AN ISOPERIMETRIC INEQUALITY FOR THE HAMMING CUBE AND SOME CONSEQUENCES

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ABSTRACT. Our basic result, an isoperimetric inequality for Hamming cube Q_n, can be written:

$$\int h_A^\beta d\mu \geq 2\mu(A)(1-\mu(A)).$$

Here μ is uniform measure on $V = \{0,1\}^n$ $(= V(Q_n))$; $\beta = \log_2(3/2)$; and, for $S \subseteq V$ and $x \in V$,

$$h_S(x) = \begin{cases} d_{V\setminus S}(x) & \text{if } x \in S, \\ 0 & \text{if } x \notin S \end{cases}$$

(where $d_T(x)$ is the number of neighbors of x in T).

This implies inequalities involving mixtures of edge and vertex boundaries, with related stability results, and suggests some more general possibilities. One application, a stability result for the set of edges connecting two disjoint subsets of V of size roughly $|V|/2$, is a key step in showing that the number of maximal independent sets in Q_n is $(1+o(1))2n\exp_2[2^{n-2}]$. This asymptotic statement, whose proof will appear separately, was the original motivation for the present work.

1. INTRODUCTION

We write Q_n for the n-dimensional Hamming cube and V for $V(Q_n)$. For $T \subseteq V$ let $d_T(x)$ be the number of neighbors of x in T $(x \in V)$ and define $h_S : V \to \mathbb{N}$ by

$$h_S(x) = \begin{cases} d_{V\setminus S}(x) & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases} \tag{1}$$

For $f : V \to \mathbb{N}$, a probability measure ν on V and $X \subseteq V$, we set

$$\int_X f d\nu = \sum_{x\in X} f(x)\nu(x).$$

We also use $\int$ for $\int_V$.

Our main result is the following isoperimetric inequality. Throughout this paper we use β for $\log_2(3/2)$ $(\approx .585)$ and μ for uniform measure on V. (A few definitions are given in Section 1.1.)

Theorem 1.1. *For any $A \subseteq V$,*

$$\int h_A^\beta d\mu \geq 2\mu(A)(1-\mu(A)). \tag{2}$$

The form of Theorem 1.1 is inspired by the following inequality of Talagrand [12].

Theorem 1.2. *For any $A \subseteq V$,*

$$\int \sqrt{h_A} d\mu \geq \sqrt{2}\mu(A)(1-\mu(A)).$$

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Notice that Theorem 1.1 is tight in two ways: it holds with equality for subcubes of codimensions 1 and 2, and for subcubes of codimension 2 it does not hold for any smaller value of β. As far as we know the $\sqrt{2}$ in Theorem 1.2 could be replaced by 2 when $\mu(A) = 1/2$ (but of course not in general). The difference between 2 and $\sqrt{2}$ wouldn't have mattered in [12], but getting the right constant when $\mu(A)$ is close to 1/2 was crucial for applications, particularly the one in [8] (Theorem 1.8 below) that was our original motivation—see the "stability" result Theorem 1.9 that is the present work's contribution to [8].

Before discussing applications we briefly recall a few basic notions.

1.1. **Definitions.** As usual $[n] = \{1, \ldots, n\}$, $\mathbb{P}$ is the set of positive integers and $x = a \pm b$ means $a - b \leq x \leq a + b$. We use A, B, C and W for subsets of V and E for $E(Q_n)$. For $x \in V$, x_i is (as usual) the ith coordinate of x, and x^i is the vertex obtained from x by flipping x_i. For any A,

$$A^i = \{x^i : x \in A\},$$

the *vertex-boundary* of A is

$$\partial A = \{x \notin A : x \sim y \text{ for some } y \in A\},$$

and the *edge-boundary* of A is

$$\nabla A = \{(x, y) : x \in A, y \notin A\}.$$

We also use

$$\nabla(A, B) = \{(x, y) : x \in A, y \in B\},$$

$$\nabla_i A = \{(x, x^i) : x \in A, x^i \notin A\},$$

$$\nabla_I A = \cup_{i \in I} \nabla A_i \quad (I \subseteq [n]),$$

and

$$\nabla_i(A, B) = \{(x, x^i) : x \in A, x^i \in B\}.$$

We say C is *a codimension k subcube* if there are $I \subseteq [n]$ of size k and $z \in \{0, 1\}^I$ such that

$$C = \{x \in V : x_i = z_i \text{ for all } i \in I\}.$$

1.2. **First application: separating the cube.** Isoperimetric inequalities beginning with Harper [4] (and for edge boundaries also Lindsey [9]) give lower bounds in terms of $|A|$ on the sizes of ∂A and ∇A; e.g.

$$|\nabla A| \geq |A| \log_2(2^n/|A|), \tag{3}$$

with equality iff A is a subcube. We are interested in hybrid versions of these. In what follows we assume (A, B, W) is a partition of V, with W thought of as small. The next two conjectures are a simple illustration of what we have in mind, followed by something general.

Conjecture 1.3. *There is a fixed K such that if $\mu(A) = 1/2$, then*

$$|\nabla(A, B)| + K\sqrt{n}\, |W| \geq 2^{n-1}.$$

With $\partial(a) = \min\{|\partial A| : |A| = a\}$ and $\nabla(a)$ defined similarly, our maximal guess in this direction is:

Conjecture 1.4. *If $|A| = a$, then*

$$|\nabla(A, B)|/\nabla(a) + |W|/\partial(a) \geq 1.$$

Results of Margulis [10] and Talagrand [12] (motivated by [10]) imply tradeoffs between $|\nabla A|$ and $|\partial A|$, but don't seem to help here. Theorem 1.1 implies a weaker version of Conjecture 1.3:

Corollary 1.5. *For A, B, W as in Conjecture 1.3,* $|\nabla(A, B)| + n^{\beta}|W| \geq 2^{n-1}$.

1.3. **Second application: stability for "almost" isoperimetric subsets.** A simple (though now suboptimal) "stability" statement for edge boundaries says:

Theorem 1.6. *For a fixed k, if $|A| = 2^{n-k}$ and $|\nabla A| < (1+\epsilon)|A|\log_2(2^n/|A|)$, then there is a subcube C with $\mu(C\Delta A) = O(\epsilon)$ (where the implied constant depends on k).*

This was proved for $k = 1$ by Friedgut, Kalai and Naor [3]; then for $k = 2, 3$ by Bollobás, Leader and Riordan, who conjectured the general statement (see [1]); and finally in full by Ellis [1]. These all based on Fourier analysis; e.g. at the heart of [1] is Talagrand's extension [11] of [7]. Even stronger, very recent results of Ellis, Keller and Lifshitz [2] are more elementary but rather involved.

Notice that if A is (sufficiently) close to a codimension k subcube then there is an $I \subseteq [n]$ of size k with $\nabla A \approx \nabla_I A$. In fact the implication goes both ways; this follows (more or less) from Theorem 1.6, but is also easy without that machine:

Proposition 1.7. *Assume $|A| = (1 \pm \epsilon)2^{n-k}$ and*

$$|\nabla A \setminus \nabla_I A| \leq \epsilon|A|,$$

where I is a k-subset of $[n]$. Then there is a (codimension k) subcube C with $|A\Delta C| = O(\epsilon)|A|$ (where the implied constant depends on k).

The original motivation for Theorem 1.1 arose in connection with our efforts to prove the following statement, which had been conjectured in [6]. Here $\mathrm{mis}(G)$ is the number of maximal independent sets in the graph G.

Theorem 1.8. $\mathrm{mis}(Q_n) \sim 2n\exp_2[2^{n-2}]$.

The proof of this is completed in [8]. What it needed from isoperimetry (see [8] for the connection) was a variant of Theorem 1.6—really, just of the original result of [3]—of the following type.

If (A, B, W) is a partition of V with $\mu(A), \mu(B) \approx 1/2$ (so W is "small") and $|\nabla(A, B)| \approx 2^{n-1}$, then
$\nabla A \approx \nabla_i A$ for some i.

Of course this depends on quantification; e.g. it can fail with $\mu(W)$ as small as $\Theta(n^{-1/2})$ (let W consist of strings of weight $\lfloor n/2 \rfloor$). Note also that here the full edge boundary of A need *not* be small, since there is no restriction (beyond $n|W|$) on $|\nabla(A, W)|$.

The following consequence of Theorem 1.1 is a (limited) statement of the desired type, the case $k = 1$ of which suffices for [8]. (Recall $\beta = \log_2(3/2)$.)

Theorem 1.9. *For $k \in \{1, 2\}$ the following holds. Suppose (A, B, W) is a partition of V with $\mu(A) = (1 \pm \epsilon)2^{-k}$, $\mu(W) \leq \epsilon n^{-\beta}$ and*

$$|\nabla(A, B)| < (1+\epsilon)k2^{n-k}. \tag{4}$$

Then there is $I \subseteq [n]$ of size k such that

$$|\nabla_i A| = (1 - O(\epsilon))2^{n-k} \;\forall i \in I. \tag{5}$$

Furthermore, there is a codimension k subcube C such that

$$\mu(C\Delta A) = O(\epsilon). \tag{6}$$

Conjecture 1.10. *The statement in Theorem 1.9 holds for all $k \in \mathbb{P}$, even with n^β replaced by $2^n/\partial(|A|)$.*

(The implied constant in (5) and (6) would necessarily depend on k.)

Note Theorem 1.9 implies an isoperimetric statement—similar to those in Section 1.2—of which it is a stability version; namely:

Corollary 1.11. *For $k \in \{1, 2\}$, the assumptions of Theorem 1.9 imply $|\nabla(A, B)| > (1 - O(\epsilon))k2^{n-k}$.*

(And of course similarly for whatever one can establish in the direction of Conjecture 1.10.)

Finally, the next observation provides a general approach to proving something like the statement in Theorem 1.9 for other values of k. (Its proof is similar to the derivation of Theorem 1.9 from Theorem 1.1 and is omitted.)

Theorem 1.12. *Fix $k \in \mathbb{P}$ and suppose there are $f, g : [0, 1] \to \Re^+$ such that (i) g is continuous with $g(2^{-k}) = k2^{-k}$ and (ii) f is increasing and strictly concave, with $f(0) = 0$, $f(k) = k$ and*

$$\int f(h_A)d\mu \geq g(\mu(A)) \;\forall A \subseteq V.$$

Then the conclusions of Theorem 1.9 hold (with implied constants depending on f and g) for A, B, W as in the theorem, except with the bound on w replaced by $w \leq \epsilon/f(n)$.

(For the cases covered by Theorem 1.9, Theorem 1.1 gives the hypothesis of Theorem 1.12 with $f(x)$ equal to x^β when $k = 1$ and $(4/3)x^\beta$ when $k = 2$.)

Theorem 1.1 is proved in Section 2. Section 3 derives the case $k = 1$ of Theorem 1.9 and then indicates the small changes needed for $k = 2$, and in passing derives Corollary 1.5 (see following Corollary 3.2). The easy proof of Proposition 1.7 is given in Section 4.

2. Proof of Theorem 1.1

Lemma 2.1. *Let $X \subseteq V$ and let f be a non-negative real-valued function on V. If*

$$\frac{1}{\mu(X)}\int_X f^\beta d\mu = T^\beta, \tag{7}$$

then

$$\frac{1}{\mu(X)}\int_X (f+1)^\beta d\mu \geq (T+1)^\beta. \tag{8}$$

Proof. Set $g(x) = f^\beta(x)$ for $x \in X$. Then the l.h.s. of (7) is $\mathbb{E}g$ and the l.h.s. of (8) is $\mathbb{E}(g^{1/\beta}+1)^\beta$, where $\mathbb{E}$ refers to uniform measure on X. But $p(x) := (x^{1/\beta}+1)^\beta$ is easily seen to be convex; so, by Jensen's inequality,

$$\mathbb{E}(g^{1/\beta}+1)^\beta \geq ((\mathbb{E}g)^{1/\beta}+1)^\beta,$$

which implies (8). □

The proof of Theorem 1.1 proceeds by induction on n. (This is also true of Theorem 1.2, but beyond this the arguments seem to be different.) It is easy to see that the theorem holds for $n = 1$, so we suppose $n \geq 2$.

Given A, fix an $i \in [n]$. Let

$$V_0 = \{x \in V : x_i = 0\},$$

$$V_1 = \{x \in V : x_i = 1\},$$

$$A_0 = A \cap V_0,$$

and

$$A_1 = (A \cap V_1)^i = \{x^i : x \in A, x_i = 1\} \subseteq V_0.$$

Let μ' be uniform measure on V_0. For simplicity, write h_0 (h_1, h, resp.) for h_{A_0} (h_{A_1}, h_A, resp.), a function on V_0 (V_0, V, resp.).

Let $\mu'(A_0) = a_0$, $\mu'(A_1) = a_1$, and $\mu(A) = a = (a_0+a_1)/2$. Then by induction hypothesis, for $i = 0, 1$,

$$\int h_i^\beta d\mu' \geq 2a_i(1-a_i). \tag{9}$$

We may assume $a_0 \geq a_1$. Note that

$$h(x) = \begin{cases} h_0(x)+1 & \text{if } x \in A_0 \setminus A_1, \\ h_0(x) & \text{if } x \in A_0 \cap A_1, \\ h_1(x^i)+1 & \text{if } x^i \in A_1 \setminus A_0, \\ h_1(x^i) & \text{if } x^i \in A_0 \cap A_1; \end{cases} \tag{10}$$

so

$$\begin{aligned} \int h^\beta d\mu &= \int_{A_0} h^\beta d\mu + \int_{(A_1)^i} h^\beta d\mu \\ &= \int_{A_0} h^\beta d\mu + \int_{A_1\setminus A_0} (h_1+1)^\beta d\mu + \int_{A_0\cap A_1} h_1^\beta d\mu \\ &\geq \int_{A_0} h^\beta d\mu + \int_{A_1} h_1^\beta d\mu \\ &\geq \int_{A_0} h^\beta d\mu + a_1(1-a_1) \end{aligned} \tag{11}$$

(the last inequality by (9)). Thus the theorem will follow if we show

$$\int_{A_0} h^\beta d\mu \geq 2a(1-a) - a_1(1-a_1) = a_0 + a_1^2 - (a_0+a_1)^2/2. \tag{12}$$

The rest of this section is devoted to the proof of (12). Let $Z = \mathrm{supp}(h_0) \setminus A_1$ and $X = \mathrm{supp}(h_0) \cap A_1$ (see Figure 1); thus

$$2\int_{A_0} h^\beta d\mu = \int_Z (h_0+1)^\beta d\mu' + \int_X h_0^\beta d\mu' + \int_{A_0\setminus(A_1\cup Z)} 1 d\mu'. \tag{13}$$

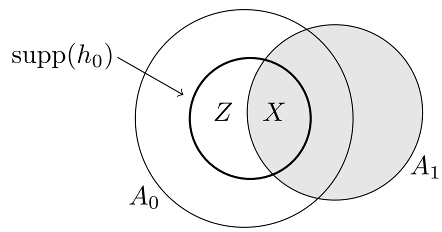

FIGURE 1.

Observation 2.2. *We may assume $A_1 \subseteq A_0$.*

Proof. If there is $x \in A_1 \setminus A_0$ then we can find $y \in A_0 \setminus A_1$ since $\mu'(A_0) \geq \mu'(A_1)$. Let $B_1 = (A_1 \setminus \{x\}) \cup \{y\}$, $B = A_0 \cup (B_1)^i$ and $B_0 = B \cap V_0$ $(= A_0)$. Notice that $|A| = |B|$, $|A_i| = |B_i|$ for $i \in \{0, 1\}$, and

$$\int_{B_0} h_B^\beta d\mu < \int_{A_0} h^\beta d\mu,$$

because: with Z_B (resp. X_B) for $\mathrm{supp}(h_{B_0}) \setminus B_1$ (resp. $\mathrm{supp}(h_{B_0}) \cap B_1$), the location of y changes either from Z to X_B or from $A_0 \setminus (A_1 \cup Z)$ to $(B_0 \cap B_1) \setminus X_B$. In either case its contribution to the r.h.s. of (13) shrinks. So if $A_1 \not\subseteq A_0$, then we can shift it to a "worse" set. □

Let $\sigma = \int_Z h_0^\beta d\mu'$, $\gamma = \int_X h_0^\beta d\mu'$, $\alpha = \sigma + \gamma$ $(=\int h_0^\beta d\mu')$ and $\mu'(Z) = z$. Since $A_1 \subseteq A_0$, the r.h.s. of (13) is

$$\begin{aligned}\int_Z (h_0+1)^\beta d\mu' + \gamma + \mu'(A_0 \setminus (A_1 \cup Z)) &\geq (\sigma^{1/\beta} + z^{1/\beta})^\beta + \gamma + (a_0 - a_1 - z)\\ &= ((\alpha-\gamma)^{1/\beta} + z^{1/\beta})^\beta + \gamma + (a_0 - a_1 - z)\\ &\geq (\alpha^{1/\beta} + z^{1/\beta})^\beta + (a_0 - a_1 - z), \end{aligned} \tag{14}$$

where the first inequality is given by Lemma 2.1 and the second holds because $((\alpha-\gamma)^{1/\beta} + z^{1/\beta})^\beta + \gamma$ is increasing in γ.

So we are done if we show that the expression in (14) is at least

$$2(a_0 + a_1^2) - (a_0 + a_1)^2, \tag{15}$$

where we are entitled to assume

$$\alpha = \int h_0^\beta d\mu' \geq 2a_0(1-a_0). \tag{16}$$

(see (9)) and

$$z \leq \min\{\alpha, a_0 - a_1\} \tag{17}$$

(where the second bound holds since $Z \subseteq A_0 \setminus A_1$). We consider two cases depending on which of $a_0 - a_1$ and the r.h.s. of (16) is smaller.

Case 1. $2a_0(1-a_0) \leq a_0 - a_1$

Equivalently,

$$a_1 \le a_0(2a_0 - 1). \tag{18}$$

Also, since $0 \le a_0(2a_0 - 1)$, we have

$$a_0 \ge 1/2. \tag{19}$$

Note that (14) is decreasing in z and $z \le \alpha$ by (17), so recalling that $2^\beta = 3/2$ and using (16), we find that (14) is at least

$$\alpha/2 + a_0 - a_1 \ge a_0(1 - a_0) + a_0 - a_1. \tag{20}$$

Subtracting (15) from (20) gives

$$-a_1^2 + (2a_0 - 1)a_1,$$

which is nonnegative since

$$f(x, y) := -y^2 + (2x - 1)y \ge 0 \quad \text{for } x \in [\tfrac{1}{2}, 1] \text{ and } y \in [0, x(2x - 1)].$$

(Because: for any $y \ge 0$, $f(x, y)$ is nondecreasing in x, so it is enough to show the inequality holds when $y = x(2x - 1)$, in which case $f(x, y) = x(1 - x)(2x - 1)^2 \ge 0$.)

Case 2. $2a_0(1 - a_0) \ge a_0 - a_1$

Equivalently,

$$a_0(2a_0 - 1) \le a_1 \ (\le a_0). \tag{21}$$

Again using the fact that (14) is decreasing in z, now with $z \le a_0 - a_1$ by (17), we find that (14) is at least

$$(\alpha^{1/\beta} + (a_0 - a_1)^{1/\beta})^\beta, \tag{22}$$

which, in view of (16) (and the fact that (22) is increasing in α), is at least

$$((2a_0(1 - a_0))^{1/\beta} + (a_0 - a_1)^{1/\beta})^\beta. \tag{23}$$

Thus the proof that (14) is at least (15) in the present case is completed by the following proposition (applied with $x = a_0$ and $y = a_1$).

Proposition 2.3. *Let*

$$g(x, y) = ((2x(1 - x))^{1/\beta} + (x - y)^{1/\beta})^\beta - 2(x + y^2) + (x + y)^2.$$

Then $g(x, y) \ge 0$ *for* $x, y \in [0, 1]$ *with* $y \in [x(2x - 1), x]$.

Proof. Observe that for $x \in [0, 1]$,

$$g(x, x(2x - 1)) = x(1 - x)(2x - 1)^2 \ge 0, \tag{24}$$

and

$$g(x, x) = 0. \tag{25}$$

Also, the partial derivative of $g(x, y)$ with respect to y is

$$g_y(x, y) = -(x - y)^{\frac{1}{\beta} - 1}((2x(1 - x))^{\frac{1}{\beta}} + (x - y)^{\frac{1}{\beta}})^{\beta - 1} + 2(x - y).$$

Now, we claim that

(26) for given $x \in [0,1]$, $g_y(x,y)$ is equal to zero for at most one $y \in [x(2x-1), x)$.

Indeed, let $A = x - y\ (> 0)$ and $B = 2x(1-x)$. Then

$$g_y(x,y) = 0 \Leftrightarrow A^{\frac{1}{\beta}} + B^{\frac{1}{\beta}} = 2^{\frac{1}{\beta-1}} A^{\frac{2\beta-1}{\beta(\beta-1)}}. \tag{27}$$

Notice that $A^{\frac{1}{\beta}} + B^{\frac{1}{\beta}}$ is increasing in A while $2^{\frac{1}{\beta-1}} A^{\frac{2\beta-1}{\beta(\beta-1)}}$ is decreasing in A (since $\frac{2\beta-1}{\beta(\beta-1)} < 0$). So we conclude that for any B, (27) holds at most once, which is (26).

Finally, we claim that

(28) for each $x \in (0,1)$, there is $c = c(x) > 0$ such that $g(x,y) > 0$ for all $y \in (x-c, x)$.

Note that Proposition 2.3 follows from the combination of (24), (25), (26), and (28).

Proof of (28). Given $x \in (0,1)$ and $c \in (0,x)$,

$$g(x, x-c) = ((2x(1-x))^{\frac{1}{\beta}} + c^{\frac{1}{\beta}})^{\beta} + 2x^2 - 2x - c^2,$$

so

$$g(x, x-c) > 0 \Leftrightarrow ((2x(1-x))^{\frac{1}{\beta}} + c^{\frac{1}{\beta}})^{\beta} > c^2 + 2x(1-x). \tag{29}$$

Now,

$$((2x(1-x))^{\frac{1}{\beta}} + c^{\frac{1}{\beta}})^{\beta} = 2x(1-x)\left(1 + \left(\frac{c}{2x(1-x)}\right)^{\frac{1}{\beta}}\right)^{\beta},$$

and if $c = c(x)$ is small enough,

$$\begin{aligned}\left(1 + \left(\frac{c}{2x(1-x)}\right)^{\frac{1}{\beta}}\right)^{\beta} &= \exp[\Theta(c^{1/\beta})\beta] \\ &= 1 + \Theta(c^{1/\beta}),\end{aligned}$$

which implies (29). □

3. Proof of Theorem 1.9

As noted at the end of Section 1.3, we prove Theorem 1.9 for $k = 1$ and then indicate what changes for $k = 2$. This seemed to us slightly clearer than proving them together, though the differences are minor. Extending to Theorem 1.12 is straightforward, though the counterpart of Proposition 3.3 is slightly more painful than the original.

As usual, $A \subseteq V$ is *increasing* if $x \in A$ and $y \geq x$ (with respect to the product order on V) imply $y \in A$ (and A is *decreasing* is defined similarly). For x, y with $x < y$, we write $x \lessdot y$ if $x \leq z \leq y$ implies $z \in \{x, y\}$. We will need Harris' Inequality [5]:

Theorem 3.1. *For any product measure ν on Q_n and increasing $A, B \subseteq V$,*

$$\nu(A \cap B) \geq \nu(A)\nu(B).$$

Recall that h_S was defined in (1) and, for disjoint $A, B \subseteq V$, set

$$h_{AB}(x) = \begin{cases} d_B(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A; \end{cases}$$

thus

$$\int_A h_{V\setminus B} d\mu = \int h_{AB}\, d\mu = 2^{-n}|\nabla(A,B)|.$$

We need the following easy consequence of Theorem 1.1.

Corollary 3.2. *If (R,S,U) is a partition of V with $\mu(R\cup U) = \alpha$, then*

$$(2^{-n}|\nabla(R,S)| = \int_R h_{R\cup U} d\mu \geq) \int_R h^\beta_{R\cup U} d\mu \geq 2\alpha(1-\alpha) - n^\beta \mu(U).$$

Proof. Theorem 1.1 gives

$$2\alpha(1-\alpha) \leq \int h^\beta_{R\cup U} d\mu = \int_R h^\beta_{R\cup U} d\mu + \int_U h^\beta_{R\cup U} d\mu \leq \int_R h^\beta_{R\cup U} d\mu + n^\beta \mu(U),$$

and the corollary follows. □

In particular, taking $(R,S,U) = (B,A,W)$ gives Corollary 1.5. □

We now assume the situation of Theorem 1.9. Note that each of $\mu(A), \mu(B)$ is $1/2 \pm O(\epsilon)$. In what follows we (abusively) use "a.e." to mean "all but an $O(\epsilon)$-fraction," so for example write "a.e. $x \in A$ satisfies Q" for "Q holds for all but an $O(\epsilon)$-fraction of the members of A."

Proposition 3.3. *For a.e. $x \in A$, $h_{AB}(x) = 1$.*

Proof. Applying Corollary 3.2 with $(R,S,U) = (A,B,W)$ (and using (4)) gives

$$(1+\epsilon)/2 \geq \int h_{AB} d\mu = \int_A h_{A\cup W} d\mu \geq \int_A h^\beta_{A\cup W} d\mu = 1/2 - O(\epsilon). \tag{30}$$

In particular, $\int (h_{AB} - h^\beta_{AB}) d\mu = O(\epsilon)$, which, since $\int (h_{AB} - h^\beta_{AB}) d\mu = \Omega\left(\mu(\{x \in A : h_{AB}(x) \notin \{0,1\}\}\right)$, implies $h_{AB}(x) \in \{0,1\}$ for a.e. $x \in A$. □

The next observation will allow us to assume that A is increasing and B is decreasing.

Proposition 3.4. *For any partition (A,B,W) of V there is another partition (A',B',W') satisfying:*

(1) $\mu(X) = \mu(X')$ for $X \in \{A,B,W\}$;
(2) A' is increasing and B' is decreasing;
(3) $|\nabla_i(A,B)| \geq |\nabla_i(A',B')|$ for all $i \in [n]$.

Proof. This is a typical "shifting" argument and we will be brief. For $i \in [n]$, the *i-shift* of a partition (A,B,W) is defined thus: let

$$V_0 = \{x \in V : x_i = 0\},\ V_1 = \{x \in V : x_i = 1\},$$

and for each $x \in V_0$ with $(x, x^i) \in (A,B), (A,W),$ or (W,B), switch the affiliations of x and x^i. This trivially does not change $|\nabla_i(A,B)|$, and it's easy to see that it does not increase $|\nabla_j(A,B)|$ for $j \in [n]\setminus\{i\}$. (Consider the contribution to $\nabla_j(A,B)$ of any quadruple $\{x, x^i, x^j, (x^i)^j\}$.)

It is also clear that no sequence of nontrivial shifts can cycle (e.g. since any such shift strictly increases $\sum_{x\in A}|x| - \sum_{x\in B}|x|$); so there is a sequence that arrives at an (A', B', W') stable under i-shifts (for all i), and this meets the requirements of the proposition. $\square$

Proof of Theorem 1.9. We first show there is an i as in (5). By Proposition 3.4, we may assume A is increasing and B is decreasing. For each $i \in [n]$, let $A_i = \{x \in A : x^i \in B\}$, and notice that

$$A_i \text{ is a decreasing subset of } A. \tag{31}$$

Indeed, given $x \in A_i$, consider any $y \in A$ satisfying $y \lessdot x$. Then $y^i \in B$ since $x^i \in B$ and B is decreasing, so $y \in A_i$.

By proposition 3.3,

$$\text{a.e. } x \in A \text{ is in exactly one } A_i; \tag{32}$$

in particular, if we let $A_0 = \{x \in A : d_B(x) = 0\}$, then $\mu(A_0) = O(\epsilon)$.

Setting $\max \mu(A_i) = \mu(A) - \delta$, we just need to show that $\delta = O(\epsilon)$.

Assume (w.l.o.g.) that $\max \mu(A_i) = \mu(A_1)$, and let $\tilde{A} = \cup_{i\neq 1} A_i$, $C_1 = A \setminus A_1$, and $\tilde{C} = A \setminus \tilde{A}$. By (32),

$$\mu(\tilde{C}) \geq \mu(A_1) - O(\epsilon), \tag{33}$$

while $C_1 \cap \tilde{C} = A_0$ implies

$$\mu(C_1 \cap \tilde{C}) = O(\epsilon).$$

Moreover, (31) and the fact that A is increasing imply that C_1 and $\tilde{C}$ are increasing (in V); so Theorem 3.1 gives

$$O(\epsilon) = \mu(C_1 \cap \tilde{C}) \geq \mu(C_1)\mu(\tilde{C}) \geq \delta(\mu(A) - \delta - O(\epsilon)), \tag{34}$$

whence

$$\delta = O(\epsilon) \ \text{ or } \ \mu(A) - \delta - O(\epsilon) = O(\epsilon).$$

But $\delta = O(\epsilon)$ is what we want, so we may assume for a contradiction that $\mu(A) - \delta - O(\epsilon) = O(\epsilon)$; equivalently, $\mu(A_1) = O(\epsilon)$. In this case, $\mu(A_i) = O(\epsilon)$ for all i, so there is a partition $[n] = I \cup J$ such that each of A_I $(:= \cup_{i\in I} A_i)$ and A_J has measure $\mu(A)/2 + O(\epsilon)$. But then, setting $C_I = A \setminus A_I$ and $C_J = A \setminus A_J$, and again using Theorem 3.1, we have

$$O(\epsilon) = \mu(C_I \cap C_J) \geq \mu(C_I)\mu(C_J) \geq \mu^2(A)/4 - O(\epsilon),$$

which is impossible. $\square$

For (6), let i be as above and for $\pi \in \{0,1\}$, let $C(i,\pi) = \{v : v_i = \pi\}$. If D is one of these subcubes then with $|A \cap D| = \delta 2^{n-1}$, Corollary 3.2 (applied in D with $R = A \cap D$ and $U = W \cap D$) gives at least $[2\delta(1-\delta) - O(\epsilon)]2^{n-1}$ edges in $\nabla(A,B) \setminus \nabla_i A$, which with (4) and (5) forces δ to be either $O(\epsilon)$ or $1 - O(\epsilon)$. So exactly one, say C, has $\delta = 1 - O(\epsilon)$, and this C satisfies (6). $\square$

Changes for $k = 2$ (briefly). The only changes are to Proposition 3.3 and the final argument(s). For the former, the statement is now:

$$\text{for a.e. } x \in A, \ \ h_{AB}(x) = 2.$$

Set $f(x) = (4/3)x^\beta$. Theorem 1.1 gives $\int f(h_{A\cup W})d\mu \geq 1/2 - O(\epsilon)$, leading to

$$\int f(h_{AB})d\mu \geq 1/2 - O(\epsilon).$$

Now let $X(x) = h_{AB}(x)$ for $x \in A$ and write $\mathbb{E}$ for expectation w.r.t. uniform measure on A. Our assumptions on $\mu(A)$ and $|\nabla(A,B)|$ give

$$\mathbb{E}X = \frac{1}{\mu(A)}\int h_{AB}d\mu = \frac{|\nabla(A,B)|}{\mu(A)2^n} \leq 2 + O(\epsilon),$$

so, using the concavity of f, we have

$$\int f(h_{AB})d\mu = \mu(A)\mathbb{E}f(X) \leq \mu(A)f(\mathbb{E}X) \leq 1/2 + O(\epsilon).$$

It's then easy to see (if somewhat annoying to write) that concavity of f, with $\mathbb{E}f(X) - f(\mathbb{E}X) = O(\epsilon)$ and $f(\mathbb{E}X) = 2 \pm O(\epsilon)$ (and $X \in \mathbb{Z}$) implies, first, that there is a c such that $f(x) = c$ for a.e. $x \in A$, and, second, that $c = 2$.

For the step leading to (5) we may as well think of a general k. Thus we assume A and B are increasing and decreasing (resp.), with $n^\beta\mu(W) \leq \epsilon$, $\mu(A) = (1 \pm \epsilon)2^{-k}$, $|\nabla(A,B)| < (1+\epsilon)k2^{n-k}$, and $h_{AB}(x) = k$ for a.e. $x \in A$, and want to show

there is $I \subseteq [n]$ of size k such that $|\nabla_i A| \geq (1 - O(\epsilon))2^{n-k}\ \forall i \in I$.

Here for each k-subset I of $[n]$ we set

$$A_I = \{x \in A : x^i \in B\ \forall i \in I\}.$$

Each A_I is decreasing in A and a.e. $x \in A$ is in exactly one A_I. We then assume $\max_I \mu(A_I) = \mu(A_{[k]}) = \mu(A) - \delta$ and continue essentially as before.

The step yielding (6) again takes no extra effort for general k: here we have 2^k subcubes corresponding to the members of $\{0,1\}^k$, and Corollary 3.2 (with (4) and (5)) shows that all but one of these meet A in sets of size $O(\epsilon)2^{n-k}$ (and the one that doesn't is the promised C).

4. Proof of Proposition 1.7

Let $|A| = a$. For $z \in \{0,1\}^I$ let $V_z = \{x : x_i = z_i\ \forall i \in I\}$, $A_z = A \cap V_z$, $a_z = |A_z|$ and $\alpha_z = a_z/a$. Assume (w.l.o.g.) that a_z is maximum when $z = \underline{0}$. We have

$$\begin{aligned}\epsilon a \geq |\nabla A \setminus \nabla_I A| &= \sum_z |\nabla(A_z, V_z \setminus A_z)| \geq \sum_z a_z \log_2(2^{n-k}/a_z)\\ &= a\left[H(\alpha_z : z \in \{0,1\}^I) + \log_2(2^{n-k}/a)\right] = aH(\alpha_z : z \in \{0,1\}^I) + O(\epsilon)a,\end{aligned}$$

where H is binary entropy and the inequality is given by (3). It follows that each α_z is either $O(\epsilon/\log(1/\epsilon))$ or $1 - O(\epsilon)$; so in fact $\alpha_{\underline{0}} = 1 - O(\epsilon/\log(1/\epsilon))$ and $V_{\underline{0}}$ is the promised subcube.

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