

INVERSE ITERATION FOR THE MONGE–AMPÈRE EIGENVALUE PROBLEM

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ABSTRACT. We present an iterative method based on repeatedly inverting the Monge–Ampère operator with Dirichlet boundary condition and prescribed right-hand side on a bounded, convex domain $\Omega \subset \mathbb{R}^n$. We prove that the iterates u_k generated by this method converge as $k \rightarrow \infty$ to a solution of the Monge–Ampère eigenvalue problem

$$\begin{cases} \det D^2u = \lambda_{MA}(-u)^n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the solutions of this problem are unique up to a positive multiplicative constant, the normalized iterates $\hat{u}_k := \frac{u_k}{\|u_k\|_{L^\infty(\Omega)}}$ converge to the eigenfunction of unit height. In addition, we show that $\lim_{k \rightarrow \infty} R(u_k) = \lim_{k \rightarrow \infty} R(\hat{u}_k) = \lambda_{MA}$, where the Rayleigh quotient $R(u)$ is defined as

$$R(u) := \frac{\int_{\Omega} (-u) \det D^2u}{\int_{\Omega} (-u)^{n+1}}.$$

Our method converges for a wide class of initial choices u_0 that can be constructed explicitly, and does not rely on prior knowledge of the Monge–Ampère eigenvalue λ_{MA} .

1. INTRODUCTION AND MAIN RESULT

Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain. The Monge–Ampère eigenvalue problem seeks to find a convex function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and a positive number λ such that

$$(1) \quad \begin{cases} \det D^2u = \lambda(-u)^n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem was first considered by Lions in [14], who proved the following result.

Theorem 1.1 (Lions 1985). *Assume $\Omega \subset \mathbb{R}^n$ is a smooth, bounded, uniformly convex domain. There exist a unique positive constant λ_{MA} and a unique (up to positive multiplicative constants) nonzero convex function $u \in C^{1,1}(\bar{\Omega}) \cap C^\infty(\Omega)$ solving the eigenvalue problem (1).*

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The constant λ_{MA} is called the Monge–Ampère eigenvalue and is defined in the following manner. Let $A(x) \in C(\Omega)$ be a symmetric, positive-definite matrix such that $\det A(x) \geq n^{-n}$ for all $x \in \Omega$. The collection of all such matrices will be denoted \mathcal{A} . Let L_A be the linear operator $L_A v = -\text{tr}(A(x)D^2v)$, and denote by λ_A^1 the (positive) first Dirichlet eigenvalue of L_A . Then the Monge–Ampère eigenvalue is defined as

$$\lambda_{MA} := \left(\inf_{A \in \mathcal{A}} \lambda_A^1 \right)^n.$$

The eigenvalue problem (1) was revisited by Tso in [20] from a variational point of view. In order to state Tso's result, we need a few definitions. Consider the class of functions

$$\mathcal{K}_2 = \{u \in C^{0,1}(\overline{\Omega}) \cap C^\infty(\Omega) : u \text{ convex and nonzero in } \Omega, u = 0 \text{ on } \partial\Omega\}.$$

Define the Rayleigh quotient of a function $u \in \mathcal{K}_2$ as

$$R(u) := \frac{\int_{\Omega}(-u) \det D^2u}{\int_{\Omega}(-u)^{n+1}}.$$

It is useful to observe that $R(cu) = R(u)$ for all $c > 0$.

Theorem 1.2 (Tso 1990). *Assume $\Omega \subset \mathbb{R}^n$ is a smooth, bounded, uniformly convex domain. Then*

$$\lambda_{MA} = \inf_{u \in \mathcal{K}_2} R(u).$$

Owing to recent work of Le [12], Theorems 1.1 and 1.2 hold for arbitrary convex domains Ω , without assuming uniform convexity. To state Le's result, we let

$$\mathcal{K} = \{u \in C(\overline{\Omega}) : u \text{ convex and nonzero in } \Omega, u = 0 \text{ on } \partial\Omega\}.$$

Given $u \in \mathcal{K}$, we denote by Mu the Monge–Ampère measure of u , defined in (8) in Section 2. The Monge–Ampère energy of u is the quantity $I(u) := \int_{\Omega}(-u) dMu$. The Rayleigh quotient of u is then defined as

$$(2) \quad R(u) := \frac{I(u)}{\|u\|_{L^{n+1}(\Omega)}^{n+1}} = \frac{\int_{\Omega}(-u) dMu}{\int_{\Omega}(-u)^{n+1}}.$$

Note that this definition coincides with the one considered by Lions and Tso when $u \in \mathcal{K}_2$.

Theorem 1.3 (Le 2018). *Assume $\Omega \subset \mathbb{R}^n$ is a bounded, convex domain. Then there exists a unique positive constant (still denoted by λ_{MA}) and a unique (up to positive multiplicative constants) function $u \in \mathcal{K} \cap C^\infty(\Omega)$ satisfying (1) with*

$$\lambda = \lambda_{MA} = \inf_{u \in \mathcal{K}} R(u).$$

There are two methods currently available for constructing a solution of (1), both relying on compactness arguments. The first, by Lions [14], considers solving the following Dirichlet problem for a convex function $u_\tau \in C^2(\overline{\Omega})$ for each $\tau \geq 0$:

$$(3) \quad \begin{cases} \det D^2u_\tau = (1 - \tau u_\tau)^n & \text{in } \Omega, \\ u_\tau = 0 & \text{on } \partial\Omega. \end{cases}$$

It is shown in [14, Theorem 1] that the quantity

$$(4) \quad \mu := \sup\{\tau > 0 : \text{there exists a solution } u_\tau \text{ of (3)}\}$$

is strictly positive, that $\lim_{\tau \rightarrow \mu^-} \|u_\tau\|_{L^\infty(\Omega)} = \infty$, and that (up to choice of a subsequence) the functions $\hat{u}_\tau := \frac{u_\tau}{\|u_\tau\|_{L^\infty(\Omega)}}$ converge to a solution of (II) as $\tau \rightarrow \mu^-$.

Furthermore, $\mu = \lambda_{MA}^{\frac{1}{n}}$; thus, (4) provides a third characterization of the Monge–Ampère eigenvalue λ_{MA} .

The second method of constructing a solution of (II), by Tso [20], is to fix constants $\sigma, p > 0$ and consider the Dirichlet problem

$$(5) \quad \begin{cases} \det D^2u = \sigma(-u)^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that the equation (5) is the Euler–Lagrange equation of the functional

$$(6) \quad J_{p,\sigma}(u) := \frac{1}{n+1} \int_{\Omega} (-u) \det D^2u - \frac{\sigma}{p+1} \int_{\Omega} (-u)^{p+1}.$$

Using variational methods, Tso proves the existence of unique minimizers in \mathcal{K}_2 of the functional $J_{p,\sigma}$ for $p < n$ and $\sigma = \lambda_{MA}$. By establishing estimates for the minimizers that are uniform in p , Tso shows there exists a sequence $p_k \nearrow n$ such that the solutions u_k of (5) with $p = p_k$ and $\sigma = \lambda_{MA}$ converge to a solution of (II).

The primary contribution of the present work is to present an iterative method for constructing a sequence of functions $u_k \in \mathcal{K}$ that converges uniformly to a solution of (II). This sequence is obtained by repeatedly inverting the Monge–Ampère operator with Dirichlet boundary condition. We show, moreover, that $\lim_{k \rightarrow \infty} R(u_k) = \lambda_{MA}$. Similar inverse iteration methods have been considered for equations in divergence form such as the p -Laplace equation [1, 2, 11]. The present work establishes the first inverse iteration result for the eigenvalue problem of a fully nonlinear degenerate elliptic equation.

Theorem 1.4. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded, convex domain. Let $u_0 \in C(\overline{\Omega})$ satisfy the following conditions:*

- (i) u_0 is convex and $u_0 \leq 0$ on $\partial\Omega$,
- (ii) $R(u_0) < \infty$,
- (iii) $Mu_0 \geq \mathcal{L}^n$ in Ω , where \mathcal{L}^n denotes n -dimensional Lebesgue measure.

For $k \geq 0$, define the sequence $u_k \in \mathcal{K}$ to be the solutions of the Dirichlet problem

$$(7) \quad \begin{cases} \det D^2u_{k+1} = R(u_k)(-u_k)^n & \text{in } \Omega, \\ u_{k+1} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\{u_k\}$ converges uniformly on $\overline{\Omega}$ to a nonzero Monge–Ampère eigenfunction u_∞ . Consequently, the sequence $\hat{u}_k := \frac{u_k}{\|u_k\|_{L^\infty(\Omega)}}$ converges uniformly on $\overline{\Omega}$ to the unique solution u of (II) satisfying $\|u\|_{L^\infty(\Omega)} = 1$. Furthermore, $\lim_{k \rightarrow \infty} R(u_k) = \lim_{k \rightarrow \infty} R(\hat{u}_k) = \lambda_{MA}$.

We briefly outline the strategy behind the proof of Theorem 1.4. The starting point is a monotonicity relation, proved in Lemma 3.1, which provides control over the Rayleigh quotients $R(u_k)$ and enables us to prove uniform Hölder estimates for the functions u_k ; see Proposition 3.2. The sequence $\{u_k\}$ is, therefore, compact; hence, there exists a subsequence $\{u_{k(j)}\}_{j \in \mathbb{N}}$ converging to a limiting function u_∞ . Comparison principle arguments using the eigenfunctions from Theorem 1.3 show that $\|u_k\|_{L^\infty(\Omega)}$ stays uniformly away from zero; see Proposition 3.3. Consequently, $u_\infty \in \mathcal{K}$ is a candidate to solve the eigenvalue problem (II). However, in

order to prove that u_∞ is an eigenfunction, it is necessary to show that the shifted subsequence $\{u_{k(j)+1}\}_{j \in \mathbb{N}}$ also converges to u_∞ . The monotonicity relation and a continuity property of the Monge–Ampère energy, Lemma 2.9, are essential to verify the aforementioned claim, as well as to establish that any convergent subsequence of $\{u_k\}$ must converge to the same eigenfunction u_∞ .

Let us point out an elementary construction of an initial function u_0 satisfying the hypotheses of Theorem 1.4 for any bounded, convex domain $\Omega \subset \mathbb{R}^n$. Let $B_R(x_0)$ be any ball centered at $x_0 \in \mathbb{R}^n$ of radius $R > 0$ such that $\Omega \Subset B_R(x_0)$. Consider the parabola $P_R(x) = \frac{1}{2}(|x - x_0|^2 - R^2)$, which satisfies $\det D^2 P_R(x) = 1$ for all $x \in \mathbb{R}^n$ and vanishes on $\partial B_R(x_0)$. Then $u_0(x) = P_R(x)$ satisfies all the properties required in the statement of Theorem 1.4.

We highlight some other noteworthy attributes of the iteration (7). First, let us point out that both the approaches of Lions and Tso outlined above for constructing a solution of (1) require a priori knowledge of the Monge–Ampère eigenvalue λ_{MA} . The iterative method (7) solves for both the eigenfunction and eigenvalue simultaneously and thus requires no advance knowledge of λ_{MA} . Additionally, (7) provides a means to estimate λ_{MA} by computing the Rayleigh quotients $R(u_k)$ for k large. An approximation of the Monge–Ampère eigenvalue is of interest, as λ_{MA} is known to satisfy analogues of the classical Brunn–Minkowski, isoperimetric, and reverse isoperimetric inequalities; we refer to the works [3, 10, 12, 18] for the exact statements of these inequalities. It has also been noted in [3, 17] that λ_{MA} should determine the rate of extinction for a class of nonparametric surfaces flowing by the n th root of their Gauss curvature.

Second, the methods of Lions and Tso necessitate solving Dirichlet problems for Monge–Ampère equations of the form $\det D^2 u = f(u)$, where the right-hand side is some function f of the unknown u . The iteration (7), on the other hand, requires solving Dirichlet problems for Monge–Ampère equations of the form $\det D^2 u = g$, where the right-hand side g depends only on the previous iterate, hence is a known function. This makes (7) appealing from the point of view of numerical analysis. There is a vast literature on numerical methods for the Dirichlet problem for the Monge–Ampère equation and, more generally, fully nonlinear elliptic equations. We refer the reader to the recent survey [16] for an extensive overview.

Finally, let us recall that the Monge–Ampère operator can also be written in divergence form:

$$\det D^2 u = \frac{1}{n} \operatorname{div}(\Phi_u \nabla u),$$

where $\Phi_u(x)$ is the cofactor matrix of $D^2 u(x)$, given by $\det D^2 u(x)(D^2 u(x))^{-1}$ when $D^2 u(x)$ is invertible. An integration by parts shows that one can write the Rayleigh quotient (2) in the more familiar manner

$$R(u) = \frac{\frac{1}{n} \int_{\Omega} \langle \Phi_u \nabla u, \nabla u \rangle}{\int_{\Omega} (-u)^{n+1}}.$$

This form of the Rayleigh quotient suggests using appropriate versions of Poincaré and Sobolev-type inequalities (see [15, 19]) to prove Theorem 1.4. However, this would require explicit control of the cofactor matrix Φ_u at each step of the iteration, which is difficult as the smallest eigenvalue of $D^2 u$ degenerates near $\partial\Omega$, due to imposing the Dirichlet boundary condition. Our proof of Theorem 1.4 thus relies heavily on techniques for tackling nondivergence form equations and makes full use

of various fundamental attributes of convex functions and solutions of the Monge–Ampère equation.

Let us mention that Theorem 1.4 does not provide an independent proof of existence and uniqueness (up to scaling of the eigenfunction) of an eigenpair (u, λ) solving (1); it merely provides a computational method for obtaining the eigenfunction u of unit height and the eigenvalue λ_{MA} . In fact, the proof of Theorem 1.4 uses Theorem 1.3.

The rest of this paper is structured as follows: in Section 2 we state some basic properties of convex functions and the Monge–Ampère equation. The proof of the main result, Theorem 1.4, is carried out in Section 3.

2. BACKGROUND ON THE MONGE–AMPÈRE EQUATION

This section is devoted to stating some basic results on convex functions and weak solutions of the Monge–Ampère equation that will be used in the proof of Theorem 1.4. From here onward, we will assume that the domain Ω is bounded and convex.

Given a function $u \in C(\bar{\Omega})$, the subdifferential of u at $x \in \Omega$ is the set

$$\partial u(x) := \{p \in \mathbb{R}^n : u(y) \geq u(x) + p \cdot (y - x) \text{ for all } y \in \Omega\}.$$

If u is differentiable at x , then $\partial u(x) = \{\nabla u(x)\}$. Given a set $E \subset \Omega$, we define

$$\partial u(E) := \bigcup_{x \in \Omega} \partial u(x).$$

The Monge–Ampère measure of u is defined as

(8) $Mu(E) := \mathcal{L}^n(\partial u(E))$ for all $E \subset \Omega$ such that $\partial u(E)$ is Lebesgue measurable, where, \mathcal{L}^n denotes n -dimensional Lebesgue measure. It is well known that Mu is a Radon measure (see [8, Lemma 1.2.2]) and that if $u \in C^2(\Omega)$,

$$Mu(E) = \int_E \det D^2 u.$$

The following result shows that Monge–Ampère measures are stable under uniform convergence.

Lemma 2.1 (Weak convergence of Monge–Ampère measures; [8, Lemma 1.2.3] and [7, Proposition 2.6]). *If u_k are convex functions in Ω converging locally uniformly to a function u , then the associated Monge–Ampère measures Mu_k converge weakly to the measure Mu ; that is,*

$$\lim_{k \rightarrow \infty} \int_{\Omega} \varphi \, dMu_k = \int_{\Omega} \varphi \, dMu \quad \text{for all } \varphi \in C_c(\Omega).$$

Given a nonnegative Borel measure ν on Ω , we say that the convex function $u \in C(\Omega)$ is an *Aleksandrov solution* of $\det D^2 u = \nu$ in Ω if $Mu = \nu$ as measures. We also write $Mu \geq \nu$ in Ω (resp., $Mu \leq \nu$ in Ω) if $Mu(E) \geq \nu(E)$ (resp., $Mu(E) \leq \nu(E)$) for all Borel sets $E \subset \Omega$. If ν is absolutely continuous with respect to n -dimensional Lebesgue measure and has a density f , then we will write $\det D^2 u = f$.

We next state the interior gradient estimate, the Aleksandrov maximum principle, and the comparison principle for Aleksandrov solutions.

Lemma 2.2 (Interior gradient estimate; [8, Lemma 3.2.1]). *Suppose $u \in C(\overline{\Omega})$ is convex and vanishes on $\partial\Omega$. Then*

$$(9) \quad |p| \leq \frac{\sup_{\Omega} |u|}{\text{dist}(x, \partial\Omega)} \quad \text{for all } x \in \Omega, p \in \partial u(x).$$

Theorem 2.3 (Aleksandrov maximum principle; [8, Theorem 1.4.2]). *Suppose $u \in C(\overline{\Omega})$ is convex and vanishes on $\partial\Omega$. Then there exists a constant $C_n > 0$ depending only on the dimension n such that*

$$(10) \quad |u(x)|^n \leq C_n \text{diam}(\Omega)^{n-1} \text{dist}(x, \partial\Omega) M u(\Omega) \quad \text{for all } x \in \Omega.$$

Lemma 2.4 (Comparison principle; [8, Theorem 1.4.6]). *Suppose $u, v \in C(\overline{\Omega})$ are convex and satisfy $u \geq v$ on $\partial\Omega$ and $M u \leq M v$ in Ω . Then $u \geq v$ in Ω .*

The following result due to Hartenstine [9] shows that the Dirichlet problem for the Monge–Ampère equation on any bounded, convex domain with zero boundary data always has a unique Aleksandrov solution; see also [7, Theorem 2.1.3].

Theorem 2.5 (Solvability of Dirichlet problem; [9, Theorem 1]). *Given a Borel measure ν with $\nu(\Omega) < \infty$, there exists a unique convex function $u \in C(\overline{\Omega})$ that is an Aleksandrov solution of the Dirichlet problem*

$$\begin{cases} \det D^2 u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Aleksandrov solutions of the Dirichlet problem with zero boundary conditions are closed under uniform limits, as shown by the following lemma.

Lemma 2.6 (Stability of Aleksandrov solutions; [7, Proposition 2.12]). *Let $\{\nu_k\}$ be a sequence of Borel measures in Ω such that $\sup_k \nu_k(\Omega) < \infty$, and let $u_k \in C(\overline{\Omega})$ be Aleksandrov solutions of the Dirichlet problem*

$$\begin{cases} \det D^2 u_k = \nu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

If ν_k converges weakly to a Borel measure ν on Ω , then u_k converges locally uniformly to the Aleksandrov solution u of the Dirichlet problem

$$\begin{cases} \det D^2 u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

A hallmark result in the theory of Monge–Ampère equations is the strict convexity and regularity of Aleksandrov solutions established by Caffarelli in the seminal works [4, 6]. We summarize these important contributions as follows.

Theorem 2.7 (Regularity results for Aleksandrov solutions; see also [7, Corollaries 4.11, 4.21, and 4.43] and [8, Theorem 5.4.8]). *Let u be an Aleksandrov solution of the Dirichlet problem*

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Suppose there exist constants $C_1, C_2 > 0$ such that $C_1 \leq f \leq C_2$ in Ω . Then the following results hold:

- (i) *u is strictly convex and $u \in C_{loc}^{1,\alpha}(\Omega)$.*
- (ii) *If $f \in C^\alpha(\Omega)$, then $u \in C_{loc}^{2,\alpha}(\Omega)$.*
- (iii) *If $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

Standard bootstrap arguments using Theorem 2.7 show that Aleksandrov solutions of the Monge–Ampère eigenvalue problem are strictly convex and smooth in the interior (see [12, Proposition 2.8]).

Proposition 2.8 (Interior regularity). *Let $\sigma, p > 0$ be fixed constants. Suppose $u \in C(\overline{\Omega})$ is a nonzero Aleksandrov solution of the Dirichlet problem*

$$\begin{cases} \det D^2u = \sigma(-u)^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then u is strictly convex and $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$.

We next prove a continuity property of the Monge–Ampère energy, $I(u) = \int_{\Omega} (-u) dMu$ along a sequence of convex functions $\{v_k\}$ converging uniformly and satisfying uniform upper bounds on Mv_k with respect to Lebesgue measure (cf. [20, Proposition 1.1]).

Lemma 2.9. *Suppose $v_k \in C(\overline{\Omega})$ are convex functions converging uniformly on $\overline{\Omega}$ to a function v , and there exists a constant $\Lambda > 0$ such that $Mv_k \leq \Lambda \mathcal{L}^n$ for all $k \geq 0$. Then $\lim_{k \rightarrow \infty} I(v_k) = I(v)$.*

Proof. Let $\varphi \in C_c(\Omega)$ be arbitrary. We have

$$\begin{aligned} & \left| \int_{\Omega} \varphi v \, dMv - \int_{\Omega} \varphi v_k \, dMv_k \right| \\ & \leq \left| \int_{\Omega} \varphi v \, dMv - \int_{\Omega} \varphi v \, dMv_k \right| + \left| \int_{\Omega} \varphi(v - v_k) \, dMv_k \right| \\ & \leq \left| \int_{\Omega} \varphi v \, dMv - \int_{\Omega} \varphi v \, dMv_k \right| + \|\varphi\|_{L^\infty(\Omega)} \|v - v_k\|_{L^\infty(\Omega)} Mv_k(\Omega) \\ & \leq \left| \int_{\Omega} \varphi v \, dMv - \int_{\Omega} \varphi v \, dMv_k \right| + \|\varphi\|_{L^\infty(\Omega)} \|v - v_k\|_{L^\infty(\Omega)} \Lambda \mathcal{L}^n(\Omega) \\ & =: A_k + B_k. \end{aligned}$$

By Lemma 2.1, we know $\lim_{k \rightarrow \infty} A_k = 0$, while $\lim_{k \rightarrow \infty} B_k = 0$ due to the uniform convergence of v_k to v . Therefore,

$$(11) \quad \lim_{k \rightarrow \infty} \int_{\Omega} \varphi v_k \, dMv_k = \int_{\Omega} \varphi v \, dMv \quad \text{for all } \varphi \in C_c(\Omega).$$

Now let $\epsilon > 0$ be fixed, and let Ω_ϵ be an open set such that $\Omega_\epsilon \Subset \Omega$ and $\mathcal{L}^n(\Omega \setminus \overline{\Omega_\epsilon}) \leq \epsilon$. Let $\psi_\epsilon \in C_c(\Omega)$ be such that $0 \leq \psi_\epsilon \leq 1$ in Ω and $\psi_\epsilon \equiv 1$ on $\overline{\Omega_\epsilon}$. Then, for any $k \geq 0$, we can write

$$\begin{aligned} & I(v_k) - I(v) \\ & = \int_{\Omega} v \, dMv - \int_{\Omega} v_k \, dMv_k \\ & = \int_{\Omega} \psi_\epsilon v \, dMv - \int_{\Omega} \psi_\epsilon v_k \, dMv_k + \int_{\Omega} (1 - \psi_\epsilon)v \, dMv - \int_{\Omega} (1 - \psi_\epsilon)v_k \, dMv_k \\ & = \int_{\Omega} \psi_\epsilon v \, dMv - \int_{\Omega} \psi_\epsilon v_k \, dMv_k + \int_{\Omega \setminus \overline{\Omega_\epsilon}} (1 - \psi_\epsilon)v \, dMv - \int_{\Omega \setminus \overline{\Omega_\epsilon}} (1 - \psi_\epsilon)v_k \, dMv_k. \end{aligned}$$

Since $Mv_k \leq \Lambda \mathcal{L}^n$ for all $k \geq 0$, the lower semicontinuity on open sets of the Monge–Ampère measure under uniform convergence (see [8, Lemma 1.2.2 (ii)]) implies $Mv(U) \leq \Lambda \mathcal{L}^n(U)$ for any open set $U \subset \Omega$. Therefore,

$$\begin{aligned} \left| \int_{\Omega \setminus \overline{\Omega}_\epsilon} (1 - \psi_\epsilon) v \, dMv \right| &\leq \|1 - \psi_\epsilon\|_{L^\infty(\Omega)} \|v\|_{L^\infty(\Omega)} Mv(\Omega \setminus \overline{\Omega}_\epsilon) \\ &\leq \|v\|_{L^\infty(\Omega)} \Lambda \mathcal{L}^n(\Omega \setminus \overline{\Omega}_\epsilon) \leq C_1 \epsilon, \end{aligned}$$

where $C_1 > 0$ is a constant independent of ϵ . Similarly,

$$\begin{aligned} \left| \int_{\Omega \setminus \overline{\Omega}_\epsilon} (1 - \psi_\epsilon) v_k \, dMv_k \right| &\leq \|1 - \psi_\epsilon\|_{L^\infty(\Omega)} \|v_k\|_{L^\infty(\Omega)} Mv_k(\Omega \setminus \overline{\Omega}_\epsilon) \\ &\leq \|v_k\|_{L^\infty(\Omega)} \Lambda \mathcal{L}^n(\Omega \setminus \overline{\Omega}_\epsilon) \leq C_2 \epsilon, \end{aligned}$$

where $C_2 > 0$ is a constant independent of ϵ and k . Therefore, there exists a constant $C > 0$ independent of k and ϵ such that

$$|I(v_k) - I(v)| \leq \left| \int_{\Omega} \psi_\epsilon v \, dMv - \int_{\Omega} \psi_\epsilon v_k \, dMv_k \right| + C\epsilon.$$

Consequently, by (II), we have

$$\limsup_{k \rightarrow \infty} |I(v_k) - I(v)| \leq C\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\lim_{k \rightarrow \infty} I(v_k) = I(v).$$

□

We conclude this section by showing that if $u \in C(\overline{\Omega})$ is convex and vanishes on $\partial\Omega$, then all L^p norms of u are comparable.

Lemma 2.10. *If $u \in C(\overline{\Omega})$ is convex and vanishes on $\partial\Omega$, then*

$$\frac{\|u\|_{L^\infty(\Omega)}}{n+1} \leq \left(\frac{1}{|\Omega|} \int_{\Omega} |u|^p \right)^{\frac{1}{p}} \leq \|u\|_{L^\infty(\Omega)} \quad \text{for all } p \geq 1.$$

Proof. The second inequality is trivial. For the first, we let K be the convex cone with base Ω , height $-\|u\|_{L^\infty(\Omega)}$, and vertex at the point where u achieves its minimum. Then $u \leq K \leq 0$ on Ω by convexity of u . It follows from Jensen's inequality that for any $p \geq 1$,

$$\left(\frac{1}{|\Omega|} \int_{\Omega} |u|^p \right)^{\frac{1}{p}} \geq \frac{1}{|\Omega|} \int_{\Omega} |u| \geq \frac{1}{|\Omega|} \int_{\Omega} |K| = \frac{\|u\|_{L^\infty(\Omega)}}{n+1}.$$

□

3. PROOF OF THEOREM 1.4

In this entire section, u_k , $k \geq 0$, will always denote the functions from the statement of Theorem 1.4. We begin the proof of Theorem 1.4 by introducing an important monotone decreasing quantity associated to the iteration (7).

Lemma 3.1.

$$(12) \quad R(u_{k+1}) \|u_{k+1}\|_{L^{n+1}(\Omega)}^n \leq R(u_k) \|u_k\|_{L^{n+1}(\Omega)}^n \quad \text{for all } k \geq 0.$$

Proof. Multiplying (7) by $-u_{k+1}$ and integrating yields

$$\int_{\Omega} (-u_{k+1}) dM u_{k+1} = R(u_k) \int_{\Omega} (-u_{k+1})(-u_k)^n.$$

Using the definition of $R(u_{k+1})$, we can rewrite the left-hand side to get

$$R(u_{k+1}) \|u_{k+1}\|_{L^{n+1}(\Omega)}^{n+1} = R(u_k) \int_{\Omega} (-u_{k+1})(-u_k)^n.$$

Then by Hölder's inequality

$$\int_{\Omega} (-u_{k+1})(-u_k)^n \leq \|u_{k+1}\|_{L^{n+1}(\Omega)} \|u_k\|_{L^{n+1}(\Omega)}^n,$$

and inequality (12) follows after dividing by $\|u_{k+1}\|_{L^{n+1}(\Omega)}$. \square

We now use the monotonicity relation (12) to prove a global Hölder estimate for the functions u_k solving (7).

Proposition 3.2. *There exists $C = C(n, \Omega, u_0) > 0$ such that for all $k \geq 1$, $u_k \in C^{0, \frac{1}{n}}(\overline{\Omega})$ with Hölder norm uniformly bounded by C .*

Proof. By Theorem 2.3 and (7), we have for any $k \geq 0$ and $x \in \Omega$

$$\begin{aligned} |u_{k+1}(x)|^n &\leq C_n \operatorname{diam}(\Omega)^{n-1} \operatorname{dist}(x, \partial\Omega) M u_{k+1}(\Omega) \\ &= C_n \operatorname{diam}(\Omega)^{n-1} \operatorname{dist}(x, \partial\Omega) R(u_k) \int_{\Omega} (-u_k)^n \\ &\leq C_n \operatorname{diam}(\Omega)^{n-1} \operatorname{dist}(x, \partial\Omega) R(u_k) \|u_k\|_{L^{n+1}(\Omega)}^n |\Omega|^{\frac{1}{n+1}} \\ &\leq \left(C_n \operatorname{diam}(\Omega)^{n-1} |\Omega|^{\frac{1}{n+1}} R(u_0) \|u_0\|_{L^{n+1}(\Omega)}^n \right) \operatorname{dist}(x, \partial\Omega), \end{aligned}$$

where we have used Hölder's inequality in the third line and the monotonicity relation (12) in the final step. In particular, there exists $C_1 = C_1(n, \Omega, u_0) > 0$ such that

$$\sup_{\Omega} |u_k| \leq C_1.$$

It follows from the interior gradient estimate Lemma 2.2 that u_k is uniformly Lipschitz on any compact subset of Ω . Next, since u_k vanishes on $\partial\Omega$, the estimate above yields a uniform $C^{0, \frac{1}{n}}$ estimate of u_k near $\partial\Omega$. Consequently, u_k is uniformly $\frac{1}{n}$ -Hölder continuous in $\overline{\Omega}$. \square

The next proposition establishes a uniform lower bound for $\|u_k\|_{L^\infty(\Omega)}$.

Proposition 3.3. $\|u_k\|_{L^\infty(\Omega)} \geq \lambda_{MA}^{-1/n}$ for all $k \geq 0$.

Proof. Let $\hat{u} \in \mathcal{K} \cap C^\infty(\Omega)$ be the solution of (11) satisfying $\|\hat{u}\|_{L^\infty(\Omega)}^n = \lambda_{MA}^{-1}$, which exists by Theorem 1.3. We prove by induction that $\hat{u} \geq u_k$ for each $k \geq 0$. To establish the base case, we recall that $M u_0 \geq \mathcal{L}^n$. Therefore, if $E \subset \Omega$ is any Borel set,

$$M \hat{u}(E) = \lambda_{MA} \int_E (-\hat{u})^n \leq \lambda_{MA} \lambda_{MA}^{-1} \mathcal{L}^n(E) \leq M u_0(E).$$

Since $\hat{u} = 0$ on $\partial\Omega$ and $u_0 \leq 0$ on $\partial\Omega$, it follows from the comparison principle Lemma 2.4 that $\hat{u} \geq u_0$ in Ω .

Now suppose $\hat{u} \geq u_k$ on Ω for some $k \geq 0$. Then for any Borel $E \subset \Omega$, we have by the characterization of λ_{MA} in Theorem 1.3,

$$Mu_{k+1}(E) = R(u_k) \int_E (-u_k)^n \geq \lambda_{MA} \int_E (-u_k)^n \geq \lambda_{MA} \int_E (-\hat{u})^n = M\hat{u}(E).$$

Since $u_{k+1} = \hat{u} = 0$ on $\partial\Omega$, it follows from the comparison principle Lemma 2.4 that $\hat{u} \geq u_{k+1}$ in Ω . \square

Applying Proposition 3.3 and Lemma 2.10 to the monotonicity relation (12) provides an upper bound for the Rayleigh quotients $R(u_k)$.

Corollary 3.4. *There exists a positive constant C depending only on $n, \mathcal{L}^n(\Omega)$, λ_{MA} , and u_0 such that $R(u_k) \leq C$ for all $k \geq 1$.*

We are now ready to prove the main theorem.

Proof of Theorem 1.4. By Proposition 3.2, the sequence $\{u_k\}_{k=1}^\infty$ is uniformly bounded and equicontinuous. Consequently, by the Arzelà–Ascoli theorem, it is possible to choose a subsequence $\{k(j)\}_{j \in \mathbb{N}}$ of indices such that $\{u_{k(j)}\}_{j=1}^\infty$ converges uniformly on $\overline{\Omega}$ to a convex function $u_\infty \in C(\overline{\Omega})$ with $u_\infty \equiv 0$ on $\partial\Omega$, while the shifted sequence $\{u_{k(j)+1}\}_{j=1}^\infty$ converges uniformly on $\overline{\Omega}$ to a convex function $w_\infty \in C(\overline{\Omega})$ with $w_\infty \equiv 0$ on $\partial\Omega$. Proposition 3.3 implies u_∞ and w_∞ are not identically zero. Therefore, $u_\infty, w_\infty \in \mathcal{K}$.

We verify that the corresponding Rayleigh quotients also converge. Indeed, Proposition 3.2 and Corollary 3.4 show that there exists a constant $\Lambda > 0$ independent of k such that $Mu_k \leq \Lambda \mathcal{L}^n$ in Ω for all $k \geq 1$. Therefore we can apply Lemma 2.9 and Proposition 3.3 to conclude that $\lim_{j \rightarrow \infty} R(u_{k(j)}) = R(u_\infty)$ and $\lim_{j \rightarrow \infty} R(u_{k(j)+1}) = R(w_\infty)$.

Next, Lemma 2.1 implies that the measures $\nu_j := R(u_{k(j)})(-u_{k(j)})^n \mathcal{L}^n$ converge weakly to the measure $\nu := R(u_\infty)(-u_\infty)^n \mathcal{L}^n$ as $j \rightarrow \infty$. Furthermore, Proposition 3.2 and Corollary 3.4 imply $\sup_j \nu_j(\Omega) < \infty$. Since $Mu_{k(j)+1} = \nu_j$ and $u_{k(j)+1}$ converge uniformly to w_∞ , we may apply Lemma 2.6 to conclude that $\det D^2 w_\infty = R(u_\infty)(-u_\infty)^n$ in the Aleksandrov sense.

We claim $w_\infty = u_\infty$. By the monotonicity relation (12), we have

$$\begin{aligned} R(u_{k(j+1)}) \|u_{k(j+1)}\|_{L^{n+1}(\Omega)}^n &\leq R(u_{k(j)+1}) \|u_{k(j)+1}\|_{L^{n+1}(\Omega)}^n \\ &\leq R(u_{k(j)}) \|u_{k(j)}\|_{L^{n+1}(\Omega)}^n, \quad j \in \mathbb{N}. \end{aligned}$$

Letting $j \rightarrow \infty$, we conclude that

$$(13) \quad R(w_\infty) \|w_\infty\|_{L^{n+1}(\Omega)}^n = R(u_\infty) \|u_\infty\|_{L^{n+1}(\Omega)}^n.$$

On the other hand, multiplying the equation $\det D^2 w_\infty = R(u_\infty)(-u_\infty)^n$ by $-w_\infty$ and integrating yields

$$\begin{aligned} R(w_\infty) \|w_\infty\|_{L^{n+1}(\Omega)}^{n+1} &= \int_{\Omega} (-w_\infty) \, dMw_\infty \\ &= R(u_\infty) \int_{\Omega} (-w_\infty)(-u_\infty)^n \\ &\leq R(u_\infty) \|w_\infty\|_{L^{n+1}(\Omega)} \|u_\infty\|_{L^{n+1}(\Omega)}^n \quad \text{by Hölder's inequality} \\ &= R(w_\infty) \|w_\infty\|_{L^{n+1}(\Omega)}^{n+1} \quad \text{by (13).} \end{aligned}$$

This shows that we have equality in Hölder's inequality, and so there exists a constant $c > 0$ such that $(-w_\infty)^{n+1} = c(-u_\infty)^{n+1}$. In particular, $R(u_\infty) = R(w_\infty)$. It follows from (I3) that $c = 1$, and consequently, $w_\infty = u_\infty$. Since $\det D^2 u_\infty = R(u_\infty)(-u_\infty)^n$ in the Aleksandrov sense, Theorem I.3 implies u_∞ is a Monge–Ampère eigenfunction and $R(u_\infty) = \lambda_{MA}$.

We next show that the full sequence $\{u_k\}_{k=1}^\infty$ converges to the same eigenfunction u_∞ . Indeed, suppose $\{u_{k_1(j)}\}_{j=1}^\infty$ and $\{u_{k_2(j)}\}_{j=1}^\infty$ are two subsequences of $\{u_k\}_{k=1}^\infty$ converging uniformly to $u_{1,\infty}$ and $u_{2,\infty}$, respectively. By the argument outlined in the preceding paragraphs, both $u_{1,\infty}$ and $u_{2,\infty}$ are eigenfunctions and $R(u_{1,\infty}) = R(u_{2,\infty}) = \lambda_{MA}$. We construct two new subsequences $\{u_{i_1(j)}\}_{j=1}^\infty$ and $\{u_{i_2(j)}\}_{j=1}^\infty$ by setting $i_1(1) = k_1(1)$, and then inductively defining

$$\begin{aligned} i_2(j) &= \min_l \{k_2(l) \mid k_2(l) > i_1(j)\}, \quad j \geq 1, \\ i_1(j) &= \min_l \{k_1(l) \mid k_1(l) > i_2(j-1)\}, \quad j \geq 2. \end{aligned}$$

Clearly $\{u_{i_1(j)}\}_{j=1}^\infty$ and $\{u_{i_2(j)}\}_{j=1}^\infty$ converge uniformly to the original limits $u_{1,\infty}$ and $u_{2,\infty}$ respectively, while $i_1(j) < i_2(j)$ and $i_2(j) < i_1(j+1)$ for all j . Thus by repeated application of the monotonicity relation (I2), we find

$$\begin{aligned} R(u_{i_2(j)}) \|u_{i_2(j)}\|_{L^{n+1}(\Omega)}^n &\leq R(u_{i_1(j)}) \|u_{i_1(j)}\|_{L^{n+1}(\Omega)}^n, \\ R(u_{i_1(j+1)}) \|u_{i_1(j+1)}\|_{L^{n+1}(\Omega)}^n &\leq R(u_{i_2(j)}) \|u_{i_2(j)}\|_{L^{n+1}(\Omega)}^n. \end{aligned}$$

Taking $j \rightarrow \infty$ in both inequalities above and then dividing by λ_{MA} yields $\|u_{1,\infty}\|_{L^{n+1}(\Omega)} = \|u_{2,\infty}\|_{L^{n+1}(\Omega)}$. Since both $u_{1,\infty}$ and $u_{2,\infty}$ are eigenfunctions, they must be multiples of each other; this shows they are equal. Since this equality holds for any arbitrary pair of subsequences $\{u_{k_1(j)}\}_{j=1}^\infty$ and $\{u_{k_2(j)}\}_{j=1}^\infty$ of $\{u_k\}_{k=1}^\infty$, the entire sequence $\{u_k\}_{k=1}^\infty$ must converge uniformly to the same eigenfunction u_∞ .

Finally, since $\|u_k\|_{L^\infty(\Omega)}$ is uniformly bounded away from zero by Proposition 3.3, we see the sequence $\{\frac{u_k}{\|u_k\|_{L^\infty(\Omega)}}\}$ converges uniformly to the unique eigenfunction with L^∞ norm equal to 1, finishing the proof. \square

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