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# Distributed Spatially Invariant Systems: An Input-Output Approach

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***Abstract*— In this paper, the Banach duality structure of the optimal $H^{\infty}$-problem for distributed spatially invariant systems is provided. Under specific assumptions, it is shown that an optimal feedback spatially invariant $H^{\infty}$ controller exists, and can be computed through the Youla parametrization. It also shown that the optimal $H^{\infty}$ cost is equal to the induced norm of a novel operator defined on a Banach projective tensor space. An operator identity is deduced to compute the optimal Youla parameter, and thus the optimal controller provided a maximizing vector exists.**

## I. INTRODUCTION

Distributed control of spatially invariant systems has received a large attention in recent years. Distributed control is very useful since centralized control is complicated to implement, requires lots of resources and computational power, and difficult to apply in real-time for a large class of interconnected subsystems. Distributed spatially invariant systems have widespread applications including micro-elector-mechanical systems, flow control, Platoons and in general systems governed by partial differential equations (PDEs) with constant coefficients and distributed actuators and sensors [1]–[5]. In addition, to a wide range of complex systems from renewable resources to large scale robotic systems, are such systems where macroscopic coherent behavior or coarse dynamics can be modeled by such PDEs [6].

The seminal paper [2] shows that optimal controllers have an inherent degree of decentralization and the same structure as the distributed spatially invariant systems they control. Moreover, it is also shown that these optimal controllers inherit the spatially invariant properties of the system. Note that, for networked system including interacted subsystems, optimal controller need to share (global) information to compute the feedback control signals [7]. However, among these systems, there are many cases with the spatial invariant property. This means that, we have a symmetric distributed system with a spatial architecture where output signals will shift equally when input signals shift [8].

In [4] it is shown that the dependence of optimal controllers on shared information decays exponentially as subsystems move away. The model-matching framework proposed as an alternative for this class of problems based on the Youla parameterization [9] where the closed-loop transfer function assumes an affine form in the Youla parameter can result in loss of convexity. In order to preserve the latter [10] introduced quadratically invariant problems as a broader class of distributed systems. Most optimal controllers can not deal with high dimensional systems and large number of input and outputs. Even if the subsystems have simple dynamics, the whole networked system usually displays complex behaviors, when the number of subsystems is large [11]. As an alternative, instead of using global information [2] and [12] propose to compute the control signals based on local communication among neighboring subsystems.

Optimal control of linear spatially invariant systems with standard linear quadratic criterion was considered in [1], [2], [4], [12] using mainly a state-space framework. Most of the early work in distributed control focuses on spatially distributed systems considering all signals as a function of both spatial and temporal variables. [13], [14] derive various system theoretic properties including stabilizabilty and detectability conditions of spatially invariant systems.

Most of the work thus far is essentially based on state-space techniques. *In this paper, we take an input-output approach to the general spatially invariant systems introduced in [2].* This is an effort to continue the work undertaken in [7], [8], [15] for the special class of spatially invariant cone causal systems, i.e., spatially invariant systems with finite communication speed. This important class of systems was introduced in [3] and further considered in [5], [16]. In our previous work [7], [8], [15], the duality structure of the $H^{\infty}$ problem was provided, and a systematic framework to compute the optimal decentralized $H^2$ controllers using the Youla parametrization.

*In this paper, the solution of the distributed $H^{\infty}$ optimal control problem for general spatially invariant systems is characterized using Banach space duality theory. A solution based on a novel operator defined on specific projective tensor spaces is provided. In particular, it is shown that under certain conditions the optimal $H^{\infty}$ controller exists and is characterized by a duality identity. In addition, the optimal $H^{\infty}$ cost is shown to be equal to the induced norm of the aforementioned operator. An operator identity is deduced for the optimal Youla parameter (and therefore the optimal controller) when a maximal vector exists.*

The rest of the paper is organized as follows. Section II contains some preliminaries and the problem formulation. In section III the Banach space duality characterization of the problem is provided. Section IV provides a solution based on operator theory, followed by the conclusion and future work in section V.

## II. PROBLEM FORMULATION

Following [2], Let $G$ be a locally compact abelian (LCA) group with the group operation denoted by $'+'$. For example,

1) $G = \mathbb{R}$

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2) $G = \partial\mathfrak{D}$ (unit circle)
3) $G = Z$ (integers)
4) $G = \mathbb{Z}_n$ (finite group of integers modulo n)

In addition, direct product of such groups $G := G_1 \times G_2 \times \ldots \times G_n$ can be considered.

Define the translation operator $T_{x_0}$ for functions $f : G \to \mathbb{C}^n$ by $(T_{x_0} f)(x) := f(x - x_0)$. Let $G$ equipped with the (translation invariant) Haar measure $dx$. Define the space of measurable, square integrable, complex valued functions $\mathbb{L}_2(G, \mathbb{C}^n)$ as [2]:

$$\mathbb{L}_2(G, \mathbb{C}^n) := \{f : G \to \mathbb{C}^n : \|f\|_2{}^2 := \int_G |f(x)|^2\, dx < \infty\} \tag{1}$$

Let $P$ denote the generalized plant assumed a linear, space/time invariant distributed system, with spatio-temporal impulse response $h(x,t)$, $x \in G$, $t \in R^+$. We seek an optimal $H^\infty$ distributed feedback controller $K$ space and time invariant so as to internally stabilize the system and reject external disturbances $w(x,t)$ see Fig. 1.

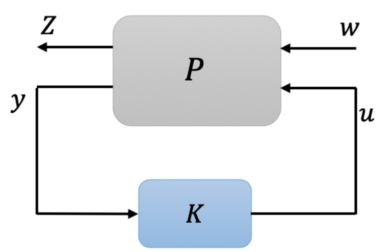


Fig. 1. Standard feedback configuration

Note as shown in [2] there is no loss in performance by restricting the controller $K$ to be spatio-temporal invariant. The closed-loop system $T_{zw}$ is then spatio-temporal invariant, and given by [2]:

$$z(x,t) = \int_G \int_0^\infty h(\xi,\tau) w(x-\xi, t-\tau)\, d\xi\, d\tau \tag{2}$$

To introduce the concept of transfer function for (2), we need to introduce the general Fourier analysis on the LCA group $G$, and define the dual group $\hat{G}$ as the set of characters, i.e., homomorphisms from $G$ to $\partial\mathfrak{D}$ [17]. For example, if $G = \mathbb{R}$, $\partial\mathfrak{D}$, $Z$, $\mathbb{Z}_n$, then $\hat{G} = \mathbb{R}$, $Z$, $\partial\mathfrak{D}$, $\mathbb{Z}_n$, respectively [2].

For $f \in \mathbb{L}^2(G)$ define its Fourier transform $F(f)$ denoted $\hat{f} : \hat{G} \to \mathbb{C}^n$ by

$$\mathbb{F}(f) := \hat{f}(\chi) = \int_G f(x)\chi^*(x)\, dx \tag{3}$$

Where $\chi(.)$ denote the character and $'*'$ the complex conjugate. The Fourier transform $F$ is an isometric isomorphism from $\mathbb{L}^2(G)$ to $\mathbb{L}^2(\hat{G})$, thus [17]:

$$\|f\|_2{}^2 = \int_G |f(x)|^2\, dx = \int_{\hat{G}} |\hat{f}(\chi)|^2 d\chi = \|\hat{f}\|_2{}^2 \tag{4}$$

A consequence is that every operator $\Gamma$ defined on $\mathbb{L}_2(G)$ can be identified with an operator $\hat{\Gamma} = F\Gamma F^{-1}$ on $\mathbb{L}_2(G)$ as pointed out in [2], where $F^{-1}$ is the inverse Fourier transform. Moreover, translation invariant operators correspond to multiplication operators in the Fourier domain. That is, if $\Gamma$ is translation invariant, then there exists a function denoted $\hat{\Gamma}(\lambda)$ such that [2]

$$(\hat{\Gamma}\hat{f})(\chi) = \hat{\Gamma}(\lambda)\hat{f}(\chi) \quad a.e. \tag{5}$$

Where, $\hat{\Gamma}(\lambda)$ is a measurable matrix-valued function.

Applying this concept to $T_{zw}$ as defined by the spatio-temporal convolution (2), we get a representation of $T_{zw}$ in terms of the transfer function

$$\hat{Z}(\chi,s) = \hat{T}_{zw}(\chi,s)\,\hat{w}(\chi,s) \tag{6}$$

Since $K$ is chosen to be internally stabilizing, this guarantees that $\hat{T}_{zw}(\chi,s)$ is bounded (and analytic in $s$) over the domain $\hat{G} \times \{Re(s) > 0\}$.

As an operator acting from $\mathbb{L}^2(G \times \mathbb{R}^+)$, $T_{zw}(x,t)$ is a well defined operator with induced norm [2]:

$$\|T_{zw}\|_\infty := ess \sup_{\chi\in\hat{G},\ \omega\in\mathbb{R}} \bar{\sigma}(\hat{T}_{zw}(\chi,\omega)) \tag{7}$$

for the continuous-time case where $\bar{\sigma}(\cdot)$ denotes the maximal singular value, and

$$\|T_{zw}\|_\infty := ess \sup_{\chi\in\hat{G},\ 0\le\theta\le 2\pi} \bar{\sigma}(\hat{T}_{zw}(\chi,e^{i\theta})) \tag{8}$$

for the discrete-time case.

Our problem is to design a distributed spatio-temporal invariant controller $K$ that is internally stabilizing and minimizes $\|T_{zw}\|_\infty$, the $H^\infty$-norm of the transmission from external disturbances $w(\cdot,\cdot)$ to controlled signals $z(\cdot,\cdot)$. Note the $\mathbb{L}^2$-norm of $\mathbb{L}^2(G \times \mathbb{R}^+)$ is for $w(x,t) \in \mathbb{L}^2(G \times \mathbb{R}^+)$ given by for the continuous-time case

$$\|w\|_2^2 = \int_G \int_0^\infty |w(x,t)|^2\, dt\, dx \tag{9}$$

and

$$\|w\|_2^2 = \int_G \sum_{t=0}^\infty |w(x,t)|^2\, dx \tag{10}$$

for the discrete-time case.

The Fourier transform in both the spatial and temporal variables of the Hilbert space $\mathbb{L}^2(G \times \mathbb{R}^+)$ yields $\mathbb{L}^2(\hat{G}, H^2(\mathbb{C}^+))$, where $\mathbb{C}^+$ denotes the right-half plane, for the continuous-time case, and $\mathbb{L}^2(\hat{G}, H^2(\partial\mathfrak{D}))$ for the discrete-time case. Under the norm

$$\|\hat{w}(\chi,\omega)\|_2^2 = \int_{\hat{G}} \int_0^\infty |\hat{w}(\chi,\omega)|^2\, d\omega\, d\chi \tag{11}$$

for the continuous-time case, and

$$\|\hat{w}(\chi,e^{j\theta})\|_2^2 = \frac{1}{2\pi} \int_{\hat{G}} \int_0^{2\pi} |\hat{w}(\chi,e^{j\theta})|^2\, d\theta\, d\chi \tag{12}$$

for the discrete-time case.

The norms defined in (7) and (8) are induced on these spaces, respectively. Henceforth, without loss of generality we will only consider the discrete-time case since as it is well-known in the $s$-domain it suffices to use a *Möbius* transformation to map the right-half plane to the unit disk. As discussed in [2], using the spatio-temporal Youla parametrization where

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \tag{13}$$

and

$$P_{22}(\chi,\omega) = N_r(\chi,\omega)D_r^{-1}(\chi,\omega) = D_\ell^{-1}(\chi,\omega)N_\ell(\chi,\omega) \quad (14)$$

is a right and left coprime factorization of $P_{22}$, respectively. It follows that the parametrization of all internally stabilizing controllers is given by:

$$K = (Y_r - QN_\ell)^{-1}(-X_r + QD_\ell) \quad (15)$$
$$= (-X_\ell + D_rQ)(Y_\ell + N_rQ)^{-1} \quad (16)$$
$$Q\, stable\, spatio-temporal\, Youla\, parameter$$

where, $Y_r$, $X_r$, $X_\ell$ and $Y_\ell$ are stable spatio-temporal invariant functions satisfying the Bezout identity [2].
With the help of the Youla parametrization, $T_{zw}(\chi,\omega)$ reduces to the following affine form [2]:

$$T_{zw}(\chi,\omega) = T_1(\chi,\omega) - T_2(\chi,\omega)Q(\chi,\omega)T_3(\chi,\omega) \quad (17)$$

Where $T_i(\chi,\omega)$, $i = 1,2,3$ are stable spatially invariant distributed functions. Here stability means that the functions belong to $L^\infty(\hat{G},H^\infty(\partial\mathcal{D}))$ the space of measurable essentially bounded function on $\hat{G}$, and bounded and analytic in the open unit disk $\mathcal{D}$. The problem becomes then the design of a distributed spatially invariant $H^\infty$ controller $K$ such that

$$\inf_{\substack{K\, internally\\ stabilizing}} \|T_{zw}\|_\infty = \inf_{Q\in L^\infty(\hat{G},H^\infty(\partial\mathcal{D}))} \|T_1 - T_2\, Q\, T_3\|_\infty \quad (18)$$

Solving problem (18) is the subject of this paper.

## III. BANACH SPACE DUALITY CHARACTERIZATION

Following [16], for each character $\chi$, perform a standard inner-outer factorization (see [9]) of $T_2(\chi,e^{j\theta})$, i.e.,

$$T_2(\chi,e^{j\theta}) = T_{2in}(\chi,e^{j\theta})\; T_{2ou}(\chi,e^{j\theta}) \quad (19)$$

where

$$T_{2in}^*(\chi,e^{j\theta})\; T_{2in}(\chi,e^{j\theta}) = 1 \quad a.e.\theta \quad (20)$$

where $T_{2in}^*$ is the complex conjugate of $T_{2in}$, that is, for each $\chi$, $T_{2in}(\chi,.)$ is an inner function. And where for each $\chi$, $T_{2ou}(\chi,.)$ is an outer function.
Likewise, for each $\chi$, perform a standard co-inner co-outer factorization (see [9]) of $T_3(\chi,e^{j\theta})$, i.e.,

$$T_3(\chi,e^{j\theta}) = T_{3cou}(\chi,e^{j\theta})\; T_{3cin}(\chi,e^{j\theta}) \quad (21)$$

Where

$$T_{3cin}^*(\chi,e^{j\theta})\; T_{3cin}(\chi,e^{j\theta}) = 1 \quad a.e.\theta \quad (22)$$

That is $T_{3cin}$ is co-inner in $\theta$ and $T_{3cou}$ is co-outer in $\theta$.

Introducing these factorizations into the optimization (18) yields

$$\mu = \inf_{Q\in L^\infty(\hat{G},H^\infty(\partial\mathcal{D}))} = \|T_1 - T_{2in}T_{2ou}\; Q\; T_{3cou}T_{3cin}\|_\infty \quad (23)$$

Note the optimization (23) is convex in $Q \in L^\infty(\hat{G},H^\infty(\partial\mathcal{D}))$ but infinite dimensional. Pre-multiplication by $T_{2in}^*$ and post-multiplication by $T_{3cin}^*$ of the argument of (23) does not change the $L^\infty(\hat{G},H^\infty(\partial\mathcal{D}))$-norm, i.e.,

$$\mu = \inf_{Q\in L^\infty(\hat{G},H^\infty(\partial\mathcal{D}))} = \|T_{2in}^*\, T_1\, T_{3cin}^* - T_{2ou}\; Q\; T_{3cou}\|_\infty \quad (24)$$

The optimization (24) is a distance minimization problem between the function $T_{2in}^*\, T_1\, T_{3cin}$ in $L^\infty(\hat{G},L^\infty(\partial\mathcal{D}))$ to the subspace $M$ defined by:

$$M = T_{2ou}\; L^\infty(\hat{G},L^\infty(\partial\mathcal{D}))\; T_{3cou} \quad (25)$$

of $L^\infty(\hat{G},L^\infty(\partial\mathcal{D}))$. To ensure that the subspace $M$ is closed in $L^\infty(\hat{G},L^\infty(\partial\mathcal{D}))$ we assume that the outer function $T_{2ou}$ and co-outer function $T_{3cou}$ are both invertible in $L^\infty(\hat{G},H^\infty(\partial\mathcal{D}))$. In this case $T_{2ou}$ and $T_{3cou}$ can be both "absorbed" into $Q$, i.e., define

$$\tilde{Q}(\chi,e^{j\theta}) := T_{2ou}(\chi,e^{j\theta})\; Q(\chi,e^{j\theta})\; T_{3cou}(\chi,e^{j\theta}) \;\in L^\infty \quad (26)$$

Then,

$$\begin{aligned}\mu &= \inf_{\tilde{Q}\in L^\infty(\hat{G},L^\infty(\partial\mathcal{D}))} \|T_{2in}^*\, T_1\, T_{3cin}^* - \tilde{Q}\|_\infty \\ &=: \mathrm{dist.}(T_{2in}^*\, T_1\, T_{3cin}^*,\; L^\infty(\hat{G},H^\infty(\partial\mathcal{D}))) \quad (27)\end{aligned}$$

and the subspace

$$M = L^\infty(\hat{G},H^\infty(\partial\mathcal{D})) \quad (28)$$

where $\simeq$ means "isomorphic to". To show existence of an optimal parameter $\tilde{Q} \in L^\infty(\hat{G},H^\infty(\partial\mathcal{D}))$, i.e., a minimizer in (27), we shall use Banach space duality theory. Let us start with some preliminaries from [18].
Let $B$ be a Banach space with norm $\|\cdot\|_B$. The space of all continuous linear functionals $f: B \to \mathbb{C}$ is known as the dual space and is denoted by $B^*$. In turn, a Banach space $\tilde{B}$, endowed with a norm $\|\cdot\|_{\tilde{B}}$, is said to be the pre-dual space of $B$ if $\tilde{B}$ is isometrically isomorphic (denoted $\cong$) to $B^*$. If $S$ is a subspace of $B$, the annihilator of $S$ in $B^*$, denoted $S^\perp$ is defined as [18]:

$$S^\perp := \{f \in B^* \;: f(g) = 0\;,\; \forall g \in S\} \quad (29)$$

The subspace $S^\perp$ is thus the set of linear continuous functionals on $B$ which vanish on $S$. Conversely, if $\tilde{S}$ is a subspace of $\tilde{B}$, then the preannihilator of $\tilde{S}$ in $B$, denoted ${}^\perp\tilde{S}$, is defined as [18]:

$${}^\perp\tilde{S} := \{b \in B \;: f(b) = 0 \;\forall f \in \tilde{S}\} \quad (30)$$

Note $({}^\perp\tilde{S})^\perp \cong \tilde{S}$ if $\tilde{S}$ is a closed subspace of $\tilde{B}$.

According to duality theory the existence of a preannihilator guarantees that the following identity holds for $\tilde{b} \in \tilde{B}/\tilde{S}$. i.e., $\tilde{b} \in \tilde{B}$ and $\tilde{b} \notin \tilde{S}$, [18],

$$\inf_{\tilde{s}\in\tilde{S}} \|\tilde{b} - \tilde{s}\|_{\tilde{B}} = \min_{\tilde{s}\in\tilde{S}} \|b - \tilde{s}\|_{\tilde{B}} = \|b - \tilde{s}_0\|_{\tilde{B}} \quad (31)$$
$$= \sup_{\substack{b\in{}^\perp\tilde{S}\\ \|b\|_B\le 1}} |\langle\tilde{b},b\rangle| \quad (32)$$

Expression (32) shows that the infimum in the distance from $\tilde{b}$ to the (closed) subspace $\tilde{S}$ is a minimum, i.e., is achieved by $\tilde{s}_0 \in \tilde{S}$, and is equal to the supremum of the functional $\langle\tilde{b},\tilde{b}\rangle$, where $\langle\cdot,\cdot\rangle$ is the duality product, over the unit ball of the preannihilator ${}^\perp\tilde{S}$.

For our problem, the identification is as follows:

$$\begin{aligned}&\tilde{B} := L^\infty(\hat{G},L^\infty(\partial\mathcal{D})),\; \tilde{S} := L^\infty(\hat{G},H^\infty(\partial\mathcal{D})) = M \quad (33)\\ &\tilde{b} := T_{2in}^*\, T_1\, T_{3cin}^* \in L^\infty(\hat{G},L^\infty(\partial\mathcal{D}))\end{aligned}$$

To compute the preannihilator of $\tilde{S}$, $^{\perp}\tilde{S}$, we need first to determine the predual space $B$ of $\tilde{B}$. Define the Banach space $L^1(G, L^1(\partial\mathcal{D}))$ of measurable and absolutely integrable function on $G$ and $\partial\mathcal{D}$ under the following $L^1$-norm for $f \in L^1(\hat{G}, L^1(\partial\mathcal{D}))$,

$$\|f\|_{L^1} := \frac{1}{2\pi}\int_{\hat{G}}\int_0^{2\pi} |f(\chi, e^{j\theta})|\, d\theta\, d\chi \tag{34}$$

Now, we shall show that

$$L^\infty(\hat{G}, L^\infty(\partial\mathcal{D})) \cong \left(L^1(\hat{G}, L^1(\partial\mathcal{D}))\right)^\star \tag{35}$$

The $L^1$-space $L^1(\hat{G}, L^1(\partial\mathcal{D}))$ can be viewed as the space of functions $f : \hat{G} \to L^1(\partial\mathcal{D})$, i.e., of $L^1(\partial\mathcal{D})$-valued $\chi$-measurable and absolutely integrable functions. This space can be written in terms of a specific tensor product that will help us in determining its dual space. First, let us write down the norm on $L^1(\hat{G})$ as for $h \in L^1(\hat{G})$ as:

$$\|h\|_{L^1(\hat{G})} := \int_{\hat{G}} |h(\chi)|\, d\chi \tag{36}$$

where $d\chi$ is a $\sigma$-finite measure. If for example, $G = \mathbb{Z}_p$ =finite group of integers modulo $n$, with the counting measure, then $\hat{G} = \mathbb{Z}_p$ [2]. In this case $L^1(\hat{G}) = \ell^1(\hat{G})$ the space of absolutely summable sequences and in this case

$$\|h\|_{L^1(\hat{G})} = \sum_{\chi} |h(\chi)| \tag{37}$$

Likewise, for $L^1(\partial\mathcal{D})$ define for $g \in L^1(\partial\mathcal{D})$ the norm:

$$\|g\|_{L^1(\partial\mathcal{D})} = \frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|\, d\theta \tag{38}$$

Then the $L^1$-space $L^1(\hat{G}, L^1(\partial\mathcal{D}))$ can be identified as the following $L^1$-tensor space:

$$L^1(\hat{G}, L^1(\partial\mathcal{D})) = L^1(\hat{G}) \otimes_\gamma L^1(\partial\mathcal{D}) \tag{39}$$

under the projective tensor norm [19] for $F \in L^1(\hat{G}) \otimes_\gamma L^1(\partial\mathcal{D})$:

$$\gamma(F) := \inf\{\sum_{i=1}^{n} \|h_i\|_{L^1(\hat{G})}\|g_i\|_{L^1(\partial\mathcal{D})} : h_i(\chi) \in L^1(\hat{G}), \quad g_i(e^{i\theta}) \in L^1(\partial\mathcal{D}), F = \sum_{i=1}^{n} h_i \otimes g_i\} \tag{40}$$

The dual space of the projective tansor space $L^1(\hat{G}) \otimes_\gamma L^1(\partial\mathcal{D})$ is isometrically isomorphic to the Banach space $L^\infty(\hat{G}, L^\infty(\partial\mathcal{D})$ by the Steinhaus-Nikodym theorem [19], that is:

$$(L^1(\hat{G}) \otimes_\gamma L^1(\partial\mathcal{D}))^* \cong L^\infty(\hat{G}, L^\infty(\partial\mathcal{D})) \tag{41}$$

Where if $f \in (L^1(\hat{G}) \otimes_\gamma L^1(\partial\mathcal{D}))^*$ then there exists a function $\tilde{f} \in L^\infty(\hat{G}, L^\infty(\partial\mathcal{D}))$ such that:

$$f(F) = \int_{\hat{G}}\int_0^{2\pi} \tilde{f}(\chi, e^{i\theta})(\sum_{i=1}^{n} h_i(\chi) \otimes g_i(e^{i\theta}))\, d\theta\, d\chi \tag{42}$$

where $F = \sum_{i=1}^{n} h_i \otimes g_i$, $h_i \in L^1(\hat{G})$, $g_i \in L^1(\partial\mathcal{D})$. In other words, to every continuous linear functionals $f$ on $L^1(\hat{G}) \otimes_\gamma L^1(\partial\mathcal{D})$, there corresponds a unique function $\tilde{f}$ in $L^\infty(\hat{G}, L^\infty(\partial\mathcal{D}))$ such that (42) holds, and vise-versa.

Now, we are in a position to compute the preannihilator of $M = L^\infty(\hat{G}, H^\infty(\partial\mathcal{D}))$. First, note that according to theory of Hardy spaces [20], the preannihilator of $H^\infty(\partial\mathcal{D})$ in $L^\infty(\partial\mathcal{D})$ is given by the Hardy space $H_0^1(\partial\mathcal{D})$ defined by [20]:

$$H_0^1(\partial\mathcal{D}) := \{g \in L^1(\partial\mathcal{D}) \ : \ \hat{g}(n) = 0, \forall n \leq 0\} \tag{43}$$

where $\hat{g}(n)$ is the nth Fourier coefficient of $g(e^{j\theta})$. That is, for all $g \in H_0^1(\partial\mathcal{D})$ and $h \in H^\infty(\partial\mathcal{D})$,

$$\int_0^{2\pi} g(e^{j\theta})h(e^{j\theta})\, d\theta = 0 \tag{44}$$

To determine the preannihilator of $M = L^\infty(\hat{G}, H^\infty(\partial\mathcal{D}))$, namely $^{\perp}M$, consider the closed subspace $L^1(\hat{G}) \otimes_\gamma H_0^1(\partial\mathcal{D})$. We shall regard formal expressions $F(\chi, e^{j\theta}) = \sum_{i=1}^{n} h_i(\chi) \otimes g_i(e^{j\theta})$, with $g_i(\cdot) \in H_0^1(\partial\mathcal{D})$, as defining an operator $A : L^1(\hat{G})^* \cong L^\infty(\hat{G}) \to H_0^1(\partial\mathcal{D})$ given by (see [19]):

$$Af = \sum_{i=1}^{n} \int_{\hat{G}} f(\chi)\ h_i(\chi)\ g_i(e^{j\theta})\ d\chi \tag{45}$$

Therefore, for each function $m(\chi, e^{j\theta}) \in L^\infty(\hat{G}, H^\infty(\partial\mathcal{D}))$, the duality product $\langle m(\chi, e^{j\theta}), F(\chi, e^{j\theta})\rangle$ is given by:

$$< m(\chi, e^{j\theta}), F(\chi, e^{j\theta}) >= \tag{46}$$

$$by(43) = \sum_{i=1}^{n} \int_{\hat{G}}\int_0^{2\pi} h_i(\chi)\, m(\chi, e^{j\theta})\, g_i(e^{j\theta})\, d\theta\, d\chi$$

$$= \sum_{i=1}^{n} \int_{\hat{G}} \underbrace{\int_0^{2\pi} m(\chi, e^{j\theta})\, g_i(e^{j\theta})\, d\theta}_{=0}\, d\chi = 0 \tag{47}$$

Since $m(\chi, \cdot) \in L^\infty(\hat{G}, H^\infty(\partial\mathcal{D}))$ and $g_i(.) \in H_0^1(\partial\mathcal{D})$, (47) holds for all $\sum_{i=1}^{n} h_i \otimes g_i \in L^1(\hat{G}) \otimes_\gamma H_0^1(\partial\mathcal{D})$ and since functions of the form $\sum_{i=1}^{n} h_i \otimes g_i$ are dense in $L^1(\hat{G}) \otimes_\gamma H_0^1(\partial\mathcal{D})$, (47) holds for all functions $F \in L^1(\hat{G}) \otimes_\gamma H_0^1(\partial\mathcal{D})$. Therefore, we have shown that

$$^{\perp}M = L^1(\hat{G}) \otimes_\gamma H_0^1(\partial\mathcal{D}) \tag{48}$$

According to Theorem 2 in [18] the existence of a predual space $L^1(\hat{G}) \otimes_\gamma L^1(\partial\mathcal{D})$ and a preannihilator $^{\perp}M$ guarantee the existence of a minimizer, i.e., an optimal $\tilde{Q}_o$ for the optimization (27). This important result is summarized in the following theorem.

**Theorem 1**: Under the invertibility assumption of the outer and co-outer functions $T_{2ou}(\chi, e^{j\theta})$ and $T_{3cou}(\chi, e^{j\theta})$ in the second variable, respectively, there exists an optimal $\tilde{Q}_o \in L^\infty(\hat{G}, H^\infty(\partial\mathcal{D}))$ for the optimal $H^\infty$-performance index $\mu$ in (27). Moreover, the identity (32) shows that the following hold:

$$\begin{aligned} \mu &= \min_{\tilde{Q} \in L^\infty(\hat{G}, H^\infty(\partial\mathcal{D}))} \|T_{2in}^* T_1 T_{3cin}^* - \tilde{Q}\|_\infty \\ &= \|T_{2in}^* T_1 T_{3cin}^* - \tilde{Q}_o\|_\infty \end{aligned} \tag{49}$$

$$= \sup_{\substack{F = \sum_{i=1}^{n} h_i \otimes g_i \in L^1(\hat{G}) \otimes_\gamma H_0^1(\partial\mathcal{D}) \\ \gamma(F) \leq 1}} \int_{\hat{G}}\int_0^{2\pi} T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta}) \sum_{i=1}^{n} h_i(\chi) \otimes g_i(e^{j\theta})\, d\theta\, d\chi \tag{50}$$

Theorem 1 implies the existence of an optimal $\tilde{Q}_o$ and therefore an **optimal Youla parameter** $Q_o \in L^\infty(\hat{G}, H^\infty(\partial\mathcal{D}))$ since from (26) we have:

$$Q_o(\chi, e^{j\theta}) = T_{2ou}^{-1}(\chi, e^{j\theta}) \tilde{Q}_o(\chi, e^{j\theta}) T_{3cou}^{-1}(\chi, e^{j\theta}) \quad (51)$$

Form (16), we deduce the **optimal $H^\infty$-controller** $K_o$ solving $\min_{K \text{ int. stabilizing}} \|T_{zw}\|_\infty$,

$$\begin{aligned} K_o &= (Y_r - Q_o N_\ell)^{-1}(-X_r + Q_o D_\ell) \\ &= (-X_\ell + D_r Q_o)(Y_\ell + N_r Q_o)^{-1} \end{aligned} \quad (52)$$

The **optimal $H^\infty$-performance cost** $\mu$ is given by:

$$\mu = \|T_1 - T_2 Q_o T_3\|_\infty \quad (53)$$

Next, we shall give a solution in terms of a tensor operator with a mixed-structure on the tensor Banach space $L^1(\hat{G}) \otimes_\alpha H^2(\partial\mathcal{D})$, under the projective norm

$$\alpha(F) := \inf\{\sum_{i=1}^n \|h_i\|_{L^1(\hat{G})} \|g_i\|_{H^2(\partial\mathcal{D})} : h_i(\chi) \in L^1(\hat{G}), \quad (54)$$
$$g_i(e^{i\theta}) \in H^2(\partial\mathcal{D}), F = \sum_{i=1}^n h_i \otimes g_i\}$$

## IV. A SOLUTION BASED ON OPERATOR THEORY

In this section, we introduce an operator denoted $\Xi$ on $L^1(\hat{G}) \otimes_\alpha H^2(\partial\mathcal{D})$ as discussed above. It is well-known that the Hilbert space $L^2(\partial\mathcal{D})$ defined on the unit circle can be decomposed as [20]

$$L^2(\partial\mathcal{D}) = H^2(\partial\mathcal{D}) \oplus H^{2^\perp}(\partial\mathcal{D}) \quad (55)$$

Where $H^{2^\perp}(\partial\mathcal{D})$ is the orthogonal complement of $H^2(\partial\mathcal{D})$ in $L^2(\partial\mathcal{D})$.
The set $\{e^{jn\theta}\}_{n=-\infty}^\infty$ is an orthogonal basis in $L^2(\partial\mathcal{D})$, i.e., $L^2(\partial\mathcal{D})$ is equal to the closure of the linear span of $\{e^{jn\theta}\}_{n=-\infty}^\infty$, that will henceforth be denoted by $\overline{span}\{e^{jn\theta}\}_{n=-\infty}^\infty$, any function $h(e^{j\theta}) \in L^2(\partial\mathcal{D})$ can be written as:

$$h(e^{j\theta}) = \sum_{n=-\infty}^\infty a_n e^{jn\theta} \quad (56)$$

Similarly,

$$\begin{aligned} H^2(\partial\mathcal{D}) &= \overline{span}\{e^{jn\theta}\}_{n=-\infty}^\infty, \\ H^{2^\perp}(\partial\mathcal{D}) &= \overline{span}\{e^{jn\theta}\}_{n=-\infty}^{-1} \end{aligned} \quad (57)$$

Next, define the positive $P_+$ and negative $P_-$ Riesz projections from $L^2(\partial\mathcal{D})$ into $H^2(\partial\mathcal{D})$ and $H^{2^\perp}(\partial\mathcal{D})$, respectively as:

$$\begin{aligned} P_+ &: L^2(\partial\mathcal{D}) \to H^2(\partial\mathcal{D}) \\ h &\longmapsto P_+ \, h(e^{j\theta}) = \sum_{n=0}^\infty a_n e^{jn\theta} \end{aligned} \quad (58)$$

$$\begin{aligned} P_- &: L^2(\partial\mathcal{D}) \to H^{2^\perp}(\partial\mathcal{D}) \\ h &\longmapsto P_- \, h(e^{j\theta}) = \sum_{-\infty}^{-1} a_n e^{jn\theta} \end{aligned} \quad (59)$$

Note $P_+$ and $P_-$ are orthogonal projections on $L^2(\partial\mathcal{D})$.

Define the identity operator $\hat{I}$ on $L^1(\hat{G})$ as:

$$\begin{aligned} \hat{I} &: L^1(\hat{G}) \to L^1(\hat{G}) \\ g &\longmapsto \hat{I}g(\chi) = g(\chi) \end{aligned} \quad (60)$$

Finally, we are in a position to define the linear operator $\Xi$ on $L^1(\hat{G}) \otimes_\alpha H^2(\partial\mathcal{D})$ as follows:

$$\begin{aligned} \Xi \; &: L^1(\hat{G}) \otimes_\alpha H^2(\partial\mathcal{D}) \to L^1(\hat{G}) \otimes_\alpha H^{2^\perp}(\partial\mathcal{D}) \\ F &\longmapsto \Xi F = (\hat{I} \otimes P_-)(T_{2in}^* T_1 T_{3cin}^* F) \end{aligned} \quad (61)$$

First, note that $\Xi$ is linear since multiplication by $T_{2in}^* T_1 T_{3cin}^*$ and applying the projection $I \otimes P_-$ are both linear operators. In addition, since $T_{2in}^* T_1 T_{3cin}^* \in L^\infty(\hat{G}, L^\infty(\partial\mathcal{D}))$ that is, $T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta})$ is (essentially) bounded in both variables $\chi$ and $\theta$, the operator $\Xi$ is bounded linear.

We shall show that the optimal $H^\infty$-performance index $\mu$ satisfies:

$$\begin{aligned} \mu &= \sup_{\substack{\alpha(F) \le 1 \\ F \in L^1(\hat{G}) \otimes H^2(\partial\mathcal{D})}} \|\, \Xi F \,\| \\ &= \|\, \Xi \,\| = \|(\hat{I} \otimes P_-)\, T_{2in}^* T_1 T_{3cin}^*\| \end{aligned} \quad (62)$$

That is, $\mu$ is equal to the operator induced norm of $\Xi$. This result is summarized in Theorem 2.
**Theorem 2**: Under the same assumptions as Theorem 1, the optimal $H^\infty$-performance index satisfies:

$$\mu = \|\, \Xi \,\| = \|(\hat{I} \otimes P_-)\, T_{2in}^* T_1 T_{3cin}^*\| \quad (63)$$

**Proof**: The proof of (63) follows from the dual formulation of Theorem 1, in particular, the identity (50) which says:

$$\mu = \sup_{\substack{\gamma(F) \le 1 \\ F \in L^1(\hat{G}) \otimes_\gamma H_0^1(\partial\mathcal{D}) \\ F = \sum_{i=1}^n h_i \otimes g_i}} \left| \int_{\hat{G}} \int_0^{2\pi} T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta}) \sum_{i=1}^n h_i(\chi) \otimes g_i(e^{j\theta}) \, d\theta \, d\chi \right| \quad (64)$$

$$= \sup_{\gamma(\sum_i h_i \otimes g_i) \le 1} \left| \sum_{i=1}^n \int_{\hat{G}} \int_0^{2\pi} h_i(\chi) T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta}) g_i(e^{j\theta}) d\theta d\chi \right| \quad (65)$$

Now since $g_i \in H_0^1(\partial\mathcal{D})$, $i = 1, 2, ..., n$, by the F. Riesz Theorem [20], there exist $g_{i1} \in H^2$, $g_{i2} \in H_0^2$ such that for $i = 1, 2, ..., n$,

$$g_i = g_{i1} \; g_{i2} \;\; and \quad \|g_i\|_{H^1} = \|g_{i1}\|_{H^2} = \|g_{i2}\|_{H^2} \quad (66)$$

Reporting in (65) yields:

$$\mu = \sup_{\gamma(\sum_{i=1}^n h_i \otimes g_i) \le 1} \left| \sum_{i=1}^n \int_{\hat{G}} h_i(\chi) \int_0^{2\pi} T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta}) g_{i1}(e^{j\theta}) g_{i2}(e^{j\theta}) \, d\theta \, d\chi \right| \quad (67)$$

Note $T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta}) g_{i1}(e^{j\theta}) \in L^2(\partial\mathcal{D})$ for each $\chi \in \hat{G}$. Therefore,

$$T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta}) g_{i1}(e^{j\theta}) = P_+ T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta}) g_{i1}(e^{j\theta}) + P_- T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta}) g_{i1}(e^{j\theta}) \tag{68}$$

But then the inner integral satisfies:

$$\int_0^{2\pi} \left[ P_+ T_{2in}^* T_1 T_{3cin}^* \, g_{i1}(e^{j\theta}) + P_- T_{2in}^* T_1 T_{3cin}^* \, g_{i1}(e^{j\theta}) \right] g_{i2}(e^{j\theta}) \, d\theta \tag{69}$$

$$= \int_0^{2\pi} \left( P_- T_{2in}^* T_1 T_{3cin}^* \, g_{i1}(e^{j\theta}) \right) g_{i2}(e^{j\theta}) \, d\theta \tag{70}$$

Thus (67) becomes

$$\mu = \sup_{\gamma(\sum_{i=1}^n h_i \otimes g_{i1} g_{i2}) \leq 1} \Big| \int_{\hat{G}} \int_0^{2\pi} \sum_{i=1}^n h_i(\chi) \otimes P_- T_{2in}^* T_1 T_{3cin}^*(\chi, e^{j\theta}) g_{i1}(e^{j\theta}) \, g_{i2}(e^{j\theta}) d\theta d\chi \Big| \tag{71}$$

$$= \|(\hat{I} \otimes P_-) \, T_{2in}^* T_1 T_{3cin}^*\| = \|\Xi\| \tag{72}$$

A consequence of Theorem 2 is that the optimal Youla parameter satisfies:

$$\begin{aligned} \mu &= \|T_{2in}^* T_1 T_{3cin}^* - T_{2ou} Q_0 T_{3cou}\|_\infty \\ &= \|\hat{I} \otimes P_- T_{2in}^* T_1 T_{3cin}^*\| = \|\Xi\| \end{aligned} \tag{73}$$

and if $F \in L^1(\hat{G}) \otimes_\alpha H^2(\partial\mathcal{D})$ is a maximizing vector for $\Xi$, i.e.,

$$\begin{aligned} \|\Xi F\| &= \|(\hat{I} \otimes P_-) T_{2in}^* T_1 T_{3cin}^*) F\| \\ &= \|(\hat{I} \otimes P_-) T_{2in}^* T_1 T_{3cin}^*\| = \|\Xi\| \end{aligned} \tag{74}$$

Then,

$$T_{2in}^* T_1 T_{3cin}^* F - T_{2ou} Q_o T_{3cou} F = (\hat{I} \otimes P_- T_{2in}^* T_1 T_{3cin}^*) F \tag{75}$$

In other words,

$$T_{2ou} Q_o T_{3cou} F = T_{2in}^* T_1 T_{3cin}^* F - (\hat{I} \otimes P_- T_{2in}^* T_1 T_{3cin}^*) F \tag{76}$$

which yields **an operator equation for $Q_o$**.

## V. CONCLUSIONS AND FUTURE WORKS

In this paper, the duality structure of the optimal $H^\infty$-problem for distributed spatially invariant systems is provided. Under specific assumptions it is shown that an optimal feedback spatially invariant $H^\infty$ controller exists, and can be computed through the Youla parametrization for this class of systems. It is also shown that the optimal $H^\infty$ cost is equal to the operator induced norm of a novel operator defined on a Banach projective tensor space with a mixed $L^1$ defined on the dual group and $H^2$ structure. An operator identity was given to compute the optimal Youla parameter, and thus the optimal controller, provided a maximizing vector exists.

Future works include studying the properties of the optimal solution and deriving numerical algorithms to compute the optimal solution within desired tolerance.

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