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Optimal excess-of-loss reinsurance contract with ambiguity aversion in the principal-agent model

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ABSTRACT
We discuss an optimal excess-of-loss reinsurance contract in a continuous-time principal-agent framework where the surplus of the insurer (agent/he) is described by a classical Cramér-Lundberg (C-L) model. In addition to reinsurance, the insurer and the reinsurer (principal/she) are both allowed to invest their surpluses into a financial market containing one risk-free asset (e.g. a short-rate account) and one risky asset (e.g. a market index). In this paper, the insurer and the reinsurer are ambiguity averse and have specific modeling risk aversion preferences for the insurance claims (this relates to the jump term in the stochastic models) and the financial market's risk (this encompasses the models' diffusion term). The reinsurer designs a reinsurance contract that maximizes the exponential utility of her terminal wealth under a worst-case scenario which depends on the retention level of the insurer. By employing the dynamic programming approach, we derive the optimal robust reinsurance contract, and the value functions for the reinsurer and the insurer under this contract. In order to provide a more explicit reinsurance contract and to facilitate our quantitative analysis, we discuss the case when the claims follow an exponential distribution; it is then possible to show explicitly the impact of ambiguity aversion on the optimal reinsurance.

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Principal-agent; excess-of-loss reinsurance; robust control; optimal investment strategy; dynamic programming

1. Introduction

Reinsurance and investments are important measures for insurers and reinsurers to manage their companies' risk positions. An insurer can employ reinsurance to control the amount of risk exposure, and can use risk-free and risky investments to improve his profit levels. A reinsurer generates income and controls risk by adjusting the premium for reinsurance contracts, and by investing in markets like the insurer. In this paper we simply assume that the insurer pays a premium to the reinsurer at an agreed-upon rate for a given level of risk retention, and the reinsurer pays a proportion of the claims amounts faced by the insurer, where that proportion is the complement of the insurer's retention level. Typically, the distribution of premia and claims is agreed to in advance, between the insurer and reinsurer. The distribution of claims depends on the actual claims amounts. In calculating the risk premium, i.e. the price of the reinsurance contract, the reinsurer can and should take into account any

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historical knowledge of claims in prior years, as well as any expectations for future losses, depending on the kind of risks involved. But the reinsurer must also consider the demand of the insurer, since an arbitrary supply may not be needed. Said differently, unlike typical stochastic models for equity prices in financial markets, one should not look at the pricing question for reinsurers by assuming an infinitely liquid reinsurance market. Instead, in financial terms, the reinsurance contract price should be determined in a highly micro-economic fashion where the inherent risks of the insurance and financial markets are external forces used to determine a mutually optimal agreement between a single reinsurer and a single insurer. To do this, assumptions about each of the parties' risk aversions are needed. This agent-demand framework is that on which we place our emphasis in this paper. We discuss the impact of this demand on the mutually optimal reinsurance premium. We also investigate the optimal reinsurance-investment strategy for the insurer, and the optimal investment strategy for the reinsurer, depending on their levels of ambiguity aversion. This notion of aversion is distinct from risk aversion: it represents a player's insecurity regarding their ability to work with a well-specified and estimated model. It leads to increased robustness in decision making, which we take advantage of in this paper.

Reinsurance and investment problems for the insurer have attracted many scholars' attention. They mostly work on various optimization problems with different objectives from the insurer's perspective. For example, some scholars focus on maximizing the expected utility of insurers' terminal wealth, we refer readers to Browne (1995), Bai & Guo (2008), Gu et al. (2012) and so on; some focus on mean-variance criteria and ruin-probability criteria, such as Bäuerle (2005), Zeng et al. (2010), Zeng & Li (2011), Li et al. (2012), Luo et al. (2008), Luo (2009), Chen et al. (2010) and Peng & Wang (2018) for ruin probability criteria. All the aforementioned works assume that the premia for insurance and reinsurance are constant. Dynamic pricing for insurance is also discussed. Young & Zariphopoulou (2002) employ dynamic prices for insurance risk by applying the principle of equivalent utility; Emms & Haberman (2005) and Emms et al. (2007) study the optimal premium pricing policies in a competitive insurance environment by using approximation methods and simulation of sample paths, with the premium rate charged by the insurer as a control variable; Henriet et al. (2016) develop a continuous-time general-equilibrium model to rationalize the dynamics of insurance prices in a competitive insurance market, where they use the variance principle to calculate the price of insurance, and employ dynamic safety loading for the insurance prices. These works all focus on insurance pricing. Even though some of them refer to reinsurance pricing as well, they all take the perspective of the insurer. This perspective is a legitimate one; for instance, it is documented in a report by Swiss (2002), where they show that ceding companies react to reinsurance price increases by buying less coverage and inversely, when reinsurance rates fall, buyers reduce retention, extend their ceded lines, and increase coverage for their clients.

To the best of our knowledge, little attention, if any, is paid to using the perspective of the reinsurer in computing the price of reinsurance. In agreement with Henriet et al. (2016) and Swiss (2002), we consider that the safety loading parameter, i.e. the reinsurance premium, represents the price of reinsurance. Herein we investigate the reaction by the reinsurer to the demand for reinsurance. We interpret the latter as the insurer's reinsurance strategy. This allows us to look at a single pair of insurer and reinsurer while still capturing the notion of supply and demand. No aggregation is needed to represent a market; this reflects the fact that the reinsurance market is not a public market, but rather one in which client-specific agreements are developed for each contract. The reinsurer should decrease prices when demand is low, i.e. when the insurer wants to increase their insurance-risk exposure, and she should raise her prices of providing reinsurance coverage when demand is high. Similar research has been done by few scholars. Chen & Shen (2018) employ a new continuous-time framework to analyze optimal reinsurance, in which an insurer and a reinsurer are two players in a stochastic Stackelberg differential game; Hu et al. (2018a, 2018b) employ a principal-agent model to study the optimal reinsurance premium from the viewpoint of the reinsurer, where proportional reinsurance and excess-of-loss reinsurance are discussed. We work in a similar framework, since we also consider a principal-agent model, which is essentially a Stackelberg game between the insurer and the reinsurer.

In this paper, we incorporate ambiguity aversion for both the insurer and the reinsurer. As alluded to above, the notion of ambiguity was developed as a way of addressing modeling uncertainty. This was designed originally to face the fact that mean rates of return and other drift parameters in stochastic models for risky assets are difficult to estimate, and their misspecification has measurable impacts on investment strategies. Ambiguity aversion is a way to define the investors' attitude toward this uncertainty. The bigger the ambiguity aversion is, the less confident is the investor that they are working with an appropriate model. As a result, ambiguity forces the investor to make a different decision compared to the ambiguity-neutral investor. Typically, this means forfeiting some level of utility in favor of guarding against the risk of following a significantly misguided strategy. Uppal & Wang (2003) develop a framework which allows investors to consider their level of ambiguity. Later, a systematic method in quantitative investment finance for portfolio selection and asset pricing with model uncertainty or model misspecification is developed. We refer readers to Maenhout (2004, 2006): they optimize an inter-temporal consumption problem with ambiguity, and derive closed-form expressions for the optimal strategies which are highly robust to drift misspecifications. Similar investigations on ambiguity is performed for the insurance market, because the same ambiguity exists in the expected value of the insurer's surplus due to uncertainty on how to specify the claim dynamics. Under a mean-variance criterion, Yi et al. (2015) study the optimal proportional reinsurance-investment strategy; Zeng et al. (2016) study the equilibrium strategy of a robust optimal reinsurance-investment problem; using an expected utility criterion, Gu et al. (2017, 2018) investigate the optimal proportional reinsurance-investment problem with mispricing; Li et al. (2018) consider the optimal excess-of-loss reinsurance-investment. These papers are all written from the perspective of the insurer; they discuss optimal reinsurance strategies, but always assume that the reinsurance premium is constant. Recently, Hu et al. (2018a, 2018b) take the reinsurer's perspective and study the optimal reinsurance premium based on insurer demand. Hu et al. (2018a) study the proportional reinsurance contract with the insurer or the reinsurer being ambiguity-averse; Hu et al. (2018b) discuss the proportional reinsurance contract and the excess-of-loss reinsurance contract only in the case where the reinsurer is ambiguity-averse or not. But as we said, for excess-of-loss reinsurance, Hu et al. (2018b) do not consider that the insurer might be averse to ambiguity. In this sense, one cannot have confidence that those strategies are robust to all misspecifications in drift parameters. It should also be noted that Hu et al. (2018a, 2018b) do not allow surplus investments into risky markets.

Following the research of Hu et al. (2018a, 2018b), but in an effort to guard against the lack of robustness due to all misspecifications that we can handle, we incorporate ambiguity aversion into the problem, we focus on the excess-of-loss reinsurance and we study the optimal reinsurance-investment strategy for the insurer, the optimal reinsurance premium, and the optimal investment strategy for the reinsurer. Sequentially, to solve the mathematics behind these questions, first, the insurer defines his optimization problem and obtains his optimal reinsurance-investment strategy. Based on the insurer's optimal reinsurance strategy, the reinsurer computes an optimal reinsurance premium in order to maximize the expected utility of her terminal wealth. Based on the reality of similar investment patterns in highly liquid publicly-traded equities, we assume that the insurer and the reinsurer invest their surpluses into the risk-free asset and a same risky asset. To quantify the effect of ambiguity aversion on the insurer's and the reinsurer's optimal decisions, we assume that there exists uncertainty in the intensity of the claim and drift misspecifications in the risky asset. Also, the insurer and the reinsurer have different ambiguity levels, to signify their possibly distinct aversions to, and/or skills in, modeling uncertainty. In our conclusions, we find that the insurer will pay little attention to the reinsurer's ambiguity aversion level. Inversely, the reinsurer will pay rather close attention to the insurer's ambiguity aversion, and to her own aversion, making both levels critical factors in the reinsurer's decisions. This in itself is a justification for providing our framework, as an extension of some results of Hu et al. (2018b) with important practical implications, in which ambiguity aversion should not be ignored.

To summarize, compared to the existing literature, the contributions of this paper are twofold. First, we discuss the excess-of-loss reinsurance in a principal-agent model, where the insurer and the reinsurer are ambiguity averse in an effort to devise more robust strategies; we consider their optimal investment strategies where the financial market is composed of a risk-free asset and a risky asset. Second, we discuss the effect of the insurer's ambiguity aversion on the reinsurer's optimal decision, and the impact of the reinsurer's ambiguity aversion on the insurer's optimal reinsurance strategy. Under this new framework some useful results are uncovered. For example, the insurer need to concern himself more with his own ambiguity aversion, but the reinsurer must pay close attention to the insurer's ambiguity aversion, more than to her own.

This paper is structured as follows. In Section 2, we introduce our insurance model and the financial market, and we form the agent's problem and the principal's problem with ambiguity aversion. Optimal excess-of-loss reinsurance contracts with claims following the exponential distribution are derived in Section 3. In Section 4, we provide a numerical analysis to show the impact of ambiguity aversion and other parameters on the reinsurance strategies and premium, as well as some loss-of-utility calculations in some cases. All technical proofs are relegated to an Appendix.

2. Model setup

In this section, we formulate optimal excess-of-loss reinsurance-investment problems for the insurer and the reinsurer in a principal-agent framework where the insurer is the agent and the reinsurer is the principal. The framework is related to a Stackelberg game, where the reinsurer and the insurer are the leader and the follower, respectively. We adopt this framework to study the optimal reinsurance contact. Assuming first that the insurer and the reinsurer are both ambiguity averse, it is then immediate to cover the special cases where only one player is ambiguity averse or neither players are ambiguity averse. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ be a complete probability space satisfying the usual condition, where T is a finite and positive constant representing the investment-time horizon, and the filtration $\mathcal{F}_t$ stands for the information available until time t. All stochastic processes introduced below are assumed to be adapted processes in this space, which means that we assume full and immediate observation for all of them.

Throughout, we use the classical Cramér-Lundberg (C-L) model (see for example Bäuerle 2005) to describe the insurer's surplus $R(t)$, i.e.

$$R(t) = x + (1+\eta)\lambda\mu_\infty t - \sum_{i=1}^{N_t} Z_i,$$

where $x \geq 0$ is the initial surplus; $\{N_t\}_{0\leq t\leq T}$ is a homogeneous Poisson process with intensity $\lambda > 0$ and $\sum_{i=1}^{N_t} Z_i$ is a compound Poisson process representing the cumulative amount of claims in time interval $[0, t]$. Here Z_i represents the size of the ith claim and N_t represents the number of claims up to time t. We assume that $Z_1, Z_2, \ldots$ are independent and identically distributed (i.i.d.) positive random variables with common distribution $F : [0, +\infty) \to [0, 1]$, and have finite first-order moment μ_∞ and second-order moment σ_∞^2. Under the expected value principle, $(1+\eta)\lambda\mu_\infty$ is the premium rate for the insurance, where $\eta > 0$ is the relative safety loading of the insurer.

We assume throughout that the insurer has the option to purchase excess-of-loss reinsurance. Let $a(t) \geq 0$ be an excess-of-loss retention level at any time t, and $Z_i^{a(t)} := \min\{Z_i, a(t)\}$ for $i = 1, 2, \ldots$, denote the value of the ith claim retained by the insurer if that claim occurs at time t. Then the reserve of the cedent becomes

$$\overline{R}_t^{a(t)} = x + p^{a(t)} t - \sum_{i=1}^{N_t} Z_i^{a(t)},$$

in which the net premium rate $p^{a(t)}$ is given by

$$\begin{aligned} p^{a(t)} &:= (1+\eta)\lambda\mu_\infty - (1+\theta(t))\lambda(\mu_\infty - E[Z_i^{a(t)}]) \\ &= (\eta - \theta(t))\lambda\mu_\infty + \lambda(1+\theta(t))E[Z_i^{a(t)}], \end{aligned}$$

where $\theta(t)$ denotes the safety loading of the reinsurer at time t. Here, we assume that it always satisfies the condition $\theta(t) \geq \eta$ for the insurer's safety loading η, that is, reinsurance is not arbitrarily inexpensive or reinsurance is non-cheap. This lower bound constraint is needed to ensure that the reinsurer accepts the insurance business from the insurer. It can be interpreted as a level below which the reinsurer would be operating at an unacceptable business loss level. Motivated by Emms (2007a, 2007b) and Hu et al. (2018a), we interpret the traditional relative safety loading of reinsurance to represent the reinsurance price (or premium). As alluded to in the introduction, it is important to note that in our framework, the reinsurance premium is no longer a constant, and is expected to change over time instead. Moreover, the reinsurance safety loading $\theta(t)$ will be seen as a decision variable (control) for the reinsurer, as we will see.

For the reinsurance business, the stochastic jump dynamics for the reinsurer's surplus are given by

$$\mathrm{d}W(t) = (1+\theta(t))\lambda(\mu_\infty - E[Z_i^{a(t)}])\,\mathrm{d}t - d\sum_{i=1}^{N_t}(Z_i - Z_i^{a(t)}).$$

In addition to reinsurance, we assume that the insurer and the reinsurer are allowed to invest their surpluses in a financial market consisting of one risk-free asset (i.e. a short-rate account) and one risky asset (i.e. a stock).

The price process $S_0 := \{S_0(t)\}_{0\leq t\leq T}$ of the risk-free asset is described by

$$\mathrm{d}S_0(t) = rS_0(t)\,\mathrm{d}t,$$

where $r > 0$ is the risk-free interest rate. The price process $S := \{S(t)\}_{0\leq t\leq T}$ of the risky asset is given by

$$\frac{\mathrm{d}S(t)}{S(t)} = (r+\mu)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \tag{1}$$

where $\mu > 0$ is the expected excess return rate of the risky asset and $\sigma > 0$ is the instantaneous volatility. The process $\{B(t)\}_{0\leq t\leq T}$ is a standard Brownian motion, which is assumed to be independent of $\sum_{i=1}^{N_t} Z_i$. Denote by $u(t)$ and $\hat{u}(t)$ the proportions of the wealth invested in the risky asset of the insurer and the reinsurer, respectively. The remainders $1-u(t)$ and $1-\hat{u}(t)$ are invested into the risk-free asset. Thus we call $\pi(t) := (a(t), u(t))$ and $\hat{\pi}(t) := (\theta(t), \hat{u}(t))$ the reinsurance-investment strategy for the insurer and the reinsurer, respectively, where the excess-of-loss retention level $a(t) \geq 0$ and the reinsurer's safety loading $\theta(t) \geq \eta$.

As mentioned, we assume that the insurer and the reinsurer are ambiguity averse, implying that they are concerned about the accuracy of statistical estimation, and possible misspecification errors, in their reference models. In order to describe and incorporate this ambiguity aversion into our model, we define a set of prior probability measures parameterized by ψ and $\hat{\psi}$, where $\psi := \{\psi(t) = (h(t), \phi(t))\}_{0\leq t\leq T}$ and $\hat{\psi} := \{\hat{\psi}(t) = (\hat{h}(t), \hat{\phi}(t))\}_{0\leq t\leq T}$. Denote by Ψ and $\hat{\Psi}$ the spaces of ψ and $\hat{\psi}$

satisfying

$$\exp\left\{\int_t^T \frac{h^2(s)}{2}\,\mathrm{d}s + \int_t^T\int_0^{+\infty}(\phi(s)\ln\phi(s)-\phi(s)+1)\lambda\,\mathrm{d}F(z)\,\mathrm{d}s\right\} < \infty,$$

and

$$\exp\left\{\int_t^T \frac{\hat{h}^2(s)}{2}\,\mathrm{d}s + \int_t^T\int_0^{+\infty}(\hat{\phi}(s)\ln\hat{\phi}(s)-\hat{\phi}(s)+1)\lambda\,\mathrm{d}F(z)\,\mathrm{d}s\right\} < \infty,$$

for any $t \in [0,T]$, respectively. As the two inequalities are the same, the two spaces are identical, i.e. $\Psi = \hat{\Psi}$.

For any $\psi = (h,\phi) \in \Psi$ and $\hat{\psi} = (\hat{h},\hat{\phi}) \in \hat{\Psi}$, we can define new alternative measures $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ equivalent to the reference model $\mathbb{P}$, via their Radon-Nykodym derivatives, as follows:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \xi(T), \quad \frac{\mathrm{d}\hat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \hat{\xi}(T)$$

where $\xi(t)$ and $\hat{\xi}(t)$ satisfy

$$\begin{aligned}\frac{\mathrm{d}\xi(t)}{\xi(t)} &= -h(t)\,\mathrm{d}B(t) + \int_0^{+\infty}(\phi(t)-1)\tilde{N}(\mathrm{d}t,\mathrm{d}z),\\ \frac{\mathrm{d}\hat{\xi}(t)}{\hat{\xi}(t)} &= -\hat{h}(t)\,\mathrm{d}B(t) + \int_0^{+\infty}(\hat{\phi}(t)-1)\tilde{N}(\mathrm{d}t,\mathrm{d}z).\end{aligned}$$

Whenever needed, we use $\mathbb{Q}(\psi)$ and $\hat{\mathbb{Q}}(\hat{\psi})$ to highlight the dependence of $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ on ψ and $\hat{\psi}$. Denote by $E^{\psi}[\cdot] = E^{\mathbb{Q}(\psi)}[\cdot]$ and $E^{\hat{\psi}}[\cdot] = E^{\hat{\mathbb{Q}}(\hat{\psi})}[\cdot]$ the expectations taken under $\mathbb{Q}$ and $\hat{\mathbb{Q}}$, respectively.

Above, $\tilde{N}(\cdot,\cdot)$ is the compensated measure of $N(\cdot,\cdot)$. In fact, we can rewrite the total retained losses at time t, and its mean value, as

$$\begin{aligned}\sum_{i=1}^{N_t} Z_i^{a(t)} &= \int_0^t\int_0^{+\infty}\min(z,a(t))N(\mathrm{d}t,\mathrm{d}z), \quad \forall\, t\in[0,T],\\ E\left[\sum_{i=1}^{N_t} Z_i^{a(t)}\right] &= \int_0^t\int_0^{+\infty}\min(z,a(t))\vartheta(\mathrm{d}t,\mathrm{d}z), \quad \forall\, t\in[0,T],\end{aligned}$$

where $\vartheta(\cdot,\cdot)$ is the compensator of the random measure $N(\cdot,\cdot)$, so the compensated measure $\tilde{N}(\mathrm{d}t,\mathrm{d}z) := N(\mathrm{d}t,\mathrm{d}z) - \vartheta(\mathrm{d}t,\mathrm{d}z)$ is a martingale, where $\vartheta(\mathrm{d}t,\mathrm{d}z)$ is simply equal to $\lambda\,\mathrm{d}t\,\mathrm{d}F(z)$.

Based on Girsanov's Theorem, under the new measures $\mathbb{Q}$ and $\hat{\mathbb{Q}}$, the following processes $\{B^{\mathbb{Q}}(t)\}_{0\le t\le T}$ and $\{B^{\hat{\mathbb{Q}}}(t)\}_{0\le t\le T}$ are two standard Brownian motions and satisfy

$$\mathrm{d}B^{\mathbb{Q}} = \mathrm{d}B(t) + h(t)\,\mathrm{d}t, \quad \mathrm{d}B^{\hat{\mathbb{Q}}} = \mathrm{d}B(t) + \hat{h}(t)\,\mathrm{d}t;$$

the Poisson random measures under $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ differ from $N(\mathrm{d}t,\mathrm{d}z)$, i.e. that under the reference measure $\mathbb{P}$ only via their respective claim intensities $\lambda^{\mathbb{Q}} := \lambda\phi(t)$ and $\lambda^{\hat{\mathbb{Q}}} := \lambda\hat{\phi}(t)$; the probability law of Z_i remains the same under $\mathbb{P}$, $\mathbb{Q}$ and $\hat{\mathbb{Q}}$.

We may now interpret the measures $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ as the alternative measures for the insurer and the reinsurer, respectively. By this we mean that, since these players are ambiguity averse, they are both wary of the consequences of what would happen if the financial and insurance markets followed these alternative measures instead of their original reference measure $\mathbb{P}$. Here, we concentrate on the following spaces of alternative probability measures:

$$\mathcal{Q} := \{\mathbb{Q}(\psi) \mid \mathbb{Q}(\psi) \sim \mathbb{P} \text{ and } \psi \in \Psi\}, \quad \hat{\mathcal{Q}} := \{\hat{\mathbb{Q}}(\hat{\psi}) \mid \hat{\mathbb{Q}}(\hat{\psi}) \sim \mathbb{P} \text{ and } \hat{\psi} \in \hat{\Psi}\}.$$

Note that the notational difference between $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ does not imply they are mathematically different; the different notations for $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ just show they are the spaces from which the two different players, i.e. the insurer and the reinsurer, choose alternative probability measures. Since $\Psi = \hat{\Psi}$, indeed, the two spaces are also the same, i.e. $\mathcal{Q} = \hat{\mathcal{Q}}$.

Therefore, under the new measure $\mathbb{Q}(\psi)$, with the reinsurance-investment strategy π, the surplus of the insurer can be described by the following equation:

$$\begin{aligned} dX^{\pi,\psi}(t) &= X^{\pi,\psi}(t)(r + \mu u(t))\,\mathrm{d}t + X^{\pi,\psi}(t)u(t)\sigma(\mathrm{d}B^{\mathbb{Q}}(t) - h(t)\,\mathrm{d}t) \\ &\quad + \left[(\eta - \theta(t))\lambda\mu_\infty + \lambda(1+\theta(t))\int_0^{a(t)} \overline{F}(z)\,\mathrm{d}z\right]\mathrm{d}t - d\sum_{i=1}^{N_t} Z_i^{a(t)} \\ &= \left[(\eta - \theta(t))\lambda\mu_\infty + \lambda(1+\theta(t))\int_0^{a(t)} \overline{F}(z)\,\mathrm{d}z\right]\mathrm{d}t - \int_0^{+\infty} \min\{z, a(t)\}N^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}z) \\ &\quad + X^{\pi,\psi}(t)(r + \mu u(t) - u(t)\sigma h(t))\,\mathrm{d}t + X^{\pi,\psi}(t)u(t)\sigma\,\mathrm{d}B^{\mathbb{Q}}(t), \end{aligned} \tag{2}$$

in which $\overline{F}(x) := 1 - F(x)$. Under the new measure $\hat{\mathbb{Q}}(\hat{\psi})$, with the reinsurance-investment strategy $\hat{\pi}$, the surplus of the reinsurer can be described by the following equation:

$$\begin{aligned} \mathrm{d}W^{\hat{\pi},\hat{\psi}}(t) &= W^{\hat{\pi},\hat{\psi}}(t)(r + \hat{u}(t)\mu)\,\mathrm{d}t + W^{\hat{\pi},\hat{\psi}}(t)\hat{u}(t)\sigma(\mathrm{d}B^{\hat{\mathbb{Q}}}(t) - \hat{h}(t)\,\mathrm{d}t) \\ &\quad + (1+\theta(t))\lambda(\mu_\infty - E[Z_i^{a(t)}])\,\mathrm{d}t - d\sum_{i=1}^{N_t}(Z_i - Z_i^{a(t)}) \\ &= W^{\hat{\pi},\hat{\psi}}(t)(r + \hat{u}(t)\mu - \hat{u}(t)\sigma\hat{h}(t))\,\mathrm{d}t + W^{\hat{\pi},\hat{\psi}}(t)\hat{u}(t)\sigma\,\mathrm{d}B^{\hat{\mathbb{Q}}}(t) \\ &\quad + (1+\theta(t))\lambda\left(\mu_\infty - \int_0^{a(t)} \overline{F}(z)\,\mathrm{d}z\right)\mathrm{d}t - \int_0^{+\infty}(z - \min\{z, a(t)\})N^{\hat{\mathbb{Q}}}(\mathrm{d}t, \mathrm{d}z), \end{aligned} \tag{3}$$

where $N^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}z)$ and $N^{\hat{\mathbb{Q}}}(\mathrm{d}t, \mathrm{d}z)$ are Poisson measures under the alternative measures $\mathbb{Q}$ and $\hat{\mathbb{Q}}$, respectively.

In the following, we will introduce the optimal reinsurance-investment problems in the principal-agent framework. We incorporate two kinds of preferences towards uncertainty into our models: one is via an ambiguity aversion parameter for the modeling uncertainty, the other is a risk-return preference via a risk aversion utility function. As is well known, an investor's risk-return preference is often described by a utility function $U(t)$. Arrow (1965) suggests to use $-U''(x)/U'(x)$ to represent the risk aversion function. In this paper, we assume that U is an exponential utility, i.e. constant absolute risk aversion (CARA) utility, and thus the risk aversion function is a constant, which is often referred to as a CARA parameter. In fact, the CARA utility plays a vital role in actuarial mathematics and insurance practice. It is the only utility function under the principle of 'zero utility' giving a fair premium that is independent of an insurer's level of reserves (see Gerber 1979). Finally, we derive the optimal reinsurance contracts by maximizing the expected utility of the terminal wealth for the insurer and the reinsurer.

Definition 2.1: A strategy $\pi = \{a(t), u(t)\}_{0 \le t \le T}$ is called an admissible reinsurance-investment strategy if the following conditions are satisfied

(i) π is predictable with respect to $\{\mathcal{F}_t\}$;
(ii) $\forall\, t \in [0, T], a(t) > 0$ and $E^{\psi}[\int_0^T u^2(t)X^2(t)\, dt] < \infty$, for any $\psi \in \Psi$;
(iii) $\forall\, (t, x) \in [0, T] \times R^+$ and $\psi \in \Psi$, Equation (2) has a unique solution $\{X^{\pi,\psi}(t)\}_{0\le t\le T}$ such that

$$E^{\psi}_{t,x}\left[|U(X^{\pi,\psi}(T))| + \left|\int_t^T G^{\psi}(J(s, X^{\pi,\psi}(s), \pi, \psi))\, ds\right|\right] < \infty,$$

where $U(\cdot)$ is the CARA utility function and $\int_t^T G^{\psi}(J(s, X^{\pi,\psi}(s), \pi, \psi))\, ds$ is a penalty term to be defined in Subsection 2.1.

Denote by Π the set of all admissible strategies of the insurer. The notion of worst-case measure defined above, which is a minimization over all models, is expected to have a non-trivial solution despite the set of models being non-compact under most ordinary metrics on probability measures, because of the presence of the penalty term. Similarly, we can define the set of admissible strategies $\hat{\Pi}$ for the reinsurer, where the admissible reinsurance-investment strategy $\hat{\pi} = \{\theta(t), \hat{u}(t)\}_{0\le t\le T}$ satisfies $\theta(t) \ge \eta$ and other regularity and measurability conditions as in Definition 2.1.

2.1. The agent's problem

The insurer is uncertain about the risk from the claims and the financial market, so he regards the reference model as a possible misspecification, i.e. the insurer is suspicious about the reference model's accuracy. However, he must trust certain features of the reference model, particulary the use of an aggregate claims process, and of a standard diffusion model for the financial market. He also will not admit that alternative models can deviate very far from the reference model. As such, he incorporates the distance between the reference measure $\mathbb{P}$ and the measure $\mathbb{Q}$ in a penalty term, to avoid having to consider wildly different scenarios. This is particularly important to ensure that the optimization problem under this robustness framework is well posed and has an interior solution. As mentioned in the abstract, and in agreement with our reference to Maenhout (2004, 2006) on page 4, we employ an objective function which was introduced and developed in those papers, which achieves an optimization under a worst-case scenario in the following sense. Since we want to maximize an exponential utility, for any fixed strategy, we first search for the worst-case-scenario model among all the models which are equivalent to the original model, hence the presence of an infimum in the objective function below. Since the space of all models $\mathbb{Q}$ is non-compact (and infinite-dimensional), the minimization part of the problem requires a penalization to compactify the problem and provide an interior solution. As mentioned, this penalization uses the distance between the alternative models and the original model $\mathbb{P}$, giving less weight to those which are far from $\mathbb{P}$, with tuning parameters which allow the agent to choose their level of aversion to modeling ambiguity. Once a worst-case model $\mathbb{Q}$ has been identified, the agent optimizes their decisions by maximizing the worst-case utility over all strategies, hence the supremum as the second step in the optimization. Following Maenhout (2004), Zeng et al. (2016) and Gu et al. (2017), we build the following optimization problem:

$$(PI): \quad J(t, x) = \sup_{\pi\in\Pi} \inf_{\psi\in\Psi} J(t, x, \pi, \psi) \tag{4}$$

where

$$J(t, x, \pi, \psi) := E^{\psi}_{t,x}\left[-\frac{1}{\gamma}\, e^{-\gamma X^{\pi,\psi}(T)} + \int_t^T \left(\frac{\frac{1}{2}h^2(s)}{\varphi_1^{\pi,\psi}(s, X^{\pi,\psi}(s))} + \frac{\lambda(\phi(s)\ln\phi(s) - \phi(s) + 1)}{\varphi_2^{\pi,\psi}(s, X^{\pi,\psi}(s))}\right) ds\right]. \tag{5}$$

Here the second term is the aforementioned penalty term if the measure $\mathbb{Q}$ deviates far from the reference measure $\mathbb{P}$. Following Maenhout (2004), for mathematical convenience and for the economic reasons stated in that reference, we assume that $\varphi_1^{\pi,\psi}(t,x)$ and $\varphi_2^{\pi,\psi}(t,x)$ are non-negative and state-dependent functions which are inversely proportional to the objective function (i.e. the penalized utility function):

$$\varphi_1^{\pi,\psi}(t,x) = -\frac{\alpha_1}{\gamma J(t,x,\pi,\psi)}, \quad \varphi_2^{\pi,\psi}(t,x) = -\frac{\alpha_2}{\gamma J(t,x,\pi,\psi)}, \tag{6}$$

where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ represent the insurer's ambiguity aversion levels to the diffusion modeling risk of the financial market and the jump modeling risk of the aggregate claims process, respectively. Here the insurer's preference is modeled by a CARA utility, where $\gamma > 0$ is a constant representing the absolute risk aversion coefficient.

For notational simplicity, we denote by

$$\begin{aligned} G^{\psi}(J(s,X^{\pi,\psi}(s),\pi,\psi)) &:= \frac{\frac{1}{2}h^2(s)}{\varphi_1^{\pi,\psi}(s,X^{\pi,\psi}(s))} + \frac{\lambda(\phi(s)\ln\phi(s) - \phi(s) + 1)}{\varphi_2^{\pi,\psi}(s,X^{\pi,\psi}(s))} \\ &= -\gamma\left(\frac{\frac{1}{2}h^2(s)}{\alpha_1} + \frac{\lambda(\phi(s)\ln\phi(s) - \phi(s) + 1)}{\alpha_2}\right) J(s,X^{\pi,\psi}(s),\pi,\psi), \end{aligned}$$

where the notation $G^{\psi}(J(s,X^{\pi,\psi}(s),\pi,\psi))$ highlights that the penalty term is recursive in the (penalized) utility function.

From (2), for $\forall\,(t,x) \in [0,T] \times R$, $\forall f(t,x) \in C^{1,2}([0,T] \times R)$, the infinitesimal generator of $X^{\pi,\psi}$ under measure $\mathbb{Q}$ is given by

$$\begin{aligned} \mathcal{A}_0^{\pi,\psi} f(t,x) &= \lim_{\varepsilon \downarrow 0} \frac{E_{t,x}^{\psi}[f(t+\varepsilon, X^{\pi,\psi}(t+\varepsilon)) - f(t,x)]}{\varepsilon} \\ &= f_x\left[xr + x\mu u(t) - x\sigma u(t)h(t) + (\eta - \theta(t))\lambda\mu_\infty + \lambda(1+\theta(t))\int_0^{a(t)} \overline{F}(x)\,\mathrm{d}x\right] \\ &\quad + f_t(t,x) + \frac{1}{2}f_{xx}(t,x)x^2u^2(t)\sigma^2 + \lambda\phi(t)E^{\psi}[f(t,x-\min\{Z,a(t)\}) - f(t,x)]. \end{aligned}$$

To simplify our presentation, we define the following operator:

$$\mathcal{A}^{\pi,\psi} f(t,x) = \mathcal{A}_0^{\pi,\psi} f(t,x) - \gamma\left(\frac{h^2(s)}{2\alpha_1} + \frac{\lambda(\phi(s)\ln\phi(s) - \phi(s) + 1)}{\alpha_2}\right) f(t,x). \tag{7}$$

The recursive structure in the penalized utility function (5) makes it difficult to apply standard theory to derive the HJB equation directly. To overcome this difficulty, we show that the penalized recursive utility function (5) can be transformed to an additive utility.

Lemma 2.1: *For any $\pi \in \Pi$ and $\psi \in \Psi$, the objective function* (5) *is equivalent to*

$$J(t,x,\pi,\psi) = E_{t,x}^{\psi}\left[e^{-\int_t^T \rho(\psi(s))\,\mathrm{d}s}\left(-\frac{1}{\gamma}e^{-\gamma X^{\pi,\psi}(T)}\right)\right], \tag{8}$$

where

$$\rho(\psi(s)) := \gamma\left(\frac{h^2(s)}{2\alpha_1} + \frac{\lambda(\phi(s)\ln\phi(s) - \phi(s) + 1)}{\alpha_2}\right).$$

Moreover, the insurer's optimization problem (PI) *is equivalent to*

$$J(t,x) = \sup_{\pi\in\Pi}\inf_{\psi\in\Psi}\left\{E_{t,x}^{\psi}\left[e^{-\int_t^T \rho(\psi(s))\,\mathrm{d}s}\left(-\frac{1}{\gamma}e^{-\gamma X^{\pi,\psi}(T)}\right)\right]\right\}. \tag{9}$$

Proof: See Appendix. ∎

Though we have proved that the objective function (5) is equivalent to (8), we prefer not to formulate the problem by using the equivalent objective function (8) from the very beginning. This is because the original objective function (5) is more informative and can be decomposed into two parts of profound economic meaning. The first part shows that the objective is to maximize the expected exponential utility from the insurer's terminal wealth; the second part is the penalty term when the measure $\mathbb{Q}$ deviates far from the reference measure P. Indeed, the structure of (5) is adopted by most literature on robust optimization problems in finance and insurance. One may refer to, for instance, Maenhout (2004, 2006), Yi et al. (2015), and Zeng et al. (2016).

Therefore, we can follow Theorem 3.4 in Talay & Zheng (2002) and Theorem 3.2 in Mataramvura & Øksendal (2008) to establish and solve the following HJB equation for the insurer's optimization problem (PI)

$$\sup_{\pi\in\Pi}\inf_{\psi\in\Psi}\{\mathcal{A}^{\pi,\psi}J(t,x)\}=0 \tag{10}$$

with boundary condition

$$J(T,x)=-\frac{1}{\gamma}\,e^{-\gamma x}. \tag{11}$$

It will turn out that the reinsurer's optimal safety loading θ^* is deterministic. To solve (10), we only need to focus on the deterministic function $\theta(t)$ in the infinitesimal generator, which reduces Problem (PI) to a standard optimal control problem in the worst-case model and allows us to apply the HJB equation to solve Problem (PI). Indeed, the derivation of the HJB equation (10) relies on using Itô's formula/Dynkin's formula and localization techniques. For more discussion, one may also refer to Appendices A and B in Chen & Shen (2018), particularly, Remark B1 therein.

According to the second step of a Stackelberg game, the insurer attempts to maximize the expected utility of his terminal wealth given the parameter $\theta(t)$, and then derives his optimal reinsurance and investment strategy.

Proposition 2.2: *For the optimization problem* (PI), $S(t,x)$ *given below is the solution of HJB equation* (10) *with boundary condition* (11):

$$S(t,x)=-\frac{1}{\gamma}\exp\{-\gamma(x+g(t))\},$$

where

$$g(t)=\int_t^T e^{r(T-s)}\left[(\eta-\theta(s))\lambda\mu_\infty+\lambda(1+\theta(s))\int_0^{a^*(s)}\overline{F}(z)\,\mathrm{d}z\right]\mathrm{d}s$$
$$-\frac{\lambda}{\alpha_2}\int_t^T\phi^*(s)\,\mathrm{d}s+\left(\frac{\lambda}{\alpha_2}+\frac{\mu^2}{2(\alpha_1+\gamma)\sigma^2}\right)(T-t).$$

Moreover, the maximum point π^* *is given by*

$$\pi^*=(a^*(t),\quad u^*(t))=\left(\frac{\ln(1+\theta(t))-\ln\phi^*(t)}{\gamma\,e^{r(T-t)}},\frac{\mu}{x\sigma^2(\alpha_1+\gamma)\,e^{r(T-t)}}\right), \tag{12}$$

and the worst-case measure $\mathbb{Q}^*=\mathbb{Q}(\psi^*)$ *is determined by*

$$\psi^*(t)=(h^*(t),\phi^*(t))=\left(\frac{\alpha_1\mu}{(\alpha_1+\gamma)\sigma},\exp\left\{\alpha_2\,e^{r(T-t)}\int_0^{a^*(t)}e^{\gamma z\,e^{r(T-t)}}\overline{F}(z)\,\mathrm{d}z\right\}\right). \tag{13}$$

Proof: See Appendix. ■

The following theorem shows that the solution of HJB equation (10) given in Proposition 2.2 is indeed the solution of problem (4).

Theorem 2.3: *For the optimization problem* (PI), *if* $S(t,x)$ *is the solution of HJB equation* (10) *with boundary condition* (11), *the optimal value function* $J(t,x) = S(t,x)$, *the optimal strategy* π^* *and the worst-case measure* ψ^* *are given in Proposition* 2.2.

To prove Theorem 2.3, we only need to follow the method in the proof of Theorem 3.1 in Li et al. (2018). The only difference is that our insurance model is described by C-L model and their insurance model is described by an approximated diffusion model. Due to the jump term $\sum_{i=1}^{N_t}(Z_i - Z_i^{a(t)})$ is bounded, so it does not create any difficulties when applying the proof method of Li et al. (2018). Thus, we omitted the proof here.

2.2. The principal's problem

In this subsection, as in Subsection 2.1, we incorporate the reinsurer's ambiguity aversion into her optimization problem. So she considers a new measure $\hat{\mathbb{Q}}$ as an alternative to the reference measure, and is penalized if the alternative deviates far from the reference. As mentioned above, the reinsurer can invest all her wealth into the financial market. Moreover, she accepts the insurance business ceded from the insurer (recall that we assumed a lower bound to the reinsurance loading factor, which justifies this acceptance). As mentioned, similar to Hu et al. (2018a, 2018b), we regard the reinsurance premium $\theta(t)$ as the reinsurance price at time t. Thus the reinsurance premium $\theta(t)$ and the investment strategy $\hat{u}(t)$ form the decision variable (control) $\hat{\pi}(t) = (\theta(t), \hat{u}(t))$. Based on the Stackelberg game, the reinsurer will make her final decision under the strategy $a^*(t)$ given by Proposition 2.2. Again, this is consistent with the assumption that the reinsurer picks up the insurer's ceded risk. Under the new measure $\hat{\mathbb{Q}}$ and a reinsurance-investment strategy $\hat{\pi}$, the surplus of the reinsurer is given by:

$$\begin{aligned} \mathrm{d}W^{\hat{\pi},\hat{\psi}}(t) &= W^{\hat{\pi},\hat{\psi}}(t)(r + \hat{u}(t)\mu)\,\mathrm{d}t + W^{\hat{\pi},\hat{\psi}}(t)\hat{u}(t)\sigma(\mathrm{d}B^{\hat{\mathbb{Q}}}(t) - \hat{h}(t)\,\mathrm{d}t) \\ &\quad + (1+\theta(t))\lambda(\mu_\infty - E[Z_i^{a^*(t)}])\,\mathrm{d}t - d\sum_{i=1}^{N_t}(Z_i - Z_i^{a^*(t)}) \\ &= W^{\hat{\pi},\hat{\psi}}(t)(r + \hat{u}(t)\mu - \hat{u}(t)\sigma\hat{h}(t))\,\mathrm{d}t + W^{\hat{\pi},\hat{\psi}}(t)\hat{u}(t)\sigma\,\mathrm{d}B^{\hat{\mathbb{Q}}}(t) \\ &\quad + (1+\theta(t))\lambda\left(\mu_\infty - \int_0^{a^*(t)}\overline{F}(z)\,\mathrm{d}z\right)\mathrm{d}t - \int_0^\infty (z - \min(z, a^*(t)))N^{\hat{\mathbb{Q}}}(\mathrm{d}t, \mathrm{d}z). \end{aligned}$$

We assume that the reinsurer aims to maximize the expected exponential utility under the worst-case scenario, and seeks the optimal strategy $\hat{\pi}$ to form the optimal robust reinsurance contract. Thus, we build the optimization problem:

$$(PR): \quad V(t,w) = \sup_{\hat{\pi}\in\hat{\Pi}} \inf_{\hat{\psi}\in\hat{\Psi}} V(t,w,\hat{\pi},\hat{\psi}), \tag{14}$$

where

$$V(t,w,\hat{\pi},\hat{\psi}) := E_{t,w}^{\hat{\psi}}\left[-\frac{1}{m}e^{-mW^{\hat{\pi},\hat{\psi}}(T)} + \int_t^T \left(\frac{\frac{1}{2}\hat{h}^2(s)}{\hat{\varphi}_1^{\hat{\pi},\hat{\psi}}(s, W^{\hat{\pi},\hat{\psi}}(s))}\right.\right.$$

$$+\frac{\lambda(\hat{\phi}(s)\ln\hat{\phi}(s)-\hat{\phi}(s)+1)}{\hat{\varphi}_2^{\hat{\pi},\hat{\psi}}(s,W^{\hat{\pi},\hat{\psi}}(s))}\Bigg)\,\mathrm{d}s\Bigg]. \tag{15}$$

Here the reinsurer views the second term as a penalty term if the measure $\hat{\mathbb{Q}}$ deviates from the reference measure $\mathbb{P}$. As described in Subsection 2.1, we also assume that $\hat{\varphi}_1^{\hat{\pi},\hat{\psi}}(t,w)$ and $\hat{\varphi}_2^{\hat{\pi},\hat{\psi}}(t,w)$ are non-negative and state-dependent functions which are inversely proportional to the objective/penalized utility function:

$$\hat{\varphi}_1^{\hat{\pi},\hat{\psi}}(t,w)=-\frac{\beta_1}{mV(t,w,\hat{\pi},\hat{\psi})},\quad \hat{\varphi}_2^{\hat{\pi},\hat{\psi}}(t,w)=-\frac{\beta_2}{mV(t,w,\hat{\pi},\hat{\psi})}, \tag{16}$$

where $\beta_1\geq 0$ and $\beta_2\geq 0$ quantify the reinsurer's ambiguity levels to financial risk modeling and claims risk modeling, specifically at the level of drift and jump rate parameters. The larger β_1 and β_2 are, the less modeling confidence the reinsurer has. We can also interpret this as saying that they have less modeling information. When $\beta_1=\beta_2=0$, the reinsurer believes completely in the financial market and insurance models. Just as for the insurer, the reinsurer has a CARA utility function, and $m>0$ is her absolute risk aversion coefficient.

In the following, we introduce the use of subscripts ij to show whether the insurer or the reinsurer is ambiguity-averse: $i=1$ ($j=1$) indicates that the insurer (the reinsurer) is ambiguity-averse, and $i=2$ ($j=2$) indicates that the insurer (the reinsurer) is not ambiguity-averse, i.e. is ambiguity-neutral. For example, a_{12} represents the excess-of-loss strategy under the reinsurance contract, where the insurer is ambiguity-averse and the reinsurer is ambiguity-neutral.

Following a similar method used in Subsection 2.1, we can derive the optimal strategy and value function for the reinsurer when the insurer and the reinsurer are both ambiguity-averse. First, we give the HJB equation for the optimization problem (PR) with value function $V_{11}(t,w)$. For convenience, we define a variational operator $\mathcal{B}$ with $\hat{\pi}_{11}(t)=(\theta_{11}(t),\hat{u}_{11}(t))$ and $\hat{\psi}_{11}(t)=(\hat{h}_{11}(t),\hat{\phi}_{11}(t))$:

$$\begin{aligned}\mathcal{B}^{\hat{\pi}_{11},\hat{\psi}_{11}}f(t,w)&=f_t(t,w)+f_w[wr+w\mu\hat{u}_{11}(t)-w\sigma\hat{h}_{11}(t)\hat{u}_{11}(t)]\\&\quad+f_w\left[(1+\theta(t))\lambda(\mu_\infty-\int_0^{a^*(t)}\overline{F}(z)\,\mathrm{d}z)\right]+\frac{1}{2}f_{ww}w^2\sigma^2\hat{u}_{11}^2(t)\\&\quad+\lambda\hat{\phi}_{11}(t)E^{\hat{\psi}_{11}}[f(t,w-(Z_i-\min(Z_i,a^*(t))))-f(t,w)]\\&\quad-m\left(\frac{\hat{h}_{11}^2(t)}{2\beta_1}+\frac{\lambda\big(\hat{\phi}_{11}(t)\ln\hat{\phi}_{11}(t)-\hat{\phi}_{11}(t)+1\big)}{\beta_2}\right)f(t,w).\end{aligned} \tag{17}$$

As in Subsection 2.1, we can transform the optimization problem (PR) to the following equivalent problem without recursive structure:

$$V_{11}(t,w)=\sup_{\hat{\pi}_{11}\in\hat{\Pi}}\inf_{\hat{\psi}_{11}\in\hat{\Psi}}\left\{E_{t,w}^{\hat{\psi}_{11}}\left[e^{-\int_t^T\hat{\rho}(\hat{\psi}_{11}(s))\,\mathrm{d}s}\left(-\frac{1}{m}e^{-mW^{\hat{\pi},\hat{\psi}}(T)}\right)\right]\right\} \tag{18}$$

where

$$\hat{\rho}(\hat{\psi}_{11}(s)):=m\left(\frac{\hat{h}_{11}^2(s)}{2\beta_1}+\frac{\lambda\big(\hat{\phi}_{11}(s)\ln\hat{\phi}_{11}(s)-\hat{\phi}_{11}(s)+1\big)}{\beta_2}\right).$$

It can again be shown as Theorem 3.4 in Talay & Zheng (2002) and Theorem 3.2 in Mataramvura & Øksendal (2008) that the HJB equation for the optimization problem (PR) is

$$\sup_{\hat{\pi}_{11}\in\hat{\Pi}}\inf_{\hat{\psi}_{11}\in\hat{\Psi}}\{\mathcal{B}^{\hat{\pi}_{11},\hat{\psi}_{11}}V_{11}(t,w)\}=0, \tag{19}$$

with boundary condition

$$V_{11}(T,w) = -\frac{1}{m}\, e^{-mw}. \tag{20}$$

From Proposition 2.2, we see that the insurer's optimal retention level a^* is deterministic. As discussed after Equation (11), the optimization problem (PR) also reduces to a standard optimal control problem in the worst-case model. Indeed, for both the insurer's and reinsurer's optimization problems, it is the special structure of exponential utilities that separates the states from optimal strategies and guarantees the applicability of HJB equations in these problems.

Proposition 2.4: *For the optimization problem* (PR), *when both the insurer and reinsurer are ambiguity averse to modeling the financial market and the insurance business claims, $H(t,w)$ given below is the solution of HJB equation* (19) *with boundary condition* (20):

$$H(t,w) = -\frac{1}{m}\exp\{-m(w+\hat{g}_{11}(t))\},$$

where

$$\hat{g}_{11}(t) = \int_t^T \left[\lambda\, e^{r(T-s)}\left(\mu_\infty - \int_0^{a_{11}^*(s)} \overline{F}(z)\,\mathrm{d}z\right)(1+\theta_{11}^*(s)) - \frac{\lambda\hat{\phi}_{11}^*(s)}{\beta_2}\right]\mathrm{d}s$$
$$+\left(\frac{\mu^2}{2\sigma^2(\beta_1+m)} + \frac{\lambda}{\beta_2}\right)(T-t),$$

the maximum point $\hat{\pi}^* = (\theta_{11}^*(t), \hat{u}_{11}^*(t))$ is given by

$$1+\theta_{11}^*(t) = \begin{cases} \dfrac{\hat{\phi}_{11}^*(t)[\overline{F}(a_{11}^*(t)) + m\, e^{r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{+\infty} \overline{F}(z)\, e^{mz\, e^{r(T-t)}}\,\mathrm{d}z]}{\overline{F}(a_{11}^*(t)) - (\mu_\infty - \int_0^{a_{11}^*(t)} \overline{F}(z)\,\mathrm{d}z)\gamma\, e^{r(T-t)}}, & t\in O,\\ 1+\eta, & t\in\overline{O},\end{cases}$$

$$\hat{u}_{11}^*(t) = \frac{\mu}{w\sigma^2(\beta_1+m)\, e^{r(T-t)}},$$

and the worst-case measure $\hat{\mathbb{Q}}^* = \hat{\mathbb{Q}}(\hat{\psi}_{11}^*)$ is determined by

$$\hat{\psi}_{11}^*(t) = (\hat{h}_{11}^*(t), \hat{\phi}_{11}^*(t)) = \left(\frac{\beta_1\mu}{\sigma(\beta_1+m)}, \exp\left\{\beta_2\, e^{r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{\infty} \overline{F}(z)\, e^{mz\, e^{r(T-t)}}\,\mathrm{d}z\right\}\right)$$

where

$$a_{11}^*(t) = \frac{\ln(1+\theta_{11}^*(t)) - \ln\phi_{11}^*(t)}{\gamma\, e^{r(T-t)}},$$

$$O := \left\{t \,\middle|\, \frac{\hat{\phi}_{11}^*(t)[\overline{F}(a_{11}^*(t)) + m\, e^{r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{+\infty} \overline{F}(z)\, e^{mz\, e^{r(T-t)}}\,\mathrm{d}z]}{\overline{F}(a_{11}^*(t)) - (\mu_\infty - \int_0^{a_{11}^*(t)} \overline{F}(z)\,\mathrm{d}z)\gamma\, e^{r(T-t)}} > 1+\eta\right\}$$

and

$$\overline{O} := \left\{t \,\middle|\, \frac{\hat{\phi}_{11}^*(t)[\overline{F}(a_{11}^*(t)) + m\, e^{r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{+\infty} \overline{F}(z)\, e^{mz\, e^{r(T-t)}}\,\mathrm{d}z]}{\overline{F}(a_{11}^*(t)) - (\mu_\infty - \int_0^{a_{11}^*(t)} \overline{F}(z)\,\mathrm{d}z)\gamma\, e^{r(T-t)}} \le 1+\eta\right\}.$$

Proof: See Appendix. ■

The following theorem shows the solution of problem (PR) and gives the reinsurance contract between the insurer and the reinsurer.

Theorem 2.5: *When both the insurer and reinsurer are ambiguity averse to modeling the financial market and the insurance business claims, if $H(t,w)$ is the solution of HJB equation* (19) *with boundary condition* (20), *then for the optimization problem* (PR) *the optimal value function $V(t,w) = H(t,w)$ and the robust optimal reinsurance contract $(a_{11}^*(t), \theta_{11}^*(t))$ are given by Proposition 2.4. Under this optimal reinsurance contract, the insurer's value function is*

$$J(t,x) = -\frac{1}{\gamma}\exp\{-\gamma(e^{r(T-t)}x + g_{11}(t))\},$$

where

$$g_{11}(t) = \int_t^T e^{r(T-s)}\left[(\eta - \theta_{11}^*(s))\lambda\mu_\infty + \lambda(1+\theta_{11}^*(s))\int_0^{a_{11}^*(s)} \overline{F}(z)\,dz\right] ds$$
$$- \frac{\lambda}{\alpha_2}\int_t^T \phi_{11}^*(s)\,ds + \left(\frac{\lambda}{\alpha_2} + \frac{\mu^2}{2(\alpha_1+\gamma)\sigma^2}\right)(T-t),$$

and the worst-case scenario for the insurer is $\psi_{11}^*(t) = (h_{11}^*(t), \phi_{11}^*(t))$, and

$$h_{11}^*(t) = \frac{\alpha_1\mu}{(\alpha_1+\gamma)\sigma};$$
$$\phi_{11}^*(t) = \exp\left\{\alpha_2\, e^{r(T-t)}\int_0^{a_{11}^*(t)} e^{\gamma z\, e^{r(T-t)}}\overline{F}(z)\,dz\right\}.$$

On the one hand, the optimal excess-of-loss retention level $a_{11}^*(t)$ depends on the optimal reinsurance premium $\theta_{11}^*(t)$. The bigger $\theta_{11}^*(t)$ is, the greater $a_{11}^*(t)$ is. On the other hand, the reinsurance premium $\theta_{11}^*(t)$ depends on $a_{11}^*(t)$, the insurer's absolute risk aversion γ and the distribution of the claim. Therefore, the insurer and the reinsurer are expected to negotiate to produce a fair reinsurance contract. In order to better understand the trade-off between $a_{11}^*(t)$ and $\theta_{11}^*(t)$, in the next section, we assume the claim size follows the exponential distribution with parameter $\hat{\lambda}$, and we derive explicit expressions for the reinsurance contract and the value functions.

3. Reinsurance contract with claim following the exponential distribution

In order to solve explicitly the optimization problems, we make the simplifying assumption that the claim Z follows an exponential distribution with parameter $\hat{\lambda}$. It turns out that our framework runs into mathematical difficulties if the claims intensity is low compared to the two players' risk-aversion coefficients. To avoid this problem, for the sake of a simpler exposition, we assume that $\hat{\lambda} > \max\{\gamma\, e^{rT}, m\, e^{rT}\}$. According to the results given in the last section, we can derive the following propositions. For convenience, and as done above, we denote the strategies as a_{ij}, u_{ij} and ψ_{ij} for the insurer and θ_{ij}, $\hat{u}_{ij}$ and $\hat{\psi}_{ij}$ for the reinsurer, where the indices $i = 1$ or $j = 1$ represent the AAI (ambiguity-averse insurer) or AAR (ambiguity-averse reinsurer), and $i = 2$ or $j = 2$ represent the ANI (ambiguity-neutral insurer) or ANR (ambiguity-neutral reinsurer).

Proposition 3.1: *When the claim follows the exponential distribution with parameter $\hat{\lambda}$, and both the insurer and the reinsurer have ambiguity aversion and risk aversion, the optimal robust reinsurance contract is given by*

$$a_{11}^*(t) = \frac{\ln \hat{\phi}_{11}^*(t) + \ln \frac{\hat{\lambda}^2}{(\hat{\lambda} - \gamma\, e^{r(T-t)})(\hat{\lambda} - m\, e^{r(T-t)})} - \ln \phi_{11}^*(t)}{\gamma\, e^{r(T-t)}},$$

$$1 + \theta_{11}^*(t) = \begin{cases} \dfrac{\hat{\phi}_{11}^*(t)\hat{\lambda}^2}{(\hat{\lambda} - \gamma\, e^{r(T-t)})(\hat{\lambda} - m\, e^{r(T-t)})}, & 0 \le t < t_0, \\ 1 + \eta, & t_0 \le t \le T. \end{cases} \tag{21}$$

where t_0 is a root to the following equation

$$\frac{\hat{\phi}_{11}^*(t_0)\hat{\lambda}^2}{(\hat{\lambda} - \gamma\, e^{r(T-t_0)})(\hat{\lambda} - m\, e^{r(T-t_0)})} = 1 + \eta.$$

With this reinsurance contract, for the insurer, the worst-case distortion $\psi_{11}^*(t) = (h_{11}^*(t), \phi_{11}^*(t))$ and the optimal investment strategy $u_{11}^*(t)$ are:

$$h_{11}^*(t) = \frac{\alpha_1 \mu}{(\alpha_1 + \gamma)\sigma}$$

$$\phi_{11}^*(t) = \exp\left\{ \frac{\alpha_2\, e^{r(T-t)}(e^{a_{11}^*(t)(\gamma\, e^{r(T-t)} - \hat{\lambda})} - 1)}{\gamma\, e^{r(T-t)} - \hat{\lambda}} \right\} \tag{22}$$

and

$$u_{11}^*(t) = \frac{\mu}{x\sigma^2(\alpha_1 + \gamma)\, e^{r(T-t)}}. \tag{23}$$

Then the insurer's corresponding value function $J_{11}(t,x)$ is:

$$J_{11}(t,x) = -\frac{1}{\gamma} \exp\{-\gamma(e^{r(T-t)}x + g_{11}(t))\},$$

where

$$g_{11}(t) = \int_t^T e^{r(T-s)} \left[(\eta - \theta_{11}^*(s))\lambda\mu_\infty + \lambda(1 + \theta_{11}^*(s)) \int_0^{a_{11}^*(s)} \overline{F}(z)\, dz \right] ds - \frac{\lambda}{\alpha_2} \int_t^T \hat{\phi}_{11}^*(s)\, ds$$
$$+ \left(\frac{\lambda}{\alpha_2} + \frac{\mu^2}{2(\alpha_1 + \gamma)\sigma^2} \right)(T - t).$$

With this reinsurance contract, for the reinsurer, the worst-case distortion $\hat{\psi}_{11}^*(t) = (\hat{h}_{11}^*(t), \hat{\phi}_{11}^*(t))$ and the optimal investment strategy $\hat{u}_{11}^*(t)$ are:

$$\hat{h}_{11}^*(t) = \frac{\beta_1 \mu}{(\beta_1 + m)\sigma}$$

$$\hat{\phi}_{11}^*(t) = \exp\left\{ \frac{-\beta_2\, e^{r(T-t)}\, e^{-a_{11}^*(t)\hat{\lambda}}}{m\, e^{r(T-t)} - \hat{\lambda}} \right\} \tag{24}$$

and

$$\hat{u}^*_{11}(t) = \frac{\mu}{w\sigma^2(\beta_1 + m)\, e^{r(T-t)}}. \tag{25}$$

The reinsurer's corresponding value function $V_{11}(t, w)$ *is:*

$$V_{11}(t, w) = -\frac{1}{m} \exp\{-m(e^{r(T-t)} w + \hat{g}_{11}(t))\}$$

where

$$\hat{g}_{11}(t) = \int_t^T \left[\lambda\, e^{r(T-s)} \left(\mu_\infty - \int_0^{a^*_{11}(s)} \overline{F}(z)\, dz\right) (1 + \theta^*_{11}(s)) - \frac{\lambda \hat{\phi}^*_{11}(s)}{\beta_2}\right] ds$$
$$+ \left(\frac{\mu^2}{2\sigma^2(\beta_1 + m)} + \frac{\lambda}{\beta_2}\right)(T - t).$$

Proposition 3.1 implies that the more ambiguity averse the insurer is (or the reinsurer is), the more distortions there are. Equation (21) shows that the insurer pays attention not only to his own ambiguity aversion α_2 but also to the reinsurer's ambiguity aversion β_2. Moreover, the reinsurer's ambiguity aversion has a positive effect on the insurer's risk position in his reinsurance strategy: with the increase of β_2, the corresponding distortion $\hat{\phi}^*_{11}(t)$ will increase, which causes the insurer to pay more for the reinsurance, and as a direct result, the insurer will decrease his reinsurance.

Proposition 3.2: *When the claims follow the exponential distribution with parameter* $\hat{\lambda}$, *the retention level* $a^*_{11}(t)$ *for the insurer increases w.r.t.* β_2 *and decreases w.r.t.* α_2; *the reinsurance premium* $\theta^*_{11}(t)$ *increases w.r.t.* α_2 *and* β_2, *i.e.*

$$\frac{da^*_{11}(t)}{d\alpha_2} < 0; \quad \frac{da^*_{11}(t)}{d\beta_2} > 0; \quad \frac{d\theta^*_{11}(t)}{d\alpha_2} > 0; \quad \frac{d\theta^*_{11}(t)}{d\beta_2} > 0.$$

Remark 3.1: When both parties to the contract are ambiguity averse, Proposition 3.2 shows that the AAI would like to buy more reinsurance with the increase of his ambiguity level and tends to decrease his reinsurance with the increase of the reinsurer's ambiguity level; the AAR would raise the reinsurance price with the increase of the insurer's ambiguity aversion or her own. This phenomenon will be further explained in Subsection 4.1.

Proposition 3.3: *When the claims follow the exponential distribution with parameter* $\hat{\lambda}$, *the distortion* $\psi^*_{11}(t)$ *and* $\hat{\psi}^*_{11}(t)$ *will increase w.r.t. the ambiguity aversion coefficients* α_2 *and* β_2, *i.e.* $d\psi^*_{11}(t)/d\alpha_2 > 0$ *and* $d\hat{\psi}^*_{11}(t)/d\beta_2 > 0$.

The proof can be given by differentiating $\psi^*_{11}(t)$ and $\hat{\psi}^*_{11}(t)$ w.r.t α_2 and β_2. Obviously, $a^*_{11}(t)$ depends on not only α_2 and β_2, but also on γ, m and $\hat{\lambda}$. In order to understand these parameters' effects more explicitly, we will present a numerical analysis of these effects on the $a^*_{11}(t)$ and $\theta^*_{11}(t)$ in Section 4.

Corollary 3.4: *Assume the claims follow the exponential distribution with parameter* $\hat{\lambda}$, *the insurer is an ANI, i.e. no ambiguity aversion to the financial market and the insurance business, and the reinsurer is an AAR. Then the robust reinsurance contract is the following: the optimal robust reinsurance contract* *is given by*

$$a_{21}^*(t) = \frac{\ln \hat{\phi}_{21}^*(t) + \ln \frac{\hat{\lambda}^2}{(\hat{\lambda}-\gamma\, e^{r(T-t)})(\hat{\lambda}-m\, e^{r(T-t)})}}{\gamma\, e^{r(T-t)}},$$

$$1+\theta_{21}^*(t) = \begin{cases} \dfrac{\hat{\phi}_{21}^*(t)\hat{\lambda}^2}{(\hat{\lambda}-\gamma\, e^{r(T-t)})(\hat{\lambda}-m\, e^{r(T-t)})}, & 0 \le t < t_0, \\ 1+\eta, & t_0 \le t \le T, \end{cases}$$

where t_0 is the root to the following equation

$$\frac{\hat{\phi}_{21}^*(t_0)\hat{\lambda}^2}{(\hat{\lambda}-\gamma\, e^{r(T-t_0)})(\hat{\lambda}-m\, e^{r(T-t_0)})} = 1+\eta,$$

and

$$\hat{\phi}_{21}^*(t) = \exp\left\{\frac{-\beta_2\, e^{r(T-t)}\, e^{-a_{21}^*(t)\hat{\lambda}}}{m\, e^{r(T-t)} - \hat{\lambda}}\right\}.$$

With this reinsurance contract, for the insurer, the optimal investment strategy is

$$u_{21}^*(t) = \frac{\mu}{x\sigma^2\gamma\, e^{r(T-t)}}.$$

The insurer's corresponding value function $J_{21}(t,x)$ is:

$$J_{21}(t,x) = -\frac{1}{\gamma}\exp\{-\gamma(e^{r(T-t)}x + g_{21}(t))\},$$

where

$$g_{21}(t) = \int_t^T e^{r(T-s)}\left[(\eta-\theta_{21}^*(s))\lambda\mu_\infty + \lambda(1+\theta_{21}^*(s))\int_0^{a_{21}^*(s)} \overline{F}(z)\,\mathrm{d}z\right]\mathrm{d}s + \frac{\mu^2}{2\gamma\sigma^2}(T-t)$$
$$\lambda\int_t^T e^{r(T-s)}\int_0^{a_{21}^*(s)} e^{\gamma z e^{r(T-s)}}\overline{F}(z)\,\mathrm{d}z\,\mathrm{d}s.$$

With this reinsurance contract, for the reinsurer, the worst-case distortion, the optimal strategy, and the optimal value function have the same form as that in Proposition 3.1:

$$V_{21}(t,w) = -\frac{1}{m}\exp\{-m(e^{r(T-t)}w + \hat{g}_{21}(t))\}$$

where

$$\hat{g}_{21}(t) = \int_t^T \left[\lambda\, e^{r(T-s)}\left(\mu_\infty - \int_0^{a_{21}^*(s)} \overline{F}(z)\,\mathrm{d}z\right)(1+\theta_{21}^*(s)) - \frac{\lambda\hat{\phi}_{21}^*(s)}{\beta_2}\right]\mathrm{d}s$$
$$+\left(\frac{\mu^2}{2\sigma^2(\beta_1+m)} + \frac{\lambda}{\beta_2}\right)(T-t).$$

Remark 3.2: Theorem 4 in Hu et al. (2018b) is a special case of Corollary 3.4. If the investment strategy $u(t)\equiv 0$ and $\hat{u}(t)\equiv 0$, Corollary 3.4 degenerates into Theorem 4 in Hu et al. (2018b). Indeed, this reference does not consider the possibility of risky investments.

Corollary 3.5: *When the claims follow the exponential distribution with parameter $\hat{\lambda}$, the insurer is an AAI and the reinsurer is an ANR, then the reinsurance contract is given by the following*:

$$a_{12}^*(t) = \frac{\ln \frac{\hat{\lambda}^2}{(\hat{\lambda}-\gamma\, e^{r(T-t)})(\hat{\lambda}-m\, e^{r(T-t)})} - \ln \phi_{12}^*(t)}{\gamma\, e^{r(T-t)}},$$

$$1+\theta_{12}^*(t) = \begin{cases} \dfrac{\hat{\lambda}^2}{(\hat{\lambda}-\gamma\, e^{r(T-t)})(\hat{\lambda}-m\, e^{r(T-t)})}, & 0 \le t < t_0, \\ 1+\eta, & t_0 \le t \le T, \end{cases}$$

where t_0 is the root of the following equation

$$\frac{\hat{\lambda}^2}{(\hat{\lambda}-\gamma\, e^{r(T-t_0)})(\hat{\lambda}-m\, e^{r(T-t_0)})} = 1+\eta,$$

and

$$\phi_{12}^*(t) = \exp\left\{ \frac{\alpha_2\, e^{r(T-t)}(e^{a_{12}^*(t)(\gamma\, e^{r(T-t)}-\hat{\lambda})} - 1)}{\gamma\, e^{r(T-t)} - \hat{\lambda}} \right\}.$$

With this reinsurance contract, for the insurer, the worst-case distortion $\psi_{12}^*(t) = (h_{12}^*(t), \phi_{12}^*(t))$ and the insurer's optimal investment strategy $u_{12}^*(t)$ are:

$$h_{12}^*(t) = \frac{\alpha_1 \mu}{(\alpha_1+\gamma)\sigma}$$

$$\phi_{12}^*(t) = \exp\left\{ \frac{\alpha_2\, e^{r(T-t)}(e^{a_{12}^*(t)(\gamma\, e^{r(T-t)}-\hat{\lambda})} - 1)}{\gamma\, e^{r(T-t)} - \hat{\lambda}} \right\}$$

and

$$u_{12}^*(t) = \frac{\mu}{x\sigma^2(\alpha_1+\gamma)\, e^{r(T-t)}}.$$

The insurer's corresponding value function $J_{12}(t,x)$ is:

$$J_{12}(t,x) = -\frac{1}{\gamma}\exp\{-\gamma(e^{r(T-t)}x + g_{12}(t))\},$$

where

$$g_{12}(t) = \int_t^T e^{r(T-s)}\left[(\eta-\theta_{12}^*(s))\lambda\mu_\infty + \lambda(1+\theta_{12}^*(s))\int_0^{a_{12}^*(s)} \overline{F}(z)\,\mathrm{d}z\right]\mathrm{d}s - \frac{\lambda}{\alpha_2}\int_t^T \phi_{12}^*(s)\,\mathrm{d}s$$
$$+ \left(\frac{\lambda}{\alpha_2} + \frac{\mu^2}{2(\alpha_1+\gamma)\sigma^2}\right)(T-t).$$

With this reinsurance contract, for the reinsurer, the optimal investment strategy $\hat{u}_{12}^*(t)$ is:

$$\hat{u}_{12}^*(t) = \frac{\mu}{w\sigma^2 m\, e^{r(T-t)}}.$$

The reinsurer's corresponding value function $V_{12}(t,w)$ is:

$$V_{12}(t,w) = -\frac{1}{m}\exp\{-m(e^{r(T-t)}w + \hat{g}_{12}(t))\}$$

where

$$\hat{g}_{12}(t) = \int_t^T \left[\lambda\, e^{r(T-s)} \left(\mu_\infty - \int_0^{a_{12}^*(s)} \overline{F}(z)\,\mathrm{d}z \right) (1 + \theta_{12}^*(s)) \right] \mathrm{d}s + \frac{\mu^2}{2\sigma^2 m}(T-t)$$
$$- \lambda \int_t^T e^{r(T-s)}\, e^{-m a_{12}^*(s)\, e^{r(T-s)}} \int_{a_{12}^*(s)}^\infty \overline{F}(z)\, e^{m\, e^{r(T-s)} z}\,\mathrm{d}z\,\mathrm{d}s.$$

Remark 3.3: According to Corollaries 3.5 and 3.4, we know that $\theta_{12}^*(t)$ is different from $\theta_{21}^*(t)$. Even though they are both time-dependent functions and depend on the absolute risk aversion γ and m, $\theta_{12}^*(t)$ is independent of the ambiguity aversion of the insurer while $\theta_{21}^*(t)$ is a function of the ambiguity aversion of the reinsurer. Can we conclude that the reinsurance premium depends only on the ambiguity of the reinsurer but not on the ambiguity of the insurer? The answer is No. Proposition 3.1 reveals that the reinsurance premium depends on ambiguity aversion levels of both the insurer and the reinsurer. Consequently, it is critical for the two parties of the contract to make clear if the opposite side is ambiguity averse. After all, the insurer's retention level is always a function of the ambiguity aversion levels for the insurer and the reinsurer.

If the insurer's and the reinsurer's ambiguity aversion coefficients equal 0, this means that they believe the reference model, or they are ambiguity-neutral. So the robust excess-of-loss reinsurance contract will reduce to the standard excess–of-loss reinsurance contract.

Corollary 3.6: *When the claims follow the exponential distribution with parameter $\hat{\lambda}$, and both the insurer and the reinsurer are ambiguity-neutral, the reinsurance contract is given by*

$$a_{22}^*(t) = \frac{\ln \frac{\hat{\lambda}^2}{(\hat{\lambda} - \gamma\, e^{r(T-t)})(\hat{\lambda} - m\, e^{r(T-t)})}}{\gamma\, e^{r(T-t)}},$$

$$1 + \theta_{22}^*(t) = \begin{cases} \dfrac{\hat{\lambda}^2}{(\hat{\lambda} - \gamma\, e^{r(T-t)})(\hat{\lambda} - m\, e^{r(T-t)})}, & 0 \le t < t_0, \\ 1 + \eta, & t_0 \le t \le T, \end{cases}$$

where t_0 is the root of the following equation

$$\frac{\hat{\lambda}^2}{(\hat{\lambda} - \gamma\, e^{r(T-t_0)})(\hat{\lambda} - m\, e^{r(T-t_0)})} = 1 + \eta.$$

With this reinsurance contract, for the insurer, the optimal investment strategy $u_{22}^*(t)$ is:

$$u_{22}^*(t) = \frac{\mu}{x\sigma^2 \gamma\, e^{r(T-t)}}.$$

The insurer's corresponding value function $J_{22}(t,x)$ is:

$$J_{22}(t,x) = -\frac{1}{\gamma} \exp\{-\gamma (e^{r(T-t)} x + g_{22}(t))\},$$

where

$$g_{22}(t)=\int_t^T e^{r(T-s)}\left[(\eta-\theta_{22}^*(s))\lambda\mu_\infty+\lambda(1+\theta_{22}^*(s))\int_0^{a_{22}^*(s)}\overline{F}(z)\,\mathrm{d}z\right]\mathrm{d}s+\frac{\mu^2}{2\gamma\sigma^2}(T-t)$$
$$-\lambda\int_t^T e^{r(T-s)}\int_0^{a_{22}^*(s)} e^{\gamma z\, e^{r(T-s)}}\overline{F}(z)\,\mathrm{d}z\,\mathrm{d}s.$$

With this reinsurance contract, for the reinsurer, the optimal investment strategy $\hat{u}_{22}^*(t)$ *is*:

$$\hat{u}_{22}^*(t)=\frac{\mu}{w\sigma^2 m\, e^{r(T-t)}}.$$

The reinsurer's corresponding value function $V_{22}(t,w)$ *is*:

$$V_{22}(t,w)=-\frac{1}{m}\exp\{-m(e^{r(T-t)}w+\hat{g}_{22}(t))\}$$

where

$$\hat{g}_{22}(t)=\int_t^T\left[\lambda\, e^{r(T-s)}\left(\mu_\infty-\int_0^{a_{22}^*(s)}\overline{F}(z)\,\mathrm{d}z\right)(1+\theta_{22}^*(s))\right]\mathrm{d}s+\frac{\mu^2}{2\sigma^2 m}(T-t)$$
$$-\lambda\int_t^T e^{r(T-s)}\, e^{-ma_{22}^*(s)\, e^{r(T-s)}}\int_{a_{22}^*(s)}^\infty \overline{F}(z)\, e^{m\, e^{r(T-s)}z}\,\mathrm{d}z\,\mathrm{d}s.$$

Remark 3.4: As we can see, the reinsurance contract does not depend on the financial volatility and return parameters, while the investment strategies $u(t)$ and $\hat{u}(t)$ only depend on these parameters and the risk-aversion parameters. This shows that the reinsurance strategy is independent of the financial market, and the investment strategy is independent of the insurance model.

4. Numerical analysis

In this section, we focus on the effect of the insurer and reinsurer's ambiguity aversion levels on the reinsurance strategies and the reinsurance price. Using exponential utility, we analyze the actions which the insurer or the reinsurer would take depending on their aversion or neutrality to ambiguity. We use the acronyms AAR, AAI, ANR, and ANI, for the ambiguity averse or neutral reinsurer or insurer. We find that in some cases the ambiguity aversion parameters have no significant influence on the utility functions, but in some cases they have an active impact on the value functions. Finally, we analyze utility loss functions, i.e. the proportion of utility that is given up as a consequence of adopting a robust strategy; we focus particularly on how these loss functions are impacted by the framework's important parameters, especially those measuring ambiguity preference. Throughout this section, unless otherwise stated, the basic parameters are given by $r=0.03, \gamma=0.5, m=0.4, T=4, \lambda=1, \mu=0.15, \sigma=0.35, \eta=0.15, \beta_1=4, \beta_2=4, \alpha_1=4, \alpha_2=4, x=1$. Moreover, we assume that the claim size Z_i follows the exponential distribution with parameter $\hat{\lambda}=6$ (an average of one claim every two months, corresponding to the idea of an insurer with a small number of large clients), and therefore the claim size density function is $f(z)=6\,e^{-6z}$.

4.1. Impact of ambiguity on the insurer's excess-of-loss reinsurance strategies, and on the reinsurance premiums

From the discussion in Section 3, we know that the excess-of-loss reinsurance strategies a_{ij} and the prices or premiums for reinsurance θ_{ij} only depend on the ambiguity aversion of the insurer and

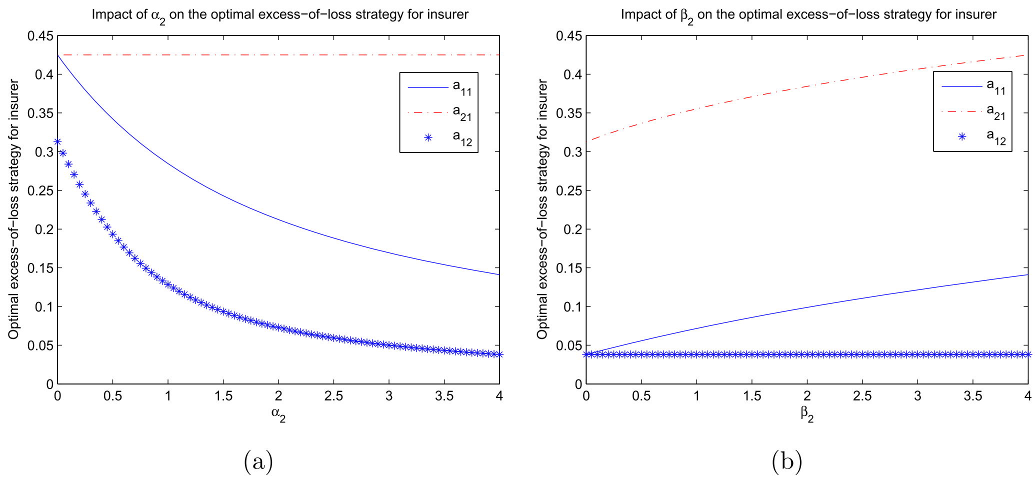
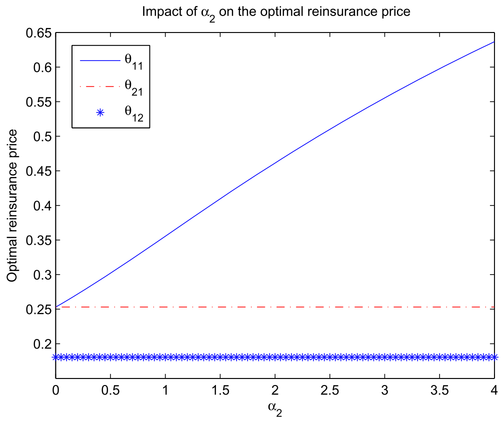


Figure 1. Impact of α_2 and β_2 on the excess-of loss reinsurance strategies of the insurer at time $t = 2$. Note: in the figure a_{21} is red.

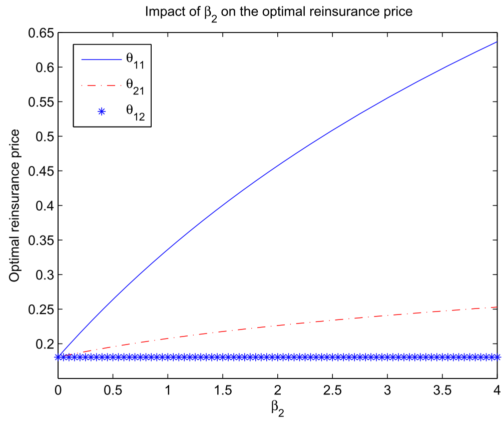
the reinsurer for the negative claims, i.e. α_2 and β_2. First, for the impact of α_2 and β_2 on the reinsurance strategies, Figure 1(a) shows that the retention levels a_{11} and a_{12} will decrease quickly with an increase in the ambiguity-aversion level α_2 in the beginning, but this sensitivity is weaker when α_2 is greater than 2. In other words, as one would expect intuitively, when the insurer feels more uncertain about the negative claims, he would prefer to depend more on reinsurance; certainly, an AAI depends more heavily on reinsurance than an ANI. He decreases heavily his retention level with the initial increase of α_2, but increases smoothly his retention level with the increase of the reinsurer's ambiguity aversion β_2, particularly for larger values of β_2, resulting in more expensive reinsurance. These effects are illustrated in Figure 1(b). Comparing Figure 1(a,b), we notice that α_2 takes a more important role than β_2 in the reinsurance strategies, because the reinsurance strategies slowly increase with β_2 and a_{21} is always greater than a_{11}. In all cases, the reinsurance strategies are sensitive to parameter α_2, i.e. the insurer pays close attention to his own ambiguity to claims-risk modeling.

Similarly to the analysis of the reinsurance strategy, we study the impact of ambiguity aversion parameters α_2 and β_2 on the price of reinsurance. Intuitively, the reinsurer may be tempted to increase the value of the reinsurance price θ if her own ambiguity level increases, to make up that added uncertainty. However, the ANR would not do so for $\theta^*_{12} = \theta^*_{22}$. Even in the case of AAR and ANI, the AAR's price increase is negligible as her ambiguity level β_2 rises. We only find notable sensitivities of reinsurance prices θ on ambiguity levels in the case that both parties are ambiguity averse. As Figure 2 shows, the price of the reinsurance will rise with the increase of α_2 and β_2. Also, the reinsurance prices rise with β_2 when the insurer is neutral on claims modeling ambiguity. In other cases, the reinsurance price is nearly insensitive to β_2 and α_2. In Section 3 we saw that $\theta_{12} = \theta_{22}$. We get the result that if the reinsurer is ambiguity-neutral (ANR), she will have no concern over whether the insurer is ANI, the reinsurance price tends to be a function of time t only. Similarly, if the reinsurer is an AAR, and the insurer is an ANR, the reinsurer will tend to ignore her own ambiguity in making decision when her ambiguity aversion level is not too big. The situation changes completely when both players are ambiguity averse. In particular, the reinsurer would pay close attention to the insurer's ambiguity, i.e. she would increases the price of reinsurance when α_2 and β_2 increase.

Summarizing the discussion above, we conclude that the ambiguity levels for the claims are key factors in decision-making, which should be taken into account carefully in designing a contract when both parties are ambiguity averse.



(a)



(b)

Figure 2. Impact of α_2 and β_2 on the reinsurance price of reinsurer at time $t = 2$.

4.2. Impact of ambiguity on the value functions

We have already determined above that financial market model's parameters do not affect the reinsurance strategy and the price of reinsurance, but they have an active effect on both the insurer's and the reinsurer's utility functions because the opportunity for risky investments is a main measure that they would use to improve profits from their reserves. Accordingly, in this subsection, the impact of ambiguity aversion α_1, α_2 and β_2 will be discussed. From Figure 3, we find that all forms of the insurer's utility function decrease with respect to time t, α_1, α_2 and β_2, except J_{12} and J_{21} which are largely insensitive to the claims-risk modeling preference parameters α_2 and β_2. In other words, the insurer's utility function will not be sensitive to ambiguity levels about claims modeling when one part of the reinsurance contract is ambiguity-averse, and the other part is ambiguity-neutral.

As mentioned in the above Subsection 4.1, not only does the insurer pay close attention to his own ambiguity, but the reinsurer is also very sensitive to her client's ambiguity. This has important practical applications when writing reinsurance contracts, since it suggests that a discussion between both parties is critically needed, so that the reinsurer can understand the insurer's confidence level about his own risk modeling.

Let us be qualitatively specific. On the one hand, α_1 and α_2 have a negative effect on the insurer's utility function. The reason is that the increase of α_1 and α_2 will force the insurer to decrease his financial investment and his reinsurance level, and those would have positive effects on his utility function, as can be seen in Figure 6 of Gu et al. (2017). On the other hand, α_2 has a positive effect on the reinsurer's utility function, and the reinsurer's optimal value function is more sensitive to α_2. This fact is quite clearly illustrated in Figure 4(a). Somewhat surprisingly, V_{12} is most sensitive to α_2 in all cases. This tells us that the reinsurer must pay very close attention to the insurer's ambiguity level, or face a significant loss in her utility function. We will have more to say about these kinds of losses in the next subsection. We notice that V_{11} increases slightly with β_2, even though it is not very sensitive to this parameter. In other words, the reinsurer can attempt to raise her utility by increasing her ambiguity aversion level, but this is not a very effective means of increasing profits. This can be viewed as reassuring, since, in the asymmetric framework of a principal-agent contract negotiation, for the principal to increase arbitrarily her ambiguity level for the sole purpose of squeezing out more profit, could be seen as an inefficient way of hiding information from the agent. In any case, the most effective measure for the reinsurer to have a good handle on her value function, and to make sure it is as high as possible, is to estimate the insurer's ambiguity level and adopt the corresponding optimal reinsurance price.

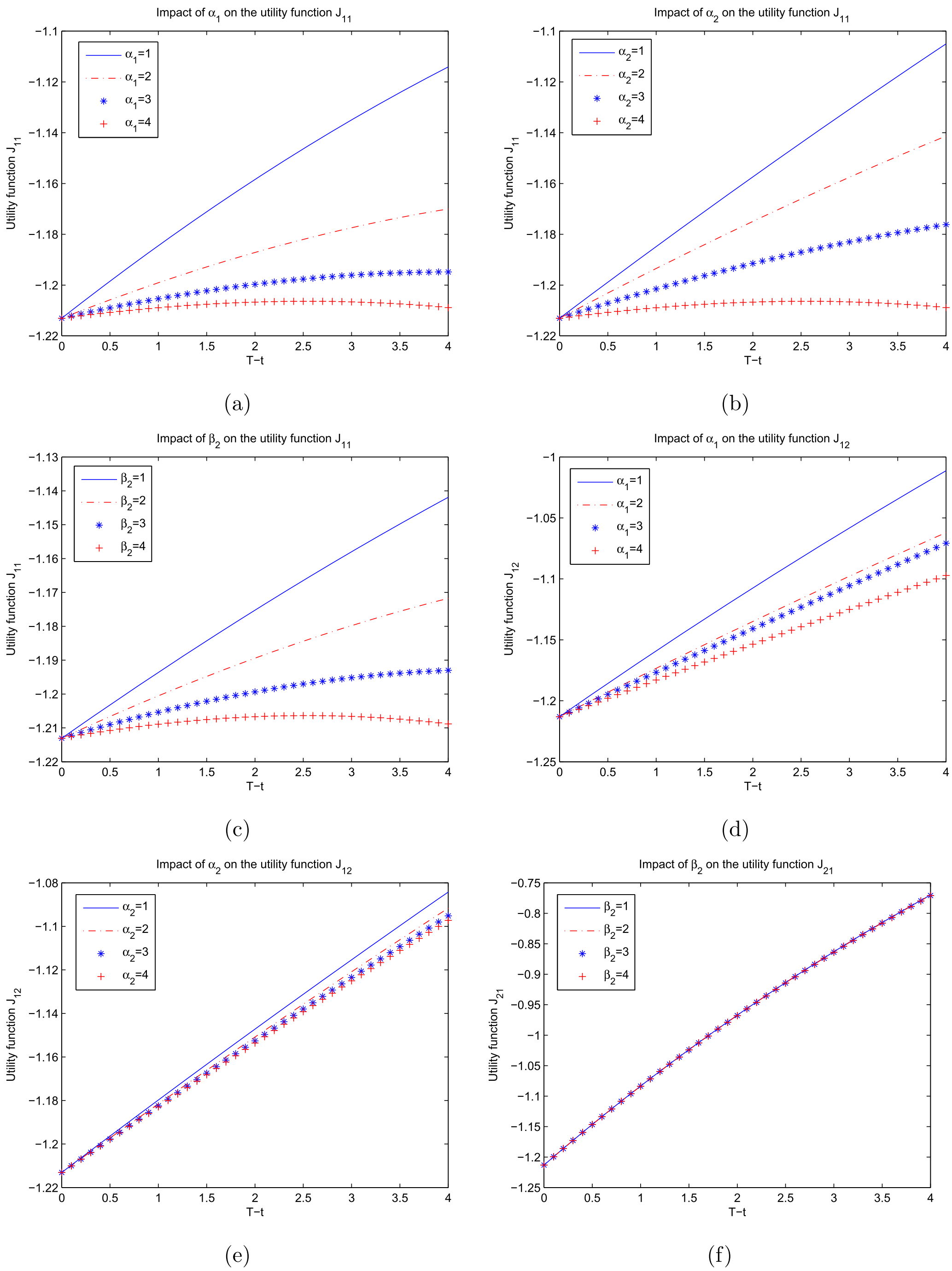


Figure 3. Impact of α_1, α_2 and β_2 on the insurer's utility functions.

Summarizing the above analysis: when the insurer is an ANI, it is not necessary for the reinsurer to consider ambiguity when she makes her decisions; but when the insurer is an AAI, the reinsurer has to understand both her own ambiguity aversion level and the insurer's in order to follow an optimal strategy; and in this case, understanding the insurer's ambiguity aversion level is quantitatively very significant.

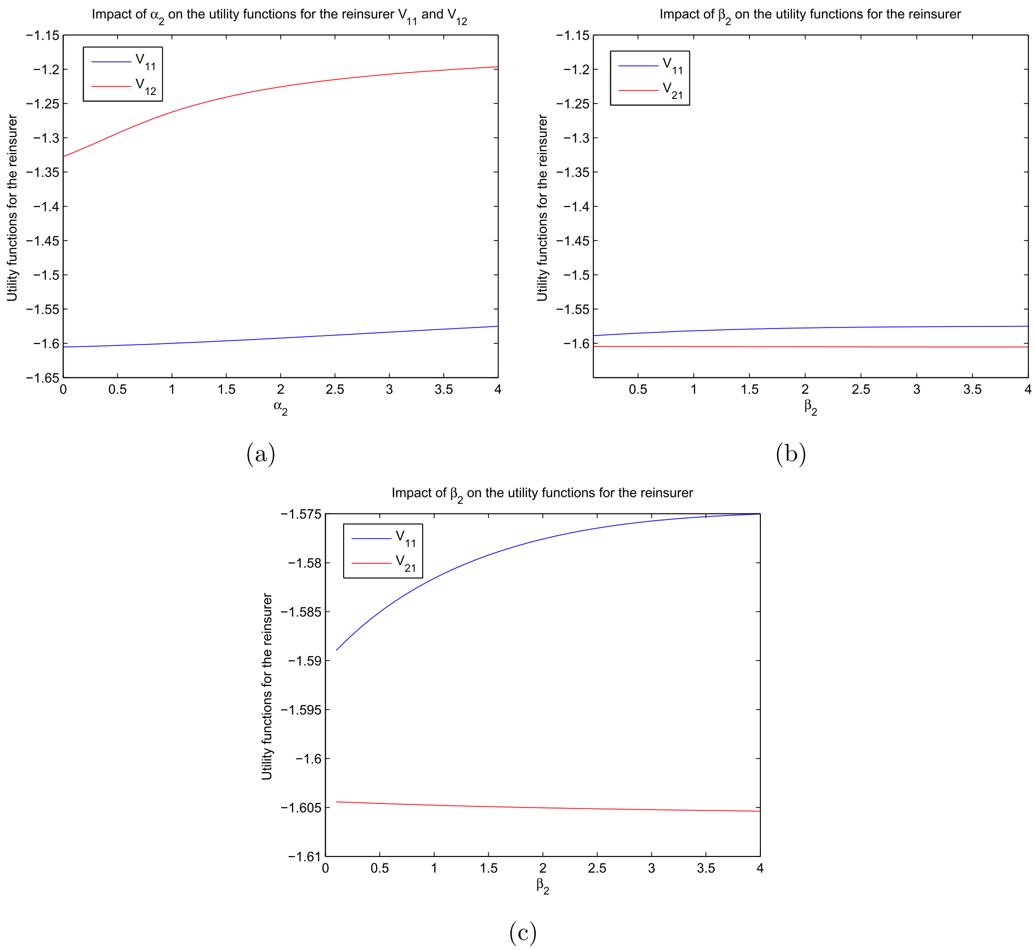


Figure 4. Impact of α_2 and β_2 on the reinsurer's utility functions.

4.3. Utility loss functions

Firstly, we discuss the utility loss functions for the insurer and the reinsurer, which are caused by adopting a robust strategy to deal with ambiguity aversion, in comparison to the (overly optimistic) situation in which the estimated models are assumed to be correctly specified. We strive to present the positive and negative effects of ambiguity aversion on both parties in the reinsurance contract.

First, we define the utility loss functions for the insurer

$$L_{ij} = 1 - \frac{J_{22}}{J_{ij}}, \quad i, j = 1, 2,$$

where J_{ij} is the value function of the insurer under the reinsurance-investment strategy, in which the reinsurance strategy is determined by the reinsurance contract with the insurer in the state i and the reinsurer in the state j. Recall that i or $j = 1$ indicates that the insurer or the reinsurer is ambiguity averse (is an AAI or AAR), while i or $j = 2$ indicates corresponding neutrality towards ambiguity (ANI or ANR). Since J_{22} is the utility assuming that the estimated models are correct, thus L_{ij} represents the proportion of loss one must accept for following a strategy that is robust towards the ambiguity scenario (i, j). Similarly, using the value function V_{ij}, we define the utility loss functions

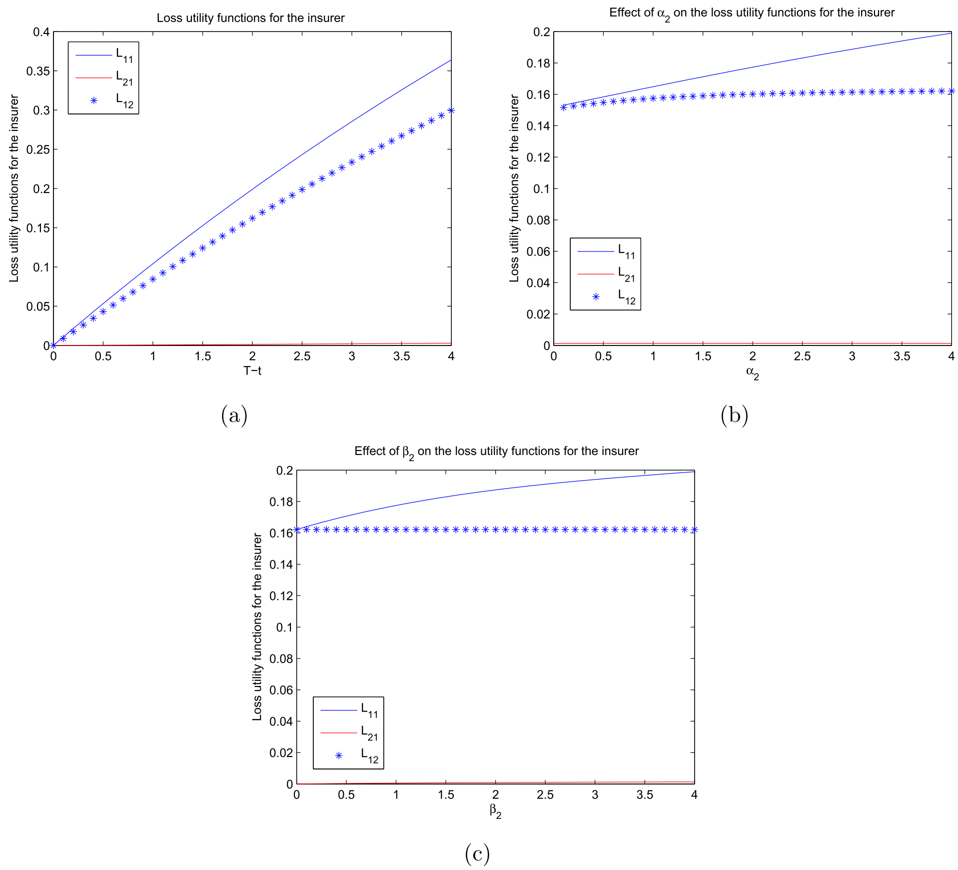


Figure 5. Impact of time t, α_2 and β_2 on the insurer's utility loss functions.

for the reinsurer

$$RL_{ij} = 1 - \frac{V_{22}}{V_{ij}}.$$

Figure 5(a) with $\alpha_i = \beta_i = 4,\ i = 1, 2$ shows that the utility loss functions for the insurer will decrease with time t. Figure 5(b,c) illustrates that the utility loss changes with α_2 and β_2. Surprisingly, the utility loss $L_{21} = 0$, which means that $J_{21} = J_{22}$, and the ANI will not see any difference in utility whether the reinsurer is an AAR or ANR; he only needs to focus on his ambiguity-aversion. This is an aspect of the contract asymmetry in a principal-agent framework. However, when the insurer is an AAI, Figure 5 tells us that $L_{11} > L_{12} > L_{21} = 0$, which shows that ambiguity aversion always causes the insurer to get less utility no matter whether the reinsurer is ambiguity averse or not; on the other hand, the reinsurer's ambiguity aversion has a positive effect on the utility loss (as we hinted earlier), but the impact is small: the difference between L_{11} and L_{12} is not large and disappears as time closing to maturity T. All these points towards saying that the reinsurer's ambiguity has hardly any effect on the insurer's strategy and utility; the insurer only needs to concern himself about his own ambiguity.

Figure 6 illustrates that $RL_{21} > RL_{11} > 0 > RL_{12}$. We notice that $RL_{12} < 0$. This is consistent with our intuition. The insurer would like to buy more reinsurance to transfer his risk to the reinsurer when he is averse to ambiguity, and as a result, the reinsurer will pick up more business, causing her utility function to improve. From Figure 6(b), the utility loss RL_{11} keeps close to RL_{21} when time t is close to maturity T. This means that an ambiguity-averse reinsurer would pay little attention to the insurer's ambiguity, but an ambiguity-neutral reinsurer would like to do more business with the

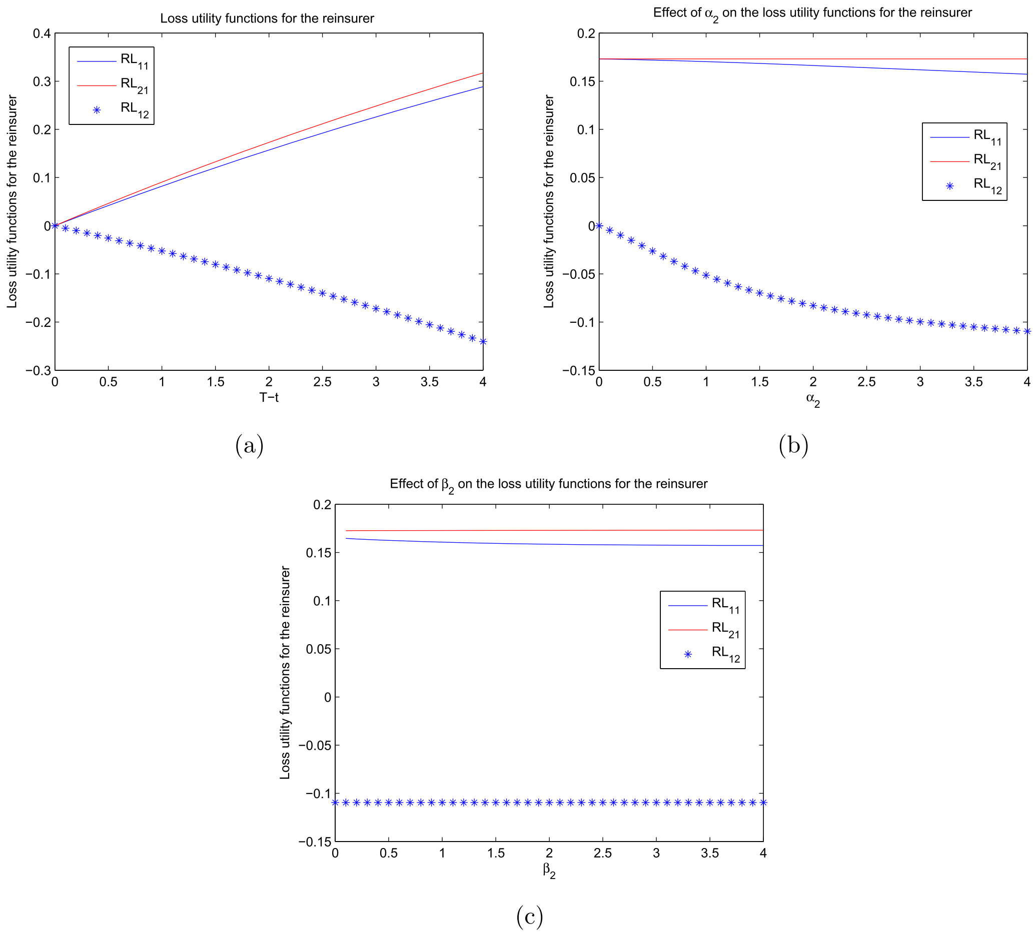


Figure 6. Impact of time t, α_2 and β_2 on the reinsurer's utility loss functions.

ambiguity-averse insurer. All these figures show that the levels of utility losses are all moderate. Thus, broadly speaking, ambiguity for the insurer or the reinsurer will cause some utility loss but the loss is not substantial. Moreover, if we assume that the ambiguity aversion stays unchanged with time t, the reinsurer's ambiguity will cause very little or no concern to either party, while the insurer should care about his own ambiguity.

From the discussion above, we know that the utilities of the insurer or the reinsurer see little difference between the cases with ambiguity or no ambiguity. This is an excellent point in favor of using robust optimal strategies. It is reasonable for both the insurer and the reinsurer to follow strategies, since they do not need to worry much at all about their utility losses.

But there is a second way of seeing how important it is to follow a strategy that is robust to modeling ambiguity: what happens if one is overly confident in one's model, and one follows the strategy it dictates, not a robust strategy, but the model turns out to be wrong. To address this question, we define the notion of real utility loss function for the AAR (ambiguity averse reinsurer), and the reason causing this real loss is that she uses a strategy which is non-robust with respect to her own ambiguity and the insurer's. We can assume that the insurer is ambiguity neutral (ANI) or ambiguity averse (AAI). Thus in both cases, the reinsurer believes that $(a^*_{22}(t), \theta^*_{22}(t), \hat{u}^*_{22}(t))$ is her optimal strategy, but the optimal strategy is a robust one, either with subscript $(1, 1)$ or $(2, 1)$. With the $(2, 2)$ strategy, we can easily calculate the value functions $\overline{V}_{11}$ and $\overline{V}_{21}$ corresponding to what the insurer will do in

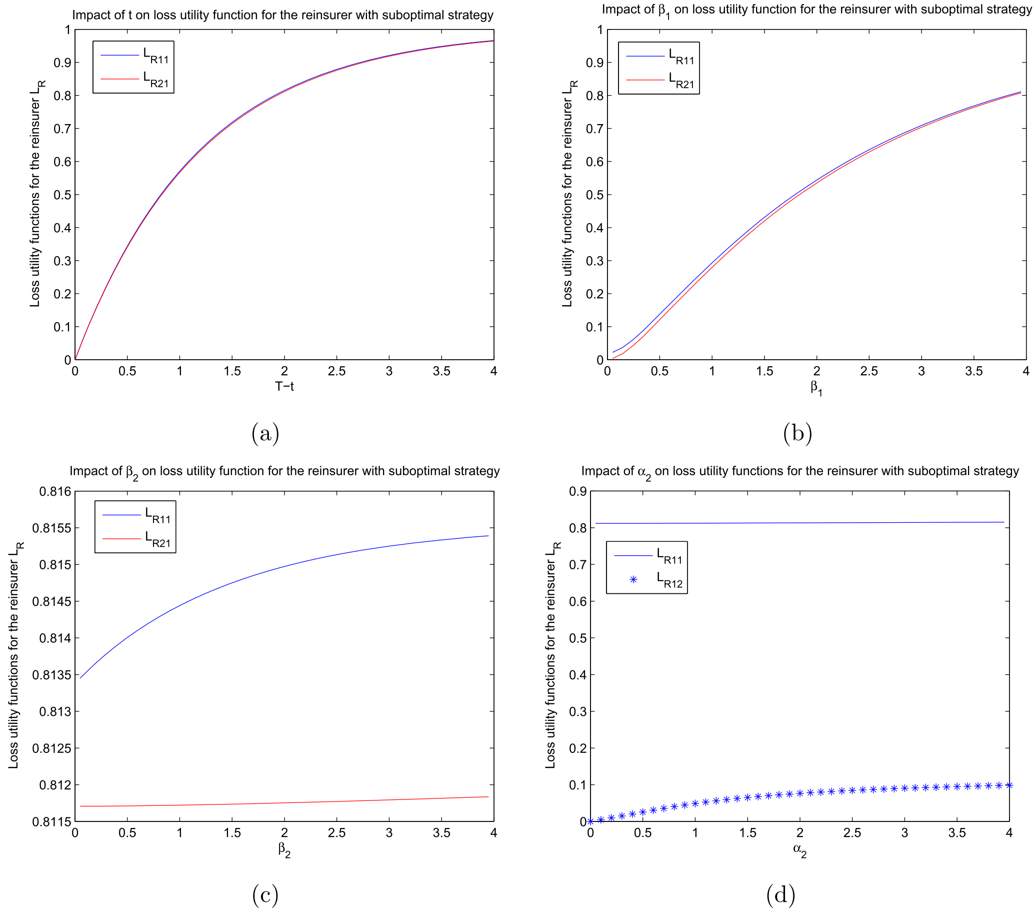


Figure 7. Impact of time t, β_1 and β_2 on the utility loss functions of the reinsurer with suboptimal strategy.

our principal-agent framework. In fact, we have

$$\overline{V}_{11} = \overline{V}_{21} = -\frac{1}{m}\exp\{-m(e^{r(T-t)}w + \overline{g}(t))\}$$

where

$$\overline{g}(t) = \int_t^T \left[\lambda\, e^{r(T-s)}(\mu_\infty - \int_0^{a_{22}^*(s)} \overline{F}(z)\,\mathrm{d}z)(1+\theta_{22}^*(s)) - \frac{\lambda\overline{\phi}^*(s)}{\beta_2}\right]\mathrm{d}s + \left[\frac{\mu^2}{2\sigma^2}\left(\frac{1}{m} - \frac{\beta_1}{m^2}\right) + \frac{\lambda}{\beta_2}\right](T-t)$$

and $\overline{\phi}^*(t) = \exp\{-\beta_2\, e^{r(T-t)}\, e^{-a_{22}^*(t)\hat{\lambda}}/m\, e^{r(T-t)-\hat{\lambda}}\}$.

Now we define the utility loss function for the reinsurer $\overline{L}_{R11} = 1 - V_{11}/\overline{V}_{11}$ and $\overline{L}_{R21} = 1 - V_{21}/\overline{V}_{21}$. Again, these are the proportions of utility loss that the reinsurer will experience for following the non-robust optimal strategy, when in fact she is ambiguity-averse, whether she deals with an AAI or an ANI. In Figure 7, as we can observe, the utility loss will decrease with time t and increase with β_1 and β_2. It is especially striking to see that ignoring β_2, i.e. for an AAR to follow the strategy of an ANR mistakenly, causes a heavy utility loss, about 80%, even when the value of β_2 is small.

Comparing Figures 6 and 7, we can conclude that the reinsurer's ambiguity aversion will not cause much loss in utility function, but ignoring ambiguity for the AAR will cause heavy utility loss. Also, if the reinsurer is ANR, and she adopts the suboptimal strategy $(a^*_{22}(t), \theta^*_{22}(t), \hat{u}^*_{22}(t))$ as her optimal strategy when dealing with an ambiguity averse client (AAI), her value function $\overline{V}_{12}$ will equal V_{22}. Because of this, by ignoring the insurer's ambiguity aversion, the ANR will lose the chance to gain more utility, as one can see in the curve labeled as RL_{12} in Figure 6.

In conclusion, in order to avoid utility loss, the reinsurer should pay close attention to the ambiguity of both parties to the contract, and design her optimal robust strategy as corresponding to the appropriate ambiguity levels.

5. Conclusion

In this paper, we discuss an optimal excess-of-loss reinsurance contract in a continuous-time principal-agent framework where the surplus of the insurer is described by a C-L model. Here, two parties of the contract are ambiguity averse and have specific modeling risk aversion preferences for the insurance claims and the financial market's risk. In addition to reinsurance, they put their surpluses into the financial market containing one risk-free asset and one risky asset. The reinsurer designs a reinsurance contract that maximizes the exponential utility of her terminal wealth under a worst-case scenario which depends on the retention level of the insurer. By employing the dynamic programming approach, we derive the optimal robust reinsurance contract, and the value functions for the reinsurer and the insurer under this contract. At last, we discuss the case when the claims follow an exponential distribution and find some interesting results. In the robust framework, ignoring the ambiguity aversion of the insurer and the reinsurer will not cause much utility loss, but ignoring ambiguity for the AAR will cause heavy utility loss if she mistakes the suboptimal strategy as the optimal strategy.

One limitation of the current paper is focusing on the ambiguity on the claim intensity. Keeping the ambiguity to the claim intensity only allows us to work with the robust optimization method proposed in Maenhout (2004, 2006). If we also consider the ambiguity on the claim size, this would lead to different expectations with each different probability law and make our study mathematically more difficult. In order not to mask the efficiency of the method and the new robust game theory framework presented in the paper, we leave this interesting topic of claim size ambiguity as our future research direction, for which additional mathematical methodology (see, e.g. Jin et al. 2017) would be required.

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Appendix

Proof of Lemma 2.1: Consider any $\pi \in \Pi$ and $\psi \in \Psi$. Condition (iii) in Definition 2.1 implies the following integrability result

$$E^{\psi}\left[\left|-\frac{1}{\gamma}\,e^{-\gamma X^{\pi,\psi}(T)}+\int_0^T G^{\psi}(J(s,X^{\pi,\psi}(s),\pi,\psi))\,\mathrm{d}s\right|\right]<\infty,$$

So, the following process

$$M(t) := J(t, X^{\pi,\psi}(t), \pi, \psi) + \int_0^t G^{\psi}(J(s, X^{\pi,\psi}(s), \pi, \psi))\,ds$$
$$= E_t^{\psi}\left[-\frac{1}{\gamma}e^{-\gamma X^{\pi,\psi}(T)} + \int_0^T G^{\psi}(J(s, X^{\pi,\psi}(s), \pi, \psi))\,ds\right]$$

is a $\mathbb{Q}$-martingale. By the product rule, we have

$$d\big[e^{-\int_0^t \rho(\psi(s))\,ds}J(t, X^{\pi,\psi}(t), \pi, \psi)\big] = -e^{-\int_0^t \rho(\psi(s))\,ds}J(t, X^{\pi,\psi}(t), \pi, \psi)\rho(\psi(t))\,dt$$
$$+ e^{-\int_0^t \rho(\psi(s))\,ds}\big[dM(t) - G^{\psi}(J(t, X^{\pi,\psi}(t), \pi, \psi))\,dt\big]$$
$$= e^{-\int_0^t \rho(\psi(s))\,ds}\,dM(t). \tag{A1}$$

Moreover, for any $\psi \in \Psi$, we know

$$\rho(\psi(s)) = \gamma\left(\frac{h^2(s)}{2\alpha_1} + \frac{\lambda(\phi(s)\ln\phi(s) - \phi(s) + 1)}{\alpha_2}\right) \geq 0.$$

Hence, associated with any $\pi \in \Pi$ and $\psi \in \Psi$, the process $\int_0^t e^{-\int_0^s \rho(\psi(v))dv}dM(v)$ is also a $\mathbb{Q}$-martingale. Integrating from t to T in Equation (A1) and taking conditional expectations on both sides, we obtain

$$E_t^{\psi}\left[e^{-\int_0^T \rho(\psi(s))\,ds}J(T, X^{\pi,\psi}(T), \pi, \psi)\right] - e^{-\int_0^t \rho(\psi(s))\,ds}J(t, X^{\pi,\psi}(t), \pi, \psi) = 0.$$

Then, the penalized recursive utility is equivalent to an additive utility

$$J(t, X^{\pi,\psi}(t), \pi, \psi) = E_t^{\psi}\left[e^{-\int_t^T \rho(\psi(s))\,ds}J(T, X^{\pi,\psi}(T), \pi, \psi)\right] = E_t^{\psi}\left[e^{-\int_t^T \rho(\psi(s))\,ds}\left(-\frac{1}{\gamma}e^{-\gamma X^{\pi,\psi}(T)}\right)\right].$$

Therefore, the original insurer's optimization problem can be transformed to a robust control problem without recursive utility as follows:

$$J(t,x) = \sup_{\pi\in\Pi}\inf_{\psi\in\Psi}\left\{E_{t,x}^{\psi}\left[e^{-\int_t^T \rho(\psi(s))\,ds}\left(-\frac{1}{\gamma}e^{-\gamma X^{\pi,\psi}(T)}\right)\right]\right\},$$

where $\rho(\psi(s))$ is considered as a discount rate. ■

Proof of Proposition 2.2: We assume that $S(t,x)$ is the solution of the HJB equation (10). Since we do not handle model uncertainty on the claim size random variable Z_i, its probability law remains the same under $\mathbb{P}$ and $\mathbb{Q}$. Therefore, in the HJB equation (10) the expectation involving Z_i under $\mathbb{Q}$ can be replaced by that under $\mathbb{P}$, that is,

$$E^{\psi}[S(t, x - \min(Z_i, a(t))) - S(t,x)] = E[S(t, x - \min(Z_i, a(t))) - S(t,x)], \quad \forall\, \psi \in \Psi,$$

where $E^{\psi}[\cdot] = E^{\mathbb{Q}}[\cdot]$ denotes the expectation under $\mathbb{Q}$, and $E[\cdot]$ represents the expectation taken under the reference probability measure $\mathbb{P}$. In the following, we always use the expectation taken under $\mathbb{P}$.

According to the first-order condition for $h(t)$ and $\phi(t)$ in HJB equation (10), we have

$$S_x(-x)\sigma u(t) - \frac{\gamma S}{2\alpha_1}2h(t) = 0$$

and

$$\lambda E[S(t, x - \min(Z_i, a(t))) - S(t,x)] - \frac{\gamma S\lambda}{\alpha_2}\ln\phi(t) = 0.$$

By some simple algebra, we obtain

$$h^*(t) = -\frac{\alpha_1 x\sigma u(t)}{\gamma S}S_x \tag{A2}$$

and

$$\phi^*(t) = \exp\left\{\frac{\alpha_2}{\gamma S}E[S(t, x - \min(Z_i, a(t))) - S(t,x)]\right\}. \tag{A3}$$

Then the optimal distortion $\psi^*(t) = (h^*(t), \phi^*(t))$ determines the worst-case measure $\mathbb{Q}^*$. Inserting $\psi^*(t)$ into the above HJB equation, we obtain

$$\sup_{\pi\in\Pi}\left\{S_t + S_x(xr + x\mu u(t)) + (\eta - \theta(t))\lambda\mu_\infty + \lambda(1+\theta)\int_0^{a(t)} \overline{F}(z)\,\mathrm{d}z \right.$$
$$\left. + \frac{1}{2}S_{xx}x^2\sigma^2u^2(t) + \frac{\gamma\lambda S}{\alpha_2}(\phi^* - 1) + \frac{S_x^2x^2\sigma^2u^2(t)\alpha_1}{2\gamma S}\right\} = 0. \tag{A4}$$

We try the following ansatz for the value function

$$S(t,x) = -\frac{1}{\gamma}\exp\{-\gamma(e^{r(T-t)}x + g(t))\}$$

where $g(t)$ satisfies the condition $g(T) = 0$.

Based on the equation above, we know that

$$E[S(t, x - \min(Z_i, a(t))) - S(t,x)] = S(t,x)\gamma\, e^{r(T-t)}\int_0^{a(t)} e^{\gamma z\, e^{r(T-t)}}\overline{F}(z)\,\mathrm{d}z.$$

Therefore, according to (A4) and the first-order condition for $u(t)$, we derive that

$$S_x x\mu + S_{xx}x^2\sigma^2u(t) + \frac{\alpha_1x^2\sigma^2S_x^2}{\gamma S}u(t) = 0.$$

So, the optimal investment strategy for the insurer reads

$$u^*(t) = -\frac{x\mu S_x}{x^2\sigma^2(S_{xx} + \frac{\alpha_1S_x^2}{\gamma S})} = \frac{\mu}{x\sigma^2(\alpha_1+\gamma)\,e^{r(T-t)}}. \tag{A5}$$

According to the first-order condition for $a(t)$, the following is satisfied

$$S_x\lambda(1+\theta(t))\overline{F}(a(t)) + \frac{\gamma S\lambda}{\alpha_2}\frac{\mathrm{d}\phi^*(t)}{\mathrm{d}a(t)} = 0.$$

Recalling $\mathrm{d}\phi^*(t)/\mathrm{d}a(t) = \phi^*(t)\alpha_2\, e^{r(T-t)}\, e^{\gamma a(t)\, e^{r(T-t)}}\overline{F}(a(t))$, we know that $a(t)$ satisfies the equation

$$(\phi^*\, e^{\gamma a(t)\, e^{r(T-t)}} - 1 - \theta(t))\overline{F}(a(t)) = 0.$$

Due to $0 \le F(z) < 1$ for claim size z, we have $\overline{F}(a(t)) \neq 0$. Thus,

$$1 + \theta(t) = \phi^*(t)\, e^{\gamma\, e^{r(T-t)}a(t)}. \tag{A6}$$

According to Equation (A3), we know $\phi^*(t)\, e^{\gamma\, e^{r(T-t)}a(t)} \le 1$ if $a(t) \le 0$, this will contradict with the fact that $\phi^*(t)\, e^{\gamma\, e^{r(T-t)}a(t)} = 1 + \theta(t) > 1$ where $\theta(t) \ge \eta > 0$. Therefore, we derive the optimal reinsurance strategy

$$a^*(t) = \frac{\ln(1+\theta(t)) - \ln\phi^*(t)}{\gamma\, e^{r(T-t)}} > 0.$$

Putting $u^*(t)$ and $a^*(t)$ into Equation (A4), we have

$$S_t + S_x(xr + (\eta - \theta(t))\lambda\mu_\infty + \lambda(1+\theta(t))\int_0^{a^*(t)}\overline{F}(z)\,\mathrm{d}z + \frac{\gamma S\lambda}{\alpha_2}(\phi^* - 1) - \frac{\gamma\mu^2S}{2(\alpha_1+\gamma)\sigma^2} = 0.$$

Plugging S_t and S_x into the above equation, we derive

$$g'(t) + e^{r(T-t)}\left[(\eta - \theta(t))\lambda\mu_\infty + \lambda(1+\theta(t))\int_0^{a^*(t)}\overline{F}(z)\,\mathrm{d}z\right] - \frac{\lambda}{\alpha_2}(\phi^*(t) - 1) + \frac{\mu^2}{2(\alpha_1+\gamma)\sigma^2} = 0,$$

with boundary condition $g(T) = 0$. Thus,

$$g(t) = \int_t^T e^{r(T-s)}\left[(\eta - \theta(s))\lambda\mu_\infty + \lambda(1+\theta(s))\int_0^{a^*(s)}\overline{F}(z)\,\mathrm{d}z\right]\mathrm{d}s - \frac{\lambda}{\alpha_2}\int_t^T\phi^*(s)\,\mathrm{d}s$$
$$+ \left(\frac{\lambda}{\alpha_2} + \frac{\mu^2}{2(\alpha_1+\gamma)\sigma^2}\right)(T-t).$$

■

Proof of Proposition 2.4: Assume that $H(t,w)$ is the solution of HJB equation (19). According to the first-order condition $\hat{h}_{11}(t)$ and $\hat{\phi}_{11}(t)$ in equation (19), we have the following equations:

$$-H_w w\sigma\hat{u}_{11}(t) - \frac{mH}{\beta_1}\hat{h}_{11}(t) = 0,$$

and

$$\lambda E[H(t, w - (Z_i - \min(Z_i, a(t)))) - H(t,w)] - \frac{mH}{\beta_2}\lambda \ln \hat{\phi}_{11}(t) = 0.$$

Solving the equations above, we attain the optimal distortion $\hat{\psi}^*_{11}(t) = (\hat{h}^*_{11}(t), \hat{\phi}^*_{11}(t))$:

$$\hat{h}^*_{11}(t) = -\frac{\beta_1 w\sigma\hat{u}_{11}(t)}{m}\frac{H_w}{H},$$

and

$$\hat{\phi}^*_{11}(t) = \exp\left\{\frac{\beta_2}{mH}E[H(t, w - (Z_i - \min(Z_i, a^*(t)))) - H(t,w)]\right\}.$$

Inserting $\hat{\psi}^*_{11}(t)$ into Equation (19), we have

$$H_t + H_w(wr + w\mu\hat{u}_{11}(t)) + H_w\left[(1+\theta(t))\lambda\left(\mu_\infty - \int_0^{a^*(t)}\overline{F}(z)\,\mathrm{d}z\right)\right] + \frac{1}{2}H_{ww}w^2\hat{u}^2_{11}(t)\sigma^2 + \frac{\beta_1 w^2\sigma^2\hat{u}^2_{11}(t)H_w^2}{2mH}$$
$$+ \frac{m\lambda}{\beta_2}H(\hat{\phi}^*_{11}(t) - 1) = 0. \tag{A7}$$

By differentiating (A7) with respect to $\hat{u}_{11}(t)$, we have

$$H_w w\mu + H_{ww}w^2\sigma^2\hat{u}_{11}(t) + \frac{\beta_1 w^2\sigma^2\hat{u}_{11}(t)H_w^2}{mH} = 0.$$

Thus, the optimal investment strategy is given by

$$\hat{u}^*_{11}(t) = -\frac{w\mu H_w}{w^2\sigma^2(H_{ww} + \frac{\beta_1 H_w^2}{mH})}.$$

By differentiating (A7) with respect to $\theta(t)$, note that $\hat{\phi}^*_{11}(t)$ and $a^*(t)$ are functions of $\theta(t)$, we have

$$H_w\lambda\left(\mu_\infty - \int_0^{a^*(t)}\overline{F}(z)\,\mathrm{d}z\right) + H_w(1+\theta(t))\lambda(-1)\overline{F}(a^*(t))\frac{\mathrm{d}a^*(t)}{\mathrm{d}\theta(t)} + \frac{m\lambda}{\beta_2}H\frac{\mathrm{d}\hat{\phi}^*_{11}(t)}{\mathrm{d}a^*(t)}\frac{\mathrm{d}a^*(t)}{\mathrm{d}\theta(t)} = 0.$$

According to the expressions of $a^*(t)$ and $\hat{\phi}^*_{11}(t)$,

$$\frac{\mathrm{d}a^*(t)}{\mathrm{d}\theta(t)} = \frac{1}{\gamma\, e^{r(T-t)}(1+\theta(t))}$$

and

$$\frac{\mathrm{d}\hat{\phi}^*_{11}(t)}{\mathrm{d}a^*(t)} = \frac{\hat{\phi}^*_{11}(t)\beta_2}{mH}\frac{\mathrm{d}}{\mathrm{d}a^*(t)}E[H(t, w - (Z_i - \min(Z_i, a^*(t)))) - H(t,w)].$$

As a result, the optimal reinsurance premium $\theta^*(t) := \theta^*_{11}(t)$ is determined by the following equation:

$$\lambda\left(\mu_\infty - \int_0^{a^*(t)}\overline{F}(z)\,\mathrm{d}z\right)H_w + \left[\lambda\hat{\phi}^*_{11}(t)\frac{\mathrm{d}}{\mathrm{d}a^*(t)}E[H(t, w - (Z_i - \min(Z_i, a^*(t)))) - H(t,w)]\right.$$
$$\left. -\lambda(1+\theta(t))\overline{F}(a^*(t))H_w\right]\frac{1}{\gamma\, e^{r(T-t)}(1+\theta(t))} = 0.$$

As a direct result, the corresponding reinsurance strategy for the insurer satisfies

$$a^*_{11}(t) = a^*(\theta^*_{11}(t)) = \frac{\ln(1+\theta^*_{11}(t)) - \ln\phi^*_{11}(t)}{\gamma\, e^{r(T-t)}}.$$

Just as we solved optimization (PI), we propose an ansatz for the value function of the optimization problem (PR) :

$$H(t,w) = -\frac{1}{m}\exp\left\{-m(e^{r(T-t)}w + \hat{g}_{11}(t))\right\}$$

with $\hat{g}_{11}(T) = 0$. Therefore, we know that

$$\begin{aligned}
E[H(t, w - (Z_i - \min(Z_i, a_{11}^*(t)))) - H(t, w)] &= E[H(t, w)(\exp\{m\, e^{r(T-t)}(Z_i - \min(a_{11}^*(t)))\} - 1)] \\
&= H(t, w)E[\exp\{m\, e^{r(T-t)}(Z_i - \min(a_{11}^*(t), Z_i))\} - 1] \\
&= H(t, w)\left[F(a_{11}^*(t)) + \int_{a_{11}^*(t)}^{+\infty} \exp\{m(z - a_{11}^*(t))\, e^{r(T-t)}\}\, \mathrm{d}F(z) - 1\right] \\
&= H(t, w)\Bigg[F(a_{11}^*(t)) - 1 + \exp\{m\, e^{r(T-t)}(z - a_{11}^*(t))F(z)|_{a_{11}^*(t)}^{+\infty}\} \\
&\quad - \int_{a_{11}^*(t)}^{+\infty} F(z) m\, e^{r(T-t)} \exp\{m(z - a_{11}^*(t))e^{r(T-t)}\}\mathrm{d}z\Bigg] \\
&= Hm\, e^{r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{+\infty} \overline{F}(z)\, e^{m\, e^{r(T-t)} z}\, \mathrm{d}z,
\end{aligned}$$

and the worst-case scenario and the optimal strategy are given by

$$\begin{aligned}
\hat{h}_{11}^*(t) &= -\frac{\beta_1 w\sigma \hat{u}_{11}^*(t) H_w}{mH} = \frac{\beta_1 \mu}{\sigma(\beta_1 + m)} \\
\hat{\phi}_{11}^*(t) &= \exp\left\{\beta_2\, e^{r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{+\infty} \overline{F}(z)\, e^{mz\, e^{r(T-t)}}\, \mathrm{d}z\right\} \\
\hat{u}_{11}^*(t) &= -\frac{w\mu}{w^2\sigma^2} \frac{H_w}{H_{ww} + \frac{\beta_1 H_w^2}{mH}} = \frac{\mu}{w\sigma^2\, e^{r(T-t)}(\beta_1 + m)}.
\end{aligned}$$

Note

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a_{11}^*(t)} E\big[H\big(t, w - (Z_i - \min(a_{11}^*(t), Z_i))\big) - H(t, w)\big] \\
&= Hm\, e^{r(T-t)}\left[-m\, e^{r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{+\infty} \overline{F}(z)\, e^{mz\, e^{r(T-t)}}\, \mathrm{d}z - \overline{F}(a_{11}^*(t))\right].
\end{aligned}$$

Thus, $\theta_{11}^*(t)$ satisfies the following equation:

$$\begin{aligned}
&\left(\mu_\infty - \int_0^{a_{11}^*(t)} \overline{F}(z)\, \mathrm{d}z\right) H_w + \Bigg[-m^2 \hat{\phi}_{11}^*(t) H\, e^{2r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{+\infty} \overline{F}(z)\, e^{mz\, e^{r(T-t)}}\, \mathrm{d}z \\
&-mH\hat{\phi}_{11}^*(t)\, e^{r(T-t)} \overline{F}(a_{11}^*(t)) - (1 + \theta_{11}(t))\overline{F}(a_{11}^*(t)) H_w\Bigg] \frac{1}{\gamma\, e^{r(T-t)}(1 + \theta_{11}(t))} = 0.
\end{aligned}$$

Inserting the expression of H_w, we have

$$1 + \theta_{11}^*(t) = \frac{\hat{\phi}_{11}^*(t)[-\overline{F}(a_{11}^*(t)) - m\, e^{r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{+\infty} \overline{F}(z)\, e^{mz\, e^{r(T-t)}}\, \mathrm{d}z]}{(\mu_\infty - \int_0^{a_{11}^*(t)} \overline{F}(z)\, \mathrm{d}z)\gamma\, e^{r(T-t)} - \overline{F}(a_{11}^*(t))}. \tag{A8}$$

Note that $1 + \theta_{11}^* \geq 1 + \eta$. So, when $1 + \theta_{11}^* < 1 + \eta$, we let $1 + \theta_{11}^* = 1 + \eta$. Therefore, the robust optimal reinsurance contract is determined by

$$\begin{aligned}
a_{11}^*(t) &= \frac{\ln(1 + \theta_{11}^*(t)) - \ln \phi_{11}^*(t)}{\gamma\, e^{r(T-t)}}, \\
1 + \theta_{11}^*(t) &= \begin{cases} \dfrac{\hat{\phi}_{11}^*(t)[\overline{F}(a_{11}^*(t)) + m\, e^{r(T-t)}\, e^{-ma_{11}^*(t)\, e^{r(T-t)}} \int_{a_{11}^*(t)}^{+\infty} \overline{F}(z)\, e^{mz\, e^{r(T-t)}}\, \mathrm{d}z]}{\overline{F}(a_{11}^*(t)) - (\mu_\infty - \int_0^{a_{11}^*(t)} \overline{F}(z)\, \mathrm{d}z)\gamma\, e^{r(T-t)}}, & t \in O, \\ 1 + \eta, & t \in \overline{O}. \end{cases}
\end{aligned}$$

Inserting $a_{11}^*(t), \theta_{11}^*(t), \hat{\phi}_{11}^*(t)$ and $u_{11}^*(t)$ into the HJB equation yields,

$$H_t + H_w\left[wr + (1+\theta_{11}(t))\lambda\big(\mu_\infty - \int_0^{a_{11}^*(t)} \overline{F}(z)\,\mathrm{d}z\big)\right] + \frac{m\lambda H}{\beta_2}(\hat{\phi}_{11}^*(t)-1) - \frac{mH\mu^2}{2\sigma(\beta_1+m)} = 0.$$

Plugging H_w, H_t into the above equation, we have

$$\hat{g}_{11}'(t) + e^{r(T-t)}(1+\theta_{11}^*(t))\lambda\left(\mu_\infty - \int_0^{a_{11}^*(t)} \overline{F}(z)\,\mathrm{d}z\right) + \frac{\mu^2}{2\sigma^2(\beta_1+m)} - \frac{\lambda(\hat{\phi}_{11}^*(t)-1)}{\beta_2} = 0. \tag{A9}$$

Combining with the boundary condition $\hat{g}_{11}(T) = 0$, we derive

$$\hat{g}_{11}(t) = \int_t^T \left[e^{r(T-s)}(1+\theta_{11}^*(s))\lambda\left(\mu_\infty - \int_0^{a_{11}^*(s)} \overline{F}(z)\,\mathrm{d}z\right) - \frac{\lambda\hat{\phi}_{11}^*(s)}{\beta_2}\right]\mathrm{d}s + \left(\frac{\mu^2}{2\sigma^2(\beta_1+m)} + \frac{\lambda}{\beta_2}\right)(T-t). \tag{A10}$$

■