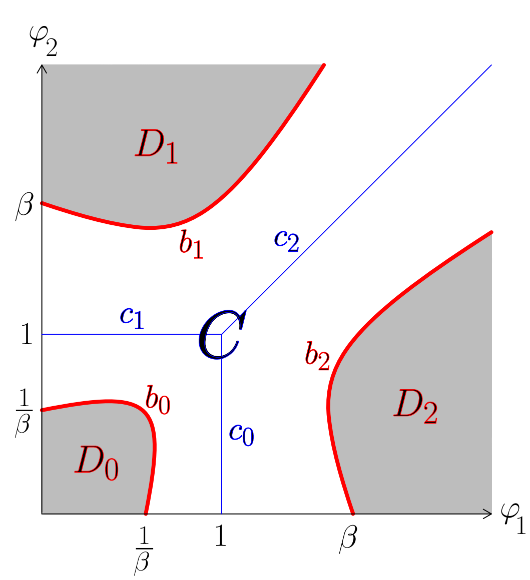

FIG. 1. *Location of the continuation set C, the stopping sets D_0, D_1, D_2, and the optimal stopping boundaries b_0, b_1, b_2, recalling that $b_1(\varphi_1) = b_2(\varphi_2)$ for $\varphi_1 = \varphi_2$ in $[0,\infty)$.*

by c_1 and c_2 (from below and the right), and $\Delta_2 := \{ (\varphi_1, \varphi_2) \in [0, \infty)^2 \mid \varphi_1 \geq \varphi_2 \geq 1 \}$ is a subset of the state space surrounded by c_0 and c_2 (from the left and above).

PROPOSITION 3. *The value function $\hat{V}$ from (4.18) can be expressed as*

$$\hat{V}(\varphi_1, \varphi_2) = \inf_{\tau} \mathsf{E}^0_{\varphi_1,\varphi_2}\Big[\int_0^\tau (1+\Phi_t^1+\Phi_t^2)\,dt - \frac{c}{2}\big(\ell_\tau^{c_0}(\Phi)+\ell_\tau^{c_1}(\Phi)+\ell_\tau^{c_2}(\Phi)\big)\Big] + \hat{M}(\varphi_1, \varphi_2) \tag{6.5}$$

for $(\varphi_1, \varphi_2) \in [0, \infty)^2$ where $\ell^{c_i}(\Phi)$ is the local time of Φ at c_i for $i = 0, 1, 2$ given by

$$\ell_\tau^{c_0}(\Phi) = \mathsf{P}\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^\tau I(1-\varepsilon < \Phi_t^1 < 1+\varepsilon)\, I(0 \leq \Phi_t^2 \leq 1)\, d\langle \Phi^1, \Phi^1\rangle_t \tag{6.6}$$

$$\ell_\tau^{c_1}(\Phi) = \mathsf{P}\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^\tau I(1-\varepsilon < \Phi_t^2 < 1+\varepsilon)\, I(0 \leq \Phi_t^1 \leq 1)\, d\langle \Phi^2, \Phi^2\rangle_t \tag{6.7}$$

$$\ell_\tau^{c_2}(\Phi) = \mathsf{P}\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^\tau I(-\varepsilon < \Phi_t^2 - \Phi_t^1 < \varepsilon) \times I(\Phi_t^1 \geq 1, \Phi_t^2 \geq 1)\, d\langle \Phi^2-\Phi^1, \Phi^2-\Phi^1\rangle_t \tag{6.8}$$

and the infimum in (6.5) is taken over all stopping times τ of Φ.

PROOF. It is evident from (6.4) that $\hat{M}$ restricted to $\Delta_0 \cup \Delta_2$ can be extended to a twice continuously differentiable function $\hat{F}$ on $[0, \infty)^2 \backslash c_0$. Then $\hat{M} = \hat{F} + (\hat{M} - \hat{F})$ and $\hat{M}^1 := \hat{F}$ is not smooth at c_0 while $\hat{M}^2 := \hat{M} - \hat{F}$ is not smooth at c_1 and c_2. Since c_0 is the graph of a (linear) function of φ_2, and c_1 and c_2 are the graphs of (linear) functions of φ_1, we see that the change-of-variable formula with local time on surfaces [15], Theorem 2.1, is applicable to $\hat{M}^1$ and $\hat{M}^2$ composed with Φ, where we note that $\hat{M}^1_{\varphi_1}(\varphi_1+, \varphi_2) - \hat{M}^1_{\varphi_1}(\varphi_1-, \varphi_2) = -c$ for $(\varphi_1, \varphi_2) \in c_0$ and $\hat{M}^2_{\varphi_2}(\varphi_1, \varphi_2+) - \hat{M}^2_{\varphi_2}(\varphi_1, \varphi_2-) = -c$ for $(\varphi_1, \varphi_2) \in c_1 \cup c_2$. Hence the formula is also applicable to $\hat{M}$ composed with Φ and this gives

$$\hat{M}(\Phi_t^1, \Phi_t^2) = \hat{M}(\varphi_1, \varphi_2) + c\int_0^t I(\Phi_s \in \Delta_0 \cup \Delta_1)\, d\Phi_s^1 + c\int_0^t I(\Phi_s \in \Delta_0 \cup \Delta_2)\, d\Phi_s^2 - \frac{c}{2}\big(\ell_t^{c_0}(\Phi)+\ell_t^{c_1}(\Phi)+\ell_t^{c_2}(\Phi)\big) \tag{6.9}$$

for $(\varphi_1, \varphi_2) \in [0, \infty)^2$ and $t \geq 0$ where the local times are defined in (6.6)–(6.8) above. Since Φ^1 and Φ^2 are continuous martingales under P_0 we see that the two integrals on the right-hand side of (6.9) are continuous martingales under P_0 as well. By the optional sampling theorem we therefore find from (6.9) that

$$\mathsf{E}^0_{\varphi_1,\varphi_2}[\hat{M}(\Phi_\tau^1, \Phi_\tau^2)] = \hat{M}(\varphi_1, \varphi_2) - \frac{c}{2}\mathsf{E}^0_{\varphi_1,\varphi_2}[\ell_\tau^{c_0}(\Phi)+\ell_\tau^{c_1}(\Phi)+\ell_\tau^{c_2}(\Phi)] \tag{6.10}$$

for all $(\varphi_1, \varphi_2) \in [0, \infty)^2$ and all stopping times τ of Φ. Inserting (6.10) into (4.18) we obtain (6.5) as claimed and the proof is complete. □

The Lagrange reformulation (6.5) of the optimal stopping problem (4.18) reveals the underlying rationale for continuing vs stopping in a clearer manner. Indeed, recalling that the local time process $t \mapsto \ell_t^{c_i}(\Phi)$ strictly increases only when Φ is at c_i, and that $\ell_t^{c_i}(\Phi) \sim \sqrt{t}$ is strictly larger than $\int_0^t (1+\Phi_s^1+\Phi_s^2)\, ds \sim t$ for small t, we see from (6.5) that it should never be optimal to stop at c_i and the incentive for stopping should increase the further away Φ gets from c_i for $= 0, 1, 2$. We will see in Section 8 below that these informal conjectures can be formalised and this will give a proof of the fact that the three straight lines c_0, c_1, c_2 are contained in the continuation set of the optimal stopping problem (4.18).

2. We next consider the Mayer reformulation of the optimal stopping problem (4.18). For this, in addition to $\hat{M}$ in (4.13) above, define

$$\check{M}(\varphi_1,\varphi_2)=\frac{1}{\mu^2}\Big(\varphi_1(\log\varphi_1-1)+\varphi_2(\log\varphi_2-1)-\frac{1}{2}\log(\varphi_1\varphi_2)\Big) \tag{6.11}$$

and set $M(\varphi_1,\varphi_2)=\check{M}(\varphi_1,\varphi_2)+\hat{M}(\varphi_1,\varphi_2)$ for $(\varphi_1,\varphi_2)\in(0,\infty)^2$.

PROPOSITION 4. *The value function $\hat{V}$ from* (4.18) *can be expressed as*

$$\hat{V}(\varphi_1,\varphi_2)=\inf_{\tau}\mathsf{E}^0_{\varphi_1,\varphi_2}[M(\Phi^1_\tau,\Phi^2_\tau)]-\check{M}(\varphi_1,\varphi_2) \tag{6.12}$$

for $(\varphi_1,\varphi_2)\in(0,\infty)^2$ where the infimum is taken over all stopping times τ of Φ.

PROOF. Recalling the closed-form expression for $\mathbb{L}_\Phi$ in (5.9) it is easily verified that

$$\mathbb{L}_\Phi\check{M}(\varphi_1,\varphi_2)=1+\varphi_1+\varphi_2 \tag{6.13}$$

for $(\varphi_1,\varphi_2)\in(0,\infty)^2$. By Itô's formula we thus find using (5.6)+(5.7) above that

$$\begin{aligned}\check{M}(\Phi^1_t,\Phi^2_t)&=\check{M}(\varphi_1,\varphi_2)+\int_0^t\check{M}_{\varphi_1}(\Phi^1_s,\Phi^2_s)\,d\Phi^1_s\\&\quad+\int_0^t\check{M}_{\varphi_2}(\Phi^1_s,\Phi^2_s)\,d\Phi^2_s+\int_0^t\mathbb{L}_\Phi\check{M}(\Phi^1_s,\Phi^2_s)\,ds\\&=\check{M}(\varphi_1,\varphi_2)+\int_0^t\frac{\mu}{\sqrt{2}}\Big(\Phi^1_s\log\Phi^1_s-\frac{1}{2}\Big)(\sqrt{3}\,dW^1_s+dW^2_s)\\&\quad+\int_0^t\frac{\mu}{\sqrt{2}}\Big(\Phi^2_s\log\Phi^2_s-\frac{1}{2}\Big)(\sqrt{3}\,dW^1_s-dW^2_s)\\&\quad+\int_0^t(1+\Phi^1_s+\Phi^2_s)\,ds\end{aligned} \tag{6.14}$$

for $(\varphi_1,\varphi_2)\in(0,\infty)^2$ and $t\ge0$ where the two integrals on the right-hand side define continuous local martingales under P_0. Making use of a localisation sequence of stopping times for these two local martingales if needed, and applying the optional sampling theorem, we find from (6.14) that

$$\mathsf{E}^0_{\varphi_1,\varphi_2}[\check{M}(\Phi^1_\tau,\Phi^2_\tau)]=\check{M}(\varphi_1,\varphi_2)+\mathsf{E}^0_{\varphi_1,\varphi_2}\Big[\int_0^\tau(1+\Phi^1_t+\Phi^2_t)\,dt\Big] \tag{6.15}$$

for all $(\varphi_1,\varphi_2)\in(0,\infty)^2$ and all (bounded) stopping times τ of Φ. Inserting (6.15) into (4.18) we obtain (6.12) as claimed and the proof is complete. □

7. Two dimensions. The observed process X in the initial sequential testing problem (3.3) is three-dimensional. In this section we consider the analogue of (3.3) and the resulting optimal stopping problem (4.18) when X is two-dimensional. The reduction of dimension three to dimension two corresponds to either Φ^1 or Φ^2 becoming 0 which is a natural boundary point for both processes (cf. [6]). This shows that Φ is a one-dimensional Markov/diffusion process when X is two-dimensional so that standard arguments enable us to solve the problem (4.18) in a closed form. The derived results for the one-dimensional optimal stopping problem (4.18) when X is two-dimensional will be used in the subsequent analysis of the two-dimensional optimal stopping problem (4.18) when X is three-dimensional.

1. Using the same arguments as above, it is easily seen that the sequential testing problem (3.3) when X is two-dimensional reduces to the optimal stopping problem (4.18) with Φ^2 being formally equal to zero. Omitting the subscript 1 from φ_1 for simplicity, we thus see that the optimal stopping problem (4.18) reads

$$\hat{V}(\varphi) = \inf_{\tau} \mathsf{E}_{\varphi}^{0}\Big[\int_0^{\tau}(1+\Phi_t)\,dt + c(1\wedge\Phi_{\tau})\Big] \tag{7.1}$$

for $\varphi \in [0,\infty)$ with $\mathsf{P}_{\varphi}^{0}(\Phi_0=\varphi)=1$ where the infimum in (7.1) is taken over all stopping times τ of Φ. From (5.1) and (5.2) we see that

$$\Phi_t = \varphi\, e^{\sqrt{2}\mu W_t - \mu^2 t} \tag{7.2}$$

$$d\Phi_t = \sqrt{2}\mu\,\Phi_t\,dW_t \tag{7.3}$$

under P_0 for $t \geq 0$ with $\Phi_0 = \varphi$ in $[0,\infty)$ where $W := (B^1 - B^0)/\sqrt{2}$ is a standard Brownian motion. From (7.3) we see that the infinitesimal generator of Φ is given by

$$\mathbb{L}_{\Phi} = \mu^2\varphi^2\frac{d^2}{d\varphi^2} \tag{7.4}$$

which also follows formally by setting $\varphi_2 = 0$ in (5.9) above.

Recognising the loss function in (7.1) as $\hat{M}(\varphi) = c(1\wedge\varphi)$ for $\varphi \in [0,\infty)$, standard arguments imply (see, e.g., [17]) that $\hat{V}$ should solve the free-boundary problem

$$\mathbb{L}_{\Phi}\hat{V}(\varphi) = -(1+\varphi)\ \text{ for }\ \varphi \in (\varphi_0^*, \varphi_1^*) \tag{7.5}$$

$$\hat{V}(\varphi_i^*) = \hat{M}(\varphi_i^*)\ \text{ for }\ i=0,1\ \text{ (instantaneous stopping)} \tag{7.6}$$

$$\hat{V}'(\varphi_i^*) = \hat{M}'(\varphi_i^*)\ \text{ for }\ i=0,1\ \text{ (smooth fit)} \tag{7.7}$$

where $0 < \varphi_0^* < 1 < \varphi_1^* < \infty$ are the optimal stopping/boundary points to be found and we have $\hat{V}(\varphi) = \hat{M}(\varphi)$ for $\varphi \in [0,\varphi_0^*) \cup (\varphi_1^*,\infty)$ as well (in addition to (7.6) above).

The general solution to the ordinary differential equation (7.5) is given by

$$\hat{V}(\varphi) = A\varphi + B + \frac{1}{\mu^2}(1-\varphi)\log\varphi \tag{7.8}$$

for $\varphi > 0$ where A and B are two undetermined real constants. Boundary conditions (7.6) and (7.7) then read as follows

$$A\varphi_0^* + B + \frac{1}{\mu^2}(1-\varphi_0^*)\log\varphi_0^* = c\varphi_0^* \tag{7.9}$$

$$A\varphi_1^* + B + \frac{1}{\mu^2}(1-\varphi_1^*)\log\varphi_1^* = c \tag{7.10}$$

$$A + \frac{1}{\mu^2}\Big(\frac{1}{\varphi_0^*} - \log\varphi_0^* - 1\Big) = c \tag{7.11}$$

$$A + \frac{1}{\mu^2}\Big(\frac{1}{\varphi_1^*} - \log\varphi_1^* - 1\Big) = 0. \tag{7.12}$$

It is a matter of routine to verify that the system (7.9)–(7.12) has a unique solution given by

$$A^* = c - \frac{1}{\mu^2}\Big(\frac{1}{\varphi_0^*} - \log\varphi_0^* - 1\Big)\ \ \&\ \ B^* = c - \frac{1}{\mu^2}(\varphi_1^* + \log\varphi_1^* - 1) \tag{7.13}$$

where φ_0^* and φ_1^* are the unique solution to

$$\frac{1}{\mu^2}\Big(\frac{1}{\varphi_0^*}-\frac{1}{\varphi_1^*}+\log\Big(\frac{\varphi_1^*}{\varphi_0^*}\Big)\Big)=c \;\;\&\;\; \frac{1}{\mu^2}\Big(\varphi_1^*-\varphi_0^*+\log\Big(\frac{\varphi_1^*}{\varphi_0^*}\Big)\Big)=c \tag{7.14}$$

satisfying $0<\varphi_0^*<1<\varphi_1^*<\infty$.

By symmetry we may conclude that $\varphi_0^*=1/\varphi_1^*$ so that (7.13) and (7.14) reduce to

$$A^*=B^*=c-\frac{1}{\mu^2}(\varphi_1^*+\log\varphi_1^*-1) \tag{7.15}$$

$$\varphi_1^*-\frac{1}{\varphi_1^*}+2\log\varphi_1^*=c\mu^2 \tag{7.16}$$

respectively. It follows from (7.8) and (7.15) that

$$\begin{aligned}\hat{V}^*(\varphi)&=\Big(c-\frac{1}{\mu^2}(\varphi_1^*+\log\varphi_1^*-1)\Big)(1+\varphi)\\&\quad+\frac{1}{\mu^2}(1-\varphi)\log\varphi \;\text{ for } \varphi\in(1/\varphi_1^*,\varphi_1^*)\\&=\hat{M}(\varphi)\;\text{ for } \varphi\in[0,1/\varphi_1^*]\cup[\varphi_1^*,\infty)\end{aligned} \tag{7.17}$$

defines a candidate value function for the optimal stopping problem (7.1).

Applying the Itô–Tanaka formula (cf. [19], p. 223) to $\hat{V}^*$ composed with Φ, which reduces to Itô's formula due to smooth fit (7.7), and making use of the optional sampling theorem, it is easily verified that $\hat{V}^*$ from (7.17) coincides with the value function $\hat{V}$ from (7.1) and the optimal stopping time (at which the infimum in (7.1) is attained) is given by

$$\tau_*=\inf\{\,t\ge 0 \mid \Phi_t\notin(1/\varphi_1^*,\varphi_1^*)\,\} \tag{7.18}$$

where φ_1^* is the unique solution to (7.16) on $(1,\infty)$.

To avoid a possible confusion with subscripts we will set $\beta:=\varphi_1^*$ in the sequel. Thus $\beta\in(0,\infty)$ is the unique solution to

$$\beta-\frac{1}{\beta}+2\log\beta=c\mu^2 \tag{7.19}$$

and the stopping time

$$\tau=\inf\{\,t\ge 0\mid \Phi_t\notin(\alpha,\beta)\,\} \tag{7.20}$$

is optimal in (7.1) where we set $\alpha=1/\beta$. These facts will be used in the subsequent analysis of the optimal stopping problem (4.18) when X is three-dimensional.

8. Properties of the optimal stopping boundaries. In this section we establish the existence of an optimal stopping time in (4.18) when the observed process X is three-dimensional and derive basic properties of the optimal stopping boundaries. These results will be further refined in Section 10 below.

1. Looking at (4.18) we may conclude that the (candidate) continuation and stopping sets in this problem are respectively given by

$$C=\{\,(\varphi_1,\varphi_2)\in[0,\infty)^2\mid \hat{V}(\varphi_1,\varphi_2)<\hat{M}(\varphi_1,\varphi_2)\,\} \tag{8.1}$$

$$D=\{\,(\varphi_1,\varphi_2)\in[0,\infty)^2\mid \hat{V}(\varphi_1,\varphi_2)=\hat{M}(\varphi_1,\varphi_2)\,\} \tag{8.2}$$

where $\hat{M}$ is defined in (4.13) above. Recalling (5.8) we see that the expectation in (6.12) defines a continuous function of the initial point (φ_1,φ_2) in $[0,\infty)^2$ for every (bounded) stopping time τ of Φ given and fixed. Taking the infimum over all (bounded) stopping time τ of Φ we can conclude that the value function $\hat{V}$ is upper semicontinuous on $[0,\infty)^2$. From (4.13) and (6.11) we see that the loss function $M = \check{M} + \hat{M}$ is continuous and hence lower semicontinuous on $[0,\infty)^2$. It follows therefore by [17], Corollary 2.9, that the first entry time of the process Φ into the closed set D defined by

$$\tau_D = \inf\{\, t \geq 0 \mid \Phi_t \in D \,\} \tag{8.3}$$

is optimal in (6.12), and hence in (4.18) as well, whenever $\mathsf{P}^0_{\varphi_1,\varphi_2}(\tau_D < \infty) = 1$ for all $(\varphi_1,\varphi_2) \in [0,\infty)^2$. In the sequel we will establish this and other properties of τ_D by analysing the boundary of D. We first turn to global properties of the value function $\hat{V}$ itself.

PROPOSITION 5. *For the value function $\hat{V}$ from* (4.18) *we have*

$$(\varphi_1,\varphi_2) \mapsto \hat{V}(\varphi_1,\varphi_2) \;\textit{ is concave on }\; [0,\infty)^2 \tag{8.4}$$

$$(\varphi_1,\varphi_2) \mapsto \hat{V}(\varphi_1,\varphi_2) \;\textit{ is continuous on }\; [0,\infty)^2 . \tag{8.5}$$

PROOF. We first show that (8.4) is satisfied. Combining (5.8) with the concavity of the loss function $\hat{M}$ from (4.13) we see that the expectation in (4.18) defines a concave function of the initial point (φ_1,φ_2) in $[0,\infty)^2$ for every (bounded) stopping time τ of Φ given and fixed. Taking the infimum over all (bounded) stopping times τ of Φ we find that the value function $\hat{V}$ itself is concave as claimed in (8.4) above.

We next show that (8.5) is satisfied. From the concavity of $\hat{V}$ on the open set $(0,\infty)^2$ we can conclude that $\hat{V}$ is continuous on $(0,\infty)^2$. Recall that there are concave functions F defined on a convex subset S of $\mathbb{R}^2$ and taking values in $\mathbb{R}$, such that the limit of $F(x_n)$ may not exist when x_n belonging to the interior of S converges to a point x_0 at the boundary of S as $n \to \infty$. However, if S is closed then it is well known (and easily verified) that such a function F must be lower semicontinuous. Applying this implication to $F = \hat{V}$ and $S = [0,\infty)^2$ we can conclude that $\hat{V}$ is lower semicontinuous on $[0,\infty)^2$. At the same time we know that $\hat{V}$ is upper semicontinuous (as established following (8.2) above) and hence we can conclude that $\hat{V}$ is continuous as claimed in (8.5) above. □

2. We show that the three straight lines c_0, c_1, c_2 defined in (6.1)–(6.3) above are contained in the continuation set C. The proof of this fact uses the Lagrange reformulation (6.5) of the optimal stopping problem (4.18) combined with the fact that the local times in (6.5) have a square-root growth at the three straight lines while the integral in (6.5) grows linearly.

PROPOSITION 6. *The straight lines c_0, c_1, c_2 from* (6.1)–(6.3) *are contained in the continuation set C of the optimal stopping problem* (4.18).

PROOF. We claim that

$$\mathsf{E}^0_{\varphi_1,\varphi_2}[\ell^{c_i}_{t\wedge\tau_{R^c}}] \geq \kappa_i \sqrt{t} \tag{8.6}$$

for all $t \in (0,t_i)$ with some $\kappa_i > 0$ and $t_i > 0$ for $i = 0,1,2$ where $\tau_{R^c} = \inf\{t \geq 0 \mid (\Phi^1_t, \Phi^2_t) \notin R\}$ is the first exit time of Φ from a bounded rectangle R containing the given point $(\varphi_1,\varphi_2) \in c_0 \cup c_1 \cup c_2$ in its interior. Indeed, this follows by a direct application of Lemma 15 in [16] when (φ_1,φ_2) belongs to $c_0 \cup c_1$, while the same lemma is applicable to $(\hat{\Phi}^1, \hat{\Phi}^2) := (\Phi^2 - \Phi^1, \Phi^2 + \Phi^1)$ obtained by a (bijective) clockwise rotation of (φ_1,φ_2) for 45° when (φ_1,φ_2) belongs to c_2. Note that the case when $(\varphi_1,\varphi_2) = (1,1) \in c_0 \cap c_1 \cap c_2$ presents no difficulty as the proof of Lemma 15 in [16] extends plainly to cover this case as well. Having (8.6) in place we can then proceed as follows.

For $(\varphi_1, \varphi_2) \in c_0 \cup c_1 \cup c_2$ given and fixed, set $R = [0, 2\varphi_1] \times [0, 2\varphi_2]$ and consider the stopping time $\tau := t \wedge \tau_{R^c}$ for $t \in (0, t_i)$ if (φ_1, φ_2) belongs to c_i for $i = 0, 1, 2$. Inserting this τ under the expectation sign in (6.5) and making use of (8.6) we find that

$$\hat{V}(\varphi_1, \varphi_2) \leq (1+2\varphi_1+2\varphi_2)t - \frac{c}{2}\kappa_i \sqrt{t} + \hat{M}(\varphi_1, \varphi_2) \tag{8.7}$$

for all $t \in (0, t_i)$ if (φ_1, φ_2) belongs to c_i for $i = 0, 1, 2$. Taking t in (8.7) sufficiently small we see that $\hat{V}(\varphi_1, \varphi_2) < \hat{M}(\varphi_1, \varphi_2)$ which shows that (φ_1, φ_2) belongs to C as claimed. □

3. The three straight lines c_0, c_1, c_2 naturally split the stopping set D into the three subsets

$$D_0 = \{ (\varphi_1, \varphi_2) \in D \mid \varphi_1, \varphi_2 \in [0, 1] \} \tag{8.8}$$

$$D_1 = \{ (\varphi_1, \varphi_2) \in D \mid 1 \leq \varphi_1 \leq \varphi_2 \} \tag{8.9}$$

$$D_2 = \{ (\varphi_1, \varphi_2) \in D \mid 1 \leq \varphi_2 \leq \varphi_1 \}. \tag{8.10}$$

Note that the set D_0 is surrounded by the straight lines c_0 and c_1, the set D_1 is surrounded by the straight lines c_1 and c_2, and the set D_2 is surrounded by the straight lines c_0 and c_2. Clearly $D = D_0 \cup D_1 \cup D_2$ and the sets D_0, D_1, D_2 are disjoint (see Figure 1).

PROPOSITION 7. *The sets D_0, D_1, D_2 are convex.*

PROOF. We will show that the set D_2 is convex and the same arguments can be used to show that the sets D_0 and D_1 are convex. For this, let (φ_1', φ_2') and $(\varphi_1'', \varphi_2'')$ belonging to D_2 and $\lambda \in (0, 1)$ be given and fixed. Firstly, note that

$$\begin{aligned} \hat{V}\big(\lambda(\varphi_1', \varphi_2')+(1-\lambda)(\varphi_1'', \varphi_2'')\big) &= \hat{V}\big(\lambda\varphi_1'+(1-\lambda)\varphi_1'', \lambda\varphi_2'+(1-\lambda)\varphi_2''\big) \\ &\leq \hat{M}\big(\lambda\varphi_1'+(1-\lambda)\varphi_1'', \lambda\varphi_2'+(1-\lambda)\varphi_2''\big) \\ &= c\big(1+\lambda\varphi_2'+(1-\lambda)\varphi_2''\big), \end{aligned} \tag{8.11}$$

where we use (4.13) to infer that $\hat{M}(\varphi_1, \varphi_2) = c(1+\varphi_2)$ for (φ_1, φ_2) belonging to the subset of $[0, \infty)^2$ surrounded by c_0 and c_2. Secondly, using that $\hat{V}$ is concave on $[0, \infty)^2$ as established in (8.4) above, we find that

$$\begin{aligned} \hat{V}\big(\lambda(\varphi_1', \varphi_2')+(1-\lambda)(\varphi_1'', \varphi_2'')\big) &\geq \lambda\hat{V}(\varphi_1', \varphi_2')+(1-\lambda)\hat{V}(\varphi_1'', \varphi_2'') \\ &= \lambda\hat{M}(\varphi_1', \varphi_2')+(1-\lambda)\hat{M}(\varphi_1'', \varphi_2'') \\ &= c\big(1+\lambda\varphi_2'+(1-\lambda)\varphi_2''\big) \end{aligned} \tag{8.12}$$

where in the first equality we use that (φ_1', φ_2') and $(\varphi_1'', \varphi_2'')$ belong to $D_2 \subseteq D$. Combining (8.11) and (8.12) we see that $\hat{V}(\lambda(\varphi_1', \varphi_2')+(1-\lambda)(\varphi_1'', \varphi_2'')) = \hat{M}(\lambda(\varphi_1', \varphi_2')+(1-\lambda)(\varphi_1'', \varphi_2''))$ showing that $\lambda(\varphi_1', \varphi_2')+(1-\lambda)(\varphi_1'', \varphi_2'')$ belongs to D_2 as needed. □

4. To describe the shape of the stopping sets D_0, D_1, D_2 we may recall from Section 7 that the subsets $([0, 1/\beta] \cup [\beta, \infty)) \times \{0\}$ and $\{0\} \times ([0, 1/\beta] \cup [\beta, \infty))$ of $[0, \infty)^2$ are contained in D where $\beta \in (1, \infty)$ solves (7.19) uniquely. Symmetry arguments to be addressed shortly below show that it is sufficient to focus on the set D_2 as the conclusions will directly extend to the sets D_0 and D_1 as well. Moving from the straight lines c_0 and c_2 in C to the right, let us formally define the (least) boundary between C and D_2 by setting

$$b_2(\varphi_2) = \inf \{ \varphi_1 > 1 \vee \varphi_2 \mid (\varphi_1, \varphi_2) \in D_2 \} \tag{8.13}$$

for $\varphi_2 \in [0, \infty)$. Clearly the infimum in (8.13) is attained since D_2 is closed. We now show that b_2 constitutes the entire boundary of D_2 in $[0, \infty)^2$ (see Figure 1).

PROPOSITION 8. *The mapping $\varphi_2 \mapsto b_2(\varphi_2)$ is finite valued on $[0,\infty)$ and we have*

$$D_2 = \{ (\varphi_1,\varphi_2) \in [0,\infty)^2 \mid \varphi_1 \geq b_2(\varphi_2) \} \tag{8.14}$$

with $b_2(0) = \beta \in (1,\infty)$ and $b_2(\varphi_2) \to \infty$ as $\varphi_2 \to \infty$.

PROOF. To derive (8.14) we show that

$$(\varphi_1', \varphi_2') \in D_2 \Rightarrow (\varphi_1'', \varphi_2') \in D_2 \tag{8.15}$$

for all $\varphi_1'' \geq \varphi_1'$. For this, recall from (8.4) that $\varphi_1 \mapsto \hat{V}(\varphi_1, \varphi_2')$ is concave on $[0,\infty)$ while $\varphi_1 \mapsto \hat{M}(\varphi_1,\varphi_2') = c(1+\varphi_2')$ is constant for $\varphi_1 \geq \varphi_2' \geq 1$. Hence if $\hat{V}(\varphi_1',\varphi_2') = \hat{M}(\varphi_1',\varphi_2')$ due to $(\varphi_1',\varphi_2') \in D_2$ with $\hat{V}(\varphi_1'',\varphi_2') < \hat{M}(\varphi_1'',\varphi_2')$ meaning that $(\varphi_1'',\varphi_2') \notin D_2$ for some $\varphi_1'' > \varphi_1'$, then $\hat{V}(\varphi_1,\varphi_2')$ must converge to $-\infty$ as φ_1 converges to ∞. This however contradicts the fact that $\hat{V}$ is nonnegative and hence (8.15) must hold as claimed. Combining (8.13) and (8.15) we see that (8.14) is satisfied as claimed.

To establish that b_2 is finite valued we first show that

$$(\varphi_1', \varphi_2') \in D_2 \Rightarrow (\varphi_1', \varphi_2'') \in D_2 \tag{8.16}$$

for all $\varphi_2'' \in [0, \varphi_2']$ when $\varphi_1' \geq \beta$. (Note that the latter inequality cannot be omitted and (8.16) may fail when $\varphi_1' < \beta$ as we will see in Section 10 below.) For this, recall from (8.4) that $\varphi_2 \mapsto \hat{V}(\varphi_1', \varphi_2)$ is concave on $[0,\infty)$ while $\varphi_2 \mapsto \hat{M}(\varphi_1', \varphi_2) = c(1+\varphi_2)$ is linear for $\varphi_2 \in [1, \varphi_1']$. By the results of Section 7 we know that $(\varphi_1', 0)$ belongs to D_2 so that $\hat{V}(\varphi_1', 0) = \hat{M}(\varphi_1', 0)$ when $\varphi_1' \geq \beta$. Hence if $\hat{V}(\varphi_1',\varphi_2') = \hat{M}(\varphi_1',\varphi_2')$ due to $(\varphi_1',\varphi_2') \in D_2$ then $\hat{V}(\varphi_1',\varphi_2) = \hat{M}(\varphi_1',\varphi_2)$ for all $\varphi_2 \in [0,\varphi_2']$. This shows that (8.16) holds as claimed.

From (8.16) we see that if $b_2(\varphi_2') \geq \beta$ for some $\varphi_2' > 0$ then $\varphi_2 \mapsto b_2(\varphi_2)$ is increasing on $[\varphi_2', \infty)$. In particular, this means that if $b_2(\varphi_2'') = \infty$ for some $\varphi_2'' > 0$ then $b_2(\varphi_2) = \infty$ for all $\varphi_2 \geq \varphi_2''$. We will now use this fact to show that b_2 is finite valued as claimed.

Assuming that $b_2(\varphi_2'') = \infty$ for some $\varphi_2'' > 0$, and fixing $b > a > \varphi_2''$, it follows from the previous argument that the rectangle $R_N = (N,\infty) \times (a,b)$ is contained in C for every $N \geq N_0$ with some $N_0 \geq 1$ large enough. Consider the stopping time

$$\tau_{R_N^c}^{\varphi_1,\varphi_2} = \inf\{ t \geq 0 \mid (\Phi_t^{\varphi_1}, \Phi_t^{\varphi_2}) \notin R_N \} \tag{8.17}$$

for $(\varphi_1,\varphi_2) \in R_N$. Since $R_N \subseteq C$ we see that $\tau_{R_N^c}^{\varphi_1,\varphi_2} \leq \tau_D^{\varphi_1,\varphi_2}$ and hence it follows that

$$\begin{aligned} c(1+\varphi_2) &\geq \hat{M}(\varphi_1,\varphi_2) \geq \hat{V}(\varphi_1,\varphi_2) \\ &= \mathsf{E}^0_{\varphi_1,\varphi_2}\left[\int_0^{\tau_D} (1+\Phi_t^1+\Phi_t^2)\,dt + \hat{M}(\Phi^1_{\tau_D}, \Phi^2_{\tau_D}) \right] \\ &\geq \mathsf{E}_0\left[\int_0^{\tau_{R_N^c}^{\varphi_1,\varphi_2}} \Phi_t^{\varphi_1}\,dt \right] \geq N\,\mathsf{E}_0[\tau_{R_N^c}^{\varphi_1,\varphi_2}] \end{aligned} \tag{8.18}$$

for all $N \geq N_0$. Noting that $\mathsf{E}_0[\tau_{R_N^c}^{\varphi_1,\varphi_2}] \to \mathsf{E}_0[\tau_{(a,b)^c}^{\varphi_2}]$ where $\tau_{(a,b)^c}^{\varphi_2} = \inf\{ t \geq 0 \mid \Phi_t^{\varphi_2} \notin (a,b) \}$ as $N \to \infty$, we see from (8.18) that

$$c(1+\varphi_2) \geq \frac{N}{2}\,\mathsf{E}_0[\tau_{(a,b)^c}^{\varphi_2}] \tag{8.19}$$

for all $N \geq N_1$ with some $N_1 \geq 1$ large enough. Letting $N \to \infty$ and using that $\mathsf{E}_0[\tau_{(a,b)^c}^{\varphi_2}] > 0$ we obtain a contradiction. Thus there is no $\varphi_2'' > 0$ such that $b_2(\varphi_2'') = \infty$ and hence b_2 is finite valued as claimed.

Finally, the fact that $b_2(0) = \beta \in (1,\infty)$ was established in Section 7 above. Moreover, since $b_2(\varphi_2) > \varphi_2$ for all $\varphi_2 > 0$ due to c_0 and c_2 being contained in C, we see that $b_2(\varphi_2) \to \infty$ as $\varphi_2 \to \infty$ and the proof is complete. □

PROPOSITION 9. *The mapping* $\varphi_2 \mapsto b_2(\varphi_2)$ *is convex and continuous on* $[0, \infty)$.

PROOF. Convexity of the mapping $\varphi_2 \mapsto b_2(\varphi_2)$ on $[0, \infty)$ follows from the convexity of the stopping set D_2 as established in Proposition 7 above. Hence the mapping $\varphi_2 \mapsto b_2(\varphi_2)$ is continuous on $(0, \infty)$ while b_2 cannot make a jump at 0 due to the fact that the stopping set D_2 is closed. This completes the proof. □

We will show in Section 10 below that $b_2(\varphi_2) < \beta$ for $\varphi_2 \in (0, \kappa)$ with $\kappa > 0$ such that $b_2(\kappa) = \beta$. This fact combined with the convexity of b_2 on $[0, \infty)$ means that the mapping $\varphi_2 \mapsto b_2(\varphi_2)$ is (firstly) decreasing on $[0, \kappa']$ and (then) increasing on $[\kappa', \infty)$ with some $\kappa' \in (0, \kappa)$. In addition to these facts about b_2 around zero we will conclude this section by evaluating the asymptotic behaviour of b_2 at infinity. Before we do that we will turn to the remaining two stopping sets D_0 and D_1 including their boundaries.

5. Symmetry arguments enable us to extend the setting and results of Proposition 8 and Proposition 9 from the stopping set D_2 to the remaining two stopping sets D_0 and D_1. For this, recall from (4.3) that $\Phi^1 = \Pi^1/\Pi^0$ and $\Phi^2 = \Pi^2/\Pi^0$. Since Π^0, Π^1, Π^2 play a symmetric role in the optimal stopping problem (3.8) we see that any permutation of the three coordinates should yield the same result. There are two generic permutations which generate all the others (six in total). The first generic permutation is obtained by swapping Π^1 and Π^2 while keeping Π^0 intact. This yields $\Phi^1 = \Pi^1/\Pi^0 \sim \Pi^2/\Pi^0 = \Phi^2$ and $\Phi^2 = \Pi^2/\Pi^0 \sim \Pi^1/\Pi^0 = \Phi^1$ showing that

$$(\varphi_1, \varphi_2) \in \partial C \iff (\varphi_2, \varphi_1) \in \partial C \tag{8.20}$$

where ∂C can also be replaced by C or D. The second generic permutation is obtained by swapping Π^0 and Π^1 while keeping Π^2 intact. This yields $\Phi_1 = \Pi^1/\Pi^0 \sim \Pi^0/\Pi^1 = 1/\Phi^1$ and $\Phi_2 = \Pi^2/\Pi^0 \sim \Pi^2/\Pi^1 = \Phi^2/\Phi^1$ showing that

$$(\varphi_1, \varphi_2) \in \partial C \iff \left(\frac{1}{\varphi_1}, \frac{\varphi_2}{\varphi_1}\right) \in \partial C \tag{8.21}$$

where ∂C can also be replaced by C or D. The remaining four equivalencies can be obtained by combining (8.20) and (8.21). For example, applying first (8.20) and then (8.21) we find that $(\varphi_1, \varphi_2) \in \partial C \Leftrightarrow (1/\varphi_2, \varphi_1/\varphi_2) \in \partial C$ (where ∂C can also be replaced by C or D as above) which is obtained by swapping Π^0 and Π^2 while keeping Π^1 intact.

6. Having understood the symmetry relations we now move to extending the setting and results of Proposition 8 and Proposition 9 from D_2 to D_0 and D_1. We first address the case of D_1 which in view of (8.20) is a mirror image of D_2 across the main diagonal in $[0, \infty)^2$. In analogy with (8.13) we thus define the (least) boundary between C and D_1 by setting

$$b_1(\varphi_1) = \inf\{\,\varphi_2 > 1 \vee \varphi_1 \mid (\varphi_1, \varphi_2) \in D_1\,\} \tag{8.22}$$

for $\varphi_1 \in [0, \infty)$. Clearly the infimum in (8.22) is attained since D_1 is closed. Similarly to b_2 and D_2 above we now show that b_1 constitutes the entire boundary of D_1 in $[0, \infty)^2$ (see Figure 1).

PROPOSITION 10. *The mapping* $\varphi_1 \mapsto b_1(\varphi_1)$ *is finite valued on* $[0, \infty)$ *and we have*

$$D_1 = \{\,(\varphi_1, \varphi_2) \in [0, \infty)^2 \mid \varphi_2 \ge b_1(\varphi_1)\,\} \tag{8.23}$$

with $b_1(0) = \beta \in (1, \infty)$ *and* $b_1(\varphi_1) \to \infty$ *as* $\varphi_1 \to \infty$.

PROOF. This can be derived in exactly the same way as in Proposition 8 above. Alternatively Proposition 10 also follows directly from Proposition 8 using the symmetry relation (8.20) which shows that D_1 is a mirror image of D_2 across the main diagonal in $[0, \infty)^2$. □

PROPOSITION 11. *The mapping* $\varphi_1 \mapsto b_1(\varphi_1)$ *is convex and continuous on* $[0, \infty)$.

PROOF. This can be derived in exactly the same way as in Proposition 9 above. Alternatively Proposition 11 also follows directly from Proposition 9 using the symmetry relation (8.20) which shows that b_1 coincides with b_2 on $[0, \infty)$. □

Despite the fact that the functional rules of b_1 and b_2 coincide on $[0, \infty]$, we will still keep their different subscripts 1 and 2 in place to account for different arguments in $(\varphi_1, b_1(\varphi_1))$ and $(b_2(\varphi_2), \varphi_2)$ for $\varphi_1 \geq 0$ and $\varphi_2 \geq 0$ respectively.

7. We next address the case of D_0 which in view of (8.21) can similarly be linked to the case of D_2 in a one-to-one way. Moving from the point $(1, 1) \in C$ down to the point $(0, 0) \in D_0$ along the main diagonal in $[0, 1]^2$, we know that there exists the (first) point (γ, γ) that belongs to D_0. Equivalently γ can also be formally defined by

$$\gamma = \sup \{ \varphi \in [0, 1] \mid (\varphi, \varphi) \in D_0 \} . \tag{8.24}$$

Clearly the supremum in (8.24) is attained since D_0 is closed and we have $\gamma \in (0, 1)$ since $(1, 1) \in C$. Similarly to (8.13) and (8.22) we then define the (least) upper boundary between C and D_0 by setting

$$b_0^1(\varphi_1) = \sup \{ \varphi_2 \in [0, 1] \mid (\varphi_1, \varphi_2) \in D_0 \} \tag{8.25}$$

for $\varphi_1 \in [0, \gamma]$, and the (least) lower boundary between C and D_0 by setting

$$b_0^2(\varphi_2) = \sup \{ \varphi_1 \in [0, 1] \mid (\varphi_1, \varphi_2) \in D_0 \} \tag{8.26}$$

for $\varphi_2 \in [0, \gamma]$. Clearly the suprema in (8.25) and (8.26) are attained since D_0 is closed. In view of (8.20), it is clear that the graphs of b_0^1 and b_0^2 are mirror images of each other across the main diagonal in $[0, \gamma]^2$, so that $b_0^1 = b_0^2$ on $[0, \gamma]$ and we set

$$b_0(\varphi) := b_0^1(\varphi) = b_0^2(\varphi) \tag{8.27}$$

for $\varphi \in [0, \gamma]$. Similarly to Proposition 8 and Proposition 10 above, we now show that b_0 can be used to describe the entire boundary of D_0 in $[0, \infty)^2$ (see Figure 1).

PROPOSITION 12. *The following identity holds*

$$D_0 = \{ (\varphi_1, \varphi_2) \in [0, \gamma]^2 \mid \varphi_1 \leq \varphi_2 \leq b_0(\varphi_1) \ \textit{or} \ \varphi_2 \leq \varphi_1 \leq b_0(\varphi_2) \} \tag{8.28}$$

with $b_0(0) = 1/\beta$ *and* $b_0(\gamma) = \gamma \in (1/\beta, 1)$. *The mapping* $\varphi \mapsto b_0(\varphi)$ *is concave and continuous on* $[0, \gamma]$.

PROOF. All claims follow by convexity (and closeness) of D_0 established in Proposition 7 above combined with the symmetry relations (8.20) and (8.21). The latter symmetry relation links D_0 to D_2 in a one-to-one way and this enables us to conclude that $b_0(0) = 1/\beta$ as claimed. The final claim $b_0(\gamma) = \gamma \in (1/\beta, 1)$ is evident from (8.24)–(8.26) above. □

The one-to-one correspondence between D_0 and D_2 obtained by the symmetry relation (8.21) enables us to transfer the facts stated following Proposition 9 above from D_2 to D_0. In particular, this yields that the mapping $\varphi \mapsto b_0(\varphi)$ is (firstly) increasing on $[0, \delta]$ and (then) decreasing on $[\delta, \gamma]$ for some $\delta \in (0, \gamma)$ (see Figure 1).

8. Another consequence of the one-to-one correspondence between D_0 and D_2 (and hence D_1 as well) is the possibility to describe the asymptotic behaviour of b_1 and b_2 at infinity.

PROPOSITION 13. *We have*

$$\lim_{\varphi_1\to\infty}\frac{b_1(\varphi_1)}{\varphi_1}=\lim_{\varphi_2\to\infty}\frac{b_2(\varphi_2)}{\varphi_2}=\beta. \tag{8.29}$$

PROOF. The first equality follows by the symmetry relation (8.20) implying that b_1 coincides with b_2 on $[0,\infty)$ so that it is enough to establish the second equality in (8.29). For this, note that the symmetry relation (8.21) yields

$$\big(\varphi_1, b_0(\varphi_1)\big)\in\partial C \Longleftrightarrow \left(\frac{1}{\varphi_1},\frac{b_0(\varphi_1)}{\varphi_1}\right)\in\partial C \tag{8.30}$$

for $\varphi_1\in[0,\gamma]$. Note also that $(\varphi_1, b_0(\varphi_1))\in\partial D_0$ tends to $(0,1/\beta)$ and $(1/\varphi_1, b_0(\varphi_1)/\varphi_1)\in\partial D_2$ tends to (∞,∞) as $\varphi_1\to 0$. The fact that the point $(1/\varphi_1, b_0(\varphi_1)/\varphi_1)$ belongs to $\partial C\cap\partial D_2$ means that this point can be identified with $(b_2(\varphi_2),\varphi_2)$ for some $\varphi_2>0$ with $\varphi_2\to\infty$ as $\varphi_1\to 0$. This shows that

$$\frac{b_2(\varphi_2)}{\varphi_2}=\frac{\frac{1}{\varphi_1}}{\frac{b_0(\varphi_1)}{\varphi_1}}=\frac{1}{b_0(\varphi_1)}\to\frac{1}{\frac{1}{\beta}}=\beta \tag{8.31}$$

as $\varphi_2\to\infty$. This establishes (8.29) and the proof is complete. □

We will continue our study of the sets D_0, D_1, D_2 in Section 10 below.

9. Smooth fit. In this section we show that the value function $\hat V$ from (4.18) satisfies the smooth fit condition at the optimal stopping boundaries b_0, b_1, b_2. A key point in the proof is based upon the fact that the boundary points are *Green regular* for D_0, D_1, D_2 in the sense that the first entry time $\tau_{D_i}^{\varphi_1^n,\varphi_2^n}$ of $(\Phi^{\varphi_1^n},\Phi^{\varphi_2^n})$ into D_i satisfies

$$\tau_{D_i}^{\varphi_1^n,\varphi_2^n}\to 0 \tag{9.1}$$

with P_0-probability one whenever $(\varphi_1^n,\varphi_2^n)$ from C tends to (φ_1,φ_2) at the boundary $\partial C\cap D_i$ for $i=0,1,2$ as $n\to\infty$. The Green regularity follows from the fact that the boundary points are *probabilistically regular* for D_0, D_1, D_2 in the sense that $\mathsf{P}^0_{\varphi_1,\varphi_2}(\tau_{D_i}=0)=1$ for every (φ_1,φ_2) at the boundary $\partial C\cap D_i$ for $i=0,1,2$ combined with the fact that the process (Φ^1,Φ^2) is strong Feller which is evident from (5.8) above (cf. [4], Section 3). The probabilistic regularity is a consequence of the fact that the sets D_i are convex (as established in Proposition 7 above) so that in view of (5.8) each boundary point from $\partial C\cap D_i$ satisfies Zaremba's cone condition for D_i with $i=0,1,2$ (see, e.g., [11], Theorem 3.2, p. 250). These facts establish (9.1) and we can now state the main result of this section.

PROPOSITION 14 (Smooth fit). *For the value function $\hat V$ from* (4.18) *we have*

$$\hat V_{\varphi_1}(\varphi_1,\varphi_2)=\hat M_{\varphi_1}(\varphi_1,\varphi_2) \tag{9.2}$$

$$\hat V_{\varphi_2}(\varphi_1,\varphi_2)=\hat M_{\varphi_2}(\varphi_1,\varphi_2) \tag{9.3}$$

for all $(\varphi_1,\varphi_2)\in\partial C\cap D_i$ *with* $i=0,1,2$.

PROOF. We will establish (9.2) and (9.3) for D_2 and similar arguments can be used for D_0 and D_1. For this, let $\varphi_1=b_2(\varphi_2)$ with $\varphi_2>0$ be given and fixed in the sequel.

1. We show that (9.2) holds. For this, we first note that

$$\liminf_{h\downarrow 0} \frac{\hat{V}(\varphi_1-h,\varphi_2)-\hat{V}(\varphi_1,\varphi_2)}{-h} \geq \liminf_{h\downarrow 0} \frac{\hat{M}(\varphi_1-h,\varphi_2)-\hat{M}(\varphi_1,\varphi_2)}{-h} = 0 \tag{9.4}$$

since $\hat{V}(\varphi_1-h,\varphi_2) \leq \hat{M}(\varphi_1-h,\varphi_2)$ and $\hat{V}(\varphi_1,\varphi_2) = \hat{M}(\varphi_1,\varphi_2)$ with $\varphi_1' \mapsto \hat{M}(\varphi_1',\varphi) = c(1+\varphi_2)$ being constant for $\varphi_1' > \varphi_2 \geq 1$. We next show that

$$\limsup_{h\downarrow 0} \frac{\hat{V}(\varphi_1-h,\varphi_2)-\hat{V}(\varphi_1,\varphi_2)}{-h} \leq 0\,. \tag{9.5}$$

For this, let $\tau_D^{\varphi_1-h,\varphi_2}$ denote the first entry time of $(\Phi^{\varphi_1-h},\Phi^{\varphi_2})$ into D for $h>0$ given and fixed. Since $\tau_D^{\varphi_1-h,\varphi_2}$ is optimal for $\hat{V}(\varphi_1-h,\varphi_2)$ we find by (5.8) that

$$\begin{aligned}
&\hat{V}(\varphi_1-h,\varphi_2)-\hat{V}(\varphi_1,\varphi_2)\\
&\quad\geq \mathsf{E}_0\Big[\int_0^{\tau_D^{\varphi_1-h,\varphi_2}} (1+\Phi_t^{\varphi_1-h}+\Phi_t^{\varphi_2})\,dt + \hat{M}\big(\Phi^{\varphi_1-h}_{\tau_D^{\varphi_1-h,\varphi_2}},\Phi^{\varphi_2}_{\tau_D^{\varphi_1-h,\varphi_2}}\big)\Big]\\
&\qquad -\mathsf{E}_0\Big[\int_0^{\tau_D^{\varphi_1-h,\varphi_2}} (1+\Phi_t^{\varphi_1}+\Phi_t^{\varphi_2})\,dt + \hat{M}\big(\Phi^{\varphi_1}_{\tau_D^{\varphi_1-h,\varphi_2}},\Phi^{\varphi_2}_{\tau_D^{\varphi_1-h,\varphi_2}}\big)\Big]\\
&\quad= \mathsf{E}_0\Big[\int_0^{\tau_D^{\varphi_1-h,\varphi_2}} (-h\,\Phi_t^1)\,dt - ch\,\Phi^1_{\tau_D^{\varphi_1-h,\varphi_2}}\, I\big(\tau_D^{\varphi_1-h,\varphi_2}\neq \tau_{D_2}^{\varphi_1-h,\varphi_2}\big)\Big]
\end{aligned}\tag{9.6}$$

for all $h\in(0,h_0)$ with some $h_0>0$ sufficiently small, where in the final equality we use (6.4) combined with the two implications in (9.7) below which we motivate and derive first.

Recalling (6.4) and definitions of Δ_0, Δ_1, Δ_2 stated afterwards, we claim that

$$\big(\Phi^{\varphi_1-h}_{\tau_D^{\varphi_1-h,\varphi_2}},\Phi^{\varphi_2}_{\tau_D^{\varphi_1-h,\varphi_2}}\big)\in\partial C\cap D_i \Longrightarrow \big(\Phi^{\varphi_1}_{\tau_D^{\varphi_1-h,\varphi_2}},\Phi^{\varphi_2}_{\tau_D^{\varphi_1-h,\varphi_2}}\big)\in\Delta_i \tag{9.7}$$

for $h\in(0,h_0)$ with some $h_0>0$ sufficiently small and $i=0,1$.

To show (9.7) for $i=0$ recall that c_0 and c_1 are contained in C so that the continuous curve b_0 stays away from the straight line c_0 in particular. Setting $\tau_h := \tau_{D_0}^{\varphi_1-h,\varphi_2}$ to simplify the notation throughout this shows that there exists $\delta>0$ sufficiently small such that the right-hand side in (9.7) with $i=0$ implies that $(\varphi_1-h)\,\Phi^1_{\tau_h}\leq 1-\delta$ for $h\in(0,\varphi_1)$. This implies that $\Phi^1_{\tau_h}\leq(1-\delta)/(\varphi_1-h_0)$ for all $h\in(0,h_0)$ with $h_0\in(0,\varphi_1)$ given and fixed. It follows that $\Phi^{\varphi_1}_{\tau_h}=\varphi_1\Phi^1_{\tau_h}=(\varphi_1-h)\Phi^1_{\tau_h}+h\Phi^1_{\tau_h}\leq 1-\delta+h_0(1-\delta)/(\varphi_1-h_0)=[(1-\delta)(1+h_0)/(\varphi_1-h_0)]\leq 1$ if we choose $h_0>0$ small enough. This shows that (9.7) holds for $i=0$ as claimed.

To show (9.7) for $i=1$ set $\tau_h := \tau_{D_1}^{\varphi_1-h,\varphi_2}$ to simplify the notation throughout and note that (8.29) shows that there exists $\delta>0$ sufficiently small such that the right-hand side in (9.7) with $i=1$ implies that $(\varphi_1-h)\,\Phi^1_{\tau_h}\leq(1-\delta)\,\varphi_2\,\Phi^2_{\tau_h}$ for $h\in(0,\varphi_1)$. This implies that $\Phi^1_{\tau_h}\leq[(1-\delta)/(\varphi_1-h_0)]\varphi_2\,\Phi^2_{\tau_h}$ for all $h\in(0,h_0)$ with $h_0\in(0,\varphi_1)$ given and fixed. It follows that $\Phi^{\varphi_1}_{\tau_h}=\varphi_1\,\Phi^1_{\tau_h}=(\varphi_1-h)\,\Phi^1_{\tau_h}+h\,\Phi^1_{\tau_h}\leq(1-\delta)\,\varphi_2\,\Phi^2_{\tau_h}+h\,[(1-\delta)/(\varphi_1-h_0)]\,\varphi_2\,\Phi^2_{\tau_h}\leq[(1-\delta)(1+h_0)/(\varphi_1-h_0)]\Phi^{\varphi_2}_{\tau_h}\leq\Phi^{\varphi_2}_{\tau_h}$ if we choose $h_0>0$ small enough. This shows that (9.7) holds for $i=1$ as claimed.

Making now use of (6.4) and (9.7) in the middle term of (9.6) above, upon noting that $\tau_D^{\varphi_1-h,\varphi_2}$ always equals one among $\tau_{D_0}^{\varphi_1-h,\varphi_2}$, $\tau_{D_1}^{\varphi_1-h,\varphi_2}$, $\tau_{D_2}^{\varphi_1-h,\varphi_2}$ respectively, we see that the final equality in (9.6) holds as claimed. Dividing both sides of (9.6) by $-h$ we obtain

$$\begin{aligned}(9.8)\qquad &\frac{\hat{V}(\varphi_1-h,\varphi_2)-\hat{V}(\varphi_1,\varphi_2)}{-h}\\ &\le \mathsf{E}_0\Big[\int_0^{\tau_D^{\varphi_1-h,\varphi_2}} \Phi_t^1\,dt + c\,\Phi^1_{\tau_D^{\varphi_1-h,\varphi_2}}\, I\big(\tau_D^{\varphi_1-h,\varphi_2}\neq \tau_{D_2}^{\varphi_1-h,\varphi_2}\big)\Big]\end{aligned}$$

for all $h \in (0, h_0)$. Letting $h \downarrow 0$ and using that the right-hand side in (9.8) tends to zero by (9.1) and the continuity of $\hat{V}$ we see that (9.5) holds as claimed. Combining (9.4) and (9.5) with the fact that $\hat{M}_{\varphi_1}(\varphi_1,\varphi_2)=0$ we see that (9.2) holds as claimed.

2. We show that (9.3) holds. For this, we first note that

$$\begin{aligned}(9.9)\qquad &\liminf_{h\downarrow 0}\frac{\hat{V}(\varphi_1,\varphi_2-h)-\hat{V}(\varphi_1,\varphi_2)}{-h}\\ &\ge \liminf_{h\downarrow 0}\frac{\hat{M}(\varphi_1,\varphi_2-h)-\hat{M}(\varphi_1,\varphi_2)}{-h}=c\end{aligned}$$

$$\begin{aligned}(9.10)\qquad &\limsup_{h\downarrow 0}\frac{\hat{V}(\varphi_1,\varphi_2+h)-\hat{V}(\varphi_1,\varphi_2)}{h}\\ &\le \limsup_{h\downarrow 0}\frac{\hat{M}(\varphi_1,\varphi_2+h)-\hat{M}(\varphi_1,\varphi_2)}{h}=c\end{aligned}$$

depending on whether (φ_1,φ_2-h) or (φ_1,φ_2+h) belongs to C for $h>0$ respectively. In (9.8) and (9.9) we use that $\hat{V}(\varphi_1,\varphi_2\mp h)\le \hat{M}(\varphi_1,\varphi_2\mp h)$ and $\hat{V}(\varphi_1,\varphi_2)=\hat{M}(\varphi_1,\varphi_2)$ with $\varphi_2'\mapsto \hat{M}(\varphi_1,\varphi_2')=c(1+\varphi_2')$ being linear for $1\le\varphi_2'<\varphi_1$. We next show that

$$(9.11)\qquad \limsup_{h\downarrow 0}\frac{\hat{V}(\varphi_1,\varphi_2-h)-\hat{V}(\varphi_1,\varphi_2)}{-h}\le c$$

$$(9.12)\qquad \liminf_{h\downarrow 0}\frac{\hat{V}(\varphi_1,\varphi_2+h)-\hat{V}(\varphi_1,\varphi_2)}{h}\ge c$$

depending on whether (φ_1,φ_2-h) or (φ_1,φ_2+h) belongs to C for $h>0$ respectively. For this, let $\tau_D^{\varphi_1,\varphi_2\mp h}$ denote the first entry time of $(\Phi^{\varphi_1},\Phi^{\varphi_2\mp h})$ into D for $h>0$ given and fixed. Since $\tau_D^{\varphi_1,\varphi_2\mp h}$ is optimal for $\hat{V}(\varphi_1,\varphi_2\mp h)$ we find by (5.8) that

$$\begin{aligned}(9.13)\qquad &\hat{V}(\varphi_1,\varphi_2\mp h)-\hat{V}(\varphi_1,\varphi_2)\\ &\ge \mathsf{E}_0\Big[\int_0^{\tau_D^{\varphi_1,\varphi_2\mp h}}\big(1+\Phi_t^{\varphi_1}+\Phi_t^{\varphi_2\mp h}\big)\,dt + \hat{M}\big(\Phi^{\varphi_1}_{\tau_D^{\varphi_1,\varphi_2\mp h}},\Phi^{\varphi_2\mp h}_{\tau_D^{\varphi_1,\varphi_2\mp h}}\big)\Big]\\ &\quad - \mathsf{E}_0\Big[\int_0^{\tau_D^{\varphi_1,\varphi_2\mp h}}\big(1+\Phi_t^{\varphi_1}+\Phi_t^{\varphi_2}\big)\,dt + \hat{M}\big(\Phi^{\varphi_1}_{\tau_D^{\varphi_1,\varphi_2\mp h}},\Phi^{\varphi_2}_{\tau_D^{\varphi_1,\varphi_2\mp h}}\big)\Big]\\ &= \mathsf{E}_0\Big[\int_0^{\tau_D^{\varphi_1,\varphi_2\mp h}}\big(\mp h\,\Phi_t^2\big)\,dt \mp ch\,\Phi^2_{\tau_D^{\varphi_1,\varphi_2\mp h}}\, I\big(\tau_D^{\varphi_1,\varphi_2\mp h}\neq\tau_{D_1}^{\varphi_1,\varphi_2\mp h}\big)\Big]\end{aligned}$$

for all $h \in (0, h_0)$ with some $h_0 > 0$ sufficiently small, where in the final equality we use (6.4) and (9.7) similarly as in (9.6) above. Dividing both sides of (9.13) by $\mp h$ we obtain

$$\begin{aligned}\text{(9.14)}\quad & \frac{\hat{V}(\varphi_1, \varphi_2 - h) - \hat{V}(\varphi_1, \varphi_2)}{-h} \\ & \le \mathsf{E}_0\Big[\int_0^{\tau_D^{\varphi_1,\varphi_2-h}} \Phi_t^2\,dt + c\,\Phi^2_{\tau_D^{\varphi_1,\varphi_2-h}}\, I\big(\tau_D^{\varphi_1,\varphi_2-h} \neq \tau_{D_1}^{\varphi_1,\varphi_2-h}\big)\Big]\end{aligned}$$

$$\begin{aligned}\text{(9.15)}\quad & \frac{\hat{V}(\varphi_1, \varphi_2 + h) - \hat{V}(\varphi_1, \varphi_2)}{h} \\ & \ge \mathsf{E}_0\Big[\int_0^{\tau_D^{\varphi_1,\varphi_2+h}} \Phi_t^2\,dt + c\,\Phi^2_{\tau_D^{\varphi_1,\varphi_2+h}}\, I\big(\tau_D^{\varphi_1,\varphi_2+h} \neq \tau_{D_1}^{\varphi_1,\varphi_2+h}\big)\Big]\end{aligned}$$

for all $h \in (0, h_0)$. Letting $h \downarrow 0$ and using that the right-hand side in (9.14) and (9.15) tends to c by (9.1) and the continuity of $\hat{V}$ we see that (9.11) and (9.12) hold as claimed. Combining (9.9)+(9.10) and (9.11)+(9.12) respectively with the fact that $\hat{M}_{\varphi_2}(\varphi_1, \varphi_2) = c$ we see that (9.3) holds as claimed. This completes the proof. □

COROLLARY 15 (C^1 regularity). *For the value function $\hat{V}$ from (4.18) we have*

$$\text{(9.16)}\qquad (\varphi_1, \varphi_2) \mapsto \hat{V}_{\varphi_1}(\varphi_1, \varphi_2)\ \textit{ is continuous on }\ (0, \infty)^2$$

$$\text{(9.17)}\qquad (\varphi_1, \varphi_2) \mapsto \hat{V}_{\varphi_2}(\varphi_1, \varphi_2)\ \textit{ is continuous on }\ (0, \infty)^2.$$

PROOF. We have established in Proposition 14 that $\hat{V}$ is differentiable on $(0, \infty)^2$. By (8.4) we know that $\hat{V}$ is concave on $[0, \infty)^2$. The claims (9.16) and (9.17) then follow from the general fact that concave differentiable functions are continuously differentiable on open sets (see, e.g., [2], Theorem 2.2.2). This completes the proof. □

10. Nonmonotonicity of the optimal stopping boundaries. In this section we show that the optimal stopping boundaries b_0, b_1, b_2 are nonmonotone as functions of their arguments and prove the existence of a "belly" which determines their curvature/shape. In the first part of the proof we introduce the local time of Φ on a fictitious curve which enables us to decompose the two-dimensional optimal stopping problem into two one-dimensional optimal stopping problems which can be solved explicitly. In the second part of the proof we follow the general hint from [16], Remark 13, on establishing the absence of jumps of the optimal stopping boundaries and make use of Hopf's boundary point lemma to derive a contradiction with the directional smooth fit. In view of the symmetry relations (8.20)+(8.21) it is sufficient to focus on the optimal stopping boundary b_2 and these facts then extend to the optimal stopping boundaries b_0 and b_1 as discussed in Section 8 above.

1. To derive that the optimal stopping set D_2 has a "belly" as displayed on Figure 1, we first show that not only the point $(\beta, 0)$ belongs to D_2 as derived in Section 7 above but also a nontrivial vertical segment above $(\beta, 0)$ is contained in D_2.

PROPOSITION 16. *For the stopping set D_2 from (8.10) we have*

$$\text{(10.1)}\qquad \{\beta\} \times [0, \varphi_2] \subseteq D_2$$

for some $\varphi_2 > 0$ small enough.

PROOF. The idea is to introduce the local time of Φ on the line

$$c_0' = \{ (\varphi_1, \varphi_2) \in [0, \infty)^2 \mid \varphi_1 = 1 \ \& \ \varphi_2 \in [1, \infty) \} \tag{10.2}$$

and decompose the two-dimensional optimal stopping problem (4.18) into two one-dimensional optimal stopping problems that can be solved explicitly.

For this, set $\varphi_1^* := \beta$ throughout and consider the Lagrange reformulation (6.5) of the optimal stopping problem (4.18) that yields

$$\begin{aligned} &\hat{V}(\varphi_1^*, \varphi_2) - \hat{M}(\varphi_1^*, \varphi_2) \\ &\quad = \inf_\tau \mathsf{E}^0_{\varphi_1^*, \varphi_2}\Big[\int_0^\tau (1+\Phi_t^1+\Phi_t^2)\, dt - \frac{c}{2}\big(\ell_\tau^{c_0}(\Phi)+\ell_\tau^{c_1}(\Phi)+\ell_\tau^{c_2}(\Phi)\big)\Big] \end{aligned} \tag{10.3}$$

for $\varphi_2 \in [0, \infty)$ where the infimum is taken over all stopping times τ of Φ. Since the left-hand side of (10.3) is nonpositive, it is enough to show that the left-hand side of (10.3) is nonnegative for all $\varphi_2 > 0$ sufficiently small. For this, adding and subtracting $\ell_\tau^{c_0'}(\Phi)$ under the expectation sign in (10.3) and noting that

$$\ell^{c_0}(\Phi) + \ell^{c_0'}(\Phi) = \ell^1(\Phi^1) \tag{10.4}$$

we find that

$$\begin{aligned} &\inf_\tau \mathsf{E}^0_{\varphi_1^*, \varphi_2}\Big[\int_0^\tau (1+\Phi_t^1+\Phi_t^2)\, dt - \frac{c}{2}\big(\ell_\tau^{c_0}(\Phi)+\ell_\tau^{c_1}(\Phi)+\ell_\tau^{c_2}(\Phi)\big)\Big] \\ &\quad \geq \inf_\tau \mathsf{E}^0_{\varphi_1^*, \varphi_2}\Big[\int_0^\tau (1+\Phi_t^1)\, dt - \frac{c}{2}\ell_\tau^1(\Phi^1)\Big] \\ &\qquad + \inf_\tau \mathsf{E}^0_{\varphi_1^*, \varphi_2}\Big[\int_0^\tau \Phi_t^2\, dt + \frac{c}{2}\ell_\tau^{c_0'}(\Phi) - \frac{c}{2}\big(\ell_\tau^{c_1}(\Phi)+\ell_\tau^{c_2}(\Phi)\big)\Big] \\ &\quad = \inf_\tau \mathsf{E}^0_{\varphi_1^*, \varphi_2}\Big[\int_0^\tau (1+\Phi_t^1)\, dt + c(1\wedge\Phi_\tau^1)\Big] - c(1\wedge\varphi_1^*) \\ &\qquad + \inf_\tau \mathsf{E}^0_{\varphi_1^*, \varphi_2}\Big[\int_0^\tau \Phi_t^2\, dt + \tilde{M}(\Phi_\tau^1, \Phi_\tau^2)\Big] - \tilde{M}(\varphi_1^*, \varphi_2) \\ &\quad \geq \inf_\tau \mathsf{E}^0_{\varphi_1^*, \varphi_2}\Big[\int_0^\tau \Phi_t^2\, dt + c(1\wedge\Phi_\tau^2)\Big] - c(1\wedge\varphi_2) \end{aligned} \tag{10.5}$$

for $\varphi_2 \in [0, 1]$ where in the equality we use the Itô–Tanaka formula (cf. [19], p. 223) applied to $c(1 \wedge \Phi^1)$, and the change-of-variable formula with local time on surfaces [15], Theorem 2.1, applied to $\tilde{M}(\Phi^1, \Phi^2)$ similarly to (6.9) above with

$$\tilde{M} := c\big[(1 \vee \varphi_1) \wedge \varphi_2\big] \tag{10.6}$$

for $(\varphi_1, \varphi_2) \in [0, \infty)^2$, both combined with the optional sampling theorem upon using that Φ^1 and Φ^2 are martingales under P_0. In the final inequality of (10.5) we use that $\varphi_1^* = \beta$ is an optimal stopping point in the one-dimensional optimal stopping problem for Φ^1 as established in Section 7 above as well as that $\tilde{M}(\varphi_1^*, \varphi_2) = c[(1 \vee \varphi_1^*) \wedge \varphi_2] = c(\varphi_1^* \wedge \varphi_2) = c\varphi_2 = c(1 \wedge \varphi_2)$ for $\varphi_2 \in [0, 1]$ as claimed.

Motivated by the right-hand side in (10.5) above, consider the optimal stopping problem

$$\tilde{V}(\varphi) = \inf_\tau \mathsf{E}^0_\varphi\Big[\int_0^\tau \Phi_t\, dt + c(1\wedge\Phi_\tau)\Big] \tag{10.7}$$

for $\varphi \in [0, \infty)$ with $\mathsf{P}^0_\varphi(\Phi_0 = \varphi) = 1$ where the process Φ and its infinitesimal generator $\mathbb{L}_\Phi$ are given by (7.2)+(7.3) and (7.4) above, and the infimum in (10.7) is taken over all stopping times τ of Φ. The optimal stopping problem (10.7) is similar to the optimal stopping problem

(7.1) and we can use similar arguments to tackle it. Denoting the loss function in (10.7) by $\tilde{M}(\varphi) = c(1 \wedge \varphi)$ for $\varphi \in [0, \infty)$ it follows that the free-boundary problem now reads

$$\mathbb{L}_{\Phi} \tilde{V}(\varphi) = -\varphi \text{ for } \varphi \in (\tilde{\varphi}_0^*, \tilde{\varphi}_1^*) \tag{10.8}$$

$$\tilde{V}(\tilde{\varphi}_i^*) = \tilde{M}(\tilde{\varphi}_i^*) \text{ for } i = 0, 1 \text{ (instantaneous stopping)} \tag{10.9}$$

$$\tilde{V}'(\tilde{\varphi}_i^*) = \tilde{M}'(\tilde{\varphi}_i^*) \text{ for } i = 0, 1 \text{ (smooth fit)} \tag{10.10}$$

where $0 < \tilde{\varphi}_0^* < 1 < \tilde{\varphi}_1^* < \infty$ are the optimal stopping/boundary points to be found and we have $\tilde{V}(\varphi) = \tilde{M}(\varphi)$ for $\varphi \in [0, \tilde{\varphi}_0^*) \cup (\tilde{\varphi}_1^*, \infty)$ as well (in addition to (10.9) above).

The general solution to the ordinary differential equation (10.8) is given by

$$\hat{V}(\varphi) = \tilde{A}\varphi + \tilde{B} + \frac{1}{\mu^2}\varphi(1 - \log\varphi) \tag{10.11}$$

for $\varphi > 0$ where $\tilde{A}$ and $\tilde{B}$ are two undetermined real constants. Boundary conditions (10.9) and (10.10) then read as follows

$$\tilde{A}\tilde{\varphi}_0^* + \tilde{B} + \frac{1}{\mu^2}\tilde{\varphi}_0^*(1 - \log\tilde{\varphi}_0^*) = c\varphi_0^* \tag{10.12}$$

$$\tilde{A}\tilde{\varphi}_1^* + \tilde{B} + \frac{1}{\mu^2}\tilde{\varphi}_1^*(1 - \log\tilde{\varphi}_1^*) = c \tag{10.13}$$

$$\tilde{A} - \frac{1}{\mu^2}\log\tilde{\varphi}_0^* = c \tag{10.14}$$

$$\tilde{A} - \frac{1}{\mu^2}\log\tilde{\varphi}_1^* = 0. \tag{10.15}$$

It is a matter of routine to verify that the unique solution to (10.12)–(10.15) is given by

$$\tilde{\varphi}_0^* = \frac{c\mu^2}{e^{c\mu^2} - 1} \quad \& \quad \tilde{\varphi}_1^* = \frac{c\mu^2 e^{c\mu^2}}{e^{c\mu^2} - 1} \tag{10.16}$$

$$\tilde{A}^* = c + \frac{1}{\mu^2}\log\tilde{\varphi}_0^* \quad \& \quad \tilde{B}^* = -\frac{1}{\mu^2}\tilde{\varphi}_0^*. \tag{10.17}$$

Note that $\tilde{\varphi}_0^* \in (0, 1)$ and $\tilde{\varphi}_1^* \in (1, \infty)$ as needed. Inserting $\tilde{A}^*$ and $\tilde{B}^*$ from (10.17) to (10.11) we obtain a candidate value function $\tilde{V}^*$ for the optimal stopping problem (10.7). Applying the Itô–Tanaka formula (cf. [19], p. 223) to $\tilde{V}^*$ composed with Φ, which reduces to Itô's formula due to smooth fit (10.10), and making use of the optional sampling theorem, it is easily verified that $\tilde{V}^*$ coincides with the value function $\tilde{V}$ from (10.7) and the optimal stopping time (at which the infimum in (10.7) is attained) is given by

$$\tau_* = \inf\{\, t \geq 0 \mid \Phi_t \notin (\tilde{\varphi}_0^*, \tilde{\varphi}_1^*) \,\} \tag{10.18}$$

where $\tilde{\varphi}_0^*$ and $\tilde{\varphi}_1^*$ are given by (10.16) above. This in particular shows that the interval $[0, \tilde{\varphi}_0^*]$ is contained in the stopping set of the optimal stopping problem (10.7). Translating this conclusion to the right-hand side of (10.5) above we see that its value equals zero whenever φ_2 belongs to $[0, \tilde{\varphi}_0^*]$. It follows therefore from (10.3) and (10.5) that φ_2 in (10.1) can be taken to be equal to $\tilde{\varphi}_0^* = c\mu^2/(e^{c\mu^2} - 1)$ and the proof is complete. □

2. We now show that the "belly" of the optimal stopping set D_2 is not flat but curved (see Figure 1). For this, suppose that this is not the case. Then $[a, b) \times [c, d] \subseteq C$ with $\{b\} \times [c, d] \subseteq D_2$ and $\{b\} \times (d, b] \subseteq C$ for some $a < b$ with $[a, b] \subseteq [1, \beta]$ and some $c < d$ with $[c, d] \subseteq [0, a]$. The initial claim is then a direct consequence of the following fact.

PROPOSITION 17. *If the "belly" of the optimal stopping set D_2 would be flat as described above, then the horizontal smooth fit condition* (9.2) *would fail on* $\{b\} \times [c,d] \subseteq \partial C \cap D_2$.

PROOF. Suppose that the "belly" of the optimal stopping set D_2 is flat as described above. Set $R^0 = (a,b) \times (c,d)$ and $R^1 = (a,b] \times (c,d)$ with $R = [a,b] \times [c,d]$. Recalling the Lagrange reformulation (6.5) of the optimal stopping problem (4.18), and arguing as in the proof of Theorem 12 in [16], we find that the value function $\hat{V}$ from (4.18) solves the equation

$$\mathbb{L}_\Phi \hat{V} = -\hat{H} \tag{10.19}$$

on R^0 and belongs to $C^4(R^1)$ where $\mathbb{L}_\Phi$ is given by (5.9) above and we set

$$\hat{H}(\varphi_1, \varphi_2) = 1 + \varphi_1 + \varphi_2 \tag{10.20}$$

for $(\varphi_1, \varphi_2) \in [0,\infty)^2$. Differentiating both sides of (10.19) with respect to φ_2 and defining the differential operator $\tilde{\mathbb{L}}$ by setting

$$\tilde{\mathbb{L}} = \varphi_1^2 \partial^2_{\varphi_1\varphi_1} + \varphi_1\varphi_2 \partial^2_{\varphi_1\varphi_2} + \varphi_2^2 \partial^2_{\varphi_2\varphi_2} + 2\varphi_1 \partial_{\varphi_1} + 4\varphi_2 \partial_{\varphi_2} + 2 \tag{10.21}$$

we find that $\hat{V}_{\varphi_2\varphi_2}$ solves the equation

$$\tilde{\mathbb{L}} \hat{V}_{\varphi_2\varphi_2} = 0 \tag{10.22}$$

on R^0. We will now complete the proof in two steps as follows.

1. We claim that the strict inequality holds

$$\hat{V}_{\varphi_2\varphi_2}(\varphi_1, \varphi_2) < 0 \tag{10.23}$$

for all $(\varphi_1, \varphi_2) \in R^0$. For this, suppose that (10.23) fails for some $(\varphi_1, \varphi_2) \in R^0$. Recalling that $\{b\} \times (d,b] \subseteq C$, consider the ball $b(z,r)$ with centre at $z := (b,d)$ and radius $r > 0$ small enough so that $b(z,r) \subseteq \Delta_2$, where Δ_2 is defined following (6.4) above. Enlarge R^0 by setting $\tilde{R}^0 := R^0 \cup (b(z,r) \cap C)$ and note that the same arguments as above show that the equations (10.19) and (10.21) hold on $\tilde{R}^0$ too. Since the coefficients of $\tilde{\mathbb{L}}$ are continuous and the set $\tilde{R}^0$ is bounded we can conclude that $\tilde{\mathbb{L}}$ is uniformly elliptic on $\tilde{R}^0$ (cf. [9], p. 31). The hypothesis that (10.23) fails for some $(\varphi_1, \varphi_2) \in R^0$, combined with the fact that $\hat{V}_{\varphi_2\varphi_2} \le 0$ on $\tilde{R}^0$ by (8.4) above, implies that $\hat{V}_{\varphi_2\varphi_2}(\varphi_1, \varphi_2) = 0$ so that $\hat{V}_{\varphi_2\varphi_2}$ attains its maximum in the interior of $\tilde{R}^0$ (i.e., not at its boundary alone). Hence by the strong maximum principle for elliptic equations (see Theorem 3.5 in [9], p. 35, and the second sentence following its proof) we can conclude that $\hat{V}_{\varphi_2\varphi_2} = 0$ on the entire $\tilde{R}^0$. This in particular means that $\varphi_2 \mapsto \hat{V}(b, \varphi_2)$ is linear on $[d, d+r]$. Since $\varphi_2 \mapsto \hat{V}(b, \varphi_2) = c(1+\varphi_2)$ is linear on $[c,d]$ as well, and the vertical smooth fit (9.3) holds at $z = (b,d)$, it follows that $\hat{V}(b,\varphi_2) = c(1+\varphi_2)$ for all $\varphi_2 \in [c, d+r]$ so that $\{b\} \times (d, d+r] \subseteq D$ which is a contradiction. This establishes that (10.23) is satisfied as claimed.

2. Fix any point e in (c,d) and note that $\hat{V}_{\varphi_2\varphi_2}(b,e) = 0$ since $\hat{V} \in C^4(R^1) \subseteq C^2(R^1)$ and $\hat{V}(b, \varphi_2) = c\,(1+\varphi_2)$ for $\varphi_2 \in [c,d]$. Hence we see that (10.23) reads as $\hat{V}_{\varphi_2\varphi_2}(\varphi_1, \varphi_2) < \hat{V}_{\varphi_2\varphi_2}(b,e)$ for all $(\varphi_1, \varphi_2) \in R^0$. Moreover, we know that $\tilde{\mathbb{L}}$ is uniformly elliptic and $\tilde{\mathbb{L}} \hat{V}_{\varphi_2\varphi_2} \ge 0$ holds on R^0 by (10.22) above. Finally, it is evident that R^0 satisfies an interior sphere condition at $z := (b,e) \in \partial R^0$ (i.e., there exist $w \in R^0$ and $r > 0$ such that $b(w,r) \subseteq R^0$ and $z \in \partial(b(w,r))$). These facts show that Hopf's boundary point lemma for elliptic equations (see [9], Lemma 3.4, p. 34) is applicable and thus the outer normal derivative of $\hat{V}_{\varphi_2\varphi_2}$ at $z = (b,e)$ must be strictly positive. In other words, we have

$$(\hat{V}_{\varphi_2\varphi_2})_{\varphi_1}(b,e) > 0. \tag{10.24}$$

This conclusion shows that the horizontal smooth fit condition (9.2) cannot hold on $\{b\}\times[c,d]$ as claimed, since otherwise we would have $(\hat{V}_{\varphi_2\varphi_2})_{\varphi_1}(b,e) = (\hat{V}_{\varphi_1})_{\varphi_2\varphi_2}(b,e) = 0$ due to $\hat{V} \in C^4(R^1) \subseteq C^3(R^1)$, and the proof is complete. □

11. Free-boundary problem. In this section we derive a free-boundary problem that stands in one-to-one correspondence with the optimal stopping problem (4.18) and establish the fact that the value function $\hat{V}$ and the optimal stopping boundary ∂C solve the free-boundary problem uniquely. These considerations will be continued in the next section.

1. Consider the optimal stopping problem (4.18) where the strong Markov/Feller process $\Phi = (\Phi^1, \Phi^2)$ solves the system of stochastic differential equations (5.6)+(5.7). Recalling that the infinitesimal generator $\mathbb{L}_\Phi$ of Φ is given by (5.9) above, and relying on other properties of $\hat{V}$ and ∂C derived in Section 8 above, we are naturally led to formulate the following free-boundary problem for finding $\hat{V}$ and ∂C:

$$\mathbb{L}_\Phi \hat{V} = -\hat{H} \text{ on } C \tag{11.1}$$

$$\hat{V} = \hat{M} \text{ on } D \quad \text{(instantaneous stopping)} \tag{11.2}$$

$$\hat{V}_{\varphi_i} = \hat{M}_{\varphi_i} \text{ on } \partial C \text{ for } i = 1, 2 \text{ (smooth fit)} \tag{11.3}$$

where $\hat{H}$ is given by (10.20) above and $\hat{M}$ is given by (4.13) above. The continuation set C and the stopping set D are formally defined by (8.1) and (8.2) respectively. We know from the results of Section 8 that the optimal stopping boundary ∂C can be fully described by means of the functions b_0 and b_1 defined in Section 8 above via the equivalence $(\varphi_1, \varphi_2) \in \partial C$ if and only if either $(\varphi_1, \varphi_2) \in \partial C \cap D_0$ and $\varphi_i = b_0(\varphi_j)$ when $\varphi_i \geq \varphi_j$ for $i \neq j \in \{1, 2\}$ or $(\varphi_1, \varphi_2) \in \partial C \cap D_i$ and $\varphi_j = b_1(\varphi_i)$ for $i \neq j \in \{1, 2\}$ where D_0, D_1, D_2 are given by (8.8)–(8.10) above (see Figure 1). Clearly the global condition (11.2) can be replaced by the local condition $\hat{V} = \hat{M}$ on ∂C so that the free-boundary problem (11.1)–(11.3) needs to be considered on the closure of C only (extending $\hat{V}$ to the rest of D as $\hat{M}$).

2. To formulate the existence and uniqueness result for the free-boundary problem (11.1)–(11.3) we let $\mathcal{C}$ denote the class of functions $(F; a_0, a_1)$ such that

$$F \text{ is concave and continuous on } [0,\infty)^2 \text{ and belongs to } C^1\big((0,\infty)^2\big) \cap C^2(C_{a_0,a_1}) \tag{11.4}$$

$$a_0 \text{ is concave and continuous on } [0,\delta] \text{ with } a_0(0) = 1/\beta \;\&\; a_0(\delta) = \delta \text{ and } \varphi < a_0(\varphi) < 1 \text{ for } \varphi \in (0,\delta) \text{ with some } \delta \in (0,1) \tag{11.5}$$

$$a_1 \text{ is convex and continuous on } [0,\infty) \text{ with } a_1(0) = \beta \;\&\; a_1(\infty) = \infty \text{ and } a_1(\varphi) > 1 \vee \varphi \text{ for } \varphi \in [0,\infty) \tag{11.6}$$

where $C_{a_0,a_1} := \{ (\varphi_1, \varphi_2) \in [0,\infty)^2 \mid a_0(\varphi_i) < \varphi_j < a_1(\varphi_i)$ when $\varphi_i \leq \varphi_j$ and $\varphi_i \in [0,\delta]$ or $\varphi_i \leq \varphi_j < a_1(\varphi_i)$ when $\varphi_i \leq \varphi_j$ and $\varphi_i \in (\delta, \infty)$ for $i \neq j \in \{1,2\}$ and some $\delta \in (0,1) \}$ is the open set surrounded by a_0 and a_1 (applied twice).

THEOREM 18. *The free-boundary problem* (11.1)–(11.3) *has a unique solution* $(\hat{V}; b_0, b_1)$ *in the class* $\mathcal{C}$ *where* $\hat{V}$ *is given by* (4.18) *while* b_0 *and* b_1 *are defined in Section* 8 *above.*

PROOF. Combining the results of Proposition 5 and Corollary 15 with the arguments leading to (10.19) above, we see that the value function $\hat{V}$ from (4.18) satisfies (11.4) and solves the boundary value problem (11.1)–(11.3) with ∂C described by b_0 and b_1 from Section 8 as recalled above. Moreover, combining the results of Propositions 8–12 we see that

b_0 and b_1 satisfy (11.5) and (11.6) respectively. This shows that $(\hat{V}; b_0, b_1)$ solves the free-boundary problem (11.1)–(11.3) in the class $\mathcal{C}$ as claimed. To derive uniqueness of the solution we will first see in the next section that any solution $(F; a_0, a_1)$ to the free-boundary problem (11.1)–(11.3) in the class $\mathcal{C}$ admits a closed triple-integral representation of F expressed in terms of a_0 and a_1, which in turn solve a coupled system of nonlinear Fredholm integral equations, and we will see that this system cannot have other solutions satisfying the specified properties. Drawing these facts together we can conclude that there cannot exist more than one solution to the free-boundary problem (11.1)–(11.3) in the class $\mathcal{C}$ as claimed. □

12. Nonlinear integral equations. In this section we show that the optimal stopping boundaries b_0 and b_1 can be characterised as the unique solution to a coupled system of nonlinear Fredholm integral equations (recall that b_2 coincides with b_1 in terms of its functional rule). This also yields a closed triple-integral representation of the value function $\hat{V}$ expressed in terms of the optimal stopping boundaries b_0 and b_1. As a consequence of the existence and uniqueness result for the coupled system of nonlinear Fredholm integral equations we also obtain uniqueness of the solution to the free-boundary problem (11.1)–(11.3) as explained in the proof of Theorem 18 above. Finally, collecting the results derived throughout the paper we conclude our exposition at the end of this section by disclosing the solution to the initial problem.

1. To formulate the theorem below, let p denote the transition probability density function of the (time-homogeneous) Markov process $\Phi = (\Phi^1, \Phi^2)$ under P_0 in the sense that

$$\mathsf{P}^0_{\varphi_1,\varphi_2}(\Phi^1_t \le \psi_1, \Phi^2_t \le \psi_2) = \int_0^{\psi_1}\int_0^{\psi_2} p(t;\varphi_1,\varphi_2;\eta_1,\eta_2)\,d\eta_1 d\eta_2 \tag{12.1}$$

for $t > 0$ with (φ_1, φ_2) and (ψ_1, ψ_2) in $[0,\infty)^2$. A lengthy but straightforward calculation based on (5.8) shows that

$$\begin{aligned} p(t;\varphi_1,\varphi_2;\psi_1,\psi_2) = {} & \frac{1}{2\pi\sqrt{3}\,\mu^2 t\,\psi_1\psi_2}\exp\biggl[-\frac{1}{3}\biggl(\mu^2 t + \log\Bigl(\frac{\psi_1\psi_2}{\varphi_1\varphi_2}\Bigr) \\ & + \frac{1}{\mu^2 t}\Bigl[\log^2\Bigl(\frac{\psi_1\varphi_2}{\psi_2\varphi_1}\Bigr) + \log\Bigl(\frac{\psi_1}{\varphi_1}\Bigr)\log\Bigl(\frac{\psi_2}{\varphi_2}\Bigr)\Bigr]\biggr)\biggr] \end{aligned} \tag{12.2}$$

for $t > 0$ with (φ_1, φ_2) and (ψ_1, ψ_2) in $[0,\infty)^2$. Recalling from Section 8 above that the functions b_0 and b_1 are sufficient to describe the entire boundary of the continuation set, we can then evaluate the expression of interest in the theorem below as follows

$$\begin{aligned} & \mathsf{E}^0_{\varphi_1,\varphi_2}\biggl[\int_0^\infty \hat{H}(\Phi^1_s,\Phi^2_s)\,I\bigl((\Phi^1_s,\Phi^2_s)\in C\bigr)\,ds\biggr] \\ & = \int_0^\infty\biggl(\int_0^\infty\int_{b_0(\psi_1)\vee\psi_1}^{b_1(\psi_1)} \hat{H}(\psi_1,\psi_2)\,p(s;\varphi_1,\varphi_2;\psi_1,\psi_2)\,d\psi_2 d\psi_1 \\ & \quad + \int_0^\infty\int_{b_0(\psi_2)\vee\psi_2}^{b_1(\psi_2)} \hat{H}(\psi_1,\psi_2)\,p(s;\varphi_1,\varphi_2;\psi_1,\psi_2)\,d\psi_1 d\psi_2\biggr)ds \\ & = 2\int_0^\infty\int_0^\infty\int_{b_0(\psi_1)\vee\psi_1}^{b_1(\psi_1)} \hat{H}(\psi_1,\psi_2)\,p(s;\varphi_1,\varphi_2;\psi_1,\psi_2)\,d\psi_2 d\psi_1 ds \end{aligned} \tag{12.3}$$

for $(\varphi_1, \varphi_2) \in [0,\infty)^2$ where the final equality follows by symmetry relative to the main diagonal in $[0,\infty)^2$ and we recall that $\hat{H}$ is defined in (10.20) above.

THEOREM 19 (Existence and uniqueness). *The optimal stopping boundaries b_0 and b_1 in the problem (4.18) can be characterised as the unique solution to the coupled system of nonlinear Fredholm integral equations*

$$\varphi_1 + b_0(\varphi_1) = \frac{2}{c}\int_0^\infty\int_0^\infty\int_{b_0(\psi_1)\vee\psi_1}^{b_1(\psi_1)} \hat{H}(\psi_1,\psi_2) \times p(s;\varphi_1,\varphi_2;\psi_1,\psi_2)\,d\psi_2\,d\psi_1\,ds \tag{12.4}$$

$$1 + \varphi_1 = \frac{2}{c}\int_0^\infty\int_0^\infty\int_{b_0(\psi_1)\vee\psi_1}^{b_1(\psi_1)} \hat{H}(\psi_1,\psi_2) \times p(s;\varphi_1,\varphi_2;\psi_1,\psi_2)\,d\psi_2\,d\psi_1\,ds \tag{12.5}$$

in the class of functions a_0 and a_1 satisfying (11.5) *and* (11.6) *respectively*, *where φ_1 in* (12.4) *belongs to $[0,\gamma]$ with $b_0(\gamma)=\gamma$ for some $\gamma\in(0,1)$ and φ_1 in* (12.5) *belongs to $[0,\infty)$. The value function $\hat{V}$ in the problem* (4.18) *admits the following representation*

$$\hat{V}(\varphi_1,\varphi_2) = 2\int_0^\infty\int_0^\infty\int_{b_0(\psi_1)\vee\psi_1}^{b_1(\psi_1)} \hat{H}(\psi_1,\psi_2) \times p(s;\varphi_1,\varphi_2;\psi_1,\psi_2)\,d\psi_2\,d\psi_1\,ds \tag{12.6}$$

for $(\varphi_1,\varphi_2)\in[0,\infty)^2$. The optimal stopping time in the problem (4.18) *is given by*

$$\tau_{b_0,b_1} = \inf\{\, t\ge 0 \mid \Phi_t^i \ge \Phi_t^j \ \text{with}\ \Phi_t^i \le b_0(\Phi_t^j)\ \text{or}\ \Phi_t^i \ge b_1(\Phi_t^j) \ \text{for}\ i\ne j\ \text{in}\ \{1,2\}\} \tag{12.7}$$

under $\mathsf{P}^0_{\varphi_1,\varphi_2}$ with $(\varphi_1,\varphi_2)\in[0,\infty)^2$ given and fixed (*see Figure* 1).

PROOF. (I) *Existence*. We first show that the value function $\hat{V}$ in the problem (4.18) admits the representation (12.6) and that the optimal stopping boundaries b_0 and b_1 solve the system (12.4)+(12.5). Recalling that b_0 and b_1 satisfy the properties (11.5) and (11.6) this will establish the existence of a solution to the system (12.4)+(12.5).

For this, recall that by (8.4) in Proposition 5 we know that $\hat{V}$ is concave and from Corollary 15 we know that $\hat{V}$ is globally C^1 on $(0,\infty)^2$. These properties however are generally insufficient to apply a known extension of Itô's formula to $\hat{V}(\Phi^1,\Phi^2)$ due to not knowing the size of the second partial derivatives $\hat{V}_{\varphi_1,\varphi_1}$, $\hat{V}_{\varphi_1,\varphi_2}$, $\hat{V}_{\varphi_2,\varphi_2}$ close to the optimal stopping boundaries. Note that we know that the optimal stopping boundaries are convex/concave, however, this is generally insufficient to derive a local boundedness of the second partial derivatives close to the optimal stopping boundaries (without having their smoothness) using the generally theory of elliptic PDEs (see [9]). A semimartingale decomposition of $\hat{V}(\Phi^1,\Phi^2)$ obtained by Itô's formula is useful because it leads to Dynkin's formula (upon localising, taking expectations, and passing to the limit) which in turn yields the representation (12.6). We will show in the proof below that Dynkin's formula can be derived without appealing to Itô's formula and/or without formally verifying that the second partial derivatives are locally bounded close to the optimal stopping boundaries. This will be accomplished in several steps below by exploiting the underlying convexity/concavity in the problem (4.18) combined with the fact that the expectation of the running local time of (Φ^1,Φ^2) on the (approximating) optimal stopping boundaries remains uniformly bounded as the time tends to infinity (recall that (Φ^1,Φ^2) itself converges to zero so that this is rather intuitive).

1. We begin by localising the process $\Phi=(\Phi^1,\Phi^2)$. For this, let $N\ge 1$ be given and fixed (large) and consider the first exit time of Φ from the square $[1/N,N]^2$ given by

$$\rho_N = \inf\{\, t\ge 0 \mid \Phi_t \notin [1/N,N]^2 \,\}. \tag{12.8}$$

Let $\Phi^{\rho_N} = (\Phi_{t\wedge\rho_N})_{t\geq 0}$ denote the process Φ stopped at ρ_N. Clearly the process Φ^{ρ_N} stays in the square $[1/N, N]^2$ all the time while both $\hat{V}$ and $\hat{V}_{\varphi_i}$ are continuous and thus bounded on $[1/N, N]^2$ for $i = 1, 2$. As we have not established that $\hat{V}_{\varphi_1,\varphi_1}$, $\hat{V}_{\varphi_1,\varphi_2}$, $\hat{V}_{\varphi_2,\varphi_2}$ are locally bounded close to the optimal stopping boundaries, we proceed by modifying the value function $\hat{V}$ within the continuation set C close to its boundary.

2. For $n \geq 1$ given and fixed (large) define the sets $C_n := \{(\varphi_1, \varphi_2) \in [0,\infty)^2 \mid \hat{V}(\varphi_1, \varphi_2) < \hat{M}(\varphi_1, \varphi_2) - 1/n\}$ and $D_n := \{(\varphi_1, \varphi_2) \in [0,\infty)^2 \mid \hat{V}(\varphi_1, \varphi_2) \geq \hat{M}(\varphi_1, \varphi_2) - 1/n\}$. Set $C_i^n := C_n \cap \Delta_i$ and $D_i^n := D_n \cap \Delta_i$ where Δ_i are defined following (6.4) above for $i = 0, 1, 2$. Clearly $D_i^n \downarrow D_i$ as $n \to \infty$ for $i = 0, 1, 2$. Using the same arguments as in the proof of Proposition 7 above we find that each set D_i^n is convex for $i = 0, 1, 2$. Hence we can conclude that the boundary b_i^n of D_i^n restricted to the square $[1/N, N]^2$ converges uniformly to the boundary b_i restricted to the square $[1/N, N]^2$ as $n \to \infty$ for $i = 0, 1, 2$. Thus, as in the case of the sets D_0, D_1, D_2 and their boundaries b_0, b_1, b_2, the boundary of the set D_0^n restricted to the square $[1/N, N]^2$ is described by a concave/continuous function $b_0^n : [1/n, \gamma_n] \to [0, 1]$ and the boundaries of the sets D_1^n and D_2^n are described by a convex/continuous function $b_1^n : [1/N, N] \to [1, N]$ for all $n \geq n_0$ with $n_0 \geq 1$ sufficiently large.

3. We approximate the value function $\hat{V}$ by functions $\hat{V}^n$ defined as follows

$$\begin{aligned} \hat{V}^n(\varphi_1, \varphi_2) &= \hat{V}(\varphi_1, \varphi_2) \text{ if } (\varphi_1, \varphi_2) \in C_n \\ &= \hat{M}(\varphi_1, \varphi_2) - \frac{1}{n} \text{ if } (\varphi_1, \varphi_2) \in D_n \end{aligned} \tag{12.9}$$

for $(\varphi_1, \varphi_2) \in [0,\infty)^2$ with $n \geq n_0$ given and fixed. Clearly $\hat{V}^n$ is a continuous function on $[0,\infty)^2$ and moreover $\hat{V}^n$ restricted to C_n and D_n belongs to $C^2(\bar{C}_n)$ and $C^2(\bar{D}_n)$ respectively. Thus the change-of-variable formula with local time on surfaces [15], Theorem 2.1, is applicable to $\hat{V}^n$ composed with $\Phi^{\rho_N} = (\Phi^{1,\rho_N}, \Phi^{2,\rho_N})$ and this gives

$$\begin{aligned} &\hat{V}^n(\Phi_t^{\rho_N}) \\ &= \hat{V}^n(\varphi) + \int_0^t \hat{V}^n_{\varphi_1}(\Phi_s^{\rho_N})\, d\Phi_s^{1,\rho_n} + \int_0^t \hat{V}^n_{\varphi_2}(\Phi_s^{\rho_N})\, d\Phi_s^{2,\rho_n} \\ &\quad + \int_0^t \mathbb{L}_{\Phi^{\rho_N}} \hat{V}^n(\Phi_s^{\rho_N})\, ds \\ &\quad + \frac{1}{2}\int_0^t \big[\hat{V}^n_{\varphi_2}(\Phi_s^{1,\rho_N}, b_0^n(\Phi_s^{1,\rho_N})+) \\ &\quad - \hat{V}^n_{\varphi_2}(\Phi_s^{1,\rho_N}, b_0^n(\Phi_s^{1,\rho_N})-)\big]\, d\ell_s^{b_0^{1,n}}(\Phi^{\rho_N}) \\ &\quad + \frac{1}{2}\int_0^t \big[\hat{V}^n_{\varphi_1}(b_0^n(\Phi_s^{2,\rho_N})+, \Phi_s^{2,\rho_N}) \\ &\quad - \hat{V}^n_{\varphi_1}(b_0^n(\Phi_s^{2,\rho_N})-, \Phi_s^{2,\rho_N})\big]\, d\ell_s^{b_0^{2,n}}(\Phi^{\rho_N}) \\ &\quad + \frac{1}{2}\int_0^t \big[\hat{V}^n_{\varphi_2}(\Phi_s^{1,\rho_N}, b_1^n(\Phi_s^{1,\rho_N})+) \\ &\quad - \hat{V}^n_{\varphi_2}(\Phi_s^{1,\rho_N}, b_1^n(\Phi_s^{1,\rho_N})-)\big]\, d\ell_s^{b_1^n}(\Phi^{\rho_N}) \\ &\quad + \frac{1}{2}\int_0^t \big[\hat{V}^n_{\varphi_1}(b_1^n(\Phi_s^{2,\rho_N})+, \Phi_s^{2,\rho_N}) \end{aligned} \tag{12.10}$$

$$
\begin{aligned}
&- \hat{V}^n_{\varphi_1}\big(b^n_1(\Phi^{2,\rho_N}_s)-, \Phi^{2,\rho_N}_s\big)\big] \, d\ell^{b^n_2}_s(\Phi^{\rho_N}) \\
&= \hat{V}^n(\varphi) + M_t - \int_0^{t\wedge\rho_N} \hat{H}(\Phi_s) I(\Phi_s \in C_n) \, ds + \frac{1}{2} L^{n,N}_t
\end{aligned}
$$

for $\varphi \in [0,\infty)^2$ using (11.1) and (11.2) where M_t is a continuous martingale (the sum of the first two integrals in the first identity of (12.10) above) and $L^{n,N}_t$ is the sum of the final four integrals in the first identity of (12.10) above. Note that the first partial derivatives $\hat{V}^n_{\varphi_1}$ and $\hat{V}^n_{\varphi_2}$ are discontinuous over the boundary curves b^n_0 and b^n_1 because these boundary curves are not optimal. However, since $\hat{V}$ is globally C^1 on $(0,\infty)^2$ by Corollary 15, it follows that

$$
\sup_{1/N \le \varphi_1 \le N} \big|\hat{V}^n_{\varphi_2}(\varphi_1, b^n_i(\varphi_1)) - \hat{M}^n_{\varphi_2}(\varphi_1, b^n_i(\varphi_1))\big| \to 0 \tag{12.11}
$$

$$
\sup_{1/N \le \varphi_1 \le N} \big|\hat{V}^n_{\varphi_1}(\varphi_2, b^n_i(\varphi_2)) - \hat{M}^n_{\varphi_1}(\varphi_2, b^n_i(\varphi_2))\big| \to 0 \tag{12.12}
$$

for $i = 0, 1$ as $n \to \infty$. Note that the suprema in (12.11) and (12.12) for $i = 0, 1$ provide uniform upper bounds on the modulus of the integrands in the four integrals of (12.10) with respect to the local times. To obtain a control over the local times themselves in these four integrals (their integrators) we now show that their expectations remain uniformly bounded as the running time tends to infinity.

4. We first consider the case of $b^{1,n}_0$ and $b^{2,n}_0$ recalling that the two functions coincide by symmetry for $n \ge 1$ given and fixed. We thus focus on $b^{1,n}_0$ in the sequel. Since $b^{1,n}_0$ is concave we see that $\Phi^{2,\rho_N} - b^{1,n}_0(\Phi^{1,\rho_N})$ is a continuous semimartingale so that by the Itô–Tanaka formula we find that

$$
\begin{aligned}
&\big(\Phi^{2,\rho_N}_t - b^{1,n}_0(\Phi^{1,\rho_N}_t)\big)^+ \\
&\quad = \big(\varphi_2 - b^{1,n}_0(\varphi_1)\big)^+ \\
&\qquad + \int_0^t I\big(\Phi^{2,\rho_N}_s > b^{1,n}_0(\Phi^{1,\rho_N}_s)\big) \, d\big(\Phi^{2,\rho_N} - b^{1,n}_0(\Phi^{1,\rho_N})\big)_s \\
&\qquad + \frac{1}{2} \ell^{b^{1,n}_0}_t(\Phi^{\rho_N}) \\
&\quad = \big(\varphi_2 - b^{1,n}_0(\varphi_1)\big)^+ + \int_0^t I\big(\Phi^{2,\rho_N}_s > b^{1,n}_0(\Phi^{1,\rho_N}_s)\big) \, d\Phi^{2,\rho_N}_s \\
&\qquad - \int_0^t I\big(\Phi^{2,\rho_N}_s > b^{1,n}_0(\Phi^{1,\rho_N}_s)\big) \big(b^{1,n}_0\big)'(\Phi^{1,\rho_N}_s) \, d\Phi^{1,\rho_N}_s \\
&\qquad - \int_0^t I\big(\Phi^{2,\rho_N}_s > b^{1,n}_0(\Phi^{1,\rho_N}_s)\big) \int_0^\infty d\ell^{\psi_1}_s(\Phi^{1,\rho_N}) \, d\big(b^{1,n}_0\big)'(\psi_1) \\
&\qquad + \frac{1}{2} \ell^{b^{1,n}_0}_t(\Phi^{\rho_N})
\end{aligned} \tag{12.13}
$$

for $t \ge 0$ where $(b^{1,n}_0)'$ denotes the first derivative of $b^{1,n}_0$ (its existence follows by the implicit function theorem since smooth fit fails at $b^{1,n}_0$ as pointed out above). Since $b^{1,n}_0$ is concave we see that $d(b^{1,n}_0)'$ defines a nonpositive measure on $[1/N, \gamma_n]$ so that the final integral in (12.13) is nonpositive. Using this fact in (12.13) we obtain the following pathwise bound on the size of the local time

$$
\ell^{b^{1,n}_0}_t(\Phi^{\rho_N}) \le 2\big(\Phi^{2,\rho_N}_t - b^{1,n}_0(\Phi^{1,\rho_N}_t)\big)^+ - M_t \tag{12.14}
$$

where M_t is a continuous martingale (the difference between the second and the third integral in (12.13) above) for $t \geq 0$. Taking $\mathsf{E}^0_{\varphi_1,\varphi_2}$ on both sides of (12.14) above and using that $\Phi^{2,\rho_N}_t \leq N$ for all $t \geq 0$ we find that

$$\mathsf{E}^0_{\varphi_1,\varphi_2}\big[\ell_t^{b_0^{i,n}}\big] \leq 2N \tag{12.15}$$

for all $t \geq 0$ and all $n \geq n_0$ with $(\varphi_1,\varphi_2) \in [0,\infty)^2$ and $i = 1,2$ (where the case $i = 2$ follows from the case $i = 1$ by symmetry).

5. We next consider the case of b_1^n and b_2^n recalling that the two functions coincide by symmetry for $n \geq 1$ given and fixed. We thus focus on b_1^n in the sequel. Similarly, since b_1^n is convex we see that $b_1^n(\Phi^{1,\rho_N}) - \Phi^{2,\rho_N}$ is a continuous semimartingale so that by the Itô–Tanaka formula we find that

$$\begin{aligned}
&\big(b_1^n(\Phi_t^{1,\rho_N}) - \Phi_t^{2,\rho_N}\big)^+ \\
&\quad = \big(b_1^n(\varphi_1) - \varphi_2\big)^+ \\
&\qquad + \int_0^t I\big(b_1^n(\Phi_s^{1,\rho_N}) > \Phi_s^{2,\rho_N}\big)\, d\big(b_1^n(\Phi^{1,\rho_N}) - \Phi^{2,\rho_N}\big)_s + \frac{1}{2}\,\ell_t^{b_1^n}(\Phi^{\rho_N}) \\
&\quad = \big(b_1^n(\varphi_1) - \varphi_2\big)^+ + \int_0^t I\big(b_1^n(\Phi_s^{1,\rho_N}) > \Phi_s^{2,\rho_N}\big)\,(b_1^n)'(\Phi_s^{1,\rho_N})\, d\Phi_s^{1,\rho_N} \\
&\qquad + \int_0^t I\big(b_1^n(\Phi_s^{1,\rho_N}) > \Phi_s^{2,\rho_N}\big) \int_0^\infty d\ell_s^{\psi_1}(\Phi^{1,\rho_N})\, d(b_1^n)'(\psi_1) \\
&\qquad - \int_0^t I\big(b_1^n(\Phi_s^{1,\rho_N}) > \Phi_s^{2,\rho_N}\big)\, d\Phi_s^{2,\rho_N} + \frac{1}{2}\,\ell_t^{b_1^n}(\Phi^{\rho_N})
\end{aligned} \tag{12.16}$$

for $t \geq 0$ where $(b_1^n)'$ denotes the first derivative of b_1^n (its existence follows by the implicit function theorem since smooth fit fails at b_1^n as pointed out above). Since b_1^n is convex we see that $d(b_1^n)'$ defines a nonnegative measure on $[1/N, \gamma_n]$ so that the second last integral in (12.16) is nonnegative. Using this fact in (12.16) we obtain the following pathwise bound on the size of the local time

$$\ell_t^{b_1^n}(\Phi^{\rho_N}) \leq 2\big(b_1^n(\Phi_t^{1,\rho_N}) - \Phi_t^{2,\rho_N}\big)^+ - M_t \tag{12.17}$$

where M_t is a continuous martingale (the difference between the second and the final integral in (12.16) above) for $t \geq 0$. Taking $\mathsf{E}^0_{\varphi_1,\varphi_2}$ on both sides of (12.17) above and using that $b_1^n \leq b_1$ with $M_N := \sup_{1/N \leq \varphi_1 \leq N} b_1(\varphi_1) < \infty$ we find that

$$\mathsf{E}^0_{\varphi_1,\varphi_2}\big[\ell_t^{b_1^n}\big] \leq 2M_N \tag{12.18}$$

for all $t \geq 0$ and all $n \geq n_0$ with $(\varphi_1,\varphi_2) \in [0,\infty)^2$.

6. Combining (12.11)+(12.12) with (12.15)+(12.18) we find that $\mathsf{E}^0_{\varphi_1,\varphi_2}[L_t^{n,N}] \to 0$ as $n \to \infty$ for every $(\varphi_1,\varphi_2) \in [0,\infty)^2$ and $N \geq 1$ given and fixed. Taking $\mathsf{E}^0_{\varphi_1,\varphi_2}$ on both sides of (12.10), letting $n \to \infty$ and using the monotone convergence theorem due to $\hat{H} \geq 0$, we obtain the following identity

$$\mathsf{E}^0_{\varphi_1,\varphi_2}\big[\hat{V}(\Phi_t^{\rho_N})\big] = \hat{V}(\varphi_1,\varphi_2) - \mathsf{E}^0_{\varphi_1,\varphi_2}\bigg[\int_0^{t\wedge\rho_N} \hat{H}(\Phi_s)\, I(\Phi_s \in C)\, ds\bigg] \tag{12.19}$$

for $t \geq 0$ and $N \geq 1$ with $(\varphi_1,\varphi_2) \in [0,\infty)^2$. Recalling that $0 \leq \hat{V} \leq \hat{M}$ where $\hat{M}$ is defined in (4.13) above, and noting that $\mathsf{E}^0_{\varphi_1,\varphi_2}(\sup_{0\leq s\leq t} \Phi_s^i) < \infty$ for $i = 1,2$, we see by letting $N \to \infty$ that the dominated convergence theorem is applicable to the left-hand side of (12.19), while the monotone convergence theorem is applicable to the right-hand side of (12.19) since $\hat{H} \geq 0$. Letting $N \to \infty$ in (12.19) we thus obtain

$$\mathsf{E}^0_{\varphi_1,\varphi_2}\big[\hat{V}(\Phi_t)\big] = \hat{V}(\varphi_1,\varphi_2) - \mathsf{E}^0_{\varphi_1,\varphi_2}\bigg[\int_0^t \hat{H}(\Phi_s) I(\Phi_s \in C)\, ds\bigg] \tag{12.20}$$

for $t \geq 0$ and $(\varphi_1, \varphi_2) \in [0,\infty)^2$.

7. Despite the fact that neither $(\Phi^1_t)_{t\geq 0}$ nor $(\Phi^2_t)_{t\geq 0}$ is uniformly integrable (since $\Phi^i_t \to 0$ with $\mathsf{P}_{\varphi_1,\varphi_2}$-probability one as $t \to \infty$ but $\mathsf{E}_{\varphi_1,\varphi_2}(\Phi^i_t) = \varphi_i$ for all $t \geq 0$ with $i = 1, 2$ and $(\varphi_1,\varphi_2) \in (0,\infty)^2$ given and fixed) we claim that

$$\big\{\hat{M}\big(\Phi^1_t, \Phi^2_t\big) \mid t \geq 0\big\} \text{ is uniformly integrable} \tag{12.21}$$

where we recall that $\hat{M}$ is defined in (4.13) above. For this, note that $0 \leq \hat{M}(\Phi^1_t, \Phi^2_t) = c((\Phi^1_t + \Phi^2_t) \wedge (1+\Phi^1_t) \wedge (1+\Phi^2_t)) \leq c((1+\Phi^1_t) \wedge (1+\Phi^2_t)) = c(1+\Phi^1_t \wedge \Phi^2_t)$ for $t \geq 0$. A direct martingale argument based on (5.8) then gives

$$\begin{aligned} \mathsf{E}^0_{\varphi_1,\varphi_2}\big[\Phi^1_t \wedge \Phi^2_t\big] &= \varphi_1\varphi_2\, \mathsf{E}\big[e^{\frac{\mu}{\sqrt{2}}\sqrt{3}W^1_t - \mu^2 t}\big(e^{\frac{\mu}{\sqrt{2}}W^2_t} \wedge e^{-\frac{\mu}{\sqrt{2}}W^2_t}\big)\big] \\ &\leq \varphi_1\varphi_2\, e^{-\frac{1}{4}\mu^2 t}\, \mathsf{E}\big[e^{\frac{\mu}{\sqrt{2}}\sqrt{3}W^1_t - \frac{3}{4}\mu^2 t}\big] \\ &= \varphi_1\varphi_2\, e^{-\frac{1}{4}\mu^2 t} \to 0 \end{aligned} \tag{12.22}$$

as $t \to \infty$ for $(\varphi_1,\varphi_2) \in [0,\infty)^2$. Since $\Phi^1_t \wedge \Phi^2_t \to 0$ with $\mathsf{P}_{\varphi_1,\varphi_2}$-probability one as $t \to \infty$ for $(\varphi_1,\varphi_2) \in (0,\infty)^2$ given and fixed, we see from (12.22) that $\{\Phi^1_t \wedge \Phi^2_t \mid t \geq 0\}$ is uniformly integrable. Hence by the bound preceding to (12.22) we see that (12.21) holds as claimed.

8. Since $0 \leq \hat{V}(\Phi^1_t, \Phi^2_t) \leq \hat{M}(\Phi^1_t, \Phi^2_t)$ for $t \geq 0$ we see from (12.21) that $\{\hat{V}(\Phi^1_t, \Phi^2_t) \mid t \geq 0\}$ is uniformly integrable. Letting $t \to \infty$ in (12.20) and using that $\hat{V}(\Phi^1_t, \Phi^2_t) \to 0$ with $\mathsf{P}_{\varphi_1,\varphi_2}$-probability one we thus find by the extended dominated convergence theorem (applied to the left-hand side) and the monotone convergence theorem (applied to the right-hand side) that the following identity holds:

$$\hat{V}(\varphi_1,\varphi_2) = \mathsf{E}^0_{\varphi_1,\varphi_2}\bigg[\int_0^\infty \hat{H}(\Phi_s) I(\Phi_s \in C)\, ds\bigg] \tag{12.23}$$

$(\varphi_1,\varphi_2) \in [0,\infty)^2$. Combining (12.23) with (12.3) we obtain (12.6) as claimed. Evaluating $\hat{V}$ from (12.23) at the optimal stopping points $(\varphi_1, b_0(\varphi_1))$ and $(\varphi_1, b_1(\varphi_1))$ upon using that $\hat{V}(\varphi_1, b_0(\varphi_1)) = \hat{M}(\varphi_1, b_0(\varphi_1)) = \varphi_1 + b_0(\varphi_1)$ and $\hat{V}(\varphi_1, b_1(\varphi_1)) = \hat{M}(\varphi_1, b_1(\varphi_1)) = 1 + \varphi_1$ for $\varphi_1 \in [0,\gamma]$ and $\varphi_1 \in [0,\infty)$ respectively, we see that the functions b_0 and b_1 solve the integral equations (12.4) and (12.5) as claimed. This completes the proof of the existence of the solution to these equations.

(II) *Uniqueness*. To show that b_0 and b_1 are a unique solution to the system (12.4)+(12.5) one can adopt the four-step procedure from the proof of uniqueness given in [5], Theorem 4.1, extending and further refining the original arguments from [14], Theorem 3.1, in the case of a single boundary. Given that the present setting creates no additional difficulties we will omit further details of this verification and this completes the proof. □

The coupled system of nonlinear Fredholm integral equations (12.4)+(12.5) can be used to find the optimal stopping boundaries b_0 and b_1 numerically (using Picard iteration). Inserting these b_0 and b_1 into (12.6) we also obtain a closed form expression for the value function $\hat{V}$. Collecting the results derived throughout we now disclose the solution to the initial problem.

COROLLARY 20. *The value function of the initial problem* (3.3) *is given by*

$$V(\pi_0, \pi_1, \pi_2) = \pi_0 \hat{V}\left(\frac{\pi_1}{\pi_0}, \frac{\pi_2}{\pi_0}\right) \tag{12.24}$$

for $(\pi_0, \pi_1, \pi_2) \in [0, 1]^3$ *with* $\pi_0+\pi_1+\pi_2 = 1$ *where the function* $\hat{V}$ *is given by* (12.6) *above. The optimal stopping time in the initial problem* (3.3) *is given by*

$$\begin{aligned}\tau_* = \inf\Bigg\{ t \geq 0 \,\Big|\, &\frac{\pi_i}{\pi_0} e^{\mu(X_t^i - X_t^0)} \geq \frac{\pi_j}{\pi_0} e^{\mu(X_t^j - X_t^0)} \\ &\textit{with } \frac{\pi_i}{\pi_0} e^{\mu(X_t^i - X_t^0)} \leq b_0\left(\frac{\pi_j}{\pi_0} e^{\mu(X_t^j - X_t^0)}\right) \\ &\textit{or } \frac{\pi_i}{\pi_0} e^{\mu(X_t^i - X_t^0)} \geq b_1\left(\frac{\pi_j}{\pi_0} e^{\mu(X_t^j - X_t^0)}\right) \textit{ for } i \neq j \textit{ in } \{1, 2\} \Bigg\}\end{aligned} \tag{12.25}$$

where b_0 *and* b_1 *are a unique solution to the coupled system of nonlinear Fredholm integral equations* (12.4)+(12.5). *The optimal decision function* d_{τ_*} *in the initial problem* (3.3) *equals* 0 *if stopping in* (12.25) *happens at* b_0, *equals* 1 *if stopping in* (12.25) *happens at* b_1 *with* $i = 1$, *and equals* 2 *is stopping in* (12.25) *happens at* b_1 *with* $i = 2$.

PROOF. The identity (12.24) was established in (4.11) above. The explicit form (12.25) follows from (12.7) in Theorem 19 combined with (4.2)–(4.4) above. The final claim on the optimal decision function follows from (3.7) combined with the argument used in the second equality of (4.15) above completing the proof. □

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