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Stability of Nonlinear Switched Systems on Non-uniform Time Domains with Application to Multi-Agents Consensus

Fatima. Z. Taousser[1], Seddik. M. Djouadi[1], Kevin Tomsovic[1] and Mohamed Djemai[2]

***Abstract*— A recent development in Lyapunov stability theory allows for analysis of switched systems evolving on non-uniform time domains, called *Time Scales*. We will present new sufficient conditions to guarantee the stability of a special class of switched systems, between continuous-time subsystems (on intervals with variable lengths) and discrete-time subsystems (with variable discrete-step sizes). By introducing the time scales theory, the conditions are derived using the concept of Time Scale Multiple Lyapunov Functions (TSMLF). The results are applied in the problem of consensus for multi-agent systems with intermittent information transmission.**

keywords: Time scales theory; switched systems; stability analysis; multiple Lyapunov functions.

I. Introduction

Stability of switched systems has been a topic of increasing discussion over the past decade, and several methods have been developed, which can be applied to systems evolving on either continuous or discrete uniform time domains ($\mathbb{R}$ or $h\mathbb{Z}$, respectively) [3], [4], [6]. Recently, there is an increasing interest in non-uniform time domains, called *Time Scales*, which can contain non-uniformly spaced discrete points or a mixture of discrete parts with variable step sizes and continuous parts with variable lengths. There are many applications of such systems including real-time communication network [13], population model [14] and economics [15].

Our interest is a special class of switched systems on a non-uniform time domain, where the dynamical system switches between a continuous-time subsystem and discrete-time subsystems during a certain time. This class can describe a wide range of physical and engineering systems. For instance, cooperative control over networks [22], [23], such that the derived controller assumes that local information is exchanged over some disconnected time intervals due to communication obstacles or sensor failure [17]. In this case, the time domain is neither continuous nor uniformly discrete. Time scale theory was found promising to study this class of switched systems because it captures the interplay between the theories of continuous and discrete dynamics.

One of the most widely used tools for investigating the stability of switched systems is the Lyapunov stability theory. So far, a common Lyapunov function (CLF) method was adopted for all subsystems and the existence of such a function guarantees the stability of the system under arbitrary switching. In this case, it is necessary to require that all subsystems are asymptotically stable [1]. Note that, a CLF, in some situations, is not known or does not exist, and finding a CLF is not an easy task, except for certain special cases [5], [8]. To seek less conservative results, the Multiple Lyapunov Functions (MLF) (or Lyapunov-like functions) approach was introduced to analyze the stability of switched systems under constrained switching [1], [2], [7]. It was shown that switched systems, without exception, are prone to instability problems, and an arbitrary fast switching may cause large state transients at the switching points.

The objective of this paper is to introduce time scales theory to extend the MLF approach, to the special class of switched systems between continuous and discrete dynamics, by considering nonlinear uncertainties. This class of switched systems has been studied in [16] by considering the commutativity of the matrices of the subsystems so that a CLF has been computed. In [18], this problem has been studied using the solution of the switched system. In [19], an MLF approach has been introduced to study such class of linear switched systems (without uncertainties) by considering that unstable modes may exist. In this paper, we will remove the condition of commutativity of the matrices of the subsystems, we will consider the nonlinear uncertainties, and the problem has been generalized to several discrete-time subsystems. The result has been applied to the problem of consensus for multi-agent systems with intermittent information transmission.

II. Preliminaries

Time scale theory is introduced, in this paper, to analyze the stability of a special class of switched systems between a nonlinear continuous-time subsystem and a set of nonlinear discrete-time subsystems during a certain period of time.

We briefly outline the portions of the time scales theory that are needed for this paper (for more details see [9]). A time scale $\mathbb{T}$ is an arbitrary closed subset of $\mathbb{R}$. The *forward jump* operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t, t \in \mathbb{T}\}$. The *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$, is defined by $\mu(t) = \sigma(t) - t$, and it measures the distance between consecutive time instances. For $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, and $\mu(t) = 0$, while for $\mathbb{T} = h\mathbb{Z}$, we have $\sigma(t) = t + h$, and $\mu(t) = h$.

Let $f : \mathbb{T} \to \mathbb{R}$. The Δ-derivative of f is defined by, $f^{\Delta}(t) = \lim_{s\to t} \frac{f(\sigma(t))-f(s)}{\sigma(t)-s}$, which unifies the continuous derivative and the difference operator. We say that f is Δ-differentiable on $\mathbb{T}$, if its Δ-derivative exists $\forall t \in \mathbb{T}$. A

[1]Fatima Z. Taousser, Seddik M. Djouadi and Kevin Tomsovic are with Department of Electrical Engineering and Computer Science, University of Tennessee, Knoxville, TN 37996, USA {ftaousse, mdjouadi, tomsovic}@utk.edu

[2]Mohamed Djemai is with Polytechnic University of Hauts-de-France, LAMIH, 59313 Valenciennes, France mohamed.djemai@uphf.fr

function $p : \mathbb{T} \to \mathbb{R}$ is said *regressive* (resp. *positively regressive*), if $1 + \mu(t)p(t) \neq 0$ (resp. $1 + \mu(t)p(t) > 0$), $\forall t \in \mathbb{T}$. We denote the set of regressive (resp. positively regressive) functions by $\mathcal{R}$ (resp. $\mathcal{R}^+$). A matrix function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is *regressive*, if and only if all its eigenvalues are regressive and we denote $A \in \mathcal{R}(\mathbb{R}^{n\times n})$. The *generalized exponential function* of $p \in \mathcal{R}$, on $\mathbb{T}$, is expressed by $e_p(t, s) = \exp\left(\int_s^t \frac{\log(1+\mu(\tau)p(\tau))}{\mu(\tau)}\Delta\tau\right)$, $s, t \in \mathbb{T}$. For $\mathbb{T} = \mathbb{R}$ and p constant, $e_p(t, t_0) = e^{p(t-t_0)}$. For $\mathbb{T} = h\mathbb{Z}$, $e_p(t, s) = \prod_{\tau=s}^{t-h}(1 + hp(\tau))$, and $e_p(\sigma(t), t) = (1 + \mu(t)p(t))$. Let $A \in \mathbb{R}^{n\times n}$, we define $x(t) = e_A(t, t_0)x_0$ as the unique solution of the dynamical system

$$x^{\Delta}(t) = A(t)x(t), \quad x(t_0) = x_0, \quad t, t_0 \in \mathbb{T}. \tag{1}$$

Notice that, $x(t)$ is well defined if $A \in \mathcal{R}(\mathbb{R}^{n\times n})$ (see [9]).

The definitions of stability and asymptotic stability of dynamical systems on time scales, are similar to the standard stability concepts. The exponential stability is achieved by some modifications, such that, the system (1) is exponentially stable, if $\exists \beta, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$, such that $\|x(t)\| \leq \beta e_{-\lambda}(t, t_0)\|x(t_0)\|, \forall t \geq t_0$. Specifically, the condition that $-\lambda \in \mathcal{R}^+$ reduce to $\lambda > 0$, for $\mathbb{T} = \mathbb{R}$, and to $0 < \lambda < 1$, for $\mathbb{T} = \mathbb{Z}$. Since, it is difficult to determine the region of exponential stability of (1) on an arbitrary $\mathbb{T}$ [10], that is why the Hilger circle is introduced

$$\mathcal{H}_{\mu(t)} := \left\{ z \in \mathbb{C} : \ |1 + z\mu(t)| < 1, \ z \neq -\frac{1}{\mu(t)} \right\}.$$

For $0 \leq \mu(t) \leq \mu_{\max}$ and $A(t) \equiv A$, there exists a Hilger circle $\mathcal{H}_{\min}$ associated to $\mu_{\max} = \sup_{t\in\mathbb{T}} \mu(t)$, such that if all the eigenvalues of A are in $\mathcal{H}_{\min}$, then system (1) is exponentially stable (see [20]). For the general case, and to extend Lyapunov's Second Method to dynamic systems on time scales, we define a time scale Lyapunov function (TSLF) [11] A function $V : \mathbb{R}^n \to \mathbb{R}$ is called a time scale (or a generalized) Lyapunov function (TSLF) for the system (1), if it satisfies, $\forall x : \mathbb{T} \to \mathbb{R}^n$,

(i) $V(x(t)) \geq 0$, with equality if and only if $x(t) = 0$,
(ii) V is Δ-differentiable and $V^{\Delta}(x(t)) \leq 0$.

Theorem II.1. *[21]*
Consider the system (1). If there exists an associated TSLF $V(x(t))$, then the equilibrium $x = 0$ is stable. Furthermore, if $V^{\Delta}(x(t)) < 0$, then $x = 0$ is asymptotically stable.

The Lyapunov stability on time scales was studied in several works [12], [16]. To establish asymptotic stability of system (1), a standard approach is to seek an associated quadratic Lyapunov function $V(x(t)) = x^T(t)Px(t)$, with $P = P^T > 0$ is a positive definite matrix. The Δ-derivative of V along the trajectories of system (1) on an arbitrary time scale $\mathbb{T}$, is given by $V^{\Delta}(x(t)) = x^T(t)(A^TP + PA + \mu(t)A^TPA)x(t)$ [11]. Therefore, we seek a solution P to the following time scale algebraic Lyapunov equation (TSALE)

$$A^TP + PA + \mu(t)A^TPA = -Q(t), \ Q(t) = Q^T(t) > 0. \tag{2}$$

This equation is a generalization of the Lyapunov criteria for stability of discrete-time and continuous-time linear systems, such that, if $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$, and (2) reduces to the standard algebraic equation for continuous-time dynamic. If $\mathbb{T} = \mathbb{Z}$, $\mu(t) = 1$, and (2) reduces to

$$A^TP + PA + A^TPA = (I + A)^TP(I + A) - P = -Q(t), \tag{3}$$

which coincides with the algebraic Lyapunov equation of the standard recursive equation $x(t + 1) = (I + A)x(t)$. Note that, the Δ-derivative of $x(t)$ on $\mathbb{T} = \mathbb{Z}$, is given by the difference equation $x^{\Delta}(t) = \Delta x(t) = x(t + 1) - x(t) = Ax(t)$. Thus changing from difference form to recursive form just requires a unit shift on the matrix A. If $A(t) \equiv A$ has all its eigenvalues in the Hilger circle $\mathcal{H}_{\min}$, then $\forall t \in \mathbb{T}$, the matrix P is determined by (see [12])

$$P = \int_{t_0}^{t} e_{A^T}(s, t_0)Q(t)e_A(s, t_0)\Delta s, \ t \geq t_0.$$

Notice that (2) unifies the TSALE on an arbitrary $\mathbb{T}$ and it is generally time varying because of the time varying $\mu(t)$. If the TSALE (2) is satisfied for $\mu_{\max}$, then it is satisfied for all $\mu(t)$, since $A^TP + PA + \mu(t)A^TPA \leq A^TP + PA + \mu_{\max}A^TPA = -Q$, $\forall t \in \mathbb{T}$, which can simplify the computation of P (see [16]).

III. Problem statement

In this paper, we will consider the particular time scale $\mathbb{T} = \mathbb{P}_{\{\sigma_1(t_k),\sigma_2(t_k),t_{k+1}\}} = \cup_{k=0}^{\infty}\{[\sigma_2(t_k), t_{k+1}] \cup \sigma_1(t_{k+1})\}$, as shown in Fig 1, without accumulation points (i.e; There is no Zeno behavior). The forward jump operator and the corresponding graininess functions are defined, $\forall k \in \mathbb{N}$, by:

- For $t \in \cup_{k=0}^{\infty}[\sigma_2(t_k), t_{k+1}[$; $\sigma(t) = t$, $\mu(t) = 0$;
- For $t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$; $\sigma(t) = \sigma_1(t_{k+1})$,

$$\mu_1(t) = \sigma(t) - t = \sigma_1(t_{k+1}) - t_{k+1};$$

- For $t \in \cup_{k=0}^{\infty}\{\sigma_1(t_{k+1})\}$; $\sigma(t) = \sigma_2(t_{k+1})$;

$$\mu_2(t) = \sigma(t) - t = \sigma_2(t_{k+1}) - \sigma_1(t_{k+1}).$$

In the following, we suppose that $\mathbb{T}$ is unbounded above, and the graininess functions are bounded (i.e, $0 < \mu_1 \leq \mu_{1\max}$ and $0 < \mu_2 \leq \mu_{2\max}$).

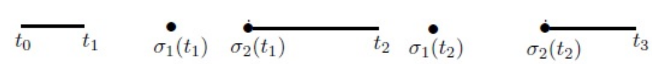

Fig. 1: Time scale $\mathbb{T} = \mathbb{P}_{\{\sigma_1(t_k),\sigma_2(t_k),t_{k+1}\}}$

Let $\{A_c, A_1, A_2\}$ be a set of a constant regressive matrices in $\mathbb{R}^{n\times n}$. The considered nonlinear switched system on $\mathbb{T} = \mathbb{P}_{\{\sigma_1(t_k),\sigma_2(t_k),t_{k+1}\}}$ is given by, for $x(t) \in \mathbb{R}^n$

$$x^{\Delta}(t) = \begin{cases} A_cx(t) + f_c(x(t)); & t \in \cup_{k=0}^{\infty}[\sigma_2(t_k), t_{k+1}[\\ A_1x(t) + f_1(x(t)); & t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \\ A_2x(t) + f_2(x(t)); & t \in \cup_{k=0}^{\infty}\{\sigma_1(t_{k+1})\}, \end{cases} \tag{4}$$

where $t_0 = \sigma_2(t_0) = 0$ is the initial time. The first equation of (4) describes the continuous-time dynamic. The second and third equations, describe the discrete-time dynamics (i.e, the states jumps) during certain periods $\mu_1(t)$ and $\mu_2(t)$, which are considered to be variable in time. Uncertainties are modeled by functions $f_c : \mathbb{R}^n \to \mathbb{R}^n$, which act on the continuous-time subsystem, and $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in \{1, 2\}$, which act on the discrete-time subsystems. Suppose that the conditions of existence and uniqueness of the solution of each subsystem in (4) are satisfied (i.e, f_i, $i \in \{c, 1, 2\}$, are rd-continuous, bounded and Lipschitz continuous [9]).

IV. Stability analysis

Stability of switched system (4), has been studied in [16], by considering one continuous-time subsystem, and one discrete-time subsystem described by matrix A_d. The matrices A_c and A_d were supposed to be commutative, such that, a CLF exists and it has been computed. In this paper, the commutativity condition of matrices of (4) is removed. In this case, the CLF may not be known, or does not exist. Therefore, one can investigate the stability of switched system (4) using TSML. The following theorem, provides a generalization of the standard MLF method [1] to the nonlinear switched systems (4) on $\mathbb{T}$.

Theorem IV.1. *Consider the switched system (4), such that all the subsystems are asymptotically stable. Let the positive definite radially unbounded functions $V_i : \mathbb{R}^n \to \mathbb{R}^+, i \in \{c, 1, 2\}$, be the corresponding time scale Lyapunov function (TSLF). Suppose that there exists a family of a positive definite continuous functions $W_i : \mathbb{R}^n \to \mathbb{R}^+, i \in \{c, 1, 2\}$, such that, $\forall k \in \mathbb{N}$,*

$$V_c(x(\sigma_2(t_{k+1}))) - V_c(x(\sigma_2(t_k))) \leq -W_c(x(\sigma_2(t_k))) \quad (5)$$

$$V_1(x(t_{k+2})) - V_1(x(t_{k+1})) \leq -W_1(x(t_{k+1})) \quad (6)$$

$$V_2(x(\sigma_1(t_{k+2}))) - V_2(x(\sigma_1(t_{k+1}))) \leq -W_2(x(\sigma_1(t_{k+1}))). \quad (7)$$

Then, the system (4) is asymptotically stable.

Inequalities (5), (6) and (7), means that, $\forall i \in \{c, 1, 2\}$, the value of V_i at every entering time of each subsystem, form a decreasing sequence.

Proof. The proof is based on Lyapunov functions theory, and is similar to the continuous case (see [1]). □

Note that, the TSMLF conditions in Theorem IV.1 can be satisfied, when the switching signal is constrained such that, the desired relationships between the values of the Lyapunov functions at switching times is ensured.

A. Main results

In this section, constraints on the switching signal are derived, to ensure the desired relationships between the values of the TSMLF at switching times, described in Theorem IV.1.

First, the stability of the linear switched system, without uncertainties (i.e; $f_i(x(t)) = 0,\ i \in \{c, 1, 2\}$) is discussed. Let us consider the switched system

$$x^{\Delta}(t) = \begin{cases} A_c x(t); & t \in \cup_{k=0}^{\infty}[\sigma_2(t_k), t_{k+1}[\\ A_1 x(t); & t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \\ A_2 x(t); & t \in \cup_{k=0}^{\infty}\{\sigma_1(t_{k+1})\}. \end{cases} \quad (8)$$

Theorem IV.2. *Consider the switched system (8), and suppose that the following conditions are satisfied,*

(i) *The matrices A_i, $i \in \{c, 1, 2\}$ are exponentially stable.*

(ii) *There exist functions $V_i : \mathbb{R}^n \to \mathbb{R}^+$, defined by*

$$V_i(x) = x^T P_i x,\ P_i = P_i^T > 0,\ i \in \{c, 1, 2\}, \quad (9)$$

such that, for a positive constants $\alpha_i, \beta_i, \gamma_i$,

$$\beta_i \|x\|^2 \leq V_i(x) \leq \alpha_i \|x\|^2, \quad (10)$$

$$V_i^{\Delta}(x) \leq -\gamma_i \|x\|^2, \quad (11)$$

$$\frac{-\gamma_i}{\alpha_i} \in \mathcal{R}^+, \quad i \in \{1, 2\}. \quad (12)$$

(iii) *The duration of each continuous-time subsystem satisfies*

$$t_{k+1} - \sigma_2(t_k) \geq \frac{\log\left[\prod_{i=1}^{2} \frac{\alpha_c \alpha_i}{\beta_c \beta_i}\left(1 + \mu_i(t_{k+1})\frac{-\gamma_i}{\alpha_i}\right)\right]}{\frac{\gamma_c}{\alpha_c}}, \quad (13)$$

for all $k \in \mathbb{N}$. Then system (8) is exponentially stable.

Proof.
Let the TSMLF $V_i : \mathbb{R}^n \to \mathbb{R}^+$, $i \in \{c, 1, 2\}$, defined by (9). Condition (10) is satisfied for $\alpha_i = \lambda_{\max}(P_i)$ and $\beta_i = \lambda_{\min}(P_i)$, the lowest and the largest eigenvalues of $P_i, i \in \{c, 1, 2\}$, respectively. According to (i), there exists a positive definite matrices Q_c, Q_1 and Q_2, such that, the Δ-derivative of V_i along the trajectories of (8), is given by (see [16])

$$V_i^{\Delta}(x(t)) = \begin{cases} x^T(A_c^T P_c + P_c A_c)x \leq -x^T Q_c x^T, \\ \quad t \in \cup_{k=0}^{\infty}[\sigma_2(t_k), t_{k+1}[, \\ x^T(A_1^T P_1 + P_1 A_1 + \mu_1(t) A_1^T P_1 A_1)x \\ \leq -x^T Q_1 x^T, \quad t \in \cup_{k=0}^{\infty}\{t_{k+1}\}, \\ x^T(A_2^T P_2 + P_2 A_2 + \mu_2(t) A_2^T P_2 A_2)x \\ \leq -x^T Q_2 x^T, \quad t \in \cup_{k=0}^{\infty}\{\sigma_1(t_{k+1})\}. \end{cases} \quad (14)$$

The Δ-derivative of $V_i(x)$ satisfies condition (11) for $\gamma_i = \lambda_{\min}(Q_i) > 0,\ i \in \{c, 1, 2\}$, the lowest eigenvalue of Q_i. From (10) and (11), we have

$$V_i^{\Delta}(x(t)) \leq \frac{-\gamma_i}{\alpha_i} V_i(x(t)), \quad \forall i \in \{c, 1, 2\}. \quad (15)$$

Using differential inequalities on time scales [9], we obtain, for $\sigma_2(t_k) \leq t \leq t_{k+1},\ \forall k \in \mathbb{N}$,

$$V_c(x(t)) \leq e^{\frac{-\gamma_c}{\alpha_c}(t - \sigma_2(t_k))}\ V_c(x(\sigma_2(t_k))). \quad (16)$$

For $t = t_{k+1},\ \forall k \in \mathbb{N}$,

$$V_1(x(\sigma_1(t_{k+1}))) \leq e_{\frac{-\gamma_1}{\alpha_1}}(\sigma_1(t_{k+1}), t_{k+1}) V_1(x(t_{k+1}))$$

$$= (1 + \mu_1(t_{k+1})(-\gamma_1/\alpha_1))V_1(x(t_{k+1})), \quad (17)$$

and for $t = \sigma_1(t_{k+1}),\ \forall k \in \mathbb{N}$,

$$V_2(x(\sigma_2(t_{k+1}))) \leq e_{\frac{-\gamma_2}{\alpha_2}}(\sigma_2(t_{k+1}), \sigma_1(t_{k+1}))V_2(x(\sigma_1(t_{k+1})))$$

$$= \left(1 + \mu_2(t_{k+1})\frac{-\gamma_2}{\alpha_2}\right)V_2(x(\sigma_1(t_{k+1}))). \quad (18)$$

From (10), (16), (17) and (18), we can derive the following relationship, for $k \in \mathbb{N}$,

$$\begin{aligned} &V_c(x(\sigma_2(t_{k+1}))) \\ &\leq \textstyle\prod_{i=1}^{2}\left(\frac{\alpha_c\alpha_i}{\beta_c\beta_i}\right)\left(1 + \mu_i(t_{k+1})\frac{-\gamma_i}{\alpha_i}\right)V_c(x(t_{k+1})) \\ &\leq \textstyle\prod_{i=1}^{2}\left(\frac{\alpha_c\alpha_i}{\beta_c\beta_i}\right)\left(1 + \mu_i(t_{k+1})\frac{-\gamma_i}{\alpha_i}\right) \\ &\times e^{\frac{-\gamma_c}{\alpha_c}(t_{k+1}-\sigma_2(t_k))}V_c(x(\sigma_2(t_k))). \end{aligned} \quad (19)$$

It is now straightforward to compute an explicit time condition, which guarantees that conditions of Theorem IV.1 are satisfied. In fact, it is sufficient to ensure that

$$V_c(x(\sigma_2(t_{k+1}))) - V_c(x(\sigma_2(t_k))) \leq -\eta\|x(\sigma_2(t_k))\|^2, \quad (20)$$

for some $\eta > 0$. According to (19), this will be true, if

$$\left[\textstyle\prod_{i=1}^{2}\left(\frac{\alpha_c\alpha_i}{\beta_c\beta_i}\right)\left(1 + \mu_i(t_{k+1})\frac{-\gamma_i}{\alpha_i}\right)e^{\frac{-\gamma_c}{\alpha_c}(t_{k+1}-\sigma_2(t_k))} - 1\right]$$

$$\times V_c(x(\sigma_2(t_k))) \leq -\eta\|x(\sigma_2(t_k))\|^2.$$

This holds, by virtue of (10), if

$$\prod_{i=1}^{2}\left(\frac{\alpha_c\alpha_i}{\beta_c\beta_i}\right)\left(1 + \mu_i(t_{k+1})\frac{-\gamma_i}{\alpha_i}\right)e^{\frac{-\gamma_c}{\alpha_c}(t_{k+1}-\sigma_2(t_k))} - 1 < 0,$$

which can be equivalently rewritten as

$$t_{k+1} - \sigma_2(t_k) > \frac{\log\left[\prod_{i=1}^{2}\left(\frac{\alpha_c\alpha_i}{\beta_c\beta_i}\right)\left(1 + \mu_i(t_{k+1})\frac{-\gamma_i}{\alpha_i}\right)\right]}{\frac{\gamma_c}{\alpha_c}}, \quad (21)$$

which concludes the proof. □

B. Stability analysis of uncertain switched system

Based on the above result, sufficient conditions are derived to guarantee the asymptotic stability of the uncertain nonlinear switched system (4).

Theorem IV.3. *Consider the nonlinear switched system (4). It is assumed that the following hold:*

(1) *The conditions (i) and (ii) of Theorem IV.2 are satisfied.*

(2) *The uncertainties are bounded as follows*

$$\|f_i(x(t))\| \leq L_i\|x(t)\|, \quad i \in \{c, 1, 2\}, \text{ such that}$$

$$L_c < \frac{\gamma_c}{2\alpha_c}, \quad (22)$$

$$2L_i(1 + \mu_{i\max}\|A_i\|)\alpha_i + \mu_{i\max}L_i^2\alpha_i < \gamma_i, \forall i \in \{1, 2\} \quad (23)$$

(3) *The duration of the continuous-time subsystems satisfy,*

$$t_{k+1} - \sigma_2(t_k) > \frac{\log\left[\prod_{i=1}^{2}\left(\frac{\alpha_c\alpha_i}{\beta_c\beta_i}\right)(1 + \mu_i(t_{k+1})\xi_i)\right]}{\xi_c}, \quad (24)$$

where $\xi_c = \frac{\gamma_c}{\alpha_c} - 2L_c$, *and for* $i = \{1, 2\}$,

$$\xi_i = \frac{-\gamma_i}{\alpha_i} + 2L_i(1 + \mu_{i\max}\|A_i\|) + \mu_{i\max}L_i^2.$$

Then the switched system (4) is asymptotically stable.

Proof.
Let the TSMLF, $V_i(x) = x^T P_i x,\ i \in \{c, 1, 2\}$ satisfying (10). The Δ-derivative of $V_c(x)$ along the trajectories of the continuous-time subsystem of (4), is given by (see [16])

$$\begin{aligned} V_c^{\Delta}(x) &= x^T(A_c^T P_c + P_c A_c)x + f_c^T(x)P_c x + x^T P_c f_c(x) \\ &\leq -\lambda_{\min}(Q_c)\|x\|^2 + 2\|P_c\|\|f_c(x)\|\|x\| \\ &\leq [-\gamma_c + 2L_c\alpha_c]\|x\|^2. \end{aligned} \quad (25)$$

Since L_c is bounded according to (22), $V_c(x)$ is a quadratic TSLF. From (10), we can derive the following inequality

$$V_c^{\Delta}(x) \leq [\frac{-\gamma_c}{\alpha_c} + 2L_c]V_c(x) = \xi_c V_c(x). \quad (26)$$

On the other hand, $V_i^{\Delta}(x),\ i \in \{1, 2\}$ along the trajectories of the discrete-time subsystems is given by (see [16])

$$\begin{aligned} &V_i^{\Delta}(x) = x^{T\Delta}P_i x(\sigma_i(t)) + x^T P_i x^{\Delta} \\ &= x^T(A_i^T P_i + P_i A_i + \mu_i(t)A_i^T P_i A_i)x \\ &+2x^T(\mu(t)A_i^T + I)P_i f_i(x) + \mu_i(t)f_i^T(x)P_i f_i(x) \\ &\leq [-\gamma_i + 2L_i(1 + \mu_{i\max}\|A_i\|)\alpha_i + \mu_{i\max}L_i^2\alpha_i]\ \|x\|^2. \end{aligned} \quad (27)$$

Since $f_i(x)$ is bounded according to (23), $V_i(x)$ is a quadratic TSLF. Similarly to the proof of Theorem (IV.2), and from (10), we can derive the following inequality:

$$V_i^{\Delta}(x) \leq [\frac{-\gamma_i}{\alpha_i} + 2L_i(1 + \mu_{\max i}\|A_i\|) + \mu_{\max i}L_i^2]V_i(x) \quad (28)$$

From (19), (26) and (28), we conclude that, $\forall k \in \mathbb{N}$,

$$V_c(x(\sigma_2(t_{k+1}))) = \\ \textstyle\prod_{i=1}^{2}\frac{\alpha_c\alpha_i}{\beta_c\beta_i}(1 + \mu_i(t_{k+1})\xi_i)\,e^{\xi_c(t_{k+1}-\sigma_2(t_k))}V_c(x(\sigma_2(t_k)))$$

Condition (20) is ensured if (24) is satisfied, which concludes the proof. □

C. Generalisation to N-discrete subsystem

We can generalize the result to N-discrete subsystems by considering $\{A_c, A_i\},\quad i = 1, 2, \ldots, N$ be a set of $N + 1$ constant regressive matrices in $\mathbb{R}^{n\times n}$, and the switched system determined as (4) on the time scale $\mathbb{T} = \mathbb{P}_{\{\sigma_i(t_k), t_{k+1}\}} = \cup_{k=0}^{\infty}\{[\sigma_N(t_k), t_{k+1}] \cup (\cup_{i=1}^{N-1}\sigma_i(t_{k+1}))\}$. Suppose that conditions (1)-(3) of Theorem IV.3 are fulfilled for $i \in \{c, 1, \ldots, N\}$, and the duration of each continuous-time subsystem satisfies,

$$t_{k+1} - \sigma_N(t_k) > \frac{\log\left[\prod_{i=1}^{N}\left(\frac{\alpha_c \alpha_i}{\beta_c \beta_i}\right)(1+\mu_i(t_{k+1})\xi_i)\right]}{\xi_c}.$$

Then, the considered switched system is asymptotically stable.

V. Consensus problem under intermittent information transmissions

The consensus problem for linear multi-agent systems (MASs) with intermittent information transmissions, has been studied in [17] by introducing time scale theory. The considered MAS consists of N agents and described by

$$\begin{array}{ll} \dot{x}_i = Ax_i + Bu_i & i \in \{1, \cdots, N\} \\ \dot{x}_0 = Ax_0, & \end{array} \tag{29}$$

$x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ are the state and the control input of agent i, respectively. $x_0 \in \mathbb{R}^n$ is the state of the leader. $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant real matrices. The communication network graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ consists of a node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, such that, each edge $(i, j) \in \mathcal{E}$ in the directed graph [24], corresponds to an information link from agent i to agent j. Let $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The Laplacian matrix of $\mathcal{G}$ is defined as $L = (m_{ij}) \in \mathbb{R}^{N\times N}$, with $m_{ii} = \sum_{j=1}^{N} a_{ij}$ and $m_{ij} = -a_{ij}$ for $i \neq j$. The topology of $\mathcal{G}$ is described by the weighted matrix $H = L + D \in \mathbb{R}^{N\times N}$, where $D = diag(d_1, ..., d_N)$, with $d_i = 1$ if the leader state is available to follower i, and $d_i = 0$ otherwise. Let z_i be the local information available for agent i, such that

$$z_i = \sum_{j\in\mathcal{N}_i}(x_j - x_i) + d_i(x_0 - x_i), \tag{30}$$

where $\mathcal{N}_i$ is the set of neighbours of agent i. It is assumed that, the local information is exchanged between neighboring agents through a communication channel over some disconnected time intervals because of possible sensor failures or communication obstacles. Suppose that the communication channel is modeled with an additive uncertainty $\delta(t) \in \mathbb{R}$, such that $|\delta(t)| \leq \delta_{max}$, where $\delta_{max} > 0$. Based on the available local information, the following distributed intermittent controller is proposed, $\forall i \in \{1, \cdots, N\}$ and $K \in \mathbb{R}^{m\times n}$

$$u_i(t) = \begin{cases} K(1+\delta(t))z_i(t), & t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ K(1+\delta(t))z_i(t_{k+1}), & t \in \cup_{k=0}^{\infty}[t_{k+1}, \sigma(t_{k+1})[. \end{cases}$$

The agents can communicate with their neighbours over the set $\cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[$. At $t_{k+1}, k \in \mathbb{N}$, the communication fails during a period $\mu(t_{k+1}) = \sigma(t_{k+1}) - t_{k+1}$, and the control is held on (i.e; does not evolve) until the communication resumes at times $t = \sigma(t_{k+1})$, and it will be updated. The state error between the leader and agent i is determined by $e_i = x_i - x_0$. Let $z = (z_1^T, \ldots, z_N^T)^T, u = (u_1^T, \ldots, u_N^T)^T$.

The dynamic of the tracking error $e = (e_1^T, \ldots, e_N^T)^T$ can be written in the compact form as ($\otimes$ is the Kronecker product)

$$\begin{aligned} \dot{e}(t) &= (I_N \otimes A)e(t) + (I_N \otimes B)u(t), \\ u(t) &= \begin{cases} -(1+\delta(t))(H \otimes K)e(t), t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ -(1+\delta(t_{k+1}))(H \otimes K)e(t_{k+1}), \\ \quad t \in \cup_{k=0}^{\infty}[t_{k+1}, \sigma(t_{k+1})[\end{cases} \end{aligned} \tag{31}$$

The closed-loop system (31) can be written as

$$\dot{e} = \begin{cases} [(I_N \otimes A) - (1+\delta(t))(H \otimes BK)]e(t), \\ \quad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[. \\ \\ (I_N \otimes A)e(t) - (1+\delta(t_{k+1}))(H \otimes BK) \\ \quad \times e(t_{k+1}),\ t \in \cup_{k=0}^{\infty}[t_{k+1}, \sigma(t_{k+1})[\end{cases} \tag{32}$$

Using the definition of the Δ-derivative and considering the specific time scale $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$, the closed-loop system (32) is written as (see [16], [17] for more details):

$$e^{\Delta}(t) = \begin{cases} A_c e(t) - \delta(t)(H \otimes BK)e(t); \\ t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ \\ A_d e(t) + \\ \delta(t)\left(\dfrac{e^{(I_N\otimes A)\mu_{\max}} - I}{\mu_{\max}}\right)(H \otimes A^{-1}BK); \\ \times e(t)\ t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \end{cases} \tag{33}$$

with $A_c = [(I_N \otimes A) - (H \otimes BK)]$ and $A_d = \left(\dfrac{e^{(I_N\otimes A)\mu_{\max}} - I}{\mu_{\max}}\right)[I - (H \otimes A^{-1}BK)]$. Notice that, if A is not invertible, we can always determine the discrete matrix via the convergence power series $\mathcal{E}(A\mu(t)) = \sum_{n=1}^{\infty}\frac{(A\mu(t))^{n-1}}{n!}$, and $A_d = \left(\dfrac{e^{(I_N\otimes A)\mu_{\max}} - I}{\mu_{\max}}\right)[\mathcal{E}(A\mu(t))(H \otimes BK)]$. The objective is to determine an upper bound of the duration of communication to be respected, such that the consensus problem is solved, even when the communication fails for some period of time.

Consider now the consensus for MAS which consists of one leader and two followers ($N = 2$, (Fig. 2)), described by $A = \begin{pmatrix} 0 & 1 \\ 0.1 & 0.05 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$.

$$A_c = \begin{pmatrix} 0 & 1.0000 & 0 & 0 \\ -0.9000 & -0.4500 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 1.0000 & 0.5000 & -1.9000 & -0.9500 \end{pmatrix}.$$

For $\mu_{\max} = 0.9$, the discrete subsystem is described by

$$A_d = \begin{pmatrix} -0.4139 & 0.8067 & 0 & 0 \\ -0.9330 & -0.4205 & 0 & 0 \\ 0.4599 & 0.2300 & -0.8739 & 0.5768 \\ 1.0367 & 0.5184 & -1.9697 & -0.9389 \end{pmatrix}.$$

The control gain, such that A_c is stable, is set as $K = \begin{pmatrix} 1 & 0.5 \end{pmatrix}$. Let us define, for $Q_c = Q_d = I_4$ (the 4×4 identity matrix), the positive definite matrices

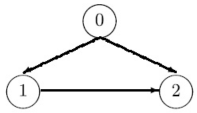

Fig. 2: Communication topology.

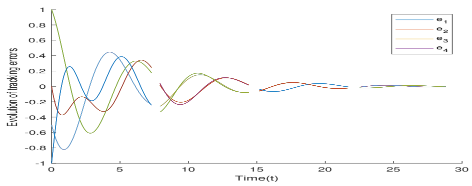

Fig. 3: Trajectories of the tracking error e.

$$P_c = \begin{pmatrix} 3.4328 & 1.0914 & -0.2435 & 0.4823 \\ 1.0914 & 3.6620 & -0.7257 & 0.1130 \\ -0.2435 & -0.7257 & 1.7763 & 0.2632 \\ 0.4823 & 0.1130 & 0.2632 & 0.8033 \\ \end{pmatrix} \text{ and}$$

$$P_d = \begin{pmatrix} 6.1883 & 0.0235 & -0.0415 & 0.0008 \\ 0.0235 & 5.3271 & -0.0032 & -0.0121 \\ -0.0415 & -0.0032 & 0.0421 & 0.0006 \\ 0.0008 & -0.0121 & 0.0006 & 0.0126 \end{pmatrix},$$

which is computed according to Remark II. We have, $\alpha_c = 4.8367, \beta_c = 0.6331, \alpha_d = 6.1892, \beta_d = 0.0126$ and $\gamma_c = \gamma_d = 1$. For $\delta_{\max} = 0.003$, we have $L_c = 0.01$ and $L_d = 0.002$, that satisfies conditions (22), (23), and we get $\xi_c = 0.9033, \xi_d = 0.9164$. Condition (24) is satisfied if

$$t_{k+1} - \sigma(t_k) \geq 7.183, \ \forall k \in \mathbb{N},$$

which guarantee the stability of switched system (33). The error signals are plotted in Fig. 3 on time scale $\mathbb{T} = \cup_{k=0}^{\infty}[8.1k + \frac{k}{1.11k+0.8}, 8.1(k+1)]$, such that $0.523 \leq \mu(t_k) = \frac{k}{1.11k+0.8} \leq 0.9$, and $7.19 \leq t_{k+1} - \sigma(t_k) = 8.1 - \frac{k}{1.11k+0.8} \leq 8.1$, which shows that they converge to zero.

VI. Conclusion

Time scale theory was introduced to derive new conditions for stability of a class of uncertain switched systems between a continuous-time subsystem and several discrete-time subsystems, using the concept of TSMLF. The proposed scheme has been applied to solve the problem of consensus for multi-agent systems with intermittent information transmissions.

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