

EXPANSIONS OF THE REAL FIELD BY DISCRETE SUBGROUPS OF $\mathrm{Gl}_n(\mathbb{C})$

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ABSTRACT. Let Γ be an infinite discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$. Then either $(\mathbb{R}, <, +, \cdot, \Gamma)$ is interdefinable with $(\mathbb{R}, <, +, \cdot, \lambda^{\mathbb{Z}})$ for some real number λ , or $(\mathbb{R}, <, +, \cdot, \Gamma)$ defines the set of integers. When Γ is not virtually abelian, the second case holds.

1. INTRODUCTION

Let $\bar{\mathbb{R}} = (\mathbb{R}, <, +, \cdot, 0, 1)$ be the real field. For $\lambda \in \mathbb{R}_{>0}$, set $\lambda^{\mathbb{Z}} := \{\lambda^m : m \in \mathbb{Z}\}$. Throughout this paper Γ denotes a discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$, and G denotes a subgroup of $\mathrm{Gl}_n(\mathbb{C})$. We identify the set $M_n(\mathbb{C})$ of n -by- n complex matrices with \mathbb{C}^{n^2} and identify \mathbb{C} with \mathbb{R}^2 in the usual way. Our main result is the following classification of expansions of $\bar{\mathbb{R}}$ by a discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$.

Theorem A. *Let Γ be an infinite discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$. Then either*

- $(\bar{\mathbb{R}}, \Gamma)$ defines \mathbb{Z} or
- there is $\lambda \in \mathbb{R}_{>0}$ such that $(\bar{\mathbb{R}}, \Gamma)$ is interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$.

If Γ is not virtually abelian, then $(\bar{\mathbb{R}}, \Gamma)$ defines \mathbb{Z} .

By Hieronymi [12, Theorem 1.3], the structure $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ defines \mathbb{Z} whenever $\log_{\lambda} \mu \notin \mathbb{Q}$, and is interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ otherwise. Therefore Theorem A extends immediately to expansions of $\bar{\mathbb{R}}$ by multiple discrete subgroups of $\mathrm{Gl}_n(\mathbb{C})$.

Corollary A. *Let \mathcal{G} be a collection of infinite discrete subgroups of various $\mathrm{Gl}_n(\mathbb{C})$. Then either*

- $(\bar{\mathbb{R}}, (\Gamma)_{\Gamma \in \mathcal{G}})$ defines \mathbb{Z} or
- there is $\lambda \in \mathbb{R}_{>0}$ such that $(\bar{\mathbb{R}}, (\Gamma)_{\Gamma \in \mathcal{G}})$ is interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$.

The dichotomies in Theorem A and Corollary A are arguably as strong as they can be. An expansion of the real field that defines \mathbb{Z} has not only an undecidable theory but also defines every real projective set in sense of descriptive set theory (see Kechris [18, 37.6]). From a model-theoretic/geometric point of view such a structure is as wild as can be. On the other hand, by van den Dries [5] the structure $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ has a decidable theory whenever λ is recursive, and admits quantifier-elimination in a suitably extended language. It satisfies combinatorical model-theoretic tameness conditions such as NIP and distality (see [10, 16]). Furthermore, it follows

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from these results that every subset of \mathbb{R}^n definable in $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ is a boolean combination of open sets, and thus $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ defines only sets on the lowest level of the Borel hierarchy. See Miller [20] for more on tameness in expansions of the real field.

Our proof of Theorem A relies crucially on the following two criteria for the definability of \mathbb{Z} in expansions of the real field.

Fact 1.1. *Suppose $D \subseteq \mathbb{R}^k$ is discrete.*

- (1) *If $(\bar{\mathbb{R}}, D)$ defines a subset of \mathbb{R} that is dense and co-dense in a nonempty open interval, then $(\bar{\mathbb{R}}, D)$ defines \mathbb{Z} .*
- (2) *If D has positive Assouad dimension, then $(\bar{\mathbb{R}}, D)$ defines \mathbb{Z} .*

The first statement is [13, Theorem E], a fundamental theorem on first-order expansions of $\bar{\mathbb{R}}$, and the second claim is proven using the first in Hieronymi and Miller [15, Theorem A]. We recall the definition of Assouad dimension in Section 5. This important metric dimension bounds more familiar metric dimensions (such as Hausdorff and Minkowski dimension) from above. We refer to [15] for a more detailed discussion of Assouad dimension and its relevance to definability theory.

The outline of our proof of Theorem A is as follows. Let Γ be a discrete, infinite subgroup of $\mathrm{GL}_n(\mathbb{C})$. Using Fact 1.1(1), we first show that $(\bar{\mathbb{R}}, \Gamma)$ defines \mathbb{Z} whenever Γ contains a non-diagonalizable matrix. It follows from a theorem of Mal'tsev that $(\bar{\mathbb{R}}, \Gamma)$ defines \mathbb{Z} when Γ is virtually solvable and not virtually abelian. In the case that Γ is not virtually solvable, we prove using Tits' alternative that Γ has positive Assouad dimension, and hence $(\bar{\mathbb{R}}, \Gamma)$ defines \mathbb{Z} by Fact 1.1(2). We conclude the proof of Theorem A by proving that whenever Γ is virtually abelian and $(\bar{\mathbb{R}}, \Gamma)$ does not define \mathbb{Z} , then $(\bar{\mathbb{R}}, \Gamma)$ is interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ for some $\lambda \in \mathbb{R}_{>0}$. Along the way we give (Lemma 3.4) an elementary proof showing that a torsion free non abelian nilpotent subgroup of $\mathrm{GL}_n(\mathbb{C})$ has a non-diagonalizable element. Since every finitely generated subgroup of $\mathrm{GL}_n(\mathbb{C})$ is of polynomial or exponential growth, every finitely generated subgroup of $\mathrm{GL}_n(\mathbb{C})$ is either virtually nilpotent or has exponential growth by Gromov's theorem [8]. As every finitely generated subgroup of $\mathrm{GL}_n(\mathbb{C})$ is virtually torsion free by Selberg [28], this yields a more direct proof of Theorem A in the case when Γ is finitely generated.

We want to make an extra comment about the case when Γ is a discrete, virtually solvable, and not virtually abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$. The **Novosibirsk theorem** [24] of Noskov (following work of Mal'stev, Ershov, and Romanovskii) shows that a finitely generated, virtually solvable and non-virtually abelian group interprets $(\mathbb{Z}, +, \cdot)$. It trivially follows that if G is finitely generated, virtually solvable, and non-virtually abelian, then $(\bar{\mathbb{R}}, G)$ interprets $(\mathbb{Z}, +, \cdot)$. However, it does not directly follow that $(\bar{\mathbb{R}}, G)$ defines \mathbb{Z} . We use an entirely different method below to show that if G is in addition discrete, then $(\bar{\mathbb{R}}, G)$ defines \mathbb{Z} . Our method also applies when G is not finitely generated, but relies crucially on the discreteness of G . Note that there are discrete subgroups of $\mathrm{GL}_2(\mathbb{C})$ that are not finitely generated. It is well-known that the free group F_2 on two generators is isomorphic to a discrete subgroup of $\mathrm{GL}_2(\mathbb{C})$ and the commutator subgroup of F_2 is a free group on \aleph_0 generators.

This paper is by no means the first paper to study expansions of the real field by subgroups of $\mathrm{Gl}_n(\mathbb{C})$. Indeed, there is a large body of work on this subject, often not explicitly mentioning $\mathrm{Gl}_n(\mathbb{C})$. Because we see this paper as part of a larger investigation, we survey some of the earlier results and state a conjecture. It is convenient to consider three distinct classes of such expansion. By Miller and Speissegger [22] every first-order expansion \mathcal{R} of $\bar{\mathbb{R}}$ satisfies at least one of the following:

- (1) \mathcal{R} is o-minimal,
- (2) \mathcal{R} defines an infinite discrete subset of \mathbb{R} ,
- (3) \mathcal{R} defines a dense and co-dense subset of \mathbb{R} .

The **open core** \mathcal{R}° of \mathcal{R} is the expansion of $(\mathbb{R}, <)$ generated by all open \mathcal{R} -definable subsets of all \mathbb{R}^k . By [22], if \mathcal{R} does not satisfy (2), then \mathcal{R}° is o-minimal.

The case when \mathcal{R} is o-minimal, is largely understood. Wilkie's famous theorem [31] that $(\bar{\mathbb{R}}, \exp)$ is o-minimal is crucial. This shows the expansion of $\bar{\mathbb{R}}$ by the subgroup

$$\left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & \lambda^t & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

is o-minimal for $\lambda \in \mathbb{R}_{>0}$, and so is the expansion of $\bar{\mathbb{R}}$ by any subgroup of the form

$$\left\{ \begin{pmatrix} t^s & 0 \\ 0 & t^r \end{pmatrix} : t \in \mathbb{R}_{>0} \right\}$$

for $s, r \in \mathbb{R}_{>0}$. Indeed, by Peterzil, Pillay, and Starchenko [26], whenever an expansion $(\bar{\mathbb{R}}, G)$ by a subgroup G of $\mathrm{Gl}_n(\mathbb{R})$ is o-minimal, then G is already definable in $(\bar{\mathbb{R}}, \exp)$. Furthermore, note that by a classical theorem of Tannaka and Chevalley [4] every compact subgroup of $\mathrm{Gl}_n(\mathbb{C})$ is the group of real points on an algebraic group defined over \mathbb{R} . Thus every compact subgroup of $\mathrm{Gl}_n(\mathbb{C})$ is $\bar{\mathbb{R}}$ -definable, and therefore the case of expansions by compact subgroups of $\mathrm{Gl}_n(\mathbb{C})$ is understood as well.

We now consider the case when infinite discrete sets are definable. Corollary A for discrete subgroups of \mathbb{C}^\times follows easily from the proof of [12, Theorem 1.6]. While Corollary A handles the case of expansions by discrete subgroups of $\mathrm{Gl}_n(\mathbb{C})$, there are examples of subgroups of $\mathrm{Gl}_n(\mathbb{C})$ that define infinite discrete sets for which the conclusion of Theorem A fails. Given $\alpha \in \mathbb{R}^\times$ the logarithmic spiral

$$S_\alpha = \{(\exp(t) \sin(\alpha t), \exp(t) \cos(\alpha t)) : t \in \mathbb{R}\}$$

is a subgroup of \mathbb{C}^\times . Let \mathfrak{s} and \mathfrak{e} be the restrictions of \sin and \exp to $[0, 2\pi]$, respectively. Then $(\bar{\mathbb{R}}, S_\alpha)$ is a reduct of $(\bar{\mathbb{R}}, \mathfrak{s}, \mathfrak{e}, \lambda^\mathbb{Z})$ when $\lambda = \exp(2\pi\alpha)$, as was first observed by Miller and Speissegger. As $(\bar{\mathbb{R}}, \mathfrak{s}, \mathfrak{e})$ is o-minimal with field of exponents \mathbb{Q} , the structure $(\bar{\mathbb{R}}, S_\alpha)$ is d-minimal¹ by Miller [20, Theorem 3.4.2] and thus does not define \mathbb{Z} . It can be checked that $(\bar{\mathbb{R}}, S_\alpha)$ defines an analytic function that is not

¹A expansion \mathcal{R} of $\bar{\mathbb{R}}$ is **d-minimal** if every definable unary set in every model of the theory of \mathcal{R} is a union of an open set and finitely many discrete sets.

semi-algebraic², and thus is not interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ for any $\lambda \in \mathbb{R}_{>0}$.

Most work in the case of expansions that define dense and co-dense sets concerns expansions by finite rank subgroups of \mathbb{C}^{\times} (see the introduction of [3] for a thorough discussion of expansions by subgroups of \mathbb{C}^{\times}). In [6] van den Dries and Günaydin showed that an expansion of $\bar{\mathbb{R}}$ by a finitely generated dense subgroup of $(\mathbb{R}_{>0}, \cdot)$ admits quantifier-elimination in a suitably extended language. Günaydin [9] and Belegradek and Zilber [1] proved similar results for the expansion of $\bar{\mathbb{R}}$ by a dense finite rank subgroup of the unit circle $\mathbb{U} := \{a \in \mathbb{C}^{\times} : |a| = 1\}$. This covers the case when G is the group of roots of unity. In all these cases the open core of the resulting expansion is interdefinable with $\bar{\mathbb{R}}$. This does not always have to be the case. In Caulfield [2] studies expansions by subgroups of \mathbb{C}^{\times} of the form

$$\{\lambda^k \exp(i\alpha l) : k, l \in \mathbb{Z}\} \quad \text{where } \lambda \in \mathbb{R}_{>0} \text{ and } \alpha \in \mathbb{R} \setminus \pi\mathbb{Q}.$$

Such an expansion obviously defines a dense and co-dense subset of \mathbb{R} , but by [2] its open core is interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$. Furthermore, even if the open core is o-minimal, it does not have to be interdefinable with $\bar{\mathbb{R}}$. By [14] there is a co-countable subset Λ of $\mathbb{R}_{>0}$ such that if $r \in \Lambda$ and H is a finitely generated dense subgroup of $(\mathbb{R}_{>0}, \cdot)$ contained in the algebraic closure of $\mathbb{Q}(r)$, then the open core of the expansion of $\bar{\mathbb{R}}$ by the subgroup

$$\left\{ \begin{pmatrix} t & 0 \\ 0 & t^r \end{pmatrix} : t \in H \right\}$$

is interdefinable with the expansion of $\bar{\mathbb{R}}$ by the power function $t \mapsto t^r : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$.

All these previous results suggest that the next class of subgroups of $\mathrm{Gl}_n(\mathbb{C})$ for which we can hope to prove a classification comparable to Theorem A, is the class of finitely generated subgroups. Here the following conjecture seems natural, but most likely very hard to prove. Let $\bar{\mathbb{R}}_{\mathrm{Pow}}$ be the expansion of $\bar{\mathbb{R}}$ by all power functions $\mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ of the form $t \mapsto t^r$ for $r \in \mathbb{R}^{\times}$.

Conjecture. *Let G be a finitely generated subgroup of $\mathrm{Gl}_n(\mathbb{C})$ such that $(\bar{\mathbb{R}}, G)$ does not define \mathbb{Z} . Then the open core of $(\bar{\mathbb{R}}, G)$ is a reduct of $\bar{\mathbb{R}}_{\mathrm{Pow}}$ or of $(\bar{\mathbb{R}}, S_{\alpha})$ for some $\alpha \in \mathbb{R}_{>0}$.*

Even when the statement “ $(\bar{\mathbb{R}}, G)$ does not define \mathbb{Z} ” is replaced by “ $(\bar{\mathbb{R}}, G)$ does not interpret $(\mathbb{Z}, +, \cdot)$ ”, the conjecture is open. However, this weaker conjecture might be easier to prove, because the Novosibirsk theorem can be used to rule out the case when G is virtually solvable and non-virtually abelian. It is worth pointing out that Caulfield conjectured that when G is assumed to be a subgroup of \mathbb{C}^{\times} , then the open core $(\bar{\mathbb{R}}, G)$ is either $\bar{\mathbb{R}}$ or a reduct of $(\bar{\mathbb{R}}, S_{\alpha})$ for some $\alpha \in \mathbb{R}_{>0}$. See [2, 3] for progress towards this latter conjecture.

²By induction on the complexity of terms it follows easily from [5, Theorem II] that the definable functions in $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ are given piecewise by finite compositions of $x \mapsto \max(\{0\} \cup (\lambda^{\mathbb{Z}} \cap [-\infty, x]))$ and functions definable in $\bar{\mathbb{R}}$. From this one can deduce that every definable function in this structure is piecewise semi-algebraic.

2. NOTATION AND CONVENTIONS

Throughout m, n range over \mathbb{N} and k, l range over \mathbb{Z} , G is a subgroup of $\mathrm{GL}_n(\mathbb{C})$, and Γ is a discrete subgroup of $\mathrm{GL}_n(\mathbb{C})$. Let $\bar{\mathbb{R}}_\Gamma$ be the expansion of $\bar{\mathbb{R}}$ by a $(2n)^2$ -ary predicate defining Γ . We set $\bar{\mathbb{R}}_\lambda := \bar{\mathbb{R}}_{\lambda^2}$. A subset of \mathbb{R}^k is **discrete** if every point is isolated. We let $\mathrm{UT}_n(\mathbb{C})$ be the group of n -by- n upper triangular matrices, $\mathrm{D}_n(\mathbb{C})$ be the group of n -by- n diagonal matrices, and \mathbb{U} be the multiplicative group of complex numbers with norm one.

All structures considered are first-order, “definable” means “definable, possibly with parameters”. Two expansions of $(\mathbb{R}, <)$ are **interdefinable** if they define the same subsets of \mathbb{R}^k for all k . If \mathbf{P} is a property of groups, then a group H is **virtually \mathbf{P}** if there is finite index subgroup H' of H that is \mathbf{P} .

3. LINEAR GROUPS

We gather some general facts on groups. Throughout this section H is a finitely generated group with a symmetric set S of generators. Let S_m be the set of m -fold products of elements of S for each m . If S' is another symmetric set of generators then there is a constant $k \geq 1$ such that

$$k^{-1}|S_m| \leq |S'_m| \leq k|S_m| \quad \text{for all } m.$$

Thus the growth rate of $m \mapsto |S_m|$ is an invariant of H . We say H has **exponential growth** if there is a $C > 1$ such that $|S_m| \geq C^m$ for all m and H has **polynomial growth** there are $k, t \in \mathbb{R}_{>0}$ such that $|S_m| \leq tm^k$ for all m . Note that finitely generated non-abelian free groups are of exponential growth. Gromov’s theorem [8] says H has polynomial growth if and only if it is virtually nilpotent. Gromov’s theorem for subgroups of $\mathrm{GL}_n(\mathbb{C})$ is less difficult and may be proven using the following two theorems:

Fact 3.1. *If G does not contain a non-abelian free subgroup, then G is virtually solvable.*

Fact 3.1 is **Tits’ alternative** [29]. Fact 3.2 is due to Milnor [23] and Wolf [32].

Fact 3.2. *Suppose H is virtually solvable. Then H either has exponential or polynomial growth. If the latter case holds then H is virtually nilpotent.*

Note that Fact 3.1 and Fact 3.2 imply that every finitely generated subgroup of $\mathrm{GL}_n(\mathbb{C})$ is of polynomial or exponential growth. This dichotomy famously does not hold for finitely generated groups in general, see for example [7].

Fact 3.3. *Let E be a nilpotent, torsion-free, and non-abelian group. Then there are $a, b, c \in E$ such that c is not the identity of E and*

$$[a, b] = c, \quad ac = ca, \quad bc = cb.$$

Proof. Let e be the identity element of E . We define the lower central series $(E_k)_{k \in \mathbb{N}}$ of E by declaring $E_0 = E$ and $E_k = [E_{k-1}, E]$ for $k \geq 1$. Nilpotency means there is an m such that $E_m \neq \{e\}$ and $[E_m, E] = \{e\}$. Moreover $m \geq 1$ as E is not abelian.

On the one hand, $[E_{m-1}, E] = E_m \neq \{e\}$ and so E_{m-1} is not contained in $Z(E)$. Thus, there exists $a \in E_{m-1} \setminus Z(E)$ and $b \in E_m$ that does not commute with a . On the other hand, $[E_m, E] = \{e\}$ implies E_m is contained in the center $Z(E)$ of E

and is thus abelian. So, $c := [a, b]$ is an element of $Z(E)$ and commutes with both a and b . \square

3.1. Non-diagonalizable elements. We show certain linear groups necessarily contain non-diagonalizable elements.

Lemma 3.4. *If G is nilpotent, torsion-free, and not abelian, then G contains a non-diagonalizable element.*

Lemma 3.4 follows from Fact 3.3 above and Lemma 3.5 below.

Lemma 3.5. *Suppose $a, b, c \in \mathrm{Gl}_n(\mathbb{C})$ satisfy*

$$[a, b] = c, \quad ac = ca, \quad bc = cb,$$

and c is not torsion. Then either a or c is not diagonalizable.

Proof. Suppose a, c are both diagonalizable. As a, c commute, they are simultaneously diagonalizable and share a basis \mathfrak{B} of eigenvectors. As c is not torsion, there is $\lambda_c \in \mathbb{C}^\times$ which is not a root of unity and $v \in \mathfrak{B}$ such that $cv = \lambda_c v$. Let $\lambda_a \in \mathbb{C}^\times$ be such that $av = \lambda_a v$.

By way of contradiction, we will show $a(b^k v) = (\lambda_a \lambda_c^k)(b^k v)$ for all $k \geq 1$. As λ_c is not a root of unity, this implies a has infinitely many eigenvalues, which is impossible for an $n \times n$ matrix. The base case holds as

$$a(bv) = bacv = (\lambda_a \lambda_c)(bv).$$

Let $k \geq 2$ and suppose $a(b^{k-1} v) = (\lambda_a \lambda_c^{k-1})(b^{k-1} v)$. As c commutes with b ,

$$a(b^k v) = ab(b^{k-1} v) = bac(b^{k-1} v) = bab^{k-1} cv = (\lambda_c)(bab^{k-1} v).$$

Applying the inductive assumption,

$$(\lambda_c)(bab^{k-1} v) = (\lambda_c)b(\lambda_a \lambda_c^{k-1} b^{k-1} v) = (\lambda_a \lambda_c^k)(b^k v).$$

\square

We now prove a slight weakening of Lemma 3.4 for solvable groups. Recall $a \in \mathrm{Gl}_n(\mathbb{C})$ is **unipotent** if some conjugate of a is upper triangular with every diagonal entry equal to one. The only diagonalizable unipotent matrix is the identity. We recall a theorem of Mal'tsev [19].

Fact 3.6. *Suppose G is solvable. Then there is a finite index subgroup G' of G such that G' is conjugate to a subgroup of $\mathrm{UT}_n(\mathbb{C})$.*

We now derive an easy corollary from Fact 3.6.

Lemma 3.7. *Suppose G is solvable and not virtually abelian. Then G contains a non-diagonalizable element.*

Proof. Suppose every element of G is diagonalizable. After applying Fact 3.6 and making a change of basis if necessary we suppose $G' = G \cap \mathrm{UT}_n(\mathbb{C})$ has finite index in G . Let $\rho : \mathrm{UT}_n(\mathbb{C}) \rightarrow \mathrm{D}_n(\mathbb{C})$ be the natural quotient map; that is the restriction to the diagonal. Every element of the kernel of ρ is unipotent. Thus the restriction of ρ to G' is injective, and so G' is abelian. \square

4. NON-DIAGONALIZABLE MATRICES

Lemma 4.1. *Suppose G contains a non-diagonalizable matrix. Then there is a rational function h on $\mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_n(\mathbb{C})$ such that $h(G \times G) \subseteq \mathbb{C}$ is dense in $\mathbb{R}_{>0}$.*

Proof. Suppose $a \in G$ is non-diagonalizable. Let $b \in \mathrm{Gl}_n(\mathbb{C})$ be such that bab^{-1} is in Jordan form, i.e.

$$bab^{-1} = \begin{pmatrix} A_1 & O & \dots & O \\ O & A_2 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_l \end{pmatrix}$$

where each A_i is a Jordan block and each O is a zero matrix of the appropriate dimensions. We have

$$ba^k b^{-1} = \begin{pmatrix} A_1^k & O & \dots & O \\ O & A_2^k & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_l^k \end{pmatrix} \quad \text{for all } k.$$

As a is not diagonalizable, A_j has more than one entry for some j . We suppose A_1 is m -by- m with $m \geq 2$. For some $\lambda \in \mathbb{C}^\times$ we have

$$A_1 = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

It is well-known and easy to show by induction that for every $k \geq 1$:

$$A_1^k = \begin{pmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \binom{k}{3}\lambda^{k-3} & \binom{k}{4}\lambda^{k-4} & \dots & \binom{k}{m}\lambda^{k-m+1} \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \binom{k}{3}\lambda^{k-3} & \dots & \binom{k}{m-1}\lambda^{k-m+2} \\ 0 & 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \dots & \binom{k}{m-2}\lambda^{k-m+3} \\ 0 & 0 & 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \dots & \binom{k}{m-3}\lambda^{k-m+4} \\ 0 & 0 & 0 & 0 & \lambda^k & \dots & \binom{k}{m-4}\lambda^{k-m+5} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & \vdots & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & 0 & 0 & \dots & \dots & 0 & \lambda^k \end{pmatrix}.$$

Let g_{ij} be the (i, j) -entry of $g \in \mathrm{Gl}_n(\mathbb{C})$. Thus, for each $k \geq 1$,

$$(ba^k b^{-1})_{12} = k\lambda^{k-1} \quad \text{and} \quad (ba^k b^{-1})_{22} = \lambda^k.$$

We define a rational function h' on $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ by declaring

$$h'(g, g') := \frac{g_{12}g'_{22}}{g'_{12}g_{22}}$$

for all $g, g' \in \mathrm{GL}_n(\mathbb{C})$ such that $g_{22}, g'_{12} \neq 0$. Then define h by declaring

$$h(g, g') := h'(bgb^{-1}, bg'b^{-1})$$

We have

$$h(a^i, a^j) = \frac{(i\lambda^{i-1})(\lambda^j)}{(j\lambda^{j-1})(\lambda^i)} = \frac{i}{j} \quad \text{for all } i, j \geq 1.$$

Thus $\mathbb{Q}_{>0}$ is a subset of the image of $G \times G$ under h . \square

Corollary 4.2. *If Γ contains a non-diagonalizable matrix, then $\bar{\mathbb{R}}_\Gamma$ defines \mathbb{Z} . In particular, if Γ is either*

- solvable and not virtually abelian, or
- torsion-free, nilpotent and non-abelian,

then $\bar{\mathbb{R}}_\Gamma$ defines \mathbb{Z} .

Proof. Applying Lemma 4.1, suppose h is a rational function on $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ such that the image of $\Gamma \times \Gamma$ under h is dense in $\mathbb{R}_{>0}$. Note Γ is countable as Γ is discrete. It follows that the image of $\Gamma \times \Gamma$ under any function is co-dense in $\mathbb{R}_{>0}$. Fact 1.1(1) implies that $\bar{\mathbb{R}}_\Gamma$ defines \mathbb{Z} . The second claim follows from the first by applying Lemma 3.4 and Lemma 3.7. \square

Corollary 4.3. *If $a \in \mathrm{GL}_n(\mathbb{C})$ is non-diagonalizable, then $(\bar{\mathbb{R}}, \{a^k : k \in \mathbb{Z}\})$ defines \mathbb{Z} .*

Proof. Set $G := \{a^k : k \in \mathbb{Z}\}$. The proof of Lemma 4.1 shows that in this case $\mathbb{Q}_{>0}$ is the intersection of $h(G \times G)$ and $\mathbb{R}_{>0}$. Thus the corollary follows by Julia Robinson's classical theorem of definability of \mathbb{Z} in $(\mathbb{Q}, +, \cdot)$ in [27]. \square

5. THE CASE OF EXPONENTIAL GROWTH

We recall the **Assouad dimension** of a metric space (X, d) . See Heinonen [11] for more information. The Assouad dimension of a subset Y of \mathbb{R}^k is the Assouad dimension of Y equipped with the euclidean metric induced from \mathbb{R}^k .

Suppose $A \subseteq X$ has at least two elements. Then A is **δ -separated** for $\delta \in \mathbb{R}_{>0}$ if $d(a, b) \geq \delta$ for all distinct $a, b \in A$, and A is **separated** if A is δ -separated for some $\delta > 0$. Let $\mathcal{S}(A) \in \mathbb{R}$ be the supremum of all $\delta \geq 0$ for which A is δ -separated. Let $\mathcal{D}(A)$ be the **diameter** of A ; that is the infimum of all $\delta \in \mathbb{R} \cup \{\infty\}$ such that $d(a, b) < \delta$ for all $a, b \in A$, and A is **bounded** if $\mathcal{D}(A) < \infty$. Note $\mathcal{S}(A) \leq \mathcal{D}(A)$ and every separated and bounded set is finite. The **Assouad dimension** of (X, d) is the infimum of the set of $\beta \in \mathbb{R}_{>0}$ for which there is a $C > 0$ such that

$$|A| \leq C \left(\frac{\mathcal{D}(A)}{\mathcal{S}(A)} \right)^\beta \quad \text{for all bounded and separated } A \subseteq X.$$

If the set of such β is empty then the Assouad dimension of (X, d) is declared to be ∞ . The proof of Fact 5.1 is an elementary computation which we leave to the reader.

Fact 5.1. *Suppose there is a sequence $\{A_m\}_{m \in \mathbb{N}}$ of bounded separated subsets of X with cardinality at least two, and $B, C > 1, t > 0$ are such that*

$$|A_m| \geq C^m \quad \text{and} \quad \frac{\mathcal{D}(A_m)}{\mathcal{S}(A_m)} \leq tB^m \quad \text{for all } m$$

then (X, d) has positive Assouad dimension.

Let $|v|$ be the usual euclidean norm of $v \in \mathbb{C}^n$. Given $g \in \mathrm{M}_n(\mathbb{C})$ we let

$$\|g\| = \inf\{t \in \mathbb{R}_{>0} : |gv| \leq t|v| \text{ for all } v \in \mathbb{C}^n\}$$

be the operator norm of g . Then $\|\cdot\|$ is a linear norm on $\mathrm{M}_n(\mathbb{C})$ and satisfies $\|gh\| \leq \|g\|\|h\|$ for all $g, h \in \mathrm{M}_n(\mathbb{C})$. As any two linear norms on $\mathrm{M}_n(\mathbb{C})$ are bi-Lipschitz equivalent the metric induced by $\|\cdot\|$ is bi-Lipschitz equivalent to the usual euclidean metric on \mathbb{R}^{n^2} .

Proposition 5.2. *Suppose Γ contains a finitely generated subgroup Γ' of exponential growth. Then Γ has positive Assouad dimension.*

Proof. Because Assouad dimension is a bi-Lipschitz invariant (see [11]), it suffices to show that Γ has positive Assouad dimension with respect to the metric induced by $\|\cdot\|$. We let I be the n -by- n identity matrix. Let S be a symmetric generating set of Γ' , and let S_m be the set of m -fold products of elements of S for $m \geq 2$. Set

$$B := \max\{\|g\| : g \in S\} \quad \text{and} \quad D := \min\{\|g - I\| : g \in S\}.$$

Note that $D > 0$. The subgroup of matrices of norm one is compact. Since Γ is infinite and discrete, there is $g \in S$ such that $\|g\| \neq 1$. Because S is symmetric, we obtain $g \in S$ such that $\|g\| > 1$. Thus $B > 1$. Induction shows that $\|g\| \leq B^m$ when $g \in S_m$. The triangle inequality directly yields $\mathcal{D}(S_m) \leq 2B^m$. Each S_m is symmetric as S is symmetric. Therefore $\|g^{-1}\| \leq B^m$ for all $g \in S_m$. Let $g, h \in \Gamma$. We have

$$\|I - g^{-1}h\| \leq \|g^{-1}\| \|g - h\|.$$

Equivalently,

$$\frac{\|I - g^{-1}h\|}{\|g^{-1}\|} \leq \|g - h\|.$$

Suppose $g, h \in S_m$ are distinct. Then $g^{-1}h \neq I$, and hence $\|I - g^{-1}h\| \geq D$. So

$$\|g - h\| \geq \frac{\|I - g^{-1}h\|}{\|g^{-1}\|} \geq \frac{D}{B^m}.$$

Hence $\mathcal{S}(S_m) \geq D/B^m$. Thus

$$\frac{\mathcal{D}(S_m)}{\mathcal{S}(S_m)} \leq \frac{2B^m}{D/B^m} = \frac{2}{D}B^{2m}.$$

As Γ' has exponential growth, there is a $C > 0$ such that $|S_m| \geq C^m$ for all m . An application of Fact 5.1 shows that Γ has positive Assouad dimension. \square

Proposition 5.3. *Suppose Γ is not virtually abelian. Then $\bar{\mathbb{R}}_\Gamma$ defines \mathbb{Z} .*

Proof. By Corollary 4.2 and Lemma 3.7 we can assume that Γ is not virtually solvable. Thus by Fact 3.1, the group Γ contains a non-abelian free subgroup. Therefore Γ has positive Assouad dimension by Proposition 5.2. We conclude that $\bar{\mathbb{R}}_\Gamma$ defines \mathbb{Z} by Fact 1.1(2). \square

6. THE VIRTUALLY ABELIAN CASE

We first reduce the virtually abelian case to the abelian case.

Lemma 6.1. *Suppose G is virtually abelian and every element of G is diagonalizable. Then there is a finite index abelian subgroup G' of G such that $(\bar{\mathbb{R}}, G)$ and $(\bar{\mathbb{R}}, G')$ are interdefinable.*

Proof. Let G'' be a finite index abelian subgroup of G . As every element of G'' is diagonalizable, G'' is simultaneously diagonalizable. Fix $g \in \mathrm{GL}_n(\mathbb{C})$ such that gag^{-1} is diagonal for all $a \in G''$. Let G' be the set of $a \in G$ such that gag^{-1} is diagonal, i.e. G' is the intersection of G and $g^{-1}\mathrm{D}_n(\mathbb{C})g$. Then G' is abelian, $(\bar{\mathbb{R}}, G)$ -definable, and is of finite index in G as $G'' \subseteq G'$. Because G' has finite index in G , we have

$$G = g_1 G' \cup \dots \cup g_m G' \text{ for some } g_1, \dots, g_m \in G.$$

So G is $(\bar{\mathbb{R}}, G')$ -definable. \square

Proposition 6.2 and an application of Corollary 4.2 finishes the proof of Theorem A.

Proposition 6.2. *Suppose Γ is abelian and $\bar{\mathbb{R}}_\Gamma$ does not define \mathbb{Z} . Then there is $\lambda \in \mathbb{R}_{>0}$ such that $\bar{\mathbb{R}}_\Gamma$ is interdefinable with $\bar{\mathbb{R}}_\lambda$.*

Let $\mathfrak{u} : \mathbb{C}^\times \rightarrow \mathbb{U}$ be the argument map and $|\cdot| : \mathbb{C}^\times \rightarrow \mathbb{R}_{>0}$ be the absolute value map. Thus $z = \mathfrak{u}(z)|z|$ for all $z \in \mathbb{C}^\times$. Let \mathbb{U}_m be the group of m th roots of unity for each $m \geq 1$. In the following proof of Proposition 6.2 we will use the immediate corollary of [12, Theorem 1.3] that the structure $(\bar{\mathbb{R}}, \lambda^\mathbb{Z}, \mu^\mathbb{Z})$ defines \mathbb{Z} whenever $\log_\lambda \mu \notin \mathbb{Q}$, and is interdefinable with $(\bar{\mathbb{R}}, \lambda^\mathbb{Z})$ otherwise.

Proof. Fact 1.1(1) implies every countable $\bar{\mathbb{R}}_\Gamma$ -definable subset of \mathbb{R} is nowhere dense. It follows that every $\bar{\mathbb{R}}_\Gamma$ -definable countable subgroup of \mathbb{U} is finite and every $\bar{\mathbb{R}}_\Gamma$ -definable countable subgroup of $(\mathbb{R}_{>0}, \cdot)$ is of the form $\lambda^\mathbb{Z}$ for some $\lambda \in \mathbb{R}_{>0}$.

Every element of Γ is diagonalizable by Corollary 4.2. Thus Γ is simultaneously diagonalizable. After making a change of basis we suppose Γ is a subgroup of $\mathrm{D}_n(\mathbb{C})$. We identify $\mathrm{D}_n(\mathbb{C})$ with $(\mathbb{C}^\times)^n$. Let Γ_i be the image of Γ under the projection $(\mathbb{C}^\times)^n \rightarrow \mathbb{C}^\times$ onto the i th coordinate for $1 \leq i \leq n$.

Each $\mathfrak{u}(\Gamma_i)$ is finite. Fix an m such that $\mathfrak{u}(\Gamma_i)$ is a subgroup of \mathbb{U}_m for all $1 \leq i \leq n$. For each $1 \leq i \leq n$, $|\Gamma_i|$ is a discrete subgroup of $\mathbb{R}_{>0}$ and is thus equal to $\alpha_i^\mathbb{Z}$ for some $\alpha_i \in \mathbb{R}_{>0}$. If $\alpha_i = 1$ for all $1 \leq i \leq n$, then Γ is trivial. Thus we may suppose that there is j such that $1 \leq j \leq n$ and $\alpha_j \neq 1$. We fix this j . By [12, Theorem 1.3] each α_i is a rational power of α_j . Let $\lambda \in \mathbb{R}_{>0}$ be a rational power of α_j such that each α_i is an integer power of λ . We show $\bar{\mathbb{R}}_\Gamma$ and $\bar{\mathbb{R}}_\lambda$ are interdefinable. Note that $\lambda^\mathbb{Z}$ is $\bar{\mathbb{R}}_\Gamma$ -definable; so it suffices to show Γ is $\bar{\mathbb{R}}_\lambda$ -definable.

Every element of Γ_i is of the form $\sigma \lambda^k$ for some $\sigma \in \mathbb{U}_m$ and $k \in \mathbb{Z}$. Thus Γ is a subgroup of

$$\Gamma' = \left\{ \begin{pmatrix} \sigma_1 \lambda^{k_1} & 0 & \dots & 0 \\ 0 & \sigma_2 \lambda^{k_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \lambda^{k_n} \end{pmatrix} : \sigma_1, \dots, \sigma_n \in \mathbb{U}_m, k_1, \dots, k_n \in \mathbb{Z} \right\}.$$

Note Γ' is $\bar{\mathbb{R}}_\lambda$ -definable. Abusing notation we let $\mathbf{u} : (\mathbb{C}^\times)^n \rightarrow \mathbb{U}^n$ and we let $|\cdot| : (\mathbb{C}^\times)^n \rightarrow (\mathbb{R}_{>0})^n$ be given by

$$\mathbf{u}(z_1, \dots, z_n) = (\mathbf{u}(z_1), \dots, \mathbf{u}(z_n)) \quad \text{and} \quad |(z_1, \dots, z_n)| = (|z_1|, \dots, |z_n|).$$

Then the map $(\mathbb{C}^\times)^n \rightarrow \mathbb{U}^n \times (\mathbb{R}_{>0})^n$ given by $\bar{z} \mapsto (\mathbf{u}(\bar{z}), |\bar{z}|)$ restricts to a $\bar{\mathbb{R}}_\lambda$ -definable isomorphism between Γ' and $\mathbb{U}_m^n \times (\lambda^\mathbb{Z})^n$. Lemma 6.3 below implies any subgroup of $\mathbb{U}_m^n \times (\lambda^\mathbb{Z})^n$ is $\bar{\mathbb{R}}_\lambda$ -definable. \square

We consider $(\mathbb{Z}/m\mathbb{Z}, +)$ to be a group with underlying set $\{0, \dots, m-1\}$ in the usual way so that $(\mathbb{Z}/m\mathbb{Z}, +)$ is a $(\mathbb{Z}, +)$ -definable group. Lemma 6.3 is folklore. We include a proof for the sake of completeness.

Lemma 6.3. *Every subgroup H of $(\mathbb{Z}/m\mathbb{Z})^l \times \mathbb{Z}^n$ for $l \geq 0$ is $(\mathbb{Z}, +)$ -definable.*

Proof. We first reduce to the case $l = 0$. The quotient map $\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is $(\mathbb{Z}, +)$ -definable, it follows that the coordinate-wise quotient $\mathbb{Z}^l \times \mathbb{Z}^n \rightarrow (\mathbb{Z}/m\mathbb{Z})^l \times \mathbb{Z}^n$ is $(\mathbb{Z}, +)$ -definable. It suffices to show the preimage of H in \mathbb{Z}^{l+n} is $(\mathbb{Z}, +)$ -definable.

Suppose H is a subgroup of \mathbb{Z}^n . Then H is finitely generated with generators β_1, \dots, β_k where $\beta_i = (b_1^i, \dots, b_n^i)$ for all $1 \leq i \leq k$. Then

$$\begin{aligned} H &= \left\{ \sum_{i=1}^k c_i \beta_i : c_1, \dots, c_k \in \mathbb{Z} \right\} \\ &= \left\{ \left(\sum_{i=1}^k c_i b_1^i, \dots, \sum_{i=1}^k c_i b_n^i \right) : c_1, \dots, c_n \in \mathbb{Z} \right\}. \end{aligned}$$

Thus H is $(\mathbb{Z}, +)$ -definable. \square

7. COUNTABLE $(\mathcal{R}, \lambda^\mathbb{Z})$ -DEFINABLE GROUPS

Fix $\lambda \in \mathbb{R}_{>0}$ and an o-minimal \mathcal{R} with field of exponents \mathbb{Q} . Since $(\mathcal{R}, \lambda^\mathbb{Z})$ does not define \mathbb{Z} by [20, Theorem 3.4.2], Theorem A implies every $(\mathcal{R}, \lambda^\mathbb{Z})$ -definable discrete subgroup of $\mathrm{GL}_n(\mathbb{C})$ is virtually abelian. We extend this result to all countable interpretable groups.

Proposition 7.1. *Every countable $(\mathcal{R}, \lambda^\mathbb{Z})$ -interpretable group is virtually abelian.*

Proposition 7.1 follows directly from several previous results. Every d-minimal expansion of $\bar{\mathbb{R}}$ admits definable selection by Miller [21] (d-minimal definable selection also follows from [17, Proposition 6.3]). Therefore an $(\mathcal{R}, \lambda^\mathbb{Z})$ -interpretable group is isomorphic to an $(\mathcal{R}, \lambda^\mathbb{Z})$ -definable group. We now recall two results of Tychonievich. The first is a special case of [30, 4.1.10].

Fact 7.2. *If $X \subseteq \mathbb{R}^k$ is $(\mathcal{R}, \lambda^\mathbb{Z})$ -definable and countable, then there is an $\bar{\mathbb{R}}_\lambda$ -definable surjection $f : (\lambda^\mathbb{Z})^m \rightarrow X$ for some m .*

Fact 7.3 is a minor rewording of [30, 4.1.2].

Fact 7.3. *Every $(\mathcal{R}, \lambda^\mathbb{Z})$ -definable subset of $(\lambda^\mathbb{Z})^m$ is $(\lambda^\mathbb{Z}, <, \cdot)$ -definable.*

Facts 7.2 and 7.3 together imply that every countable $(\mathcal{R}, \lambda^\mathbb{Z})$ -definable group is isomorphic to a $(\mathbb{Z}, <, +)$ -definable group. Now apply the following result of Onshuus and Vicaria [25] to complete the proof of Proposition 7.1.

Fact 7.4. *Every $(\mathbb{Z}, <, +)$ -definable group is virtually abelian.*

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