# Simple, Credible, and Approximately-Optimal Auctions

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We present a general framework for designing approximately revenue-optimal mechanisms for multi-item additive auctions, which applies to both truthful and non-truthful auctions. Given a (not necessarily truthful) single-item auction format $A$ satisfying certain technical conditions, we run simultaneous item auctions augmented with a personalized entry fee for each bidder that must be paid before the auction can be accessed. These entry fees depend only on the prior distribution of bidder types, and in particular are independent of realized bids. We bound the revenue of the resulting two-part tariff mechanism using a novel geometric technique that enables revenue guarantees for many common non-truthful auctions that previously had none. Our approach adapts and extends the duality framework of Cai et al [CDW16] beyond truthful auctions.

Our framework can be used with many common auction formats, such as simultaneous first-price, simultaneous second-price, and simultaneous all-pay auctions. Our results for first price and all-pay are the first revenue guarantees of non-truthful mechanisms in multi-dimensional environments, addressing an open question in the literature [RST17]. If all-pay auctions are used, we prove that the resulting mechanism is also credible in the sense that the auctioneer cannot benefit by deviating from the stated mechanism after observing agent bids. This is the first static credible mechanism for multi-item additive auctions that achieves a constant factor of the optimal revenue. If second-price auctions are used, we obtain a truthful $O(1)$-approximate mechanism with fixed entry fees that are amenable to tuning via online learning techniques.

## 1 INTRODUCTION

Imagine that you are participating in a silent auction for a piece of art. You notice that it is a sealed-bid second-price auction, which you know to be truthful, so you bid your true value which happens to be \$1400. A few days later the auctioneer contacts you to let you know that you have won, and that you owe an amount equal to the second-highest bid: \$1399. Of course, this is suspiciously convenient for the auctioneer, and you might wonder if there really was such a bid. But there is little you can do to verify the claim.

The difficulty illustrated by this scenario is that sealed-bid second-price auctions, while truthful, are not *credible*. That is, the auctioneer can benefit by deviating from the prescribed auction rules in a way that cannot be unilaterally detected through the auction's communication protocol [AL18]. This is one of many reasons why second-price auctions are typically implemented via ascending-price English auctions rather than sealed-bid.

The issue of credibility is only exacerbated for more complex auction formats, such as multi-item auctions where many different items are to be sold simultaneously. Multi-item auctions have been the subject of intense focus in the recent algorithmic mechanism design literature, in no small part because they exemplify inherent tradeoffs between optimality and simplicity. Indeed, even when valuations are additive and independent across items, revenue-optimal mechanisms are known to be highly complex: they can require lotteries [Tha04, Pav11, DDT13], can exhibit non-monotone revenue [HR15], and can be computationally difficult to compute [DDT14]. This has motivated a relaxation of revenue-optimality, leading to a search for simple and robust auction formats that approximate the Bayesian optimal revenue; see e.g. [CHK07b, CHMS10a, CMS15, BILW15, Yao14, CM16, CDW16, CZ17, CD17, CZ19]. This search has culminated in a line of work establishing that approximately optimal revenue can be obtained through two-part tariff auctions [Yao14, CM16, CZ17], wherein bidders are asked to pay an entry fee for the chance to bid on individual items. Entry fees have been a well-studied topic in the auction literature [MW82, MM87, Mey93, EW93, LS94, Arm99, CT10, CK18], and this new line of work demonstrates that they can be very useful in achieving approximately optimal revenue in multi-dimensional settings. Moreover, the aforementioned auctions are truthful and can be implemented in a computationally efficient manner.

The two-part tariff format seems quite natural at first glance, as it simply adds an entry fee to a standard auction format used in practice. However, one subtlety that bears mentioning is that in all prior works, either the mechanism is dynamic and requires multiple rounds of communication with the agents [CM16, CZ17, CD17],[1] or the entry fees are not posted in advance, but rather the fee presented to each agent is a function of the submitted bids of all other agents [Yao14, CDW16]. In the former case, the multi-round nature of the mechanisms can present implementation difficulties that static mechanisms bypass; as noted in [AL18], static mechanisms can be conducted rapidly and asynchronously, which yields several implementation benefits, supported by empirical evidence [ALS09]. In the latter case, since the entry fees are opaque functions of other player reports, the connection with the colloquial notion of "entry fee" is arguable. This introduces potential roadblocks to practical implementation, not least of which is credibility. Since all bids are provided to the mechanism in advance of fees being declared, one might naturally worry that the mechanism administrator could fudge the numbers ex post in order to raise entry fees; if this does not modify the auction outcomes, such manipulations could very easily go undetected. One idea for alleviating this complexity and credibility issue would be to post fixed entry fees in advance, independently of the realized bids and determined only based on the prior distributions from which values are to be drawn. Of course, the question is whether such mechanisms can still obtain a constant approximation to the Bayesian optimal revenue. Our main result is that they do.

We study a framework for designing static mechanisms with *fixed* entry fees, posted in advance to all agents, who choose whether or not to pay in order to participate in subsequent simultaneous single-item auctions. For example, the Entry-fee Second-price (ESP) mechanism proceeds by first posting an entry fee to each agent. The mechanism then sells each item separately using simultaneous second-price auctions, where only agents who paid their respective entry fees are eligible to win items. We emphasize that the entry fees are *lazy*, in the sense that the bids of *all* agents are entered into the single-item auctions, even those that did not pay the entry fees, but if the highest bidder on an item did not pay the entry fee then the item will not be sold. This mechanism is Bayesian incentive compatible for additive buyers in the sense that it is interim optimal for each bidder to submit truthful bids and accept the entry fee when this is smaller than her expected interim utility in the single-item auctions assuming truthful bidding by the others. Moreover, as we show, the better of this mechanism or selling all items via separate Myerson auctions yields an 8-approximation to the Bayesian optimal revenue under item independence.

One feature of the ESP mechanism is that it is amenable to online learning/tuning, in settings where the distributions of bidders' types are not known. We can imagine a setting where an auctioneer repeatedly interacts with new members from those distributions, by setting entry fees and then observing the bids and actions (i.e., accept or reject the entry fee) of each bidder. Assuming that the bidders play the truthful equilibrium, we show that the auctioneer can achieve average

[1] In particular, these works provide guarantees for *sequential posted price mechanisms with entry fees*, wherein bidders interact with the mechanism in sequence. Each bidder is shown an entry fee and individual item prices and is asked to pay the entry fee for the chance of purchasing any of the available items at its posted price. Of course, any such mechanism has a static direct-revelation implementation (i.e., that solicits bids and simulates the sequential mechanism), but such an implementation sacrifices credibility.

revenue, in a computationally efficient manner, that is a 28-approximation to the optimal revenue, less a vanishing regret that decays as $\tilde{O}(n\, m^{4/3}\, T^{-1/3})$ after $T$ rounds, with $n$ bidders and $m$ items.[2]

While fixed entry fees are simple, natural, and learnable, the ESP mechanism is still not credible in the sense of [AL18]. Indeed, it inherits all of the credibility problems of the sealed-bid second price auction, as these are the auctions used to sell items after entry fees have been paid. In fact, the characterization of [AL18] shows that even for the case of a single item, the only single-round auctions that are credible are the first-price and all-pay auctions, so credibility necessarily requires moving beyond second-price payment rules. This creates the need for a framework for analyzing revenue guarantees in non-truthful auctions. This has been an open challenge in algorithmic mechanism design for years [RST17], with progress made only for single-dimensional settings [HHT14]. Luckily, the ESP mechanism is just one instantiation of a more general framework that our work provides, for designing mechanisms via two-part tariffs. Our framework takes the form of a general construction: given an arbitrary (not necessarily truthful) single-item auction format, our mechanism proceeds by first posting a fixed entry fee to each agent. The mechanism then sells each item separately using the provided single-item auction format, where only agents who paid their respective entry fees are allowed to participate. Any agents who did not pay the entry fee are instead simulated by the mechanism, and any items won by such simulated agents are discarded. These "ghost bidders" ensure that the equilibria of the simultaneous single-item auctions are unaffected by the realization of which agents pay their respective entry fees.[3] We show that if the provided auction satisfies a certain "type-loss tradeoff property" — which essentially states that agents with higher types are sufficiently more likely to have higher allocations — then the better of this mechanism or separate Myerson auctions will obtain a constant approximation to the optimal revenue at equilibrium.

One note about our framework is that we do not require that the provided single-item auction be incentive compatible. If not, our resulting mechanism will also not be incentive compatible, in which case our revenue approximation holds at equilibrium. In such cases it will be convenient to focus on cases where the single-item auction admits a unique equilibrium, which is true for many standard auction formats. We prove that many standard auction formats, including second-price auctions, first-price auctions, and all-pay auctions, satisfy the type-loss tradeoff property. This yields a portfolio of entry-fee-based mechanisms.

Framework in hand, we can now return to the issue of credibility. When all-pay auctions are used, we prove that the resulting entry-fee mechanism (EAP) is *credible.* Recall that, roughly speaking, credibility means that the auctioneer cannot increase revenue ex post by modifying the auction rules in a way that cannot be detected by a single agent unilaterally. We show that even though the auctioneer could in principle manipulate the bids from the simulated bidders in the second stage of the EAP mechanism, such manipulations cannot increase revenue (nor can any other). But we are not quite done: recall that in order to approximate the optimal revenue, we can take the better of EAP or separate Myerson auctions. Unfortunately, Myerson single-item auctions are not credible. Instead we use simultaneous first-price auctions with reserves, which are known to approximate the Myerson optimal revenue for regular bidders [HHT14]. The end result is that the mechanism that uses the better of EAP or simultaneous first-price auctions with personalized reserves is credible and obtains a $O(1)$-approximation to the optimal revenue for additive buyers with regular value distributions.

[2]We note that this improves upon the previously best-known approximation factor of 32 achievable via PAC learning techniques [CD17], albeit the two results are not directly comparable. Their worse factor accommodates irregular distributions, but we show vanishing regret in the online learning setting.

[3]For the case of the ESP mechanism, which uses second-price auctions, these "ghost" bidders can be simulated without knowledge of the type distribution.

### 1.1 Our Techniques: Duality and Type-loss Tradeoffs

Our techniques in this paper draw heavily from [CDW16]. In that work, Cai et al. extend the notion of virtual value to the multi-item setting. They define it via a partition of the type space that is in terms of the ex post utilities bidders receive in a hypothetical second price auction. Our main idea is to redefine this partition in terms of *interim* utility rather than ex post. The benefit of our technique is that it can be applied to more general auction formats. This could be, for example, first price or all pay, as well as second price. This generalization ends up being possible because, while ex post utility is increasing in type for second price auctions specifically, interim utility is increasing in type for all auctions. Moreover, the second key idea is that if we define these regions appropriately in terms of the interim utilities achieved from each item $j$, if that item was to be sold in isolation by some single-item auction $A$ and under some equilibrium $b^j$, then we can upper bound the resulting multi-dimensional virtual value, in terms of the revenue achieved by an entry fee auction, where bidders pay a fixed entry fee in order to participate in a simultaneous $A$ auction for each item.

To prove this claim we perform a decomposition of the multidimensional virtual value in multiple terms, in a manner similar to prior work [CDW16, BILW15], and bound each of these terms by the revenue of a simple auction. Unlike prior work, when we consider a general auction class $A$ (e.g., first price or all-pay), it comes with the added difficulty of bounding a specific term in this decomposition that relates to the types of bidders that would lose in the hypothetical $A$ auction. The ease with which we can bound this term for different $A$ is what is described by the "type-loss tradeoff property" of auction $A$. While the type of a bidder who does not win a second price auction is easy to capture through the revenue of a simple auction, the same cannot be said for first price and all pay auctions. In these auctions, we often have the highest type bidder not winning, and generally, we cannot capture the highest type through revenue. Our main technical contribution is the proof of a duality whereby both of these worst cases cannot co-exist. First price and all pay auctions can only misallocate with a high frequency when the expected highest type is attainable through some revenue, and the expected highest type is unattainable when first price achieves almost optimal welfare.

In proving these results, we offer new insights on the efficiency loss in non-truthful auctions that could be of independent interest; roughly: the welfare lost in a single item first price or all-pay auction, can be achieved as revenue of a posted price mechanism (see Lemmas 6.1 and 7.1).

### 1.2 Related Work

There has been a recent flurry of results on approximately optimal mechanisms for buyers with multi-dimensional types. As discussed above, simple constant approximations are known for additive buyers with independent valuations [BILW15, Yao14, CDW16]. For unit-demand buyers, one can likewise obtain a constant approximation to the optimal mechanism with multiple buyers [CHK07a, CHMS10b, CMS10]. The ideas behind these mechanisms have since been extended to more general valuation classes, including XOS and subadditive valuations [CZ17, RW18a, CM16]. A common theme in many of these mechanisms is the combination of entry fees (or bundle prices, for a single agent) and per-item auctions or prices.

The above line of work focuses on Bayesian incentive compatible mechanisms. Less is known on approximating optimal revenue with non-truthful auctions. Hartline, Hoy, and Taggart [HHT14] develop a framework for bounding the fraction of optimal revenue obtained at equilibrium in various single-item auction formats, such as first-price and all-pay auctions. Our analysis of the type-loss trade-off for different auction formats shares inspiration from their equilibrium analysis, as well as from the literature bounding the welfare of equilibria in first-price auctions [ST13, HTW19].

Our online learning results for entry-fee mechanisms with second-price auctions relate to a recent literature on the sample complexity of approximately optimal multi-item auctions. Morgenstern and Roughgarden [MR16] presented a statistical learning theory approach to bounding the sample complexity of different classes of simple and approximately optimal auctions. Their approach bounds the pseudo-dimension of different auction classes, but this does not directly imply a polynomial sampling complexity bound for independent additive valuations. Goldner and Karlin [GK16] showed that for bidders with independent additive valuations drawn from regular distributions, one can learn an approximately optimal auction using only a single sample from each bidder's distribution. Cai and Daskalakis [CD17] extend this result to non-regular distributions and a broad class of non-additive valuations, by showing that a sequential posted pricing mechanism with entry fees yields approximately optimal revenue and has polynomial sample complexity. All of these works focus on learning from valuation samples in incentive compatible mechanisms. While we likewise restrict our attention to the sample complexity of an incentive compatible mechanism in our framework, ours is an online learning process. Learning from samples under equilibrium play in non-IC mechanisms is a more subtle task; see Hartline and Taggart [HT19] for a recent treatment and development in the context of single-parameter types.

Recently and independently, Ferreira and Weinberg [FW20] considered the design of credible and incentive compatible single-item auctions. They show how to design efficient and strategyproof auctions using cryptographic primitives. We focus on multi-item auctions and show that by relaxing incentive compatibility, it is possible to design non-truthful credible mechanisms without the use of cryptographic primitives.

## 2 MECHANISM DESIGN PRELIMINARIES AND NOTATION

We consider multi-item sealed-bid auctions with $n$ additive bidders and $m$ indivisible items. Each bidder $i$'s valuation/type for each item $j$ is drawn independently from a continuous distribution $D_{ij}$, supported on type space $T_{ij} \subseteq [0, H]$, for some constant $H$ and which admits a continuous bounded density. We will refer to the latter type of distributions as continuous, throughout the paper. The type distributions are common knowledge to the bidders and the auctioneer. The value of a player for a bundle $S$ is the sum of the values for each item in $S$. We will be using the shorthand notation $T_i = \times_j T_{ij}$, $T_{-i} = \times_{i^* \neq i} T_{i^*}$, $T = \times_i T_i$; and analogously we can define $D_i$, $D_{-i}$, $D$.

Each bidder $i$ observes their type $t_i = (t_{i1}, \cdots, t_{im})$ and chooses an action $a_i$ (e.g. a bid to submit or a total contingency plan over a multi-round auction). The auction maps the action profile $a = (a_1, \cdots, a_n)$ to ex-post feasible allocations $x(a) = (x_1(a), \cdots, x_n(a))$ and payments $p^*(a) = (p_1^*(a), \cdots, p_n^*(a))$; where $x_i(a) = (x_{i1}(a), \cdots, x_{im}(a))$ is a vector whose $j$-th entry $x_{ij}(a) \in [0, 1]$ represents the probability of bidder $i$ being allocated item $j$. An allocation is feasible if $\sum_i x_{ij}(a) \leq 1$. Bidder's have quasi-linear utility: the utility obtained by a bidder is the value they get from the items they receive minus the payment they must make. Thus, the ex-post utility of bidder $i$ is:

$$u_i^*(a) = \sum_j x_{ij}(a) \cdot t_{ij} - p_i^*(a) = x_i(a)' t_i - p_i^*(a)$$

Since we are also considering non-truthful auctions, we need to define the notion of a Bayes-Nash equilibrium. A *bid strategy* $b = (b_1, \cdots, b_n)$ is a collection of mappings $b_i$ from types $t_i$ to actions $a_i$.[4] A bid strategy forms a *pure Bayesian Nash Equilibrium* (BNE), if each bidder has no incentive to deviate conditional on his observed type $t_i$:

$$\mathbb{E}_{t_{-i} \sim D_{-i}}[u_i^*(b_i(t_i), b_{-i}(t_{-i}))] \geq \mathbb{E}_{t_{-i} \sim D_{-i}}[u_i^*(a_i', b_{-i}(t_{-i}))] \quad \forall t_i, a_i'$$

[4]We restrict attention to pure bid strategies for simplicity. All our results extend to mixed equilibria.

Given a BNE $b$, we define the *interim* utilities $u$, allocations $\pi$, and payments $p$ as:

$$u_i^b(t_i) = \mathbb{E}_{t_{-i}\sim D_{-i}}[u_i^*(b_i(t_i), b_{-i}(t_{-i}))] \qquad \pi_i^b(t_i) = \mathbb{E}_{t_{-i}\sim D_{-i}}[x_i(b_i(t_i), b_{-i}(t_{-i}))]$$
$$p_i^b(t_i) = \mathbb{E}_{t_{-i}\sim D_{-i}}[p_i^*(b_i(t_i), b_{-i}(t_{-i}))]$$

For an auction $A = (x, p^*)$ with bid equilibrium $b$, we define the total expected equilibrium utility, welfare and revenue as:

$$\mathsf{Util}^b(A) = \sum_i \mathbb{E}_{t_i\sim D_i}\left[u_i^b(t_i)\right] \quad \mathsf{Wel}^b(A) = \sum_i \mathbb{E}_{t_i\sim D_i}\left[\pi_i^b(t_i)\right] \quad \mathsf{Rev}^b(A) = \sum_i \mathbb{E}_{t_i\sim D_i}\left[p_i^b(t_i)\right]$$

When describing the utility/welfare/revenue of a truthful auction, we will omit the superscript $b$ and assume we are discussing the truthful equilibrium.

## 3 REVENUE APPROXIMATION VIA ENTRY-FEE SIMULTANEOUS AUCTIONS

Our goal is to bound the revenue achievable via simultaneous (potentially non-truthful) item-auctions with an entry fee. In this section we will consider a general class of item auctions and define the condition that leads to constant factor revenue guarantees. In the subsequent sections, we will instantiate our analysis to particular item auctions.

Let $A$ be an arbitrary single item auction, with allocation and payment rules $(x_A, p_A^*)$. Throughout we will assume that players always have an action in $A$ that gives them zero utility, so that all equilibria $b_A$ of $A$ are *interim individually rational*, i.e. $u_i^{b_A}(t_i) \geq 0$. We will require a crucial, *but non-trivial*, property that the auction $A$ will need to satisfy. This property captures the intuition that high-value bidders at the equilibrium of auction $A$ bid high enough and that losers of the auction are with significant probability not the highest type player. Therefore, the type of the losers can be achieved as revenue by some mechanism. This is a rough intuition of the property, and the formal notion is given below.

Definition 3.1 ($c$-type-loss trade-off). *Let $A$ be a single-item auction. We say that auction $A$ satisfies the $c$-*type-loss trade-off* property if for any collection of bidders participating in an $A$-auction, with any vector of type distributions $D = \times_i D_i$, with $D_i$ supported on $[0, H]$, and for any equilibrium strategy $b$:*

$$\mathbb{E}_{t\sim D}\left[\max_i t_i \left(1 - \pi_i^b(t_i)\right)\right] \leq c \cdot OPT(D)$$

*where $OPT(D)$ is the optimal revenue in a single-item auction setting with type distributions $D$.*

To achieve any reasonable approximation to revenue in multi-dimensional settings, we need to allow for our mechanisms to impose bundle prices. We achieve this by augmenting simultaneous item auctions with an entry fee. Apart from the entry fee, for technical reasons in some of our mechanisms we also need to simulate non-participating bidders with ghost bidders. In Algorithm 1 we provide a formal definition of simultaneous auctions with an entry-free and ghost bidders and below we provide an intuitive description of this class of mechanisms. For a single-item auction mechanism $A$, we define the multi-item entry-fee auction EA as follows. First, each bidder $i$ is given the option to pay a fixed entry fee $e_i$ that is a function of the type profile distribution $D$. Then, the bidders who pay the entry fee have access to separate $A$-auctions on each of the items. The bidders who do not pay the entry fee are replaced by a *ghost bidder* who bids in the auctions according to some bid distribution $D^g$. If a ghost bidders wins an item, that item is allocated to nobody and no payment is received for that item.

**ALGORITHM 1:** Simultaneous $A$-Item-Auction with Entry Fee and Ghost Bidders (EA$(e, D^g)$)

**Input:** A single-item auction $A = (x_A, p_A^*)$.

**Input:** For each bidder $i$: an entry fee $e_i$ and a distribution $D_i^g$ of ghost bids $a_i^g$ that will be submitted to the auctions if the bidder decides to not enter.

Each bidder $i$ submits a pair $(z_i, a_i)$ of the decision $z_i \in \{0, 1\}$ to enter or not the auction and if they decide to enter they also submit a bid vector $a_i$;

Let $S = \{i : z_i = 1\}$ denote the bidders that decided to enter and $a^S$ the corresponding vector of actions;

For each $i \notin S$, draw a bid vector $a_i^g \sim D_i^g$;

For each item $j$ run auction $A$, with bids $\tilde{a}_j^S$, such that $\tilde{a}_{ij}^S = a_{ij}^S$ for $i \in S$ and $\tilde{a}_{ij}^S = a_{ij}^g$ for $j \notin S$ to decide allocation $x_A(\tilde{a}_j^S)$ and payments $p_A^*(\tilde{a}_j^S)$;

For each bidder $i \in S$ return: $x_i(a^S) = (x_{A,i}(\tilde{a}_1^S), \ldots, x_{A,i}(\tilde{a}_m^S))$ and payment $p_i^*(a^S) = \sum_{j \in m} p_{A,i}^*(\tilde{a}_j^S)$;

For each bidder $i \notin S$ return zero allocation and payment;

*Remark on ghost bidders.* In our main theorem the bid distribution $D^g$ of the ghost bidders is constructed as follows. Let $b^j$ be an equilibrium bid strategy profile of item auction $A$ for item $j$ with type profile distribution $D_j = \times_i D_{ij}$. Then the *ghost bidder* for player $i$ bids as the bidder $i$ would have, had they paid the entry fee, and according to equilibrium bid strategy $b_i = (b_i^1, \ldots, b_i^j)$ on each of the separate auctions. Observing their own type, bidder $i$ will choose to pay the entry fee if it is less than their total interim utility over all the separate $A$-auctions under equilibrium profiles $b^j$. If bidder $i$ does not pay the entry fee, a *ghost bidder* is created with type sampled from $D_i$ conditioned on the event that the type $t_i$ achieves a total interim utility less than the entry fee, i.e. $\left\{\sum_j u_{ij}^{b^j}(t_{ij}) \leq e_i\right\}$. The ghost then participates in each of the auctions, playing according to $b_i^j$ in each auction $j$ for the re-sampled type. We will denote this ghost bidder distribution $D^g$ with $D^g(\{b^j\})$ and refer to it as a $\{b^j\}$*-simulating distribution.*

The key observation is that, from the perspective of the bidders who do pay the fee and participate in the auctions, their opponents in the auction have type distributed according to $D_{-i}$. They are unable to observe which opponents are ghosts. All they know is that each opponent $i'$ has type sampled from $D_{i'}$ and then, if $t_{i'}$ happens to lie in the subset of $T_{i'}$ that gives interim utility less than the fee, $t_{i'}$ is re-drawn from that subset again according to $D_{i'}$, conditional on that event. Importantly, any collection of equilibria $b^1, \ldots, b^j$ over the separate $A$ auctions gives rise to the following focal equilibrium.

Definition 3.2 (Focal Equilibrium of EA Auction). *Let $\{b^j\}_{j \in [m]}$ be a set of equilibria of the single-item $A$ auctions for type distributions $D_j = \times_i D_{ij}$ correspondingly. Then the EA$(e, D^g(\{b^j\}))$ auction, with $\{b^j\}$-simulating ghost bidder distribution admits a* focal equilibrium *$b$, where each bidder $i$ submits a bid $b^j(t_{ij})$ on each item $j$, whenever they enter, and they enter if $\sum_j u_{ij}^{b^j}(t_{ij}) > e_i$.*

Finally, for any entry fee auction we denote with $\mathsf{EF\text{-}Rev}^b$ the total revenue collected solely due to the collection of entry fees from entrant bidders, i.e.:

$$\mathsf{EF\text{-}Rev}^b(\mathrm{EA}(e, D^g)) = \sum_i \mathbb{E}_{t_i \sim D_i}\left[e_i \cdot 1\left\{i \text{ entered auction under equil. } b \text{ with type } t_i\right\}\right] \quad (1)$$

We are now ready to state our main theorem.

Theorem 3.3. *Let $A$ be any single-item auction, satisfying the $c$-type-loss trade-off and which admits an equilibrium strategy $b^j$ for type vector distribution $D_j = \times_i D_{ij}$ that is interim individually rational. Then there exists a set of player-specific entry-fees $e_i$, such that the focal equilibrium $b$ of* *the simultaneous A-item-auction with entry fees $e = (e_1, \ldots, e_n)$ and $\{b^j\}$-simulating ghost bidder distribution, $EA(e, D^g(\{b^j\})$ satisfies:*

$$OPT(D) \le (c+5) \cdot \sum_{j=1}^{m} OPT(D_j) + 2 \cdot \textsf{EF-Rev}^b(EA(e, D^g(\{b^j\})) \quad (2)$$

*where $OPT(D)$ denotes the optimal revenue in the multi-dimensional multi-item auction setting with type distributions $D = \times_i D_i$ and $OPT(D_j)$ is optimal revenue in a single item auction setting with type vector distribution $D_j = \times_i D_{ij}$.*

*Remark on $OPT(D_j)$.* The first part of the upper bound corresponds to the sum of the optimal revenues achievable in a set of single-dimensional auction settings. For each item $j$, this optimal quantity $\text{OPT}(D_j)$ is achieved by the celebrated Myerson auction [Mye81] that maps the type $t_{ij}$ of each bidder $i$ for the item to a virtual value $\tilde{\phi}_{ij}(t_{ij})$ and then allocates the item to the highest virtual value bidder. Moreover, for each of these quantities we can also use existing results on revenue guarantees of truthful and non-truthful simple auctions [HHT14] in single-dimensional settings, to show that this revenue is also achievable by simple, learnable and potentially also credible auctions. For instance, based on the results by [HHT14], if the type distributions $D_{ij}$ are *regular* (as defined by [Mye81]), then the first part of the upper bound is approximated to within a constant factor by running a first price auction with a player specific reserve price for each of the items. Similarly, for regular distributions it is also approximated [HR09] by running a second price auction with player specific reserves. Thus in that respect, our theorem states that *the best of running a separate entry fee auction for each item, or running a grand-bundle entry fee auction, where the entry fee is paid to access all the item auctions*, is a constant factor approximation to the optimal revenue.

### 3.1 Proof Outline

We defer the full proof of the main theorem to Appendix A, but we outline here the main parts of the proof and some key technical insights.

Our analysis starts with a continuous analogue of the upper bound on the optimal revenue in our multi-item auction setting, as presented in [CDW16], which is mostly of technical interest. The reason why we chose to work with continuous type spaces is primarily because we are interested in analyzing non-truthful auctions, for which there is a plethora of existing equilibrium analysis results (e.g. existence of a monotone pure equilibrium and uniqueness of equilibria) primarily under continuity assumptions on the distribution of types.

Our continuous analogue is phrased in terms of a partition of the type space of each player into $m+1$ regions, defined via a monotone preference function: for each item $j$ there exists a monotone function of the player's type, which assigns a preference score to that item as a function of the type of the player solely for that item. Then the type vector $t_i$ of player $i$ belongs to partition $j$, roughly if item $j$ is assigned the highest score. More formally:

Definition 3.4 (Monotone Preference Partition of Type Space). *For all $i$, we say that $R_{i,0}, R_{i,1}, \cdots, R_{i,m}$ is a* preference partition *of the type space $T_i$ if it is defined as follows: for each item $j$, there exists* non-decreasing preference functions $\mathcal{U}_{i,j} : T_{ij} \to \mathbb{R} \ge 0$ *such that, for all $j \neq 0$*

$$t_i \in R_{i,j} \Leftrightarrow \mathcal{U}_{i,j}(t_{ij}) \ge \mathcal{U}_{i,k}(t_{ik}), \forall k \neq j \;\; and \quad \mathcal{U}_{i,j}(t_{ij}) > \mathcal{U}_{i,k}(t_{ik}), \forall k < j \;\; and \quad \mathcal{U}_{i,j}(t_{ij}) > 0$$

*and*

$$t_i \in R_{i,0} \Leftrightarrow \mathcal{U}_{i,j}(t_{ij}) = 0 \quad \forall j$$

*i.e. the preference function assigns an index to each item that is a monotone function of that item's type $t_{ij}$ and then the type vector $t_i$ belongs to the region $j$ with the highest positive index, breaking ties lexicographically, or to region 0 if all indices are zero.*

Then we can show the following continuous type analogue of Theorem 31 of [CDW16]. The proof considers $\epsilon$-discretizations of the continuous type distribution, applies the discrete bound result and then verifies that we can take the limit as $\epsilon$ goes to zero, to get the desired theorem. This requires a careful accounting of the discretization errors and showing that the upper bound in the optimal discretized revenue, relates to its continuous analogue, up to an $O(\epsilon)$ error, whenever the partition of the type space is a monotone preference partition.

Lemma 3.5 (Revenue Bound via Monotone Preference Partitions of Type Space). *Consider a multi-item auction setting with additive bidders and independent continuous type distributions $D_{ij}$ on a bounded support $[0, H]$. Let $\{R_{i,j}\}_{i\in[n],j\in[m]}$ be a monotone preference partition of the type space and let $\mathcal{F}$ denote the space of all interim feasible allocations. Then:*

$$OPT(D) \leq \sup_{\pi\in\mathcal{F}} \sum_i \mathbb{E}_{t_i\sim D_i}\left[\sum_j \pi_{ij}(t_i)\left(t_{ij}\cdot 1\left\{t_i \notin R_{i,j}\right\} + \tilde{\varphi}^*_{ij}(t_{ij})\cdot 1\left\{t_i \in R_{i,j}\right\}\right)\right] \tag{3}$$

*where $\tilde{\varphi}^*_{ij}(t_{ij}) = \max(\tilde{\varphi}_{ij}(t_{ij}), 0)$ and $\tilde{\varphi}_{ij}(t_{ij})$ represents Myerson's ironed virtual value function [Mye81] for the distribution $D_{ij}$.*

The crucial conceptual contribution of our work is to consider monotone preference partitions of the type space that are described in terms of the interim utilities $u^{b^j}_{ij}(t_{ij})$ of the bidders at some equilibrium $b^j$ for each (potentially non-truthful) item auction $A$ for item $j$. All prior works in the area considered partitions of the type space as a function of ex-post utilities and solely based on the outcomes of a truthful auction $A$ for each item. In particular, our region definition will assign type $t_i$ to region $j \in \{1, \ldots, m\}$, if item $j$ achieves the highest non-zero interim utility $u^{b^j}_{ij}(t_{ij})$, among all items (and to region 0 if all interim utilities are zero). By monotonicity of interim utilities of any equilibrium in any single-dimensional mechanism, such a partition of the type space is a monotone preference partition. Hence, we can apply Lemma 3.5.

Subsequently, we analyze the right-hand-side of Equation (3) via a similar decomposition to prior work in the are [BILW15, CDW16] into four terms: Single, Under, Over and Surplus. Our proof shows that this type of analysis can also be carried out even when the regions are defined in terms of interim utilities of non-truthful auctions and still yield meaningful upper bounds in terms of the revenue of simple multi-item auctions. The terms Single, Under and Over can all be shown to be upper bounded by the sum of the per-item optimal auction revenues; thereby reducing the problem to independent single-dimensional settings. The final term Surplus is shown to be achievable as the revenue of the multi-item $A$-item auction with a particular entry fee.

In particular, Single corresponds to the second summand in Equation (3), which can be shown to be upper bounded by the sum across items, of the maximum ironed virtual value for each item; which in turn is the optimal per item revenue. The first summand on the right-hand-side of Equation (3) can be divided into the quantities Under, Over and Surplus. Under corresponds to the part of the event that $t_i$ is not in region $j$ because player $i$ did not bid high enough on item $j$ and hence was not allocated the item. This is exactly where we use the $c$-type-loss trade-off property to show that this quantity, which is roughly the type of the player that lost the item $j$ under equilibrium $b^j$, can be related to the revenue achievable by the optimal auction for item $j$. This property is a non-trivial property of auction $A$ and we will show that it is satisfied by many auctions of interest in the next few sections.

What remains from the first summand, is accounting for the type of the player in the event that player $i$ bids high enough to win auction $j$, but item $j$ is not player $i$'s favorite item as captured by the aforementioned interim utility score. Since in this case, the player received the item, he claims his type for item $j$ as a value, and hence we can relate his type to his utility plus the auctioneer revenue from player $i$ at item $j$. The revenue part is exactly the Over term and it is easily shown to correspond to the revenue achieved by simultaneous $A$-item-auctions without any entry fee. The utility part of this decomposition is the Surplus term, which is much harder to analyze and which roughly corresponds to sums of terms of the form:

$$\mathbb{E}_{t_i \sim D_i}\left[u^b_{ij}(t_{ij}) \cdot 1\left[\exists k \neq j, u^b_{ik}(t_{ik}) \geq u^b_{ij}(t_{ij})\right]\right] \tag{4}$$

This term can be shown to be related to the revenue achieved by an simultaneous $A$-item-auction with an entry fee $e_i$, that satisfies that the probability of entry for each player is at least $1/2$. More concretely it satisfies that:

$$\Pr_{t_i}\left[\sum_j u^b_{ij}(t_{ij}) > e_i\right] \geq \frac{1}{2} \tag{5}$$

In fact, we show that it relates to the part of the revenue stemming solely from the collection of entry fees from entrant players. The details of this part of the analysis are provided in Appendix A.5.

## 4 GUARANTEES FOR ALL EQUILIBRIA: *EA* AUCTION WITH RANDOM ENTRY-FEE

In this section we show how, via the means of randomization, we can modify the auction described in the main section so as to achieve to key improvements: i) remove the use of ghost bidders, thereby rendering the auction even simpler, ii) achieve guarantees at all equilibria, whenever the single item auction that is used has a unique equilibrium when run in isolation (a property possessed by many auction formats, such as first price and all-pay, under mild regularity conditions on the type distributions).

Consider a single-item auction $A$ which satisfies the type-loss trade off property at every equilibrium. Consider the following multi-item auction $\mathsf{rand} - EA$ in which an auctioneer collects bids from the bidders and then tosses a biased random coin and with probability $1-\delta$, chooses a set of bidder-specific entry fees $\{e_i\}$ and with probability $\delta$, charges no entry-fee. In the former case, the bidders are given a choice to pay the entry-fee and participate in the auction. The auctioneer then runs simultaneous $A$ auctions on all the items, with both the participant bids and "ghost" bids from the non-participating bidders. In the case when an auction is won by a ghost bid, the auctioneer discards the item. In Algorithm 2, we provide a more formal definition of the mechanism.

**ALGORITHM 2:** Simultaneous $A$-Item-Auction with Random Entry Fee ($\mathrm{rand} - \mathrm{EA}(e)$)

**Input:** A single-item auction $A = (x_A, p^*_A)$.
**Input:** For each bidder $i$: an entry fee $e_i$.
Each bidder $i$ submits a pair $(z_i, a_i)$ of the decision $z_i \in \{0,1\}$ to enter the auction or not if an entry fee is imposed, and the bid vector $a_i$ to submit to the auctions, whenever they participate;
The auctioneer tosses a biased coin and with probability $\delta$ charges no entry fee (sets $e_i = 0$) to the bidders and with probability $1-\delta$, charges entry fees $e = (e_1, \ldots, e_n)$ and filters the bidders based on their submitted $z_i$;
Let $S = \{i : z_i = 1 \text{ or } e_i = 0\}$ denote the bidders that enter;
For each item $j$ run auction $A$ with bids $a_j = (a_{1j}, \ldots, a_{nj})$ to decide allocation $x_A(a_j)$ and payments $p^*_A(a_j)$;
For each bidder $i \in S$ return: $x_i(a) = (x_{A,i}(a_1), \ldots, x_{A,i}(a_m))$ and payment $p^*_i(a) = \sum_{j \in m} p^*_{A,i}(a_j)$;
For each bidder $i \notin S$ return zero allocation and payment;

In this section, we show that when $\delta > 0$, every equilibrium of the rand $-$ $EA$ auction is payoff equivalent to some focal equilibrium of the $EA$ auction Let $n$ denote the number of bidders and $m$ denote the number of items. A *mixed* BNE of the rand $-$ $EA$ auction consists of a set of mappings $\mathbf{b} := \{\mathbf{b}_i\}_{i\in[n]}$, where $\mathbf{b}_i : T_i \to \Delta(\{0,1\} \times A_{i1} \times \ldots \times A_{im})$, is a mapping from the type $t_i$ of the player to a distribution over entry decisions $z_i \in \{0,1\}$ and actions $a_{ij}$ for each item $j$.

The following lemma shows that the revenue of the rand $-$ $EA$ auction at any such equilibrium is at least a constant fraction of the revenue of the $EA$ auction (with the same entry fees $\{e_i\}$) at a focal equilibrium.

Lemma 4.1. *Let $\boldsymbol{b}$, be any mixed BNE of the* rand $-$ $EA(e)$ *auction, with entry fees $e = \{e_i\}$, for type vector distribution $D = \times_{j\in[m]} D_j$, where $D_j = \times_i D_{ij}$. Let $\tilde{\boldsymbol{b}}_i^j : T_{ij} \to \Delta(A_{ij})$, denote the marginal action distribution of player $i$ on item $j$ conditional only on his type $t_{ij}$ for item $j$. Then, $\tilde{\boldsymbol{b}}^j = \{\tilde{\boldsymbol{b}}_i^j\}$ is a mixed BNE of the auction for item $j$, when run in isolation. Moreover:*

$$\textsf{EF-Rev}^{\boldsymbol{b}}(\textsf{rand} - EA(e)) \geq (1-\delta) \cdot \textsf{EF-Rev}^{\tilde{\boldsymbol{b}}}\left(EA\left(e, D^g(\{\tilde{\boldsymbol{b}}^j\})\right)\right) \tag{6}$$

*where $EA\left(e, D^g(\{\tilde{\boldsymbol{b}}^j\})\right)$, is the EA auction with $\{\tilde{b}^j\}$-simulating ghost bidders and $\tilde{\boldsymbol{b}}$ is the focal equilibrium.*

This lemma allows us to provide two corollaries that render the results of our main section more robust to equilibrium selection. In the first, we show that if the single item auction has a unique equilibrium when run in isolation, then all equilibria of the rand $-$ EA auction achieve approximately optimal revenue. In the second, we show that even if the underlying single item auction does not have a unique equilibrium, then a very natural equilibrium selection criterion is sufficient to guarantee approximately optimal revenue.

Corollary 4.2 (Item Auctions with Unique Equilibria). *Let $A$ be any single-item auction, satisfying the $c$-type-loss trade-off and which admits a unique equilibrium strategy $b^j$ for type vector distribution $D_j = \times_i D_{ij}$ that is interim individually rational. Then there exists a set of player-specific entry-fees $e_i$, such that at any mixed BNE equilibrium $\boldsymbol{b}$ of the* rand $-$ $EA(e)$:

$$OPT(D) \leq (c+5) \cdot \sum_{j=1}^{m} OPT(D_j) + \frac{2}{1-\delta} \cdot \textsf{EF-Rev}^{\boldsymbol{b}}(\textsf{rand} - EA(e)) \tag{7}$$

Corollary 4.3 (Entry-Fee Oblivious Equilibrium Selection). *Let $A$ be any single-item auction, satisfying the $c$-type-loss trade-off and such that all its mixed BNE are interim individually rational. Consider an equilibrium selection process that maps a set of entry fees $e$ to a mixed BNE $\boldsymbol{b}^e$ of the* rand $-$ $EA(e)$ *auction. Suppose that the equilibrium selection process satisfies that the marginal bid distribution $\tilde{\boldsymbol{b}}_i^j : T_{ij} \to \Delta(A_{ij})$ of each player $i$ for each item $j$ conditional on his type for item $j$, is independent of the entry fee. Then for any such equilibrium selection process, we have:*

$$OPT(D) \leq (c+5) \cdot \sum_{j=1}^{m} OPT(D_j) + \frac{2}{1-\delta} \cdot \max_{e} \textsf{EF-Rev}^{\boldsymbol{b}^e}(\textsf{rand} - EA(e)) \tag{8}$$

*Remark on credibility vs randomization.* We note that contrary to our main theorem, the mechanism presented in this auction is randomized. This can be at odds with credibility (or more generally transparency) as there is no way to verify that the auctioneer adheres to the results of the coin toss in a truthful manner. We note however, that if we are in a setting where the auctioneer is selling at least two units of each item, then randomization can be simulated in a deterministic manner. Simply charge a zero entry fee for one set of copies of each item and entry fee $e_i$ for the other set of copies and couple the bids of the players across the two auctions; this simulates the entry-fee auction with $\delta = 1/2$ in a deterministic manner.

## 5 APPROXIMATELY OPTIMAL FIXED ENTRY-FEE TRUTHFUL AUCTIONS

As a starting point we apply our main theorem to the case where the single item auction $A$ is the second-price (SP) auction; where the highest bidder for an item wins and pays the second highest bid. This will yield a simple and truthful auction that approximates the optimal revenue. Before stating the main theorem of this section we define the instantiation of the EA auction for the case of $A$ being the (SP) auction. The definition is a slight modification of the ghost bidder auction, where we don't simulate non-entrant bidders with ghost bidders, but rather we ask bidders to always report their types and use their reported types in all the auctions, regardless of whether they decided to enter the market or not.[5]

Definition 5.1 (ESP($e$): Simultaneous Second Price Auctions with Bundle Entry Fee). *In an ESP($e$), each bidder $i$ is given an entry fee $e_i$ that they have to pay to participate in a simultaneous second price item auction. Bidders submit a bid/type $b_{ij}$ for each item $j$ and their decision $z_i$ to enter or not in the market. Each item is allocated via a second price auction where all bids are included. If the item was won by a non-entrant bidder then it remains unallocated.*

The above auction admits a truthful equilibrium, in the sense that it is a weakly dominant strategy for all bidders to report their true types, i.e. $b_{ij}(t_{ij}) = t_{ij}$. This determines interim utilities in the simultaneous second price item auctions, so each bidder will choose to enter the market if and only if their interim utility under truth-telling behavior exceeds the entry fee $e_i$. That is, $z_i(t_i) = 1\left\{\sum_{ij} u_{ij}(t_{ij}) \geq e_i\right\}$, where

$$u_{ij}(t_{ij}) = \mathbb{E}_{t_{-i}\sim D_{-i}}\left[\left[t_{ij} - \max_{k\neq i} t_{kj}\right]_+\right] \tag{9}$$

Moreover, observe that under this truthful equilibrium the auction is outcome equivalent to the entry fee auction with ghost bidders.

Thus to apply Theorem 3.3 all that remains to show is that the single-item second price auction satisfies the $c$-type-loss trade-off property. We will show this for $c = 1$. In fact, we show that the type loss trade-off property is achieved by the revenue of the best posted price (PP) single-item mechanism Rev(PP) (abbreviated PP), which announce some fixed price and allocates to any bidder willing to pay it, i.e.:

$$\mathrm{PP}(D) = \max_r r \Pr_{t\sim D}\left[r \leq \max_i t_i\right] \qquad \text{(Posted Price Mechanism Revenue)}$$

Lemma 5.2 (1-type-loss trade Off of SP). *In a single-item second-price auction with type vector distribution $D = \times_i D_i$ and under the truthful equilibrium $b$, we have:*

$$t_i(1 - \pi_i^b(t_i)) \leq PP(D) \leq OPT(D)$$

*for all bidders $i$ and all possible types $t_i$ of bidder $i$.*

[5] We note that the guarantees of this section also apply to the version of the auction where we only elicit types from the entrant bidders and only use the entrant bidders in the subsequent second-price auctions. At any equilibrium of this auction, the probability of entry is weakly higher than the probability of entry in the auction with ghost bidders (equiv. when all players are used in the second stage). Since our main Theorem 3.3 accounts only the revenue collected by entry fees, the main conclusions of this section remain valid. However, in this auction, even when players report truthfully conditional on entry, the decision to enter depends on the entry strategy of other bidders and hence equilibrium of this entry game is more complex, and even equilibrium existence is not immediate when agents have continuous types.

Proof. The lemma follows by the following simple set of inequalities:

$$t_i(1-\pi_i^b(t_i)) = t_i \Pr_{t_{-i}\sim D_{-i}}\left[t_i \le \max_{j\neq i} t_j\right] \le \max_r r \Pr_{t_{-i}\sim D_{-i}}\left[r \le \max_{j\neq i} t_j\right] \le \max_r r \Pr_{t\sim D}\left[r \le \max_i t_i\right]$$

□

Thus we can invoke Theorem 3.3 to show the following result.

Corollary 5.3. *Consider a multi-item auction with additive bidders and independent types across bidders i and items j, distributed according to $D_{ij}$ and supported in $[0,H]$. There exists a set of player-specific entry-fees $e=(e_1,\ldots,e_n)$, such that under the truthful equilibrium of the ESP($e$) auction:*

$$OPT(D) \le 6 \cdot \sum_{j=1}^{m} OPT(D_j) + 2 \cdot \mathsf{EF\text{-}Rev}(ESP(e)) \tag{10}$$

*where* EF-Rev(*ESP*($e$)) *is the revenue of the ESP($e$) auction solely due to collection of entry fees.*

Finally, let SP($r$) denote the simultaneous second price auction with item and bidder specific *lazy* reserve prices $r_{ij}$; where each item is sold separately via a second price auction, the highest bidder wins and if the bid passes the player specific reserve $r_{ij}$ it is allocated the item and charged the maximum of $r_{ij}$ and the highest other bid; if not then the item remains unallocated. Then the results of [HR09] show that for regular distributions, this mechanism is a 2-approximation to OPT($D_j$). Thus we can also get the following simplifying corollary:

Corollary 5.4. *Consider a multi-item auction with additive bidders and independent types across bidders i and items j, distributed according to $D_{ij}$ and supported in $[0,H]$. Suppose that type distributions $D_{ij}$ are regular. Then, for appropriately chosen parameters r, e, the better of: i) running simultaneous second price auctions with item and bidder specific lazy reserve prices SP($r$), ii) running simultaneous second price auctions with bidder specific bundle entry fees ESP($e$), achieves a 14-factor approximation to the optimal revenue.*

## 5.1 Online learnability when prior is unknown to auctioneer

We conclude this section with a remark on the fact that Corollary 5.4 gives rise to an auction rule that is easy for an auctioneer to optimize in an online manner from historical data, even when the prior distribution of types $D$ is not known to her. We will operate under the assumptions of Corollary 5.4. Consider the following online learning setting: at each period $\tau$

(1) For all $i$, players draws their type $t_i^\tau \sim D_i$
(2) The auctioneer posts bidder-specific entry fees $e_i^\tau$ and (item, bidder)-specific reserves $r_{ij}^\tau$.
(3) Players report bids on all items $b_{ij}^\tau$ and their decision $z_i^\tau$ to enter in the entry fee mechanism.
(4) The auctioneer draws a coin and choose SP($r^\tau$) with probability 1/2 and ESP($e^\tau$) otherwise.
(5) The auctioneer runs the chosen mechanism on the reported input and receives revenue $R^\tau$

Assuming that players are myopic (or equivalently that each period corresponds to a fresh draw of players from a population), then at each period $\tau$, it is a weakly dominant strategy for all players to report their true types: $b_{ij}^\tau = t_{ij}^\tau$ and to enter the entry fee mechanism if their belief of their interim utility $u_i(t_i^\tau) = \sum_j u_{ij}(t_{ij}^\tau)$ exceeds the entry fee $e_i^\tau$.[6]

Observe that the revenue at each period $\tau$ is an un-biased estimate of the expected revenue under type distribution $D$, of the mechanism that was chosen, i.e.

$$\mathbb{E}[R^\tau \mid r^\tau, e^\tau] = \frac{1}{2}\left(\mathrm{Rev}(\mathrm{SP}(r^\tau)) + \mathrm{Rev}(\mathrm{ESP}(e^\tau))\right) := f(r^\tau, e^\tau) \tag{11}$$

[6]Given that our mechanisms are BIC we still need the players to know $D$ so as to make their entry decision. This is a minimal oracle we need from our bidders to run our auction.

Moreover, from the definition of the two mechanisms, $f$ is additively separable across parameters:

$$\mathrm{Rev}(\mathrm{SP}(r)) = \sum_{i,j} \mathbb{E}\left[\max\left\{r_{ij}, \max_{k\neq i} t_{kj}\right\} \cdot 1\left[t_{ij} \geq \max\left\{r_{ij}, \max_{k\neq i} t_{kj}\right\}\right]\right] = \sum_{i,j} g_{ij}(r_{ij}) \tag{12}$$

$$\mathrm{Rev}(\mathrm{ESP}(e)) = \sum_{i} \mathbb{E}\left[1\left\{\sum_{j=1}^{m} u_{ij}(t_{ij}) > e_i\right\}\left(e_i + \sum_{j} \max_{k\neq i} t_{kj} 1\left[t_{ij} \geq \max_{k\neq i} t_{kj}\right]\right)\right] = \sum_{i} h_i(e_i) \tag{13}$$

where $u_{ij}(t_{ij})$ is defined in Equation (9). Observe that for any entry fee $e_i \in [0, H\cdot m]$ if we consider the largest entry fee $e_i^\epsilon$ below $e_i$ that is a multiple of $\epsilon$, then we have that: $h_i(e_i^\epsilon) \geq h_i(e_i) - \epsilon$. This follows since $e_i^\epsilon$ allocates at least to all bidder types $t_i$ for which $e_i$ allocates to, and the decrease in payment from every such type is at most $\epsilon$. For an identical reason, for every $r_{ij} \in [0, H]$ the largest reserve price $r_{ij}^\epsilon$ below $r_{ij}$ that is a multiple of $\epsilon$ achieves: $g_{ij}(r_{ij}^\epsilon) \geq g_{ij}(r_{ij}) - \epsilon$. Moreover, after every period we observe unbiased estimates of each of these summands:

- if $\mathrm{SP}(r)$ was chosen, the revenue collected by bidder $i$ at item $j$ is unbiased estimate of $g_{ij}(r_{ij})$.
- if $\mathrm{ESP}(e)$ was chosen, the revenue collected by a bidder $i$ is an unbiased estimate of $h_i(e_i)$.

Thus for each parameter we can reduce the problem of learning a good parameter to an independent stochastic multi-armed bandit problem, each with at most $O(H\,m/\epsilon)$ arms (all multiples of $\epsilon$ in $[0, H\,m)$. Thus using classic results in multi-armed bandits [BCB12], we can use the Hedge algorithm for each of these problems to guarantee that:

$$\max_{r_{ij}^*\in[0,H]} \mathbb{E}_{r_{ij}^{1:T}}\left[\sum_{\tau=1}^{T} \frac{1}{2}\left(g_{ij}\left(r_{ij}^*\right) - g_{ij}\left(r_{ij}^\tau\right)\right)\right] = O\left(H\sqrt{(H/\epsilon)\log(H/\epsilon)\,T} + \epsilon\, T\right)$$

$$\max_{e_i^*\in[0,Hm]} \mathbb{E}_{e_i^{1:T}}\left[\sum_{\tau=1}^{T} \frac{1}{2}\left(h_i\left(e_i^*\right) - h_i\left(e_i^\tau\right)\right)\right] = O\left(H\sqrt{(H\,m/\epsilon)\log(H\,m/\epsilon)\,T} + \epsilon\, T\right)$$

Setting $\epsilon = H^{1/3}\,m^{1/3}\,T^{-1/3}$ and combining this with our revenue approximation theorem from Corollary 5.4 we conclude that for $\delta(n,m,H,T) := O\left(\frac{n\,H^{4/3}\,m^{4/3}\,\log(H\,m\,T)}{T^{1/3}}\right)$:[7]

$$\mathbb{E}_{e^{1:T},r^{1:T}}\left[\frac{1}{T}\sum_{\tau=1}^{T} R^\tau\right] \geq \max_{e,r} f(r,e) - \delta(n,m,H,T) \geq \frac{1}{28}\mathrm{OPT}(D) - \delta(n,m,H,T)$$

## 6 APPROXIMATELY OPTIMAL FIRST PRICE AUCTIONS

We now move to the case where the auction $A$ is a non-truthful First Price auction (FP); the highest bidder wins and pays her bid. First price single item auctions are known to admit monotone equilibria in our setup with a continuous bounded type distribution with a twice differentiable density [MR00] and under some extra assumptions these equilibria are also unique [Leb06] (e.g. if we add any non-zero reserve price). Thus as long as we can show the $c$-type-loss trade-off property for the FP auction, we can apply Theorem 3.3.

Lemma 6.1 (Type-Loss Trade-Off for FPA). *In a single-item first-price auction, with any independent continuous type distribution $D = \times_i D_i$ and under any bid equilibrium $b$, we have*

$$\mathbb{E}_{t\sim D}\left[\max_i t_i\left(1 - \pi_i^b(t_i)\right)\right] \leq 4PP(D) \leq 4OPT(D)$$

[7]We note that the constant $1/28$ should be improvable to $1/14$ by also deploying a bandit learning algorithm to adapt the probability of playing each of the two auctions over time, so as to favor the better of the two.

To state our main corollary we will define the instantiation of the entry fee simultaneous first price auction with ghost bidders as EFP$(e, b)$, parameterized by a set of entry fees $e_i$ and a set equilibrium strategies $b = (b^1, \ldots, b^m)$, each $b^j$ corresponding to an equilibrium of the single-item first price auction for item $j$. Then the ghost bidders submit a bid on each item drawn based on the equilibrium strategies $b$, conditional on the event that the player decides not to enter (i.e. that the interim utility under $b$ is smaller than the entry fee). Such a mechanism admits the following *focal equilibrium*: player $i$ with type $t_i$ submits bid $b_i^j(t_{ij})$ on each auction $j$ and decides to enter if the interim utility, i.e. $u_i^b(t_i) = \sum_j (t_{ij} - b_i^j(t_{ij})) \Pr[b_i^j(t_{ij}) \geq \max_{k \neq i} b_k^j(t_{kj})]$, is greater then $e_i$. Then by Theorem 3.3:

Corollary 6.2. *Consider a multi-item auction with additive bidders and independent types across bidders $i$ and items $j$, distributed according to $D_{ij}$ and supported in $[0, H]$. For each item $j$, let $b^j$ denote an equilibrium of the single-item first price auction with type profile distribution $D_j = \times_i D_{ij}$. Then there exists a set of player-specific entry-fees $e = (e_1, \ldots, e_n)$, such that in the focal equilibrium $b$ of the EFP$(e, b)$:*

$$OPT(D) \leq 9 \cdot \sum_{j=1}^{m} OPT(D_j) + 2 \cdot \textsf{EF-Rev}^b(EFP(e, b)) \tag{14}$$

*where* $\textsf{EF-Rev}(EFP(e, b))$ *is the revenue of the EFP$(e, b)$ auction solely due to collection of entry fees.*

Moreover, the results of [HHT14], show that in a single-item auction settings with independent types and regular distributions $D_j = \times_i D_{ij}$, a first price auction with bidder specific reserves (equal to the monopoly reserve price of each bidder), achieves revenue at least $\frac{e-1}{2e}\mathrm{OPT}(D_j)$. Thus if we denote with FP$(r)$, the simultaneous version of this auction with item and bidder specific reserves, we have:

Corollary 6.3. *Consider a multi-item auction with additive bidders and independent types across bidders $i$ and items $j$, distributed according to $D_{ij}$ and supported in $[0, H]$. Suppose that type distributions $D_{ij}$ are regular. Then, for appropriately chosen parameters $r$, $e$, the better of: i) running simultaneous first-price auctions with item and bidder specific reserve prices FP$(r)$, ii) running simultaneous first price auctions with bidder specific bundle entry fees EFP$(e, b)$ (at the focal equilibrium), achieves a $\frac{20e-2}{e-1}$-factor approximation to the optimal revenue.*

This is the *first multi-dimensional revenue approximation result in the literature that is based solely on winner-pays-bid mechanisms.* The use of first price auction based mechanisms is for instance desireable in settings with multiple competing auctioneers [PLST20] and many real-world systems rely on first price auction rules [Slu19]. Thus understanding their revenue guarantees is of practical importance.

However, EFP is still not credible in the formal sense defined in [AL18]: whenever a ghost bidder wins, the auctioneer has incentive to deviate, without the bidders noticing, and allocate the item to an entrant bidder. In the next Section 7, we show how this problem can be fixed by switching to all-pay auctions.

### 6.1 Proof of Lemma 6.1: Type-Loss Trade-Off for FPA

First we note that by the fact that utilities are quasi-linear:

$$t_i\,(1 - \pi_i^b(t_i)) \leq t_i - \left(\pi_i^b(t_i)\, t_i - p_i^b(t_i)\right) = t_i - u_i^b(t_i)$$

Thus it suffices to show that:

$$\mathbb{E}_{t \sim D}\left[\max_i \left(t_i - u_i^b(t_i)\right)\right] \leq 4\mathrm{PP}(D) \tag{15}$$

Let $B_i^b(t_{-i}) = \max_{k \neq i} b_k(t_k)$ denote the highest other bid in the FP auction as a function of the type profile of player $i$'s opponents. Then by the rules of the FP auction and the BNE condition:

$$u_i^b(t_i) \geq \max_{r \leq t_i} (t_i - r) \Pr_{t_{-i} \sim D_{-i}} \left[ B_i^b(t_{-i}) < r \right]$$

As a first step we show a structural lemma that connects a player's interim equilibrium utility with his type and the distribution of the highest other bid.

Lemma 6.4 (Box Lemma). *Let $F : \mathbb{R}_+ \to [0, 1]$ be any function and let $t \in \mathbb{R}_+$. Consider the quantities $u(t) = \max_b (t - b) F(b)$ and $a(t) = \max_{r \leq t} r (1 - F(r))$. Then, $\sqrt{u(t)} + \sqrt{a(t)} \geq \sqrt{t}$.*

Proof. By definition of $u(t), a(t)$, we must have $F(x) \leq \overline{F}(x) := \frac{u(t)}{t-x}$ and $F(x) \geq \underline{F}(x) := 1 - \frac{a(t)}{x}$ for all $x \in [0, t]$ (see Figure 1). Thus, we must have $\overline{F}(x) \geq \underline{F}(x)$ for all $x \in [0, t]$, i.e.:

$$\min_{x \in [0, t]} \left( \overline{F}(x) - \underline{F}(x) \right) = \min_{x \in [0, t]} \left( \frac{u(t)}{t - x} + \frac{a(t)}{x} \right) - 1 \geq 0 \tag{16}$$

Observe that the function $\frac{u(t)}{t-x} + \frac{a(t)}{x}$ is convex in $x$, when $x \in [0, t]$. Hence, by writing down the first order condition and solving for $x$, we find that it is minimized at: $x = \frac{t\sqrt{a(t)}}{\sqrt{u(t)} + \sqrt{a(t)}}$, yielding:

$$\min_{x \in [0, t]} \left( \overline{F}(x) - \underline{F}(x) \right) = \frac{u(t)}{t} \cdot \frac{\sqrt{u(t)} + \sqrt{a(t)}}{\sqrt{u(t)}} + \frac{a(t)}{t} \cdot \frac{\sqrt{u(t)} + \sqrt{a(t)}}{\sqrt{a(t)}} - 1 = \frac{\left( \sqrt{u(t)} + \sqrt{a(t)} \right)^2}{t} - 1$$

Thus for Equation (16) to hold it must be that $\sqrt{u(t)} + \sqrt{a(t)} \geq \sqrt{t}$, as desired. □

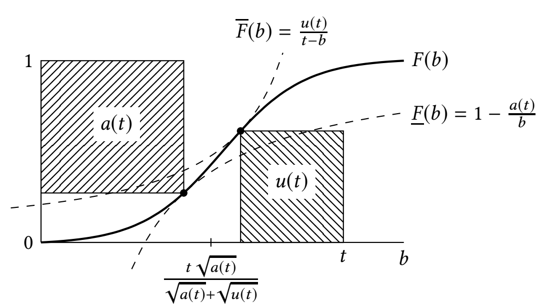


Fig. 1. Pictorial representation of quantities in Box Lemma 6.4 and its proof.

Applying this lemma with $F(r) := \Pr_{t_{-i} \sim D_{-i}}[B_i^b(t_{-i}) < r]$, i.e. the CDF of $B_i^b(t_{-i})$, gives:

$$\sqrt{u_i^b(t_i)} \geq \sqrt{t_i} - \sqrt{a_i(t_i)}$$

where $a_i(t_i) = \max_{r \leq t_i} r \Pr_{t_{-i} \sim D_{-i}}[r \leq B_i^b(t_{-i})]$. Since by definition $a_i(t_i) \leq t_i$, we can square the last inequality to obtain:

$$u_i^b(t_i) \geq \left( \sqrt{t_i} - \sqrt{a_i(t_i)} \right)^2 \geq t_i - 2\sqrt{a_i(t_i) \cdot t_i}$$

Moreover,

$$a_i(t_i) = \max_{r \leq t_i} r \Pr_{t_{-i} \sim D_{-i}}[r \leq B_i^b(t_{-i})] \leq \max_{r \leq t_i} r \Pr_{t_{-i} \sim D_{-i}}[r \leq \max_{j \neq i} t_j] \leq \max_r r \Pr_{t \sim D}[r \leq \max_i t_i] = \mathrm{PP}(D)$$

Thus,

$$\mathbb{E}_{t\sim D}\left[\max_i\left(t_i - u_i^b(t_i)\right)\right] \le \mathbb{E}_{t\sim D}\left[\max_i\left(2\sqrt{a_i(t_i)\cdot t_i}\right)\right] \le 2\sqrt{\mathrm{PP}(D)}\cdot\mathbb{E}_{t\sim D}\left[\sqrt{\max_i t_i}\right]$$

We finish with the following lemma that shows that the expected root of the highest type can be achieved to within a constant factor as the root of the revenue of the best posted price mechanism, i.e. $\mathbb{E}_{t\sim D}\left[\sqrt{\max_i t_i}\right] \le 2\sqrt{\mathrm{PP}(D)}$. Combined with the above inequality, this would conclude the overall proof of the lemma.

Lemma 6.5 (Root Lemma). *For a single item auction setting with any type profile distribution $D$:*

$$\mathbb{E}_{t\sim D}\left[\sqrt{\max_i t_i}\right] \le 2\sqrt{PP(D)}$$

Proof. By the definition of $\mathrm{PP}(D)$:

$$\Pr\left[\sqrt{\max_i t_i} \le x\right] = \Pr\left[\max_i t_i \le x^2\right] \ge \max\left\{1 - \frac{\mathrm{PP}(D)}{x^2}, 0\right\}$$

since $\mathrm{PP}(D) \ge x^2 \Pr_{t\sim D}\left[x^2 \le \max_i t_i\right]$. Hence:

$$\begin{aligned}\mathbb{E}_{t\sim D}\left[\sqrt{\max_i t_i}\right] &= \int_0^\infty \left(1 - \Pr\left[\sqrt{\max_i t_i} \le x\right]\right)dx \le \int_0^\infty\left(1 - \max\left(1 - \frac{\mathrm{PP}(D)}{x^2}, 0\right)\right)dx\\ &= \int_0^{\sqrt{\mathrm{PP}(D)}} 1dx + \int_{\sqrt{\mathrm{PP}(D)}}^\infty \frac{\mathrm{PP}(D)}{x^2}dx = 2\sqrt{\mathrm{PP}(D)}\end{aligned}$$

□

## 7 APPROXIMATELY OPTIMAL CREDIBLE AUCTION

We finally discuss the case where the auction $A$ is a non-truthful All-Pay auction (AP); the highest bidder wins and every bidder pays her bid. In our setting, with continuous type distributions and a common interval support of $[0, H]$, all-pay auctions admit pure monotone equilibria (see e.g. [AL96, Leb06, LKDT14]).

Crucially we show that all-pay auctions also satisfy the $c$-type-loss trade-off property. In fact we show a much more general statement: all sealed high-bid-wins auctions, where players do not overbid, and the auctioneer charges at most the player's bid (irrespective of allocation), satisfy that property. All-pay auctions certainly meet these criteria.

Lemma 7.1 (Type-Loss Trade-Off for General Auctions). *Consider any sealed-bid single-item auction, where the highest bidder wins and irrespective of allocation is charged at most her bid. Moreover, suppose that bidders do not bid more than their type at equilibrium. Then for any independent type distribution $D = \times_i D_i$ and under any no-overbidding bid equilibrium $b$, we have*

$$\mathbb{E}_{t\sim D}\left[\max_i t_i\left(1 - \pi_i^b(t_i)\right)\right] \le 4PP(D) \le 4OPT(D)$$

We defer the proof of Lemma 7.1 to Section 7.2. To prove the lemma, we observe that the equilibrium interim utility in any such auction is at least the largest box below the highest-other-bidder CDF curve, minus the largest box above that curve. Then we can follow similar analysis as in the first price auction to handle the first part of this decomposition and carry over an extra "largest box above curve" term; which subsequently is upper bounded by the best posted price revenue.

To state our main corollary we will define the instantiation of the entry fee simultaneous all-pay auction with ghost bidders as $\mathrm{EAP}(e, b)$, parameterized by a set of entry fees $e_i$ and a set equilibrium strategies $b = (b^1, \ldots, b^m)$, each $b^j$ corresponding to an equilibrium of the single-item all-pay auction for item $j$. Then the ghost bidders submit a bid on each item drawn based on the equilibrium strategies $b$, conditional on the event that the player decides not to enter (i.e. that the interim utility under $b$ is smaller than the entry fee). Such a mechanism admits the following *focal equilibrium*: player $i$ with type $t_i$ submits bid $b_i^j(t_{ij})$ on each auction $j$ and decides to enter if the interim utility, i.e. $u_i^b(t_i) = \sum_j t_{ij} \cdot \Pr[b_i^j(t_{ij}) \geq \max_{k \neq i} b_k^j(t_{kj})] - b_i^j(t_{ij})$, is greater then $e_i$. Then by Theorem 3.3:

Corollary 7.2. *Consider a multi-item auction with additive bidders and independent types across bidders $i$ and items $j$, distributed according to $D_{ij}$ and supported in $[0, H]$. For each item $j$, let $b^j$ denote an equilibrium of the single-item all-pay auction with type profile distribution $D_j = \times_i D_{ij}$. Then there exists a set of player-specific entry-fees $e = (e_1, \ldots, e_n)$, such that in the focal equilibrium $b$ of the EAP$(e, b)$:*

$$OPT(D) \leq 9 \cdot \sum_{j=1}^{m} OPT(D_j) + 2 \cdot \textsf{EF-Rev}^b(EAP(e, b)) \tag{17}$$

*where* $\textsf{EF-Rev}(EAP(e, b))$ *is the revenue of the EAP$(e, b)$ auction solely due to collection of entry fees.*

Combining the latter with the results of [HHT14], we have:

Corollary 7.3. *Consider a multi-item auction with additive bidders and independent types across bidders $i$ and items $j$, distributed according to $D_{ij}$ and supported in $[0, H]$. Suppose that type distributions $D_{ij}$ are regular. Then, for appropriately chosen parameters $r$, $e$, the better of: i) running simultaneous first-price auctions with item and bidder specific reserve prices FP$(r)$, ii) running simultaneous all-pay auctions with bidder specific bundle entry fees EAP$(e, b)$ (at the focal equilibrium), achieves a $\frac{20e-2}{e-1}$-factor approximation to the optimal revenue.*

This is the *first multi-dimensional revenue approximation result in the literature with a credible mechanism.* The results of [AL18] show that FP$(r)$, for any setting of the parameter $r$, is a credible mechanism. In the subsequent section, we prove that EAP$(e, b)$ is also credible, for any setting of the parameter $e$ and under any bid equilibirum $b$.

### 7.1 Credibility of entry-fee all pay auction

In this section, we formally define the criteria for a mechanism to be credible, as in [AL18], and then prove that EAP is a credible mechanism. We view a mechanism as a communication game between the auctioneer and the bidders. Let $n$ denote the number of bidders, $X$ denote the set of all possible outcomes of the mechanism and $T = \times_i T_i$ denote the type-space of the bidders. The auctioneer is viewed as a player 0 in the auction with utility (revenue) denoted by

$$u_0 : X \times T \mapsto \mathbb{R}.$$

The bidders are viewed as players indexed by the set $[n]$. At each step, player 0 contacts a player $i \in [n]$ privately. It sends a message and receives a reply. At any step, player 0 can choose an outcome $x \in X$ and end the game. Each player $i$ may have access to a part of the outcome. Let $S_i$ denote the strategy of player $i$. Let $o_j(S_0, S_1, ..., S_n, t)$ denote the observation of player $j$ when the auctioneer plays $S_0$, bidder $i$ plays $S_i$ and the type profile is $t = (t_1, t_2, \ldots, t_n)$. The observation $o_j$ includes the set of all messages received by player $j$ along with the part of the outcome it observes.

Definition 7.4. *Given a promised strategy profile $(S_0, S_1, \ldots, S_n)$, we define an auctioneer strategy $\hat{S}_0$ to be* **safe** *if for every player $i \in [n]$ and type profile $t = (t_1, t_2, \ldots, t_n)$, there exists $\hat{t}_{-i}$ such that*

$$o_i(\hat{S}_0, S_1, \ldots, S_n, t) = o_i(S_0, S_1, \ldots, S_n, (t_i, \hat{t}_{-i})),$$

*i.e., even if the auctioneer deviates from the promised strategy, there is an equivalent* innocent *explanation for each bidder's observation.*

Let $S_0^*(S_0, S_1, \ldots, S_n)$ denote the set of all safe strategies for the auctioneer. The auctioneer is restricted to play only a strategy $S \in S_0^*$ the messaging game. This is a reasonable constraint because if the auctioneer plays a strategy that is not "safe", the deviation can be easily detected by some bidder $i$.

Definition 7.5. *A mechanism with strategy profile $(S_0^G, S_1, \ldots, S_n)$ is credible if*

$$S_0^G \in \arg\max_{S_0 \in S_0^*(S_0^G, S_1, \ldots, S_n)} \mathbb{E}_t \left[ u_0(S_0, S_1, \ldots, S_n, t) \right].$$

Theorem 7.6 (Credibility of EAP). *The entry-fee all pay (EAP) auction is a credible mechanism.*

Proof. The communication protocol for the EAP mechanism proceeds as follows. Recall that the individual entry fees are fixed and known in advance to all players. Each bidder first sends a message to the auctioneer stating whether they will pay the entry fee. Those that do then provide bids to the auctioneer for each of the separate all-pay auctions. The auctioneer then stops the game and returns an outcome. In the auctioneer's promised strategy, this is done by simulating the bids of any bidders who chose not to pay their entry fees and then choosing an outcome consistent with the all-pay auction evaluated on each item separately.

Let $(S_0, S_1, \ldots, S_n)$ be the promised (non-deviating) strategy profile for the EAP mechanism. As the auctioneer's only decision point is the selection of the outcome, strategies differ only in this choice of outcome. However, note that each bidder's payment under the promised strategy $S_0$ is a deterministic function of their action, since entry fees are fixed and each bidder's payment in an all-pay auction is determined by their bid. Thus all safe strategies $\hat{S}_0 \in S_0^*$ must agree on each agent's payment, and therefore generate the same revenue for the auctioneer.[8] We conclude that $S_0$ weakly maximizes revenue over all safe strategies, and hence the EAP mechanism is credible. □

## 7.2 Proof of Lemma 7.1: Type Loss Trade-Off for General Auctions

Consider any sealed high-bid-wins auction $A$ that charges each player at most their bid (irrespective of winning or losing), i.e. $p^*_{A,i}(b) \le b_i$, and suppose the players do not overbid at equilibrium, i.e. $b_i(t_i) \le t_i$. We first note that by the fact that utilities are quasi-linear:

$$t_i \left(1 - \pi_i^b(t_i)\right) \le t_i - \left(\pi_i^b(t_i)\, t_i - p_i^b(t_i)\right) = t_i - u_i^b(t_i)$$

Therefore, it suffices to show that

$$\mathbb{E}_{t \sim D}\left[\max_i \left(t_i - u_i^b(t_i)\right)\right] \le 4\,\mathrm{PP}(D)$$

Let $B_i^b(t_{-i}) = \max_{k \neq i} b_k(t_k)$, denote the highest other equilibrium bid and let $F_i$, denote the CDF of this random variable over the randomness of $t_{-i} \sim D_{-i}$, i.e. $F_i(r) = \Pr_{t_{-i} \sim D_{-i}}\left[B_i^b(t_{-i}) < r\right]$. By the assumptions on the allocation and payment rule of $A$, the best-response equilibrium condition

[8]Note that since we assume the goods are consumed immediately and have 0 value for the seller, the seller's payoff is entirely determined by their revenue.

and the fact that bidders do not overbid, we have:

$$\begin{aligned} u_i^b(t_i) &\geq \max_{r \leq t_i} \left( t_i \cdot F_i(r) - \mathbb{E}_{t_{-i} \sim D_{-i}} \left[ p^*_{A,i}(r, b_{-i}(t_{-i})) \right] \right) \\ &\geq \max_{r \leq t_i} (t_i \cdot F_i(r) - r) \qquad \text{(by assumption that payment is at most bid)} \\ &= \max_{r \leq t_i} ((t_i - r) \cdot F_i(r) - r \cdot (1 - F_i(r))) \\ &\geq \underbrace{\max_{r \leq t_i} (t_i - r) \cdot F_i(r)}_{u_i(t_i)} - \underbrace{\max_{r \leq t_i} r \cdot (1 - F_i(r))}_{a_i(t_i)} \qquad (18) \end{aligned}$$

Applying Lemma 6.4 with $F$ being the CDF of $B_i^b(t_{-i})$, gives:

$$\sqrt{u_i(t_i)} \geq \sqrt{t_i} - \sqrt{a_i(t_i)}$$

where we note that $a_i(t_i) = \max_{r \leq t_i} r \Pr_{t_{-i} \sim D_{-i}}[r \leq B_i^b(t_{-i})]$. Since by definition $a_i(t_i) \leq t_i$, we can square the last inequality to obtain:

$$u_i(t_i) \geq \left( \sqrt{t_i} - \sqrt{a_i(t_i)} \right)^2 = t_i - 2\sqrt{a_i(t_i) \cdot t_i} + a_i(t_i)$$

Combining with Equation (18):

$$u_i^b(t_i) \geq u_i(t_i) - a_i(t_i) \geq t_i - 2\sqrt{a_i(t_i) \cdot t_i} \qquad (19)$$

Moreover, since by assumption players do not overbid at equilibrium:

$$a_i(t_i) = \max_{r \leq t_i} r \Pr_{t_{-i} \sim D_{-i}} [r \leq B_i^b(t_{-i})] \leq \max_{r \leq t_i} r \Pr_{t_{-i} \sim D_{-i}} [r \leq \max_{j \neq i} t_j] \leq \max_r r \Pr_{t \sim D} [r \leq \max_i t_i] = \mathrm{PP}(D)$$

Thus,

$$\begin{aligned} \mathbb{E}_{t \sim D} \left[ \max_i \left( t_i - u_i^b(t_i) \right) \right] \leq \mathbb{E}_{t \sim D} \left[ \max_i \left( 2\sqrt{a_i(t_i) \cdot t_i} \right) \right] &\leq 2\sqrt{\mathrm{PP}(D)} \cdot \mathbb{E}_{t \sim D} \left[ \sqrt{\max_i t_i} \right] \\ &\leq 4\,\mathrm{PP}(D) \qquad \text{(by Lemma 6.5)} \end{aligned}$$

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## A PROOF OF THEOREM 3.3

Theorem 3.3. *Let $A$ be any single-item auction, satisfying the $c$-type-loss trade-off and which admits an equilibrium strategy $b^j$ for type vector distribution $D_j = \times_i D_{ij}$ that is interim individually rational. Then there exists a set of player-specific entry-fees $e_i$, such that the focal equilibrium $b$ of the simultaneous $A$-item-auction with entry fees $e = (e_1, \ldots, e_n)$ and $\{b^j\}$-simulating ghost bidder distribution, $EA(e, D^g(\{b^j\})$ satisfies:*

$$OPT(D) \leq (c+5) \cdot \sum_{j=1}^{m} OPT(D_j) + 2 \cdot \textsf{EF-Rev}^b(EA(e, D^g(\{b^j\}))) \tag{20}$$

*where $OPT(D)$ denotes the optimal revenue in the multi-dimensional multi-item auction setting with type distributions $D = \times_i D_i$ and $OPT(D_j)$ is optimal revenue in a single item auction setting with type vector distribution $D_j = \times_i D_{ij}$.*

Our starting point is Lemma 3.5, whose proof is provided in Appendix B. Based on this lemma, we have that if can define a monotone preference partition $\{R_{i,j}\}_{i\in[n], j\in[m]}$ of the type spaces, then if we let $\mathcal{F}$ denote the set of interim feasible allocations:

$$\text{OPT}(D) \leq \sup_{\pi \in \mathcal{F}} \underbrace{\sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi_{ij}(t_i)\left(t_{ij} \cdot 1\left\{t_i \notin R_{i,j}\right\} + \tilde{\varphi}^*_{ij}(t_{ij}) \cdot 1\left\{t_i \in R_{i,j}\right\}\right)\right]}_{\text{VW}(\pi)} \tag{21}$$

where $\tilde{\varphi}^*_{ij}(t_{ij}) = \max(\tilde{\varphi}_{ij}(t_{ij}), 0)$ and $\tilde{\varphi}_{ij}(t_{ij})$ represents Myerson's ironed virtual value function [Mye81] for the distribution $D_{ij}$. We will refer to the latter bound as the multi-dimensional virtual welfare (VW).

To apply this bound it suffices to define the monotone preference functions $\mathcal{U}_{i,j} : T_{ij} \to \mathbb{R} \geq 0$ such that, for all $j \neq 0$. We define this preference function in terms of the interim utility $u_{ij}^{b^j}(t_{ij})$ that player $i$ receives in auction $j$ with type vector distribution $D_j = \times_i D_{ij}$ and equilibrium $b^j$ (as defined in the statement of the theorem). Moreover, we will denote with $b$ the bid strategy that corresponds to each player submitting to item $j$ the bid prescribed by the per-item equilibrium $b^j$ for his type $t_{ij}$ and for simplicity of notation we will denote $u_{ij}^{b^j}(t_{ij})$ with $u_{ij}^b(t_{ij})$. Then we can define the preference function as:

$$\mathcal{U}_{i,j}(t_{ij}) = u_{ij}^b(t_{ij}) \tag{22}$$

Observe that this preference function is non-decreasing in $t_{ij}$, since interim utility at any equilibrium of any mechanism in a single dimensional environment is non-decreasing in type, by standard results in single-dimensional mechanism design (see e.g. Theorem 2.2 of [Har]).

*Description of regions.* Intuitively, we define our preference regions in terms of the best interim utility item under strategy profile $b$: each type vector $t_i$ of bidder $i$ induces a ranking of the items based on interim utility: $j_1, j_2, \cdots, j_m$ such that

$$u_{ij_1}^b(t_{ij_1}) \geq u_{ij_2}^b(t_{ij_2}) \geq \cdots \geq u_{ij_m}^b(t_{ij_m}) \geq 0 \tag{23}$$

breaking ties lexicographically. Then, we assign the bidder to the highest ranked item for his type, with non-zero interim utility. Thus we can re-express the partitions as:

$$\begin{aligned} t_i \in R_{i,j} \Leftrightarrow u_{ij}^b(t_{ij}) > 0 & \\ &\text{and for all } k\text{: } u_{ik}^b(t_{ik}) \leq u_{ij}^b(t_{ij}) \\ &\text{and for all } k < j\text{: } u_{ik}^b(t_{ik}) < u_{ij}^b(t_{ij}) \end{aligned} \tag{24}$$

We say $t_i \in R_0$ if it belongs to no other regions.

## A.1 Decomposition of Upper Bound VW

We can apply Lemma 3.5 on the latter monotone preference partition regions to get Equation (21). We now further decompose the right hand side of Equation (21) into four terms, Under, Single, Over, Surplus that we will subsequently bound separately.

First, observe that by the characterization of regions described in Equation (24):

$$1\left\{t_i \notin R_{i,j}\right\} \leq 1\left\{\exists k \neq j, \text{ s.t. } u^b_{ik}(t_{ik}) \geq u^b_{ij}(t_{ij})\right\} \tag{25}$$

Let $\mathcal{Z}^b_{ij}(t_i)$ denote the event that *item $j$ is a strictly favorite item for player $i$ in terms of interim utility under equilibrium $b$*, i.e.:

$$\mathcal{Z}^b_{ij}(t_i) := \{\forall k \neq j : u^b_{ij}(t_{ij}) > u^b_{ik}(t_{ik})\} \tag{26}$$

and let $\bar{\mathcal{Z}}^b_{ij}(t_i)$ denote its complement. Thus:

$$1\left\{t_i \notin R_{i,j}\right\} = 1\left\{\bar{\mathcal{Z}}^b_{ij}(t_i)\right\} \leq \underbrace{1\left\{\bar{\mathcal{Z}}^b_{ij}(t_i)\right\} \cdot \pi^b_{ij}(t_{ij})}_{\text{Prob allocated } j\text{, but } j \text{ not strict favorite}} + \underbrace{(1 - \pi^b_{ij}(t_{ij}))}_{\text{Prob not allocated item } j \text{ in auction } A}$$

where we remind that $\pi^b_{ij}(t_{ij})$ is the interim allocation of player $i$ in a single item auction $A$ for item $j$ under equilibrium $b^j$. So, we can upper bound and decompose the virtual welfare $\mathrm{VW}(\pi)$ as

$$\mathrm{VW}(\pi) \leq \sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi_{ij}(t_i) \cdot t_{ij} \cdot 1\left\{\bar{\mathcal{Z}}^b_{ij}(t_i)\right\} \cdot \pi^b_{ij}(t_{ij})\right] \tag{NonFavorite($\pi$)}$$

$$+ \sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi_{ij}(t_i) \cdot t_{ij} \cdot (1 - \pi^b_{ij}(t_{ij})\right] \tag{Under($\pi$)}$$

$$+ \sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi_{ij}(t_i) \cdot \tilde{\varphi}^*_{ij}(t_{ij}) \cdot 1\left\{t_i \in R_{i,j}\right\}\right] \tag{Single($\pi$)}$$

We further decompose NonFavorite by invoking the quasi-linearity of player utilities in each item auction, i.e. $t_{ij} \cdot \pi^b_{ij}(t_{ij}) = u^b_{ij}(t_{ij}) + p^b_{ij}(t_{ij})$:

$$\begin{aligned}
\textsc{NonFavorite}(\pi) &= \sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi_{ij}(t_i) \cdot 1\left\{\bar{\mathcal{Z}}^b_{ij}(t_i)\right\} \cdot \left(u^b_{ij}(t_{ij}) + p^b_{ij}(t_{ij})\right)\right] \\
&\leq \underbrace{\sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi_{ij}(t_i) \cdot p^b_{ij}(t_{ij})\right]}_{\textsc{Over}(\pi)} + \underbrace{\sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi_{ij}(t_i) \cdot u^b_{ij}(t_{ij}) \cdot 1\left\{\bar{\mathcal{Z}}^b_{ij}(t_i)\right\}\right]}_{\textsc{Surplus}(\pi)}
\end{aligned}$$

Which completes our final upper bound decomposition as:

$$\mathrm{VW}(\pi) \leq \textsc{Over}(\pi) + \textsc{Surplus}(\pi) + \textsc{Under}(\pi) + \textsc{Single}(\pi) \tag{27}$$

In the next sections we will prove the following bounds, which complete the proof of our theorem.

$$\textsc{Single}(\pi) \le \sum_{j=1}^{m} \mathrm{OPT}(D_j) \qquad \textsc{Under}(\pi) \le c \cdot \sum_{j=1}^{m} \mathrm{OPT}(D_j)$$
$$\textsc{Over}(\pi) \le \sum_{j=1}^{m} \mathrm{OPT}(D_j) \qquad \textsc{Surplus}(\pi) \le 3 \sum_{j=1}^{m} \mathrm{OPT}(D_j) + 2 \cdot \textsf{EF-Rev}^{b}(EA(e, D^{g}))$$

### A.2 Upper Bounding Single

Since $\pi_{ij}$ is an interim feasible allocation, we have that there exists an ex-post feasible allocation $x_{ij}$, such that $\pi_{ij}(t_i) = \mathbb{E}_{t_{-i}\sim D_{-i}}\left[x_{ij}(t)\right]$. Invoking this fact and the fact that $\phi^{*}_{ij}(t_{ij}) \ge 0$, we have:

$$\begin{aligned}
\textsc{Single}(\pi) &= \sum_i \mathbb{E}_{t_i\sim D_i}\left[\sum_j \pi_{ij}(t_i)\cdot \tilde{\varphi}^{*}_{ij}(t_{ij})\cdot 1\left\{t_i \in R_{i,j}\right\}\right] \\
&\le \sum_i \mathbb{E}_{t_i\sim D_i}\left[\sum_j \pi_{ij}(t_i)\cdot \tilde{\varphi}^{*}_{ij}(t_{ij})\right] && \text{(since } \tilde{\varphi}^{*}_{ij}(t_{ij}) \ge 0\text{)} \\
&= \sum_i \mathbb{E}_{t_i\sim D_i}\left[\sum_j \mathbb{E}_{t_{-i}\sim D_{-i}}\left[x_{ij}(t_i, t_{-i})\right]\cdot \tilde{\varphi}^{*}_{ij}(t_{ij})\right] && \text{(by interim feasibility of } \pi_{ij}\text{)} \\
&= \sum_j \mathbb{E}_{t\sim D}\left[\sum_i x_{ij}(t)\cdot \tilde{\varphi}^{*}_{ij}(t_{ij})\right] \\
&\le \sum_j \mathbb{E}_{t\sim D}\left[\max_i \tilde{\varphi}^{*}_{ij}(t_{ij})\right] && \text{(by ex-post feasibility of } x_{ij}\text{)} \\
&= \sum_j \mathrm{OPT}(D_j) && \text{(by Myerson's [Mye81] theorem)}
\end{aligned}$$

### A.3 Upper Bounding Under

We rearrange $\textsc{Under}(\pi)$ to be in terms of the ex-post feasible allocation $x$ that gives rise to interim allocation $\pi$.

$$\begin{aligned}
\textsc{Under}(\pi) &= \sum_i \mathbb{E}_{t_i\sim D_i}\left[\sum_j \pi_{ij}(t_i)\cdot t_{ij}\cdot\left(1-\pi^{b}_{ij}(t_{ij})\right)\right] \\
&= \sum_i \mathbb{E}_{t_i\sim D_i}\left[\sum_j \mathbb{E}_{t_{-i}\sim D_{-i}}\left[x_{ij}(t_i, t_{-i})\right] t_{ij}\left(1-\pi^{b}_{ij}(t_{ij})\right)\right] \\
&= \sum_j \mathbb{E}_{t\sim D}\left[\sum_i x_{ij}(t)\cdot t_{ij}\cdot\left(1-\pi^{b}_{ij}(t_{ij})\right)\right] \\
&\le \sum_j \mathbb{E}_{t\sim D}\left[\max_i t_{ij}\left(1-\pi^{b}_{ij}(t_{ij})\right)\right] && \text{(by ex-post feasibility of } x_{ij}\text{)} \\
&\le \sum_j c\cdot \mathrm{OPT}(D_j) && \text{(by } c\text{-type-loss trade off property of } A\text{)}
\end{aligned}$$

### A.4 Upper Bounding Over

$$
\begin{aligned}
\textsc{Over}(\pi) &= \sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi_{ij}(t_i)\cdot p^b_{ij}(t_{ij})\right] \\
&\le \sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j p^b_{ij}(t_{ij})\right] && \text{(by interim feasibility: } \pi_{ij}(t_{ij}) \le 1) \\
&= \sum_j \mathrm{Rev}^{b^j}(A) \\
&\le \sum_j \mathrm{OPT}(D_j)
\end{aligned}
$$

where $\mathrm{Rev}^{b_j}(A)$ represents the revenue of the $A$-auction on item $j$ under equilibrium $b^j$ with type vector distribution $D_j$.

### A.5 Upper Bounding Surplus

By rearranging the terms in Surplus and invoking the fact that types are independent across items, we have:

$$
\begin{aligned}
\textsc{Surplus}(\pi) &= \sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi_{ij}(t_i)\cdot u^b_{ij}(t_{ij})\cdot 1\left\{\bar{\mathcal{Z}}^b_{ij}(t_i)\right\}\right] \\
&\le \sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j u^b_{ij}(t_{ij})\cdot 1\left\{\bar{\mathcal{Z}}^b_{ij}(t_i)\right\}\right] && \text{(by interim feasibility: } \pi_{ij}(t_{ij}) \le 1) \\
&= \sum_i \sum_j \mathbb{E}_{t_{ij} \sim D_{ij}}\left[u^b_{ij}(t_{ij})\cdot \mathbb{E}_{t_{i,-j} \sim D_{i,-j}}\left[1\left\{\bar{\mathcal{Z}}^b_{ij}(t_i)\right\}\right]\right] && \text{( independence across items)} \\
&= \sum_{i,j} \mathbb{E}_{t_{ij} \sim D_{ij}}\left[u^b_{ij}(t_{ij}) \Pr_{t_{i,-j} \sim D_{i,-j}}\left[\exists k \ne j, u^b_{ik}(t_{ik}) \ge u^b_{ij}(t_{ij})\right]\right] && \text{(definition of } \bar{\mathcal{Z}}^b_{ij}(t_i))
\end{aligned}
$$

Analyzing the relative size of each $u^b_{ij}(t_{ij})$ (we abbreviate as $u^b_{ij}$) will be fundamental to bounding the surplus term. Intuitively, in the event that $u^b_{ij}$ is not too large, its contribution to the Surplus sum will be not too large and therefore boundable. When $u^b_{ij}$ is very large, bounding Surplus will still be possible due to the fact that the probability there exists an even larger $u^b_{ik}$ will be small. Thus, we will analyze Surplus by splitting into an analysis of these two regimes, denoted as Core and Tail. The pivotal point that defines these two regimes is based on an interim utility threshold $r^b_i$ defined as follows.

$$
\begin{aligned}
r^b_{ij} &= \max_x \left(x \Pr_{t_{ij} \sim D_{ij}}[u^b_{ij}(t_{ij}) \ge x]\right) \\
r^b_i &= \sum_j r^b_{ij}
\end{aligned}
$$

We decompose SURPLUS based on this interim utility threshold:

$$\text{SURPLUS}(\pi) \le \sum_{i,j} \mathbb{E}_{t_{ij}\sim D_{ij}}\left[u^b_{ij}(t_{ij}) \Pr_{t_{i,-j}\sim D_{i,-j}}\left[\exists k \neq j, u^b_{ik}(t_{ik}) \ge u^b_{ij}(t_{ij})\right] \cdot 1[u^b_{ij}(t_{ij}) \ge r^b_i]\right] \quad (\text{TAIL})$$

$$+ \sum_{i,j} \mathbb{E}_{t_{ij}\sim D_{ij}}\left[u^b_{ij}(t_{ij}) \cdot 1[u^b_{ij}(t_{ij}) < r^b_i]\right] \quad (\text{CORE})$$

*Upper bounding TAIL.* We upper bound this term by $r^b = \sum_i r^b_i$. At the end of the analysis, we prove that $r^b \le \sum_j \text{OPT}(D_j)$. First, by union bound

$$\Pr_{t_{i,-j}\sim D_{i,-j}}\left[\exists k \neq j, u^b_{ik}(t_{ik}) \ge u^b_{ij}(t_{ij})\right] \le \sum_{k\neq j} \Pr_{t_{ik}\sim D_{ik}}\left[u^b_{ik}(t_{ik}) \ge u^b_{ij}(t_{ij})\right]$$

By the definition of $r^b_{ik}$, we have that:

$$r^b_{ik} \ge u^b_{ij}(t_{ij}) \Pr_{t_{ik}\sim D_{ik}}\left[u^b_{ik}(t_{ik}) \ge u^b_{ij}(t_{ij})\right] \tag{28}$$

and so we can bound TAIL as:

$$\begin{aligned}
\text{TAIL} &\le \sum_{i,j} \mathbb{E}_{t_{ij}\sim D_{ij}}\left[1\left[u^b_{ij}(t_{ij}) \ge r^b_i\right] \sum_{k\neq j} u^b_{ij}(t_{ij}) \Pr_{t_{ik}\sim D_{ik}}\left[u^b_{ik}(t_{ik}) \ge u^b_{ij}(t_{ij})\right]\right] \\
&\le \sum_{i,j} \mathbb{E}_{t_{ij}\sim D_{ij}}\left[1\left[u^b_{ij}(t_{ij}) \ge r^b_i\right] \sum_{k\neq j} r^b_{ik}\right] \\
&\le \sum_{i,j} r^b_i \cdot \mathbb{E}_{t_{ij}\sim D_{ij}}\left[1\left[u^b_{ij}(t_{ij}) \ge r^b_i\right]\right] \\
&= \sum_{i,j} r^b_i \cdot \Pr_{t_{ij}\sim D_{ij}}\left[u^b_{ij}(t_{ij}) \ge r^b_i\right] \\
&\le \sum_{i,j} r^b_{ij} && (\text{since } r^b_{ij} \ge x \Pr[u^b_{ij} \ge x] \text{ for all } x) \\
&= \sum_i r^b_i = r^b
\end{aligned}$$

*Upper bounding CORE.* For notational convenience, let

$$c^b_{ij}(t_{ij}) = u^b_{ij}(t_{ij}) \cdot 1\left[u^b_{ij}(t_{ij}) < r^b_i\right] \tag{29}$$

so that:

$$\text{CORE} = \sum_{i,j} \mathbb{E}_{t_{ij}\sim D_{ij}}\left[c^b_{ij}(t_{ij})\right] \tag{30}$$

Now, we consider the $\text{EA}(e, D^g)$ mechanism with entry fee $e_i$ for bidder $i$ defined as:

$$e^b_i = \left[\left(\sum_j \mathbb{E}_{t_{ij}}\left[c^b_{ij}(t_{ij})\right]\right) - 2r^b_i\right]_+ \tag{31}$$

where $[x]_+ := \max\{x, 0\}$. This is a valid entry fee as it is a non-negative constant that only depends on the type distributions $D_{ij}$.

We will show that each bidder $i$ accepts the entry fee with probability at least $1/2$. Bidder $i$ accepts the entry fee iff his total interim utility over the auctions exceeds the fee. Thus, if we can show

$$\Pr_{t_i \sim D_i}\left[\sum_j u_{ij}^b(t_{ij}) \geq e_i^b\right] \geq 1/2$$

then we know the expected revenue of $EA(e, D^g)$ (in equilibrium $b$ as described in the definition of the Theorem) from entry fees alone is at least

$$\frac{1}{2}\sum_i e_i^b \geq \frac{1}{2}\sum_i \left(\sum_j \mathbb{E}_{t_{ij}}[c_{ij}^b(t_{ij})] - 2r_i^b\right) = \frac{\textsc{Core}}{2} - r^b \tag{32}$$

this would imply that:

$$\textsc{Core} \leq 2\, r^b + 2\,\textsf{EF-Rev}^b(EA(e, D^g) \tag{33}$$

as desired. We make use of the following lemma, originally proved in [BILW15],

Lemma A.1. *Let $x$ be a positive single dimensional random variable drawn from $F$ of finite support, such that for any number $a$, $a \cdot \Pr_{x\sim F}[x \geq a] \leq \mathcal{B}$ where $\mathcal{B}$ is an absolute constant. Then, for any positive number $s$, the second moment of the random variable $x_s = x \cdot 1[x \leq s]$ is upper bounded by $2 \cdot \mathcal{B} \cdot s$.*

Applying this lemma with $x = u_{ij}^b(t_{ij})$, $\mathcal{B} = r_{ij}^b$ and $s = r_i^b$, we obtain:

$$\mathbb{E}\left[\left(c_{ij}^b(t_{ij})\right)^2\right] \leq 2r_i^b r_{ij}^b \tag{34}$$

Since $c_{ij}^b(t_{ij})$ are independent across items,

$$\mathrm{Var}\left[\sum_j c_{ij}^b(t_{ij})\right] = \sum_j \mathrm{Var}\left[c_{ij}^b(t_{ij})\right] \leq \sum_j \mathbb{E}\left[\left(c_{ij}^b(t_{ij})\right)^2\right] \leq 2\,\left(r_i^b\right)^2 \tag{35}$$

By Chebyshev, we know:

$$\Pr_{t_i \sim D_i}\left[\sum_j c_{ij}^b(t_{ij}) \leq \sum_j \mathbb{E}[c_{ij}^b(t_{ij})] - 2r_i^b\right] \leq \frac{\mathrm{Var}\left[\sum_j c_{ij}^b(t_{ij})\right]}{4\,\left(r_i^b\right)^2} \leq \frac{1}{2} \tag{36}$$

Moreover, since $c_{ij}^b(t_{ij})$ are non-negative, we have that in the case where the $[\cdot]_+$ binds and $e_i = 0$, then it is definitely true that:

$$\Pr_{t_i \sim D_i}\left[\sum_j c_{ij}^b(t_{ij}) \leq \left[\sum_j \mathbb{E}[c_{ij}^b(t_{ij})] - 2r_i^b\right]_+\right] \leq \frac{1}{2} \tag{37}$$

Hence, $\Pr_{t_i \sim D_i}\left[\sum_j c_{ij}^b(t_{ij}) \leq e_i\right] \leq \frac{1}{2}$. Since, we also have that $u_{ij}^b(t_{ij}) \geq c_{ij}^b(t_{ij})$, we can conclude that $\Pr\left[\sum_j u_{ij}^b > e_i^b\right] \geq 1/2$, as desired.

*Upper bounding $r^b$.* We conclude the proof of the bound on Surplus by providing an upper bound on $r^b$. We can obtain revenue $r^b$ via selling the items separately, where each item is sold via an EA$(e, D^g)$ auction solely for that item. More concretely, for each item $j$, each bidder $i$ can choose to pay an entry fee $e_{ij}$ to access an $A$-auction on item $j$. Bidder $i$ can choose whether or not to buy into the item $j$ auction totally independently of his choice for the other auctions. We will again be using ghost bidders for all bidders who do not pay the entry fee. Thus, bidder $i$'s utility for entering the item $j$ auction is $u^b_{ij}(t_{ij})$, and he will pay the entry fee iff $u^b_{ij}(t_{ij}) \geq e_{ij}$. The maximum entry fee revenue we can obtain in such an auction is equal to:

$$\max_{e_{ij}} e_{ij} \Pr_{t_{ij}\sim D_{ij}}[u^b_{ij}(t_{ij}) \geq e_{ij}] = r^b_{ij}$$

Thus, setting entry fees optimally on all items for all bidders, we obtain entry fee revenue $\sum_i \sum_j r^b_{ij} = r^b$. The revenue obtained from these separate EA auctions on each item is upper bounded by the revenue obtained from separate optimal single item auctions on each item, giving

$$r^b \leq \sum_j \mathrm{OPT}(D_j) \tag{38}$$

as desired.

*Concluding.* Combining all the above analysis, we have:

$$\begin{aligned}\textsc{Surplus}(\pi) &\leq \textsc{Tail} + \textsc{Core} \leq r^b + \left(2\,r^b + 2\,\textsf{EF-Rev}^b(EA(e, D^g)\right)\\ &\leq 3\sum_j \mathrm{OPT}(D_j) + 2\,\textsf{EF-Rev}^b(EA(e, D^g)\end{aligned}$$

## B PROOF OF LEMMA 3.5

Lemma 3.5 (Revenue Bound via Monotone Preference Partitions of Type Space). *Consider a multi-item auction setting with additive bidders and independent continuous type distributions $D_{ij}$ on a bounded support $[0, H]$. Let $\{R_{i,j}\}_{i\in[n],j\in[m]}$ be a monotone preference partition of the type space and let $\mathcal{F}$ denote the space of all interim feasible allocations. Then:*

$$OPT(D) \leq \sup_{\pi\in\mathcal{F}} \sum_i \mathbb{E}_{t_i\sim D_i}\left[\sum_j \pi_{ij}(t_i)\left(t_{ij}\cdot 1\left\{t_i \notin R_{i,j}\right\} + \tilde{\varphi}^*_{ij}(t_{ij})\cdot 1\left\{t_i \in R_{i,j}\right\}\right)\right] \tag{39}$$

*where $\tilde{\varphi}^*_{ij}(t_{ij}) = \max(\tilde{\varphi}_{ij}(t_{ij}), 0)$ and $\tilde{\varphi}_{ij}(t_{ij})$ represents Myerson's ironed virtual value function [Mye81] for the distribution $D_{ij}$.*

Our starting point is the following lemma of [CDW16] that applies to discrete types and discrete type distributions. We will subsequently provide a discretization argument that allows us to prove the continuous analogue of it presented in Lemma 3.5.

Theorem B.1 (Theorem 31 [CDW16]). *Consider a multi-item auction setting with discrete type space $T^+$ and discrete valuation distribution $D^+$. For each $v_{-i} \in T^+_{-i}$, let $R^{v_{-i}}_0, R^{v_{-i}}_1, \cdots, R^{v_{-i}}_m$ be a partition of the type space $T^+_i$ into "upwards-closed" regions. That is, for all $j \neq 0$,*

$$t_i = (t_{i1}, \cdots, t_{ij}, \cdots, t_{im}) \in R^{v_{-i}}_{i,j} \Rightarrow (t_{i1}, \cdots, t^*_{ij}, \cdots, t_{im}) \in R^{v_{-i}}_{i,j} \text{ for all } t^*_{ij} > t_{ij}$$

*Let $M$ be any BIC mechanism with values drawn from $D^+$ that has interim allocation and payment $\pi^M, p^M$ in the truthful equilibrium. The expected revenue of $M$ in the truthful equilibrium is upper* *bounded by the expected virtual welfare of the same allocation rule with respect to the canonical virtual value function $\Phi_i$. In particular,*

$$\mathsf{Rev}(M) \leq \sum_{i,j} \mathbb{E}_{t_i \sim D_i^+}\left[\pi_{ij}^M(t_i)\left(t_{ij} \cdot \Pr_{v_{-i}\sim D_{-i}^+}\left[t_i \notin R_{i,j}^{v_{-i}}\right] + \tilde{\varphi}_{ij}(t_{ij}) \cdot \Pr_{v_{-i}\sim D_{-i}^+}\left[t_i \in R_{i,j}^{v_{-i}}\right]\right)\right] \tag{40}$$

*where $\tilde{\varphi}_{ij}(t_{ij})$ represents Myerson's discrete ironed virtual value for the distribution $D_{ij}^+$.*

*Proof outline.* Our approach will be to consider a discretization of the continuous type distribution $D$: $D^\epsilon$. We define $D_{ij}^\epsilon$ to first sample $t_{ij} \sim D_{ij}$ and then output $t_{ij}^\epsilon = \epsilon^2 \cdot \lceil t_{ij}/\epsilon^2 \rceil$. We see that $D_{ij}^\epsilon$ will have finite support $T^\epsilon$, as the support of $D_{ij}$ is bounded $\in [0, H]$. Our approach is as follows. Due to the coupling of samples from $D$ and $D^\epsilon$, we will be able to show that the revenue-optimal mechanism $OPT$ for values drawn from $D$ achieves approximately the same revenue as the revenue optimal mechanism $OPT^\epsilon$ for values drawn from $D^\epsilon$: $\mathsf{Rev}^D(OPT) \approx \mathsf{Rev}^{D^\epsilon}(OPT^\epsilon)$. Since $D^\epsilon$ has finite support, we will be able to apply theorem B.1 bounding the revenue of $OPT^\epsilon$ by its virtual welfare. Then, one last argument on the coupled distributions will give that the virtual welfare upper bound for the discrete distribution is related to the desired virtual welfare bound for the continuous distribution.

*Preference partitions are upwards-closed.* Let $\{R_{i,j}\}_{i\in[n],j\in[m]}$ be a preference partition of the continuous type space $T$, which will also be a preference partition on the discrete subset $T^\epsilon$. Observe that preference partitions are always upwards-closed partitions, which is true due to the following argument: Let $t_{ij}, t'_{ij} \in T_{ij}$ with $t'_{ij} > t_{ij}$. Say for some type vector $t_i = (t_{i1}, \cdots, t_{ij}, \cdots, t_{im})$ we have $t_i \in R_{i,j}$. We want to show $t'_i = (t_{i1}, \cdots, t'_{ij}, \cdots, t_{im}) \in R_{i,j}$. We see

$$t_i \in R_{i,j} \Rightarrow \begin{Bmatrix} \mathcal{U}_{i,j}(t_{ij}) \geq \mathcal{U}_{i,k}(t_{ik}) & \forall k \neq j \\ \mathcal{U}_{i,j}(t_{ij}) > \mathcal{U}_{i,k}(t_{ik}) & \forall k < j \\ \mathcal{U}_{i,j}(t_{ij}) > 0 & \end{Bmatrix} \Rightarrow \begin{Bmatrix} \mathcal{U}_{i,j}(t'_{ij}) \geq \mathcal{U}_{i,k}(t_{ik}) & \forall k \neq j \\ \mathcal{U}_{i,j}(t'_{ij}) > \mathcal{U}_{i,k}(t_{ik}) & \forall k < j \\ \mathcal{U}_{i,j}(t'_{ij}) > 0 & \end{Bmatrix} \Rightarrow t'_i \in R_{i,j}$$

since $\mathcal{U}_{i,j}$ is non-decreasing, as desired. Thus we can apply Theorem B.1 on the discretized type space and bound $\mathsf{Rev}^{D^\epsilon}(OPT^\epsilon)$

$$\mathsf{Rev}^{D^\epsilon}(OPT^\epsilon) \leq \underbrace{\sum_{i,j} \mathbb{E}_{t_i \sim D_i^\epsilon}\left[\pi_{ij}^{OPT^\epsilon}(t_i)\left(t_{ij} \cdot 1\left\{t_i \notin R_{i,j}\right\} + \tilde{\varphi}_{ij}^+(t_{ij}) \cdot 1\left\{t_i \in R_{i,j}\right\}\right)\right]}_{\mathsf{VW}^\epsilon}$$

where $\tilde{\varphi}_{ij}^+(t_{ij})$ is Myerson's discrete ironed virtual value for the distribution $D_{ij}^\epsilon$. In the latter we also used the fact that the preference partition of a player's type space is independent of the types of other players.

We conclude by separately relating the left-hand-side $\mathsf{Rev}^{D^\epsilon}(OPT^\epsilon)$ to $\mathsf{Rev}^D(OPT)$ (Section B.1), and the right-hand-side $\mathsf{VW}^\epsilon$ to its continuous counter-part VW (Section B.2). In both cases, we show that the two quantities converge to each other as $\epsilon \to 0$, which implies the desired continuous upper bound.

## B.1 Relating $\mathsf{Rev}^D(OPT)$ to $\mathsf{Rev}^{D^\epsilon}(OPT^\epsilon)$

We make use of the following theorem.

THEOREM B.2. *[[RW18b], [DW12]] Let $M^\downarrow$ be any BIC mechanism for additive bidders with values drawn from distribution $D^\downarrow$. For all $i$, let $D_i^\downarrow$ and $D_i^\uparrow$ be any two distributions with coupled samples $t_i^\downarrow$*

*and $t_i^{\uparrow}$ such that $t_i^{\uparrow} \cdot x_i \geq t_i^{\downarrow} \cdot x_i$ for all feasible allocations $x \in F$. If $\delta_i = t_i^{\uparrow} - t_i^{\downarrow}$, then for any $\epsilon > 0$, there exists a BIC mechanism $M^{\uparrow}$ such that*

$$\mathrm{Rev}^{D^{\uparrow}}(M^{\uparrow}) \geq (1-\epsilon)\left(\mathrm{Rev}^{D^{\downarrow}}(M^{\downarrow}) - \frac{VAL(\delta)}{\epsilon}\right)$$

*where $VAL(\delta)$ denotes the expected welfare of the VCG allocation when bidder $i$'s type is drawn according to the random variable $\delta_i$.*

Using this theorem, we will be able to bound the gap between $\mathrm{Rev}^{D}(OPT)$ and $\mathrm{Rev}^{D^{\epsilon}}(OPT^{\epsilon})$. We introduce $D_{ij}^{-,\epsilon}$, defined similarly to $D_{ij}^{\epsilon}$, that first samples $t_{ij} \sim D_{ij}$ and then outputs $t_{ij}^{-,\epsilon} = \epsilon^2 \cdot \left(\lceil t_{ij}/\epsilon^2 \rceil - 1\right)$. Also, define $OPT^{-,\epsilon}$ to be the revenue-optimal mechanism for values drawn from $D^{-,\epsilon}$. We will apply Theorem B.2 with $D$ as $D^{\downarrow}$ and $D^{\epsilon}$ as $D^{\uparrow}$ and $OPT$ as $M^{\downarrow}$ as well as with $D^{-,\epsilon}$ as $D^{\downarrow}$ and $D$ as $D^{\uparrow}$ and $OPT^{-,\epsilon}$ as $M^{\downarrow}$. In both cases, due to the coupling, we will have the necessary $t_i^{\uparrow} \cdot x_i \geq t_i^{\downarrow} \cdot x_i$ for all $x$. Additionally, we will have $\delta_{ij} \leq \epsilon^2$ for all $i, j$. Thus, $VAL(\delta) \leq m\epsilon^2$ as the welfare contribution of any one item is at most $\epsilon^2$ for types $\delta_i$. So, applying this theorem in these two settings gives, for some mechanisms $M_1^{\uparrow}$,$M_2^{\uparrow}$,

$$\mathrm{Rev}^{D^{\epsilon}}(OPT^{\epsilon}) \geq \mathrm{Rev}^{D^{\epsilon}}(M_1^{\uparrow}) \geq (1-\epsilon)(\mathrm{Rev}^{D}(OPT) - m\epsilon)$$
$$\mathrm{Rev}^{D}(OPT) \geq \mathrm{Rev}^{D}(M_2^{\uparrow}) \geq (1-\epsilon)(\mathrm{Rev}^{D^{-,\epsilon}}(OPT^{-,\epsilon}) - m\epsilon)$$

Lastly, note that

$$\mathrm{Rev}^{D^{\epsilon}}(OPT^{\epsilon}) = \mathrm{Rev}^{D^{-,\epsilon}}(OPT^{-,\epsilon}) + m\epsilon$$

as every buyer values every item at exactly $\epsilon$ more in $D^{\epsilon}$ versus $D^{-,\epsilon}$. For every BIC mechanism with values drawn from $D^{-,\epsilon}$, there is an analogous mechanism for values $D^{\epsilon}$ in which every bidders payment increases by exactly $\epsilon$ times the number of items they are expost allocated. Thus, we have

$$\mathrm{Rev}^{D}(OPT) \in \left[(1-\epsilon)(\mathrm{Rev}^{D^{\epsilon}}(OPT^{\epsilon}) - 2m\epsilon), \frac{\mathrm{Rev}^{D^{\epsilon}}(OPT^{\epsilon})}{1-\epsilon} + m\epsilon\right]$$

So, as $\epsilon \longrightarrow 0$, we achieve discrete type distributions $D^{\epsilon}$ for which there exists mechanisms $OPT^{\epsilon}$ with revenue arbitrarily close to $\mathrm{Rev}^{D}(OPT)$.

## B.2 Relating $\mathrm{VW}^{\epsilon}$ to VW

We can simulate a sample of the discrete distribution as follows: first sample $t_i \sim D_i$ from the continuous distribution and then let $t_i^{+}$ be the rounded discrete type in terms of $t_i$. That is, $t_{ij}^{+} = \epsilon^2 \cdot \lceil t_{ij}/\epsilon^2 \rceil$. We can then write the upper bound on $\mathrm{Rev}^{D^{\epsilon}}(OPT^{\epsilon})$ as:

$$\mathrm{VW}^{\epsilon} = \sum_{i,j} \mathbb{E}_{t_i \sim D_i}\left[\pi_{ij}^{OPT^{\epsilon}}(t_i^{+})\left(t_{ij}^{+} \cdot 1\left\{t_i^{+} \notin R_{i,j}\right\} + \tilde{\varphi}_{ij}^{+}(t_{ij}^{+}) \cdot 1\left\{t_i^{+} \in R_{i,j}\right\}\right)\right]$$

Let $\pi_{ij}^{M}(t_i) = \pi_{ij}^{OPT^{\epsilon}}(t_i^{+})$ and observe that $\pi^{M}$ is a feasible interim allocation as $OPT^{\epsilon}$ is a feasible mechanism and sampling $t_i$ from the continuous distribution and then rounding is identical to sampling from the discrete distribution. We rewrite

$$\mathrm{VW}^{\epsilon} = \sum_{i,j} \underbrace{\mathbb{E}_{t_i \sim D_i}\left[\pi_{ij}^{M}(t_i)\, t_{ij}^{+}\, 1\left\{t_i^{+} \notin R_{i,j}\right\}\right]}_{A^{\epsilon}} + \sum_{i,j} \underbrace{\mathbb{E}_{t_i \sim D_i}\left[\pi_{ij}^{M}(t_i)\, \tilde{\varphi}_{ij}^{+}(t_{ij}^{+}) \cdot 1\left\{t_i^{+} \in R_{i,j}\right\}\right]}_{B^{\epsilon}}$$

We denote with $A, B$ the corresponding continuous type terms where all plus signs are removed from the types. Moreover, in the $B$ term, the function $\tilde{\phi}_{ij}$ is replaced by its non-negative version $\tilde{\phi}_{ij}^{*}$.

*Bounding $A^\epsilon$.* We relate $A^\epsilon$ to $A$ as:

$$A^\epsilon - A = \mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,(t^+_{ij}-t_{ij})1\left\{t^+_i \notin R_{i,j}\right\}\right] + \mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,t_{ij}\,\left(1\left\{t^+_i \notin R_{i,j}\right\} - 1\left\{t_i \notin R_{i,j}\right\}\right)\right]$$

We can bound $\mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)(t^+_{ij}-t_{ij})1\left\{t^+_i \notin R_{i,j}\right\}\right] \leq \epsilon^2$ as $\pi^M_{ij}(t_i) \leq 1$ and $t^+_{ij}-t_{ij} \leq \epsilon^2$. Moreover:

$$\left|\mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,t_{ij}\,\left(1\left\{t^+_i \notin R_{i,j}\right\} - 1\left\{t_i \notin R_{i,j}\right\}\right)\right]\right| \leq H\cdot\sum_{k\neq j}\Pr_{t_i\sim D_i}\left[t_i\in R_{i,j}\wedge t^+_i\in R_{i,k}\right]$$

To upper bound this we prove the following lemma, whose proof we defer to Section B.3.

Lemma B.3. *Let $D_i$ be an absolutely continuous distribution supported on a subset of $[0,H]$, with density upper bounded by $P$. Let $t^+_i$ denote the discrete type that corresponds to a rounded up version of each coordinate of $t_i$ to the closest multiple of $\epsilon^2$, i.e.: $t^+_{ij} = \epsilon^2\cdot\lceil t_{ij}/\epsilon^2\rceil$. If $\{R_{i,j}\}$ is a monotone preference partition of the continuous type space, then:*

$$\sum_{k\neq j}\Pr_{t_i\sim D_i}\left[t_i\in R_{i,j}\wedge t^+_i\in R_{i,k}\right] \leq m^2\,H\,(2H+\epsilon^2)\,P\,\epsilon^2 \tag{41}$$

So, in total, we bound $A^\epsilon \leq A+\epsilon^2\,(m^2\,H\,(2\,H+\epsilon^2)\,P+1) = A+o_\epsilon(1)$.

*Bounding $B^\epsilon$.* Similarly, we decompose the term $B^\epsilon$:

$$\begin{aligned}B^\epsilon &= \mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,(\tilde{\varphi}^+_{ij}(t^+_{ij})-\tilde{\varphi}_{ij}(t^+_{ij}))\,1\left\{t^+_i\in R_{i,j}\right\}\right]\\&+\mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,\tilde{\varphi}_{ij}(t^+_{ij})\left(1\left\{t^+_i\in R_{i,j}\right\}-1\left\{t_i\in R_{i,j}\right\}\right)\right]\\&+\mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,\tilde{\varphi}_{ij}(t^+_{ij})\,1\left\{t_i\in R_{i,j}\right\}\right]\end{aligned}$$

where $\tilde{\varphi}_{ij}$ represents Myerson's continuous ironed virtual value for the distribution $D_{ij}$. In [CDW16], they prove that the discrete virtual value converges to the continuous virtual value for increasingly fine discretizations (Observation 9), $\lim_{\epsilon\to 0}\varphi^+_{ij}(v) = \varphi_{ij}(v)$ for all $v$. We can easily extend this argument to ironed virtual values as $\varphi^+_{ij}$ converges to $\varphi_{ij}$ at all points. So,

$$\lim_{\epsilon\to 0}\tilde{\varphi}^+_{ij}(v) = \lim_{\epsilon\to 0}\max_{v\leq t}\varphi^+_{ij}(v) = \max_{v\leq t}\varphi_{ij}(v) = \tilde{\varphi}_{ij}(v)$$

Thus,

$$\left|\mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,(\tilde{\varphi}^+_{ij}(t^+_{ij})-\tilde{\varphi}_{ij}(t^+_{ij}))\,1\left\{t^+_i\in R_{i,j}\right\}\right]\right| = o_\epsilon(1)$$

and from Lemma B.3:

$$\left|\mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,\tilde{\varphi}_{ij}(t^+_{ij})\left(1\left\{t^+_i\in R_{i,j}\right\}-1\left\{t_i\in R_{i,j}\right\}\right)\right]\right| \leq m^2\,H\,(2\,H+\epsilon^2)\,P\,\epsilon^2$$

as $\tilde{\varphi}_{ij}(t^+_{ij}) \leq t^+_{ij}$. Thus, all that remains to show is that we can replace the $\tilde{\varphi}_{ij}(t^+_{ij})$ term in

$$\mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,\tilde{\varphi}_{ij}(t^+_{ij})\,1\left\{t_i\in R_{i,j}\right\}\right]$$

with a $\tilde{\varphi}_{ij}(t_{ij})$. Here, we make use of the relaxation of virtual value to positive virtual value: $\tilde{\varphi}^*_{ij}(t_{ij}) = \max(\tilde{\varphi}_{ij}(t_{ij}),0)$. Clearly, this upper bounds the virtual value. It will give us a weaker result, but still a meaningful bound. We have

$$\begin{aligned}\mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,\tilde{\varphi}_{ij}(t^+_{ij})\,1\left\{t_i\in R_{i,j}\right\}\right] &\leq \mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,\tilde{\varphi}^*_{ij}(t^+_{ij})\,1\left\{t_i\in R_{i,j}\right\}\right]\\&= \mathbb{E}_{t_i\sim D_i}\left[\pi^M_{ij}(t_i)\,(\tilde{\varphi}^*_{ij}(t^+_{ij})-\tilde{\varphi}^*_{ij}(t_{ij}))\,1\left\{t_i\in R_{i,j}\right\}\right]+B\\&\leq \mathbb{E}_{t_i\sim D_i}\left[\tilde{\varphi}^*_{ij}(t^+_{ij})-\tilde{\varphi}^*_{ij}(t_{ij})\right]+B\\&\leq \mathbb{E}_{t_i\sim D_i}\left[\tilde{\varphi}^*_{ij}(t^+_{ij})-\tilde{\varphi}^*_{ij}(t^-_{ij})\right]+B\end{aligned}$$

where $t^-_{ij} = \epsilon^2\cdot\left(\lceil t_{ij}/\epsilon^2\rceil - 1\right)$. This is true since the *ironed* virtual value function is non-decreasing. We can view the discretization as a breaking up of $T_{ij}$ into segments of length $\epsilon^2$ and $\tilde{\varphi}^*_{ij}(t^+_{ij})-\tilde{\varphi}^*_{ij}(t^-_{ij})$

will be the difference in the endpoints of the interval containing $t_{ij}$. Moreover, making use of the fact that $\tilde{\varphi}^*_{ij}$ is a non-decreasing function with range $\subseteq [0, H]$ as $\tilde{\varphi}_{ij}(t_{ij}) \leq t_{ij}$:

$$\mathbb{E}_{t_i \sim D_i}\left[\tilde{\varphi}^*_{ij}(t^+_{ij}) - \tilde{\varphi}^*_{ij}(t^-_{ij})\right] \leq \epsilon + H \cdot \Pr_{t_i \sim D_i}\left[\tilde{\varphi}^*_{ij}(t^+_{ij}) - \tilde{\varphi}^*_{ij}(t^-_{ij}) > \epsilon\right]$$

Moreover, due to the monotonicity of $\tilde{\varphi}^*_{ij}$, there can only be at most $H/\epsilon$ segments of $T_{ij}$ with endpoints differing by at least $\epsilon$. Again making use of the fact that $D_{ij}$ is atomless and there is some finite upper bound $P$ on its density function, we can argue that the probability of $t_{ij} \sim D_{ij}$ belonging to any specific interval is $\leq P\epsilon^2$. Thus,

$$\Pr_{t_i \sim D_i}\left[\tilde{\varphi}^*_{ij}(t^+_{ij}) - \tilde{\varphi}^*_{ij}(t^-_{ij}) > \epsilon\right] \leq (H/\epsilon) \cdot P\epsilon^2 = HP\epsilon$$

and so, $\mathbb{E}_{t_i \sim D_i}\left[\tilde{\varphi}^*_{ij}(t^+_{ij}) - \tilde{\varphi}^*_{ij}(t^-_{ij})\right] \leq (H^2P + 1)\,\epsilon$. Putting all this together, we have $B^\epsilon \leq B + o_\epsilon(1)$.

*Concluding.* Combining the facts that $A^\epsilon \leq A + o_\epsilon(1)$ and $B^\epsilon \leq B + o_\epsilon(1)$, yields:

$$VW^\epsilon \leq \sum_i \mathbb{E}_{t_i \sim D_i}\left[\sum_j \pi^M_{ij}(t_i)\left(t_{ij} \cdot 1\left\{t_i \notin R_{i,j}\right\} + \tilde{\varphi}^*_{ij}(t_{ij}) \cdot 1\left\{t_i \in R_{i,j}\right\}\right)\right] + o_\epsilon(1)$$

giving the desired upper bound as $\epsilon \longrightarrow 0$.

### B.3 Proof of Lemma B.3

In order to have $t_i \in R_{i,j}$ and $t^+_i \in R_{i,k}$, we must have $\mathcal{U}_{i,j}(t_{ij}) \geq \mathcal{U}_{i,k}(t_{ik})$ and $\mathcal{U}_{i,j}(t^+_{ij}) < \mathcal{U}_{i,k}(t^+_{ik})$ in the event $j < k$. Similarly, we must have $\mathcal{U}_{i,j}(t_{ij}) > \mathcal{U}_{i,k}(t_{ik})$ and $\mathcal{U}_{i,j}(t^+_{ij}) \leq \mathcal{U}_{i,k}(t^+_{ik})$ in the event $j > k$. We assume,without loss of generality, that $j < k$ as the argumentation is symmetric in both cases.

We think about the two-dimensional plane $T_{ij} \times T_{ik}$ of possible values $(t_{ij}, t_{ik})$. We can view the discretization $(t^+_{ij}, t^+_{ik})$ as a division of this plane into a grid of squares of side length $\epsilon^2$. Here, $(t^+_{ij}, t^+_{ik})$ represents the upper corner of whichever square $(t_{ij}, t_{ik})$ belongs to. We also consider a partitioning of this plane into the set on points for which $\mathcal{U}_{i,j}(t_{ij}) \geq \mathcal{U}_{i,k}(t_{ik})$ and the set of points for which $\mathcal{U}_{i,j}(t_{ij}) < \mathcal{U}_{i,k}(t_{ik})$. In order to have $\mathcal{U}_{i,j}(t_{ij}) \geq \mathcal{U}_{i,k}(t_{ik})$ and $\mathcal{U}_{i,j}(t^+_{ij}) < \mathcal{U}_{i,k}(t^+_{ik})$, we must have the border of this partition pass through the square containing $(t_{ij}, t_{ik})$. However, we show that only a small number of squares will contain a piece of this border, enabling us to bound the probability of such an event as $\epsilon \longrightarrow 0$.

*Proof intuition.* The border of any monotone preference partition, when projected on the two dimensional plane $T_{ij} \times T_{ik}$ of the types $(t_{ij}, t_{ik})$ for two items, must be a curve that corresponds to a monotone non-decreasing function of $t_{ij}$. Thus any two squares that are in the $x + y = u$ diagonal (for some $u$), cannot contain points from both partitions as that would imply that there is a point of the border in both squares, which would subsequently imply that these two points violate the monotonicity of the border. Since there are at most $O(H/\epsilon^2)$ diagonals, there can be at most $O(H/\epsilon^2)$ squares that can be problematic, each with density at most $P\,\epsilon^4$. In total a probability mass of types of at most $O(H\,\epsilon^2) \rightarrow 0$, can be problematic (see Figure 2).

*Formal argument.* We can index the grid of squares as an ordered pair $(x, y)$ where $x$ and $y$ are integers in the range $[1, \lceil H/\epsilon^2 \rceil]$. Square $(x, y)$ contains the points $((x-1)\epsilon^2, x\epsilon^2] \times ((y-1)\epsilon^2, y\epsilon^2]$. In the edge cases, index 1 corresponds to $[0, \epsilon^2]$ inclusive and index $\lceil H/\epsilon^2 \rceil$ corresponds to $[\epsilon(\lceil H/\epsilon^2 \rceil - 1), H]$.

We claim that, for any two squares $(x_1, y_1), (x_2, y_2)$ containing points from both sides of the partition, we must have $x_1 + y_1 \neq x_2 + y_2$. Assume for the sake of contradiction that $x_1 + y_1 = x_2 + y_2$ and

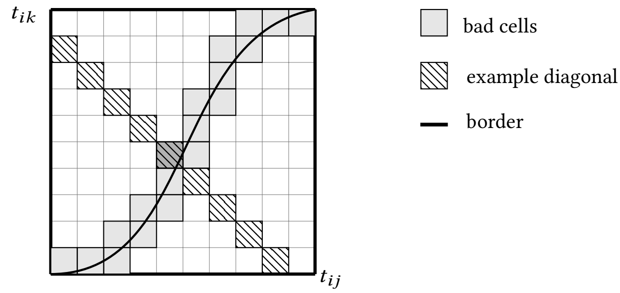


Fig. 2. Pictorial representation of proof arguments.

WLOG $x_1 < x_2, y_1 > y_2$. Say we had $(t_{ij,1}, t_{ik,1})$ in square $(x_1, y_1)$ with $\mathcal{U}_{i,j}(t_{ij,1}) \geq \mathcal{U}_{i,k}(t_{ik,1})$ and $(t_{ij,2}, t_{ik,2})$ in square $(x_2, y_2)$ with $\mathcal{U}_{i,j}(t_{ij,2}) < \mathcal{U}_{i,k}(t_{ik,2})$. We cannot have $\mathcal{U}_{i,k}(t_{ik,1}) < \mathcal{U}_{i,k}(t_{ik,2})$. We must have $t_{ik,1} > t_{ik,2}$ since $y_1 > y_2$ and $\mathcal{U}_{i,k}$ is non-decreasing. However, we cannot have $\mathcal{U}_{i,k}(t_{ik,1}) \geq \mathcal{U}_{i,k}(t_{ik,2})$ as that would imply $\mathcal{U}_{i,j}(t_{ij,1}) > \mathcal{U}_{i,j}(t_{ij,2})$. We must have $t_{ij,1} < t_{ij,2}$ since $x_1 < x_2$ and $\mathcal{U}_{i,j}$ is non-decreasing, a contradiction.

Thus, the partition border can only pass through one square along the diagonal of squares $(x, y)$ satisfying $x + y = u$. Since $x, y$ are integers in the range $[1, \lceil H/\epsilon^2 \rceil]$, we have $x + y \in [2, 2\lceil H/\epsilon^2 \rceil]$. Therefore, the partition border passes through at most $2H/\epsilon^2 + 1$ squares.

Then, since $D_i$ is a bounded distribution and is absolutely continuous with respect to the Lebesgue measure, the probability density function of $D_{ij}$ is bounded for every $j$. Thus, there is some finite $P$ that upper bounds the PDF of the joint distribution $D_{ij} \times D_{ik}$ for every pair $j, k$. So, the probability of $(t_{ij}, t_{ik})$ belonging to any specific square is at most $P(\epsilon^2)^2$. Therefore, the probability that $(t_{ij}, t_{ik})$ belongs to a square containing a piece of the partition border is $\leq 2HP\epsilon^2 + P\epsilon^4$. So,

$$\sum_{k \neq j} \Pr_{t_i \sim D_i} \left[ t_i \in R_{i,j} \wedge t_i^+ \in R_{i,k} \right] \leq \sum_{k \neq j} (2HP\epsilon^2 + P\epsilon^4) \leq m^2\, H\, (2\, H + \epsilon^2)\, P\, \epsilon^2$$

## C PROOF OF LEMMA 4.1

Proof of Lemma 4.1. Consider a mixed equilibrium strategy $\mathbf{b} = \{\mathbf{b}_i\}_{i \in [n]}$ of the rand $-$ $EA$ auction. Let $\mathbf{b}_i^j : T_i \to \Delta(A_{ij})$, [9] denote the mapping from a type $t_i \in T_i$ to the marginal distribution of actions of bidder $i$ for item $j$, conditional on type $t_i$, under mixed the mixed equilibrium $\mathbf{b}$. More concretely, the (probability) density function of distribution $\mathbf{b}_i^j(t_i)$ is given by

$$p_{\mathbf{b}_i^j(t_i)}(b) = \mathbb{E}_{(z_i, a_i) \sim \mathbf{b}_i(t_i)} \left[ 1 \left\{ a_i^j = b \right\} \right]$$

Moreover, let $\tilde{\mathbf{b}}_i^j : T_{ij} \to \Delta(A_{ij})$ denote the marginal distribution of actions on the auction for item $j$ conditional only on his type $t_{ij}$ for item $j$ and marginalizing his types for other items. More concretely, the (probability) density function of distribution $\tilde{\mathbf{b}}_i^j(t_{ij})$ is given by

$$p_{\tilde{\mathbf{b}}_i^j(t_{ij})}(b) = \mathbb{E}_{t_{i,-j} \sim D_{i,-j}} \left[ \mathbb{E}_{(z_i, a_i) \sim \mathbf{b}_i(t_i)} \left[ 1 \left\{ a_i^j = b \right\} \right] \right]$$

[9]Where $\Delta(A_{ij})$ denotes the set of distributions over actions submitted to the auction for item $j$.

The interim utility of bidder $i$ with type $t_i$, in the rand $-$ $EA$ auction, is given by

$$u_i^{\mathbf{b}}(t_i) = \sum_{j=1}^{m} \mathbb{E}_{t_{-i}\sim D_{-i}}\left[\mathbb{E}_{(z,a)\sim\mathbf{b}(t)}\left[(z_i\,(1-\delta)+\delta)\cdot u_{ij}^*(t_i;a)\right]\right] - \mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[z_i\,(1-\delta)\,e_i\right],$$

where $u_{ij}^*(t_i;a)$ denotes the ex-post utility of bidder $i$ in the auction for item $j$ under bid profile $a$. Since the ex-post utility $u_{ij}^*$ depends only on $t_{ij}$ and the bid profile $a^j$ for item $j$, we can re-write the above expression for interim utility as

$$u_i^{\mathbf{b}}(t_i) = \sum_{j=1}^{m} \mathbb{E}_{t_{-i}\sim D_{-i}}\left[\mathbb{E}_{(z,a)\sim\mathbf{b}(t)}\left[(z_i\,(1-\delta)+\delta)\cdot u_{ij}^*(t_{ij};a^j)\right]\right] - \mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[z_i\,(1-\delta)\,e_i\right]$$

For simplicity let $G_{-i}^j \in \Delta(A_{-i,j})$ denote the distribution of other player actions at the auction of item $j$ under the mixed BNE $\mathbf{b}$ of the rand $-$ $EA$ auction. Moreover, observe that this is the same distribution as first drawing a random type $t_{i',j}$ of each opponent $i'$ for item $j$ and then drawing an action for that player from the marginal distribution $\tilde{b}_{i'}^j(t_{i',j})$. Then:

$$u_i^{\mathbf{b}}(t_i) = \sum_{j=1}^{m} \mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[(z_i\,(1-\delta)+\delta)\cdot \mathbb{E}_{a_{-i}^j\sim G_{-i}^j}\left[u_{ij}^*(t_{ij};a^j)\right]\right] - \mathbb{E}_{(z,a)\sim\mathbf{b}_i(t_i)}\left[z_i\,(1-\delta)\,e_i\right]$$

Let $U_{ij}(t_{ij};a_i^j) = \mathbb{E}_{a_{-i}^j\sim G_{-i}^j}\left[u_{ij}^*(t_{ij};a^j)\right]$, then:

$$u_i^{\mathbf{b}}(t_i) = \sum_{j=1}^{m} \underbrace{\mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[(z_i\,(1-\delta)+\delta)\cdot U_{ij}(t_{ij};a_i^j)\right]}_{A_{ij}(t_i)} - \mathbb{E}_{(z,a)\sim\mathbf{b}_i(t_i)}\left[z_i\,(1-\delta)\,e_i\right]$$

Let $u_{ij}^{\mathbf{b}}(t_i) = \max_{a_i^j\in A_{ij}} U_{ij}(t_{ij};a_i^j)$. Now suppose that the distribution $b_i(t_i)$ submits with probability $\rho > 0$ actions $a_i^j$ that achieve utility $U_{ij}(t_{ij};a_i^j) \le u_{ij}^{\mathbf{b}}(t_i) - \epsilon$ for $\epsilon > 0$. Then observe that the player can deviate and strictly increase their utility by submitting action $\arg\max_{a_i^j\in A_{ij}} U_{ij}(t_{ij};a_i^j)$, whenever they would have submitted any such sub-optimal action $\tilde{a}_i^j$. This is a strictly improving deviation since, it leads to an improvement of at least $\delta\,\epsilon\,\rho$. Thus we have that, when $a_i$ is drawn from distribution $b_i(t_i)$, then with probability 1: $U_{ij}(t_{ij};a_i^j) = u_{ij}^{\mathbf{b}}(t_i)$. We can then re-write $A_{ij}(t_i)$:

$$\begin{aligned}
A_{ij}(t_i) &= \mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[(z_i\,(1-\delta)+\delta)\cdot u_{ij}^{\mathbf{b}}(t_i)\right]\\
&= \left(\mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[z_i\right]\,(1-\delta)+\delta\right)\cdot u_{ij}^{\mathbf{b}}(t_i)\\
&= \left(\mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[z_i\right]\,(1-\delta)+\delta\right)\cdot \mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[U_{ij}(t_{ij};a_i^j)\right]
\end{aligned}$$

Now observe, that since $U_{ij}(t_{ij};a_i^j)$ is independent of $t_{i,-j}$, we then have that:

$$\mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[U_{ij}(t_{ij};a_i^j)\right] = \mathbb{E}_{a_i^j\sim\tilde{\mathbf{b}}_i^j(t_{ij})}\left[U_{ij}(t_{ij};a_i^j)\right] \tag{42}$$

Now observe, that the latter is the interim utility of player $i$ in a single item auction for item $j$, where players use bid strategies $\tilde{\mathbf{b}}^j = \left\{\tilde{\mathbf{b}}_i^j\right\}_{i\in[n]}$, denoted as $u_{ij}^{\tilde{\mathbf{b}}^j}(t_{ij})$. Thus we can write a player's interim utility in the rand $-$ $EA$ auction in terms of the latter interim utility as:

$$u_i^{\mathbf{b}}(t_i) = \sum_{j=1}^{m} \left(\mathbb{E}_{(z_i,a_i)\sim\mathbf{b}_i(t_i)}\left[z_i\right]\,(1-\delta)+\delta\right)\cdot u_{ij}^{\tilde{\mathbf{b}}^j}(t_{ij}) - \mathbb{E}_{(z,a)\sim\mathbf{b}_i(t_i)}\left[z_i\,(1-\delta)\,e_i\right]$$

If we denote with $q_i^{\mathbf{b}_i}(t_i) = \mathbb{E}_{(z_i, a_i)\sim \mathbf{b}_i(t_i)}[z_i]$, the marginal probability of entry with type $t_i$ under the mixed BNE $\mathbf{b}$, then:

$$u_i^{\mathbf{b}}(t_i) = \sum_{j=1}^{m} \left(q_i^{\mathbf{b}_i}(t_i)\,(1-\delta) + \delta\right) u_{ij}^{\tilde{\mathbf{b}}^j}(t_{ij}) - q_i^{\mathbf{b}_i}(t_i)\,(1-\delta)\, e_i$$

$$= q_i^{\mathbf{b}_i}(t_i)\,(1-\delta)\left(\sum_{j=1}^{m} u_{ij}^{\tilde{\mathbf{b}}^j}(t_{ij}) - e_i\right) + \delta \sum_{j=1}^{m} u_{ij}^{\tilde{\mathbf{b}}^j}(t_{ij})$$

Now we argue that the marginal distribution mappings $\tilde{\mathbf{b}}^j = \left\{\tilde{\mathbf{b}}_i^j\right\}_{i\in[n]}$, must constitute a mixed BNE of the single item auction $A$ for item $j$, if run in isolation. Suppose that this was not the case. This means that there is some player $i$ that has a profitable deviating strategy, i.e. that has some action $\tilde{a}_i^j$, such that for some $\epsilon > 0$:

$$u_{ij}^{\tilde{\mathbf{b}}^j}(t_{ij}) \le U_{ij}(t_{ij}; \tilde{a}_i^j) - \epsilon \tag{43}$$

However, in that case there is a profitable deviation of player $i$ in the rand $-$ $EA$ auction, since if player $i$ was always submitting action $\tilde{a}_i^j$ on item $j$, instead of his prior bid, he could increase his interim utility by at least $\delta\,\epsilon$.

Finally, observe that a player enters the rand $-$ $EA$ auction deterministically whenever:

$$\sum_{j=1}^{m} u_{ij}^{\tilde{\mathbf{b}}^j}(t_{ij}) - e_i > 0$$

otherwise there is a profitable deviation. Thus the probability of entry is at least:

$$\Pr_{t_i\sim D_i}\left[\sum_{j=1}^{m} u_{ij}^{\tilde{\mathbf{b}}^j}(t_{ij}) - e_i > 0\right]$$

Observe that this is equal to the entry probability in the EA auction with $\left\{\tilde{\mathbf{b}}^j\right\}$-simulating ghost bidders and entry fees $\{e_i\}_{i\in[n]}$ at the focal equilibrium $\tilde{\mathbf{b}}$.

Thus the entry fee revenue collected by the rand $-$ $EA$ auction at *any mixed BNE* equilibrium $\mathbf{b}$ is at least:

$$\text{EF-Rev}^{\mathbf{b}}(\text{rand} - \text{EA}(e)) \ge (1-\delta)\cdot \text{EF-Rev}^{\tilde{\mathbf{b}}}\left(\text{EA}\left(e, D^g(\{\tilde{\mathbf{b}}^j\})\right)\right) \tag{44}$$

□